

# A lattice-theoretic approach to the abc conjecture

## Weighted arithmetic derivatives, a master inequality, and the structure of the residual family

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### Abstract

We develop a self-contained lattice-theoretic framework for the abc conjecture. For coprime positive integers  $a + b = c$  with  $k = \omega(ab)$ , the framework rests on a weighted arithmetic derivative  $d^\psi$  whose additivity equation  $d^\psi(a + b) = d^\psi(a) + d^\psi(b)$  is solvable modulo the modulus  $g_c = \gcd_{p \parallel c}(\frac{c}{p} v_p(c))$ , and on a Wronskian identity  $W = a d^\psi(b) - b d^\psi(a) = M_{ab} H(u)$ , where  $M_{ab} = ab / \text{rad}(ab)$  and  $H$  is an explicit linear form. A divisibility lemma  $M_a M_b M_c \mid W$  and an elementary upper bound yield the master inequality  $c \leq \mu_{ab} \cdot \frac{\Omega(ab)}{2} \cdot \text{rad}(abc)$ , in which the defect factor  $\mu_{ab}$  is the length of a short vector in a solution lattice  $\Lambda_{ab}$  of determinant  $g_c/g$ . Writing  $\eta = \log \mu_{ab} / \log c$ , geometry-of-numbers input (Hermite's bound and an orthogonal covering-radius sandwich; an earlier "min-max barrier" is falsified in Theorem 4.5) yields, conditionally on an explicit gap hypothesis  $(\star')$  isolated in Section 7, the bound  $\eta \leq \frac{1}{k} + o(1)$ , whence  $\limsup_{c \rightarrow \infty} q(a, b, c) \leq \frac{k}{k-1} \leq 2$ , i.e.  $c \lesssim \text{rad}(abc)^{2+o(1)}$ . On the subclass  $a + b = r^e$  the method closes unconditionally (Zsigmondy for  $r \geq 3$ ; cyclotomic/LTE control of valuations for  $a = 1$ ). The residual family is localized by a power-congruence bridge:  $q^V \parallel b$  forces  $e$  into a single residue class modulo  $d q^{V-1}$  ( $d = \text{ord}_q(r)$ ), of convergent total density; the unique unclosed layer is equivalent to Pillai-type equations  $r^e - q^V = a'$  with  $a > 1$ . We also prove a structural ceiling: the main term of  $\eta$  is at most  $\frac{1}{k} + o(1)$  and attains  $\frac{1}{k}$  on the family  $c = r^e$ , so the full abc conjecture ( $q \leq 1 + \varepsilon$ ) is unreachable inside this framework. A final section determines the reach and limits of the method elementarily: almost all  $c$  have every coprime decomposition with  $q \leq 1 + o(1)$  (density theorem, by one-variable counting), while any argument within the method class  $\mathcal{M}$  cannot prove  $\limsup q \leq \alpha$  for any  $\alpha < 2$ , and the only remaining channel is equivalent to a sparse growth-difference Pillai core (obstacle theorem; one clause conditional on the recorded working hypothesis). All statements are proved in full or are explicitly labelled as working hypotheses; numerical illustrations are included, and no claim of a proof of the abc conjecture is made.

**Keywords:** abc conjecture; lattice method; weighted arithmetic derivative; Wronskian; density theorem; obstacle theorem; Pillai equation; errata

## 1 Introduction

### 1.1 The abc conjecture and the quality

For a positive integer  $n$  let  $\text{rad}(n) = \prod_{p \mid n} p$  be its radical. The abc conjecture of Oesterlé and Masser [1, 2] asserts that for every  $\varepsilon > 0$  there is a constant  $K_\varepsilon$  such that for all coprime positive integers  $a + b = c$ ,

$$c \leq K_\varepsilon \cdot \text{rad}(abc)^{1+\varepsilon}.$$

Define the quality of a coprime triple by

$$q(a, b, c) = \frac{T}{R}, \quad T = \log c, \quad R = \log \text{rad}(abc).$$

Then (1) is equivalent to  $q \leq 1 + \varepsilon$  outside a finite exceptional set. The strongest unconditional bound is due to Stewart and Yu [3, 4]:  $c < \exp(CR^{1/3}(\log R)^3)$ , a sub-exponential bound. Any polynomial bound  $c \ll R^{O(1)}$  would already be a qualitative improvement.

## 1.2 Results of this paper

We develop a lattice-theoretic framework whose principal outputs are the following. All undefined notation is introduced in Sections 2 and 3; in particular  $k = \omega(ab)$ ,  $\Omega(ab)$  counts prime factors of  $ab$  with multiplicity, and the defect factor  $\mu = \min\{\|u\|_\infty : u \in \Lambda, H(u) \neq 0\}$  is the shortest non-kernel weight of an explicit solution lattice  $\Lambda$ .

**Theorem 1.1** (Master inequality, Theorem 2.11). *For coprime  $a + b = c$ ,*

$$c \leq \mu \cdot \frac{\Omega(ab)}{2} \cdot \text{rad}(abc).$$

**Theorem 1.2** (Sandwich, Theorem 4.3). *With the density parameter  $x = \det(\Lambda)^{(k-1)/k}/\Delta$ ,*

$$\frac{x}{\sqrt{k}} \leq \kappa \leq K(x) = \sqrt{x^2 + Q^2 x^{-2/(k-1)}}.$$

*The earlier “min-max barrier”  $\kappa \leq Q^{(k-1)/k}$  is false (Theorem 4.5: the triple  $1 + 242 = 3^5$  has  $\kappa = 1.3416 > 0.7071$ ); the two branches are orthogonal components and must combine by quadrature.*

**Theorem 1.3** (Complete solution for  $k = 2$ , Theorems 5.1 and 5.3). *For  $\omega(ab) = 2$ ,*

$$\kappa \leq \max(\sqrt{\gamma_2}, \gamma_2 x) = \max(1.0746, 1.1547x),$$

*and on the bad branch the sharp geometric bound  $\kappa/x \leq \max(|\cos \varphi|, |\sin \varphi|)/\sin \psi \leq \gamma_2$  holds. A 4461-sample scan violates neither bound.*

**Theorem 1.4** (Pointwise exact bound; falsification of pointwise grading, Theorems 6.1 and Proposition 6.3). *Pointwise and unconditionally,  $q \leq 1/(1 - \eta - \omega/T)$  with  $\eta = \log \mu / \log c$ . The pointwise bound  $\eta \leq 1/k$  is false (certified counterexamples:  $1 + 242 = 3^5$ ,  $1 + 2047 = 2^{11}$ ,  $49 + 531392 = 531441$ ).*

**Theorem 1.5** (Conditional asymptotic, Theorem 6.5). *Under the gap condition  $(\star')$  — equivalently  $\text{rad}(ab) \geq c^{1/2-o(1)}$ , the unique remaining gap (Theorem 7.1) — one has  $\eta \leq \frac{1}{k} + O(\frac{\log \log c}{\log c})$ , hence  $\limsup_{c \rightarrow \infty} q \leq \frac{k}{k-1} \leq 2$ , i.e.  $c \lesssim \text{rad}(abc)^{2+o(1)}$ . The condition is false for high-exponent prime-power pairs ( $1/u + 1/v < 1/2$ ): exactly the Beal/Pillai core.*

**Theorem 1.6** (Unconditional subclass, Section 8). *On the subclass  $a + b = r^e$  with  $a = 1$  (any  $r$ ) and for  $r \geq 3$  generally, the gap condition holds unconditionally (Zsigmondy, LTE, Catalan), so the conclusion  $\eta \leq \frac{1}{k} + o(1)$  is unconditional there.*

**Theorem 1.7** (Structure of the residual family, Section 9). *Triples failing the gap condition split into three layers by  $V = \max_q v_q(ab)$ . Layers I and II close unconditionally. In the unique residual layer III, a power-congruence bridge (Theorem 9.1) forces the exponent  $e$  into a single residue class modulo  $dq^{V-1}$  ( $d = \text{ord}_q(r)$ ), of convergent total density; layer III is equivalent to Pillai-type equations  $r^e - q^V = a'$ , with  $a = 1$  closed by Catalan [5] and  $a > 1$  open.*

**Theorem 1.8** (Structural ceiling, Section 10). *The main term of  $\eta$  is  $\Pi = \frac{1}{k} \frac{\log \det \Lambda}{\log c}$ , and  $\det \Lambda \asymp c\theta_c = c^{1+o(1)}$ ; hence  $\Pi \rightarrow 1/k \neq 0$  and the full abc conjecture (1) is structurally unreachable inside this framework, independently of any improvement of the constants.*

**Theorem 1.9** (Density theorem, Theorem 11.7). *Elementary counting, independent of the lattice framework, gives  $E(X) := \#\{c \leq X : \exists \text{ coprime } a+b=c, q>2\} \ll X^{1/2+\varepsilon(X)}$  with  $\varepsilon(X) = o(1)$ ; for density 1 of integers  $c$ , every coprime decomposition satisfies  $q \leq 1+o(1)$ .*

**Theorem 1.10** (Obstacle theorem, Theorem 11.12). *For the method class  $\mathcal{M}$  of this paper (divisibility identities, geometry of numbers, effective transcendence bounds), every  $\mathcal{M}$ -argument yields at best  $\limsup q \leq k/(k-1) \leq 2$ , the value 2 is tight inside  $\mathcal{M}$ , and full abc is unreachable in  $\mathcal{M}$  given the working hypothesis of Appendix C on the transcendence layer. Equivalently,  $\sup\{\alpha : \exists \mathcal{M}\text{-argument proving } \limsup q \leq \alpha\} = 2$  (Corollary 11.13).*

### 1.3 Honest framing

We emphasize three points. First, this paper does not prove the abc conjecture, and Theorem 1.8 explains precisely why the present framework cannot. Second, two earlier claims of the project are false and are falsified in the body of the paper with certified counterexamples (the min-max barrier, Theorem 4.5; pointwise  $\eta \leq 1/k$ , Proposition 6.3); a complete list of falsified or withdrawn statements is Appendix A. The single remaining hypothesis  $(\star')$  is isolated exactly and is not proved (Appendix C). Third, we observe strict circularity discipline: results that themselves follow from abc — e.g. the finiteness of solutions of the Nagell–Ljunggren equation in general (Shorey–Tijdeman [6]) or the Beal conjecture [7, 8] — are never used; only unconditional subclass results (Ljunggren [9], Nagell [10], Mihăilescu [5], Zsigmondy [11]) enter the proofs.

## 1.4 Notation

| Symbol                                                                            | Meaning                                          |
|-----------------------------------------------------------------------------------|--------------------------------------------------|
| $\text{rad}(n)$ , $M_n = n/\text{rad}(n)$                                         | radical of $n$ ; powerful part of $n$            |
| $\omega(n)$ , $\Omega(n)$                                                         | prime factors of $n$ without/with multiplicity   |
| $k = \omega(ab)$ , $q_{\min}$                                                     | number of distinct primes of $ab$ ; smallest one |
| $T = \log c$ , $R = \log \text{rad}(abc)$ , $q = T/R$                             | logarithmic height, radical, quality             |
| $d^\psi(n) = n \sum_{p n} \frac{v_p(n)}{p} u_p$                                   | weighted arithmetic derivative                   |
| $g_c = \gcd_{p c} \left( \frac{c}{p} v_p(c) \right)$                              | modulus of the additivity equation               |
| $\theta_c = g_c/c$ , $F_c = \theta_c \text{rad}(c)$                               | normalized modulus                               |
| $W = a d^\psi(b) - b d^\psi(a)$                                                   | Wronskian; $W = M_{ab}H(u)$                      |
| $D = \text{rad}(ab)$ , $M_{ab} = ab/D$                                            | radical and powerful part of $ab$                |
| $h, A \in \mathbb{Z}^k$ ; $H(z)$ , $A \cdot z$                                    | coefficient vectors and forms, Section 3         |
| $g_h = \gcd(h_i)$ , $a' = h/g_h$                                                  | content of $h$ ; primitive normal vector         |
| $\Lambda$ , $\Lambda_0 = \Lambda \cap \ker H$                                     | solution lattice; bad sublattice                 |
| $g = \gcd(g_c, A_1, \dots, A_k)$                                                  | index factor; $\det \Lambda = g_c/g$             |
| $N = [\ker H : \Lambda_0]$ , $\Delta = N a' _2$                                   | index; determinant of $\Lambda_0$                |
| $\mu = \min\{\ u\ _\infty : H(u) \neq 0\}$                                        | shortest non-kernel weight                       |
| $\kappa = \mu/\det^{1/k}$                                                         | normalized defect                                |
| $x = \det^{(k-1)/k}/\Delta$ , $\text{dist} = \det/\Delta$                         | density parameter; layer spacing                 |
| $\text{ani} = \Delta^{1/(k-1)}/\lambda_1(\Lambda_0)$                              | anisotropy defect                                |
| $\eta = \log \mu / \log c$ , $\Pi = \frac{1}{k} \frac{\log \det \Lambda}{\log c}$ | defect exponent; its main term                   |
| $\delta_c = 1 - \log \text{rad}(c) / \log c$                                      | power defect of $c$                              |
| $\rho_{\text{orth}}$ , $\mathcal{Q}$ , $C_k = (k-1)2^{k-4}$                       | covering bound; anisotropy factor                |
| $V = \max_q v_q(ab)$ , $d = \text{ord}_q(r)$                                      | maximal valuation; multiplicative order          |

## 1.5 Organization

Section 2 introduces the weighted arithmetic derivative, the Wronskian, and proves the master inequality and the surplus identity. Section 3 builds the solution lattice in the  $D = \text{rad}(ab)$  parametrization and proves the core identity (4) and the three determinant formulas. Section 4 proves the sandwich inequality, falsifies the min–max barrier, and gives the explicit kernel vectors. Section 5 solves  $k = 2$  completely. Section 6 proves the pointwise exact bound, falsifies pointwise grading, and proves the conditional asymptotic. Section 7 isolates the remaining gap. Section 8 treats  $c = r^e$ . Section 9 proves the power–congruence bridge and the three-layer sieve. Section 10 proves the ceiling. Section 11 determines the reach and the limits elementarily: the density theorem (almost all decompositions have  $q \leq 1 + o(1)$ ), the obstacle theorem (no  $\mathcal{M}$ -argument proves  $\limsup q \leq \alpha$  for  $\alpha < 2$ ), and the sparsity of the residual growth-difference Pillai core. Section 12 gives numerical illustrations. Section 13 lists open problems. Appendix A records the full errata history; Appendix B collects auxiliary estimates; Appendix C lists every unproved statement.

## 2 Weighted arithmetic derivatives and the master inequality

Throughout,  $a + b = c$  are coprime positive integers. For a prime  $p$  we write  $v_p$  for the  $p$ -adic valuation. We set

$$M_n := \frac{n}{\text{rad}(n)}, \quad k = \omega(ab), \quad \Omega(n) = \sum_{p|n} v_p(n).$$

## 2.1 The weighted arithmetic derivative

**Definition 2.1.** Let  $\psi = (u_p)_p$  be a family of integer weights indexed by primes. The **weighted arithmetic derivative** is

$$d^\psi(n) = n \sum_{p|n} \frac{v_p(n)}{p} u_p.$$

For  $u_p \equiv 1$  this is the classical arithmetic derivative  $n'$ . The point of the weights is the additivity equation: we seek weights making

$$d^\psi(a+b) = d^\psi(a) + d^\psi(b)$$

hold, i.e.

$$c \sum_{p|c} \frac{v_p(c)}{p} u_p = \sum_{p|a} \frac{a}{p} v_p(a) u_p + \sum_{p|b} \frac{b}{p} v_p(b) u_p. \quad (3)$$

**Lemma 2.2** (The correct modulus). For weights supported on the primes of  $c$ , the linear form

$$L_c(u) = \sum_{p|c} \frac{c}{p} v_p(c) u_p$$

represents exactly the multiples of  $g_c = \gcd_{p|c}(\frac{c}{p} v_p(c))$ ; that is,  $\{L_c(u) : u \in \mathbb{Z}^{\omega(c)}\} = g_c \mathbb{Z}$ . Consequently the additivity equation (3) is solvable for the right-hand side determined by  $a, b$  whenever that right-hand side lies in  $g_c \mathbb{Z}$ , and  $g_c$  is the natural modulus of the solution lattice. For  $c = r^e$  one has  $g_c = r^{e-1}e = M_c \cdot e$ ; in general  $g_c \mid \prod_{p|c} e_p p^{e_p-1} = c'$  (the arithmetic derivative), with equality exactly when  $c$  is a prime power.

*Proof.* The image of a linear form  $\sum_i m_i u_i$  over  $\mathbb{Z}$  is the ideal  $\gcd(m_1, \dots, m_s) \mathbb{Z}$ : it contains the gcd by Bézout, and every value is a multiple of the gcd. Apply this with  $m_p = \frac{c}{p} v_p(c)$ . For  $c = r^e$  there is a single term  $m_r = r^{e-1}e$ , so the gcd is  $r^{e-1}e = M_c \cdot e$ . In general each  $m_p$  divides  $\prod_p e_p p^{e_p-1}$  up to the cofactor, and the gcd divides the product; for a prime power the product collapses to the single term, giving equality.  $\square$

**Example 2.3** (Computing  $g_c$ ). (a)  $c = 243 = 3^5$ : a single term,  $g_c = \frac{243}{3} \cdot 5 = 405 = 3^4 \cdot 5 = M_c \cdot e$ . (b)  $c = 360 = 2^3 3^2 5$ : the three terms are  $\frac{360}{2} \cdot 3 = 540$ ,  $\frac{360}{3} \cdot 2 = 240$ ,  $\frac{360}{5} \cdot 1 = 72$ , and  $g_c = \gcd(540, 240, 72) = 12$ ; note  $12 < 72 = \prod e_p p^{e_p-1}$ . (c)  $c = 2^{18}$ :  $g_c = 2^{17} \cdot 18 = 2359296$ ,  $\theta_c = 9$ . (d)  $c = 121 = 11^2$ :  $g_c = 11 \cdot 2 = 22$ .

**Remark 2.4.** The  $g_c$  above is a gcd, not the product  $\prod e_p p^{e_p-1}$ : the two coincide for prime powers but diverge for multi-prime  $c$  (e.g.  $c = 360 = 2^3 3^2 5$ :  $g_c = \gcd(540, 240, 72) = 12$ , while  $\prod e_p p^{e_p-1} = 72$ ). The gcd form is the correct one; it makes  $\theta_c = g_c/c$  smaller and every subsequent margin larger (see Example 12.2).

**Proposition 2.5** (Size of  $\theta_c$ ). Write  $c = \prod_p p^{e_p}$ . Then

$$\theta_c = \frac{g_c}{c} \leq \min_{p|c} \frac{e_p}{p} \leq \frac{\log_2 c}{2},$$

so  $\theta_c = c^{o(1)}$  and  $\log \theta_c = O(\log \log c)$ .

*Proof.* Each term  $(c/p)e_p$  in the gcd is divisible by  $g_c$ , hence  $g_c \leq (c/p)e_p$  for every  $p \mid c$ , giving  $\theta_c \leq e_p/p$ . For the second inequality,  $e_p \leq \log_2 c$  and  $p \geq 2$ .  $\square$

## 2.2 The Wronskian

**Definition 2.6.** The **Wronskian** of the pair  $(a, b)$  relative to a weight family  $\psi$  is

$$W = a d^\psi(b) - b d^\psi(a).$$

**Lemma 2.7** (Factored form of  $W$ ). Write  $D = \text{rad}(ab)$  and  $M_{ab} = ab/D$ . Then

$$W = M_{ab} H(u), \quad H(u) = \sum_{q|ab} h_q u_q, \quad h_q = \begin{cases} -\frac{D}{q} v_q(a), & q \mid a, \\ +\frac{D}{q} v_q(b), & q \mid b. \end{cases}$$

*Proof.* Substitute Definition 2.1 and collect the coefficient of  $u_q$ : for  $q \mid a$  the coefficient is  $-b \cdot \frac{a}{q} v_q(a) = -\frac{ab}{q} v_q(a) = M_{ab} \cdot \left(-\frac{D}{q} v_q(a)\right)$ , and symmetrically for  $q \mid b$ .  $\square$

**Lemma 2.8** (Independence). Let  $\Lambda \subseteq \mathbb{Z}^k$  be the solution lattice of the additivity equation (Definition 3.4;  $\Lambda$  has full rank  $k$  since its index in  $\mathbb{Z}^k$  is  $g_c/g < \infty$ ). Then for every  $u \in \Lambda \setminus \ker H$  one has  $W = M_{ab} H(u) \neq 0$ ; such  $u$  exist, because  $\Lambda_0 = \Lambda \cap \ker H$  has rank at most  $k - 1$ . Moreover the forms  $H_a$  and  $H_b$  are linearly independent whenever  $a, b > 1$ .

*Proof.*  $\Lambda$  has rank  $k$  (finite index in  $\mathbb{Z}^k$ ) while  $\Lambda_0 = \Lambda \cap \ker H$  lies in the hyperplane  $\ker H$ , hence has rank  $\leq k - 1$ ; therefore  $\Lambda \setminus \ker H \neq \emptyset$ , and for any  $u$  in the difference,  $H(u) \neq 0$ , so  $W = M_{ab} H(u) \neq 0$  by Lemma 2.7. For the last statement,  $H_a$  and  $H_b$  have disjoint supports (the primes of  $a$  and of  $b$ ), so a linear dependence would force both to vanish identically;  $H_a \equiv 0$  would mean  $v_q(a) = 0$  for all  $q \mid a$ , i.e.  $a = 1$ .  $\square$

**Lemma 2.9** (Divisibility lemma).  $M_a M_b M_c \mid W$ .

*Proof.* We show  $M_a \mid W$ ,  $M_b \mid W$ ,  $M_c \mid W$  separately; the three factors are pairwise coprime (since  $a, b, c$  are), so their product divides  $W$ .

*Step 1:*  $M_a \mid W$ . Fix  $p^e \parallel a$ , so  $p^{e-1} \mid M_a$ ; we show  $p^{e-1} \mid W$ . Since  $\gcd(a, b) = 1$ ,  $p \nmid b$ . In  $W = a d^\psi(b) - b d^\psi(a)$  the first term contains the factor  $a$ , hence  $p^e \mid a d^\psi(b)$ . In the second term,

$$d^\psi(a) = a \sum_{\ell \mid a} \frac{v_\ell(a)}{\ell} u_\ell = a \cdot \frac{e}{p} u_p + a \sum_{\ell \neq p} \frac{v_\ell(a)}{\ell} u_\ell;$$

the first summand is divisible by  $p^{e-1}$  (as  $p^e \mid a$ ), and each remaining summand  $a \cdot \frac{v_\ell(a)}{\ell} u_\ell$  with  $\ell \neq p$  is divisible by  $p^e$ . Hence  $p^{e-1} \mid d^\psi(a)$ , so  $p^{e-1} \mid b d^\psi(a)$ . Both terms of  $W$  are divisible by  $p^{e-1}$ , and since this holds for every  $p \mid a$ ,  $M_a \mid W$ .

*Step 2:*  $M_b \mid W$ . Symmetric to Step 1, exchanging the roles of  $a$  and  $b$  and using  $W = -(b d^\psi(a) - a d^\psi(b))$ .

*Step 3:*  $M_c \mid W$ . Using additivity (3),  $d^\psi(c) = d^\psi(a) + d^\psi(b)$ , so

$$W = a d^\psi(b) - b d^\psi(a) = a d^\psi(c) - (a + b) d^\psi(a) = a d^\psi(c) - c d^\psi(a).$$

Fix  $p^e \parallel c$ ; then  $p \nmid a$  (as  $\gcd(a, c) = 1$ ). The term  $c d^\psi(a)$  is divisible by  $p^e$ . In  $d^\psi(c) = c \sum_{\ell \mid c} \frac{v_\ell(c)}{\ell} u_\ell$ , the  $\ell = p$  summand is  $c \cdot \frac{e}{p} u_p$ , divisible by  $p^{e-1}$ , and every other summand is divisible by  $p^e$ ; hence  $p^{e-1} \mid d^\psi(c)$  and therefore  $p^{e-1} \mid a d^\psi(c)$ . Both terms are divisible by  $p^{e-1}$ ; varying  $p \mid c$  gives  $M_c \mid W$ .  $\square$

**Lemma 2.10** (Upper bound). *With  $\mu_{ab}$  denoting the maximal weight magnitude needed in a minimal solution of (3) (formalized as a lattice norm in Section 3),*

$$|W| \leq ab \cdot \mu_{ab} \cdot \frac{\Omega(ab)}{2}.$$

*Proof.* By Lemma 2.7,  $|W| = M_{ab}|H(u)|$  and

$$|H(u)| \leq \max_q |u_q| \sum_{q|ab} \frac{D}{q} v_q \leq \mu_{ab} D \frac{\Omega(ab)}{2},$$

using  $\sum_q \frac{v_q}{q} \leq \frac{1}{2} \sum_q v_q = \frac{\Omega(ab)}{2}$  (since  $q \geq 2$ ). Multiplying by  $M_{ab}$  gives the claim, as  $M_{ab}D = ab$ .  $\square$

### 2.3 The master inequality and the surplus identity

**Theorem 2.11** (Master inequality).

$$c \leq \mu_{ab} \cdot \frac{\Omega(ab)}{2} \cdot \text{rad}(abc).$$

*Proof.* Lemma 2.9 and  $W \neq 0$  (Lemma 2.8) give  $|W| \geq M_a M_b M_c$ . Combining this with Lemma 2.10,

$$M_a M_b M_c \leq ab \mu_{ab} \frac{\Omega(ab)}{2} = M_a M_b \text{rad}(ab) \mu_{ab} \frac{\Omega(ab)}{2},$$

since  $ab = M_a M_b \text{rad}(ab)$  for coprime  $a, b$ . Cancelling  $M_a M_b$  and using  $c = M_c \text{rad}(c)$ ,

$$c = M_c \text{rad}(c) \leq \mu_{ab} \frac{\Omega(ab)}{2} \text{rad}(ab) \text{rad}(c) = \mu_{ab} \frac{\Omega(ab)}{2} \text{rad}(abc).$$

$\square$

**Theorem 2.12** (Surplus identity). *The logarithmic surplus  $X = T - R$  satisfies*

$$X = \log M_c - \log \text{rad}(ab).$$

*Proof.*  $T - R = \log c - \log \text{rad}(abc) = \log \frac{c}{\text{rad}(c)} - \log \text{rad}(ab) = \log M_c - \log \text{rad}(ab)$ .  $\square$

**Corollary 2.13** (Quality relation). *Taking logarithms in Theorem 2.11 and writing  $\omega = \log \frac{\Omega(ab)}{2} = O(\log T)$ ,*

$$q \leq \frac{1}{1 - \eta - \omega/T}, \quad \eta = \frac{\log \mu_{ab}}{\log c}.$$

*In particular, any bound  $\eta \leq \alpha + o(1)$  with  $\alpha < 1$  yields  $\limsup q \leq 1/(1 - \alpha)$ .*

*Proof.*  $T \leq R + \log \mu_{ab} + \omega$ , so  $q = T/R \leq 1 + (\log \mu_{ab} + \omega)/R$ ; dividing  $T \leq R + (\eta + \omega/T)T$  by  $T$  gives  $1 \leq 1/q + \eta + \omega/T$ .  $\square$

### 3 The solution lattice

#### 3.1 The $D = \text{rad}(ab)$ parametrization

Let the distinct prime factors of  $ab$  be  $q_1, \dots, q_k$  and put  $v_i = v_{q_i}(ab)$ ,  $D = \text{rad}(ab) = \prod_i q_i$ . Define two integer vectors  $h, A \in \mathbb{Z}^k$  by

$$h_i = \begin{cases} -\frac{D}{q_i}v_i, & q_i \mid a, \\ +\frac{D}{q_i}v_i, & q_i \mid b, \end{cases} \quad A_i = \begin{cases} \frac{a}{q_i}v_i, & q_i \mid a, \\ \frac{b}{q_i}v_i, & q_i \mid b, \end{cases}$$

and the linear forms

$$H(z) = \sum_{i=1}^k h_i z_i = H_a(z) + H_b(z), \quad A \cdot z = \sum_{i=1}^k A_i z_i,$$

where  $H_a, H_b$  collect the summands with  $q_i \mid a$  and  $q_i \mid b$  respectively. The vector  $h$  is exactly the coefficient vector of the Wronskian form of Lemma 2.7.

#### 3.2 The core identity

**Theorem 3.1** (Core identity).

$$D(A \cdot z) = b H_b(z) - a H_a(z) \quad \text{for all } z \in \mathbb{Z}^k. \quad (4)$$

*Proof.* For each coordinate,  $DA_i = D \cdot \frac{a}{q_i}v_i = a \cdot \frac{D}{q_i}v_i = -ah_i$  if  $q_i \mid a$  and  $DA_i = +bh_i$  if  $q_i \mid b$ : indeed, for  $q_i \mid a$ ,  $DA_i = D \cdot \frac{a}{q_i}v_i$  and symmetrically for  $q_i \mid b$ . Multiplying by  $z_i$  and summing over  $i$  gives (4).  $\square$

**Remark 3.2.** Identity (4) is the structural pivot of the whole framework: it identifies the two natural linear forms on  $\mathbb{Z}^k$  up to the scalars  $a, b, D$ . It is independent of  $g_c$ , and it is coordinatewise checkable (Example 3.9).

#### 3.3 The arithmetic meaning of $N$

**Proposition 3.3.**  $N$  measures the proportional deviation of  $A$  from  $h$ . From (4),  $A_i/h_i = -a/D$  for  $q_i \mid a$  and  $+b/D$  for  $q_i \mid b$ . Hence

$$A \parallel h \iff -a/D = b/D \iff a, b \text{ are not both } > 1.$$

*Consequently:* if  $a = 1$  (or  $b = 1$ ) then  $\ker A = \ker H$  and the index  $N$  below equals 1 (the typical case); if  $a, b > 1$  then  $A$  and  $h$  are not proportional,  $\ker A \neq \ker H$ , and typically  $N > 1$ .

*Proof.* The ratios are read off the coordinate identities  $DA_i = -ah_i$  ( $q_i \mid a$ ) and  $DA_i = +bh_i$  ( $q_i \mid b$ ) proved in Theorem 3.1. A proportionality  $A = \lambda h$  would force  $\lambda = -a/D$  on the support of  $a$  and  $\lambda = +b/D$  on the support of  $b$ ; if both supports are nonempty (i.e.  $a, b > 1$ ) this requires  $-a/D = b/D$ , impossible for positive  $a, b$ . If  $a = 1$ , all coordinates satisfy  $DA_i = bh_i$ , so  $A = (b/D)h$  and  $\ker A = \ker H$ ; the index  $N = [\ker H : \Lambda \cap \ker H]$  then coincides with  $[\ker A : \Lambda \cap \ker A]$ , and the defining congruence  $A \cdot z \equiv 0 \pmod{g_c}$  pulls back to a condition on the single functional  $H$ , giving  $N = 1$  in the generic case. The deviation of the two ratios  $-a/D$  and  $+b/D$  is exactly what  $N$  records; this explains the empirical pattern “ $a = 1 \Rightarrow N = 1$  typical;  $a, b > 1 \Rightarrow N > 1$  typical” observed in the manuscripts.  $\square$

### 3.4 The three determinant formulas

**Definition 3.4.** *The solution lattice, the kernel lattice, and the bad sublattice are*

$$\Lambda = \{z \in \mathbb{Z}^k : A \cdot z \equiv 0 \pmod{g_c}\}, \quad \Lambda_0 = \Lambda \cap \ker H,$$

and  $g_h = \gcd(h_1, \dots, h_k)$ ,  $a' = h/g_h$ ,  $N = [\ker H : \Lambda_0]$ .

**Theorem 3.5** (Determinant formulas). (i)  $\det(\Lambda) = \frac{g_c}{g}$  where  $g = \gcd(g_c, A_1, \dots, A_k)$ . In particular  $\det(\Lambda) \leq g_c$ , with equality when  $a$  is prime.

(ii)  $\det(\ker H) = |a'|_2 = \frac{\|h\|_2}{g_h}$ .

(iii)  $\det(\Lambda_0) = N|a'|_2$ .

*Proof.* (i) The homomorphism  $\varphi : \mathbb{Z}^k \rightarrow \mathbb{Z}/g_c$ ,  $z \mapsto A \cdot z$ , has image generated by  $g = \gcd(g_c, A_1, \dots, A_k)$ , so  $\det(\Lambda) = [\mathbb{Z}^k : \Lambda] = |\operatorname{im} \varphi| = g_c/g$ . (ii)  $\ker H \subset \mathbb{Z}^k$  is a primitive sublattice of rank  $k-1$ ; its  $(k-1)$ -dimensional covolume equals the length of the primitive normal vector  $h/g_h = a'$ . (iii)  $\det(\Lambda_0) = [\ker H : \Lambda_0] \cdot \det(\ker H) = N|a'|_2$ .  $\square$

**Remark** (computing  $N$ ). For a  $\mathbb{Z}$ -basis  $b_1, \dots, b_{k-1}$  of  $\ker H$ ,

$$N = \frac{g_c}{\gcd(g_c, A \cdot b_1, \dots, A \cdot b_{k-1})},$$

since  $z \mapsto A \cdot z \pmod{g_c}$  embeds  $\ker H/\Lambda_0$  into  $\mathbb{Z}/g_c\mathbb{Z}$  with image generated by the  $A \cdot b_j$ . Anchors:  $3 + 125 = 128$  gives  $N = 7$ ; the Reyssat triple gives  $N = 5$  (Table 4).

**Lemma 3.6** (Exact formula for  $g_h$ ). *For each prime  $p$ ,*

$$v_p(g_h) = \min_{1 \leq i \leq k} (\varepsilon_p(i) + v_p(v_i)), \quad \varepsilon_p(i) = v_p(D/q_i).$$

*Proof.*  $h_i = \pm \frac{D}{q_i} v_i$ , so  $v_p(h_i) = v_p(D/q_i) + v_p(v_i)$ , and the valuation of a gcd is the minimum of the valuations.  $\square$

**Example 3.7** (Computing  $g_h$ ). (a)  $31 + 329 = 360$ :  $h = (-329, 1457, 217)$ . Since  $329 = 7 \cdot 47$ ,  $1457 = 31 \cdot 47$ ,  $217 = 7 \cdot 31$ , no prime divides all three components, so  $\gcd(329, 1457, 217) = 1$ ; hence  $g_h = 1$ ,  $a' = h$ , and  $|a'|_2 = \sqrt{329^2 + 1457^2 + 217^2} = \sqrt{2278179} = 1509.4\dots$  (b) Reyssat  $2 + 3^{10} \cdot 109 = 23^5$ :  $h = (-327, 2180, 6)$ ,  $\gcd(327, 2180, 6) = 1$ ;  $g_h = 1$ . (c)  $1 + 242 = 3^5$ :  $h = (11, 4)$ ,  $g_h = 1$ ,  $|a'|_2 = \sqrt{121 + 16} = \sqrt{137}$ .

**Lemma 3.8** (Corrected lower bound for  $|a'|_2$ ).

$$|a'|_2 \geq \frac{D}{g_h q_{\min}}, \quad q_{\min} = \min_i q_i.$$

*Proof.*  $\|h\|_2^2 = \sum_i (\frac{D}{q_i} v_i)^2 \geq (\frac{D}{q_{\min}})^2$  since some  $v_i \geq 1$ ; divide by  $g_h^2$  and take square roots.  $\square$

### 3.5 The duality

For later use we record the duality parametrization: with  $\Delta = \det(\Lambda_0) = N|a'|_2$  and  $\det = \det(\Lambda)$  set

$$x \cdot y = 1, \quad y = \frac{\Delta}{\det^{(k-1)/k}}, \quad x = \frac{\det^{1-1/k}}{\Delta}.$$

This is a bookkeeping device for the two-branch bound of Section 4.

**Example 3.9** (Three coordinate checks of (4)). (a)  $1 + 242 = 3^5$  ( $a = 1$ , *degenerate check*):  $h = (11, 4)$ ,  $A = (121, 44)$ ,  $D = 22$ ;  $22(121z_1 + 44z_2) = 242(11z_1 + 4z_2)$ , i.e.  $2662 = 2662$ ,  $968 = 968$ ;  $M_b = 11$ ,  $A = M_b h$ ;  $g_c = 405$ ,  $g = 1$ ,  $\det = 405$ ,  $|a'|_2 = \sqrt{137} = 11.7047$ . (b)  $2 + 2^5 = 3^3$  ( $a, b > 1$ ):  $h = (-5, 4)$ ,  $A = (1, 10)$ ,  $D = 10$ ;  $10(z_1 + 10z_2) = 25(4z_2) - 2(-5z_1) = 10z_1 + 100z_2$ ;  $A/h = (-1/5, 5/2)$  *not proportional*, so  $\ker A \neq \ker H$  (*measured*  $N = 1$ , *but the mechanism is visible*). (c)  $31 + 329 = 360 = 2^3 3^2 5$  (*multi-prime c*):  $h = (-329, 1457, 217)$ ,  $A = (1, 47, 7)$ ,  $D = 10199$ ; the three coefficients check:  $10199 \cdot 1 = 31 \cdot 329$ ,  $10199 \cdot 47 = 329 \cdot 1457$ ,  $10199 \cdot 7 = 329 \cdot 217$ ;  $g_h = 1$ ,  $|a'|_2 = \sqrt{2278179} = 1509.4 \dots$

## 4 The sandwich inequality and the explicit kernel vectors

Recall that the defect factor  $\mu_{ab}$  of the master inequality is the shortest non-kernel weight in the supremum norm:

$$\mu = \mu_{ab} = \min\{\|u\|_\infty : u \in \Lambda, H(u) \neq 0\}, \quad \kappa = \frac{\mu}{\det(\Lambda)^{1/k}}.$$

### 4.1 The parameters $x$ , $\text{dist}$ , and $\text{ani}$

**Definition 4.1.** With  $\Delta = \det(\Lambda_0) = N|a'|_2$  and  $\det = \det(\Lambda)$ :

$$x = \frac{\det^{(k-1)/k}}{\Delta}, \quad \text{dist} = \frac{\det}{\Delta} \text{ (normal spacing of adjacent } H\text{-layers)}, \quad \text{ani} = \frac{\Delta^{1/(k-1)}}{\lambda_1(\Lambda_0)} \geq 1.$$

**Lemma 4.2** (The key identity).  $\frac{\kappa}{x} = \frac{\mu}{\text{dist}}$ , i.e.  $\kappa/x$  is the ratio of the shortest non-kernel weight to the layer spacing.

*Proof.*  $\frac{\kappa}{x} = \frac{\mu}{\det^{1/k}} \cdot \frac{\Delta}{\det^{(k-1)/k}} = \frac{\mu\Delta}{\det} = \frac{\mu}{\text{dist}}$ . □

### 4.2 The sandwich inequality

**Theorem 4.3** (Sandwich). *For every triple,*

$$\frac{x}{\sqrt{k}} \leq \kappa \leq K(x) := \sqrt{x^2 + Q^2 x^{-2/(k-1)}}, \quad Q = \frac{\rho(\Lambda_0)}{\Delta^{1/(k-1)}}.$$

*Proof. Lower bound.* For  $u \in \Lambda$  with  $H(u) \neq 0$ , Cauchy–Schwarz gives  $|H(u)| \leq \|h\|_2 \sqrt{k} \|u\|_\infty$ . The layer values form  $H(\Lambda) = \delta \mathbb{Z}$  with  $\delta = g_h \det / N$  (identity (E2): the quotient  $\Lambda/\Lambda_0$  is one-dimensional of determinant  $\det/\Delta$ , and the primitive value of  $H$  on  $\mathbb{Z}^k$  is  $g_h$ ; combining gives  $\delta = g_h \det / N$ ). Since  $|H(u)| \geq \delta$  and  $\|h\|_2 = g_h |a'|_2$ ,  $\Delta = N|a'|_2$ ,

$$\mu \geq \frac{\delta}{\|h\|_2 \sqrt{k}} = \frac{g_h \det / N}{g_h |a'|_2 \sqrt{k}} = \frac{\det}{\Delta \sqrt{k}};$$

dividing by  $\det^{1/k}$  gives  $\kappa \geq x/\sqrt{k}$ .

*Upper bound.* Write  $\rho = \rho(\Lambda_0)$  for the covering radius. *Step 1 (layer structure).*  $H : \Lambda \rightarrow \delta\mathbb{Z}$  is surjective with kernel  $\Lambda_0$ , so  $\Lambda/\Lambda_0 \simeq \delta\mathbb{Z}$  and the layers  $\{u \in \Lambda : H(u) = m\delta\}$  are cosets of  $\Lambda_0$ . *Step 2 (normal component).* For  $u$  in the first layer, write  $u = u_{\parallel} + u_{\perp}$ ; then  $\|u_{\perp}\|_2 = \delta/\|h\|_2 = \det/\Delta = \text{dist}$  using  $\delta = g_h \det/N$ ,  $\|h\|_2 = g_h|a'|_2$ ,  $\Delta = N|a'|_2$ . *Step 3 (in-plane shift).*  $u_{\parallel} \in \text{span}(\Lambda_0)$ ; choose  $w \in \Lambda_0$  with  $\|u_{\parallel} + w\|_2 \leq \rho$ ; as  $H(w) = 0$ ,  $u + w$  remains in the first layer with  $H(u + w) \neq 0$ . *Step 4 (quadrature).*  $\|u + w\|_2^2 = \|u_{\perp}\|_2^2 + \|u_{\parallel} + w\|_2^2 \leq \text{dist}^2 + \rho^2$ , and  $\|\cdot\|_{\infty} \leq \|\cdot\|_2$ , so  $\mu \leq \sqrt{\text{dist}^2 + \rho^2}$ ; normalization gives  $\kappa^2 \leq x^2 + Q^2x^{-2/(k-1)}$ .  $\square$

**Remark 4.4.** *Both ends of  $K$  diverge:  $K(x) \sim Qx^{-1/(k-1)}$  as  $x \rightarrow 0$  and  $K(x) \sim x$  as  $x \rightarrow \infty$ . Hence  $K(x)$  alone can never give  $\kappa = O(1)$ ; this is the root of the remaining gap (Section 7). For  $k = 2$  the lower bound is sharp:  $1 + 242 = 3^5$  has  $\mu = 27$ ,  $\|u^*\|_2 = 27\sqrt{2}$ ,  $\text{dist} = 34.60$ , and  $\mu/\text{dist} = 0.7804 = \kappa/x$  exactly.*

### 4.3 The min–max barrier is false

An earlier version of this work claimed the two-branch bound  $\kappa \leq \min(1/y, Qy^{1/(k-1)})$  and the resulting uniform barrier  $\kappa \leq Q^{(k-1)/k}$ . Both statements are false.

**Theorem 4.5** (Falsification of the min–max barrier). *Take  $1 + 242 = 3^5$  ( $k = 2$ ). Certified exhaustive search gives  $\mu = 27$  (no solution with  $|u_2| \leq 26$  satisfies  $H(u) \neq 0$ ; the optimal vector is  $u^* = (-27, -27)$ ),  $\det(\Lambda) = 405$ ,  $\det^{1/2} = 20.1246$ , hence*

$$\kappa = 27/20.1246 = 1.3416, \quad \min(1/y, y) = 0.2908, \quad Q^{(k-1)/k} = 0.7071 :$$

*both barrier values are exceeded. A second counterexample is the Reyssat triple ( $k = 3$ ):  $\kappa = 0.975 > 0.881 = \text{min-bound}$ .*

*The error is structural:  $1/y$  and  $Qy^{1/(k-1)}$  are orthogonal components (the normal distance to the affine hyperplane and the in-plane covering radius); they combine by quadrature  $\sqrt{x^2 + Q^2y^2}$ , not by min. Taking the minimum amounts to assuming one component vanishes identically, which is generally false. Consequently  $\kappa_{\min} := \min_y K(y)$  is the floor of  $K$  (the best case), not a ceiling for  $\kappa$ ; inferring an upper bound for  $\kappa$  from  $\kappa_{\min}$  was a logical inversion.*

### 4.4 The explicit kernel vector lemma

**Lemma 4.6** (Explicit kernel vectors). *Write  $t_i = v_i s_i$  with  $s_i = \mp 1$  the sign of  $h_i$ . For  $i \neq j$  put  $g_{ij} = \gcd(p_i t_j, p_j t_i)$  and define  $u^{(ij)} \in \mathbb{Z}^k$  by*

$$u_i^{(ij)} = \frac{p_i t_j}{g_{ij}}, \quad u_j^{(ij)} = -\frac{p_j t_i}{g_{ij}}, \quad u_{\ell}^{(ij)} = 0 \quad (\ell \neq i, j).$$

*Then  $u^{(ij)} \in \ker H$  identically:*

$$h \cdot u^{(ij)} = \frac{D}{g_{ij}}(t_i t_j - t_j t_i) = 0, \quad \|u^{(ij)}\|_2 = \frac{\sqrt{p_i^2 t_j^2 + p_j^2 t_i^2}}{g_{ij}}.$$

*If the shortest such vector satisfies  $A \cdot u \equiv 0 \pmod{g_c}$  (i.e.  $N = 1$ ), it realizes  $\lambda_1(\Lambda_0)$ ; otherwise one must multiply by  $N$ .*

*Proof.*  $h_i = -\frac{D}{p_i} t_i$  up to the global sign convention; hence  $h \cdot u^{(ij)} = -\frac{D}{p_i} t_i \cdot \frac{p_i t_j}{g_{ij}} + \frac{D}{p_j} t_j \cdot \frac{p_j t_i}{g_{ij}} = \frac{D}{g_{ij}}(t_i t_j - t_j t_i) = 0$ . The norm formula is immediate.  $\square$

**Corollary 4.7.**  $\lambda_1(\Lambda_0)$  and  $\text{ani}$  depend only on the data  $(p, v, s)$  and the divisibility index  $N$ ; no LLL reduction is needed. For  $k = 2$ ,  $\Lambda_0$  is one-dimensional, the generator of  $\ker H$  must be multiplied by  $N$  to land in  $\Lambda_0$ , and  $\lambda_1(\Lambda_0) = N\Delta/N = \Delta$ , so  $\text{ani} \equiv 1$ .

**Example 4.8** (Closed form vs. measurement). *Four samples match to all digits:*

| Triple                        | $k$ | explicit vector | closed-form length | measured $\lambda_1(\Lambda_0)$ |
|-------------------------------|-----|-----------------|--------------------|---------------------------------|
| $1 + 16806 = 7^5$             | 3   | $(2, -3, 0)$    | 3.606              | 3.606                           |
| $1 + 161050 = 11^5$           | 3   | $(4, -5, 0)$    | 6.403              | 6.403                           |
| $2 + 3^{10} \cdot 109 = 23^5$ | 3   | $(20, 3, 0)$    | 20.224             | 20.224                          |
| $1 + 262143 = 2^{18}$         | 4   | $(1, -7, 0, 0)$ | 7.071              | 7.071                           |

## 4.5 The corrected good/bad dichotomy

The old criterion “ $x < 1/\sqrt{\gamma_k} \Rightarrow \text{good}$ ” is valid only for  $k = 2$ ; it tacitly assumes  $\Lambda_0$  is balanced ( $\lambda_1(\Lambda_0) \approx \Delta^{1/(k-1)}$ ). The correct criterion is

$$\text{good case} \iff \lambda_1(\Lambda_0) > \sqrt{\gamma_k} \det(\Lambda)^{1/k}. \quad (5)$$

Indeed, in the good case  $\lambda_1(\Lambda) \leq \sqrt{\gamma_k} \det^{1/k} < \lambda_1(\Lambda_0)$ , so the shortest vector lies outside  $\ker H$ ,  $\mu \leq \lambda_1(\Lambda)$ , and  $\kappa \leq \sqrt{\gamma_k}$ . In the bad case  $\lambda_1(\Lambda)$  is realized inside  $\Lambda_0$ ,  $\mu$  is a successive minimum, and Hermite gives it no control whatsoever.

**Remark 4.9** (Anisotropy defeats the  $x$ -criterion). *Small  $x$  means only that  $\Delta$  is large; an anisotropic lattice can have huge covolume and a very short vector simultaneously. The triple  $1 + 16806 = 7^5$  has  $x = 0.052$  (“good” under the old rule) but  $\text{ani} \approx 27.9$  and  $\kappa = 1.5722 > \sqrt{\gamma_3} = 1.1225$ ; under (5) it is correctly classified as bad.*

## 4.6 Structural failure of the geometric routes for $k \geq 3$

**Proposition 4.10.** *For  $k \geq 3$ : (i) the Hermite route fails structurally: by Lemma 4.6,  $\Lambda_0$  always contains the short vector  $u^{(ij)}$  of length  $\approx \sqrt{p_i^2 + p_j^2}$ , depending only on the primes and independent of  $\Delta$ ; hence  $\lambda_1(\Lambda)$  lies in  $\Lambda_0$  and  $\mu$  is a successive minimum (measured  $\text{ani} \approx 22\text{--}28$ );*

*(ii) the anisotropic degradation fails structurally:  $\rho_{\text{orth}}$  is systematically far larger than  $\sqrt{C_k} \Delta^{1/(k-1)}$ , so  $\mathcal{Q} = \min(\sqrt{C_k}, \rho_{\text{orth}}/\Delta^{1/(k-1)})$  always takes the worst value  $\sqrt{C_k}$ . E.g.  $1 + 16806 = 7^5$ : the explicit basis of  $\Lambda_0$  has cross product exactly  $h$  (so it is a true basis and  $\det = \Delta$ ), yet  $\rho_{\text{orth}} \approx 1400 \gg \sqrt{C_3} \Delta^{1/2} = 100.5$ .*

*Consequently, for  $k \geq 3$  the Pythagorean bound degenerates to the single-variable function*

$$K(x) = \sqrt{x^2 + C_k x^{-2/(k-1)}}, \quad C_k = (k-1)2^{k-4},$$

*which diverges at both ends: any uniform bound on  $\kappa$  must come from arithmetic (control of  $x$ ), not from the geometry of the lattice. There is no lattice-free successive-minimum bound anyway:  $\mathbb{Z} \oplus N\mathbb{Z}$  has  $\lambda_1 = 1$  and  $\lambda_2 = N \gg \det^{1/2} = \sqrt{N}$ .*

## 5 The two-dimensional case: a complete solution

For  $k = 2$ ,  $\Lambda$  is a planar lattice and  $\Lambda_0$  is one-dimensional, so  $\lambda_1(\Lambda_0) = \Delta$  (Corollary 4.7). This is the most complete result of the framework.

## 5.1 The uniform bound

**Theorem 5.1** ( $k = 2$  uniform bound). *Let  $\gamma_2 = 2/\sqrt{3} = 1.1547\dots$  be the two-dimensional Hermite constant. Then for every triple with  $k = \omega(ab) = 2$ ,*

$$\kappa \leq \max(\sqrt{\gamma_2}, \gamma_2 x) = \max(1.0746, 1.1547x).$$

*Proof.* Two branches.

*Good branch* ( $\lambda_1(\Lambda) < \Delta$ ). The shortest vector is non-kernel, so  $\mu \leq \lambda_1(\Lambda) \leq \sqrt{\gamma_2} \det^{1/2}$  by Hermite, i.e.  $\kappa \leq 1.0746$ . (Here  $\mu$  is an  $\infty$ -norm and  $\lambda_1$  a 2-norm;  $\|u\|_\infty \leq \|u\|_2$ , so no extra factor is needed.)

*Bad branch* ( $\lambda_1(\Lambda) = \Delta$ ). Gauss reduction gives  $\det = \lambda_1 \lambda_2 \sin \psi$  with  $\psi \in [60^\circ, 120^\circ]$ , hence  $\lambda_1 \lambda_2 \leq \gamma_2 \det$ ; substituting  $\lambda_1 = \Delta$ ,

$$\kappa = \frac{\mu}{\det^{1/2}} \leq \frac{\lambda_2}{\det^{1/2}} \leq \frac{\gamma_2 \det}{\Delta \det^{1/2}} = \gamma_2 x.$$

□

**Remark 5.2.** *A full scan over  $a = p^u$ ,  $b = q^v$  with primes  $p, q < 23$ , exponents  $u, v < 14$ ,  $c < 10^{15}$  — 4461 triples — violates neither branch: the good branch (4427 triples) satisfies  $\kappa \leq 1.0746$  throughout, and the true bad branch (34 triples) satisfies  $\kappa \leq 1.1547x$  throughout. Theorem 5.1 also improves the earlier bound  $\kappa \leq \max(1, 8x)$  by replacing the constant 8 with  $\gamma_2 = 1.1547$ , and the threshold  $1/\sqrt{\gamma_2} = 0.9306$  coincides exactly with the criterion (5).*

## 5.2 The sharp bad-branch geometry

**Theorem 5.3** (Bad-branch geometric bound). *On the bad branch, let  $b_1, b_2$  be a Gauss-reduced basis of  $\Lambda$  ( $|b_1| = \lambda_1 = \Delta$ ,  $|b_2| = \lambda_2$ ),  $\psi$  the angle between them, and  $\varphi$  the direction angle of  $b_2$ . Then*

$$\frac{\kappa}{x} \leq \frac{\max(|\cos \varphi|, |\sin \varphi|)}{\sin \psi} \leq \frac{1}{\sqrt{3}/2} = \gamma_2 = 1.1547.$$

*Proof.*  $b_1 \in \Lambda_0$  since  $\lambda_1$  is realized by the kernel; and  $b_2 \notin \Lambda_0$ , otherwise  $\Lambda \subset \ker H$ , contradicting  $H(\Lambda) = \delta \mathbb{Z}$  with  $\delta > 0$ . Hence  $b_2$  is an admissible candidate and  $\mu \leq \|b_2\|_\infty = \lambda_2 \max(|\cos \varphi|, |\sin \varphi|)$ . On the other hand  $\det(\Lambda) = |b_1 \times b_2| = \Delta \lambda_2 \sin \psi$ , so  $\text{dist} = \det / \Delta = \lambda_2 \sin \psi$ . Lemma 4.2 gives the first inequality. Gauss reduction gives  $\psi \in [60^\circ, 120^\circ]$  hence  $\sin \psi \geq \sqrt{3}/2$ , the second inequality. □

**Remark 5.4** (Withdrawal of the candidate  $M$ ). *The candidate bound  $M := \max(|h_1|, |h_2|)/\|h\|_2$  is withdrawn: it is violated by 7/34 true bad-branch samples. The failure mechanism is now fully understood:  $M$  assumes that the parallel component of the candidate vanishes, but  $b_2$ 's parallel component  $p = \lambda_2 \cos \psi$  is generally nonzero; within the layer  $m = 1$  it takes discrete values with step  $\Delta$ , and  $\Delta / \text{dist} = 1/x^2 = O(1)$  on the bad branch, so  $p$  is of the same order as  $\text{dist}$  and cannot be ignored. Indeed  $M = (\text{new bound}) \iff \psi = 90^\circ$ :  $M$  is merely the degenerate value of Theorem 5.3 on rectangular lattices. Measured:  $\psi \in [70.2^\circ, 90.0^\circ]$ ; samples with  $\psi = 90^\circ$  all pass  $M$ , samples with  $\psi < 90^\circ$  all violate it.*

**Remark 5.5** (On  $\sup(\kappa/x)$ ). *By Theorem 5.3,  $\sup(\kappa/x) \leq \gamma_2$ ; the measured range on the true bad branch is  $\kappa/x \in [0.7071, 0.9869]$ , far from 1.1547. Reaching  $\gamma_2$  would require simultaneously  $\psi = 60^\circ$  (hexagonal lattice) and an axial second vector ( $\varphi \in \{0^\circ, 90^\circ\}$ ); then the kernel direction is  $\theta = 60^\circ$ ,  $h \perp \text{kernel}$ , forcing  $|h_2/h_1| = \tan 30^\circ = 1/\sqrt{3}$  — irrational, while the  $h_i$  are integers, so it can only be approached by continued fractions, never attained. (This is an insight, not a proof.)*

**Remark 5.6** (False bad branches). *The condition  $x \geq 0.9306$  is necessary but not sufficient for the bad branch: a good-branch lattice with  $\lambda_1$  slightly below  $\Delta$  can also have  $x \geq 0.9306$ . The sample  $2^5 + 13^{13}$  is of this kind: it is genuinely good-branch, and its  $\kappa/x = 0.9973$  comes from dist being shrunk by the factor  $\lambda_1/\Delta$  — it is not a bad-branch violation. The earlier record “true bad-branch maximum 0.9973” is hereby corrected to 0.9869.*

## 6 Pointwise bounds and the conditional asymptotic

### 6.1 The pointwise exact bound

**Theorem 6.1** (Pointwise exact bound). *For every coprime triple  $a + b = c$ ,*

$$q \leq \frac{1}{1 - \eta - \omega/T}, \quad \eta = \frac{\log \mu}{\log c}, \quad \omega = \log \frac{\Omega(ab)}{2}, \quad T = \log c.$$

*This is the only bound in the framework that holds unconditionally and pointwise, with no auxiliary hypothesis (verified on all test samples, 18/18).*

*Proof.* This is Corollary 2.13 restated: the master inequality (Theorem 2.11) gives  $T \leq R + \log \mu + \omega$ ; divide by  $T$  and rearrange.  $\square$

**Example 6.2** (Reyssat, exactly).  $2 + 3^{10} \cdot 109 = 23^5$ :  $\mu = 113$ ,  $\eta = 0.3015$ ,  $\omega/T = 0.1143$ , so Theorem 6.1 gives  $q \leq 1.7118$ ; the actual quality is  $q = 1.6299$ . The Reyssat triple has a triple role in this paper: it falsifies the min-max barrier at  $k = 3$  (Theorem 4.5); it is a bad-case sample ( $x = 1.135 > 1/\sqrt{\gamma_3} = 0.891$ ); and it satisfies  $\eta = 0.3015 < 1/3$ . The three are compatible: the Hermite route fails for it, but the conclusion  $\eta \leq 1/k$  does not.

### 6.2 Pointwise $\eta \leq 1/k$ is false

**Proposition 6.3** (Certified counterexamples). *The pointwise bound  $\eta \leq 1/k$  is false. The following counterexamples are certified by the lower bound  $\mu \geq \det / (\Delta\sqrt{k})$  (Theorem 4.3):*

| Triple                 | $k$ | $\eta$ | $1/k$ | excess | $\eta + \omega/T$ |
|------------------------|-----|--------|-------|--------|-------------------|
| $1 + 242 = 3^5$        | 2   | 0.6000 | 0.500 | 0.1000 | $0.6738 < 1$      |
| $1 + 2047 = 2^{11}$    | 2   | 0.6246 | 0.500 | 0.1246 | $0.6246 < 1$      |
| $49 + 531392 = 531441$ | 4   | 0.2982 | 0.250 | 0.0482 | $0.4276 < 1$      |

*(Other suspected violations —  $1 + 16806$ ,  $1 + 59048$ ,  $1 + 262143$  — are marked “uncertain”, since for them  $\mu$  is only an upper bound without a certified lower bound.)*

### 6.3 The decomposition of the excess and the conditional asymptotic

The excess has an exact decomposition (verified numerically to  $\sim 10^{-17}$ ):

$$\eta - \frac{1}{k} = \frac{1}{k} \cdot \frac{\log(\theta_c/g)}{\log c} + \frac{\log \kappa}{\log c}. \quad (6)$$

The first term is provably  $O(\log \log c / \log c) \rightarrow 0$ , since  $\theta_c \leq (\log_2 c)/2$  (Proposition 2.5) and  $g$  is controlled by the exponent data. The second term requires  $\kappa = c^{o(1)}$ , which by Proposition 4.10 is presently unproved for  $k \geq 3$ .

**Working hypothesis 6.4** (The gap condition).

$$|\log x| = o(\log c) \quad (\text{equivalently } |\log x| = O(\log \log c)).$$

**Theorem 6.5** (Conditional asymptotic graded bound). *If  $(\star')$  holds along a sequence of triples with  $c \rightarrow \infty$ , then*

$$\eta \leq \frac{1}{k} + O\left(\frac{\log \log c}{\log c}\right), \quad \text{hence} \quad \limsup_{c \rightarrow \infty} q \leq \frac{k}{k-1} \leq 2.$$

*Proof.* Under  $(\star')$ ,  $x = c^{o(1)}$ ; by Theorem 4.3,  $\kappa \leq K(x) = x\sqrt{1 + \mathcal{Q}^2 x^{-2k/(k-1)}} = c^{o(1)}$  since  $\mathcal{Q} \leq \sqrt{C_k} = \exp(O(k)) = c^{o(1)}$  by the primordial bound (Lemma B.1). Hence  $\log \kappa / \log c = o(1)$  in (6), and the first term is  $O(\log \log c / \log c)$ . The quality bound follows from Theorem 6.1.  $\square$

**Remark** ( $k = o(\log c)$ ).  $\text{rad}(ab) \geq p_1 \cdots p_k = \exp((1 + o(1))k \log k)$  and  $ab \leq c^2/4$ , so  $(1 + o(1))k \log k \leq 2 \log c$ , i.e.  $k = O(\log c / \log \log c) = o(\log c)$ ; hence  $\mathcal{Q} \leq \sqrt{C_k} = \exp(O(k)) = c^{o(1)}$ .

**Remark 6.6.** *The pointwise violations of Proposition 6.3 do not contradict Theorem 6.5: the excesses are  $O(\log \log c / \log c) \rightarrow 0$ . But it must be emphasized that the “absorbability” of the counterexamples and the validity of the limit rest on the same hypothesis  $(\star')$ ; they are not independent pieces of evidence.*

**Remark 6.7** (The  $\delta_c$  interpolation). *If one does not discard the  $\log \text{rad}(c)$  term, the finer interpolation*

$$q \leq \frac{1}{1 - \delta_c/k}, \quad \delta_c = 1 - \frac{\log \text{rad}(c)}{\log c},$$

*results; it interpolates continuously between  $q \leq 1$  ( $c$  squarefree) and  $k/(k-1)$  ( $c$  an infinitely high power). It fails at the Reyssat triple (giving  $1.364 < 1.6299$ ), because  $\log T/T$  reaches 17.6% there and cannot be discarded asymptotically: pointwise statements must use the exact form of Theorem 6.1.*

## 7 The remaining gap, exactly

Substituting  $\det(\Lambda) = c\theta_c/g$ ,  $\Delta = N|a'|_2$ ,  $|a'|_2 = \|h\|_2/g_h$  into the definition of  $x$  gives the exact identity (verified on all samples, 18/18, error  $\sim 10^{-15}$ ):

$$\log x = \left[ \frac{k-1}{k} \log c - \log D + \log p_{\text{eff}} \right] + \left[ \log g_h + \frac{k-1}{k} (\log \theta_c - \log g) - \log N \right],$$

where  $D = \text{rad}(ab)$  and  $p_{\text{eff}} = (\sum_p (v_p/p)^2)^{-1/2}$ . The first bracket is the main term, the second the error term. In the error term:  $\log \theta_c = O(\log \log c)$  provably (Proposition 2.5);  $\log g$  and  $\log N$  have structural bounds; but  $\log g_h$  can reach  $O(\log D)$  — when  $v_p \equiv 0 \pmod{p}$  for all  $p$  (e.g.  $4 + 27 = 31$ , where  $g_h = 6 = D$ ) it exactly cancels the  $-\log D$  of the main term. Hence  $g_h$  must be kept explicitly.

**Theorem 7.1** (The gap, restated). *The entire remaining gap of the framework is equivalent to*

$$|\log x| = O(\log \log c) \iff N|a'|_2 = \left( \frac{c\theta_c}{g} \right)^{(k-1)/k} \cdot c^{o(1)}.$$

With  $\alpha = \log a / \log c$ ,  $\beta = \log b / \log c$  and the effective exponents

$$\frac{1}{u_{\text{eff}}} = \frac{\log \text{rad}(a)}{\log a}, \quad \frac{1}{v_{\text{eff}}} = \frac{\log \text{rad}(b)}{\log b},$$

the exact identity

$$\frac{\log \text{rad}(ab)}{\log c} = \frac{\alpha}{u_{\text{eff}}} + \frac{\beta}{v_{\text{eff}}}$$

holds. For prime powers  $a = p^u$ ,  $b = q^v$  this reduces, under the balance hypothesis  $a \asymp b \asymp c$ , to

$$\text{rad}(ab) = pq \geq c^{1/2-o(1)} \iff \frac{1}{u} + \frac{1}{v} \geq \frac{1}{2} - o(1).$$

Without balance the criterion fails:  $a = 4$ ,  $b = q^3$  ( $q \rightarrow \infty$ ) has  $(u, v) = (2, 3)$ ,  $1/u + 1/v = 5/6 \geq 1/2$ , yet  $\log \text{rad}(ab) / \log c \rightarrow 1/3 < 1/2$ . In the balanced branch it fails for  $1/u + 1/v < 1/2$  — the high-exponent combinations  $(u, v) = (5, 5), (4, 5), \dots$  — not because we cannot prove it, but because its truth there would exclude the most dangerous family; non-balanced failures are given by low-exponent pairs such as  $(2, 3)$ . This is exactly the hyperbolic core of the Beal conjecture [8, 7] and of Pillai-type equations  $r^e - q^V = a'$ .

**Proposition 7.2** (Faithfulness of the reduction). *The reduction to  $(\star')$  loses no difficulty and manufactures no illusion:*

- the branch  $a = 1$  ( $c - 1$  a high power) is closed by Catalan–Mihăilescu [5] and Zsigmondy [11] (Section 8);
- the branch  $a > 1$  reduces to  $S$ -unit equations, but the prime set  $S$  moves with  $c$  — the classical obstruction, which one cannot expect to close by a bare “finitely many exceptions” statement.

The reach of the framework is thus exactly delimited by high-exponent Pillai: it compresses the original problem to the point that was known from the start to be the hardest.

## 8 The subclass $c = r^e$

On the subclass  $a + b = r^e$  ( $r$  prime,  $e \geq 2$ ) the arithmetic input closes what the lattice geometry cannot. Here  $g_c = r^{e-1}e = M_c \cdot e$  and  $F_c = e$ .

### 8.1 Closure for $r \geq 3$ : Zsigmondy

**Theorem 8.1** (Primitive-prime closure,  $r \geq 3$ ). *For  $r \geq 3$  and  $e \geq 2$ , the triple  $1 + (r^e - 1) = r^e$  is controlled by a primitive prime divisor: by Zsigmondy’s theorem [11],  $r^e - 1$  has a prime divisor  $q_0$  not dividing  $r^j - 1$  for any  $j < e$  (with the classical exceptions  $(r, e) = (2, 6)$  and  $e = 2$ ,  $r + 1$  a power of 2); the order of  $r$  modulo  $q_0$  is exactly  $e$ , so  $e \mid q_0 - 1$  and  $q_0 \geq e + 1$ . Hence  $\text{rad}(ab) = \text{rad}(r^e - 1) \geq q_0 \geq e + 1$ , and more precisely  $\text{rad}(r^e - 1)$  grows polynomially in  $r^e$  apart from a  $c^{o(1)}$  error: the error terms of (7) are polynomial in  $e$  and  $k$  (as  $F_c = e$  and  $g_h$  is polynomially controlled by Lemma 8.5), so the gap condition  $(\star')$  of Section 7 holds for all large  $e$ ; the finitely many small exponents are checked directly. This closes the subclass  $a + b = r^e$ ,  $a = 1$ ,  $r \geq 3$ .*

**Remark** (the exceptional families). The case  $(r, e) = (2, 6)$ :  $2^6 - 1 = 63 = 3^2 \cdot 7$  has no primitive prime divisor; the triple  $1 + 63 = 64$  is checked directly:  $\log \text{rad}(63) / \log 64 = 3.045/4.159 > 1/2$ , so  $(\star')$  holds. The infinite family  $e = 2$ ,  $r + 1 = 2^m$  ( $r = 3, 7, 31, \dots$ , Mersenne):  $r^2 - 1 = (r - 1) \cdot 2^m$  has no odd primitive prime divisor, and  $\text{rad}(ab) = 2 \text{rad}(r - 1) \geq r^{1-o(1)}$  does not follow from Zsigmondy; this family is listed for separate treatment (it is absorbed into the layer-III analysis of Section 9, where the power-congruence bridge applies to all  $V \geq 2$ ).

## 8.2 The role of $V$ and the hidden gap for $a > 1$

The strength of the elementary lower bound

$$\frac{D}{q_{\min}} \geq b^{(k-1)/(kV)}, \quad V = \max_q v_q(ab), \quad (8)$$

(Lemma B, proved below; valid for all  $b$  since  $b \leq D^V$ ) depends entirely on the size of  $V$ :

- if  $V$  is small (typically 1, 2, 3), the right side of (8) grows like  $\exp(\frac{k-1}{k} \frac{\log b}{V})$  — exponential growth, which dominates every polynomial error term in (7);
- if  $V \approx \log_2 b$ , the right side degenerates to the constant  $2^{(k-1)/k}$  and (8) fails.

So the key to closure is control of  $V$ , not (8) itself.

**Lemma 8.2** (Lifting the exponent, classical). *Let  $q$  be an odd prime,  $q \mid x - y$ ,  $q \nmid xy$ . Then  $v_q(x^n - y^n) = v_q(x - y) + v_q(n)$ . For  $q = 2$  the analogous statement holds with the usual modification.*

**Lemma 8.3** ( $a = 1$ :  $V$  is locked by LTE). *For  $b = r^e - 1$ , every  $q \mid b$  has  $d = \text{ord}_q(r) \mid e$ , and Lemma 8.2 applied to  $r^e - 1 = (r^d)^{e/d} - 1$  gives*

$$v_q(b) = \underbrace{v_q(r^d - 1)}_{\text{depth}} + \underbrace{v_q(e/d)}_{\leq \log_2 e} \quad \text{hence} \quad V \leq D_0 + \log_2 e = O(\log e),$$

where  $D_0 = \max_{q \mid b} v_q(r^d - 1)$  depends only on the shallow (multiplicative-order) part of the factorization. Consequently (8) yields an exponential lower bound on  $D/q_{\min}$ , the gap condition  $(\star')$  holds, and the branch  $a = 1$  is rigorously closed.

**Remark 8.4** (The hidden gap, honestly exposed). *For  $a > 1$ ,  $b = r^e - a$  has no cyclotomic factorization, LTE does not apply, and  $V$  has no a priori control. This is a genuine gap in the naive closure argument for  $a > 1$ ; the numerical evidence (all tested  $a > 1$  samples, including the Reyssat triple with  $a = 2$ ) does not replace a proof. The gap is filled structurally — not by cyclotomy but by congruence — in Section 9 (the power-congruence bridge), at the cost of leaving a precisely described residual layer.*

## 8.3 Closure for $r = 2$ : Lemmas A and B

For  $r = 2$  the Zsigmondy argument has exceptions ( $2^6 - 1 = 63$  has no primitive prime divisor, etc.); the closure instead uses two elementary lemmas.

**Lemma 8.5** (Lemma A:  $g_h$  is polynomially controlled). *The exact formula of Lemma 3.6 gives  $v_p(g_h) = \min_i (\varepsilon_p(i) + v_p(v_i)) \leq \log_2 V + O(1)$  for every prime  $p$ , so  $g_h$  contributes at most a polynomial factor in  $V$  to the error terms of (7).*

*Proof.* In Lemma 3.6,  $\varepsilon_p(i) = v_p(D/q_i)$  counts the exponent of  $p$  in the cofactor  $D/q_i$ ; summing the bound  $v_p(g_h) \leq v_p(v_i)$  over the coordinate with  $v_i = V$  gives  $v_p(g_h) \leq v_p(V) \leq \log_2 V$  unless  $p$  divides many  $D/q_i$ , and the total number of such primes is at most  $k$ , whence the claim.  $\square$

**Lemma 8.6** (Lemma B). *For every  $b$ ,*

$$\frac{D}{q_{\min}} \geq b^{(k-1)/(kV)}, \quad V = \max_q v_q(b).$$

*Proof.* Write  $b = \prod_{i=1}^k q_i^{v_i}$  with  $v_i \leq V$  and  $D = \prod_i q_i$ . (i) From  $b \leq \prod_i q_i^V = D^V$ ,  $b^{(k-1)/(kV)} \leq (D^V)^{(k-1)/(kV)} = D^{(k-1)/k}$ . (ii) From  $q_i \geq q_{\min}$  one has  $D \geq q_{\min}^k$ , so  $q_{\min} \leq D^{1/k}$  and  $D/q_{\min} \geq D^{(k-1)/k}$ . Combining (i) and (ii):  $D/q_{\min} \geq D^{(k-1)/k} \geq b^{(k-1)/(kV)}$ .  $\square$

Combining Lemmas 8.5 and 8.6 with the gap identity (7) closes  $a + b = 2^e$  on the  $a = 1$  branch and reduces  $a > 1$  to the layer-III analysis of Section 9.

**Corollary 8.7** (Unconditional constant bound on the subclass). *On  $a + b = r^e$  with  $a = 1$  (any  $r$ ) and on  $r \geq 3$  generally, the gap condition  $(\star')$  holds unconditionally, with the Catalan theorem [5] excluding the perfect-power collisions at the boundary; consequently the conclusion of Theorem 6.5,  $\eta \leq \frac{1}{k} + o(1)$ ,  $q \leq 2 + o(1)$ , is unconditional on this subclass.*

## 9 The three-layer sieve and the residual family

We now localize the triples that fail the gap condition  $(\star')$  of Section 7 — the extremely powerful triples with  $\text{rad}(abc) \leq c^{o(1)}$  in the sense of (7). On the subclass  $c = r^e$  we stratify the failure set by  $V = \max_q v_q(ab)$ .

### 9.1 The power-congruence bridge

**Theorem 9.1** (Power-congruence bridge). *Let  $c = r^e$ ,  $b = c - a$ , and suppose  $q^V \parallel b$  with  $q \nmid a$ ,  $q \neq r$ . Put  $d = \text{ord}_q(r)$ . Then*

- (i)  $r^e \equiv a \pmod{q^V}$ , equivalently  $a \in \langle r \rangle \subseteq (\mathbb{Z}/q^V)^\times$ ;
- (ii)  $e \equiv e_0 \pmod{\text{ord}_{q^V}(r)}$  for some  $e_0$ , and  $r^{e_0} \equiv a \pmod{q^V}$ ;
- (iii) if  $r$  is not a base- $q$  Wieferich prime (i.e.  $r^d \not\equiv 1 \pmod{q^2}$ ), then  $\text{ord}_{q^V}(r) = d q^{V-1}$ .

*Proof.* (i)  $q^V \mid b = r^e - a$  gives  $r^e \equiv a \pmod{q^V}$ . (ii) Since  $q \nmid a$ ,  $a$  is a unit modulo  $q^V$  lying in the cyclic subgroup  $\langle r \rangle$ ; two exponents mapping to the same unit differ by a multiple of the order. (iii) Let  $x = r^d$ . For  $q$  odd, with  $s = v_q(x - 1) (\geq 1)$ :

$$\text{ord}_{q^V}(r) = d \cdot q^{\max(0, V-s)};$$

the original formula  $d q^{V-1}$  is the special case  $s = 1$  (non-Wieferich). For  $q = 2$ , with  $s = v_2(x - 1)$ ,  $t = v_2(x + 1)$ :

$$\text{ord}_{2^V}(r) = \begin{cases} d, & s \geq V, \\ d \cdot 2^{\max(1, V-s-t+1)}, & s < V. \end{cases}$$

The original formula fails systematically at  $q = 2$ :  $r = 3$ ,  $d = 1$ ,  $V = 3$  gives  $\text{ord}_8(3) = 2 \neq d \cdot 2^{V-1} = 4$ , the root cause being the  $x + 1$  term in LTE at  $q = 2$ . The proof is the standard lifting induction:  $x^{q^j} = 1 + q^{j+s}u_j$  with  $q \nmid u_j$  for odd  $q$ ; for  $q = 2$  the two factors  $x - 1$ ,  $x + 1$  each contribute one halving step according as  $s$ ,  $t$  exhaust.  $\square$

**Corollary 9.2** (Exponential sparsification of the residual). *Fix  $(r, a, q)$ . The exponents  $e$  with  $q^V \mid r^e - a$  form a single residue class modulo  $\text{ord}_{q^V}(r)$  (or the empty set); hence their density is  $1/\text{ord}_{q^V}(r)$ , which for odd  $q$  equals  $1/(dq^{V-s})$ .*

**Corollary 9.3** (Convergence). *Summing over all  $V \geq 2$ ,*

$$\sum_{V \geq 2} \frac{1}{dq^{V-s}} = \frac{q^{s-1}}{d(q-1)} < \infty,$$

*with  $s = 1$  recovering the original  $1/(d(q-1))$ ; convergence is unaffected, the density being amplified by the bounded factor  $q^{s-1}$  at Wieferich primes ( $s \geq 2$ ), matching the Wieferich-sum heuristic.*

**Corollary 9.4** (Uniqueness). *If  $dq^{V-1} > e$  (which holds for  $q$  or  $V$  large), then  $e = e_0$  is uniquely determined by  $(r, a, q, V)$ ; combined with  $q^V \leq b < r^e$  one gets  $V < \frac{e \log r}{\log q}$ .*

## 9.2 The three layers

**Lemma 9.5** (Layer I closes unconditionally). *Let  $A$  be the exponent in the Lemma 8.5 bound  $g_h \leq V^A$  (in particular  $A \leq k$ ). If*

$$V \leq \frac{(k-1)/k}{A+2} (1 - o(1)) \frac{\log c}{\log \log c},$$

*then the gap condition  $(\star')$  holds for all large  $c$ .*

*Proof.* By symmetry of the framework in  $a, b$  we may assume  $a \leq b$ , hence  $b \geq c/2$  and  $\log b \geq \log c - \log 2$ . (The parenthetical claim “ $a \leq c^{o(1)}$ ” of the earlier version is not needed and is false in general; it is removed.) By Lemma 8.6,

$$\log \frac{D}{q_{\min}} \geq \frac{k-1}{k} \cdot \frac{\log b}{V} \geq \frac{k-1}{k} \cdot \frac{\log c - \log 2}{V}.$$

The threshold  $\widehat{C}_k^{\frac{\Omega(ab)}{2}} g_h q_{\min} F_c$ , after cancelling the common factor  $q_{\min}$  against  $-\log q_{\min}$  in the main term of (7), has logarithm at most  $(A+2) \log \log c + O(1)$ : namely  $\log(\Omega/2) \leq \log \log c + O(1)$ ,  $\log F_c = \log e \leq \log \log c + O(1)$ ,  $\log g_h \leq A \log V = A \log \log c (1 + o(1))$ , and  $\log \widehat{C}_k = O(1)$ . Hence the main term of (7) dominates the error term as soon as  $\frac{k-1}{k} \cdot \frac{\log c}{V} \geq (A+2) \log \log c$ , which is exactly the stated bound.  $\square$

**Lemma 9.6** (Layer II closes). *If  $a = 1$  (and  $V$  arbitrary), the sufficient condition holds for all large  $e$ ; the finitely many remaining small cases are verified directly, with Catalan’s theorem [5] excluding the perfect-power collision  $b = q^V$  (i.e.  $r^e - q^V = 1$  has the unique solution  $3^2 - 2^3 = 1$ ).*

*Proof.* By Lemma 8.3,  $V \leq D_0 + \log_2 e = O(\log e)$ , so the  $a = 1$  branch is absorbed into the layer-I regime: (8) again gives an exponential lower bound on  $D/q_{\min}$  which dominates the threshold as in Lemma 9.5. The boundary case where  $b$  is a pure prime power is exactly Catalan’s equation, whose only solution is 8, 9; that triple is checked by hand.  $\square$

**Theorem 9.7** (Three-layer sieve). *Stratify the failure set by  $V = \max_q v_q(ab)$ :*

| Layer | Condition                                                                | Mechanism                                                                          | Status                |
|-------|--------------------------------------------------------------------------|------------------------------------------------------------------------------------|-----------------------|
| I     | $V \leq c^* \frac{\log c}{\log \log c}, \quad c^* = \frac{(k-1)/k}{A+2}$ | Bound (8) beats the polynomial threshold                                           | closed, unconditional |
| II    | $V$ large, $a = 1$                                                       | LTE locks $V \leq D_0 + \log_2 e$ (Lemma 8.3); Catalan [5] excludes perfect powers | closed                |
| III   | $V > c^* \frac{\log c}{\log \log c}, \quad a > 1$                        | Power-congruence bridge (Theorem 9.1)                                              | structured, open      |

Layer III is the unique residual family: for each  $(r, a, q, V)$  the exponent  $e$  lies in at most one residue class modulo  $dq^{V-1}$ . Unlike the Wieferich sparsity, this sparsity is per  $(r, a, q)$ , not uniform in  $r$ ; and since  $a$  ranges over all values coprime to  $c$ , the total number of classes can still be large. The residual is thus structured — an enumerable family of congruence conditions — rather than an unknown inequality.

### 9.3 Reduction to Pillai-type equations

**Theorem 9.8** (Layer III: exact form, Pillai as the extremal case). Write  $b = \prod_{i=1}^t q_i^{V_i}$  with  $V_1 = V = \max_i V_i \geq 3$ . A layer-III triple satisfies

$$r^e - \prod_{i=1}^t q_i^{V_i} = a', \quad |a'| \leq c^{o(1)}, \quad \text{rad}(b) = \prod_i q_i = c^{o(1)}, \quad \sum_i V_i \log q_i = \log b.$$

The extremal case  $t = 1$  is the Pillai equation  $r^e - q^V = a'$  (with  $a = 1$  closed by Catalan [5]); for  $t \geq 2$  the equation involves several prime powers and does not reduce to a Pillai equation, so Pillai's finiteness results do not apply. Thus Pillai-type equations characterize the extremal subfamily, not the whole of layer III.

*Proof.* In layer III the gap condition fails, so  $\text{rad}(abc) \leq c^{o(1)}$  in the sense of (7); in particular  $\text{rad}(b) \leq c^{o(1)}$ , and the cofactor  $m = b / \prod_i q_i^{V_i}$  is  $c^{o(1)}$ -smooth and  $c^{o(1)}$ -bounded, hence is absorbed into  $a' = a + q^V(m-1)$  in the extremal configuration, giving  $|a'| \leq c^{o(1)}$ . (The earlier form of this theorem asserted  $q^V \geq b^{1-o(1)}$ , which is false —  $b = 2^V 3^V$  has  $\text{rad}(b) = 6 = c^{o(1)}$  but  $3^V = b^{0.613}$ ; the statement above is exact as written. On the subclass  $c = r^e$  no such counterexample was found, and the extremal configuration  $q^V = r^{e/2}$  would force  $q = r$  by unique factorization.)  $\square$

**Remark 9.9** (Circularity discipline). The finiteness of the Nagell–Ljunggren equation  $\frac{x^n-1}{x-1} = y^q$  in general follows from  $abc$  (Shorey–Tijdeman [6]) and may not be used here; likewise the Beal conjecture [7, 8] follows from  $abc$  and is excluded. Only the unconditional subclass results of Ljunggren [9] ( $q = 2$ ), Nagell [10] ( $3 \mid n, 4 \mid n$ ), and Mihăilescu [5] enter.

## 10 The structural ceiling

**Theorem 10.1** (Ceiling). The main term of  $\eta$  satisfies

$$\Pi := \frac{1}{k} \frac{\log \det(\Lambda)}{\log c} \leq \frac{1}{k} + O\left(\frac{\log \log c}{\log c}\right),$$

and this bound is optimal: on the family  $c = r^e$ ,  $e \rightarrow \infty$ , one has  $\Pi \rightarrow 1/k$ . Consequently the best possible output of the present framework is a constant polynomial bound  $c \lesssim \text{rad}(abc)^{k/(k-1)}$ , and

the full abc conjecture (1) is structurally unreachable inside it. The corrections from  $\Omega(ab)/2$  and from the gcd form of  $g_c$  affect only the  $o(1)$  term.

*Proof.* From  $\det(\Lambda) = g_c/g \leq g_c = c\theta_c$  and  $\theta_c \leq (\log_2 c)/2$  (Proposition 2.5),  $\log \det(\Lambda) \leq \log c + \log \theta_c \leq \log c + O(\log \log c)$ ; dividing by  $k \log c$  gives the upper bound. *Optimality.* For  $c = r^e$ ,  $g_c = r^{e-1}e$ , so (with  $g = 1$ , the typical case)

$$\frac{\log \det(\Lambda)}{\log c} = \frac{(e-1) \log r + \log e}{e \log r} \rightarrow 1 \quad (e \rightarrow \infty),$$

i.e.  $\Pi \rightarrow 1/k$ . Prime  $c$  ( $e = 1$ ) gives  $\det(\Lambda) = 1$  and  $\Pi = 0$ ; this family is simply better (its  $q$  is small) and does not affect the limsup.  $\square$

**Corollary 10.2.** *Tightening the constants (e.g. improving  $\gamma_2$  to its geometric optimum in Section 5) improves only the finite- $c$  margins; it does not change the ceiling.*

**Remark 10.3** (What the ceiling means). *The framework proves  $\eta \leq \frac{1}{k} + o(1)$  and cannot prove  $\eta = o(1)$  because  $\eta$ 's main term is  $1/k$ . To go below the constant  $k/(k-1)$  in the exponent of  $\text{rad}(abc)$  one would need an input that sees  $\text{rad}(c)$  directly rather than through  $g_c$  — for instance a genuine  $r$ -uniform (rather than per- $(r, a, q)$ ) sparsity in layer III, or an effective Pillai theorem uniform in the difference  $a'$ . As  $k \rightarrow \infty$  the bound  $q \leq k/(k-1)$  tends to 1, but this is trivial: large  $k$  already forces  $\text{rad}(ab)$  to be large.*

## 11 The reach and limits of the method

This section records two kinds of results. The counting and density statements (Subsections 11.2–11.3) are elementary: they use nothing from the lattice framework and no transcendence theory, only standard analytic number theory (partial summation, Rankin's method, Mertens' theorem). The obstacle statements (Subsection 11.4) are statements about the framework of this paper: they combine the structural ceiling (Theorem 10.1) with the information budget of the three currently available tool layers, and one clause is conditional on the working hypothesis on the transcendence layer recorded in Appendix C.

Throughout this section we write

$$\varepsilon(X) := C_0 \frac{(\log \log X)^2}{\log X} = o(1),$$

with  $C_0$  a sufficiently large absolute constant; the same symbol is used in every bound, allowing  $C_0$  to be enlarged finitely many times.

### 11.1 Genesis: two withdrawn drafts and the repair

The results of this section were reached through a three-layer genealogy of errors in earlier drafts. We record it in compressed form; the complete disposal record of the withdrawn draft statements is maintained in Appendix A (entries E24–E29).

*Layer 1: density is not an exponent (E24).* The first attempted route to a density theorem for the quality invoked the Dickman–de Bruijn asymptotic  $\Psi(X, X^{1/u}) \sim X\rho(u)$ . This is an asymptotic for the *density* of smooth numbers, not a power-saving exceptional-set exponent: since  $\Psi(X, \sqrt{X}) \asymp X$ , the claimed exceptional-set exponent 0.3069 (the value  $\rho(2) = 1 - \log 2$ ) was withdrawn.

*Layer 2: coupling is not decoupling (E25).* The counting problem for coprime pairs with  $\text{rad}(a)\text{rad}(b) \leq T$  was first treated by dropping the coupling condition  $rs \leq T$  from the joint parametrization, which announced the exponent 1; that exponent was withdrawn. The correct coupled exponent is  $1/2$  (Theorem 11.3 below). A shorter elementary route, which bypasses the coupling problem entirely, follows from the trivial observation  $\text{rad}(abc) \geq \text{rad}(c)$ : any triple with  $q > 2$  forces  $\text{rad}(c) < c^{1/2}$ , so the density theorem for the quality (with the  $1/2$ -exponent replaced by  $1 - \varepsilon$ ) follows from the one-variable counting lemma alone. We give the full coupled analysis anyway, because it records the sharp exponent for the genuinely two-variable problem.

## 11.2 Elementary counting

For integers  $M \geq 1$  and  $r \geq 1$  write

$$F(M, r) = \#\{m \leq M : m \mid r^\infty\},$$

and for real  $X, Y \geq 1$  write

$$S(X, Y) = \#\{n \leq X : \text{rad}(n) \leq Y\} = \sum_{\substack{r \leq Y \\ r \text{ squarefree}}} F(X, r),$$

where the last identity groups  $n$  by  $r = \text{rad}(n)$ .

**Lemma 11.1** (One-variable bound). *Let  $r$  be squarefree with  $\omega(r) = t \geq 1$  and let  $M \geq 2$ . Then*

$$F(M, r) \leq (3 \log M)^t, \quad \text{and, refined,} \quad F(M, r) \leq \left( \frac{C \log M}{t} \right)^t$$

for an absolute constant  $C$ , valid for  $t \leq \log_2 M$ .

*Proof.* If  $m \mid r^\infty$  then  $m = \prod_{p \mid r} p^{e_p}$  with  $\sum_{p \mid r} e_p \log p \leq \log M$ , hence  $\sum_p e_p \leq \log_2 M =: E$ . The number of exponent tuples is

$$\binom{E+t}{t} \leq \left( \frac{2eE}{t} \right)^t$$

by  $\binom{E+t}{t} \leq \left( \frac{e(E+t)}{t} \right)^t \leq \left( \frac{2eE}{t} \right)^t$  for  $t \leq E$ , giving the refined bound; the coarse bound follows from  $\binom{E+t}{t} \leq (E+t)^t \leq (3 \log M)^t$  for  $M \geq 2$ .  $\square$

**Lemma 11.2** (The one-variable sum). *For  $X \geq X_0$  and  $1 \leq Y \leq X^{1/2}$ ,*

$$S(X, Y) \leq Y \cdot X^{\varepsilon(X)}.$$

*Proof.* Set  $L = \log X$ ,  $W = 10 \log \log Y$ , and split the defining sum according as  $\omega(r) \leq W$  or  $\omega(r) > W$ .

If  $\omega(r) \leq W$ , the coarse bound of Lemma 11.1 gives  $F(X, r) \leq (3L)^W$  and summing over the at most  $Y$  admissible  $r$  yields

$$Y(3L)^W \leq Y \cdot X^{\varepsilon(X)/2}$$

for  $X \geq X_0$ , since  $W \log(3L) = O((\log \log X)^2)$ .

If  $\omega(r) = t > W$  we first count the  $r$ 's. Rankin's method with  $z = t / \log(C \log Y) \geq 1$  gives

$$\#\{r \leq Y : r \text{ squarefree}, \omega(r) = t\} \leq z^{-t} \sum_{\substack{r \leq Y \\ r \text{ squarefree}}} z^{\omega(r)} \leq z^{-t} Y \prod_{p \leq Y} \left( 1 + \frac{z}{p} \right),$$

using  $\sum_r z^{\omega(r)}/r \leq \prod_{p \leq Y} (1 + z/p) \leq \exp(z \sum_{p \leq Y} 1/p) \leq (C \log Y)^z$  and Mertens' theorem, hence

$$\#\{r \leq Y : r \text{ squarefree}, \omega(r) = t\} \leq Y \left( \frac{e^{1+o(1)} \log \log Y}{t} \right)^t.$$

Consequently

$$\sum_{\substack{r \leq Y \\ \omega(r) > W}} F(X, r) \leq Y \sum_{t > W} \left( \frac{A}{t} \right)^t, \quad A = 3e^{1+o(1)} L \log \log Y.$$

Since  $r$  is squarefree with  $\omega(r) = t$  one has  $Y \geq r \geq \prod_{p \leq p_t} p = \exp((1 + o(1))t \log t)$ , so the sum runs over  $t \leq (1 + o(1)) \log Y / \log \log Y =: T^*$ ; moreover  $(A/t)^t$  is increasing on  $[W, T^*]$  because  $T^* \leq A/e$  for  $X \geq X_0$ . Hence

$$\sum_{t > W} \left( \frac{A}{t} \right)^t \leq T^* \left( \frac{A}{T^*} \right)^{T^*}.$$

Writing  $s = \log Y$  we have  $\log(A/T^*) = \log \log X - \log s + O(\log \log s)$  and  $T^* = s / \log s$ , so

$$T^* \log \frac{A}{T^*} = \frac{s(\log \log X - \log s)}{\log s} + O\left(\frac{s \log \log s}{\log s}\right) = o(\log X),$$

the maximum of  $s \mapsto s(\log \log X - \log s) / \log s$  on  $[\log 2, \log X]$  being  $O(\log X / \log \log X)$ . Therefore the tail is  $\leq Y \cdot X^{\varepsilon(X)/2}$ , and the lemma follows.  $\square$

**Theorem 11.3** (Global coupled counting). *For  $T = X^{1/2-\delta}$ ,  $0 \leq \delta \leq 1/2$ ,*

$$P(X, T) := \#\{(a, b) : 1 \leq a < b, a + b \leq X, \text{rad}(a) \text{rad}(b) \leq T\} \ll X^{1/2-\delta+\varepsilon(X)}.$$

*Proof.* Group the pairs by  $r = \text{rad}(a)$ . For each squarefree  $r \leq T$  there are  $F(X, r)$  choices of  $a$ , and  $b$  ranges over  $\{b \leq X : \text{rad}(b) \leq T/r\}$ , of which there are  $S(X, T/r)$ . Hence

$$P(X, T) \leq \sum_{\substack{r \leq T \\ r \text{ squarefree}}} F(X, r) S(X, T/r) \leq T X^{\varepsilon(X)} \sum_{r \leq T} \frac{F(X, r)}{r}$$

by Lemma 11.2 (applicable since  $T/r \leq T \leq X^{1/2}$ ). Partial summation with  $W(u) = S(X, u)$  gives

$$\sum_{r \leq T} \frac{F(X, r)}{r} = \frac{W(T)}{T} + \int_1^T \frac{W(u)}{u^2} du \leq X^{\varepsilon(X)} + X^{\varepsilon(X)} \int_1^T \frac{du}{u} \leq X^{2\varepsilon(X)},$$

and absorbing the constant into  $C_0$  yields the claim.  $\square$

**Proposition 11.4** (Sharpness of the exponent). *The exponent  $1/2 - \delta$  in Theorem 11.3 is optimal for elementary counting: taking distinct primes  $p, q \leq X^{1/4-\delta/2}$  and  $k, \ell \in \{1, 2\}$ , the pairs  $(a, b) = (p^k, q^\ell)$  satisfy  $a + b \leq X$  and  $\text{rad}(a) \text{rad}(b) = pq \leq T$ , and there are  $\asymp \frac{X^{1/2-\delta}}{(\log X)^2}$  such pairs.*

**Theorem 11.5** (Per- $c$  version). *Fix  $c$  and  $T = c^\beta$  with  $0 \leq \beta \leq 1/2$ . The number of all representations*

$$\#\{(a, b) : a + b = c, \text{rad}(a) \text{rad}(b) \leq T\} \ll c^{\beta+\varepsilon(c)}.$$

*The coprime subfamily  $(a, b) = 1$  therefore also satisfies this bound.*

*Proof.* By symmetry assume  $b \geq c/2$ ; this restriction at most doubles the count. Since  $\text{rad}(b) \leq \text{rad}(a) \text{rad}(b) \leq T$ , the parameter  $b$  satisfies  $\text{rad}(b) \leq T$ , and for each such  $b$  the value  $a = c - b$  satisfies  $\text{rad}(a) \leq T/\text{rad}(b)$ . Hence the count is at most

$$\sum_{\substack{c/2 \leq b \leq c \\ \text{rad}(b) \leq T}} \#\{a \leq X : \text{rad}(a) \leq T/\text{rad}(b)\} \leq X^{\varepsilon(X)} T \sum_{b \leq X} \frac{1}{\text{rad}(b)},$$

by Lemma 11.2 applied with  $X$  in place of the ambient height (and  $T/\text{rad}(b) \leq X^{1/2}$ ). Finally, grouping  $b$  by its radical,

$$\sum_{b \leq X} \frac{1}{\text{rad}(b)} \leq \sum_{\substack{r \leq X \\ r \text{ squarefree}}} \frac{F(X, r)}{r} \leq X^{2\varepsilon(X)}$$

by the same partial-summation step as in Theorem 11.3. Since  $c \leq X$ , this is  $\leq c^{\beta+\varepsilon(c)}$  after enlarging  $C_0$ .  $\square$

**Remark 11.6** (( $\star'$ )-failure counting). *The framework identity expresses the main term of the quality as  $\frac{k-1}{k} \log c - \log \text{rad}(ab) + (\text{effective-prime correction})$ , so ( $\star'$ )-failure forces  $\text{rad}(ab) \leq c^{(k-1)/k-\eta}$ . Theorem 11.5 with  $\beta = \frac{k-1}{k} - \eta$  bounds the number of such exceptional  $c \leq X$  by*

$$\ll_k X^{(k-1)/k-\eta+o(1)} = o(X) \quad \text{for each fixed } k.$$

*In the alternative branch  $\text{rad}(abc) \geq \text{rad}(ab)$  one has  $R \geq \log \text{rad}(ab) + o(\log c)$  and  $q \leq \frac{k}{k-1} + o(1)$  trivially, so the nontrivial exceptional sets are exactly the sparse ones counted above.*

### 11.3 Density consequences

For  $\eta \geq 0$  let

$$E_\eta(X) := \#\{c \leq X : \exists \text{ coprime } a + b = c \text{ with } q(a, b, c) > 2 + \eta\}, \quad E(X) = E_0(X).$$

**Theorem 11.7** (Density theorem). *With  $\varepsilon(X)$  as above:*

(i)  $\#\{c \leq X : \text{rad}(c) \leq c^{1-\varepsilon}\} \ll X^{1-\varepsilon/2+\varepsilon(X)} = o(X)$  for every fixed  $\varepsilon > 0$ , hence for density 1 of integers  $c$ , every coprime decomposition  $a + b = c$  has  $q \leq 1 + o(1)$ .

(ii)  $E_\eta(X) \ll X^{\frac{1}{2} - \frac{\eta}{2(2+\eta)} + \varepsilon(X)}$ , in particular  $E(X) \ll X^{1/2+\varepsilon(X)}$ .

(iii) The superset  $\{c \leq X : \text{rad}(c) \leq \sqrt{c}\}$  has size  $X^{1/2 \pm o(1)}$ , and the exponent  $1/2$  is sharp.

(iv) For random coprime decompositions of a random  $c$ ,  $q \rightarrow 1$  in probability.

*Proof.* (i) By dyadic decomposition and Lemma 11.2 with  $Y = (2^j)^{1-\varepsilon} \leq 2^j$ ,

$$\#\{c \leq X : \text{rad}(c) \leq c^{1-\varepsilon}\} \leq \sum_{j \leq \log_2 X} S(2^j, 2^{j(1-\varepsilon)}) \leq \sum_{j \leq \log_2 X} 2^{j(1-\varepsilon+\varepsilon(2^j))},$$

and  $\varepsilon(2^j) < \varepsilon/2$  for  $j \geq j_0(\varepsilon)$ , so the sum is  $\ll_\varepsilon X^{1-\varepsilon/2+\varepsilon(X)}$ . The second clause of (i) uses  $\text{rad}(abc) \geq \text{rad}(c)$ : a decomposition with  $q > 1 + \varepsilon'$  forces  $\text{rad}(c) \leq c^{1-\varepsilon'}$ .

(ii) If  $q(a, b, c) > 2 + \eta$  then  $\text{rad}(c) \leq \text{rad}(abc) < c^{1/(2+\eta)}$ , so  $E_\eta(X) \leq S(X, X^{1/(2+\eta)}) \leq X^{1/(2+\eta)+\varepsilon(X)}$  by Lemma 11.2, and  $\frac{1}{2+\eta} = \frac{1}{2} - \frac{\eta}{2(2+\eta)}$ .

(iii) The upper bound is the case  $\eta = 0$  of (ii) (or the lemma with  $Y = \sqrt{X}$ ); the lower bound comes from the  $\pi(\sqrt{X})$  squares  $p^2 \leq X$ , all of which satisfy  $\text{rad}(p^2) = p = \sqrt{p^2}$ .

(iv) The elementary estimate

$$\sum_{n \leq X} \log \frac{n}{\text{rad}(n)} = \sum_p \log p \sum_{k \geq 2} \left\lfloor \frac{X}{p^k} \right\rfloor \leq X \sum_p \frac{\log p}{p(p-1)} = C X, \quad C = 0.755366 \dots,$$

shows that  $\log \text{rad}(c) \geq \log c - O(1)$  for density 1 of  $c$ ; combined with (i) this gives  $q \rightarrow 1$  in probability over random coprime decompositions.  $\square$

**Remark 11.8** (Lattice-free). *Theorems 11.3–11.7 are completely independent of the lattice framework: no solution lattice, defect factor, or  $(\star')$  enters their statements or proofs.*

**Remark 11.9** (The pointwise weight  $\omega$  is negligible (E26)). *The  $\omega = \log(\Omega(ab)/2)$  in Theorem 6.1 satisfies  $\Omega(ab) \leq \log_2(ab) < 2 \log_2 c$ , hence  $\omega = O(\log \log c)$ , whence  $\omega/T \rightarrow 0$  pointwise since  $T = \log c$ . A draft sentence claiming a uniform  $\omega/T$  bound on a parameter range is withdrawn; the pointwise statement is what Theorem 6.1 uses, and it is correct.*

**Remark 11.10** (The exceptional-set exponent). *Define*

$$\theta_0 := \inf\{\theta : E(X) \ll X^\theta\}.$$

*Theorem 11.7(ii) gives  $0 \leq \theta_0 \leq 1/2$ ; no lower bound beyond  $\theta_0 \geq 0$  is currently known, and the exact value is open (Problem G8).*

## 11.4 The obstacle

**Definition 11.11** (The method class  $\mathcal{M}$ ). *An  $\mathcal{M}$ -argument is any argument deriving abc-type estimates through the three layers used in this paper:*

- (M1) divisibility identity layer: *the Wronskian factorization and the master inequality  $c \leq \mu \frac{\Omega(ab)}{2} \text{rad}(abc)$ ;*
- (M2) geometric control layer: *Minkowski/Hermite-type bounds  $\mu \leq \sqrt{\gamma_k} \det(\Lambda)^{1/k}$  with  $\det(\Lambda) = g_c \theta_c / g$ ;*
- (M3) transcendence layer: *effective Baker-type lower bounds for logarithmic forms  $|\Lambda_{\text{LL}}|$  with  $|\Lambda_{\text{LL}}| \leq \exp(c^{O(1)})$  (cf. the bridge, Section 9).*

**Theorem 11.12** (Obstacle theorem). *Let  $\mathcal{M}$  be the method class of Definition 11.11.*

- (a) Ceiling. *Every  $\mathcal{M}$ -argument yields at best  $\limsup q \leq k/(k-1) \leq 2$ ; for  $k=2$  the value 2 is the optimal reachable ceiling inside  $\mathcal{M}$  (this is Theorem 10.1).*
- (b) Tightness. *Even with the best-case geometry ( $\mu = \det^{1/k}(1 + o(1))$  achieved) and the maximal information ( $\kappa$  upper-controlled), the provable exponent remains  $\eta \leq 1/k + o(1)$ , i.e.  $q \leq 2 + o(1)$ : the ceiling is not an artifact of poor constant control.*
- (c) Full abc unreachable in  $\mathcal{M}$ . *Full abc would require  $\eta = o(1)$ , pressing  $\mu$  below  $\det^{1/k} \approx c^{1/k}$  by a full power of  $c$ ; pure geometry of numbers cannot do this. The only remaining channel is (M3), which by the working hypothesis of Appendix C on the shape of explicit Baker bounds in the double-fat window is equivalent to a growth-difference Pillai problem (Theorem 11.14(D2) below). No other channel is available.*

**Corollary 11.13.**  $\sup\{\alpha : \exists \mathcal{M}\text{-argument proving } \limsup q \leq \alpha\} = 2$ . *The value 2 is reached under  $(\star')$  (Theorem 6.5), and is reached unconditionally in the density and probability senses of Theorem 11.7.*

## 11.5 Remaining difficulty

**Theorem 11.14** (Remaining-difficulty theorem). *Let  $R_c = \log \text{rad}(c)$  and  $T = \log c$ .*

*(D1) The literal claim is false (E27). “ $R_c \geq (1 - o(1))T$  for almost all triples” fails already on the closed family  $c = 2^e$ :  $R_c = \log 2 = o(T)$ , and this class was closed unconditionally in Section 8. More sharply, every  $c$  divisible by 4 (density  $1/4$ ) satisfies  $\text{rad}(c) \leq c/2$ , so  $R_c \leq T - \log 2$ .*

*(D1') Restricted characterization. On the  $(\star')$ -failure set one has  $\text{rad}(c) \geq c^{1-o(1)}$ , hence  $R_c \geq (1 - o(1))T$ ; (D1') is correct exactly on this set.*

*(D2) The growing-difference Pillai core is sparse. The only remaining route (channel (M3) of Definition 11.11) reduces to a growth-difference Pillai problem, and by Theorem 11.5 the relevant per- $c$  families are  $\ll c^{\beta+o(1)}$  with  $\beta \leq 1/2$ ; in particular the growing-difference core has counting function  $o(X)$ .*

*Proof.* (D1) For  $c = 2^e$ ,  $\text{rad}(c) = 2$ . For  $4 \mid c$ , write  $c = 4m$ ; then  $\text{rad}(c) = 2 \text{rad}(m) \leq 2m = c/2$ . (D1') Restates the definition of  $(\star')$ -failure via  $\text{rad}(c) \geq c^{1-o(1)}$ . (D2) This is Theorem 11.5 together with Remark 11.6.  $\square$

**Remark 11.15.** *The constants 2 and  $k/(k-1)$  in the ceiling and in the master inequality can only be improved on thin sets; the barrier is effective, not a density artifact.*

## 11.6 Numerical cross-checks

All counts below are exhaustive.

**Table 1:** Exceptional set  $E(X)$ , powerful set  $C(X) = \{c \leq X : \text{rad}(c) \leq \sqrt{c}\}$ ; exponents  $\log C / \log X$ . All entries verified by exhaustive computation.

| $X$               | $10^2$ | $10^3$ | $10^4$ | $10^5$ | $10^6$ | $10^7$ |
|-------------------|--------|--------|--------|--------|--------|--------|
| $E(X)$            | 0      | 0      | 0      | 0      | 0      | 0      |
| $C(X)$            | 17     | 76     | 338    | 1397   | 5781   | 23368  |
| $\log C / \log X$ | 0.615  | 0.627  | 0.632  | 0.629  | 0.627  | 0.624  |

Self-check:  $1 + 4374 = 4375$  with  $4374 = 2 \cdot 3^7$ ,  $4375 = 5^4 \cdot 7$ ,  $\text{rad}(abc) = 210$ ,  $q = \log 4375 / \log 210 = 1.5679 < 2$  and  $c - \text{rad}(abc)^2 = 4375 - 210^2 < 0$ , confirming that the family is not detectable by the  $q$ -threshold in either direction.

**Table 2:** Coupled-pair counts  $P_\delta(X) = \#\{1 \leq a < b, a + b \leq X : (a, b) = 1, \text{rad}(a) \text{rad}(b) \leq X^{1/2-\delta}\}$ ; parenthesized:  $\log P / \log X$ . All entries verified by exhaustive enumeration.

| $\delta$ | $X = 10^4$   | $X = 10^5$    | $X = 10^6$     |
|----------|--------------|---------------|----------------|
| 0.00     | 3772 (0.894) | 28750 (0.892) | 223604 (0.892) |
| 0.10     | 1481 (0.793) | 7506 (0.775)  | 47604 (0.780)  |
| 0.25     | 268 (0.607)  | 657 (0.564)   | 4140 (0.603)   |

A second implementation (independent code path; its counting convention differs from the one above — ordered pairs, possibly including non-coprime pairs — and is used only as an order-of-magnitude check) reported

$$(X; P) = (4000; 10166), (16000; 37194), (64000; 154614),$$

consistent with a  $\sqrt{X}(\log X)^{2.5}$  fit to within  $\pm 35\%$  (best at  $X = 16000$ : 0.5%). Note that the tabulated  $P_\delta$  above are coprime, unordered counts; the conventions are not identical.

**Honest reading of the numerics.** The  $X^{o(1)}$  slack in the theorems is numerically indistinguishable from a constant shift of exponent: at  $X = 10^6$  the quantity  $\varepsilon(X)$  is about 0.5 with the normalization used above, and the measured deviations from the clean exponents are about 0.39. The tables therefore support the qualitative picture (no exceptional  $c$  up to  $10^7$ ; coupled exponents near  $1/2$ ) but cannot certify sharp exponents at this height.

**Table 3:** One-variable counts  $S(\alpha) = \#\{n \leq 10^6 : \text{rad}(n) \leq 10^{6\alpha}\}$ ; exponents  $\log S / \log X$  versus the withdrawn draft prediction  $\rho(1/\alpha)$  (Dickman–de Bruijn, E24).

| $\alpha$               | 0.10        | 0.25  | 1/3   | 0.50  |
|------------------------|-------------|-------|-------|-------|
| $\log S / \log X$      | 0.251       | 0.477 | 0.546 | 0.684 |
| draft $\rho(1/\alpha)$ | $\approx 0$ | 0.005 | 0.049 | 0.307 |

## 11.7 Conclusion of the analysis

Three statements summarize the reach and the limits. First, the reach of the method ends at  $q \leq 2$  under  $(\star')$ , and this ceiling is structurally stable. Second, on the density side the elementary counting of this section shows that almost all  $c$  have every coprime decomposition with  $q \leq 1 + o(1)$ . Third, the difficulty beyond 2 is precisely the effective sparsity of the growing-difference Pillai core (Theorem 11.14(D2)): not a density obstruction, but an information obstruction.

## 12 Numerical illustrations

All quantities below are computed directly from the definitions;  $\mu$  values are certified (exhaustive verification of minimality), and the examples are chosen to exercise the falsified barrier, the sandwich, the explicit kernel vectors, and the pointwise-vs.-limsup distinction.

**Example 12.1** (Falsification data).  $1 + 242 = 3^5$ :  $\mu = 27$  (*certified: no solution with  $|u_2| \leq 26$  has  $H(u) \neq 0$ ; optimal vector  $u^* = (-27, -27)$* ),  $\det = 405$ ,  $\kappa = 1.3416$ , while the falsified barriers give 0.2908 (*min*) and 0.7071 (*min-max*). The sandwich is tight:  $\mu / \text{dist} = 27/34.60 = 0.7804 = \kappa/x$  (Remark 4.4).

**Example 12.2** (Multi-prime  $c$  and the gcd modulus).  $31 + 329 = 360 = 2^3 3^2 5$ :  $k = 3$ ,  $h = (-329, 1457, 217)$ ,  $A = (1, 47, 7)$ ,  $D = 10199$ ,  $g_h = 1$ ,  $|a'|_2 = \sqrt{2278179} = 1509.4\dots$ ,  $\text{rad}(abc) = 305970$ . With the correct modulus  $g_c = \gcd(540, 240, 72) = 12$  one has  $\theta_c = 1/30$  and  $F_c = 1$ ; the erroneous product modulus would give  $g_c = 72$  and  $\theta_c = 0.2$ , shrinking every subsequent margin fourfold.

**Example 12.3** (Core sample data). Table 4 collects the certified core samples. Note  $3 + 125 = 2^7$  with  $N = 7$  (*the index is visible*), and  $4 + 27 = 31$  with  $g_h = 6 = D$  — the case where  $\log g_h$  cancels  $-\log D$  in (7), forcing  $g_h$  to be kept explicitly.

**Example 12.4** (The  $k = 2$  full-family scan). Scanning  $a = p^u$ ,  $b = q^v$  for primes  $p, q < 23$ , exponents  $u, v < 14$ ,  $c < 10^{15}$  gives 4461 triples:

| Item                                           | Result                                              |
|------------------------------------------------|-----------------------------------------------------|
| Good branch ( $\lambda_1 < \Delta$ )           | 4427 samples, $\kappa \leq 1.0746$ : 4427/4427 pass |
| True bad branch ( $\lambda_1 = \Delta$ )       | 34 samples, $\kappa \leq 1.1547x$ : 34/34 pass      |
| False bad branches ( $x \geq 0.9306$ but good) | 2 (including $2^5 + 13^{13}$ )                      |
| True bad-branch $\kappa/x$ range               | [0.7071, 0.9869]                                    |
| Geometric bound (Theorem 5.3) violations       | 0/34                                                |
| Candidate $M$ violations                       | 7/34 (hence $M$ withdrawn, Remark 5.4)              |
| $\psi$ range                                   | [70.2°, 90.0°] (hexagonal 60° not attained)         |

**Table 4:** Core sample data;  $\mu$  certified throughout.

| Triple                        | $k$ | $\mu$ | det     | $\Delta$ | $N$ | $\kappa$ | $x$   | $\eta$ |
|-------------------------------|-----|-------|---------|----------|-----|----------|-------|--------|
| $1 + 78124 = 5^7$             | 2   | 28    | 109375  | 19531.0  | 1   | 0.0847   | 0.017 | 0.2958 |
| $1 + 2186 = 3^7$              | 2   | 28    | 5103    | 1093.0   | 1   | 0.3920   | 0.065 | 0.4333 |
| $1 + 242 = 3^5$               | 2   | 27    | 405     | 11.7     | 1   | 1.3416   | 1.719 | 0.6000 |
| $3 + 125 = 2^7$               | 2   | 6     | 448     | 72.1     | 7   | 0.2835   | 0.294 | 0.3693 |
| $1 + 15 = 16$                 | 2   | 4     | 32      | 5.8      | 1   | 0.7071   | 0.970 | 0.5000 |
| $2 + 3^{10} \cdot 109 = 23^5$ | 3   | 113   | 1399205 | 11022.0  | 5   | 1.0103   | 1.135 | 0.3015 |
| $1 + 16806 = 7^5$             | 3   | 36    | 12005   | 10099.2  | 1   | 1.5722   | 0.052 | 0.3683 |
| $1 + 161050 = 11^5$           | 3   | 29    | 14641   | 20624.5  | 1   | 1.1854   | 0.029 | 0.2809 |
| $1 + 2400 = 7^4$              | 3   | 5     | 343     | 76.6     | 1   | 0.7143   | 0.640 | 0.2068 |
| $1 + 59048 = 3^{10}$          | 3   | 45    | 98415   | 2027.9   | 1   | 0.9747   | 1.051 | 0.3465 |
| $1 + 2047 = 2^{11}$           | 2   | 117   | 11264   | 91.9     | 1   | 1.1024   | 1.155 | 0.6246 |
| $1 + 262143 = 2^{18}$         | 4   | 28    | 262144  | 9821.8   | 1   | 1.2374   | 1.180 | 0.2671 |
| $49 + 531392 = 531441$        | 4   | 51    | 1062882 | 18450.2  | 2   | 1.5884   | 1.794 | 0.2982 |
| $4 + 27 = 31$                 | 2   | 1     | 1       | 1.4      | 1   | 0.3780   | 0.435 | 0.0000 |

**Example 12.5** (The power-congruence bridge, three instances).

| $r$ | $a$ | $q$ | $V$ | residue class of $e$ | modulus $dq^{V-1}$ |
|-----|-----|-----|-----|----------------------|--------------------|
| 3   | 1   | 11  | 2   | $e \equiv 5$         | 55                 |
| 2   | 1   | 7   | 2   | $e \equiv 0$         | 21                 |
| 2   | 5   | 3   | 3   | $e \equiv 5$         | 18                 |

The first row is  $1 + 242 = 3^5$ :  $11^2 \parallel 242$ ,  $e = 5$ ,  $d = \text{ord}_{11}(3) = 5$ ,  $dq^{V-1} = 5 \cdot 11 = 55 > 5$ , so  $e = 5$  is the unique solution (Corollary 9.4) — the extremal point of the  $a = 1$  closure is exactly this mechanism.

## 13 Open problems

**Problem (G1).** Close layer III. Prove an effective Pillai-type theorem for  $r^e - q^V = a'$  uniform for  $|a'|$  up to the  $c^{(1)}$  threshold, or an  $r$ -uniform version of the power-congruence sparsity (Corollary 9.2). This is the unique residual family and the true location of the difficulty.

**Problem (G2).** Pointwise vs. limsup. Determine whether  $q \leq \frac{k}{k-1}$  holds pointwise outside an explicit finite list; the Reyssat triple shows the list is nonempty.

**Problem (G3).** *Effective constants. Make the  $o(1)$  terms in Theorem 6.5 effective, with explicit dependence on  $k$ .*

**Problem (G4).** *Joint optimization of  $g_h$  and  $V$ . The exact formula (Lemma 3.6) and the bridge bounds suggest a combined inequality that could shrink the layer-III family further.*

**Problem (G5).** *Beyond  $c = r^e$ . Extend the sieve of Section 9 to multi-prime  $c$ , where  $F_c$  replaces  $E_c$  and the bridge must be built prime-by-prime.*

**Problem (G6).** *Break the ceiling. Any input bounding  $\eta$  by  $o(1)$  would escape Theorem 10.1; within the determinant parametrization this requires seeing  $\text{rad}(c)$  rather than  $g_c$ .*

**Problem (G7).** *Scale of computation. The underlying manuscripts report machine checks (several hundred triples, including the known high-quality examples); a systematic computation up to a fixed height, with the scripts and data published, would sharpen every constant in this paper.*

**Problem (G8).** *The exceptional-set exponent. Determine  $\theta_0 := \inf\{\theta : E(X) \ll X^\theta\}$  for the exceptional set  $E(X)$  of Theorem 11.7; presently  $0 \leq \theta_0 \leq 1/2$  (Remark 11.10), and even  $\theta_0 = 0$  (the empirical evidence of Table 1) is consistent with the known bounds.*

## Acknowledgements

This paper is a synthesis of a series of working manuscripts (Volumes I–X and their revised second editions) developing the present framework; the computational verifications appended to those manuscripts were used as independent checks of the examples in Section 12.

## A Revision history and errata

The framework was developed across ten working volumes, a uniform second revision, and nineteen further revision batches. For the referee’s convenience we record the errata that shaped the present text; every statement in the body of this paper already incorporates the corrected form. The later batches contain two genuine falsifications (E12, E13) and several withdrawals; they are listed with the same candour.

### First generation (volumes I–X and the second edition)

- E1 (danger direction). In the duality  $x \cdot y = 1$ , the dangerous regime is  $x \rightarrow \infty$ , not  $x \rightarrow 0$ .
- E2 (kernel bound).  $\lambda_1(\ker H)$  is a lower bound for the geometry of  $\Lambda_0$ , not an upper bound; the correct input is the determinant formula  $\det(\ker H) = |a'|_2$  (Theorem 3.5(ii)).
- E3 (good-case criterion). For  $a = 1$ , divisibility of the bad case is necessary but not sufficient; superseded by (5).
- E4 (determinant).  $\det \Lambda = g_c/g$  exactly; there is no power law in  $D$  (Theorem 3.5(i)).
- E5, E6 (bounds). The corrected lower bound  $|a'|_2 \geq D/(g_h q_{\min})$  (Lemma 3.8) replaces two earlier, weaker estimates.
- E7 (literature). An earlier draft misattributed the relevant regular-prime class to Vandiver; the correct attribution is Chen–Mihăilescu. No mathematical statement was affected.

- E8, E9 ( $\kappa_{\min}$  formula). An omitted factor  $(k-1)^{1/(2k)}$  was restored. Subsequently the entire  $\kappa_{\min}$  concept was reclassified: it is the floor of  $K$ , not a ceiling for  $\kappa$  — see E12.
- E10 ( $g_c$  is a gcd). The modulus is  $g_c = \gcd_{p|c}(\frac{c}{p}v_p(c))$ , not the product  $\prod e_p p^{e_p-1}$  (Lemma 2.2). The two agree on prime powers, so the subclass  $c = r^e$  was unaffected; for multi-prime  $c$  the correction strengthens every margin and yields the two-line bound  $\theta_c = O(\log c)$  (Proposition 2.5).
- E11 ( $\Omega/2$  and limsup). The factor  $\frac{\Omega(ab)}{2}$  in the master inequality had been dropped in an intermediate version; restoring it demotes  $q \leq \frac{k}{k-1}$  from a pointwise to a limsup bound.

## Second generation (the nineteen later batches)

- E12 (the min-max barrier is false). The claims  $\kappa \leq \min(1/y, Qy^{1/(k-1)})$  and  $\kappa \leq Q^{(k-1)/k}$  are falsified by  $1 + 242 = 3^5$  ( $\kappa = 1.3416 > 0.7071$ ) and by the Reyssat triple ( $\kappa = 0.975 > 0.881$ ); the two branches are orthogonal components and combine by quadrature, giving the Pythagorean form  $K(x)$  of Theorem 4.3. The inference “ $\kappa_{\min} = \min_y K(y)$  bounds  $\kappa$  above” was a logical inversion and is withdrawn.
- E13 (pointwise  $\eta \leq 1/k$  is false). Certified counterexamples in Proposition 6.3; the correct pointwise statement is Theorem 6.1, and the graded bound  $\eta \leq 1/k + o(1)$  is conditional on  $(\star')$  (Theorem 6.5).
- E14 (good/bad criterion). The rule  $x < 1/\sqrt{\gamma_k}$  tacitly assumed  $\Lambda_0$  balanced; it fails for  $k \geq 3$  ( $1 + 16806 = 7^5$ :  $x = 0.052$  yet  $\kappa = 1.5722$ ). Replaced by (5) together with the anisotropy defect ani.
- E15 ( $\gamma_3$ ). An earlier table filled in  $\gamma_2$ 's value for  $\gamma_3$ ; corrected to  $\gamma_3 = 2^{1/3} = 1.2599$  (threshold 0.891).
- E16 (the index  $N$ ). The formula  $N = g_c / \gcd(A \cdot B_H)$  missed  $g_c$  itself in the gcd, truncating  $N$  to 0 at the Reyssat triple and silently skipping 10/18 samples; corrected to  $N = g_c / \gcd(g_c, A \cdot (B_H)_1, \dots, A \cdot (B_H)_{k-1})$ .
- E17 ( $k = 2$  anisotropy). The kernel generator had not been multiplied by  $N$  ( $3 + 125 = 2^7$  reported ani = 7.00); corrected: ani  $\equiv 1$  for  $k = 2$  (Corollary 4.7).
- E18 (layer spacing). The quantity  $\det^{(k-1)/k} / \Delta$  is  $x$  itself; the layer spacing is  $\text{dist} = \det / \Delta$  (Definition 4.1).
- E19 ( $K^*$  and the in-plane bounds). The candidate  $K^* = \sqrt{\text{dist}^2 + \lambda_1(\Lambda_0)^2}$  and the chain  $\lambda_1(\Lambda_0) \leq (\text{in-plane}) \leq \rho(\Lambda_0)$  are both false ( $1 + 78124 = 5^7$ : in-plane  $7 \ll \lambda_1 = 19531$ ); withdrawn.
- E20 (the candidate  $M$ ).  $M = \max |h_i| / \|h\|_2$  is violated by 7/34 true bad-branch samples;  $M$  is merely the degenerate value of Theorem 5.3 at  $\psi = 90^\circ$ ; withdrawn (Remark 5.4).
- E21 (old Theorem 19.1).  $\kappa \leq \max(1, 8x)$  superseded by Theorem 5.1 (constant  $8 \rightarrow \gamma_2 = 1.1547$ ).
- E22 ( $\sup(\kappa/x)$ ). The value  $\gamma_2$  would require  $\psi = 60^\circ$  and  $|h_2/h_1| = 1/\sqrt{3}$  (irrational), hence is not attained; the empirical supremum is 0.9869. The earlier record 0.9973 came from the false bad branch  $2^5 + 13^{13}$  and is corrected (Remark 5.6).

E23 (“Reyssat breaks the graded bound”). Withdrawn: Reyssat has  $\eta = 0.3015 < 1/3$ ; the pointwise excess is explained by  $\omega/T$  (Example 6.2). Failure of the Hermite route is not failure of the conclusion.

The ceiling (Theorem 10.1) is untouched by all errata, and the falsifications E12–E13 lower the unconditional reach of the framework: the honest main statement is now the pair (Theorem 6.1 pointwise, Theorem 6.5 conditional on the unique gap  $(\star')$ ).

### Third generation (the reach-and-limits chapter, Section 11)

E24 (density is not an exponent). A draft claimed the exceptional-set exponent 0.3069 from the Dickman–de Bruijn asymptotic  $\Psi(X, X^{1/u}) \sim X\rho(u)$ . That asymptotic controls the *density* of smooth numbers, not a power-saving exceptional-set exponent; since  $\Psi(X, \sqrt{X}) \asymp X$ , the claimed exponent is withdrawn (it was the value  $\rho(2) = 1 - \log 2$ ). Table 3 records the measured discrepancy. The corrected density statements are Theorem 11.7(i)–(iii).

E25 (coupling is not decoupling). A draft treated the coupled counting problem for  $\text{rad}(a)\text{rad}(b) \leq T$  by dropping the coupling condition and announced the exponent 1; withdrawn. The correct coupled exponent is  $1/2$  (Theorem 11.3, sharp by Proposition 11.4); the shorter route via  $\text{rad}(abc) \geq \text{rad}(c)$  bypasses coupling entirely (Subsection 11.1).

E26 ( $\omega/T$  sentence). A draft sentence claimed a uniform  $\omega/T$  bound over a parameter range; withdrawn. The pointwise statement used by Theorem 6.1 is correct:  $\omega = O(\log \log c)$ , hence  $\omega/T \rightarrow 0$  (Remark 11.9).

E27 (literal remaining-difficulty claim (D1)). The draft claim “ $R_c \geq (1 - o(1))T$  for almost all triples” is false:  $c = 2^e$  has  $R_c = \log 2$ , and every  $c$  divisible by 4 has  $\text{rad}(c) \leq c/2$ . The restricted characterization (D1′) on the  $(\star')$ -failure set is correct (Theorem 11.14).

E28 (typographical, Rankin display). In the pre-merge version of the one-variable counting lemma, the Rankin step was displayed without its factor  $Y$ :  $\#\{r \leq Y : \omega(r) = t\} \leq Y(e^{1+o(1)} \log \log Y/t)^t$ . The factor is restored in the proof of Lemma 11.2; the statement and the final bounds were unaffected.

E29 (“all even  $c$ ”). A draft sentence claimed that every even  $c$  satisfies  $\text{rad}(c) \leq 2\sqrt{c}$ ; this is false already for  $c = 2p$  ( $\text{rad}(c) = c$ ). The corrected density statement — every  $c$  divisible by 4 satisfies  $\text{rad}(c) \leq c/2$  — is what Theorem 11.14(D1) uses.

E30 (Layer I boundary overestimate). Lemma 9.5 as originally stated claimed closure up to  $V \leq \log c / \log \log c$ ; the correct constant is  $c^* = \frac{(k-1)/k}{A+2} \leq \frac{k-1}{k(k+2)} (\leq 1/8 \text{ for } k=2, A=2)$ : the threshold’s logarithm carries the coefficient  $A+2$ , not 1. The false parenthetical “failing triples have  $a \leq c^{o(1)}$ ” is removed (WLOG  $a \leq b$  suffices and strengthens the bound). The Layer I/III dividing line moves down to  $c^* \log c / \log \log c$ ; the uncovered band is absorbed into layer III, which the power–congruence bridge governs for all  $V \geq 2$ .

The complete draft disposal table underlying E24–E30 is not reproduced in this volume; the entries above record the mathematical content of every disposal decision.

## B Auxiliary estimates

### B.1 Hermite constants

Hermite's constant  $\gamma_k$  is the supremum of  $\lambda_1(L)^2/\det(L)^{2/k}$  over rank- $k$  lattices. The values used in this paper are  $\gamma_2 = 2/\sqrt{3} = 1.1547\dots$ ,  $\gamma_3 = 2^{1/3} = 1.2599\dots$ , and the general bound  $\gamma_k \leq k$  (in fact  $\gamma_k \leq \frac{2}{\pi}\Gamma(2 + \frac{k}{2})^{2/k} \sim k/(\pi e)$  by Minkowski's convex-body theorem; see [13]). All occurrences of  $\sqrt{\gamma_k}$  in the good-case bound are therefore  $\leq \sqrt{k}$ ; for  $k \geq 3$  the good case is rare (Proposition 4.10), which is why the  $k = 2$  solution of Section 5 does not extend.

### B.2 The Wieferich analogy

Corollary 9.3 shows that the layer-III residual has convergent natural density  $\sum_{V \geq 2} q^{-(V-1)} = 1/(d(q-1))$ . This is structurally identical to the heuristic for Wieferich primes [14]: the condition  $r^{q-1} \equiv 1 \pmod{q^2}$  has “probability”  $1/q$  per prime, and  $\sum_q 1/q^2$  converges, so the dangerous locus is expected to be sparse — here, provably sparse at the level of residue classes. The difference is that our sparsity is per  $(r, a, q)$  rather than uniform in  $r$ ; upgrading it to an  $r$ -uniform statement is precisely open problem G1 (Section 13).

### B.3 Comparison with Stewart–Yu

The best unconditional bound toward abc is  $c < \exp(CR^{1/3}(\log R)^3)$  [3, 4] obtained from Diophantine approximation (linear forms in logarithms). Our conditional consequence (Theorem 6.5),  $c \lesssim R^{2+o(1)}$  under  $(\star')$ , would be polynomial; three caveats are essential: (i) it is a limsup statement — the  $o(1)$  absorbs constants and finitely many exceptional regimes, and the pointwise version fails already at  $1 + 242 = 3^5$  (Proposition 6.3); (ii) it is conditional on the gap  $(\star')$ , which is equivalent to  $\text{rad}(ab) \geq c^{1/2-o(1)}$  and false for high-exponent prime-power pairs (Theorem 7.1); (iii) the residual family of Section 9 (layer III, Pillai-type) is exactly where  $(\star')$  fails. The comparison is therefore between an unconditional sub-exponential bound for all triples and a polynomial bound whose hypothesis pinpoints the hard core. The two methods are complementary: linear forms in logarithms see the archimedean size, the lattice method sees the  $p$ -adic valuations.

### B.4 A primorial bound for $C_k$

**Lemma B.1.** *The covering-radius constant  $C_k = (\rho(\Lambda_0)/\Delta^{1/(k-1)})^2$  satisfies  $\sqrt{C_k} = \exp(O(k))$ , hence  $\sqrt{C_k} = c^{o(1)}$  whenever  $k = o(\log c)$ .*

*Proof.*  $\Lambda_0$  is a sublattice of  $\mathbb{Z}^{k-1}$  in  $k$  coordinates whose covering radius is bounded by  $\frac{k-1}{2}$  times the largest coordinate vector needed to span it; the explicit kernel vectors of Lemma 4.6 have sup-norm  $\leq \max_{i,j} \sqrt{p_i^2 + p_j^2}$ , and  $\max_i p_i \leq p_k \leq 2k \log k$  by the Chebyshev (primorial) bound. Hence  $\rho(\Lambda_0) \leq \frac{k}{2} \cdot 2k \log k \cdot \Delta^0$ , and after normalizing by  $\Delta^{1/(k-1)} \geq 1$  one gets, absorbing the  $\log k$  into the constant,  $\sqrt{C_k} = \exp(O(k))$ .  $\square$

### B.5 The $\Omega(ab)/2$ factor

The factor  $\omega_0 = \frac{\Omega(ab)}{2}$  enters the master inequality through the upper bound for the Wronskian (Lemma 2.10): the coefficient vector  $h$  satisfies  $\sum_i |h_i|/(D/q_i) = \sum_i v_i = \Omega(ab)$ , and each  $|h_i| = (D/q_i)v_i \leq (D/2)v_i$ . Since  $\Omega(ab) \leq \log_2(ab) \leq 2 \log_2 c$ , one has  $\omega_0 = O(\log c)$ , so the factor costs

only an  $o(1)$  in every exponent — but it is exactly what turns the pointwise bound  $q \leq \frac{k}{k-1}$  into a limsup bound, and it cannot be removed without improving Lemma 2.10 itself.

## C Statements not proved in this paper

The following items are used as hypotheses or discussed heuristically; none is proved here, and we list them explicitly to prevent misunderstanding.

- The gap  $(\star')$ :  $|\log x| = O(\log \log c)$ . This is the unique remaining gap, unproved; it is equivalent to  $\text{rad}(ab) \geq c^{1/2-o(1)}$  and is at risk of being false in high-exponent regimes (Theorem 7.1).
- The Pillai reduction for  $a > 1$ :  $S$ -unit finiteness does not apply with a moving prime set  $S$ ; no closure is claimed (Proposition 7.2).
- Bad-case  $\kappa = O(1)$  for  $k \geq 3$ ; empirically true (measured  $\kappa \leq 1.59$  on all certified samples), unproved.
- The integrality obstruction to  $\sup(\kappa/x) = \gamma_2$ : an insight, not a proof.
- The abc conjecture itself: unproved; by Theorem 10.1 it is structurally unreachable within the present framework.
- The working hypothesis on the transcendence layer (used in Theorem 11.12(c)): explicit Baker-type lower bounds for logarithmic forms in the double-fat window (both variables carrying full radical weight) have the standard shape with no sub-multiplicative gain, so that channel (M3) of Definition 11.11 is equivalent to a growth-difference Pillai problem rather than a shortcut past it. This is a hypothesis about the shape of known transcendence estimates, not a theorem.

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