

ANALYTIC SYMMETRIC FLOATING BODIES

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ABSTRACT. In every dimension $d \geq 3$, we construct nonspherical, origin-symmetric, strictly convex bodies of revolution with real-analytic boundary that float in equilibrium in every orientation. Their densities tend to the unit-ball cap fraction at height $(d+4)^{-1/2}$, where we prove that a degree-four spherical resonance is isolated. More generally, our construction applies at isolated even-degree resonances whose waterline angle is an irrational multiple of π . Varying the density gives formal deformations. To obtain actual bodies, we use a finite-order diagonal reduction to establish inverse estimates away from small divisors. Newton iteration then corrects fixed-order truncations on closed parameter sets of asymptotically full measure.

1. INTRODUCTION

A homogeneous body of relative density $\rho \in (0, 1)$ floats in equilibrium when the centroid of its submerged cap of volume $\rho \operatorname{vol} K$ lies on the vertical line through the centroid of the body. Ulam asked whether equilibrium in every orientation characterizes the ball. Nonspherical bodies of revolution exist at density $1/2$ in every dimension $d \geq 3$ [4]; at that density, origin symmetry forces a convex floating body to be a ball [2]. Wegner constructed formal centrally symmetric branches in dimension three at other densities [5]. We obtain analytic bodies near isolated zonal resonances and verify the required conditions in every dimension.

Equilibrium is equivalent to constancy of cap volume and energy. At the ball, the Funk–Hecke formula diagonalizes the linearized energy operator. At an isolated zero, the kernel is one-dimensional. Varying the density resolves the resonant equation and gives approximate solutions to any fixed order. We prove isolation for degree four in every dimension.

To correct these approximations, we write the cap integrals over fixed spherical caps and parametrize the meridian so that waterline endpoints have a constant parameter difference. Matveev’s theorem bounds the small divisors of the tangential equation at the limiting angle. For the normal equation, we diagonalize to finite order and invert the noncritical part by a Neumann series wherever the diagonal satisfies the required lower bounds. We apply Newton iteration on nested closed parameter sets, estimating the measure lost as the diagonal changes.

Write $v_d = \pi^{d/2}/\Gamma(1+d/2)$, $\lambda = (d-2)/2$, and $Z_l = C_l^\lambda/C_l^\lambda(1)$. Thus $Z_l(\omega_1)$ is the normalized degree- l zonal harmonic on S^{d-1} . For $0 < h < 1$ put

$$(1) \quad \rho_d(h) = \frac{v_{d-1}}{v_d} \int_h^1 (1-s^2)^{(d-1)/2} ds.$$

For an axial profile u , $[u]_l$ denotes its coefficient of Z_l . We use the same coefficient notation for its angular representative $u(\cos x)$.

Theorem 1. Fix $d \geq 3$, an even integer $L \geq 4$, and $h_* \in (0, 1)$ such that

$$(2) \quad Z_L''(h_*) = 0, \quad Z_l''(h_*) \neq 0 \quad (l \geq 2 \text{ even}, l \neq L), \quad a_* := \arccos h_* \notin \pi\mathbb{Q}.$$

For any $0 < c_1 < c_2$ and all sufficiently small $t > 0$, there is a closed set $G_t \subset I_t = [a_* + c_1 t, a_* + c_2 t]$ with $|I_t \setminus G_t| = o(t)$. Each $a \in G_t$ determines a nonspherical, origin-symmetric, strictly convex body of revolution $K_a \subset \mathbb{R}^d$ of volume v_d , with real-analytic boundary, floating in equilibrium in every orientation. Writing $\varepsilon = [R_a]_L > 0$, its radial function and relative density satisfy

$$(3) \quad R_a = 1 + \varepsilon Z_L + O(\varepsilon^2), \quad \rho_a = \rho_d(h_*) + \rho_1 \varepsilon + O(\varepsilon^2), \quad a - a_* = a_1 \varepsilon + O(\varepsilon^2).$$

Here $a_1 > 0$ and $\rho_1 < 0$. The meridian has an analytic parametrization γ in which each waterline chord joins $\gamma(x - a)$ to $\gamma(x + a)$. The remainders are uniform on a fixed complex angular strip. The same bodies float at density $1 - \rho_a$.

Corollary 2. *Theorem 1 applies with $L = 4$ and $h_* = (d + 4)^{-1/2}$ for every integer $d \geq 3$. In particular,*

$$\rho_{*,3} = \frac{1}{2} - \frac{5}{7\sqrt{7}}, \quad \rho_{*,4} = \frac{\arccos(1/\sqrt{8})}{\pi} - \frac{19\sqrt{7}}{96\pi}, \quad \rho_{*,d} := \rho_d((d + 4)^{-1/2}).$$

Figure 1 shows the leading quartic deformation in dimension three.

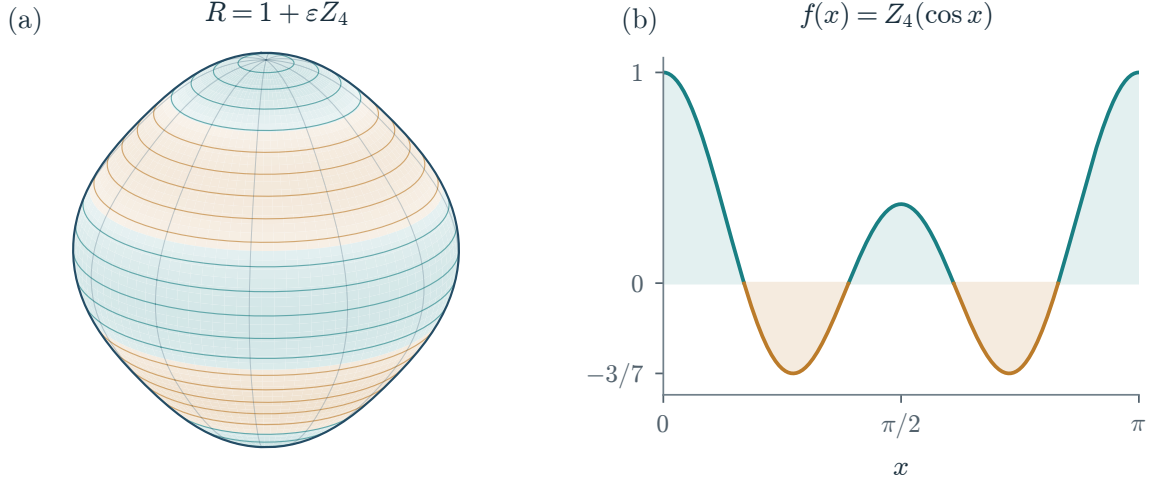


FIGURE 1. The first-order quartic profile $R = 1 + 0.08Z_4$ in dimension three and its angular harmonic. Teal and ochre indicate positive and negative displacement.

For an origin-symmetric body, set $V(n, q) = \text{vol}\{z \in K : n \cdot z > q\}$ and $J(n, q) = \int_K (n \cdot z - q)_+ dz$. If $V(n, q(n)) = \nu$, then $J_q = -V$ shows that the tangential derivative of $J(n, q(n)) + \nu q(n)$ is the transverse cap moment. Equilibrium in every orientation is therefore equivalent to constancy of this energy. Central symmetry fixes the centroid and gives the complementary-density assertion.

2. THE SPHERICAL CALCULATION

Let C_h, T_h be spherical convolutions with kernels $\mathbf{1}_{s>h}$ and $(s - h)_+$, using unnormalized surface measure. Put $\Lambda_l = l(l + d - 2)$ and $\omega_{d-2} = |S^{d-2}|$. Funk–Hecke gives their degree- l eigenvalues

$$c_l(h) = \frac{\omega_{d-2}(1 - h^2)^{(d-1)/2} Z'_l(h)}{\Lambda_l}, \quad t_l(h) = \frac{\omega_{d-2}(1 - h^2)^{(d+1)/2} Z''_l(h)}{\Lambda_l(\Lambda_l - d + 1)}.$$

The formulas apply for $l \geq 1$ and $l \geq 2$, respectively; lower degrees are given by the integrals. Also $t'_l = -c_l$. Thus the isolation condition for the quartic application reduces to excluding all other zeros at the same cap height.

Lemma 3. *For $d \geq 3$ and $l \geq 2$, $Z''_l((d + 4)^{-1/2}) = 0$ if and only if $l = 4$.*

Proof. Put $D = d + 3$, $T_* = 4(d + 2)$, and choose a square root h of $1/(D + 1)$. With $y = (1 - h)/2$, Taylor expansion at one gives

$$(4) \quad Z''_l(h) = \frac{\Lambda_l(\Lambda_l - d + 1)}{(d - 1)(d + 1)} S_M(\Lambda_l), \quad S_M(T) = \sum_{j=0}^M e_j \prod_{r=2}^{j+1} (r(r + d - 2) - T),$$

$$e_j = \frac{(2y)^j}{j! \prod_{r=0}^{j-1} (D + 2r)}, \quad M \geq \max(5, l - 2).$$

To prove isolation, we show that $S_M(T)/(T - T_*)$ is a local unit at every spectral value. This quotient is polynomial because $Z_4''(h) = 0$.

For an odd prime $p \mid D$, put $a = v_p(D)$ and choose $h \equiv 1 \pmod{p}$. Then $v_p(y) = a$ and $e_1 \equiv 1/2 \pmod{p}$. If $v_p(e_j) > 0$ for $j \geq 2$, monic division over $\mathbb{Z}_p$ gives $S_M(T)/(T - T_*) \equiv -1/2 \pmod{p}$. Write $s_p(n)$ for the base- p digit sum and $A_p = a - 2/(p - 1)$. The factorial valuation formula gives

$$(5) \quad \begin{aligned} v_p(e_j) &= a(j - 1) - v_p(j!) - v_p((j - 1)!) > 0 & (2 \leq j \leq p^a), \\ v_p(e_j) &\geq \begin{cases} jA_p + 2/(p - 1) - \log_p(D/2), & D \text{ even,} \\ jA_p + 1/(p - 1) - \log_p D, & D \text{ odd.} \end{cases} \end{aligned}$$

For the second line, express the rising factorial in e_j as a ratio of ordinary factorials and use $s_p(u + v) \leq s_p(u) + s_p(v)$ and $s_p(u) \leq (p - 1) \log_p(u + 1)$. Thus $A_p > 0$ and $\log D \leq p^a A_p \log p$ suffice.

If $b = v_2(D) \geq 3$, choose $h \equiv 1 \pmod{4}$, set $U = T/2$, and put $w_j = 2^j e_j$. All factors $r(r + d - 2)/2 - U$ have integral coefficients. Here w_1 is odd, and positivity of $v_2(w_j)$ for $j \geq 2$ gives $S_M(2U)/(U - T_*/2) \equiv 1 \pmod{2}$. The corresponding bounds are positivity for $2 \leq j \leq 2^{b-1}$ and

$$(6) \quad v_2(w_j) = j(b - 3) + s_2(j) + s_2(D/2 + j - 1) - s_2(D/2 - 1) \geq j(b - 3) + 2 - \log_2(D/2).$$

In particular $2^{b-1}(b - 3) \geq \log_2 D$ suffices.

These criteria cover all but four values of D . To check this, let P be its largest exact prime-power factor and $L_n = \text{lcm}(1, \dots, n)$. We use $D \mid L_P$ and $L_n \leq 4^{n-1}$, the latter following by induction from central binomial coefficients. If $P = p^a \geq 11$ is odd, then $A_p \log p > \log 4$, so the odd-prime criterion follows. If $P = 2^b \geq 128$, then $2^{b-1}(b - 3) \geq 2P > \log_2 D$; for $P = 64$ use $L_{64} < 2^{96}$. Otherwise $D \mid L_{32}$. If $Q = p^a$ is the largest odd prime-power factor, the remaining checks are

$$\begin{array}{l|l} Q \geq 11 & D \leq 4^Q, \quad A_p \log p > \log 4 \\ Q = 9 & D \leq 10080 < 3^9 \\ Q = 7 & D \leq 3360, \quad 3360^3 < 7^{14} \\ Q = 5 & D/2 \leq 240 < 5^{7/2} \text{ if } D \text{ is even; otherwise } D = 15. \end{array}$$

In the last row (5) applies for $j \geq 6$, and its initial range covers $j \leq 5$. In the first row, the odd part divides the odd part of L_Q and $v_2(D) \leq 5$, giving $D \leq L_Q 2^{5 - \lfloor \log_2 Q \rfloor} \leq 4^Q$; the inequality for A_p is immediate. With $Q \leq 3$ or no odd factor, $D = 2^b$ or $3 \cdot 2^b$, and (6) applies for $b \geq 4$. Only $D = 6, 8, 12, 24$ remain.

Their exact valuations and reduced quotients are listed below. In the odd-prime rows the quotient is $S_M(T)/(T - T_*)$; in the other rows it is $S_M(2U)/(U - T_*/2)$.

D	coefficient	valuation	unit indices	quotient
6	e_j ($p = 3$)	$(s_3(j) + s_3(j + 2) - 2)/2$	0, 1	$-1/2$
8	w_j ($p = 2$)	$s_2(j) + s_2(j + 3) - 2$	0, 1	1
12	e_j ($p = 3$)	$(s_3(j) + s_3(j + 5) - 3)/2$	0, 1, 4	$(1 - T)^3$
24	w_j ($p = 2$)	$s_2(j) + s_2(j + 11) - 3$	0, 1, 5	$1 + U(1 + U)^3$

The first two unit-index assertions reduce to differences of powers equal to 2 or 3. For $D = 12$, the minimum digit-sum pair has types (1, 2) and (2, 1): $j = 3^r$ gives $j = 1$, while $s_3(3^r - 5) = 2r - 2$ gives $j = 4$. For $D = 24$, a total of two would make powers of two differ by eleven; a total of three gives $j = 1$ or, using $s_2(2^r - 11) = r - 2$, $j = 5$. This proves the listed valuations are nonnegative with exactly those units. Substitution in (4) gives the quotients. For $D = 12$, $\Lambda_l = l(l + 7) \equiv 0, 2 \pmod{3}$; for $D = 24$, the quotient is one at both residues modulo two. The quotients are therefore units at all spectral values. Since Λ_l is strictly increasing, only $l = 4$ can vanish. $\square$

For the quartic resonance irrationality follows already from the rational nonintegral trace $2 \cos(2a_*) = -2 + 4/(d + 4)$. Together with Lemma 3, this verifies the hypotheses of Theorem 1

for $L = 4$ in every dimension $d \geq 3$. The density formulas in Corollary 2 follow from (1); Figure 2 shows the resulting resonant heights and limiting densities as d varies.

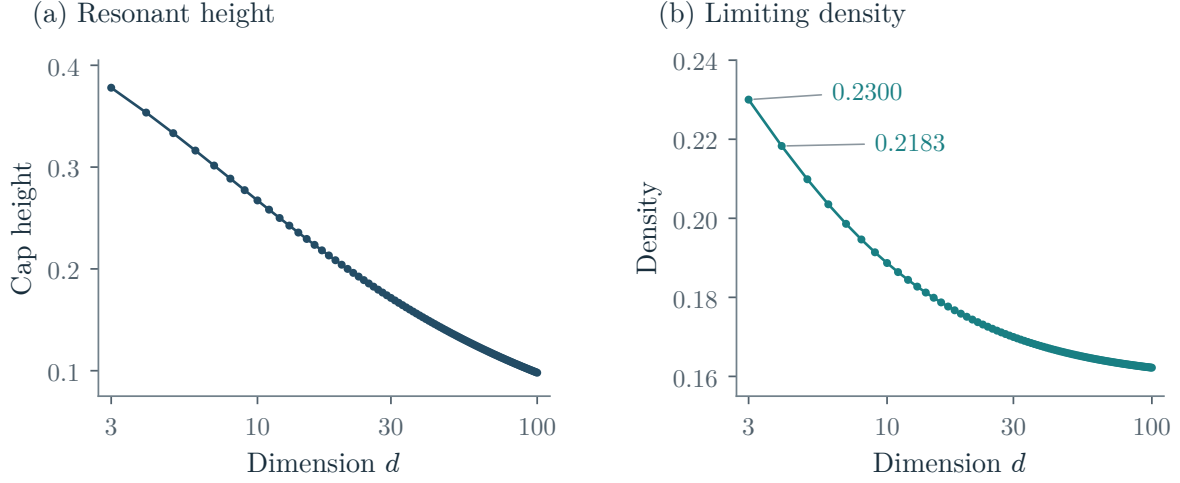


FIGURE 2. Quartic resonant heights and limiting densities for integer $3 \leq d \leq 100$, as in Corollary 2. The horizontal axes are logarithmic.

We return to a general resonance satisfying (2). Irrationality suffices for polynomial translation equations; analytic estimates require a quantitative bound. The numbers h_* and e^{2ia_*} are algebraic, and the latter is not a root of unity. The theorem on linear forms in logarithms [3], applied to $m \log(e^{2ia_*}) - 2j \log(-1)$, gives constants $\gamma_*, \tau_* > 0$ such that

$$(7) \quad |2 \sin(ma_*)| \geq \gamma_*(1 + |m|)^{-\tau_*} \quad (m \in \mathbb{Z} \setminus \{0\}).$$

We next construct formal solutions whose truncations will start the Newton iteration. Their first coefficients determine the changes in density and chord separation. In the rest of the proof, unmarked spherical quantities are evaluated at h_* . Put $f = Z_L$, $\sigma_* = (1 - h_*^2)^{1/2}$, $A = v_{d-1}\sigma_*^{d-1}$, and

$$(8) \quad \begin{aligned} b &= c_L/A, & B &= [f^2]_L, & \mu &= \frac{1}{\pi} \int_0^\pi f(\cos x) dx, \\ h_1 &= \frac{1}{2}(h_* - b)B, & \rho_1 &= -Ah_1/v_d, & a_1 &= \frac{h_* - b}{\sigma_*}(\mu - B/2), & \beta &= Abh_1. \end{aligned}$$

Solvability requires $b \neq 0$, and the angle parametrization requires $a_1 > 0$. Write $L = 2m$. The normalized polynomial $b(h) = C_{L-1}^{d/2}(h)/C_{L-1}^{d/2}(1)$ satisfies $|b(h)| \leq h$ for $0 < h < 1$, by the quadratic Jacobi transformation and endpoint bound [1, 18.7.1, 18.7.14, 18.14.1]. Equality at h_* would force $b'(h_*) = \pm 1$, whereas resonance gives $b'(h_*) = 0$. Thus $|b| < h_*$; also $b \neq 0$, by uniqueness for the Gegenbauer equation for Z'_L . The generating function gives $\mu > 0$, and the product formula [1, 18.18.22], with $v = \lambda + m$, gives

$$\frac{B}{\mu} = \frac{(2m)!}{m!} \prod_{j=0}^{m-1} \frac{2v+j}{(v+j)(v+m+1+j)} < \frac{4}{3}.$$

Every factor decreases for $v \geq m$, and the product equals $4/3$ at $v = m$. Hence $B, h_1, a_1 > 0$ and $\rho_1 < 0$. In angular formulas write $f(x) = Z_L(\cos x)$, $\langle u \rangle = (2\pi)^{-1} \int_0^{2\pi} u(x) dx$, and $T_a u(x) = u(x+a)$.

Proposition 4. *There are formal fixed-volume floating bodies with radial function $R_\varepsilon = 1 + \varepsilon f + O(\varepsilon^2)$, $[R_\varepsilon]_L = \varepsilon$, density $\rho_d(h_*) + \rho_1 \varepsilon + O(\varepsilon^2)$, and cap height $h_* + \varepsilon(h_1 + bf) + O(\varepsilon^2)$. Their meridians admit a formal parametrization for which each cap chord joins the parameters $x - a_\varepsilon$ and $x + a_\varepsilon$, where $a_\varepsilon = a_* + a_1 \varepsilon + O(\varepsilon^2)$. Every coefficient is a finite trigonometric polynomial. For each fixed N , truncation gives analytic seeds on any fixed strip with residual $O(t^N)$ in $\sup + t \text{Lip}$ on I_t .*

Proof. Eliminate the cap height using $V(R, H) = v_d \rho$ and $V_H = -A$. For $R = 1 + \varepsilon(f + w)$, $[w]_L = 0$, divide the nonconstant energy and volume equations by ε . The linearization has nonzero coefficients t_l on the noncritical modes, dv_d for volume, and $v_d b$ for the critical density variation. Funk–Hecke shows that the cap Taylor coefficients preserve finite spherical polynomials. The formal implicit function theorem gives the radial series in this class.

For $R = 1 + \varepsilon f + \varepsilon^2 g$, the quadratic nonconstant energy is

$$T_h g + \frac{1}{2}(dT_h + hC_h)(f^2) - \frac{(C_h f)^2}{2A} + \frac{v_d \rho_1}{A} C_h f.$$

Its degree- L coefficient is $\beta + v_d b \rho_1$, giving (8); the volume row fixes $[g]_0$. The height variation is $h_1 + b f$.

To give the chord endpoints constant parameter separation, let $B_{\pm}(\theta, \varepsilon)$ solve $R(B_{\pm}) \cos(B_{\pm} - \theta) = H(\theta)$, with $B_{\pm} = \theta \pm a_* + O(\varepsilon)$. The map $B_+ B_-^{-1}$ is reversible under $x \mapsto -x$ and commutes with translation by π . Conjugating it formally to translation by $2a_*$ requires, at each order, $u(x + 2a_*) - u(x) = G(x) - \langle G \rangle$. Irrationality determines the nonzero Fourier coefficients, the mean determines the angle, and reversibility gives an odd conjugacy. The first-order mean separation is $a_1 = ((h_* - b)\mu - h_1)/\sigma_*$. Since $a_1 > 0$, invert a_{ε} formally and truncate in $a - a_*$. Taylor’s theorem on a slightly wider strip gives the residual estimate for each fixed truncation order. $\square$

3. CAP COORDINATES

For convergence we keep the endpoint separation fixed and solve for the meridian. We put the moving caps on a fixed spherical cap to estimate the nonlinear equations and their derivatives on common analytic strips. Let $\gamma = (X, Y)$ be a meridian near $\gamma_*(u) = (\cos u, \sin u)$, with X even, Y odd, and $\gamma(u + \pi) = -\gamma(u)$. Put $S = (X'^2 + Y'^2)^{1/2}$ and $N = (Y', -X')/S$. For a near a_* , put $h = \cos a$ for the reference spherical cap, $\gamma_{\pm}(x) = \gamma(x \pm a)$, and define the chord normal and height shown in Figure 3 by

$$L_c^2 = (\gamma_+ - \gamma_-) \cdot (\gamma_+ - \gamma_-), \quad n = \frac{((\gamma_+ - \gamma_-)_2, -(\gamma_+ - \gamma_-)_1)}{L_c}, \quad q = n \cdot \gamma_+.$$

All square roots use their branches at the circle. Revolution gives boundary points $(X(u), Y(u)\eta)$, $\eta \in S^{d-2}$.

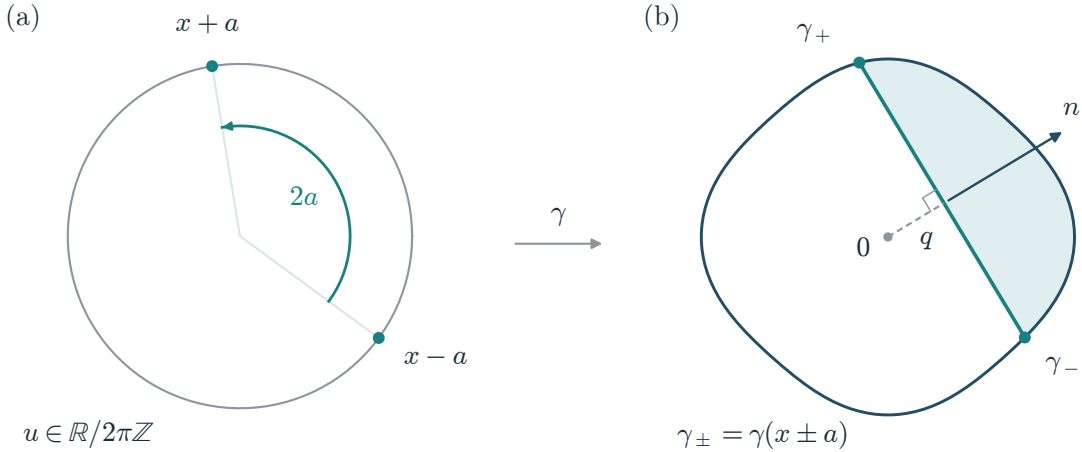


FIGURE 3. Waterline geometry for a near-circular meridian. The endpoints $\gamma_{\pm} = \gamma(x \pm a)$ determine n and q ; shading marks the cap $n \cdot z > q$.

The apparent poles in the cap formulas are removable. Set $\xi = \cos x$, $y = \cos u$, and $H = \sin x \sin u$. The endpoint zeros and reflection give analytic functions e, o , close to 1, 0, such that

$$(9) \quad \frac{n_1 X(u) + n_2 Y(u) - q}{\cos(u - x) - h} = e(\xi, y) + Ho(\xi, y).$$

The divided-difference and sine-division identities give a fixed derivative loss. Even angular functions descend analytically to cosine variables at both endpoints.

To transform the moving cap into $s > h$, put $\chi = e^2 - H^2 o^2$, $W = e - Ho\eta_1$, and $s = \xi y + H\eta_1$. The change of variables on the sphere

$$\eta_{1,\text{old}} = \frac{e\eta_1 - Ho}{W}, \quad \eta_{\perp,\text{old}} = \frac{\sqrt{\chi}}{W}\eta_{\perp}$$

satisfies $n \cdot (X, Y\eta_{\text{old}}) - q = (\chi/W)(s - h)$ and has Jacobian $\chi^{(d-2)/2} W^{-(d-2)}$. For the Cartesian extension, put $g(y) = Y(u)/\sin u$. In coordinates $\omega = (y, \omega_2, \omega_{\perp})$, the first component is $\hat{X}(y) = X(u)$, while the transverse components have common denominator $e - \sin x o \omega_2$ and numerators $g(e\omega_2 - \sin x(1 - y^2)o)$ and $g\sqrt{\chi}\omega_{\perp}$. These are analytic Cartesian expressions, and the denominator is close to one.

The fixed-plane normal variations of volume and energy are

$$(10) \quad \begin{aligned} C_{\gamma}\psi(x) &= \int_{S^{d-1}} \psi(u) b_C(\xi, y, s) \mathbf{1}_{s>h} d\omega, \\ T_{\gamma}\psi(x) &= \int_{S^{d-1}} \psi(u) b_T(\xi, y, s) (s - h)_+ d\omega, \\ b_C &= g^{d-2} S \chi^{(d-2)/2} W^{-(d-2)}, \quad b_T = g^{d-2} S \chi^{d/2} W^{-(d-1)}. \end{aligned}$$

Since $W = e - o(s - \xi y)$, both amplitudes are analytic in (ξ, y, s) . The component of the exterior normal along the cutting normal, at the transformed boundary point, is $b_N = (n_1 Y' - n_2 X' \eta_{1,\text{old}})/S$. The quotients $n_2/\sin x$ and $X'/\sin u$ are analytic by parity, so b_N is analytic in the same variables. Apply the divergence theorem to $(n \cdot z - q)n$ and $\frac{1}{2}(n \cdot z - q)^2 n$. Their fluxes vanish on the cutting plane, giving

$$(11) \quad V = \int_{S^{d-1}} b_N b_T (s - h)_+ d\omega, \quad J = \frac{1}{2} \int_{S^{d-1}} b_N b_T \frac{\chi}{W} (s - h)_+^2 d\omega.$$

Also $V_q = -A_{\gamma}$ and $J_q = -V$. Dilation gives $A_{\gamma} = (C_{\gamma}(\gamma \cdot N) - dV)/q$, an analytic expression since q stays near $h_* > 0$. Total volume is $-v_{d-1} \int_0^{\pi} Y^{d-1} X' du$.

We use the analytic Wiener norms $\|u\|_{r,s} = \sum_m (1 + |m|)^s e^{r|m|} |u_m|$. For a parameter family on a closed subset of I_t , write $\|u\|_{r,s}^{(t)} = \sum_m (1 + |m|)^s e^{r|m|} (\sup |u_m| + t \text{Lip } u_m)$, with the same coefficientwise convention for matrices and tensors. These are algebra norms, and $\|u\|_{r-\delta,s} \leq C_s \delta^{-s} \|u\|_{r,0}$.

For $z = (\gamma, \nu, c_E)$, set $F(z, a) = (F_V, F_E, F_0) = (V - \nu, J + \nu q - c_E, \text{vol } K - v_d)$. Fix a sufficiently large integer s_0 and call $\|\cdot\|_{r,s_0}^{(t)}$ the strong norm. The cap formulas give analytic maps with a fixed derivative loss, through three derivatives and for differences of parameter families. To bound the integrals, expand the amplitudes in $Z_p(\xi)Z_q(y)$ and apply Funk–Hecke in y . Normalized Gegenbauer products have nonnegative coefficients of sum one; conversion from cosine to Gegenbauer coefficients costs at most $d - 1$ derivatives per variable, by orthogonality. Thus the integrals have polynomial strip costs. The Taylor remainder of F is bounded by $C\|v\|_{r,s_0}^{(t)2}$ on a small meridian ball. All derivative losses below are fixed before the seed order is chosen.

We separate normal displacement from reparametrization and omit terms proportional to the residual, whose contribution to a Newton step is quadratic. Take the lift $\theta = x + O(t)$ with $n = (\cos \theta, \sin \theta)$, and put $e_{\theta} = (-n_2, n_1)$, $q_{\theta} = q'/\theta'$, and $\zeta = \delta q - q_{\theta} \delta \theta$. For $\delta \gamma = \psi N + w \gamma'$, split $\zeta = \zeta_{\psi} + \zeta_w$. If $\tau_{\pm} = e_{\theta} \cdot \gamma_{\pm} - q_{\theta}$, endpoint differentiation gives $\zeta_w = (\theta' \tau_+ \tau_- / L_c)(\mathbf{T}_a - \mathbf{T}_{-a})w$.

Put $c_V = -A_\gamma \theta' \tau_+ \tau_- / L_c$, positive near the circle, where $c_V = A\sigma_*/2$. The operator

$$(12) \quad \begin{aligned} (A_z \delta z)_V &= C_\gamma \psi - A_\gamma \zeta_\psi + c_V (\mathsf{T}_a - \mathsf{T}_{-a}) w - \delta \nu, \\ (A_z \delta z)_E &= T_\gamma \psi + q \delta \nu - \delta c_E, \quad (A_z \delta z)_0 = \delta \operatorname{vol} K(\psi) \end{aligned}$$

satisfies the exact identity

$$(13) \quad D_z F \delta z = A_z \delta z + (F'_V \delta \theta / \theta', F'_E \delta \theta / \theta' - F_V \zeta, 0).$$

This follows by varying the plane, using $V_q = -A_\gamma$ and $J_q = -V$, and differentiating along x . Endpoint variation bounds $\delta \theta, \zeta$ by the displacement, with one extra derivative for parameter-dependent translations. The error in (13) is therefore bounded by $C \|F\|_{r,1}^{(t)} \|\delta \gamma\|_{r,1}^{(t)}$.

4. A FINITE-ORDER REDUCTION

We reduce T_γ to a diagonal operator and a remainder of fixed negative order. This will allow inversion by a Neumann series once small diagonal divisors are excluded.

Put $\alpha = (d+2)/2$ and $\Lambda_+ = \lambda + 2\mathbb{N}_0$. Represent the degree- l coefficient by $\mathcal{H}Z_l(\cos x) = e^{ikx}$, where $k = l + \lambda$. On this coefficient space $D = -i\partial_x$ has positive spectrum. For matrices define

$$(14) \quad |B|_r = \sup_{k \in \Lambda_+} \sum_{j \in \Lambda_+} e^{r|j-k|} |B_{jk}|, \quad |R|_{r,8} = \sup_{k \in \Lambda_+} (1+k)^8 \sum_{j \in \Lambda_+} e^{r|j-k|} |R_{jk}|.$$

The norm $|\cdot|_r$ is submultiplicative. Left multiplication preserves $|\cdot|_{r,8}$; right multiplication across a strip gap δ costs $C\delta^{-8}$, since $(1+k)/(1+l) \leq 1 + |k-l|$. Finite rows and columns satisfy the latter bound after a positive strip loss, with polynomial cost. The same estimates hold in the coefficientwise family norms.

The map $\mathcal{H}^{-1}$ is bounded from coefficient to angular norms, since the Fourier coefficients of $Z_l(\cos x)$ are nonnegative and sum to one. Conversely, orthogonality gives $\|\mathcal{H}u\|_r \leq C\|u\|_{r,d-1}$. We may therefore change representations across an arbitrarily small strip gap.

The reduction requires inversion of $\mathsf{T}_a - \mathsf{T}_{-a}$. We therefore retain parameters satisfying

$$(15) \quad |2 \sin(ma)| \geq \gamma_0 (1 + |m|)^{-\tau_h} \quad (m \in 2\mathbb{Z} \setminus \{0\}),$$

where γ_0, τ_h are fixed. We first equalize the two principal amplitudes of (10), $\beta_\pm(x) = b_T(\cos x, \cos(x \pm a), h)$. They are positive and satisfy $\beta_+(-x) = \beta_-(x)$, so $\log \beta_- - \log \beta_+$ is odd. Solve

$$(\mathsf{T}_a - \mathsf{T}_{-a})u = \log \beta_- - \log \beta_+, \quad \langle u \rangle = 0, \quad w_b = e^u, \quad e_b = \beta_+ \mathsf{T}_a w_b = \beta_- \mathsf{T}_{-a} w_b.$$

The zero-mean solution u is even. Condition (15) bounds the inverse with a fixed derivative loss, including parameter differences. For sufficiently large fixed s_0 , the strong bound $\|\gamma - \gamma_*\|_{r,s_0}^{(t)} = O(t)$ gives $w_b, e_b = 1 + O(t)$ at the same width. Set

$$(16) \quad A_d(a) = (2\pi)^\lambda \sin(a)^{\lambda+1}, \quad \mathcal{L}_\gamma = A_d(a)^{-1} \mathcal{H} M_{e_b}^{-1} T_\gamma M_{w_b} \mathcal{H}^{-1} D^\alpha.$$

Here M_v denotes multiplication by v . The principal outgoing coefficients are the constants $c_\pm = e^{\mp i\pi\alpha/2}$.

Lemma 5. *There are width-independent operators Q, E , diagonal coefficients $M_\pm(k, a)$, and a remainder R such that*

$$(17) \quad \mathcal{L}_\gamma Q = E\mathcal{D} + R, \quad d_k = c_+ M_+(k, a) e^{ika} + c_- M_-(k, a) e^{-ika}, \quad \mathcal{D} = \operatorname{diag}(d_k).$$

At the sphere, $Q = E = I$, $R = 0$, and $d_k = d_k^(a) = k^\alpha t_l(\cos a)/A_d(a)$, $k = l + \lambda$. For some fixed P, C and $r' = r - C\delta > 0$,*

$$(18) \quad |Q - I|_{r'}^{(t)} + |E - I|_{r'}^{(t)} + |R|_{r',8}^{(t)} + \sup_k k \sum_{\pm} |M_\pm - P_\pm^*|^{(t)} \leq C\delta^{-P} \|\gamma - \gamma_*\|_{r,s_0}^{(t)}.$$

The same bound holds for differences of two meridians. The reference coefficients $P_-^ = \overline{P_+^*}$ are nonvanishing. On each fixed inner strip, the real diagonal has the form*

$$(19) \quad d_k = 2\mathcal{A}_k \cos(ka + \phi_k), \quad 0 < c \leq \mathcal{A}_k \leq C, \quad \operatorname{Lip} \phi_k \leq C/k, \quad \operatorname{Lip} d_k \leq C(1+k).$$

In particular $d_k - d_k^(a) = O(t/k)$ there. All constructions preserve real meridians.*

Proof. For the energy operator K_b with amplitude $b = \sum_{p,q} b_{pq}(s) Z_p(\xi) Z_q(y)$, Funk–Hecke gives the exact identity

$$(20) \quad \mathcal{H} K_b \mathcal{H}^{-1} D^\alpha = \sum_{p,q} G_p d_{(s-h)b_{pq}}(D) G_q D^\alpha, \quad G_p = \mathcal{H} M_{Z_p} \mathcal{H}^{-1}.$$

The factors act on $\lambda + \mathbb{N}_0$; only the complete tensors, for which $p + q$ is even, preserve Λ_+ . The Gegenbauer product formula [1, 18.18.22] gives G_p bandwidth p , nonnegative entries of column sum one, and rational entries whose fixed-order expansions in k^{-1} have polynomial bounds in p for $p \leq c\sqrt{k}$. The Pochhammer ratios have $O(p)$ factors $1 + O(p/k)$, whose products are bounded in this range; each fixed-order remainder costs a fixed power of p .

Repeated Sturm–Liouville integration by parts, with zero flux at $s = 1$, reduces $d_{(s-h)b_{pq}}$ to endpoint values and a bounded integral times an arbitrarily high negative power of Λ_l . The interior Darboux expansion [1, 18.15(i)], with derivatives supplied by [1, 18.9.15], gives phases $e^{\pm ika}$ and leading amplitudes $A_d(a) c_\pm b(\cos x, \cos(x \pm a), h)$. Only the cap endpoint $h = \cos a$, which stays away from ± 1 , enters this asymptotic. For the remainder norm $|\cdot|_{r',8}$, expand through order eleven. When $p + q \leq c\sqrt{k}$, the error is bounded by $C(1 + p + q)^C k^{-12}$ times the tensor norm; otherwise bandwidth and positivity give $Ck^\alpha e^{-c\delta\sqrt{k}}$ across a strip gap δ . Both contributions satisfy the required bound with cost $C\delta^{-P}$, also for a parameter difference, which costs one power of k and one transfer moment.

Retain the spherical diagonal exactly. Write its Darboux expansion as $d_k^* = c_+ e^{ika} P_+^{[11]} + \text{conjugate} + r_k$, where $P_+^{[11]} = 1 + \sum_{n=1}^{11} p_{+,n}^* k^{-n}$, $r_k = O(k^{-12})$ and $\partial_a r_k = O(k^{-11})$. Set $p_{-,n}^* = \overline{p_{+,n}^*}$, $P_+^* = P_+^{[11]} + \frac{1}{2} c_+^{-1} e^{-ika} r_k$ and $P_-^* = \overline{P_+^*}$. For the finitely many smaller frequencies use $c_+ e^{ika} P_+^* = d_k^*/2 + i$ instead. The branches are nonzero and, for large k , $P_+^* = 1 + O(k^{-1})$, $\partial_a P_+^* = O(k^{-1})$.

After balancing, the two branch symbols $P_\pm$ have leading coefficient one. Use $p \# q = \sum_j \partial_k^j p D_x^j q / j!$ through order eleven. For $Q = 1 + \sum_{n=1}^{11} q_n k^{-n}$ and $E = 1 + \sum_{n=1}^{11} e_n k^{-n}$, let $f_{\pm,n}$ be the coefficient of k^{-n} of $P_\pm \# Q_{<n}(x \pm a, k) - E_{<n} M_{\pm, <n}$. Set

$$m_{\pm,n} = \langle f_{\pm,n} \rangle, \quad \langle q_n \rangle = 0, \\ (\mathsf{T}_a - \mathsf{T}_{-a}) q_n = -f_{+,n} + f_{-,n} + \langle f_{+,n} - f_{-,n} \rangle, \quad e_n = \mathsf{T}_a q_n + f_{+,n} - m_{+,n}.$$

Define $M_\pm = P_\pm^* + \sum_{n=1}^{11} (m_{\pm,n} - p_{\pm,n}^*) k^{-n}$. At the sphere $q_n = e_n = 0$, $m_{\pm,n} = p_{\pm,n}^*$. The finitely many homogeneous inversions, derivatives and products give (18), including differences.

To obtain operators on Λ_+ , use Taylor expansion for transfers $|\nu| \leq k/2$ and the factor $e^{-\delta k/2}$ for the rest. Differentiate only the inverse-power polynomials, treating the exact reference correction as a remainder. On $\lambda + 2\mathbb{Z}$, retain all Fourier output rows; below λ , extend perturbative input columns by zero, references boundedly, and identities by the identity. Let Π project onto Λ_+ . The compression defect $-\Pi A(I - \Pi) B \Pi$ requires a transfer of at least $k - \lambda$, so a strip gap absorbs every fixed input weight; exceptional columns satisfy the same bound. Every defect vanishes at the sphere. This proves (17) and (18) with width-independent operators. Finally $\mathcal{J}v(x) = \overline{v(-x)}$ exchanges the branches and preserves the recursion. Thus $M_- = \overline{M_+}$; nonvanishing and $\text{Lip}(M_+ - P_+^*) \leq C/k$ give (19). $\square$

5. THE INVERSE AND CONVERGENCE

After imposing the volume constraint, we invert the noncritical energy modes; the critical energy row and tangential mean then determine amplitude and density. Figure 4 illustrates the quartic resonance and the effect of a strip loss on the spherical reciprocals. Remove the two indices $\lambda, \lambda + L$ from Λ_+ and denote the complement by Λ' . Choose a sufficiently large fixed M and require

$$(21) \quad |d_k| \geq M t(1 + k)^{-2} \quad (k \in \Lambda').$$

The approximate derivative (12) has the following inverse.

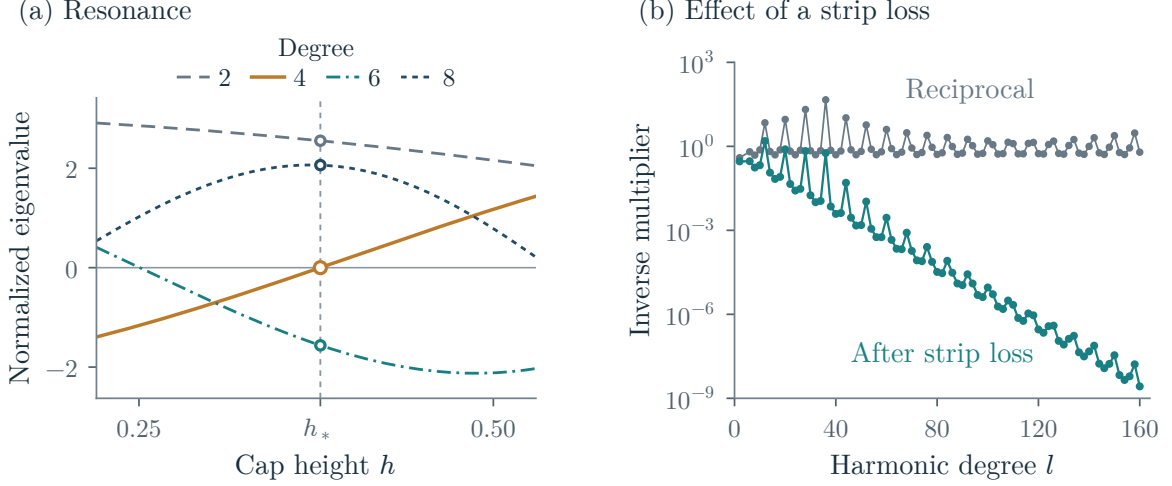


FIGURE 4. The spherical diagonal in dimension three. (a) Selected eigenvalues $d_{l+1/2}^*(\arccos h)$ near $h_* = 1/\sqrt{7}$. (b) $|d_k^*(a_*)|^{-1}$ before and after multiplication by $e^{-0.12k}$, for even $2 \leq l \leq 160$, $l \neq 4$, with $k = l + 1/2$.

Proposition 6. Suppose that (15) and (21) hold, that z is $O(t^2)$ from a seed of Proposition 4 in the strong norm, and that $Q - I, E - I$ and the Schur remainder $\hat{R}$ defined in (23) are $O(t)$ in their respective working-strip family norms. Then A_z has a real symmetry-preserving right inverse N_z , independent of the strips used to estimate it, with

$$(22) \quad \|N_z g\|_{r-C\delta, s_0}^{(t)} \leq Ct^{-P} \delta^{-P} \|g\|_{r,0}^{(t)}.$$

The same type of bound holds for differences, multiplied by the strong norm of the difference of the base meridians. The exponents are fixed independently of the seed order.

Proof. Let $S_z g = \psi$ solve

$$T_\gamma \psi - \delta c_E - \eta f = g, \quad \delta \text{vol } K(\psi) = [\psi]_L = 0.$$

At the sphere the constraints fix ψ_0, ψ_L , and the constant and critical energy rows then fix $\delta c_E, \eta$. Apply (16)–(17) and eliminate this block after multiplying by E^{-1} . The complement is

$$(23) \quad \mathcal{D}_{\Lambda'} + \hat{R}, \quad \hat{R} = (E^{-1}R)_{\Lambda'\Lambda'} - CA^{-1}B.$$

Here A^{-1} is bounded and the finite couplings $B, C = O(t)$ have analytic transfer bounds. Across fixed margins, (14) therefore gives $|\hat{R}|_{r,8}^{(t)} \leq Ct$. By (19) and (21),

$$(24) \quad |d_k^{-1}| \leq (Mt)^{-1}(1+k)^2, \quad t \text{Lip}(d_k^{-1}) \leq C(M^2t)^{-1}(1+k)^5.$$

Consequently $|\hat{R}\mathcal{D}^{-1}|_r^{(t)} \leq C/M + C/M^2 < 1/2$ and

$$(25) \quad (\mathcal{D}_{\Lambda'} + \hat{R})^{-1} = \mathcal{D}_{\Lambda'}^{-1} \sum_{n \geq 0} (-\hat{R}\mathcal{D}_{\Lambda'}^{-1})^n.$$

The Neumann series costs no strip width; $\mathcal{D}_{\Lambda'}^{-1}$ has polynomial cost across a gap. Recovering the finite variables gives the bound for S_z , and uniqueness on smaller strips makes it width-independent. Also $S_z 1 = S_z f = 0$ and $S_z T_\gamma \xi = \xi$ on the constrained subspace.

For the finite system, we need the following limit across every fixed positive strip gap:

$$(26) \quad t \|S_z\|^{(t)} \longrightarrow 0.$$

For fixed $k \in \Lambda'$, $d_k \rightarrow d_k^*(a_*) \neq 0$ and $\text{Lip } d_k = O(1)$ by (19). Hence $t \sup |d_k^{-1}|, t^2 \text{Lip}(d_k^{-1}) \rightarrow 0$. Across a fixed gap δ , (24) gives the summable majorant $C((1+k)^2 + (1+k)^5)e^{-\delta k}$. Dominated convergence proves the limit for the diagonal inverse. The bounded Neumann series acts first; recovering the finite variables and changing coordinates preserve the limit after further fixed strip losses. Use these gaps for the scalar pairings below.

To solve the volume and noncritical energy equations, put $v_0 = 1/\delta \text{vol } K(1)$, $v_L = f - v_0 \delta \text{vol } K(f)$ and

$$\begin{aligned} \psi_g &= v_0 g_0 + S_z(g_E - T_\gamma v_0 g_0), & \psi_a &= v_L - S_z T_\gamma v_L, & \psi_\rho &= -v_d S_z q, \\ \psi &= \psi_g + \delta \varepsilon \psi_a + \delta \rho \psi_\rho, & \delta \nu &= v_d \delta \rho. \end{aligned}$$

The mean of the tangential equation uses the functional

$$\mathbf{m}_z(\psi) = \frac{\int (C_\gamma \psi - A_\gamma \zeta_\psi) c_V^{-1} dx}{v_d \int c_V^{-1} dx}, \quad m = \mathbf{m}_*(f) = \frac{A}{v_d} (b - h_*) \mu,$$

where the integrals are over the circle. At the sphere $\zeta_\psi = h_*(\mathbb{T}_{a_*} + \mathbb{T}_{-a_*})\psi/2$. Since $q = h_* + \varepsilon(h_1 + bf) + O(t^2)$, (26) gives $\psi_a = f + o(1)$ and $\psi_\rho = o(t)$. To compute the critical coefficient, differentiate $T_\gamma f$ at fixed normal angle:

$$\dot{T}f = (dT_* + h_* C_*)(f^2) - (h_1 + bf)C_* f, \quad [\dot{T}f]_L = c_L((h_* - b)B - h_1) = \beta.$$

The kernel vanishes on the cutting plane, so there is no boundary term. Coordinate changes lie in the range of T_* or differentiate $T_* f = 0$, and therefore have zero degree- L coefficient. Thus the critical energy and mean rows, in $(\delta \varepsilon, \delta \rho)$, are

$$\begin{pmatrix} \varepsilon \beta + o(t) & \varepsilon v_d b + O(t^2) \\ -m + o(1) & 1 + o(t) \end{pmatrix}, \quad \det = -\varepsilon A b \sigma_* a_1 + o(t).$$

Since $\varepsilon \asymp t$, $b \neq 0$ and $a_1 > 0$, the determinant is bounded below in absolute value by ct .

The constant energy row fixes δc_E . The remaining equation

$$(\mathbb{T}_a - \mathbb{T}_{-a})w = c_V^{-1}(g_V - C_\gamma \psi + A_\gamma \zeta_\psi + \delta \nu)$$

has even, zero-mean right side; (15) gives its unique odd solution. Set $\delta \gamma = \psi N + w \gamma'$. All operations preserve the symmetries and have fixed polynomial costs in t^{-1} and the strip gap. The inverse difference identity gives the same bounds for differences, proving (22). $\square$

Proof of Theorem 1. The translation bound depends only on a and is imposed once. The energy bounds depend on the meridian and are imposed after each correction. Choose $\tau_h > \tau_* + 1$ and sufficiently small fixed γ_0 . By (7), all $|m| \leq J_t = c't^{-1/(\tau_*+1)}$ satisfy (15) throughout I_t . The closed set $G^h \subset I_t$ satisfying (15) consequently has complement of length at most

$$C\gamma_0 \sum_{m > J_t} (t + m^{-1})(1 + m)^{-\tau_h} \leq C\gamma_0 (tJ_t^{1-\tau_h} + J_t^{-\tau_h}) = o(t).$$

Take $\delta_j = \delta_0(j+1)^{-2}$ and strips $r_j = R_j - \delta_j$, $R_{j+1} = r_j - C\delta_j$, with positive limiting width and fixed inner margins for (26). Start with the order- N seed z_0 on R_0 and initialize the operators on r_0 . Define $v_j = -N_{z_j} F(z_j, a)$ and $z_{j+1} = z_j + v_j$ recursively on the nested sets

$$\begin{aligned} \Gamma_j &= Mt(1 + 2^{-j}), & G_0 &= \{a \in G^h : |d_{0,k}| \geq \Gamma_0(1+k)^{-2}, \ k \in \Lambda'\}, \\ G_{j+1} &= \{a \in G_j : |d_{j+1,k}| \geq \Gamma_{j+1}(1+k)^{-2}, \ k \in \Lambda'\}. \end{aligned}$$

Each correction is computed on G_j before restriction to G_{j+1} ; Lipschitz continuity makes these sets closed. Set $e_j = \|F(z_j, a)\|_{r_j, 0}^{(t)}$. The Taylor estimate, (13), and Proposition 6 give

$$(27) \quad e_{j+1} \leq K_j e_j^2, \quad \|v_j\|_{R_{j+1}, s_0}^{(t)} \leq K_j e_j, \quad K_j \leq Ct^{-P}(j+1)^P.$$

Since $\sum_j 2^{-j} \log K_j \leq P \log(1/t) + C$, choosing N sufficiently large makes both errors and corrections at most $t^{\kappa 2^j}$ for any prescribed fixed κ .

Proposition 6 remains applicable. On $r_{j+1} = R_{j+1} - \delta_{j+1}$, the difference bounds in Lemma 5 and (23) give increments of $Q, E, \hat{R}$ bounded by $Ct^{-P} \delta_{j+1}^{-P} \|v_j\|_{R_{j+1}, s_0}^{(t)}$, reserving part of the strip gap for the finite couplings. For large N , their sum is $O(t^\kappa)$, preserving the $O(t)$ bounds and $O(t^2)$ seed neighborhood. Restrictions decrease norms, and on a common fixed strip

$$(28) \quad \sup_k |d_{j+1,k} - d_{j,k}| \leq Ct^{\kappa 2^j}.$$

The phase bounds in (19) remain uniform.

It remains to estimate the excluded measure. For large k , the phase in (19) has lower and upper Lipschitz slopes $k/2, 2k$ on its closed domain. Linear interpolation across gaps preserves these bounds. There are $O(kt + 1)$ cosine zeros, so a threshold $Mt \leq \Gamma \leq 2Mt$ excludes length at most

$$(29) \quad C\Gamma(t + k^{-1})(1 + k)^{-2}.$$

Each fixed $k \in \Lambda'$ satisfies the G_0 divisor bound for small t , since $d_{0,k} \rightarrow d_k^*(a_*) \neq 0$. The summable majorant in (29), divided by t , gives $|G^h \setminus G_0| = o(t)$ by dominated convergence.

To bound the new exclusions, put $K_j^* = \lfloor t^{-\kappa 2^j/4} \rfloor$. For $k \leq K_j^*$, the ratio of the increment in (28) to $(\Gamma_j - \Gamma_{j+1})(1 + k)^{-2}$ is at most $C2^j t^{(\kappa/2)2^j-1} < 1$. These modes retain their divisor bounds. Summing (29) over $k > K_j^*$ gives

$$|G_j \setminus G_{j+1}| \leq CM\{t^2(K_j^*)^{-1} + t(K_j^*)^{-2}\}, \quad \sum_j |G_j \setminus G_{j+1}| = o(t).$$

Thus $G_t = \bigcap_j G_j$ is closed, $|I_t \setminus G_t| = o(t)$, and (27) converges there on a positive strip to $F = 0$.

The correction can be made smaller than any prescribed fixed power of t , so the seed gives (3). The meridian is C^2 -close to the circle, so the body is strictly convex. At the poles, parity and $Y' \neq 0$ give the analytic graph $X = \chi(Y^2)$, so revolution preserves analyticity. The radial angle is an analytic diffeomorphism on a fixed narrower strip, giving the uniform radial expansion and nonsphericity. Finally $F = 0$ fixes total and cap volumes and makes $J + \nu q$ constant; the meridian normals and transverse rotations cover every orientation. The energy characterization proves the floating assertion and its complementary-density version. $\square$

DISCLOSURE OF AI CONTRIBUTIONS

The original argument, establishing the result in dimension $d = 4$, was developed by GPT-6 Pro in ChatGPT. GPT models working in Codex subsequently extended the argument to every dimension $d \geq 3$ and completed the proof in this generality. The Codex models also assisted with drafting and revising the proof, refining the exposition, and proofreading the manuscript.

REFERENCES

- [1] F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, et al., eds., *NIST Digital Library of Mathematical Functions*, <https://dlmf.nist.gov/>, Chapter 18.
- [2] D. I. Florentin, C. Schütt, E. M. Werner, and N. Zhang, *Convex floating bodies of equilibrium*, Proc. Amer. Math. Soc. **150** (2022), 3037–3048. doi:10.1090/proc/15697.
- [3] E. M. Matveev, *An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers. II*, Izv. Math. **64** (2000), 1217–1269. doi:10.1070/IM2000v064n06ABEH000314.
- [4] D. Ryabogin, *A negative answer to Ulam’s Problem 19 from the Scottish Book*, Ann. of Math. (2) **195** (2022), 1111–1150. doi:10.4007/annals.2022.195.3.5.
- [5] F. Wegner, *Floating bodies of equilibrium in three dimensions. The central symmetric case*, arXiv:0803.1043v2, 2009.