

C_2 -cofiniteness of finite-group orbifolds

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Abstract

Let V be a simple vertex algebra with a connected nonnegative integer grading, and let G be a finite group of grading-preserving automorphisms. We prove that C_2 -cofiniteness of V implies C_2 -cofiniteness of V^G . The proof bounds the length and height of monomials in strong generators representing classes in a relative C_2 quotient. The length bound follows from a good filtration and finite-group invariant theory. The height bound follows from differential equations for products of fields, together with identities for invariants and vacuum coefficients. The height bound also shows that finite strong generation passes to V^G , without assuming C_2 -cofiniteness. For strongly rational vertex operator algebras, the cofiniteness result and McRae's orbifold theorem imply strong rationality of every finite-group fixed-point algebra.

1 Introduction

Taking invariants under a finite group of automorphisms, or forming *orbifolds*, is one of the most basic constructions in the theory of vertex operator algebras and conformal field theory [2, 3, 6, 7]. A longstanding conjecture predicts that finite-group fixed-point subalgebras of strongly rational vertex operator algebras are again strongly rational.

The C_2 condition, introduced by Zhu in his work on modular invariance [14], is central to this conjecture. For a vertex algebra A , write

$$C_2(A) = \text{span}_{\mathbb{C}}\{a_{-2}b : a, b \in A\}, \quad R_A = A/C_2(A).$$

The quotient R_A is a commutative algebra under $[a][b] = [a_{-1}b]$, and A is C_2 -cofinite when R_A is finite dimensional. For vertex operator algebras, this condition implies that Zhu's algebra is finite dimensional [3, Proposition 3.6], and hence that there are only finitely many irreducible admissible modules up to isomorphism [14, Theorem 2.2.2].

Substantial progress on the conjecture was made by Miyamoto, who proved C_2 -cofiniteness for cyclic orbifolds of simple C_2 -cofinite vertex operator algebras of CFT type, and hence for all finite solvable groups [11]. Carnahan and Miyamoto subsequently established regularity of fixed-point subalgebras under finite solvable groups for simple nonnegatively graded regular vertex operator algebras with a nondegenerate invariant bilinear form [1]. These results settled the solvable case of the conjecture, while the general nonsolvable case remained open.

For arbitrary finite groups, McRae proved that a fixed-point algebra of a strongly rational vertex operator algebra is strongly rational as soon as it is C_2 -cofinite [10, Corollary 4.23]. This reduces the strong-rationality problem to C_2 -cofiniteness.

We prove that C_2 -cofiniteness passes to fixed points for every finite group. We work over $\mathbb{C}$ throughout. More precisely, let V be a simple vertex algebra with a *connected nonnegative integer grading*, meaning

$$V = \bigoplus_{n \geq 0} V_n, \quad V_0 = \mathbb{C}\mathbf{1}, \quad a_m V_b \subseteq V_{h+b-m-1} \quad (a \in V_h),$$

with $V_n = 0$ for $n < 0$. Let G be a finite group of grading-preserving automorphisms. Under these hypotheses, we prove the following.

Theorem 1.1. *If V is C_2 -cofinite, then V^G is C_2 -cofinite.*

In particular, the theorem applies to simple C_2 -cofinite vertex operator algebras of CFT type. McRae's theorem then implies that every finite-group orbifold of a strongly rational vertex operator algebra is strongly rational (Corollary 5.1).

We also prove that finite strong generation passes to fixed points, without assuming C_2 -cofiniteness.

Theorem 1.2. *Let V be a simple, strongly finitely generated vertex algebra with a connected nonnegative integer grading. For every finite group G of grading-preserving automorphisms, V^G is strongly finitely generated.*

The proof is given in Section 5. This resolves the finite-group case of the vertex algebra Hilbert problem in this setting [9, Problem 7.1].

Notation. For a formal power or Laurent series with vector coefficients, we write

$$[z^\alpha] \left(\sum_{\gamma} v_{\gamma} z^{\gamma} \right) = v_{\alpha}, \quad z^{\alpha} = z_1^{\alpha_1} \cdots z_m^{\alpha_m}.$$

Coefficient extraction leaves any unextracted variables unchanged. In particular, $[t^j]$ extracts the coefficient of t^j , and $[d^0]$ extracts the constant coefficient in d . For $\alpha \in \mathbb{Z}_{\geq 0}^m$, we use $|\alpha| = \sum_i \alpha_i$, $\alpha! = \prod_i \alpha_i!$, and $\partial_z^{\alpha} = \prod_i \partial_{z_i}^{\alpha_i}$; inequalities between multi-indices are componentwise. Write $[n] = \{1, \dots, n\}$ and $I^c = [n] \setminus I$ for $I \subseteq [n]$. Linear maps on coefficient spaces act coefficientwise on formal series.

Proof strategy. The proof has four steps.

1. **Pass to a relative C_2 -quotient.** Regard V as a V^G -module and consider

$$Q = R_{V^G}(V) = V / \text{span}\{b_{-2}v : b \in V^G, v \in V\}.$$

Taking G -invariants gives

$$Q^G \simeq R_{V^G}(V^G) = V^G / C_2(V^G).$$

Thus it suffices to prove that Q is finite dimensional.

2. **Choose homogeneous strong generators.** Since V is C_2 -cofinite, choose homogeneous positive-weight strong generators $u^1, \dots, u^d$ with G -stable span U . Their monomials give the spanning classes

$$[u_{-n_1}^{i_1} \cdots u_{-n_\ell}^{i_\ell} \mathbf{1}] \in Q, \quad \ell \geq 0, \quad 1 \leq i_j \leq d, \quad n_j \geq 1,$$

where $\ell = 0$ gives the vacuum class. We need two bounds:

- (a) **Length.** Find L such that classes with $\ell \leq L$ already span Q .
- (b) **Height.** For nonempty monomials of length at most L , find H such that their classes vanish whenever $\max_j n_j > H$.

3. **Bound the length.** This is the more direct part of the argument. A good filtration, defined by the sum of the generating states' weights and independent of the mode indices, reduces it to a calculation in a commutative differential algebra. Modulo the ideal generated by derivatives of invariants, integration by parts moves all derivatives off a chosen factor in each projected monomial. Nilpotence supplied by C_2 -cofiniteness, together with finite-group invariant theory, then gives the length bound.

4. **Bound the height.** This is the main difficulty. We proceed in the following order.

- (a) **Differential equations.** For $n \geq 1$, consider the n -point functions $Y(a_1, z_1) \cdots Y(a_n, z_n) \mathbf{1}$, with the fields labeled by $[n] := \{1, \dots, n\}$. Multiplying by suitable powers of $z_i - z_j$ and passing to Q gives a power series $H_Q(z_1, \dots, z_n)$. For each nonempty subset $I \subseteq [n]$, we separate the fields labeled by I from those labeled by I^c . Weak associativity combines the selected fields at a common coordinate t , with their relative positions fixed. The factors joining I to I^c remain as a scalar polynomial, which we call the *cross factor*. After taking invariants in the selected factors, the relative C_2 relation kills the positive powers of t in the iterated field. The cross factor therefore bounds the powers of t in the full expression from above, while regularity excludes negative powers. Its degree determines the differential equation

$$\left(\sum_{i \in I} \partial_{z_i} \right)^{d_I} (H_Q \circ P_I) = 0,$$

where P_I projects onto invariants in the selected factors and d_I is one more than the t -degree of the cross factor.

- (b) **Vacuum pairs.** We may take the G -action to be faithful. For each $g \neq 1$, choose homogeneous a_g with $u_g = (1 - g)a_g \neq 0$. Simplicity gives a homogeneous w_g with $(u_g)_{\text{wt } u_g + \text{wt } w_g - 1} w_g = \mathbf{1}$. To each tensor product of homogeneous components of U , we adjoin sufficiently many copies of the G -spans of a_g and w_g .
- (c) **A polynomial module and its support.** Form the universal module M for the differential relations on this enlarged tensor product over $R = \mathbb{C}[p_1, \dots, p_n]$, with p_i representing differentiation. Let $\mathcal{N}$ be the submodule generated by tensors with the chosen auxiliary states fixed. Finite-group identities show that $\mathcal{N}$ is supported at the origin. Since $\mathcal{N}$ is finitely generated, some power of the maximal ideal at that point annihilates it:

$$\mathfrak{m}_0^{q_0} \mathcal{N} = 0, \quad \mathfrak{m}_0 = (p_1, \dots, p_n).$$

- (d) **Recover the height bound.** The differential equations and this annihilation relation force the enlarged series to have bounded total degree when evaluated on tensors with the auxiliary states fixed. Taking the vacuum term of each auxiliary pair removes the extra fields and transfers the resulting weight bound to the original products. The mode weight formula then bounds $\max_j n_j$.

The length and height bounds leave only finitely many choices of ℓ , i_j , and n_j . Hence Q is finite dimensional, and taking invariants gives C_2 -cofiniteness of V^G .

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2 The length bound

Throughout, V is a simple vertex algebra with a connected nonnegative integer grading, and G is a finite group of grading-preserving automorphisms. We may replace G by its image in $\text{Aut}(V)$ without changing V^G , and hence assume the action is faithful. Write $r = |G|$, $B = V^G$, and $P_G = r^{-1} \sum_{g \in G} g$ for the Reynolds projection. We use the mode convention $Y(a, z) = \sum_{m \in \mathbb{Z}} a_m z^{-m-1}$ and write T for translation. For homogeneous $a \in V_h$, write $\text{wt } a = h$. Normally ordered products are associated to the right.

In this section, assume that R_V is finite dimensional.

2.1 Strong generators

A subset $S \subset V$ *strongly generates* V if V is spanned by the monomials

$$W = u_{-n_1}^1 \cdots u_{-n_\ell}^\ell \mathbf{1}, \quad \ell \geq 0, \quad u^i \in S, \quad n_i \geq 1. \quad (1)$$

The empty product is the vacuum. The elements of S are called *strong generators*. If S can be chosen finite, V is *strongly finitely generated*. A subspace strongly generates V if its elements do. For homogeneous positive-weight u^i , the monomial (1) has *length* ℓ and weight

$$\text{wt } W = \sum_i \text{wt } u^i + \sum_i (n_i - 1). \quad (2)$$

For a nonempty monomial, its *height* is $\max_i n_i$; in particular, $\text{ht}(W) \leq \text{wt } W$. The identity $(T^{n-1}u)_{-1} = (n-1)!u_{-n}$ expresses these monomials as normally ordered products of derivatives of the generators, up to nonzero scalar factors; see [9, Section 3].

Write $R_{V,+} = \bigoplus_{w>0} (R_V)_w$. Let s be a graded linear section of the quotient map on $R_{V,+}$. Taking invariants in the space of linear maps gives the equivariant section

$$s_G(x) = \frac{1}{r} \sum_{g \in G} g s(g^{-1}x), \quad U = s_G(R_{V,+}) \subset V_{>0}.$$

Thus U is a finite-dimensional graded G -stable space of lifts. Let

$$C_1(V) = TV + \text{span}\{a_{-1}b : a, b \in V_{>0}\}.$$

The identities $a_{-2}b = (Ta)_{-1}b$ and $T\mathbf{1} = 0$ give $C_2(V) \subseteq C_1(V)$. Hence $V_{>0} = U + C_1(V)$, and induction on weight shows that U strongly generates V ; see [8, Theorem 4.10]. In particular, each V_w is finite dimensional: after choosing a finite homogeneous basis of U , both the length and the sum of the derivative orders of a monomial of weight w are bounded.

2.2 The relative quotient

For a vertex algebra W and a W -module M , write

$$R_W(M) = M / \text{span}\{w_{-2}m : w \in W, m \in M\}.$$

In particular, $R_W(W) = R_W$. We regard V as a module over $B = V^G$ and denote $R_B(V) = R_{V^G}(V)$ by Q for brevity. Thus

$$D_B = \text{span}\{b_{-2}v : b \in B, v \in V\}, \quad Q = V/D_B, \quad q : V \longrightarrow Q. \quad (3)$$

Both D_B and Q are graded. Translation gives $(T^{m-2}b)_{-2} = (m-1)!b_{-m}$, so

$$q(b_{-m}v) = 0 \quad (b \in B, m \geq 2). \quad (4)$$

Applying P_G to $\sum_j b_{-2}^j v_j \in D_B \cap B$ replaces each v_j by $P_G v_j \in B$. Hence

$$D_B \cap B = C_2(B), \quad Q^G \simeq R_B. \quad (5)$$

The second identification uses exactness of finite-group invariants. It is therefore enough to show that Q is finite dimensional.

2.3 The length estimate

Lemma 2.1. *Let A be a commutative G -algebra and J a G -stable ideal. Then*

$$J^r \subseteq AJ^G,$$

where AJ^G is the ideal generated by the invariant elements of J .

Proof. For $x \in J$, the nonleading coefficients of $\prod_{g \in G}(t - gx)$ lie in J^G . Evaluating at $t = x$ therefore gives $x^r \in AJ^G$. Replace x by $t_1 x_1 + \cdots + t_r x_r$ and compare the coefficient of $t_1 \cdots t_r$. It is $r!x_1 \cdots x_r$, and $r! \neq 0$. $\square$

The positive-weight ideal $R_{V,+}$ of the finite-dimensional, connected graded algebra R_V is nilpotent. Choose $\nu \geq 1$ with

$$R_{V,+}^\nu = 0. \quad (6)$$

We filter by the sum of the weights of the generating states, with no contribution from the mode indices. For $p \geq 0$, define

$$F_p V = \text{span} \left\{ u_{-n_1}^1 \cdots u_{-n_\ell}^\ell \mathbf{1} : \sum_i \text{wt } u^i \leq p \right\}. \quad (7)$$

Here $\ell \geq 0$, the $u^i \in U$ are homogeneous, and $n_i \geq 1$. Set $F_{-1}V = 0$, so $F_0V = \mathbb{C}\mathbf{1}$. The sum $\sum_i \text{wt } u^i$ differs from both the length ℓ and the weight $\sum_i \text{wt } u^i + \sum_i (n_i - 1)$ of the monomial. To check that translation preserves this filtration, use $T\mathbf{1} = 0$ and $[T, u_{-n}] = nu_{-n-1}$:

$$T(u_{-n_1}^1 \cdots u_{-n_\ell}^\ell \mathbf{1}) = \sum_{j=1}^{\ell} n_j u_{-n_1}^1 \cdots u_{-n_{j-1}}^j \cdots u_{-n_\ell}^\ell \mathbf{1}.$$

Each summand uses the same generating states, so $\sum_i \text{wt } u^i$ is unchanged. Hence $TF_p V \subseteq F_p V$, although translation raises the weight in V by one. For example, $T^j u \in F_h V \cap V_{h+j}$ for $u \in U \cap V_h$. The length bound below concerns the number ℓ of factors.

By [8, Theorem 4.11], this is a good increasing filtration:

$$(F_p V)_m F_q V \subseteq \begin{cases} F_{p+q} V, & m < 0, \\ F_{p+q-1} V, & m \geq 0. \end{cases}$$

Thus $A = \text{gr}_F V$ is a commutative differential algebra. For homogeneous $u \in U$, let $\bar{u}$ be its class in $F_{\text{wt } u} V / F_{\text{wt } u-1} V$. The elements $T^j \bar{u}$, $j \geq 0$, generate A , where T denotes the induced derivation. Each $F_p V$ is graded by the original weight grading and is G -stable. The two degrees are related by

$$V_w \subseteq F_w V, \quad C_2(V) \cap V_w \subseteq F_{w-1} V. \quad (8)$$

The first inclusion follows from (2). For $a \in V_h$ and $b \in V_k$, the filtration properties give $a_{-2}b = (Ta)_{-1}b \in F_{h+k} V$. This vector lies in V_{h+k+1} , proving the second inclusion.

Lemma 2.2 (Length bound). *Every class in Q is represented by a linear combination of monomials (1) in U of length at most*

$$L = r\nu - 1.$$

Proof. Let $\mathfrak{a} \subset A$ be the ideal generated by the symbols of elements of U . We first show

$$\mathfrak{a}^\nu = 0. \quad (9)$$

Take ν homogeneous elements of U , of total weight w . Their normally ordered product lies in $C_2(V)$ by (6), so its symbol in filtration degree w vanishes by (8). This proves (9).

Consider the commutative algebra

$$\bar{A} = A/AT(A^G), \quad J = (\bar{A})_{>0},$$

where the subscript denotes positive filtration degree. We claim

$$(J^G)^\nu = 0. \quad (10)$$

Taking invariants gives

$$\bar{A}^G \simeq A^G / (A^G T(A^G)),$$

since projecting the coefficients in $\sum_j a_j T(f_j)$, $f_j \in A^G$, onto invariants shows that $(AT(A^G))^G = A^G T(A^G)$. Every positive-degree invariant is obtained by applying P_G to a sum of monomials in $T^j \bar{u}$. The identity

$$P_G((Ta)b) = TP_G(ab) - P_G(a(Tb))$$

gives, by iteration,

$$P_G((T^k \bar{u})b) \equiv (-1)^k P_G(\bar{u} T^k b) \pmod{AT(A^G)}.$$

Since U is G -stable, the right side belongs to $\mathfrak{a} \cap A^G$. Thus every element of J^G has a representative in $\mathfrak{a} \cap A^G$. Products of ν such representatives vanish by (9), proving (10).

Lemma 2.1 now gives

$$J^\nu \subseteq \bar{A}(J^G)^\nu = 0.$$

Thus $\bar{A}$ is spanned by monomials in $T^j \bar{u}$ of length at most $r\nu - 1$.

Give D_B and Q the induced filtrations. Taking invariants gives $\text{gr}_F B = A^G$. If $b \in B \cap F_p V$, $v \in F_s V$, and σ_j denotes the class in $F_j V / F_{j-1} V$, then

$$\sigma_s(v) T \sigma_p(b) = \sigma_{p+s}(b_{-2} v) \in \text{gr}_F D_B.$$

Consequently

$$AT(A^G) \subseteq \text{gr}_F D_B, \quad \bar{A} \twoheadrightarrow \text{gr}_F Q \quad (11)$$

as graded vector spaces. Lift a homogeneous spanning expression in $\bar{A}$ to normally ordered products of derivatives of elements of U . Subtracting these lifts from a representative in Q lowers its filtration degree. Induction on filtration degree proves the length bound; the lifts may be chosen to preserve the original weight. $\square$

It remains to bound the heights of these monomials. We do this by bounding the weights in Q of coefficients in products of a fixed number of fields.

3 The differential equations

3.1 n -point functions

Fix $n \geq 1$ and finite-dimensional homogeneous G -stable subspaces $E_i \subset V_{h_i}$, $1 \leq i \leq n$, and put $E = \bigotimes_{i=1}^n E_i$. For the products $Y(a_1, z_1) \cdots Y(a_n, z_n) \mathbf{1}$, set

$$\Delta(z) = \prod_{i < j} (z_i - z_j)^{h_i + h_j}. \quad (12)$$

For $e = a_1 \otimes \cdots \otimes a_n$, define

$$H_V(z)(e) = \Delta(z) Y(a_1, z_1) \cdots Y(a_n, z_n) \mathbf{1}. \quad (13)$$

Write $H_Q = q \circ H_V$. The ordered product belongs to the iterated Laurent space $V((z_1)) \cdots ((z_n))$. The factor Δ makes it a power series in all variables.

Lemma 3.1 (Regularity). *The series H_V belongs to $\text{Hom}_{\mathbb{C}}(E, V[[z_1, \dots, z_n]])$. Every coefficient of total degree d has weight*

$$\sum_i h_i + d - \deg \Delta. \quad (14)$$

Consequently, the projection of H_V to each V_w is a homogeneous polynomial.

Proof. The grading gives $a_m b = 0$ for $m \geq \text{wt } a + \text{wt } b$, so $h_i + h_j$ is a locality exponent for the i th and j th fields. After multiplication by Δ , we may reorder the fields. Moving the i th field next to the vacuum gives $Y(a_i, z_i) \mathbf{1} = e^{z_i T} a_i$, excluding negative powers of z_i . Doing this for each i proves regularity. The mode weight rule gives (14); projection to V_w fixes the total degree, leaving finitely many monomials. $\square$

3.2 Formal spaces and expansion maps

For a vector space W , let $W[[z_1, \dots, z_m]]$ denote formal power series and $W((t)) = W[[t]][t^{-1}]$ Laurent series. The simultaneous Laurent space is

$$\mathcal{L}_W(z_1, \dots, z_m) = W[[z_1, \dots, z_m]][z_1^{-1}, \dots, z_m^{-1}].$$

An element of this space has a fixed lower bound for the exponent of each variable. This condition is stronger than membership in an iterated Laurent space such as $W((z_1))((z_2))$.

Fix $\emptyset \neq I = \{i_1 < \cdots < i_m\} \subseteq [n]$ and introduce a common coordinate t by

$$z_i = t + x_i \quad (i \in I), \quad x_{i_m} = 0.$$

Here $x = (x_{i_1}, \dots, x_{i_{m-1}})$ records the relative positions, and z_{I^c} remains unchanged. The substitution defines a linear isomorphism

$$\sigma_I : W[[z_1, \dots, z_n]] \longrightarrow W[[t, x, z_{I^c}]]. \quad (15)$$

The substitution preserves total degree, so every coefficient is a finite sum. Its inverse is $t = z_{i_m}$, $x_i = z_i - z_{i_m}$. It satisfies

$$\partial_t \sigma_I(f) = \sigma_I(\partial_I f), \quad \partial_I = \sum_{i \in I} \partial_{z_i}.$$

For simultaneous Laurent series in the selected coordinates, define

$$\tau_I : \mathcal{L}_W(z_I) \longrightarrow W((t))[[x]] \quad (16)$$

by the binomial convention

$$(t + x_i)^k = \sum_{a \geq 0} \binom{k}{a} t^{k-a} x_i^a \quad (k \in \mathbb{Z}).$$

If the exponents of z_i in f are bounded below by ℓ_i , the coefficient of $t^s x^\alpha$ in $\tau_I(f)$ uses only indices satisfying

$$k_i \geq \ell_i, \quad \sum_{i \in I} k_i = s + |\alpha|.$$

There are finitely many such indices, and for fixed α the t -exponent is bounded below by $\sum_i \ell_i - |\alpha|$. Thus τ_I is well defined in $W((t))[[x]]$. The lower bound may depend on α . For example,

$$(t + x)^{-1} = \sum_{a \geq 0} (-1)^a x^a t^{-a-1} \quad \text{in } \mathbb{C}((t))[[x]];$$

there is no uniform lower bound on its t -powers as a varies.

Multiplying by a monomial in the z_i reduces injectivity of τ_I to that of σ_I : the two maps agree on power series, and the images of the monomial factors are units in $\mathbb{C}((t))[[x]]$. Both maps commute with linear maps on W , including q . Extending τ_I coefficientwise gives

$$\mathcal{L}_W(z_I)[[z_{I^c}]] \longrightarrow W((t))[[x, z_{I^c}]].$$

Here the lower bounds may depend on the coefficient of z_{I^c} . On $W[[z_1, \dots, z_n]]$, the extension agrees with σ_I .

3.3 Product–iterate associativity

Separate the insertions indexed by I from those indexed by I^c and write $\Delta = \Delta_I \Delta_{I^c} \Delta_{\text{cross}}$. The first two factors contain differences within each group; the *cross factor* contains those joining the two groups. Every difference retains its original order $z_i - z_j$, $i < j$. Write $\Delta_I(x) = \sigma_I(\Delta_I)$, which depends only on the relative coordinates. We absorb the internal factors into the following two series, leaving the cross factor explicit.

For $e = a_1 \otimes \dots \otimes a_n$ and $I^c = \{l_1 < \dots < l_{n-m}\}$, set

$$\begin{aligned} C_I(x) &= \Delta_I(x) Y(a_{i_1}, x_{i_1}) \cdots Y(a_{i_{m-1}}, x_{i_{m-1}}) a_{i_m}, \\ W_{I^c}(z) &= \Delta_{I^c}(z) Y(a_{l_1}, z_{l_1}) \cdots Y(a_{l_{n-m}}, z_{l_{n-m}}) \mathbf{1}. \end{aligned}$$

Lemma 3.1 shows that both are power series with homogeneous coefficients; for C_I , use the creation axiom to set the last selected coordinate to zero. Write

$$C_I(x) = \sum_{\alpha \geq 0} c_\alpha x^\alpha, \quad W_{I^c}(z) = \sum_{\gamma \geq 0} v_\gamma z^\gamma.$$

For $I = \{i\}$ set $C_I = a_i$; for $I^c = \emptyset$ set $W_{I^c} = \mathbf{1}$.

Lemma 3.2 (Product–iterate associativity). *With the preceding definitions,*

$$(\sigma_I H_V)(e) = \sigma_I(\Delta_{\text{cross}}) \sum_{\alpha, \gamma} x^\alpha z^\gamma Y(c_\alpha, t) v_\gamma \quad (17)$$

in $V((t))[[x, z_{I^c}]]$. Here the cross factor is

$$\sigma_I(\Delta_{\text{cross}}) = \prod_{\substack{i \in I, j \notin I \\ i < j}} (t + x_i - z_j)^{h_i + h_j} \prod_{\substack{i \in I, j \notin I \\ j < i}} (z_j - t - x_i)^{h_i + h_j},$$

where $x_{i_m} = 0$. This is a polynomial in t, x, z_{I^c} , with

$$\deg_t \sigma_I(\Delta_{\text{cross}}) = \sum_{i \in I, j \notin I} (h_i + h_j).$$

The coefficients of $x^\alpha z^\gamma$ in the sum in (17) are Laurent series in t . After multiplication by the cross factor, the full expression equals the left-hand side and belongs to $V[[t, x, z_{I^c}]]$. Thus the complete expression has no negative powers of t .

Proof. For a homogeneous $v \in V_b$, put

$$L_v(z_I) = \Delta_I(z_I) Y(a_{i_1}, z_{i_1}) \cdots Y(a_{i_m}, z_{i_m}) v.$$

Locality moves each selected field next to v . The grading then bounds its coordinate exponent below by $-h_i - b$, so $L_v \in \mathcal{L}_V(z_I)$. The product-iterate form of weak associativity gives

$$\tau_I L_v = \sum_{\alpha \geq 0} x^\alpha Y(c_\alpha, t) v \quad \text{in } V((t))[[x]]. \quad (18)$$

Appendix A proves this identity with the expansion map (16); see also [5] and [4, Propositions 2.6 and 2.7].

After multiplication by the full polynomial Δ , locality lets us put the fields indexed by I before the remaining fields. Expanding W_{I^c} gives the identity

$$H_V(z)(e) = \Delta_{\text{cross}}(z) \sum_{\gamma \geq 0} z^\gamma L_{v_\gamma}(z_I)$$

in $\mathcal{L}_V(z_I)[[z_{I^c}]]$. Apply τ_I coefficientwise in z_{I^c} . On the regular left side it agrees with σ_I ; on the right, (18) applies to each homogeneous v_γ . The cross factor makes only finitely many shifts of each coefficient. This proves (17). $\square$

3.4 Taking invariants

For $I \subseteq [n]$, let P_I be the Reynolds projection onto invariants for the simultaneous G -action on the factors indexed by I , leaving the other factors fixed:

$$P_I(a_1 \otimes \cdots \otimes a_n) = \frac{1}{r} \sum_{g \in G} b_1(g) \otimes \cdots \otimes b_n(g), \quad b_i(g) = \begin{cases} ga_i, & i \in I, \\ a_i, & i \notin I. \end{cases}$$

Lemma 3.3 (Partial-translation equations). *For $\emptyset \neq I \subseteq [n]$, put*

$$\partial_I = \sum_{i \in I} \partial_{z_i}, \quad d_I = 1 + \sum_{i \in I, j \notin I} (h_i + h_j).$$

Then

$$\partial_I^{d_I} (H_Q(z) \circ P_I) = 0. \quad (19)$$

Proof. Precompose (17) with P_I and apply q . Equivariance replaces c_α by $\bar{c}_\alpha = P_G c_\alpha \in B$. For every $j \geq 1$, equation (4) gives

$$[t^j] qY(\bar{c}_\alpha, t) v_\gamma = q((\bar{c}_\alpha)_{-j-1} v_\gamma) = 0.$$

Together with lower truncation, this gives $qY(\bar{c}_\alpha, t) v_\gamma \in Q[t^{-1}]$.

Put $D = d_I - 1$. By Lemma 3.2, the cross factor has t -degree D . This factor can still contribute positive powers of t . Multiplying it by the Q -valued field series leaves none above D . Thus the full

right-hand side, after applying P_I and q , lies in $t^D Q[t^{-1}][[x, z_{I^c}]]$. The same lemma identifies it with a power series in all variables, so it has no negative powers of t . Inside $Q((t))[[x, z_{I^c}]]$,

$$Q[[t, x, z_{I^c}]] \cap t^D Q[t^{-1}][[x, z_{I^c}]] = Q[[x, z_{I^c}]] [t]_{\leq D}.$$

The right side consists of polynomials in t of degree at most D , with coefficients in $Q[[x, z_{I^c}]]$. Differentiating $d_I = D + 1$ times gives $\partial_t^{d_I} \sigma_I(H_Q \circ P_I) = 0$. The chain rule and injectivity of σ_I give (19). $\square$

For example, take homogeneous states a, b, c of weights h_1, h_2, h_3 , put $e = a \otimes b \otimes c$, and let $I = \{1, 3\}$. With $z_1 = t + x$, $z_2 = y$, and $z_3 = t$, we have

$$C_I(x) = x^{h_1+h_3} Y(a, x) c, \quad W_{I^c}(y) = e^{yT} b,$$

and

$$\sigma_I(\Delta_{\text{cross}}) = (t + x - y)^{h_1+h_2} (y - t)^{h_2+h_3}.$$

The factor $y - t$ retains the order $z_2 - z_3$. Equation (17) becomes

$$(\sigma_I H_V)(e) = \sigma_I(\Delta_{\text{cross}}) Y(C_I(x), t) e^{yT} b \quad \text{in } V((t))[[x, y]],$$

with the field on the right applied coefficientwise. Taking invariants in the first and third factors makes every coefficient of C_I invariant. After passage to Q , the cross factor bounds the t -degree by $h_1 + 2h_2 + h_3$, while regularity excludes negative powers. Hence

$$(\partial_{z_1} + \partial_{z_3})^{h_1+2h_2+h_3+1} (H_Q \circ P_{\{1,3\}}) = 0.$$

4 The height bound

We prove the following uniform weight bound. For monomials in homogeneous positive-weight states, it also bounds height by (2).

Proposition 4.1. *Let $X \subset V$ be a finite-dimensional graded G -stable subspace and let $s \geq 1$. There is an integer $H = H(X, s) \geq 0$ such that, for every $1 \leq k \leq s$ and $a_1, \dots, a_k \in X$, all coefficients of*

$$q(Y(a_1, z_1) \cdots Y(a_k, z_k) \mathbf{1})$$

belong to $\bigoplus_{w=0}^H Q_w$.

4.1 Two finite-group identities

For a tensor product of G -modules, P_I denotes the projection onto invariants for the simultaneous action on the factors indexed by I .

Lemma 4.2. *For r finite-dimensional G -modules $F_1, \dots, F_r$,*

$$\bigotimes_{i=1}^r F_i = \sum_{\emptyset \neq I \subseteq [r]} \text{im } P_I. \quad (20)$$

Further, let F_0 and F_g for $g \neq 1$ be G -modules. For $v \in F_0$ and $a_g \in F_g$,

$$v \otimes \bigotimes_{g \neq 1} (1 - g) a_g \in \sum_{J \subseteq G \setminus \{1\}} \text{im } P_{\{0\} \cup J}. \quad (21)$$

Both identities remain valid after adjoining arbitrary other factors.

Proof. Apply Lemma 2.1 to the ideal of positive-degree elements in $A = \text{Sym}(\bigoplus_i F_i)$ and take multidegree $(1, \dots, 1)$. An invariant factor in this degree uses a nonempty subset of the F_i ; multiplying by the remaining factors gives (20).

For (21), use the identity

$$\sum_{h \in G} hv \otimes \bigotimes_{g \neq 1} (ha_g - ga_g) = v \otimes \bigotimes_{g \neq 1} (a_g - ga_g). \quad (22)$$

Every summand with $h \neq 1$ vanishes in its factor indexed by $g = h$. Expand the left side according to the subset J on which ha_g is chosen. The corresponding sum over h is, up to sign, $rP_{\{0\} \cup J}$ applied to a tensor whose remaining factors are fixed. This proves (21). $\square$

4.2 Pairs with a vacuum coefficient

The following lemma provides the pairs of states used in the proof.

Lemma 4.3 (Vacuum coefficients). *For every nonzero $u \in V_h$, $h > 0$, there is a homogeneous $w \in V_k$, $k > 0$, such that*

$$u_{h+k-1}w \neq 0 \quad \text{in } \mathbb{C}\mathbf{1}.$$

Proof. Let $\pi_0 : V \rightarrow V_0 = \mathbb{C}\mathbf{1}$ be the graded projection, and put

$$J = \{v \in V : \pi_0 Y(a, z)v = 0 \text{ for all } a \in V\}.$$

The grading shows that J is graded. Let a, b, v be homogeneous with $v \in J$, and set $N = \text{wt } a + \text{wt } b$. Every coefficient of $\pi_0 Y(Y(a, x)b, t)v$ vanishes. The two-field case of (18), and the fact that π_0 commutes with $\tau_{\{1,2\}}$, give

$$\tau_{\{1,2\}}(\pi_0(z_1 - z_2)^N Y(a, z_1)Y(b, z_2)v) = 0.$$

The expression inside the map belongs to $\mathcal{L}_{V_0}(z_1, z_2)$. Injectivity of the map makes it zero. The scalar factor $(z_1 - z_2)^N$ is invertible in the original ordered space $V_0((z_1))((z_2))$, so $\pi_0 Y(a, z_1)Y(b, z_2)v = 0$ there. Taking coefficients shows that $b_m v \in J$ for all b, m . Also $\pi_0 T = 0$ and translation covariance give

$$\pi_0 Y(a, z)Tv = -\partial_z \pi_0 Y(a, z)v = 0.$$

Thus J is stable under all left modes and T , hence is an ideal by skew symmetry. It does not contain $\mathbf{1}$, and simplicity gives $J = 0$.

Thus $a_{h+k-1}u \in \mathbb{C}^\times \mathbf{1}$ for some homogeneous $a \in V_k$. Necessarily $k > 0$, since $Y(\mathbf{1}, z)u = u$. Skew symmetry $Y(u, z)a = e^{zT}Y(a, -z)u$ gives

$$u_{h+k-1}a = (-1)^{h+k}a_{h+k-1}u.$$

The terms involving positive powers of T have zero vacuum component. Take $w = a$. $\square$

Assume $r > 1$. For each $g \neq 1$, choose a homogeneous a_g with $ga_g \neq a_g$, and set $u_g = (1 - g)a_g$. By Lemma 4.3, we can choose and rescale a homogeneous w_g so that

$$(u_g)_{h_g+k_g-1}w_g = \mathbf{1}, \quad h_g = \text{wt } u_g, \quad k_g = \text{wt } w_g. \quad (23)$$

All higher modes vanish by the grading. Let

$$U_g = \text{span}\{ha_g : h \in G\}, \quad W_g = \text{span}\{hw_g : h \in G\}.$$

These are finite-dimensional homogeneous G -spaces, and $a_g, u_g \in U_g$.

Fix finite-dimensional homogeneous G -stable subspaces $E_i \subset V_{h_i}$, $1 \leq i \leq k$, and put $E = \bigotimes_i E_i$. We adjoin $2r - 1$ copies of each pair (u_g, w_g) by setting

$$\begin{aligned}\tilde{E} &= E \otimes \bigotimes_{g \neq 1} (U_g \otimes W_g)^{\otimes (2r-1)}, \\ \xi &= \bigotimes_{g \neq 1} (u_g \otimes w_g)^{\otimes (2r-1)}.\end{aligned}$$

Write the pairs in ξ as (u_t, w_t) , $1 \leq t \leq N$, where

$$N = (r - 1)(2r - 1). \quad (24)$$

4.3 The polynomial relation module

We encode the differential equations of Lemma 3.3 as relations over a polynomial ring. The tensor product $\tilde{E}$ has $n = k + 2N$ factors; let d_I be the corresponding orders. Put $R = \mathbb{C}[p_1, \dots, p_n]$ and $p_I = \sum_{i \in I} p_i$. Define

$$M = \frac{R \otimes \tilde{E}}{\sum_{\emptyset \neq I \subseteq [n]} R p_I^{d_I} P_I \tilde{E}}. \quad (25)$$

The variable p_i represents differentiation in the variable of the i th field. Let $\mathcal{N}$ be the R -submodule of M generated by the image of $E \otimes \xi$. It is finitely generated because E is finite dimensional.

Lemma 4.4 (Support at the origin). *For some integer $q_0 \geq 1$,*

$$(p_1, \dots, p_n)^{q_0} \mathcal{N} = 0. \quad (26)$$

Proof. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$, write $\mathfrak{m}_\lambda = (p_1 - \lambda_1, \dots, p_n - \lambda_n)$ for the corresponding maximal ideal of R . We show that the localization $\mathcal{N}_{\mathfrak{m}_\lambda}$ vanishes whenever $\lambda \neq 0$. Put $S = \{i : \lambda_i \neq 0\}$.

Suppose first that $|S| \geq 2r - 1$. Choose a line through the origin which contains none of the nonzero λ_i . One of its open half-planes contains at least r of these numbers; let J be the corresponding set of r indices. Every nonempty subset sum of these numbers is nonzero, because its projection onto a common real direction is strictly positive. Hence p_I is a unit in $R_{\mathfrak{m}_\lambda}$ for every nonempty $I \subseteq J$. The relations in (25) kill $P_I \tilde{E}$ in $M_{\mathfrak{m}_\lambda}$ for all such I . Equation (20), with the remaining tensor factors fixed, shows that these images span $\tilde{E}$. Thus $M_{\mathfrak{m}_\lambda} = 0$.

Now suppose $1 \leq |S| \leq 2r - 2$, and choose $j \in S$. For each $g \neq 1$, one of the $2r - 1$ copies of u_g in ξ has an index $i_g \notin S$. The indices i_g are distinct. Apply (21) to the factor indexed by j and these copies of $u_g = (1 - g)a_g$, leaving the remaining factors fixed. By linearity, for every $e \in E$ this gives

$$e \otimes \xi \in \sum_{I \subseteq S^c} P_{\{j\} \cup I} \tilde{E}.$$

For each such I , $p_{\{j\} \cup I}(\lambda) = \lambda_j \neq 0$. The defining relations therefore kill all the indicated images in $M_{\mathfrak{m}_\lambda}$. The generators of $\mathcal{N}$ vanish there, so $\mathcal{N}_{\mathfrak{m}_\lambda} = 0$.

Since $\mathcal{N}$ is finitely generated, its support is the zero set of its annihilator. The preceding localizations and the Nullstellensatz imply

$$(p_1, \dots, p_n) \subseteq \sqrt{\text{Ann}_R \mathcal{N}};$$

see [13, Tags 00L2 and 00FV]. Choose $e_i \geq 1$ with $p_i^{e_i} \mathcal{N} = 0$. Every monomial of total degree at least $q_0 = 1 + \sum_i (e_i - 1)$ is divisible by one of these powers, which proves (26). If $\mathcal{N} = 0$, take $q_0 = 1$. $\square$

4.4 Proof of the weight bound

Proof of Proposition 4.1. By linearity, it suffices to take a_i in homogeneous components $E_i = X \cap V_{h_i}$. We fix k and $h_1, \dots, h_k$ and prove a bound independent of the a_i and the mode indices. Since $k \leq s$ and X has finitely many weights, the maximum of these bounds and 0 gives $H(X, s)$.

If $r = 1$, every singleton projection is the identity. The equations in Lemma 3.3 bound the degree in each variable of H_Q , so (14) gives the result after division by Δ in the ordered Laurent expansion.

Assume $r > 1$ and form $\tilde{E}$ as above. Give the original fields variables $z_1, \dots, z_k$ and the t th auxiliary pair variables x_t, y_t . Put $H_t = \text{wt } u_t$ and $K_t = \text{wt } w_t$. For auxiliary states b_t, c_t in their respective tensor factors, list all $n = k + 2N$ states and variables in tensor order:

$$(A_1, \dots, A_n) = (a_1, \dots, a_k, b_1, c_1, \dots, b_N, c_N),$$

$$(\zeta_1, \dots, \zeta_n) = (z_1, \dots, z_k, x_1, y_1, \dots, x_N, y_N).$$

Let η_i be the weight of the i th tensor factor and write $\tilde{e} = A_1 \otimes \dots \otimes A_n$. Define

$$\tilde{\Delta}(z, x, y) = \prod_{1 \leq i < j \leq n} (\zeta_i - \zeta_j)^{\eta_i + \eta_j} \quad (27)$$

and

$$\tilde{H}_Q(z, x, y)(\tilde{e}) = \tilde{\Delta}(z, x, y) q(Y(A_1, \zeta_1) \dots Y(A_n, \zeta_n) \mathbf{1}). \quad (28)$$

The product in (27) runs over every pair of original and auxiliary fields. Thus it includes $\Delta(z)$, the factors $(x_t - y_t)^{H_t + K_t}$, and all factors between distinct auxiliary pairs or between original and auxiliary fields. Lemma 3.1 gives $\tilde{H}_Q \in \text{Hom}_{\mathbb{C}}(\tilde{E}, Q[[z, x, y]])$, and Lemma 3.3 gives the differential equations for these $k + 2N$ fields.

Using the tensor-order coordinates $\zeta_1, \dots, \zeta_n$, write $\partial^\alpha = \prod_i \partial_{\zeta_i}^{\alpha_i}$. Differentiation at zero defines a linear map

$$R \otimes \tilde{E} \longrightarrow Q, \quad p^\alpha \otimes v \longmapsto (\partial^\alpha \tilde{H}_Q)(0)(v) \quad (29)$$

that factors through M . Indeed, the image of $p^\alpha p_I^{d_I} P_I v$ is obtained by differentiating (19) and evaluating at zero, and is therefore zero. Taylor coefficients differ from these derivatives by the nonzero factors $\alpha!$. Lemma 4.4 therefore shows that

$$F(z, x, y)(e) = \tilde{H}_Q(z, x, y)(e \otimes \xi) \quad (30)$$

is a polynomial of total degree less than q_0 . Here the auxiliary states in (28) have been fixed to $b_t = u_t$ and $c_t = w_t$.

Put $\tilde{D} = \deg \tilde{\Delta}$ and $\tilde{h} = \sum_{i=1}^k h_i + \sum_{t=1}^N (H_t + K_t)$. By (14), the projection F_w of F to Q_w is homogeneous of total degree $w - \tilde{h} + \tilde{D}$. Consequently

$$F_w = 0 \quad \text{if} \quad w > H := \max\{0, \tilde{h} + q_0 - 1 - \tilde{D}\}. \quad (31)$$

It remains to transfer this vanishing to the original product.

Equation (23) and the grading give

$$d^{H_t + K_t} Y(u_t, d) w_t = \mathbf{1} + d A_t(d), \quad A_t(d) \in V[[d]].$$

Apply Lemma 3.2 to this auxiliary pair, with common coordinate y_t and relative coordinate $d = x_t - y_t$. The full factor $\tilde{\Delta}$ remains present. The internal series for the pair is $\mathbf{1} + d A_t(d)$, so the constant coefficient in d of its iterated field is $Y(\mathbf{1}, y_t) = \text{id}$. The cross factors are polynomials in d with nonnegative powers, and their constant coefficients are their values at $x_t = y_t$. Thus taking $[d^0]$

removes the pair and evaluates these factors on that diagonal. On the left this is the ordinary polynomial substitution $x_t = y_t$, since F is already a polynomial. Apply this argument to the auxiliary pairs in turn. Previously extracted scalar factors remain polynomials with nonnegative powers of each new relative coordinate. We obtain

$$F(z, y, y) = C(z, y)H_Q(z), \quad (32)$$

where H_Q is defined by $E_1, \dots, E_k$ and

$$C(z, y) = \prod_{i=1}^k \prod_{t=1}^N (z_i - y_t)^{2h_i + H_t + K_t} \prod_{1 \leq t < v \leq N} (y_t - y_v)^{2(H_t + K_t + H_v + K_v)}. \quad (33)$$

The original factors $\Delta(z)$ remain in $H_Q(z)$. The factors between original and auxiliary fields, and between distinct auxiliary pairs, in (27) give $C(z, y)$. In particular, $C \neq 0$.

For $w > H$, project (32) to Q_w . Equation (31) gives $C(z, y)(H_Q)_w(z) = 0$. Multiplication by the nonzero scalar polynomial C is injective on vector-valued polynomials, so $(H_Q)_w = 0$. Finally, Δ is invertible in the original ordered space $Q((z_1)) \cdots ((z_k))$. Dividing by it shows that every coefficient of weight w in the original product vanishes. $\square$

5 Cofiniteness and its consequences

Proof of Theorem 1.1. Let U be the finite-dimensional graded G -stable strong generating space of Section 2.1, and choose ν with $R_{V,+}^\nu = 0$. Lemma 2.2 spans Q by monomials of length at most $L = r\nu - 1$. Apply Proposition 4.1 with $X = U$ and $s = \max\{1, L\}$. With $H = H(U, s)$, we obtain $Q_w = 0$ for $w > H$. Each V_w is finite dimensional, and hence so is $Q = \bigoplus_{w \leq H} Q_w$. Equation (5) now gives $\dim B/C_2(B) < \infty$.

More explicitly, for a homogeneous basis $u^1, \dots, u^d$ of U ,

$$Q = \text{span} \left\{ q(\mathbf{1}), q(u_{-n_1}^{i_1} \cdots u_{-n_\ell}^{i_\ell} \mathbf{1}) : 1 \leq \ell \leq L, 1 \leq i_j \leq d, 1 \leq n_j \leq H \right\}.$$

The height bound follows from (2). $\square$

Recall that a vertex operator algebra is *strongly rational* if it is simple, of CFT type, self-contragredient, rational, and C_2 -cofinite. Here rationality means complete reducibility of admissible modules.

Corollary 5.1. *Let V be a strongly rational vertex operator algebra and G a finite group of automorphisms. Then V^G is strongly rational.*

Proof. Theorem 1.1 gives C_2 -cofiniteness of V^G . McRae's theorem [10, Corollary 4.23] then gives strong rationality. $\square$

Proof of Theorem 1.2. Choose a finite-dimensional graded G -stable strong generating space $X \subset V_{>0}$ and use the filtration (7), with X in place of U : each generating state contributes its weight, independently of its mode index. Put $S = \text{Sym}(\bigoplus_{j \geq 0} X^{(j)})$, where $X^{(j)}$ is a copy of X representing $T^j X$. The equivariant surjection $S \twoheadrightarrow \text{gr}_F V$ induces

$$S^G \twoheadrightarrow (\text{gr}_F V)^G = \text{gr}_F B$$

by exactness of taking G -invariants. Each polynomial involves only finitely many copies $X^{(j)}$. Applying Noether's degree bound [12] to their direct sum shows that S^G is generated by the elements $P_G(m)$, where m ranges over monomials with at most r factors.

Choose a finite homogeneous basis $\mathcal{X}$ of X . Lift these invariant elements by applying P_G to the corresponding right-associated normally ordered products of derivatives of elements of $\mathcal{X}$. Induction on filtration degree expresses every state of B as a linear combination of normally ordered products of these lifts. Their classes therefore generate R_B as an algebra.

By Proposition 4.1 with $s = r$, every lift of weight greater than $H_X = H(X, r)$ belongs to $D_B \cap B = C_2(B)$. Only finitely many lifts have weight at most H_X : each uses at most r elements of the finite basis $\mathcal{X}$, and their derivative orders are bounded. Their positive-weight members span $B_{>0}/C_1(B)$: products of two positive-weight states and $C_2(B)$ lie in $C_1(B)$. Weight induction, or [8, Theorem 4.10], now gives finite strong generation. $\square$

Remark 5.2. In Theorems 1.1 and 1.2, simplicity can be replaced by the following condition: for each $g \in G$ acting nontrivially on V , there are homogeneous a_g, w_g such that $u_g = (1 - g)a_g \neq 0$ and

$$(u_g)_{\text{wt } u_g + \text{wt } w_g - 1} w_g \in \mathbb{C}^\times \mathbf{1}.$$

These pairs replace those supplied by Lemma 4.3.

A Proof of product–iterate associativity

We verify the identity used in (18) directly from locality. Let $a_i \in V_{h_i}$, $1 \leq i \leq m$, and put

$$\Delta_m(z) = \prod_{i < j} (z_i - z_j)^{h_i + h_j}, \quad D = \deg \Delta_m.$$

With $x_m = 0$, write

$$\Delta_m(x) Y(a_1, x_1) \cdots Y(a_{m-1}, x_{m-1}) a_m = \sum_{\alpha \geq 0} c_\alpha x^\alpha.$$

For $m = 1$, the assertion is immediate. Suppose $m \geq 2$. For a homogeneous $v \in V_b$, expand

$$L_v(z) = \Delta_m(z) Y(a_1, z_1) \cdots Y(a_m, z_m) v = \sum_{\mathbf{k}} A_{\mathbf{k}}(v) z^{\mathbf{k}}.$$

Locality moves each field next to v , giving the coordinate bounds $k_i \geq -h_i - b$. In particular, $L_v \in \mathcal{L}_V(z)$, so the expansion map $\tau = \tau_{\{1, \dots, m\}}$ of (16) applies. Define $F_\alpha(t)v = [x^\alpha] \tau L_v$ and extend linearly in v . Explicitly,

$$[t^s] F_\alpha(t)v = \sum_{\substack{k_i \geq -h_i - b \\ \sum_i k_i = s + |\alpha|}} \left(\prod_{i < m} \binom{k_i}{\alpha_i} \right) A_{\mathbf{k}}(v). \quad (\text{A.1})$$

The sum is finite. Its value has weight $b + h_\alpha + s$, where $h_\alpha = \sum_i h_i + |\alpha| - D$. Thus $F_\alpha(t)$ is a field: it is lower truncated on every vector.

We next show that F_α is local with every vertex operator. Fix a homogeneous $a \in V_h$ and put

$$P(z, y) = \prod_{i=1}^m (z_i - y)^{h_i + h}, \quad N = \sum_i (h_i + h).$$

Locality of the original fields gives

$$P(z, y) \Delta_m(z) [Y(a_1, z_1) \cdots Y(a_m, z_m), Y(a, y)] = 0.$$

Apply this identity to a homogeneous $v \in V$. For each fixed power of y , the two products after multiplication by Δ_m have lower bounds in all the z_i . For the first product, apply the bound for L_v to the corresponding coefficient of $Y(a, y)v$; for the second, apply that mode of $Y(a, y)$ to the coefficients of L_v . Thus (A.1) defines the substitution coefficientwise in y on both sides. It commutes with those fixed modes. Multiplication by the polynomial P introduces only finitely many index shifts, so it also commutes with the expansion. The resulting equality lies in $\mathcal{D}_V(t, y)[[x]]$, where

$$\mathcal{D}_V(t, y) = \prod_{p, q \in \mathbb{Z}} V t^p y^q$$

is the space of formal distributions. Multiplication by a scalar polynomial is defined by finite sums in this space. We use no cancellation of $t - y$. Write

$$P(t + x, y) = \sum_{\nu} p_{\nu} (t - y)^{N - |\nu|} x^{\nu}, \quad p_0 = 1.$$

Since this holds for every v , taking the coefficient of x^{α} yields

$$\sum_{\nu \leq \alpha} p_{\nu} (t - y)^{N - |\nu|} [F_{\alpha - \nu}(t), Y(a, y)] = 0.$$

Induction on $|\alpha|$, after multiplication by $(t - y)^{|\alpha|}$, gives

$$(t - y)^{N + |\alpha|} [F_{\alpha}(t), Y(a, y)] = 0.$$

The exponent is independent of v .

The series L_1 lies in $V[[z_1, \dots, z_m]]$, where τ agrees with the regular substitution $\sigma = \sigma_{[m]}$. Translation covariance and the translation invariance of Δ_m give $\partial_t(\sigma L_1) = T(\sigma L_1)$. At $t = 0$, the creation axiom gives $(\sigma L_1)|_{t=0} = \sum_{\alpha} c_{\alpha} x^{\alpha}$. Solving this formal differential equation coefficientwise yields

$$F_{\alpha}(t) \mathbf{1} = e^{tT} c_{\alpha}.$$

The field $D_{\alpha}(t) = F_{\alpha}(t) - Y(c_{\alpha}, t)$ therefore annihilates the vacuum and is local with every $Y(a, y)$. Apply locality to $\mathbf{1}$ and take the constant coefficient in y . For a sufficiently large integer L , this gives $t^L D_{\alpha}(t) a = 0$, and hence $D_{\alpha}(t) a = 0$ for every a . Thus $F_{\alpha}(t) = Y(c_{\alpha}, t)$.

It follows that

$$\tau L_v = \sum_{\alpha \geq 0} x^{\alpha} Y(c_{\alpha}, t) v \quad \text{in } V((t))[[x]],$$

which is precisely (18). Every substitution used above is defined by the finite coefficient formula (A.1).

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