

A BOOLEAN LÉVY GROUP WITH A FREE NONSINGULAR ACTION

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ABSTRACT. We construct a Boolean Lévy group with a free ergodic spatial nonsingular action on a standard probability space, answering a question of Glasner, Tsirelson and Weiss. The action admits no equivalent invariant σ -finite measure. The group is a quotient of a weighted summable Boolean group by a discrete subgroup. By the nonsingular fixed-point theorem of Avraham-Re'em and Peterzil, the resulting group is not locally Wasserstein.

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1. INTRODUCTION

A topological group is *Polish* if its topology is separable and admits a complete compatible metric. A group is *Boolean* if every element has order at most two. It is automatically abelian: $xy = (xy)^{-1} = y^{-1}x^{-1} = yx$. Thus we write its operation additively. A *standard Borel space* is a measurable space whose σ -algebra is the σ -algebra of Borel sets for some Polish topology. Our reference for descriptive set theory is Kechris [Kec95].

Let G be a Polish group and (X, μ) a standard Borel probability space. A *spatial nonsingular action* is a jointly Borel map $G \times X \rightarrow X$, denoted $(g, x) \mapsto gx$, satisfying the action identities at every point and such that $g_*\mu \sim \mu$ for every g . Here $g_*\mu(A) = \mu(g^{-1}A)$, and equivalence of measures, denoted by $\sim$, means that they have the same null sets. Such a measure is also called *quasi-invariant*. The action is *measure-preserving* if $g_*\mu = \mu$ for every g . The term *spatial* distinguishes an actual pointwise action from an action on the measure algebra, whose elements are measurable sets modulo null sets. An action is *free* if $gx = x$ implies that g is the identity, and *ergodic* if every Borel set invariant modulo null sets has measure zero or one. A measure is *nonatomic* if it has no atoms; for a probability measure on a standard Borel space this is equivalent to every singleton having measure zero. A

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set is *conull* if its complement is null. For a Borel map T and a measure α , we use $T_*\alpha$ for the pushforward measure $T_*\alpha(A) = \alpha(T^{-1}(A))$.

Concentration and Lévy groups. The concentration framework and its connection with fixed-point properties originate in Gromov–Milman [GM83]. For each $t \in \mathbb{N}$, let (K_t, d_t) be a compact metric space and let h_t be a Borel probability measure on K_t . For a nonempty set $A \subseteq K_t$, write $d_t(x, A) = \inf_{a \in A} d_t(x, a)$ and $B_{d_t}(A, \varepsilon) = \{x : d_t(x, A) < \varepsilon\}$. The sequence (K_t, d_t, h_t) is a *Lévy family* if for every sequence of Borel sets $A_t \subseteq K_t$ with $\liminf_t h_t(A_t) > 0$ and every $\varepsilon > 0$,

$$h_t(B_{d_t}(A_t, \varepsilon)) \longrightarrow 1.$$

A Polish group G is a *Lévy group* if there is an increasing sequence $(K_t)_t$ of compact subgroups with dense union, $\overline{\bigcup_t K_t} = G$, and a compatible right-invariant metric d on G such that $(K_t, d|_{K_t}, h_t)$ is a Lévy family, where h_t is the normalized Haar measure of K_t . For a finite group, normalized Haar measure is the uniform probability measure. The construction below uses finite compact subgroups and a two-sided invariant metric.

A nontrivial Lévy group is never locally compact. Indeed, Lévy groups are extremely amenable: every continuous action on a nonempty compact Hausdorff space has a fixed point [GM83]; see also [GTW05, Remark 1.4]. On the other hand, Veech’s theorem gives a free continuous action on a nonempty compact Hausdorff space for every locally compact group [Vee77].

The fixed-point question. Glasner, Tsirelson and Weiss proved that every measure-preserving spatial action of a Lévy group is concentrated on its global fixed points [GTW05, Theorem 1.1]. They asked whether such a group can nevertheless have a nontrivial nonsingular Borel action [GTW05, Question 1.2]. Pestov included this question in his list of open problems about large topological groups [Pes07, Question 21].

Extensions of [GTW05, Theorem 1.1] using spatial Poisson suspensions were obtained by Avraham-Re’em and Roy [ARR25, Theorem 5] and, independently in the measure-preserving case, Hoareau and Le Maître [HLM26, Theorem E]. In particular, a continuous action of a Lévy group on a Polish space preserving an atomless σ -finite locally finite Borel measure is supported on fixed points. Here locally finite means that every point has a neighborhood of finite measure. Avraham-Re’em and Roy also treat nonsingular actions under an additional regularity hypothesis: a positive pointwise Radon–Nikodym cocycle that is continuous in the space variable; see [ARR25, Definition 6.2 and Theorem 5]. These results require topological regularity beyond the Borel and quasi-invariance assumptions of the original question.

We answer [GTW05, Question 1.2] affirmatively with the following theorem.

Theorem 9. *There is a Boolean Lévy group Q with a free ergodic spatial nonsingular action on a standard Borel probability space (Z, ν) . There is no Q -invariant σ -finite Borel measure equivalent to ν .*

Avraham-Re’em and Peterzil introduced the class of *locally Wasserstein groups* and proved a nonsingular fixed-point theorem for this class [AP26, Theorem 1]. Together with their result, Theorem 9 gives an example of a Lévy group that is not locally Wasserstein.

Acknowledgement of AI use. The example presented in this paper was constructed by AI using OpenAI GPT-6-Astra and OpenAI GPT-6.1-sol. These models also assisted with the writing, and Claude Fable-5.1 and Claude Opus-5.5 were used for proofreading. I have substantially revised and edited the draft to improve readability.

2. THE QUOTIENT GROUP

Throughout, $\mathbb{N} = \{0, 1, 2, \dots\}$, and all sequences are indexed from 0. In this section we define the group Q and show that it is a Boolean Lévy group. All additions of subsets are symmetric differences, or equivalently coordinatewise additions of their indicators in the field $\mathbb{F}_2 = \{0, 1\}$.

2.1. Blocks and weights

We first specify the features of the construction that the proof uses. The numerical choices will be made in the next subsection. There are two kinds of indices: i denotes a coordinate, whereas n labels a block and also the coordinate that is the head of the relation introduced below. In particular, the weight w_n of the head is generally *not* the common weight of the coordinates in B_n .

Block structure. Choose positive integers m_n , and set

$$M_{-1} = 0, \quad M_n = \sum_{k=0}^n m_k, \quad B_n = \{M_{n-1} + 1, \dots, M_n\}.$$

Thus $m_n = |B_n|$ is the length of B_n , and M_n is its right endpoint. The blocks are pairwise disjoint finite intervals that partition $\mathbb{N} \setminus \{0\}$. Because $m_k \geq 1$, $M_{n-1} \geq n$, so every element of B_n is strictly larger than n :

$$\mathbb{N} = \{0\} \sqcup \bigsqcup_{n \geq 0} B_n, \quad |B_n| < \infty, \quad \min B_n > n. \quad (1)$$

We call n the parent of each coordinate $i \in B_n$. Coordinate 0 has no parent. Every other coordinate has exactly one parent, and repeatedly passing to the parent leads to 0 because indices decrease by (1).

Figure 1 shows the blocks, their lengths, and their endpoints.

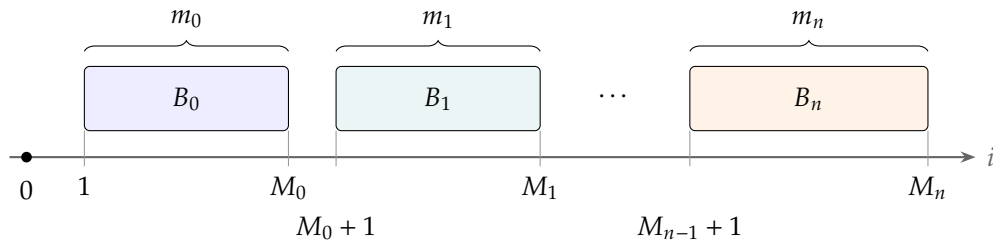


FIGURE 1. The blocks $B_n = \{M_{n-1} + 1, \dots, M_n\}$ on the coordinate axis, with lengths $m_n = M_n - M_{n-1}$ and right endpoints M_n (not to scale).

Weights and block sums. Set $w_0 = 1$. On each block B_n choose a constant weight a_n , so $w_i = a_n$ for $i \in B_n$. Define the total weight W_n and the squared-weight sum V_n :

$$W_n := \sum_{i \in B_n} w_i = m_n a_n, \quad V_n := \sum_{i \in B_n} w_i^2 = m_n a_n^2 = W_n a_n.$$

We impose the following requirements on the weights a_n .

(W1) **Bounded positive weights:** $0 < a_n \leq 1$ for all $n \in \mathbb{N}$.

(W2) **Total block weight and drift:** $W_n \geq 2$ for all $n \in \mathbb{N}$, and $W_n \sqrt{a_n} \rightarrow 1$.

(W3) **Squared-weight budget:** $\sum_{n=0}^{\infty} V_n < \infty$.

The requirements imply $V_n \rightarrow 0$ and

$$V_n W_n = (W_n \sqrt{a_n})^2 \rightarrow 1.$$

Hence $W_n \rightarrow \infty$, and $a_n = V_n/W_n \rightarrow 0$. Since the blocks are finite and partition $\mathbb{N} \setminus \{0\}$, it follows that

$$w_i \rightarrow 0, \quad W_n \rightarrow \infty, \quad \sum_i w_i = \infty, \quad \sum_i w_i^2 = 1 + \sum_n V_n < \infty. \quad (2)$$

2.2. Explicit parameters

The specific values of a_n and m_n do not matter as long as they satisfy the requirements above. One possible choice is

$$m_n = 8^{n+1}, \quad a_n = 4^{-(n+1)}.$$

The endpoints and blocks are then

$$M_n = \sum_{k=0}^n 8^{k+1} = \frac{8(8^{n+1} - 1)}{7}, \quad B_n = \{M_{n-1} + 1, \dots, M_n\}.$$

As above, $w_0 = 1$ and $w_i = a_n$ for $i \in B_n$. The first three blocks illustrate the distinction between a head's weight and its children's weight:

Head n	Block B_n	m_n	Head weight w_n	Child weight a_n
0	$\{1, \dots, 8\}$	8	1	1/4
1	$\{9, \dots, 72\}$	64	1/4	1/16
2	$\{73, \dots, 584\}$	512	1/4	1/64

In particular, $1, 2 \in B_0$, but their own child blocks B_1, B_2 lie farther along the coordinate axis and are disjoint from B_0 and from each other. We have

$$W_n = 2^{n+1} \geq 2, \quad V_n = 2^{-(n+1)}, \quad W_n \sqrt{a_n} = 1.$$

Moreover, $\sum_n V_n = 1$, so all three requirements hold. In particular, $\sum_i w_i^2 = 2$. We use this explicit choice for the rest of the construction, indicating where the general properties are used.

2.3. The weighted group

Define

$$G_w = \left\{ A \subseteq \mathbb{N} : L(A) := \sum_{i \in A} w_i < \infty \right\}, \quad d_w(A, A') = L(A + A').$$

Finite subsets are dense in this metric. For a Cauchy sequence (A_k) , each coordinate eventually stabilizes, defining a set A . Fatou's lemma gives $L(A_k + A) \leq \liminf_m L(A_k + A_m)$, so $L(A_k + A) \rightarrow 0$ and $A \in G_w$. Thus G_w is a Polish group.

Put $e_i = \{i\}$ and define

$$r_n = e_n + \sum_{j \in B_n} e_j, \quad N = \text{span}_{\mathbb{F}_2} \{r_n : n \geq 0\}, \quad Q = G_w / N.$$

The vector r_n has value 1 at the head n and at every child in B_n , and zero elsewhere. In the quotient it becomes the relation $e_n + N = \sum_{j \in B_n} (e_j + N)$. Figure 2 illustrates how a child coordinate is itself the head of a later block.

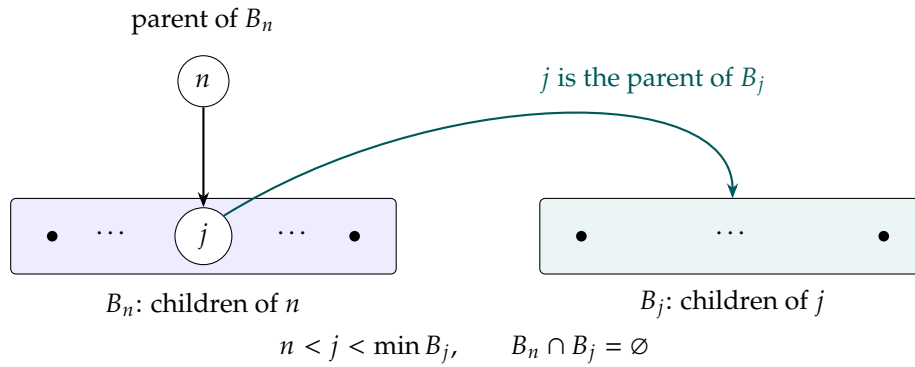


FIGURE 2. Parenthood refers to coordinates, not to inclusion of blocks. The coordinate $j \in B_n$ is a child of n and the parent of its own later block B_j . Vertical placement is schematic; coordinate order is expressed by the inequalities below the diagram.

We call the generators r_n rows.

Proposition 1. *The subgroup N is countable and closed in G_w .*

Proof. Every element of N is a sum of finitely many rows, so N is countable. For a nonempty finite set $F \subseteq \mathbb{N}$, put

$$z_F = \sum_{k \in F} r_k, \quad n = \max F.$$

The row r_n contains every coordinate of B_n . No other selected row can cancel any of these coordinates: its child block is disjoint from B_n , and its head is at most n , whereas every coordinate of B_n is larger than n . Thus $B_n \subseteq z_F$, and

$$L(z_F) \geq \sum_{j \in B_n} w_j = W_n \geq 2.$$

In particular, no nonempty finite sum of rows vanishes, so the rows are linearly independent. Distinct elements of N have distance at least 2 in d_w . The group N is therefore discrete and closed. $\square$

The quotient $Q = G_w/N$ is a Polish group in the quotient topology. Write $q : G_w \rightarrow Q$ for the quotient map. We use the compatible invariant quotient metric

$$\vec{d}(q(A), q(A')) = \inf_{z \in N} L(A + A' + z).$$

Proposition 2. *The quotient $Q = G_w/N$ is a Boolean Lévy group.*

Proof. Only the Lévy property remains to be proved. For $t \geq 0$ set

$$J_t = \left(\bigcup_{n \leq t} B_n \right) \setminus \{0, \dots, t\}, \quad K_t = \text{span}_{\mathbb{F}_2} \{q(e_j) : j \in J_t\}.$$

Here $J_t = \{t+1, \dots, M_t\}$, and $J_{t+1} = (J_t \setminus \{t+1\}) \cup B_{t+1}$. Figure 3 shows this replacement step.

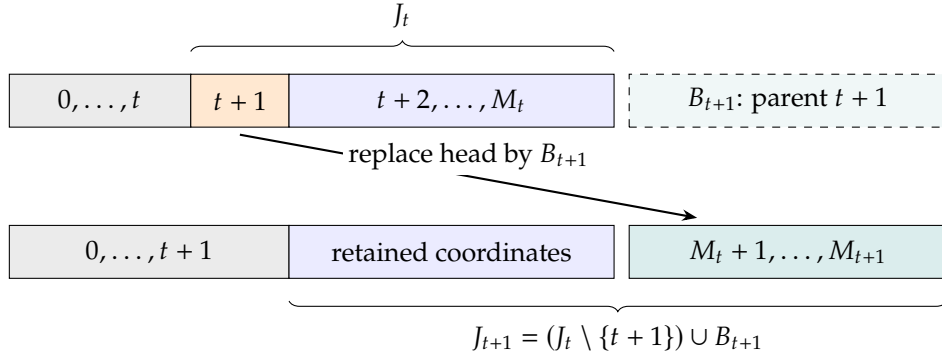


FIGURE 3. One step in the boundary construction. Gray coordinates have been processed. The head $t+1$ leaves the boundary, its children enter, and the blue coordinates remain. Although the sets J_t are not nested, their generated quotient subgroups K_t are nested.

The relation $q(e_{t+1}) = \sum_{j \in B_{t+1}} q(e_j)$ therefore shows that $K_t \subseteq K_{t+1}$. The set J_t is the remaining boundary after replacing heads $0, \dots, t$. Every $q(e_i)$ with $i \leq t$ can be expressed using J_t : recursively replace indices at most t by their children. Thus $\bigcup_t K_t$ is dense in Q .

The finite probability space and the quotient map. Fix t and give

$$\Omega_t = \{0, 1\}^{J_t}$$

the uniform probability measure $\mathbb{P}_t$, under which the coordinate bits are independent and each takes the values 0 and 1 with probability 1/2. Define the surjective group homomorphism

$$\phi_t : \Omega_t \longrightarrow K_t, \quad \phi_t(v) = \sum_{j \in J_t} v_j q(e_j).$$

Each fiber of ϕ_t is a coset of its kernel, so all fibers have the same cardinality. Therefore $(\phi_t)_* \mathbb{P}_t = h_t$, where h_t is the normalized Haar measure, or equivalently the uniform probability measure, on the finite group K_t . For the remainder of this calculation, $\mathbb{P}$ and $\mathbb{E}$ refer to $\mathbb{P}_t$ and its expectation.

Let $A \subseteq K_t$ satisfy $h_t(A) \geq b > 0$, and define the nonnegative random variable

$$f : \Omega_t \longrightarrow \mathbb{R}, \quad f(v) = \text{dist}_{\bar{d}}(\phi_t(v), A).$$

Let $j \in J_t$ and let $v, v' \in \Omega_t$ differ only in the j th bit. Then $\phi_t(v)$ and $\phi_t(v')$ differ by $q(e_j)$. The triangle inequality for $\bar{d}$ implies that distance to A is a 1-Lipschitz function, so

$$|f(v) - f(v')| \leq \bar{d}(\phi_t(v), \phi_t(v')) \leq L(e_j) = w_j. \quad (3)$$

This bound holds for every choice of the other bits.

An elementary variance bound. We claim that

$$\text{Var}(f) \leq \frac{1}{4} \sum_{j \in J_t} w_j^2 = \frac{1}{4} \sigma_t^2, \quad \sigma_t^2 := \sum_{j \in J_t} w_j^2. \quad (4)$$

This is the bounded-differences form of the Efron–Stein inequality. We prove this bound by induction on the number of bits, assuming only the coordinate bounds. With no bits, the function is constant and its variance is zero. For the induction step, fix $j \in J_t$ and split the cube into the two equally likely halves $v_j = 0$ and $v_j = 1$. Expanding the definition of variance gives

$$\text{Var}(f) = \frac{1}{2} \text{Var}(f \mid v_j = 0) + \frac{1}{2} \text{Var}(f \mid v_j = 1) + \frac{1}{4} (\mathbb{E}[f \mid v_j = 0] - \mathbb{E}[f \mid v_j = 1])^2.$$

Here the conditional means and variances are simply those on the indicated halves with their uniform measures. On each half the coordinate bounds (3) still hold for all the remaining bits, so induction bounds each conditional variance by $\frac{1}{4} \sum_{i \in J_t \setminus \{j\}} w_i^2$. Pairing points in the two halves that agree in every other bit, we use (3) and the triangle inequality to bound the difference between the two conditional means by w_j . Substitution proves (4).

From variance to concentration. Because $(\phi_t)_* \mathbb{P} = h_t$, we have $\mathbb{P}(f = 0) = h_t(A) \geq b$. Since f vanishes outside $\{f > 0\}$, the Cauchy–Schwarz inequality gives

$$(\mathbb{E} f)^2 = (\mathbb{E}[f \mathbf{1}_{\{f > 0\}}])^2 \leq \mathbb{P}(f > 0) \mathbb{E}[f^2] \leq (1 - b) \mathbb{E}[f^2].$$

Consequently, $\text{Var}(f) = \mathbb{E}[f^2] - (\mathbb{E} f)^2 \geq b \mathbb{E}[f^2]$. For $r > 0$, Markov’s inequality applied to f^2 , together with (4), gives

$$\begin{aligned} h_t(K_t \setminus B_{\bar{d}}(A, r)) &= \mathbb{P}(f \geq r) \\ &\leq \frac{\mathbb{E}[f^2]}{r^2} \leq \frac{\text{Var}(f)}{br^2} \leq \frac{\sigma_t^2}{4br^2}. \end{aligned} \quad (5)$$

Finally, the inclusion $J_t \subseteq \{j : j > t\}$ and the squared-weight budget (W3) imply

$$\sigma_t^2 \leq \sum_{j > t} w_j^2 \longrightarrow 0.$$

For any sequence $A_t \subseteq K_t$ with $\liminf_t h_t(A_t) > 0$, choose $b > 0$ such that $h_t(A_t) \geq b$ for all sufficiently large t . The bound in (5) then tends to zero for each fixed $r > 0$, which is the Lévy-family condition. Since the finite subgroups K_t are nested and have dense union, Q is a Lévy group. $\square$

3. THE PRODUCT MEASURE

Let $Y = \{0, 1\}^{\mathbb{N}}$, with its product topology and Borel sets. The group G_w acts on Y by $(A, y) \mapsto y + A$. Given a finite binary block, its bias is the imbalance between the number of zeros and the number of ones. We use the weighted, signed score

$$S_n(y) = \sum_{j \in B_n} w_j(2y_j - 1). \quad (6)$$

Thus S_n counts weights corresponding to 1 positively and those corresponding to 0 negatively. Since all weights in B_n equal a_n , it equals a_n times the difference between the number of ones and the number of zeros.

We choose a probability measure on Y under which these scores converge almost surely to $-\sqrt{2}$. The negative limit records a persistent weighted excess of zeros. For the later likelihood argument, what matters is that the limit is strictly negative. Define the Borel set

$$D = \{y \in Y : S_n(y) \longrightarrow -\sqrt{2}\}. \quad (7)$$

Lemma 3. *There is a nonatomic product probability measure μ on Y under which every translation by G_w is nonsingular. Moreover, the set D satisfies $\mu(D) = 1$ and $D + A = D$ for every $A \in G_w$.*

Proof. For $u > 0$ define

$$b(u) = \sqrt{1 - e^{-2u}}, \quad p(u) = \frac{1 - b(u)}{2}.$$

The choice of $p(u)$ makes the affinity of a Bernoulli law and its flipped law especially simple:

$$2\sqrt{p(u)(1 - p(u))} = \sqrt{1 - b(u)^2} = e^{-u}.$$

The product $\prod_{i \in A} e^{-w_i}$ over a summable set A is $e^{-L(A)} > 0$. This positivity is what matters for nonsingularity; other marginal choices giving positive affinity products over all summable sets are possible. For the rest of the construction, the chosen marginals also provide the drift and variance estimates below.

Put $p_i = p(w_i)$ and let

$$\mu_i = (1 - p_i)\delta_0 + p_i\delta_1, \quad \mu = \bigotimes_i \mu_i.$$

Thus $\mu\{y_i = 1\} = p_i$, and $p_i \rightarrow 1/2$ since $w_i \rightarrow 0$. The action $(A, y) \mapsto y + A$ of G_w on Y is jointly continuous and hence Borel.

Fix $A \in G_w$ and let $g_{A,i}(t) = t + \mathbf{1}_A(i)$ modulo two. The marginals of $A_*\mu$ are

$$v_i = (g_{A,i})_*\mu_i = \begin{cases} \mu_i, & i \notin A, \\ p_i\delta_0 + (1 - p_i)\delta_1, & i \in A. \end{cases}$$

All these laws assign positive mass to both points. Their Radon–Nikodym derivatives are

$$\frac{dv_i}{d\mu_i}(t) = \begin{cases} 1, & i \notin A, \\ (p_i/(1 - p_i))^{1-2t}, & i \in A, \end{cases} \quad t \in \{0, 1\}.$$

Kakutani's criterion [Kak48] states that, for equivalent probability measures $\mu_i \sim \nu_i$ on each coordinate space,

$$\bigotimes_i \mu_i \sim \bigotimes_i \nu_i \iff \prod_i \int \sqrt{\frac{d\nu_i}{d\mu_i}} d\mu_i > 0.$$

The integrals are the Hellinger affinities of the marginal laws. Here they equal 1 for $i \notin A$ and $2\sqrt{p_i(1-p_i)} = e^{-w_i}$ for $i \in A$. Kakutani's criterion gives $A_*\mu \sim \mu$, since

$$\prod_i \int \sqrt{\frac{d\nu_i}{d\mu_i}} d\mu_i = \prod_{i \in A} e^{-w_i} = e^{-L(A)} > 0.$$

A conull G_w -invariant domain. Under μ , the variables y_j are independent Bernoulli variables. For $j \in B_n$ they all have parameter $p(a_n)$, so

$$\mathbb{E}(2y_j - 1) = 2p(a_n) - 1 = -b(a_n), \quad \text{Var}(2y_j - 1) = 4p(a_n)(1 - p(a_n)) = e^{-2a_n}.$$

Summing the weighted expectations gives $\mathbb{E}S_n = -m_n a_n b(a_n) = -W_n b(a_n)$. Independence gives $\text{Var}(S_n) = m_n a_n^2 e^{-2a_n} = V_n e^{-2a_n}$. Thus the scores in (6) satisfy

$$\mathbb{E}S_n = -W_n \sqrt{1 - e^{-2a_n}} \longrightarrow -\sqrt{2}, \quad \text{Var}(S_n) = V_n e^{-2a_n} \leq V_n = 2^{-(n+1)}. \quad (8)$$

The mean limit follows from $W_n \sqrt{a_n} \rightarrow 1$ and $\sqrt{1 - e^{-2a_n}}/\sqrt{a_n} \rightarrow \sqrt{2}$. The variance bound is summable by (W3). By Tonelli's theorem,

$$\mathbb{E} \left[\sum_n (S_n - \mathbb{E}S_n)^2 \right] = \sum_n \text{Var}(S_n) < \infty.$$

The sum is therefore finite almost surely, so $S_n - \mathbb{E}S_n \rightarrow 0$ almost surely. Together with the mean limit in (8), this shows that D has measure one.

For every $A \in G_w$ and every $y \in Y$,

$$|S_n(y + A) - S_n(y)| \leq 2L(A \cap B_n) \longrightarrow 0. \quad (9)$$

The last limit follows from the disjointness of the blocks and $L(A) < \infty$. Thus $y \in D$ if and only if $y + A \in D$, proving $D + A = D$ for every $A \in G_w$.

The probabilities $p(w_i)$ tend to $1/2$, so the product measure is nonatomic. $\square$

Proposition 4. *No G_w -invariant σ -finite Borel measure on Y is equivalent to the measure μ of Lemma 3.*

Proof. It suffices to exclude invariance under finite-support flips, since every such flip belongs to G_w . For a Bernoulli product with parameters $0 < p_i \leq 1/2$, Moore's criterion [Moo67, Theorem 2] says that there is no equivalent σ -finite Borel measure invariant under all finite-coordinate flips if and only if

$$\sum_i p_i \min \left\{ \frac{1-p_i}{p_i} - 1, 1 \right\}^2 = \infty. \quad (10)$$

This is the two-point case of Moore's criterion with the larger marginal mass being $1 - p_i \geq 1/2$.

Here $p_i = p(w_i) \rightarrow 1/2$. Thus $(1 - p_i)/p_i - 1 \rightarrow 0$, so the truncation by 1 is inactive for all sufficiently large i . Using $1 - 2p(w_i) = b(w_i)$, the summand then becomes

$$p_i \left(\frac{1 - p_i}{p_i} - 1 \right)^2 = \frac{(1 - 2p_i)^2}{p_i} = \frac{1 - e^{-2w_i}}{p(w_i)} \sim 4w_i.$$

By (2), $\sum_i w_i = \infty$, so the series in (10) diverges. Moore's criterion therefore excludes an equivalent σ -finite Borel measure invariant under G_w . $\square$

The Radon–Nikodym derivatives of the translates of μ have an explicit form. For $u > 0$ put

$$c(u) = \log \frac{1 - p(u)}{p(u)} = 2 \operatorname{artanh} b(u).$$

Figure 4 shows c on the range $0 \leq u \leq 1$ containing all the coordinate weights, with the continuous extension $c(0) = 0$. The curve is increasing and has square-root behavior near zero: $c(u) \sim 2\sqrt{2u}$.

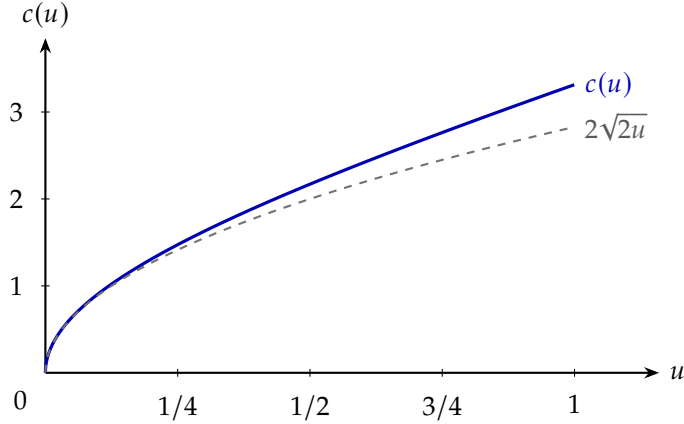


FIGURE 4. The increasing log-odds function $c(u)$ (solid) and its approximation $2\sqrt{2u}$ near zero (dashed). In particular, $c(w_i) \leq c(1)$ for all coordinate weights.

4. THE QUOTIENT ACTION

We now realize the orbit quotient by N as a standard Borel space. An *orbit selector* on D is a map $R : D \rightarrow D$ with $R(y) \in y + N$ and $R(y + n) = R(y)$ for all $n \in N$. Its fixed-point set contains exactly one representative of each orbit; such a set is called a *transversal*. Every $z \in N$ has finite support; write $z_i \in \{0, 1\}$ for its indicator at i . Flipping coordinate i multiplies its marginal density by

$$\left(\frac{p_i}{1 - p_i} \right)^{1 - 2y_i} = e^{-c(w_i)(1 - 2y_i)}.$$

Thus the finite product of these factors defines an everywhere positive version of the likelihood ratio $\rho_z = d(z_*\mu)/d\mu$:

$$\rho_z(y) = \exp(-E_z(y)), \quad E_z(y) = \sum_i z_i c(w_i)(1 - 2y_i).$$

These formulas satisfy, for every $u, z \in N$ and every $y \in Y$,

$$\rho_z(y + u) = \frac{\rho_{z+u}(y)}{\rho_u(y)}. \quad (11)$$

Lemma 5. *For every $y \in D$, the set $\{z \in N : \rho_z(y) \geq 1\}$ is finite.*

Proof. The function c is increasing, since both b and artanh are increasing. Put $C = c(1)$; then $c(w_i) \leq C$ because $w_i \leq 1$. Define

$$T_n(y) = \sum_{j \in B_n} c(w_j)(1 - 2y_j) = -\frac{c(a_n)}{a_n} S_n(y).$$

If $F \subseteq \mathbb{N}$ is a finite set of row indices, put $z_F = \sum_{n \in F} r_n$. The disjoint child blocks contribute $\sum_{n \in F} T_n(y)$. Adding each selected head changes the energy by at most C . Consequently,

$$E_{z_F}(y) \geq \sum_{n \in F} (T_n(y) - C). \quad (12)$$

For $y \in D$, we have $-S_n(y) \rightarrow \sqrt{2}$. Since $c(u) \sim 2\sqrt{2u}$ as $u \rightarrow 0$ and $a_n \rightarrow 0$,

$$T_n(y) = -\frac{c(a_n)}{a_n} S_n(y) \sim \frac{4}{\sqrt{a_n}} \rightarrow \infty.$$

Thus the numbers $T_n(y) - C$ tend to $+\infty$. Consequently, there are only finitely many $z \in N$ with $E_z(y) \leq 0$, or equivalently $\rho_z(y) \geq 1$. $\square$

Proposition 6. *On D , the action of N has a Borel orbit selector R . The induced action of $Q = G_w/N$ on the Borel set $Z = \{y \in D : R(y) = y\}$ is free, jointly Borel, and nonsingular for $\nu = R_*(\mu|D)$.*

Proof. By Lemma 5, for each $y \in D$ there are only finitely many $z \in N$ with $\rho_z(y) \geq 1$. This set is nonempty because $\rho_0(y) = 1$. The global maximum of the $\rho_z(y)$ is therefore attained in this finite set, and there are only finitely many maximizers. We can pick one such maximizer in a Borel way. For instance, order binary sequences lexicographically, comparing them at their first differing coordinate with $0 < 1$. The Borel map R is given by taking $R(y)$ to be the least point in the finite set

$$\{y + z : z \in N, \rho_z(y) = \max_{v \in N} \rho_v(y)\}.$$

The cocycle identity (11) ensures that the set of maximizing points is unchanged along each N -orbit. Hence

$$R(y + u) = R(y) \quad (u \in N), \quad R(y) \in y + N.$$

In particular, R is a Borel orbit selector on D , and

$$Z = \{y \in D : R(y) = y\}$$

is a Borel subset of the standard Borel space D , hence a standard Borel transversal; see [Kec95, Section 12].

Descent to a free spatial nonsingular Q -action. For $g \in G_w$ and $z \in Z$ put $U_g(z) = R(z + g)$. This is jointly Borel. Since $R(z + h)$ differs from $z + h$ by an element of N , its invariance gives

$$U_g(U_h(z)) = R(R(z + h) + g) = R(z + h + g) = U_{g+h}(z).$$

Moreover, $U_{g+n} = U_g$ for $n \in N$, since $R(z + g + n) = R(z + g)$. Thus N acts trivially, and the action descends to an action $(q(g), z) \mapsto U_g(z)$ of Q , with all action identities holding pointwise. To verify joint Borelness on $Q \times Z$, write $\pi = q \times \text{id} : G_w \times Z \rightarrow Q \times Z$ for the continuous surjection between standard Borel spaces, and let $U : Q \times Z \rightarrow Z$ be the action map. For a Borel set $E \subseteq Z$ put $S = U^{-1}(E)$. Then $\pi^{-1}(S) = \{(g, z) : R(z + g) \in E\}$ is Borel, so $S = \pi(\pi^{-1}(S))$ and $(Q \times Z) \setminus S = \pi(\pi^{-1}((Q \times Z) \setminus S))$ are both analytic. By Souslin's theorem S is Borel [Kec95, Theorem 14.11].

This action of Q is free. If $R(z + g) = z$, then $z + g = z + n$ for some $n \in N$, and cancellation of binary sequences gives $g = n$. Finally, let $\nu = R_*(\mu|D)$. For a Borel set $A \subseteq Z$,

$$R^{-1}(U_g^{-1}A) = R^{-1}(A) + g$$

inside the invariant set D . The quasi-invariance of μ under G_w therefore makes ν quasi-invariant under every U_g , and hence under every element of Q . This completes the construction. $\square$

5. ERGODICITY AND INVARIANT MEASURES

It remains to check that the action of Q on (Z, ν) is ergodic and admits no equivalent σ -finite invariant measures.

Proposition 7. *The action on (Z, ν) is ergodic and ν is nonatomic.*

Proof. Let $\Gamma \leq G_w$ be the countable group of finite-support flips. If a Borel set $A \subseteq Z$ is invariant modulo ν -null sets, put $B = R^{-1}(A) \subseteq D$. Pointwise equivariance of R gives $\mu((B + f) \triangle B) = 0$ for every $f \in \Gamma$. Consequently

$$B^{\text{inv}} = \bigcap_{f \in \Gamma} (B + f)$$

is pointwise Γ -invariant and agrees with B modulo a null set. Membership in B^{inv} is unchanged by arbitrary alterations of any finite initial segment, so B^{inv} belongs, for every k , to $\sigma(y_i : i > k)$, the sigma-algebra generated by the coordinates after k . Independence of the Bernoulli coordinates makes B^{inv} independent of every finite-coordinate cylinder, hence of every Borel set by a monotone class argument. In particular, $\mu(B^{\text{inv}}) = \mu(B^{\text{inv}})^2$, proving $\nu(A) = \mu(B) \in \{0, 1\}$.

To check that ν is nonatomic, note that Lemma 3 implies that μ assigns zero mass to each point. Every fiber of R is one countable N -orbit; therefore $\nu(\{z\}) = \mu(R^{-1}(\{z\})) = 0$ for all $z \in Z$. $\square$

Proposition 8. *No Q -invariant σ -finite Borel measure on Z is equivalent to ν .*

Proof. Suppose $\lambda \sim \nu$ were such an invariant measure. We first lift it to D by counting points in each N -orbit. For Borel $E \subseteq D$, define

$$F_E(z) = \sum_{n \in N} \mathbf{1}_E(z + n), \quad \tilde{\mu}(E) = \int_Z F_E(z) d\lambda(z).$$

The functions F_E are Borel, and countable additivity follows from Tonelli's theorem. Choose Borel sets Z_k covering Z with $\lambda(Z_k) < \infty$. Since Z meets each orbit once, $\tilde{\mu}(Z_k + n) = \lambda(Z_k)$, and the sets $Z_k + n$, for $k \in \mathbb{N}$ and $n \in N$, cover D . Hence $\tilde{\mu}$ is σ -finite.

For Borel $E \subseteq D$, the set $R(E) = \bigcup_{n \in N} (E + n) \cap Z$ is Borel, and $R^{-1}(R(E)) = \bigcup_{n \in N} (E + n)$. Since F_E is positive exactly on $R(E)$, we have

$$\tilde{\mu}(E) = 0 \iff \lambda(R(E)) = 0 \iff \nu(R(E)) = 0 \iff \mu(E) = 0.$$

The last equivalence uses the quasi-invariance of μ under the countable group N . Thus $\tilde{\mu} \sim \mu|_D$. For $g \in G_w$, the points $U_g z$ and $z + g$ differ by an element of N , so reindexing the sum over N gives $F_{E+g}(z) = F_E(U_g z)$. Invariance of λ therefore yields

$$\tilde{\mu}(E + g) = \int_Z F_E(U_g z) d\lambda(z) = \tilde{\mu}(E).$$

Extending $\tilde{\mu}$ by zero off the invariant conull set D gives a G_w -invariant σ -finite Borel measure on Y equivalent to μ , contradicting Proposition 4. $\square$

The preceding propositions now give the theorem announced in the introduction.

Theorem 9. *There is a Boolean Lévy group Q with a free ergodic spatial nonsingular action on a standard Borel probability space (Z, ν) . There is no Q -invariant σ -finite Borel measure equivalent to ν .*

Proof. Proposition 2 shows that Q is a Boolean Lévy group. Proposition 6 supplies its free spatial nonsingular action on the standard Borel probability space (Z, ν) . This action is ergodic by Proposition 7, and Proposition 8 excludes an equivalent Q -invariant σ -finite Borel measure. $\square$

Remark 10. A related construction also yields a Boolean Lévy group with a free ergodic spatial action preserving a nonatomic infinite σ -finite Borel measure on a standard Borel space. The construction uses the same pattern of block relations, with different block metrics and measures. Details will be presented in a subsequent version of this paper.

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