

A lower bound for the sixth moment of the Riemann zeta function

Niklas Fraccaro

Abstract

We prove that the lower limit of the sixth moment of the Riemann zeta function, normalized by $(a_3/9!)T(\log T)^9$, is greater than 41.007. Our approach extends the two-piece amplifier construction used by Durkan and Page by allowing independent profiles, centered derivatives, and finite real linear combinations of two projected test families. An asymmetric twisted fourth-moment estimate and their amplified moment formula yield a common numerator kernel. Explicit balanced and unbalanced kernels determine the projected Gram form. A rational certificate with 250 tests, followed by approximation by admissible profiles, gives the stated bound.

2020 Mathematics Subject Classification. Primary 11M06.

Keywords. Riemann zeta function, sixth moment, amplified moments, twisted fourth moment.

Introduction

Write

$$M_3(T) = \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^6 dt, \quad a_3 = \prod_{\ell \text{ prime}} (1 - \ell^{-1})^9 \sum_{j=0}^{\infty} \binom{j+2}{2}^2 \ell^{-j}.$$

Equivalently,

$$a_3 = \prod_{\ell \text{ prime}} (1 - \ell^{-1})^4 (1 + 4\ell^{-1} + \ell^{-2}).$$

This product converges to a positive number. The conjecture of Conrey and Ghosh [2], also consistent with the random matrix prediction of Keating and Snaith [6], is

$$M_3(T) \sim 42 \frac{a_3}{9!} T(\log T)^9.$$

Durkan and Page [4, Theorem 1] prove the lower bound with constant 34.4 in this normalization, using two-piece amplifiers. We obtain the following improvement.

Theorem 1. *We have*

$$\liminf_{T \rightarrow \infty} \frac{M_3(T)}{(a_3/9!)T(\log T)^9} > 41.007.$$

The construction is a finite-dimensional approximation to ζ^3 . We use two families of tests and project each onto the real line in $\mathbb{C}$ selected by the functional equation for ζ^3 . The projection preserves its real correlation with ζ^3 without increasing its norm. Allowing the profiles and centered derivatives to vary independently provides additional directions beyond the two-piece test. Once the mixed moments and the Gram matrix have been evaluated, Cauchy–Schwarz reduces the lower bound to a quotient of two explicit quadratic expressions. The certificate in Section 6 makes this quotient greater than 41.007 with a strict margin.

There are two distinct analytic inputs for the numerator. For the first family we retain the two polynomial lengths separately in the argument of Bettin, Bui, Li and Radziwiłł [1]; the resulting range is $3a + 5b < 2$ for $a \leq b$. For the second family we use Durkan–Page [4, Theorem 16, $k = 1$], with common cutoff $\eta < 1/4$. The denominator requires only two-zeta moment calculations. The essential compatibility is that any pair of admissible tests has total support exponent less than one.

The rational endpoint data are accompanied by an exact Lean 4 certificate and a formalization of the sixth-moment bound [5]. The Lean development establishes the result conditional on a single explicit analytic hypothesis, `DivisorProof.Watt41`, formulated on the basis of Proposition 4.1 of Watt [7]. This hypothesis is stated but not proved in Lean; the repository documents the regularity and uniformity conventions needed to relate it to Watt’s published estimate. The formalization uses some different technical formulations and intermediate arguments and is not intended as a line-by-line translation of the present proof.

Section 1 sets up the tests and the quotient. Section 2 establishes the asymmetric fourth-moment input, and Sections 3 and 4 derive the numerator and denominator kernels. Section 5 passes from admissible profiles to endpoint values. The certificate and the passage to the unweighted moment follow in Sections 6 and 7. The contour pairing, reflection identity and rational integration formulas are recorded in the appendices.

1 The projected test space

1.1 Notation and test functions

Let $T > 1$, $L = \log T$ and $s = \frac{1}{2} + it$. We use the functional equation in the convention

$$\zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) = \pi^{s-1/2} \frac{\Gamma((1-s)/2)}{\Gamma(s/2)}.$$

On the critical line, $1-s = \bar{s}$ and $|\chi(s)| = 1$, so that $\zeta(s)^3 = \chi(s)^3 \overline{\zeta(s)^3}$. For a real compactly supported profile f , define

$$A_{f,T}(w) = \sum_{n \geq 1} f\left(\frac{\log n}{L}\right) n^{-w}.$$

If $\text{supp } f \subseteq [0, a]$, this is a Dirichlet polynomial of length at most T^a . For $p \in \mathbb{R}[x]$, put

$$\begin{aligned} C_{p,T}(w) &= p\left(\frac{1}{2} + \frac{1}{L} \frac{d}{dw}\right) \zeta(w) \\ &= p(\partial_z) [e^{z/2} \zeta(w + z/L)] \Big|_{z=0}, \quad p^-(x) = p(-x). \end{aligned}$$

At fixed T we suppress the subscript T . Since the profiles and polynomial coefficients are real,

$$A_f(1-s) = \overline{A_f(s)}, \quad C_p(1-s) = \overline{C_p(s)}.$$

The certificate uses test polynomials p of degree at most four; this cutoff describes the chosen finite certificate, rather than an intrinsic restriction of the construction. All profiles and test polynomials remain fixed as $T \rightarrow \infty$. The degrees of polynomial profiles need not be at most four.

1.2 Two families and the two-piece construction

The two raw test families are

$$U_{p,f,g}(t) = \chi(s)C_p(s)A_f(s)A_g(1-s), \quad V_{p,f,g}(t) = C_p(s)A_f(s)A_g(s).$$

We project them by

$$\Pi H(t) = \frac{H(t) + \chi(s)^3 \overline{H(t)}}{2}.$$

Pointwise, this is the orthogonal projection for the real inner product $\Re(z\bar{w})$ onto the line

$$\{w \in \mathbb{C} : w = \chi(s)^3 \bar{w}\},$$

which also contains $\zeta(s)^3$. For $p = 1$, $f = g$ and $A = A_f$,

$$2\Pi(U_{1,f,f} + V_{1,f,f}) = \zeta(s)(A(s) + \chi(s)A(1-s))^2. \quad (1.1)$$

The expression on the right is the two-piece amplifier test used by Durkan and Page [4, Section 1.3 and equation (1.9)]. Thus their test is recovered algebraically. The identity alone does not assert that the same profiles are analytically admissible for both families.

For the moment calculations we use

$$\begin{aligned} \zeta(s)^3 \overline{\Pi H} &= \Re(\zeta(s)^3 \overline{H}), \\ (\Pi H) \overline{\Pi J} &= \frac{1}{2} \Re(H \bar{J} + \overline{\chi(s)^3 H J}). \end{aligned} \quad (1.2)$$

Also $|\Pi H| \leq |H|$. The second term in the Gram identity explains why the ordinary moments $H \bar{J}$ alone do not suffice.

1.3 Centering and shift conventions

The centering is adapted to the functional equation. Exactly,

$$e^{z/2} \zeta(s + z/L) = \chi(s) H_T(t, z) e^{-z/2} \zeta(1 - s - z/L), \quad H_T(t, z) = e^z \frac{\chi(s + z/L)}{\chi(s)}.$$

For $T \leq t \leq 2T$ and z in a fixed complex disc, $H_T(t, z) = 1 + O(L^{-1})$ uniformly. This produces the polynomial reflection $p \mapsto p^-$ below. The replacement is justified inside normalized moments, not used as an exact identity between centered derivatives. We first prove the moment limits locally uniformly in the scaled complex shifts, including coincident shifts after their apparent poles have been removed. Fixed differential operators may then be applied by Cauchy's integral formula.

1.4 Admissible profiles and the order of limits

The numerator formulas use the following classes:

$$\begin{aligned} U : \quad & f \in C_c^\infty((0, a)), \quad g \in C_c^\infty((0, b)), \quad 0 < a \leq b, \quad 3a + 5b < 2; \\ V : \quad & f, g \text{ are real polynomials on } [0, \eta], \text{ extended by zero,} \quad 0 < \eta < \frac{1}{4}. \end{aligned}$$

For a finite V -family, η is chosen in common. To apply the Gram formulas we additionally take the V -profiles to vanish at 0 and η , so their zero extensions are continuous. The beta profiles in Section 5 have this property at all sufficiently large indices.

For each test, the sum of its two support exponents is less than $1/2$. For U , this follows from $4(a+b) \leq 3a+5b < 2$; for V , from $2\eta < 1/2$. Consequently every pair of tests, including a mixed pair, has total support exponent less than one. This is the common length condition for the Gram asymptotics.

All profiles, polynomials and linear combinations remain fixed as $T \rightarrow \infty$. Point masses enter only as limits of admissible profiles: a sufficiently large profile index is fixed first, and then $T \rightarrow \infty$ is taken. No estimate uniform in a profile index growing with T is required.

1.5 The quotient to be evaluated

For real $\Phi \in C_c^\infty((1, 2))$, write $\widehat{\Phi}(0) = \int_{\mathbb{R}} \Phi(x) dx$. The moment formulas will also be used for signed Φ . In Cauchy–Schwarz we choose $0 \leq \Phi \leq 1$ and $\widehat{\Phi}(0) > 0$. Choose finitely many raw tests $B_1, \dots, B_m$ from U, V , and let $c \in \mathbb{R}^m$. Set

$$F_c = \Pi \left(\sum_{j=1}^m c_j B_j \right).$$

We shall determine a real vector k and a real symmetric matrix G , independent of T and Φ , such that

$$\begin{aligned} \frac{1}{a_3 T L^9} \int_{\mathbb{R}} \Phi(t/T) \zeta(s)^3 \overline{\Pi B_j(t)} dt &\longrightarrow \widehat{\Phi}(0) k_j, \\ \frac{1}{a_3 T L^9} \int_{\mathbb{R}} \Phi(t/T) (\Pi B_j(t)) \overline{\Pi B_k(t)} dt &\longrightarrow \widehat{\Phi}(0) G_{jk}. \end{aligned} \quad (1.3)$$

Thus $c^\top k$ and $c^\top G c$ describe the correlation of the test with ζ^3 and its squared norm. The off-diagonal entries of G record the interaction between its components. When $c^\top G c > 0$, the denominator below is positive for all sufficiently large T , and Cauchy–Schwarz gives

$$\int_{\mathbb{R}} \Phi(t/T) |\zeta(s)|^6 dt \geq \frac{\left| \int_{\mathbb{R}} \Phi(t/T) \zeta(s)^3 \overline{F_c(t)} dt \right|^2}{\int_{\mathbb{R}} \Phi(t/T) |F_c(t)|^2 dt}.$$

Together with (1.3), this yields

$$\liminf_{T \rightarrow \infty} \frac{\int_{\mathbb{R}} \Phi(t/T) |\zeta(s)|^6 dt}{(a_3/9!) T L^9} \geq \widehat{\Phi}(0) \frac{9! (c^\top k)^2}{c^\top G c}. \quad (1.4)$$

It remains to establish the two moment limits and certify an admissible choice with positive denominator and quotient greater than 41.007. A finite dyadic decomposition will then recover $M_3(T)$.

2 An asymmetric twisted fourth moment

The U -numerator requires a twisted fourth moment in which the two Dirichlet polynomials may have different lengths. We obtain the needed range by retaining this distinction in the proof of Bettin–Bui–Li–Radziwiłł [1]. We first specify the complete main term, so that the assertion also has an unambiguous meaning when shifts coincide.

For $\lambda = (\alpha, \beta, \gamma, \delta)$, put

$$\tau_{u,v}(n) = \sum_{de=n} d^{-u} e^{-v}, \quad \lambda^* = (\gamma, \delta, \alpha, \beta),$$

and define

$$\begin{aligned} \mathcal{A}_\lambda &= \frac{\zeta(1+\alpha+\gamma) \zeta(1+\alpha+\delta) \zeta(1+\beta+\gamma) \zeta(1+\beta+\delta)}{\zeta(2+\alpha+\beta+\gamma+\delta)}, \\ \mathcal{B}_\lambda(h) &= \prod_{\ell^\nu \parallel h} \frac{\sum_{r \geq 0} \tau_{\alpha,\beta}(\ell^r) \tau_{\gamma,\delta}(\ell^{r+\nu}) \ell^{-r}}{\sum_{r \geq 0} \tau_{\alpha,\beta}(\ell^r) \tau_{\gamma,\delta}(\ell^r) \ell^{-r}}, \\ Z_\lambda(h, k) &= \mathcal{A}_\lambda \mathcal{B}_\lambda(h) \mathcal{B}_{\lambda^*}(k) \quad ((h, k) = 1). \end{aligned}$$

Here ℓ denotes a prime. With $d = (h, k)$, let

$$\mathcal{M}_\lambda(T; f, g) = \sum_{h, k \geq 1} f\left(\frac{\log h}{L}\right) g\left(\frac{\log k}{L}\right) \frac{d}{hk} Z_\lambda(h/d, k/d). \quad (2.1)$$

This is a finite coefficient sum and does not depend on t . The six shifts and their phase exponents are

j	$\lambda^{(j)}$	$e_j(\lambda)$
0	$(\alpha, \beta, \gamma, \delta)$	0
1	$(-\gamma, -\delta, -\alpha, -\beta)$	$-\alpha - \beta - \gamma - \delta$
2	$(-\gamma, \beta, -\alpha, \delta)$	$-\alpha - \gamma$
3	$(-\delta, \beta, \gamma, -\alpha)$	$-\alpha - \delta$
4	$(\alpha, -\gamma, -\beta, \delta)$	$-\beta - \gamma$
5	$(\alpha, -\delta, \gamma, -\beta)$	$-\beta - \delta$

(2.2)

Define

$$I_\lambda(T; f, g) = \int_{\mathbb{R}} \Phi(t/T) \zeta(s + \alpha) \zeta(s + \beta) \zeta(\bar{s} + \gamma) \zeta(\bar{s} + \delta) A_f(s) A_g(\bar{s}) dt, \quad (2.3)$$

$$\mathfrak{M}_\lambda(T; f, g) = \int_{\mathbb{R}} \Phi(t/T) \sum_{j=0}^5 \left(\frac{t}{2\pi}\right)^{e_j(\lambda)} \mathcal{M}_{\lambda^{(j)}}(T; f, g) dt. \quad (2.4)$$

At collisions, the sum in (2.4) is interpreted by its removable continuation; no individual singular term is evaluated separately.

Theorem 2.1 (Asymmetric twisted fourth moment). *Let $0 < a \leq b$, $3a + 5b < 2$, $f \in C_c^\infty((0, a))$, $g \in C_c^\infty((0, b))$, and let $\Phi \in C_c^\infty((1, 2))$ be real. For every compact $K \subset \mathbb{C}^4$, with $\lambda = \boldsymbol{\sigma}/L$,*

$$I_\lambda(T; f, g) = \mathfrak{M}_\lambda(T; f, g) + o(T) \quad (2.5)$$

uniformly for $\boldsymbol{\sigma} \in K$. More precisely, for every $\varepsilon > 0$, the error is

$$O_{K, f, g, \Phi, \varepsilon} \left(T^{1/2+a+b+\varepsilon} + T^{3/4+(3a+5b)/8+\varepsilon} \right).$$

The same assertion holds with f and g interchanged in the moment and the main term. Thus the shorter polynomial may occupy either side of the critical line.

Proof. We use the sharper intermediate estimates in the proof of [1, Proposition 3.1]; the main-term calculation is that of [1, Sections 2 and 4]. In the proof, we suppress f, g in I_λ , $\mathcal{M}_\lambda$ and $\mathfrak{M}_\lambda$. All profiles and the weight remain fixed.

Write $\lambda = \boldsymbol{\sigma}/L$. We first work on compact sets on which

$$-\sigma_1, \quad -\sigma_2, \quad \sigma_3, \quad \sigma_4$$

are pairwise distinct, and remove this restriction at the end.

1. *The quadratic-divisor estimate.* After the approximate functional equation and a smooth dyadic decomposition, the off-diagonal terms involve

$$hm_1m_2 - kn_1n_2 = \pm\ell, \quad h \asymp A, \quad k \asymp B, \quad m_i \asymp M_i, \quad n_i \asymp N_i, \quad \ell \asymp H.$$

Put $M = M_1M_2$, $N = N_1N_2$ and $P = ABMN$. The relevant blocks satisfy $AM \asymp BN$. Order the factors on each side and, if necessary, interchange the sides so that

$$M_1 \ll M_2P^\eta, \quad N_1 \ll N_2P^\eta, \quad BN_1 \leq AM_1.$$

These are the ordering and orientation conditions in [1, Proposition 3.1, equation (13)]. Interchanging the sides does not affect the later bounds in $Q = AB$ and $R = \max\{A, B\}$. The smooth dyadic weights satisfy the derivative bounds required there.

For $H \ll (AB)^{1/2+\eta}$, with arbitrarily small fixed $\eta > 0$, the two ranges in the proof of [1, Proposition 3.1, pp. 16–17] give

$$E_I \ll_\varepsilon T^\varepsilon (PH^2)^{1/4} \left(AB + H^{1/4} A^{3/8} B^{1/8} P^{1/8} \right), \quad (2.6)$$

$$E_{II} \ll_\varepsilon T^\varepsilon (PH^2)^{1/4} \left(AB + (ABH)^{1/4} P^{1/8} \right). \quad (2.7)$$

Here range I is the large- d range for $d = (hm_1, kn_1)$, treated by equation (15) of that paper, and range II is its complement, treated by equation (16). We have retained the displayed bounds before their dependence on A, B is combined into $A + B$. The remaining Poisson and weight-replacement error is $O_\varepsilon(T^\varepsilon H^2)$. All dyadic parameters are bounded by fixed powers of T , so the small-power losses in those estimates have been absorbed into T^ε .

Set

$$Q = AB, \quad R = \max\{A, B\}, \quad X = (MN)^{1/2}.$$

Since $A^{3/8} B^{1/8} \leq Q^{1/8} R^{1/4}$, $Q^{1/4} \leq Q^{1/8} R^{1/4}$ and $P = QX^2$, we obtain

$$E \ll_\varepsilon T^\varepsilon \left(H^{1/2} Q^{5/4} X^{1/2} + H^{3/4} Q^{1/2} R^{1/4} X^{3/4} + H^2 \right), \quad (2.8)$$

where E is the discrepancy between the dyadic quadratic-divisor sum and its zero-frequency contribution.

2. Conversion to the moment estimate. For arbitrarily small positive parameters ρ and u , the approximate functional equation and decay of the oscillatory integral restrict the relevant blocks to

$$X \ll T^{1+u/2}, \quad H \ll T^\rho \frac{\sqrt{Q} X}{T}. \quad (2.9)$$

This is the dyadic form of the truncations in [1, Section 2.4]: the approximate functional equation restricts MN , while integration by parts restricts the frequency $\log(hm_1 m_2 / (kn_1 n_2)) \asymp H/(\sqrt{Q} X)$. The discarded ranges are negligible.

The compact supports of f, g inside $(0, a), (0, b)$ bound A, B below by fixed positive powers of T . Choosing ρ, u sufficiently small in (2.9) therefore gives $H \ll Q^{1/2+\eta}$. Moreover, a nonempty off-diagonal block has $H \gg 1$, so the same inequalities give the required lower bound on MN relative to a sufficiently small fixed power of AB . Thus the hypotheses used in (2.8) hold.

The coefficient normalization is $(ABMN)^{-1/2} = (\sqrt{Q} X)^{-1}$, and the t -integral has length $O(T)$. Hence the conversion factor from the quadratic-divisor discrepancy to the moment is $T/(\sqrt{Q} X)$. Writing $\Lambda = T^\rho$, multiplication of (2.8) by this factor gives

$$E_{\text{block}} \ll_\varepsilon T^\varepsilon \left(\Lambda^{1/2} T^{1/2} Q + \Lambda^{3/4} T^{1/4} Q^{3/8} R^{1/4} X^{1/2} + \Lambda^2 \frac{\sqrt{Q} X}{T} \right). \quad (2.10)$$

Using $Q \ll T^{a+b}$, $R \ll T^b$ and $X \ll T^{1+u/2}$, the three terms are bounded, respectively, by

$$T^{1/2+a+b+\rho/2+\varepsilon}, \quad T^{3/4+(3a+5b)/8+3\rho/4+u/4+\varepsilon}, \quad T^{(a+b)/2+2\rho+u/2+\varepsilon}.$$

The third is smaller than the first. There are only logarithmically many dyadic blocks. Taking ρ, u sufficiently small and absorbing this summation into an arbitrarily small power of T , we obtain

$$E_{\text{off}} \ll_\varepsilon T^{1/2+a+b+\varepsilon} + T^{3/4+(3a+5b)/8+\varepsilon}. \quad (2.11)$$

This is where the asymmetric condition $3a + 5b < 2$ enters.

3. Evaluation of the main terms. The diagonal and zero-frequency terms are evaluated by the Mellin and contour argument of [1, Sections 2 and 4]. The two coefficient sequences in that argument are now $f(\log h/L)$ and $g(\log k/L)$; keeping their lengths separate changes the error bookkeeping, but not the arithmetic identities.

Choose an even auxiliary function G , rapidly decreasing in vertical strips, normalized by $G(0) = 1$, and with zeros at the shifted poles crossed in the contour shifts. The residues at zero in the two diagonal integrals give the unswapped and fully swapped terms in $\mathfrak{M}_\lambda(T)$. Pairing each original zero-frequency integral with its dual leaves a contour difference whose residue at zero is the corresponding one-swap term. The other crossed poles are cancelled by G . For the (β, δ) swap the residue is

$$\left(\frac{t}{2\pi}\right)^{-\beta-\delta} \mathcal{M}_{(\alpha, -\delta, \gamma, -\beta)}(T),$$

and permutation gives the other three terms. Appendix A records the incoming BBLR expression and the contour pairing in our notation.

The gamma-factor and dual-weight replacements contribute $O_\varepsilon(T^{1/2+a+b+\varepsilon})$. Moving the diagonal contours to $\Re z = -1/8$ contributes $O(T^{3/4+(a+b)/2}(\log T)^{48})$, which is absorbed in (2.11), since $(a+b)/2 \leq (3a+5b)/8$. Consequently,

$$I_\lambda(T) - \mathfrak{M}_\lambda(T) \ll_\varepsilon T^{1/2+a+b+\varepsilon} + T^{3/4+(3a+5b)/8+\varepsilon} \quad (2.12)$$

uniformly on compact sets of separated normalized shifts.

4. The exponent gap and colliding shifts. Let $\Delta = 2 - 3a - 5b > 0$. The exponents in (2.12) satisfy

$$\frac{3}{4} + \frac{3a+5b}{8} = 1 - \frac{\Delta}{8}, \quad \frac{1}{2} + a + b = 1 - \frac{\Delta + b - a}{4} \leq 1 - \frac{\Delta}{4}.$$

Thus $0 < \varepsilon < \Delta/16$ gives $o(T)$ on compact separated shift sets.

Although individual terms in $\mathfrak{M}_\lambda(T)$ have apparent poles at collisions, their sum is holomorphic. On each collision hyperplane the paired terms have equal remaining factors and opposite residues; for example, on $\alpha + \gamma = 0$ the residues of $\zeta(1 + \alpha + \gamma)$ and $\zeta(1 - \alpha - \gamma)$ cancel. Multiplying by the product of the six differences among $-\alpha, -\beta, \gamma, \delta$ clears the possible poles. The resulting holomorphic numerator is divisible by every factor, hence by their product, so intersections of collision hyperplanes are removable as well.

Fix a compact $K \subset \mathbb{C}^4$ of normalized shifts, and choose a polydisc centred at the origin containing K , with pairwise distinct coordinate radii. On its distinguished boundary, the four signed coordinates $-\sigma_1, -\sigma_2, \sigma_3, \sigma_4$ are uniformly separated: the absolute value of each difference is at least the difference of the corresponding radii. Both the moment and the complete six-term expression are holomorphic in this polydisc for sufficiently large T . Applying the maximum-modulus principle successively in the four variables therefore extends (2.12), and in particular its $o(T)$ consequence, from that boundary to K . This proves (2.5) with the asserted uniform error. Finally, complex conjugation and interchange of the two shift pairs give the version with f, g reversed. $\square$

3 Mixed moments and a common numerator kernel

3.1 The kernel and the two moment limits

The six swapped main terms admit a determinant representation. When three shift parameters coincide, the resulting divided difference becomes an average over a three-dimensional

simplex. This gives the kernel below. The V -calculation produces the same kernel with its second endpoint reflected, although its analytic input is different. In this section a, b denote logarithmic positions averaged against the profiles, rather than fixed support bounds.

Let $X = (X_0, X_1, X_2, X_3)$ be uniform on the simplex $X_j \geq 0$, $\sum_j X_j = 1$. We normalize this probability measure by

$$\mathbb{E}F(X) = 6 \int_{\substack{x_1, x_2, x_3 \geq 0 \\ x_1 + x_2 + x_3 \leq 1}} F(1 - x_1 - x_2 - x_3, x_1, x_2, x_3) \, dx_1 \, dx_2 \, dx_3, \quad \mathbb{E}1 = 1.$$

For $a, b \geq 0$, put

$$\begin{aligned} P(u, v) &= \frac{uv(1-u)(1-v)(1-u-v)}{12}, \\ q(a, b, t) &= \left(-\frac{1}{2}, a-t-\frac{1}{2}, \frac{1}{2}-b+t, \frac{1}{2}\right), \\ K_z(a, b) &= \int_0^{\min(a, b)} P(b-t, a-t) \mathbb{E}e^{zq(a, b, t) \cdot X} \, dt. \end{aligned} \quad (3.1)$$

For a real polynomial p , define

$$K_p(a, b) = p(\partial_z) K_z(a, b)|_{z=0} = \int_0^{\min(a, b)} P(b-t, a-t) \mathbb{E}p(q(a, b, t) \cdot X) \, dt. \quad (3.2)$$

The generator K_z is entire in z , including when its simplex nodes coincide.

Proposition 3.1 (Numerator limits). *For an admissible U - or V -test and a real polynomial p of degree at most four, the first limit in (1.3) holds with*

$$k_U(p; f, g) = \int_0^\infty \int_0^\infty f(a)g(b)K_p(a, b) \, da \, db, \quad (3.3)$$

$$k_V(p; f, g) = \int_0^\eta \int_0^\eta f(a)g(b)K_p(a, 1-b) \, da \, db. \quad (3.4)$$

There is no extra projection factor in these formulas.

We prove the corresponding locally uniform limits for

$$\begin{aligned} \mathcal{U}_T(z) &= \int_{\mathbb{R}} \Phi(t/T) \zeta(s)^3 \overline{\chi(s)} A_f(\bar{s}) A_g(s) e^{z/2} \zeta(\bar{s} + z/L) \, dt, \\ \mathcal{V}_T(z) &= \int_{\mathbb{R}} \Phi(t/T) \zeta(s)^3 A_f(\bar{s}) A_g(\bar{s}) e^{z/2} \zeta(\bar{s} + z/L) \, dt. \end{aligned}$$

Differentiation at zero gives the unprojected correlations. The projection identity (1.2) then takes their real parts, and the limiting kernels are real.

3.2 From the six U -main terms to the simplex

Since $\overline{\chi(s)}\zeta(s) = \zeta(\bar{s})$,

$$\mathcal{U}_T(z) = e^{z/2} I_{(0,0,z/L,0)}(T; g, f). \quad (3.5)$$

The order g, f is essential. Theorem 2.1 permits this reversal, so it remains to extract the leading L^9 coefficient of its main term.

First take separated scaled shifts $\sigma = (\sigma_1, \dots, \sigma_4)$. Write

$$D(\sigma) = \prod_{i=1}^2 \prod_{j=3}^4 (\sigma_i + \sigma_j), \quad E_{r,s}(x) = \frac{e^{-rx} - e^{-sx}}{s - r}, \quad E_{r,r}(x) = xe^{-rx}.$$

Use the transformations $\sigma^{(j)}$ and the linear forms $e_j(\sigma)$ from (2.2). If A, B are the logarithmic positions of the first and second profiles in I_λ , the leading kernel is

$$\int_0^{\min(A,B)} \sum_{j=0}^5 \frac{e^{e_j(\sigma)}}{D(\sigma^{(j)})} E_{\sigma_3^{(j)}, \sigma_4^{(j)}}(A-t) E_{\sigma_1^{(j)}, \sigma_2^{(j)}}(B-t) dt. \quad (3.6)$$

The main term is $a_3 T L^9 \widehat{\Phi}(0)$ times the profile average of (3.6), up to $o(T L^9)$.

The normalization and profile positions can be read directly from Mellin inversion. The principal factor of the unswapped term is

$$\frac{1}{D(\sigma)(u+v)(u+\sigma_3)(u+\sigma_4)(v+\sigma_1)(v+\sigma_2)}. \quad (3.7)$$

The nine denominator factors are precisely the pairwise sums of the row shifts (σ_1, σ_2, u) and the column shifts (σ_3, σ_4, v) . Each corresponds to a polar zeta factor; the remaining Euler factor tends to a_3 , as in Lemma 4.1. With $\tilde{f}(u) = \int f(x) e^{ux} dx$, our inversion convention is

$$f(\log n/L) = \frac{1}{2\pi i} \int_{(c)} \tilde{f}(u) n^{-u/L} du.$$

Replacing $L^{-1}\zeta(1+w/L)$ by $1/w$ therefore gives (3.7). Finally,

$$\frac{1}{u+v} = \int_0^\infty e^{-t(u+v)} dt, \quad \mathcal{L}^{-1}\left(\frac{1}{(u+r)(u+s)}\right)(x) = E_{r,s}(x),$$

which yields (3.6) term by term. The phase $(t/2\pi)^{e_j(\sigma)/L}$ tends uniformly to $e^{e_j(\sigma)}$ on the support of Φ .

For completeness, the following bound justifies the limiting Mellin calculation, including its tails. On a compact separated shift set K , choose R with $|\sigma_i| \leq R$ and take $c > R+1$. On the fixed scaled contours $u = c + i\xi$, $v = c + iv$, the five arguments involving u, v are

$$\mathcal{W} = \{u+v, u+\sigma_3, u+\sigma_4, v+\sigma_1, v+\sigma_2\}.$$

For $w \in \mathcal{W}$, $\Re w \geq 1$, and absolute convergence gives, for $L \geq 1$,

$$|L^{-1}\zeta(1+w/L)| \leq L^{-1}\zeta(1+\Re w/L) \leq L^{-1} + (\Re w)^{-1} \leq 2.$$

The four factors depending only on the shifts are uniformly bounded after division by L , since their scaled arguments stay away from zero. The regular Euler factor is bounded on these contours: its local factors are $1 + O(\ell^{-2+\delta})$, uniformly in the imaginary parts, for some fixed $0 < \delta < 1$ and all sufficiently large T . Smooth compact support also gives $|\tilde{f}(c+i\xi)| \ll_f (1+|\xi|)^{-2}$, and similarly for g . Thus the normalized integrand is bounded by

$$\frac{C_{K,f,g}}{(1+|\xi|)^2(1+|v|)^2}. \quad (3.8)$$

On bounded contour segments the nine normalized zeta factors converge uniformly to their reciprocal arguments and the regular factor converges to a_3 . The majorant controls the tails uniformly, giving the locally uniform $o(T L^9)$ remainder. No further power saving in T is needed here. Holomorphic continuation and the distinguished-boundary argument of Section 2 then include coincident shifts.

There is a useful closed form for the six summands. Put

$$x = (-\sigma_1, -\sigma_2, \sigma_3, \sigma_4), \quad \rho(u, v) = (-1, v-1, -u, 0), \quad \Delta(x) = \prod_{0 \leq i < j \leq 3} (x_j - x_i).$$

Then

$$\begin{aligned} \sum_{j=0}^5 \frac{e^{e_j(\sigma)}}{D(\sigma(j))} E_{\sigma_3^{(j)}, \sigma_4^{(j)}}(A-t) E_{\sigma_1^{(j)}, \sigma_2^{(j)}}(B-t) \\ = e^{x_0+x_1} \frac{\det(e^{x_i \rho_j(A-t, B-t)})_{i,j=0}^3}{\Delta(x)}. \end{aligned} \quad (3.9)$$

Expand the determinant along its first two columns: the six choices of rows give the six swaps. Since the numerator is alternating in x , the quotient extends jointly holomorphically across $\Delta(x) = 0$.

In (3.5), $A = b$, $B = a$ and $x = (0, 0, z, 0)$. The confluent form of the Hermite–Genocchi divided-difference identity gives

$$e^{z/2} \frac{\det(e^{x_i \rho_j})_{i,j=0}^3}{\Delta(x)} \Big|_{x=(0,0,z,0)} = \frac{\Delta(\rho)}{12} \mathbb{E} e^{z(\rho + \frac{1}{2}\mathbf{1}) \cdot X}. \quad (3.10)$$

One can check the normalization without invoking a general formula: for distinct ρ_j and $z \neq 0$,

$$\mathbb{E} e^{z\rho \cdot X} = \frac{6}{z^3} \sum_{j=0}^3 \frac{e^{z\rho_j}}{\prod_{k \neq j} (\rho_j - \rho_k)},$$

while the three coincident rows on the left of (3.10) supply the factor $1/(2z^3)$. Continuity removes the restrictions. For the general divided-difference identity, see [3, formula (83)]. Since

$$\Delta(\rho(u, v)) = 12P(u, v), \quad \rho(b-t, a-t) + \frac{1}{2}\mathbf{1} = q(a, b, t),$$

we obtain, locally uniformly in z ,

$$\frac{\mathcal{U}_T(z)}{a_3 T L^9} \longrightarrow \widehat{\Phi}(0) \int_0^\infty \int_0^\infty f(a) g(b) K_z(a, b) \, da \, db.$$

Cauchy’s formula proves (3.3).

3.3 The V -moment and the reflection identity

The V -moment uses the short-polynomial, single- χ fourth moment of Durkan–Page [4, Theorem 16, $k = 1$, and Section 5.2]. We spell out the orientation and the residue formula, since both matter for the signs in the reflected kernel.

For scaled shifts σ , define the conjugate orientation

$$\mathcal{J}_\sigma^-(T; f, g) = \int_{\mathbb{R}} \Phi(t/T) \chi(s) \prod_{i=1}^2 \zeta(s + \sigma_i/L) \prod_{j=3}^4 \zeta(\bar{s} + \sigma_j/L) A_f(\bar{s}) A_g(\bar{s}) \, dt.$$

Relative to the displayed definition of $J_{1,1}$ in [4], this conjugates the integral and interchanges the two shift pairs; it is the orientation used after reversal of conjugation in their Section 5.2. We read the formula below from that reversed-orientation calculation, with the shifts labelled as in $\mathcal{J}_\sigma^-$, rather than identifying $\mathcal{J}_\sigma^-$ with their displayed $J_{1,1}$. Our generator satisfies the exact identity

$$\mathcal{V}_T(z) = e^{z/2} \mathcal{J}_{(0,0,z)}^-(T; f, g), \quad (3.11)$$

because $\chi(s)\zeta(\bar{s}) = \zeta(s)$. The usual polynomial convention $P(\log(T^\eta/n)/\log T^\eta)$ corresponds to $f(a) = P(1 - a/\eta)$. A formula for one squared profile extends to f, g by $A_f A_g = (A_{f+g}^2 - A_{f-g}^2)/4$, using their common cutoff η .

Here is the normalized residue expression from that calculation in our variables. Set

$$Q_\sigma(w) = (w + \sigma_1)(w + \sigma_2)(w - \sigma_3)(w - \sigma_4), \quad F_\sigma(a, \omega) = \frac{1}{2\pi i} \oint \frac{e^{au}(u - \omega)}{Q_\sigma(u)} du,$$

and

$$R_\sigma(a, b) = \frac{e^{-\sigma_3 - \sigma_4}}{2\pi i} \oint \frac{e^\omega F_\sigma(a, \omega) F_\sigma(b, \omega)}{Q_\sigma(\omega)} d\omega. \quad (3.12)$$

All contours are positively oriented circles containing the four roots $-\sigma_1, -\sigma_2, \sigma_3, \sigma_4$, with radii fixed on the compact shift set under consideration. The $k = 1$ instance of the four-residue calculation gives

$$\frac{\mathcal{J}_\sigma^-(T; f, g)}{a_3 T L^9} = \widehat{\Phi}(0) \int_0^\eta \int_0^\eta f(a) g(b) R_\sigma(a, b) da db + o(1). \quad (3.13)$$

To match (3.12) to the four terms of the source, take, for example, the outer residue at $\omega = -\sigma_1$. The factor $u + \sigma_1$ cancels in each inner contour, leaving the three factors $u + \sigma_2, u - \sigma_3, u - \sigma_4$. Its outer coefficient is

$$\frac{e^{-\sigma_1 - \sigma_3 - \sigma_4}}{(\sigma_2 - \sigma_1)(\sigma_1 + \sigma_3)(\sigma_1 + \sigma_4)}.$$

This is the first residue in [4, Theorem 16]; the other three roots give the remaining terms. Each profile inversion has scale L^3 , and the outer residue contributes L^3 , explaining L^9 . Equivalently, the two factors $(\log T^\eta)^3$ in the source lose their η^3 factors when its simplex variables are rescaled to our logarithmic positions. The arithmetic factor is a_3 . The source error is $O(TL^8)$ for fixed polynomial profiles in the range $\eta < 1/4$. Its uniformity on separated compact shift sets, followed by holomorphic continuation as in Section 2, gives the locally uniform limit in (3.13). The contour representation also makes removability at colliding shifts explicit.

Define the entire functions

$$\begin{aligned} B_a(z) &= \int_0^a (a - u) e^{-zu} du, & C_a(z) &= -\frac{1}{2} \int_0^a (a - u)^2 e^{-zu} du, \\ H_m(z) &= \frac{1}{m!} \int_0^1 (1 - u)^m e^{-zu} du \quad (m = 0, 1, 2), \end{aligned}$$

and put

$$J(a, b; z) = B_a B_b H_2 + (B_a C_b + C_a B_b) H_1 + C_a C_b H_0, \quad (3.14)$$

with every factor on the right evaluated at z . The elementary inner inverse transform is

$$\frac{1}{2\pi i} \oint e^{au} \frac{u - \omega}{u^3(u + z)} du = B_a(z) + \omega C_a(z).$$

Thus

$$J(a, b; z) = \frac{1}{2\pi i} \oint \frac{e^\omega}{\omega^3(\omega + z)} (B_a(z) + \omega C_a(z))(B_b(z) + \omega C_b(z)) d\omega, \quad (3.15)$$

where the contours enclose $0, -z$. Expanding the numerator gives H_2, H_1, H_0 , respectively, and proves (3.14). At $\sigma = (0, 0, 0, z)$, (3.12) becomes $R_\sigma(a, b) = e^{-z} J(a, b; -z)$. In view of (3.11), we have therefore proved

$$\frac{\mathcal{V}_T(z)}{a_3 T L^9} \longrightarrow \widehat{\Phi}(0) \int_0^\eta \int_0^\eta f(a) g(b) e^{-z/2} J(a, b; -z) da db, \quad (3.16)$$

locally uniformly near zero.

The connection with the U -kernel is now an identity of entire functions.

Lemma 3.2 (Reflection). *If $a, b > 0$ and $a + b < 1$, then*

$$e^{-z/2} J(a, b; -z) = K_z(a, 1 - b) \quad (z \in \mathbb{C}). \quad (3.17)$$

Proof. For $z \neq 0$, the translated simplex nodes of $K_z(a, 1 - b)$ are $0, a - t, b + t, 1$, with $0 \leq t \leq a$. The divided-difference formula reduces this kernel to four elementary exponential integrals. Their sum agrees with the result of integrating the definitions of B, C, H in $e^{-z/2} J(a, b; -z)$; Appendix B gives the common numerator. Entirety includes $z = 0$. $\square$

Combining (3.16) and (3.17), then applying $p(\partial_z)$ by Cauchy's formula, proves (3.4) and completes Proposition 3.1. The reflection is an identity of limiting kernels: the analytic V -asymptotic still uses its own short-length range $\eta < 1/4$.

4 The denominator kernels and the projected Gram form

The projection identity (1.2) splits each Gram entry into an ordinary product and a reflected product. After applying the functional equation, these reduce to the balanced and unbalanced moments below; the remaining reflected VV product is oscillatory and contributes no main term.

Throughout this section the real profiles f, g, h, k are continuous and supported in $[0, A_i]$, for $i = 1, 2, 3, 4$, respectively, with

$$S_* := A_1 + A_2 + A_3 + A_4 < 1.$$

This includes every pair of tests from Section 1. The endpoint variables a, b, c, d below are integration variables, not support bounds. The polynomials p, q have degree at most four.

4.1 The common arithmetic diagonal

Lemma 4.1 (Nine-edge diagonal limit). *Let F be bounded, measurable and compactly supported on $\mathbb{R}_{\geq 0}^3 \times \mathbb{R}_{\geq 0}^3$, and suppose that its discontinuity set has zero measure for the limiting measure below. Writing $\log \mathbf{n}/L = (\log n_1/L, \dots, \log n_3/L)$, we have*

$$\frac{1}{L^9} \sum_{n_1 n_2 n_3 = m_1 m_2 m_3} \frac{F(\log \mathbf{n}/L, \log \mathbf{m}/L)}{n_1 n_2 n_3} \longrightarrow a_3 \int_{x_{ij} \geq 0} F \left(\left(\sum_j x_{ij} \right)_i, \left(\sum_i x_{ij} \right)_j \right) \prod_{i,j=1}^3 dx_{ij}. \quad (4.1)$$

Proof. The Dirichlet series of the diagonal factors as

$$\sum_{\prod_i n_i = \prod_j m_j} \frac{\prod_i n_i^{-u_i} \prod_j m_j^{-v_j}}{\prod_i n_i} = E(\mathbf{u}, \mathbf{v}) \prod_{i,j=1}^3 \zeta(1 + u_i + v_j).$$

At each prime the nine terms of first degree agree with the nine zeta factors. Their quotient is therefore $1 + O(\ell^{-2+\varepsilon})$ uniformly near the origin, and E is regular there. At zero its local factor is

$$(1 - \ell^{-1})^9 \sum_{r \geq 0} \binom{r+2}{2} \ell^{-r} = (1 - \ell^{-1})^4 (1 + 4\ell^{-1} + \ell^{-2}),$$

so $E(\mathbf{0}, \mathbf{0}) = a_3$. Substituting $u_i = \alpha_i/L$ and $v_j = \beta_j/L$, with positive real parts, gives the limiting Laplace transform

$$a_3 \prod_{i,j} (\alpha_i + \beta_j)^{-1}.$$

This is the transform of the pushforward of Lebesgue measure on the nine nonnegative variables x_{ij} in (4.1). Exponentially tilting makes the measures finite, and the continuity theorem for Laplace transforms gives weak convergence. Removing the tilt on a compact set proves the assertion, including bounded tests whose discontinuities have limiting measure zero. $\square$

4.2 The balanced kernel

For $a + b + c + d < 1$, let $\mathcal{R}(a, b; c, d)$ consist of $r = (r_{11}, r_{12}, r_{21}, r_{22}) \geq 0$ with row sums at most a, b and column sums at most c, d . Put

$$t_r = \sum_{i,j} r_{ij}, \quad \Delta = \frac{a + b - c - d}{2}, \quad w_r = \frac{1 - a - b - c - d + 2t_r}{2},$$

and define

$$B_{p,q}(a, b; c, d) = \int_{\mathcal{R}(a,b;c,d)} \int_{-w_r}^{w_r} p(\Delta - x)q(-\Delta - x) \, dx \, dr. \quad (4.2)$$

Proposition 4.2 (Balanced moment). *The normalized integral*

$$\frac{1}{a_3 T L^9} \int_{\mathbb{R}} \Phi(t/T) C_p(s) C_q(\bar{s}) A_f(s) A_g(s) A_h(\bar{s}) A_k(\bar{s}) \, dt$$

converges to

$$\hat{\Phi}(0) \int f(a)g(b)h(c)k(d)B_{p,q}(a, b; c, d) \, da \, db \, dc \, dd.$$

Proof. Apply the smoothed two-zeta approximate functional equation, first with two scaled shifts; compare [8, Lemmas 4 and 5]. Write $H = h_1 h_2$ and $K = k_1 k_2$ for the two amplifier products. The relevant length condition in that argument is $HK \leq T^{S_*}$, so its parameter 2θ is $S_* < 1$. The direct and dual sums are effectively restricted to $mn \leq T^{1+\delta}$, where $0 < \delta < 1 - S_*$. If $Hm \neq Kn$ and their ratio is near one, integrality gives

$$T \left| \log \frac{Hm}{Kn} \right| \gg T^{(1-S_*-\delta)/2}.$$

Outside that range the frequency is larger. Repeated integration by parts against the smooth time weight therefore makes the off-diagonal negligible. The tails and gamma-factor replacements cost $O_\varepsilon(T^{1/2+S_*/2+2\varepsilon})$, with logarithmic losses absorbed into T^ε ; choose $\varepsilon < (1 - S_*)/4$.

To specify the smoothing, one may start with the pole-corrected Gaussian

$$G_{s_1, s_2}(u) = e^{u^2} \prod_{\nu=1}^2 \left(1 - \frac{u^2}{s_\nu^2} \right) \left(1 - \frac{u^2}{(1 - s_\nu)^2} \right), \quad s_1 = s + z/L, \quad s_2 = \bar{s} + w/L.$$

It is even, equals one at zero, and cancels the poles crossed in the approximate functional equation. Removing its correction polynomial and replacing the gamma ratios are included in the preceding error bound; the estimates are uniform for z, w in fixed complex discs.

We give the limiting-cutoff step explicitly, since the smoothing still depends on T . After these replacements the weight is

$$V_0(x) = \frac{1}{2\pi i} \int_{(1)} e^{u^2} x^{-u} \frac{du}{u} = \frac{1}{\sqrt{\pi}} \int_{\frac{1}{2} \log x}^{\infty} e^{-v^2} \, dv.$$

For $t = T\tau$, $1 \leq \tau \leq 2$, and $Q = \log(mn)/L$, this is

$$V_0(2\pi mn/t) = \frac{1}{\sqrt{\pi}} \int_{\{L(Q-1)+\log(2\pi/\tau)\}/2}^{\infty} e^{-v^2} \, dv.$$

It lies between zero and one and converges uniformly to $\mathbf{1}_{Q \leq 1}$ outside every fixed strip $|Q - 1| < \epsilon$, uniformly in τ . On a fixed compact set, bound the strip by a continuous function which is one on $|Q - 1| \leq \epsilon$ and zero on $|Q - 1| \geq 2\epsilon$. Lemma 4.1 applies to this majorant; its limiting integral tends to zero as $\epsilon \downarrow 0$, because the preimage of $Q = 1$ is a null hyperplane in the nine edge variables. Thus the smoothing can be replaced by the sharp cutoff on compact sets. For completeness, no mass is lost at infinity: under the exponential tilt used in the lemma, the shifted profile weight grows at most exponentially in Q , and the Gaussian decay bounds its product with the smoothing by Ce^{-Q^2} for $Q \geq 2$ and all sufficiently large L . The tilted measures have bounded total mass, so truncating at $Q \leq R$ has error $O(e^{-R})$, uniformly in the fixed shift discs. Letting first $T \rightarrow \infty$, then $\epsilon \downarrow 0$ and $R \rightarrow \infty$, justifies the cutoff $\log(mn)/L \leq 1$.

Apply Lemma 4.1 and disintegrate its nine variables by the four profile positions. The remaining matrix of edge variables is

$$\begin{pmatrix} u & c - r_{11} - r_{21} & d - r_{12} - r_{22} \\ a - r_{11} - r_{12} & r_{11} & r_{12} \\ b - r_{21} - r_{22} & r_{21} & r_{22} \end{pmatrix}, \quad r \in \mathcal{R}(a, b; c, d).$$

All changes of coordinates have Jacobian one. The two zeta lengths are the first row and first column sums,

$$R_0 = u + c + d - t_r, \quad C_0 = u + a + b - t_r.$$

The cutoff $R_0 + C_0 \leq 1$ is exactly $0 \leq u \leq w_r$. For the direct term the polynomial weight is $p(1/2 - R_0)q(1/2 - C_0)$; setting $x = u - w_r$ gives the integrand of (4.2) on $[-w_r, 0]$. For the dual term the weight is $p(C_0 - 1/2)q(R_0 - 1/2)$; setting $x = w_r - u$ gives the same integrand on $[0, w_r]$. Adding the two halves proves the kernel formula.

The shifted estimates are uniform on fixed complex discs. Cauchy's formula therefore justifies the fixed polynomial derivatives used above. $\square$

4.3 The unbalanced kernel

Put $H = a + b + c - d$ and $\tau = (1 - H)/2$. If $H \leq 0$, define $M_{p,q}(a, b, c; d) = 0$. For $H > 0$, let

$$\mathcal{S}(a, b, c; d) = \{(r_1, r_2) : 0 \leq r_1 \leq a, 0 \leq r_2 \leq b, 0 \leq r_3 := H - r_1 - r_2 \leq c\},$$

and set

$$M_{p,q}(a, b, c; d) = \int_{\mathcal{S}(a, b, c; d)} \int_{\prod_{i=1}^3 [-r_i/2, r_i/2]} p(y_1 + y_2 + y_3 - \tau) \cdot q(y_1 + y_2 + y_3 + \tau) \, \mathbf{d}\mathbf{y} \, dr_1 \, dr_2. \quad (4.3)$$

Proposition 4.3 (Unbalanced moment). *The normalized integral*

$$\frac{1}{a_3 T L^9} \int_{\mathbb{R}} \Phi(t/T) \chi(s)^{-1} C_p(s) C_q(\bar{s}) A_f(s) A_g(s) A_h(s) A_k(\bar{s}) \, dt$$

converges to

$$\widehat{\Phi}(0) \int f(a) g(b) h(c) k(d) M_{p,q}(a, b, c; d) \, da \, db \, dc \, dd.$$

Proof. For the shifted generator, the functional equation moves the first zeta factor to the negative side. Retain its centering factor during the estimates:

$$H_T(t, z) = \left(\frac{t}{2\pi T} \right)^{-z/L} + O(T^{-1})$$

uniformly on fixed z -discs. Expand the two negative-side zeta factors by the one-sided smoothed series

$$\sum_{m,n \geq 1} m^{-w_1} n^{-w_2} e^{-mn/T^3}, \quad w_1 = \bar{s} - z/L, \quad w_2 = \bar{s} + w/L.$$

To justify the replacement, Mellin inversion gives

$$\sum_{m,n \geq 1} m^{-w_1} n^{-w_2} e^{-mn/T^3} = \frac{1}{2\pi i} \int_{(2)} \Gamma(u) T^{3u} \zeta(w_1 + u) \zeta(w_2 + u) du.$$

Truncate at $|\Im u| \leq T/4$ and shift to $\Re u = -1/2 - \varepsilon$, with $0 < \varepsilon < 1/4$. Only the pole of $\Gamma(u)$ at zero is crossed: the zeta poles have imaginary part $t + O(L^{-1})$, outside this rectangle. Stirling's formula makes the horizontal edges and the discarded right-hand tails exponentially small. On the new vertical edge, the functional equation and the absolutely convergent zeta series on $\Re s > 1$ give

$$|\zeta(w_1 + u) \zeta(w_2 + u)| \ll_\varepsilon T^{1+2\varepsilon}.$$

Since $|T^{3u}| = T^{-3/2-3\varepsilon}$ and $\Gamma(u)$ is integrable along that line, the pointwise replacement error is $O_\varepsilon(T^{-1/2+2\varepsilon})$, uniformly on fixed shift discs. Multiplication by the four amplifiers costs at most $O(T^{S_*/2})$, and integration over $t \asymp T$ gives $O_\varepsilon(T^{1/2+S_*/2+2\varepsilon}) = o(T)$ after choosing $\varepsilon < (1 - S_*)/4$. The factor $H_T(t, z)$ is retained throughout: its pure-power part has the derivative bounds needed below and tends to one only in the diagonal limit. Its $O(T^{-1})$ remainder is absorbed in the same error budget.

The diagonal equation is $h_1 h_2 h_3 = kmn$. Away from it, the near-diagonal logarithmic frequency is at least a constant times $T^{-(A_1+A_2+A_3)}$. Its product with T therefore tends to infinity by a fixed power. The exponentially smoothed series has polynomially bounded total coefficient mass, so sufficiently many integrations by parts discard the off-diagonal uniformly in the shifts. On the diagonal $mn \leq T^{A_1+A_2+A_3}$, and hence $e^{-mn/T^3} \rightarrow 1$ uniformly. There is no additional condition analogous to $R_0 + C_0 \leq 1$ in the balanced case.

Apply Lemma 4.1, now with the three positive profiles fixing the row sums a, b, c and the fourth profile fixing the last column sum d . The matrix can be written

$$\begin{pmatrix} v_1 & r_1 - v_1 & a - r_1 \\ v_2 & r_2 - v_2 & b - r_2 \\ v_3 & r_3 - v_3 & c - r_3 \end{pmatrix}, \quad r_1 + r_2 + r_3 = H, \quad 0 \leq v_i \leq r_i.$$

Again the Jacobian is one. After reflection the polynomial weight is

$$p(v_1 + v_2 + v_3 - 1/2) q(1/2 - H + v_1 + v_2 + v_3).$$

Putting $y_i = v_i - r_i/2$ gives (4.3). Uniform holomorphic convergence and Cauchy's formula justify the differentiation. $\square$

4.4 Reflection errors and all projected Gram blocks

We now assemble the projected Gram form. First, we control the error in the centered functional equation. Cauchy's formula applied to $H_T(t, z) - 1$ gives, for fixed q ,

$$\left| \chi(s)^{-1} C_q(s) - C_{q-}(\bar{s}) \right| \ll_q L^{-1} \sum_{j=0}^4 |C_{x^j}(\bar{s})|. \quad (4.4)$$

After multiplication by the two profile polynomials belonging to one test, the squared weighted L^2 norm of this error is $O(TL^7)$ by Proposition 4.2. The other factor has squared norm

$O(TL^9)$. Cauchy–Schwarz therefore makes each resulting Gram error $O(TL^8) = o(TL^9)$. A nonnegative smooth majorant of $|\Phi|$ suffices. This argument uses only second moments of the test functions.

Second, the reflected VV term is negligible. After one reflection it has the form

$$\int_{\mathbb{R}} \Phi(t/T) \chi(s)^{-2} C_p(s) C_q(\bar{s}) A_f(s) A_g(s) A_h(s) A_k(s) \, dt.$$

Move the positive-side zeta to the negative side and use the same one-sided expansion as above. The remaining χ^{-1} has phase derivative $\log(t/2\pi) + O(t^{-1})$. The full phase derivative is

$$\log(t/2\pi) + \log \frac{mn}{h_1 h_2 h_3 h_4} + O(t^{-1}) \geq (1 - S_*) \log T + O(1).$$

There is no stationary point. Integration by parts, with the replacement bound from Proposition 4.3, gives $o(TL^9)$.

For two tests with polynomial/profile data $(p; a, b)$ and $(q; c, d)$, define the point kernels

$$\begin{aligned} G_{UU}(p, q; a, b, c, d) &= \frac{1}{2} \{ B_{p,q}(a, d; b, c) + B_{p,q^-}(a, c; b, d) \}, \\ G_{UV}(p, q; a, b, c, d) &= \frac{1}{2} \{ M_{q,p}(b, c, d; a) + M_{p,q^-}(a, c, d; b) \}, \\ G_{VV}(p, q; a, b, c, d) &= \frac{1}{2} B_{p,q}(a, b; c, d), \\ G_{VU}(p, q; a, b, c, d) &= G_{UV}(q, p; c, d, a, b). \end{aligned} \tag{4.5}$$

For profiles f, g, h, k , the Gram entry is their fourfold average against the appropriate kernel in (4.5). This proves the second limit in (1.3).

For example, $U_{p;f,g} \overline{U_{q;h,k}}$ has positive profiles f, k and negative profiles g, h , giving the first UU term. The real part of $\chi(s)^3 U_{p;f,g} U_{q;h,k}$, followed by (4.4), gives the second term with q^- . For UV , conjugating its ordinary product gives the unbalanced ordering $(g, h, k; f)$ with polynomials (q, p) ; its reflected product gives $(f, h, k; g)$ with (p, q^-) . The reflected VV product is the nonstationary term just treated. These calculations also account for every factor $1/2$ in (4.5).

All formulas hold for real signed Φ . Wherever positivity was used in a bound, dominate $|\Phi|$ by a smooth nonnegative weight, or write Φ as a difference of two such weights. Symmetry and nonnegativity of the resulting Gram form follow directly from its origin as a limit of squared norms.

5 From admissible profiles to endpoint kernels

The rational certificate uses point evaluations, whereas the moment theorems require actual profiles. We therefore take two successive limits, in the order specified in Section 1.

For a U -endpoint $a > 0$, choose a nonnegative $\psi \in C_c^\infty((0, 1))$ with integral one and put

$$f_{a,n}(x) = \frac{1}{h_n} \psi\left(\frac{a-x}{h_n}\right), \quad h_n \downarrow 0, \quad h_n < a.$$

This is a probability density supported strictly inside $(0, a)$, and $f_{a,n}(x) \, dx$ converges weakly to δ_a . For a pair $0 < a \leq b$ with $3a + 5b < 2$, use these profiles at a, b . The strict support bounds remain valid at every index.

For the V -family, fix the common cutoff

$$\eta = \frac{19999}{80000} < \frac{1}{4}.$$

For each endpoint $0 < a < \eta$, set $k_n = \lfloor na/\eta \rfloor$ and use the polynomial probability density on $[0, \eta]$

$$b_{a,n}(x) = \frac{n+1}{\eta} \binom{n}{k_n} \left(\frac{x}{\eta}\right)^{k_n} \left(1 - \frac{x}{\eta}\right)^{n-k_n}, \quad (5.1)$$

extended by zero. Its mean and variance are respectively

$$\frac{\eta(k_n + 1)}{n + 2} \longrightarrow a, \quad \frac{\eta^2(k_n + 1)(n - k_n + 1)}{(n + 2)^2(n + 3)} \leq \frac{\eta^2}{4(n + 3)}.$$

Thus $b_{a,n}(x) dx \Rightarrow \delta_a$. For sufficiently large n , $1 \leq k_n \leq n - 1$, so its zero extension is continuous at both ends. Its support bound is η , irrespective of a . Its degree may grow with n , but n stays fixed during every $T \rightarrow \infty$ limit.

All the required kernels are continuous in the endpoint variables on their admissible domains. For K_p this follows from its compact integral representation. For $B_{p,q}$ and $M_{p,q}$, write the integrals on a fixed surrounding box, using indicators for their defining inequalities. As the endpoints vary, the indicators converge away from finitely many affine hyperplanes, and the integrands have a common compact bound. Dominated convergence applies; when the unbalanced height H approaches zero, its integration region shrinks to a null set. The same argument applies to all the projected Gram kernels.

Weak convergence of the product profile measures now gives, for any fixed finite collection of endpoint tests,

$$\begin{aligned} k_{j,n} &\longrightarrow k_j, & G_{jk,n} &\longrightarrow G_{jk}, \\ N_n := \sum_j c_j k_{j,n} &\longrightarrow N := \sum_j c_j k_j, & D_n := \sum_{j,k} c_j c_k G_{jk,n} &\longrightarrow D := \sum_{j,k} c_j c_k G_{jk}. \end{aligned} \quad (5.2)$$

Here $k_j = K_{p_j}(a_j, b_j)$ for a U -row and $k_j = K_{p_j}(a_j, 1 - b_j)$ for a V -row, while G_{jk} is the corresponding projected point kernel from Section 4. Since there are finitely many entries, one common sufficiently large profile index works for all of them. Only the denominator D_n of the chosen vector must eventually be positive; invertibility of the whole Gram matrix is unnecessary.

6 The explicit certificate

At fixed endpoints and polynomial degrees, the kernels determine k and G , and coefficient selection is governed by $(c^\top k)^2 / (c^\top G c)$. On a subspace where G is positive definite and $k \neq 0$, Cauchy–Schwarz in the G -inner product gives the maximizing direction $G^{-1}k$. This describes the finite optimization problem; the proof uses only the rational vector specified below. Neither its 250 tests nor the degree-four cutoff assert optimality.

The certificate consists of the 250 rows (type, A, B, r, c) in `certificate_250.tsv` at the fixed repository version [5]. A row denotes endpoints $a = A/10^8$, $b = B/10^8$, coefficient c , and test polynomial $p_r(x) = T_r(2x)$, where T_r is the Chebyshev polynomial:

r	0	1	2	3	4
$p_r(x)$	1	$2x$	$8x^2 - 1$	$32x^3 - 6x$	$128x^4 - 32x^2 + 1$.

There are 220 U -rows and 30 V -rows. The data satisfy

$$0 < a \leq b, \quad \min_{U\text{-rows}} (2 - 3a - 5b) = \frac{19997}{10^8} > 0, \quad \max_{V\text{-rows}} \{a, b\} = 0.249975 < \eta.$$

Thus Section 5 applies. Rows of the same type and with the same endpoints can be combined into one polynomial, leaving 178 groups without changing N or D .

6.1 Exact evaluation

Rational endpoints and polynomial tests make every kernel entry rational. For the numerator, simplex moments reduce the calculation to polynomial integration of degree at most nine. For the balanced kernel, slicing by the sum of its four edge variables gives a piecewise cubic density, and the resulting integrand has degree at most twelve. For the unbalanced kernel, box integration followed by inclusion–exclusion leaves polynomial simplex integrals. Appendix C records the integration formulas, breakpoints and exact interpolation rule used in this finite computation.

6.2 The certified inequalities

Let N, D be the endpoint values in (5.2). Exact rational evaluation of the explicit kernels gives

$$N > 46159, \quad 0 < D < 18854000000000. \quad (6.1)$$

Consequently,

$$R := \frac{9!N^2}{D} > \frac{9! \cdot 46159^2}{18854000000000} = \frac{1208080410327}{29459375000} = 41.00835168 \dots > 41.007. \quad (6.2)$$

The strict margin allows for profile approximation and smoothing. At the snapshot [5], the exact numerator and denominator bounds are proved in `SixthMoment/Certificate/Exact/NumeratorBound.lean` and `SixthMoment/Certificate/Exact/DenominatorBound.lean`; `SixthMoment/Certificate/Bounds.lean` records the final rational comparison. Together with the endpoint table and the formulas in Appendix C, these specify the numerical part of the proof reproducibly.

7 Passage to the sixth moment

By (5.2), $N_n \rightarrow N$ and $D_n \rightarrow D > 0$, so $R_n := 9!N_n^2/D_n \rightarrow R$. Fix $q < R$. Choose a sufficiently large fixed n , a weight $0 \leq \Phi \leq 1$ in $C_c^\infty((1, 2))$, and a fixed integer $J \geq 1$ such that

$$mR_n(1 - 2^{-J}) > q, \quad m = \int_{\mathbb{R}} \Phi.$$

This is possible because m may be arbitrarily close to one. The moment limits and (1.4) give

$$\liminf_{Y \rightarrow \infty} \frac{\int_{\mathbb{R}} \Phi(t/Y) |\zeta(\frac{1}{2} + it)|^6 dt}{(a_3/9!)Y(\log Y)^9} \geq mR_n.$$

For $Y_j = T/2^{j+1}$, $0 \leq j < J$, the weights $\Phi(t/Y_j)$ have disjoint supports inside $[0, T]$. Summing these finitely many lower bounds and using $\log Y_j / \log T \rightarrow 1$ yields

$$\liminf_{T \rightarrow \infty} \frac{M_3(T)}{(a_3/9!)T(\log T)^9} \geq mR_n \sum_{j=0}^{J-1} 2^{-j-1} = mR_n(1 - 2^{-J}) > q.$$

Letting $q \uparrow R$, we conclude that

$$\liminf_{T \rightarrow \infty} \frac{M_3(T)}{(a_3/9!)T(\log T)^9} \geq R > 41.007,$$

which proves Theorem 1.

A The crossed main terms

We make explicit the source-to-notation step in the contour calculation of Theorem 2.1. Write $f_h = f(\log h/L)$ and $g_k = g(\log k/L)$, and put

$$C_\lambda(z) = (\alpha + z, -\delta - z, \gamma + z, -\beta - z).$$

In [1, Section 2.4, pp. 11–12], use the coefficient sequences f_h, g_k . After the change of variable called $s \mapsto \beta + \delta + 2s$ there, the arithmetic part of the unnumbered formula for $M_{\alpha, \beta, \gamma, \delta}^{**}$ is

$$\begin{aligned} & \frac{\zeta(1 + \alpha - \beta)\zeta(1 + \gamma - \delta)}{\zeta(2 + \alpha - \beta + \gamma - \delta)} \zeta(1 + \alpha + \gamma + 2z) \zeta(1 - \beta - \delta - 2z) \\ & \times \sum_{d \geq 1} \sum_{(h_0, k_0)=1} \frac{f_{dh_0} g_{dk_0}}{d h_0^{1-\beta-z} k_0^{1-\delta-z}} \eta_{\lambda, h_0, k_0}(0, 0, \beta + \delta + 2z) = \mathcal{M}_{C_\lambda(z)}(T). \end{aligned} \quad (\text{A.1})$$

Here η is the finite Euler factor defined in [1, equation (23)]. The equality follows from the prime-power identity [1, equation (9)], with all four shifts increased by z : the factors $h_0^{\beta+z}$ and $k_0^{\delta+z}$ convert the two η factors into $\mathcal{B}_{C_\lambda(z)}(h_0)$ and $\mathcal{B}_{C_\lambda(z)^*}(k_0)$. This also identifies the coefficient normalization $1/(dh_0k_0)$ with $d/(hk)$ in (2.1).

Before integration against $\Phi(t/T) dt$, the original and dual zero-frequency terms corresponding to the (β, δ) swap therefore combine into

$$\left(\frac{t}{2\pi}\right)^{-\beta-\delta} \frac{1}{2\pi i} \left(\int_{(c)} - \int_{(-c)} \right) \frac{G(z)}{z} \mathcal{M}_{C_\lambda(z)}(T) dz, \quad (\text{A.2})$$

where $c > 0$ is fixed and sufficiently small. The gamma-factor replacement has already been included in the error in Theorem 2.1.

Indeed, if $R_\lambda = (-\delta, -\gamma, -\beta, -\alpha)$, then $C_{R_\lambda}(z) = C_\lambda(-z)$ up to permutations within the two shift pairs. The arithmetic factor is invariant under these permutations. Changing z to $-z$ in the dual integral and using the evenness of G gives (A.2). Besides $z = 0$, the possible poles in this strip are

$$z = -\frac{\alpha + \gamma}{2}, \quad z = -\frac{\beta + \delta}{2}.$$

They are cancelled by the prescribed zeros of G . The residue theorem thus evaluates (A.2) as

$$\left(\frac{t}{2\pi}\right)^{-\beta-\delta} \mathcal{M}_{(\alpha, -\delta, \gamma, -\beta)}(T).$$

Permuting α, β and γ, δ gives the other three crossed terms. Together with the two diagonal residues, they are exactly the six terms in (2.4).

B The reflection calculation

Proof of Lemma 3.2. It suffices to check $z \neq 0$. Write $E(a, z) = e^{az} - 1 - az$ and

$$\begin{aligned} N(a, b, z) &= z(1 - z/2)E(a, z)E(b, z) \\ &\quad - \frac{z^3}{2}(a^2E(b, z) + b^2E(a, z)) + \frac{a^2b^2z^4}{4}(e^z - 1). \end{aligned}$$

Elementary integration in the definitions of B, C, H gives $z^7 J(a, b; -z) = N(a, b, z)$. For the other side, $\min(a, 1 - b) = a$ and the translated simplex nodes are $0, a - t, b + t, 1$. The

divided-difference formula gives

$$\begin{aligned}
2z^3 e^{z/2} K_z(a, 1-b) = \int_0^a [& -(b-a+2t)(1-a+t)(1-b-t) \\
& + e^{z(a-t)}(b+t)(1-b-t) \\
& - e^{z(b+t)}(a-t)(1-a+t) \\
& + e^z(a-t)(b+t)(b-a+2t)] dt.
\end{aligned}$$

Integrating the polynomial and exponential-quadratic terms yields $2N(a, b, z)/z^4$. Thus both sides of (3.17) equal $N(a, b, z)/(z^7 e^{z/2})$. Their integral definitions are entire, so the identity also holds at zero. $\square$

C Exact rational evaluation

For the numerator, the simplex moment formula is

$$\mathbb{E}(q \cdot X)^r = \frac{6r!}{(r+3)!} \sum_{i_0+i_1+i_2+i_3=r} q_0^{i_0} q_1^{i_1} q_2^{i_2} q_3^{i_3}. \quad (\text{C.1})$$

For $r \leq 4$, this reduces the numerator kernel to integration of a polynomial in t of degree at most nine over a rational interval.

For the balanced kernel, the dependence on the four edge variables is only through $t = \sum r_{ij}$. Set

$$e = \min(a+b, c+d), \quad \ell = (t-b)_+, \quad u = \min(a, t), \quad m = (t-d)_+, \quad v = \min(c, t),$$

where $x_+ = \max(x, 0)$. The density of this slice is

$$\begin{aligned}
W(t) &= \int_\ell^u \int_m^v (\min(r, s) - \max(0, r+s-t)) ds dr, \\
6W(t) &= F_t(u, v) - F_t(\ell, v) - F_t(u, m) + F_t(\ell, m), \\
F_t(r, s) &= 3r^2 s + (r-s)_+^3 - (r+s-t)_+^3.
\end{aligned} \quad (\text{C.2})$$

With $\Delta = (a+b-c-d)/2$, it follows that

$$B_{p,q}(a, b; c, d) = \int_0^e W(t) \int_{-(1-a-b-c-d+2t)/2}^{(1-a-b-c-d+2t)/2} p(\Delta-x)q(-\Delta-x) dx dt.$$

Split $[0, e]$ at the points of $\{a, b, c, d, a+c, a+d, b+c, b+d\}$ that lie in the interval. On each resulting interval, W is cubic and the inner integral has degree at most nine, so the product has degree at most twelve. Rational antiderivatives therefore suffice. Equivalently, the implementation uses the exact 13-node interpolation rule

$$\int_\ell^u Q(t) dt = (u-\ell) \sum_{j=0}^{12} w_j Q\left(\ell + \frac{j}{12}(u-\ell)\right), \quad w_j = \int_0^1 \prod_{\substack{0 \leq k \leq 12 \\ k \neq j}} \frac{12x-k}{j-k} dx.$$

The weights are rational, and the identity is exact for $\deg Q \leq 12$.

For the unbalanced kernel, first expand the integrand and use

$$\int_{-r/2}^{r/2} y^{2j} dy = \frac{r^{2j+1}}{2^{2j}(2j+1)}, \quad \int_{-r/2}^{r/2} y^{2j+1} dy = 0.$$

Evaluate the remaining capped-simplex polynomial integral by inclusion–exclusion over $r_1 \leq a$, $r_2 \leq b$, $r_3 \leq c$. After translating each violated cap, expand the polynomial and use, for $H \geq 0$,

$$\int_{\substack{r_1, r_2 \geq 0 \\ r_1 + r_2 \leq H}} r_1^i r_2^j (H - r_1 - r_2)^k dr_1 dr_2 = \frac{i!j!k!}{(i+j+k+2)!} H^{i+j+k+2}. \quad (\text{C.3})$$

Terms with negative translated H are zero. These rules, the projected Gram formulas and the rational input rows completely specify the finite computation.

References

- [1] S. Bettin, H. M. Bui, X. Li and M. Radziwiłł, *A quadratic divisor problem and moments of the Riemann zeta-function*, J. Eur. Math. Soc. **22** (2020), 3953–3980. doi:10.4171/JEMS/999. The equation, proposition and page numbers used here refer to arXiv:1609.02539v1.
- [2] J. B. Conrey and A. Ghosh, *A conjecture for the sixth power moment of the Riemann zeta-function*, Int. Math. Res. Not. **1998**, no. 15, 775–780. doi:10.1155/S1073792898000476. arXiv:math/9807187.
- [3] C. de Boor, *Math/CS 887: Approximation Theory*, class notes, Spring 2003, version of 21 November 2009, pp. 52–53, formula (83). <https://pages.cs.wisc.edu/~deboor/887/notes.pdf>.
- [4] B. Durkan and T. Page, *Amplified moments of the Riemann zeta function*, arXiv:2606.27323v2 (2026).
- [5] N. Fraccaro, *Sixth moment: Lean formalization and exact certificate*, repository snapshot 950bfaff8724d4adb1c37535634a139b70228df0. <https://github.com/Niklas-Fraccaro/sixth-moment-lean/tree/950bfaff8724d4adb1c37535634a139b70228df0>. The endpoint table is `certificate_250.tsv`; the analytic limits and exact evaluations are in `SixthMoment/Numerator`, `SixthMoment/Denominator` and `SixthMoment/Certificate`.
- [6] J. P. Keating and N. C. Snaith, *Random matrix theory and $\zeta(1/2 + it)$* , Comm. Math. Phys. **214** (2000), 57–89. doi:10.1007/s002200000261.
- [7] N. Watt, *Kloosterman sums and a mean value for Dirichlet polynomials*, J. Number Theory **53** (1995), 179–210. doi:10.1006/jnth.1995.1086.
- [8] M. P. Young, *A short proof of Levinson’s theorem*, Arch. Math. **95** (2010), 539–548. doi:10.1007/s00013-010-0199-9. The lemma numbers used here refer to arXiv:1002.4403v1.