

Cowling's conjecture: the almost connected case

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Abstract

We prove that a locally compact group G with amenable group G/G^0 of components has the Haagerup property if and only if it is weakly amenable with constant one. This proves Cowling's conjecture for connected Lie groups and almost connected locally compact groups. We also characterize weak amenability for arbitrary connected Lie groups and prove the product formula for the Cowling–Haagerup constant in terms of the simple factors of a Levi decomposition. Consequently, the constant is invariant under local isomorphism and decreases under quotients of connected locally compact groups. For groups with amenable group of components, it is unchanged by every closed central quotient. We deduce applications to lattices and their central quotients. The proof combines Ozawa's invariant-state obstruction with averaging over central characters and the tensor identity for induced representations.

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1 Introduction

Weak amenability and the Haagerup property are two extensions of amenability that are expressed in terms of approximation by functions on a locally compact group. Weak amenability requires a bounded approximation of the constant function 1 by elements of the Fourier algebra, where boundedness is measured in the Herz–Schur norm. The optimal bound is the Cowling–Haagerup constant $\Lambda_{\text{WA}}(G)$. The Haagerup property requires approximation by normalized continuous positive definite functions vanishing at infinity. Cowling's conjecture [Ver24, Conjecture 9.5] [CCJJV01, p. 7] asks whether the optimal bound one forces this positive approximation.

Conjecture 1.1 *Let G be a locally compact group. If $\Lambda_{\text{WA}}(G) = 1$, then G has the Haagerup property.*

2020 *Mathematics subject classification*: 22E15, 43A22, 43A35, 46L07.

Key words: weak amenability, Cowling–Haagerup constant, Haagerup property, connected Lie groups, central extensions.

For background on weak amenability and Cowling's conjecture, see Vergara's survey [Ver24, Sections 5 and 9]. For connected simple Lie groups, the weak amenability constants are known from the works [DCH85], [Cow83], [CoH89], [Dor96], [Haa16] and [Han90] of de Cannière, Cowling, Dorofaeff, Haagerup and Hansen. Recall that a connected Lie group is said to be simple [HiN12, Definition 12.1.15 p. 467] if its Lie algebra $\mathfrak{g}$ is simple, that is, if the only ideals of $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$ and further $\mathfrak{g}$ is non-abelian.

Theorem 1.2 ([DCH85], [Cow83], [CoH89], [Dor96], [Haa16], [Han90])

A connected simple Lie group G is weakly amenable if and only if the real rank of G is zero or one. In this case, the weak amenability constant is

$$(1.1) \quad \Lambda_{\text{WA}}(G) = \begin{cases} 1 & \text{when } G \text{ has real rank zero,} \\ 1 & \text{when } G \approx \text{SO}_0(1, n), \ n \geq 2, \\ 1 & \text{when } G \approx \text{SU}(1, n), \ n \geq 2, \\ 2n - 1 & \text{when } G \approx \text{Sp}(1, n), \ n \geq 2, \\ 21 & \text{when } G \approx \text{F}_{4(-20)}. \end{cases}$$

Here $G \approx H$ means that the group G is locally isomorphic to H .

Recall that by [Var84, Theorem 2.11.4 p. 94] two Lie groups are locally isomorphic if and only if their Lie algebras are isomorphic. Muruganandam gave a simpler proof of the constants for groups locally isomorphic to $\text{SU}(1, n)$ and $\text{Sp}(1, n)$ with finite centre, and for $\text{F}_{4(-20)}$, using spherical Fourier analysis, see [Mur08, Theorem 4.4 p. 643]. We recall the following classification [CCJJV01, Theorem 4.0.1 p. 41] of connected Lie groups with the Haagerup property.

Theorem 1.3 ([CCJJV01])

A connected Lie group G has the Haagerup property if and only if it is locally isomorphic to a direct product

$$H \times \text{SO}_0(1, n_1) \times \cdots \times \text{SO}_0(1, n_k) \times \text{SU}(1, m_1) \times \cdots \times \text{SU}(1, m_l),$$

where H is a connected amenable Lie group.

The non-compact simple factors appearing in this classification are precisely those with weak amenability constant one in Theorem 1.2. The additional structural requirement is that these factors commute with the radical at the Lie algebra level. This leads us to study the action of the non-compact simple ideals of a Levi subalgebra on the radical.

For connected simply connected Lie groups, Knudby proved in [Knu16b, Theorem 4 p. 691] that weak amenability is equivalent to the requirement that every non-compact simple ideal $\mathfrak{s}_i$ of a Levi subalgebra has real rank one and satisfies $[\mathfrak{s}_i, \mathfrak{r}] = 0$, where $\mathfrak{r}$ denotes the radical. He also obtained the product formula for the weak amenability constant and explicitly expected the theorem to remain valid without the assumption of simple connectedness.

The case where the connected analytic Levi subgroup has finite centre was treated by Cowling, Dorofaeff, Seeger and Wright in [CDSW05, Main Theorem p. 435]. The general semisimple case reduces to the case of connected simple Lie groups by [Knu18, Theorem 3.1 p. 102]. More precisely, if G is a connected semisimple Lie group, i.e., the Lie algebra of G is semisimple then G is locally isomorphic to a direct product $S_1 \times \cdots \times S_n$ of connected simple Lie groups $S_1, \dots, S_n$ and G is weakly amenable if and only if each S_i is weakly amenable. In this case, we have

$$\Lambda_{\text{WA}}(G) = \prod_{i=1}^n \Lambda_{\text{WA}}(S_i).$$

For an arbitrary connected Lie group, the analytic Levi subgroup need not be closed, and the universal covering homomorphism can have an infinite central kernel. Both points have to be taken into account. Example 5.2 illustrates the first issue.

Our first result settles Conjecture 1.1 in this setting. We write G^0 for the identity component of a locally compact group G . Recall that G is said to be almost connected (or connected-by-compact) [Pal01, Definition 12.2.1 p. 1305] if G/G^0 is compact.

Theorem 1 *Let G be a locally compact group such that G/G^0 is amenable. Then*

$$\Lambda_{\text{WA}}(G) = 1 \iff G \text{ has the Haagerup property.}$$

In particular, Conjecture 1.1 holds for any connected Lie group and for any almost connected locally compact group.

The theorem is stated without a countability assumption. To remove countability assumptions, we pass to compactly generated open subgroups and then to second countable quotients by compact normal subgroups. This allows us to use the permanence of the Haagerup property under extensions with amenable quotient in its usual second countable form.

The second result gives the full weak amenability classification. Let $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{s}$ be a Levi decomposition [Kna02, Theorem B.2 p. 660] of the Lie algebra $\mathfrak{g}$ of a connected Lie group G , with radical $\mathfrak{r}$. In particular, $\mathfrak{s}$ is a semisimple subalgebra of $\mathfrak{g}$ and with [Bou89, Chapter I, Section 6.2, Proposition 2 p. 53] we can write $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ as a sum of simple ideals. Let S_i be the connected analytic subgroup of G with Lie algebra $\mathfrak{s}_i$ provided by [HiN12, Theorem 9.4.8], equipped with its intrinsic Lie group topology.

Theorem 2 *Let G be a connected Lie group. With the previous notations then G is weakly amenable if and only if each $\mathfrak{s}_i$ is compact, or has real rank one and satisfies $[\mathfrak{s}_i, \mathfrak{r}] = 0$. In this case, we have*

$$(1.2) \quad \Lambda_{\text{WA}}(G) = \prod_{i=1}^n \Lambda_{\text{WA}}(S_i).$$

Consequently, the constant Λ_{WA} is invariant under local isomorphism of connected Lie groups.

Empty products in (1.2) are understood to be one. The constants of the simple factors depend only on their Lie algebras. By Theorem 1.2, they equal one in compact type and for the Lie algebras $\mathfrak{so}(m, 1)$ and $\mathfrak{su}(m, 1)$, equal $2m - 1$ for the Lie algebra $\mathfrak{sp}(m, 1)$ with $m \geq 2$, and equal 21 for $\mathfrak{f}_{4(-20)}$. According to [Knu18, Theorem 1.1 p. 102], simple factors of real rank at least two are not weakly amenable.

The classification also determines the behaviour under quotients. For a connected locally compact group G and a closed normal subgroup N , we obtain $\Lambda_{\text{WA}}(G/N) \leq \Lambda_{\text{WA}}(G)$ in Corollary 7.1. For central quotients, equality holds under the more general assumptions of Theorem 1, as the following result shows.

Theorem 3 *Let G be a locally compact group such that the group G/G^0 is amenable, and let Z be a closed central subgroup of G . Then*

$$(1.3) \quad \Lambda_{\text{WA}}(G/Z) = \Lambda_{\text{WA}}(G).$$

The conclusion also holds if the group $(G/Z)/(G/Z)^0$ is amenable. In particular, it holds if either G or G/Z is almost connected.

No compactness assumption is imposed on the group Z . The equality in (1.3) includes infinite constants. For lattices in connected locally compact groups, it gives the corresponding equality for quotients by subgroups contained in the centre of the ambient group, together with a Haagerup conclusion at constant one; see Corollary 7.4. We also record in Corollary 5.4 that the finite constants of connected Lie groups are precisely the positive odd integers.

Approach of the paper For the necessity in Theorem 2, we apply Ozawa's theorem [Oza12, Theorem A p. 333] (see Theorem 2.1 below) on invariant states to the closed normal nilradical of G . A non-trivial action of a non-compact simple Levi ideal survives on a vector group quotient of this nilradical. The resulting state contradicts the amenability criterion of Cornuier and Valette [CoV15, Proposition 2.2, (i) $\Leftrightarrow$ (iii)] for affine actions on vector groups. The argument takes place in G itself and does not require a closed Levi subgroup.

For the converse, we use continuous families of positive definite functions transforming under central characters. These families are supplied by the generalized Haagerup property studied in [CCJJV01, Section 4.2 pp. 46–61], whose rank-one input is the positive definiteness of covariant spherical functions of Flensted-Jensen [FJ77]. Averaging over the central characters gives a positive definite function whose representation is absolutely continuous along the centre. The tensor identity for induction then shows that multiplying a lifted Fourier algebra function by this average produces an element of the Fourier algebra $A(G)$ of the group G . This yields the required upper bound. The lower bound uses the closed semisimple subgroup generated by the factors with constants greater than one.

Structure of the paper Section 2 recalls the notation and permanence results. Section 3 proves the obstruction associated with the action on the radical. Section 4 establishes the central lifting argument. We prove Theorem 2 in Section 5 and Theorem 1 in Section 6. Section 7 proves Theorem 3 and the consequences for quotients and lattices.

2 Preliminaries

All locally compact groups are Hausdorff. Haar measures are left Haar measures, and scalar products are linear in the first variable. Representations on Hilbert spaces are strongly continuous. We denote by $C_0(G)$ the algebra of continuous complex-valued functions on a locally compact group G vanishing at infinity and by $C_c(G)$ the algebra of those with compact support. Convergence uniformly on compact subsets is used throughout.

Positive definite functions A continuous function $\varphi: G \rightarrow \mathbb{C}$ is positive definite [BHV08, Definition C.4.1 p. 351] if for any $s_1, \dots, s_n \in G$ and complex numbers $c_1, \dots, c_n \in \mathbb{C}$ we have $\sum_{i,j=1}^n \overline{c_j} c_i \varphi(s_j^{-1} s_i) \geq 0$. We denote by $P(G)$ the cone of continuous positive definite functions on G . We write $P_1(G) \stackrel{\text{def}}{=} \{\varphi \in P(G) : \varphi(e) = 1\}$. For any function $\varphi \in P(G)$, its cyclic GNS representation [BHV08, Theorem C.4.10 p. 354] is denoted by $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$, so that $\varphi(s) = \langle \pi_\varphi(s) \xi_\varphi, \xi_\varphi \rangle_{\mathcal{H}_\varphi}$ for any $s \in G$ with $\|\xi_\varphi\|_{\mathcal{H}_\varphi}^2 = \varphi(e)$. By [BHV08, Proposition C.4.2 p. 351], we have the inequality $|\varphi(s)| \leq \varphi(e)$ for any $s \in G$.

Haagerup property The locally compact group G has the Haagerup property if there exists a net in $P_1(G) \cap C_0(G)$ converging to one uniformly on compact subsets. We use this net formulation for arbitrary locally compact groups, as in [CoV15, Definition 1.1]. For second countable groups it agrees with the usual Haagerup property and its characterization by metrically proper

affine isometric actions [CCJJV01, Definition 1.1.1 p. 2 and Theorem 2.1.1 pp. 15–16]. Outside that setting, all assertions below refer to the net formulation.

For the final extension argument, we record two facts about the Haagerup property. If K is a compact normal subgroup of G , the quotient map $q: G \rightarrow G/K$ is proper by [Bou66, Chapter III, Section 4.1, Corollary 2 to Proposition 1 p. 251]. In particular, it is closed, and the inverse image of any compact subset of G/K is compact [Bou66, Chapter I, Section 10.2, Proposition 6 p. 104]. Consequently, if $f \in C_0(G/K)$ and $\varepsilon > 0$, then

$$\{s \in G : |f(q(s))| \geq \varepsilon\} = q^{-1}(\{y \in G/K : |f(y)| \geq \varepsilon\})$$

is compact. Thus pullback under q preserves vanishing at infinity, as well as positive definiteness and normalization.

Conversely, the Haagerup property passes from G to G/K . Indeed, let $\varphi_i \in P_1(G) \cap C_0(G)$ converge to one uniformly on compact subsets, and put

$$v_i(s) \stackrel{\text{def}}{=} \int_K \int_K \varphi_i(ksk') dk dk', \quad s \in G,$$

where Haar measure on K is normalized. In the GNS representation of φ_i , this is the coefficient associated with the orthogonal projection of ξ_{φ_i} onto the K -fixed vectors. Thus v_i is positive definite and K -bi-invariant. It belongs to $C_0(G)$: if $|\varphi_i| < \varepsilon$ outside a compact set F , then $|v_i| \leq \varepsilon$ outside KFK . Moreover, $v_i \rightarrow 1$ uniformly on compact subsets and $v_i(e) \rightarrow 1$. For i sufficiently large, let $\dot{v}_i$ be the continuous function on G/K such that $\dot{v}_i \circ q = v_i/v_i(e)$. It is positive definite and normalized. For any $\varepsilon > 0$, its level set satisfies

$$\{y \in G/K : |\dot{v}_i(y)| \geq \varepsilon\} = q(\{s \in G : |v_i(s)| \geq \varepsilon v_i(e)\}),$$

which is compact. Hence $\dot{v}_i \in C_0(G/K)$. For any compact subset F of G/K , its inverse image $q^{-1}(F)$ is compact, and uniform convergence of $v_i/v_i(e)$ on this inverse image gives uniform convergence of $\dot{v}_i$ on F . Hence G has the Haagerup property if and only if G/K has it.

If H is an open subgroup of G , the extension by zero of $\varphi \in P(H)$ belongs to $P(G)$. To verify positivity, partition a finite family $s_1, \dots, s_n$ according to the left cosets $s_i H$; the positive-definiteness matrix is block diagonal, with each block obtained from φ . Continuity follows because H is open and closed. Extension by zero also preserves normalization and vanishing at infinity. Consequently, a directed union of open subgroups with the Haagerup property has the Haagerup property. Indeed, any compact subset of the union is covered by finitely many of these subgroups and is therefore contained in one of them, by directedness. We choose a positive definite approximant on that subgroup and extend it by zero.

For second countable groups we will use the following permanence result [CCJJV01, Proposition 6.1.5 and Example 6.1.6, pp. 90–91]: if N is a closed normal subgroup, G/N is amenable and N has the Haagerup property, then G has the Haagerup property. The countability assumption in Theorem 1 will be removed explicitly in Section 6.

Herz-Schur multipliers Let $\lambda_G: G \rightarrow B(L^2(G))$ be the left regular representation on the complex Hilbert space $L^2(G)$. Let G be a locally compact group equipped with a left Haar measure. According to [KaL18, Theorem 2.4.3 p. 59] and [KaL18, Lemma 2.3.6 p. 52], the Fourier algebra $A(G)$ can be seen as the set of all functions $\bar{f} * \check{g}$, where $f, g \in L^2(G)$. Here $\check{g}(s) \stackrel{\text{def}}{=} g(s^{-1})$ for any $s \in G$. For any $s \in G$, note that

$$(2.1) \quad (\bar{f} * \check{g})(s) = \int_G \overline{f(t)} \check{g}(t^{-1}s) dt = \int_G \overline{f(t)} g(s^{-1}t) dt = \langle \lambda_G(s)g, f \rangle_{L^2(G)},$$

where $\langle g, f \rangle_{L^2(G)} \stackrel{\text{def}}{=} \int_G g(t) \overline{f(t)} dt$. In other words, the Fourier algebra $A(G)$ is the set of matrix coefficients of the left regular representation $\lambda: G \rightarrow B(L^2(G))$ of the locally compact group G . The Fourier algebra $A(G)$ is equipped with the norm

$$\|u\|_{A(G)} = \inf \{ \|f\|_{L^2(G)} \|g\|_{L^2(G)} : u(s) = \langle \lambda_G(s)g, f \rangle_{L^2(G)}, s \in G \}.$$

It is a Banach algebra contained in $C_0(G)$, see [KaL18, Proposition 2.3.3 and Corollary 2.3.5 pp. 51–52]. A continuous function $u: G \rightarrow \mathbb{C}$ is a Herz–Schur multiplier if there are bounded continuous maps $a, b: G \rightarrow \mathcal{H}$ into a complex Hilbert space $\mathcal{H}$ such that $u(t^{-1}s) = \langle a(s), b(t) \rangle_{\mathcal{H}}$ for any $s, t \in G$. Its norm $\|u\|_{B_2(G)}$ is the infimum of $\|a\|_{\infty} \|b\|_{\infty}$ over these factorizations. This agrees with the completely bounded Fourier multiplier norm [Knu16a, Proposition 3.1 p. 3475] [Jol92, Theorem p. 311]. We will use

$$(2.2) \quad \|u\|_{\infty} \leq \|u\|_{B_2(G)} \leq \|u\|_{A(G)} \quad (u \in A(G)), \quad \|\varphi\|_{B_2(G)} = \varphi(e) \quad (\varphi \in P(G)).$$

Herz–Schur multipliers form a Banach algebra [Knu18, Section 2.1 p. 104]. The norm inequality for pointwise products also follows directly by tensoring the defining factorizations. If $q: G \rightarrow H$ is a continuous group homomorphism, pulling back a factorization gives

$$(2.3) \quad \|u \circ q\|_{B_2(G)} \leq \|u\|_{B_2(H)},$$

see also [Spr04, Corollary 6.3 p. 185].

Weak amenability The group G is said to be weakly amenable [CoH89, (0.4) p. 509] if there exists a net (u_i) in $A(G)$ converging to one uniformly on compact subsets with $\sup_i \|u_i\|_{B_2(G)} < \infty$. The infimum of these bounds is $\Lambda_{WA}(G)$. It is set equal to ∞ if no such net exists. By [CoH89, Proposition 1.1 p. 510 and Corollary 1.2 p. 511], the same constant is obtained by requiring the approximating functions to belong to $A(G) \cap C_c(G)$. Moreover, we have

$$B_2(G) \cap C_c(G) = A(G) \cap C_c(G).$$

Indeed, if $u \in B_2(G)$ has compact support, choose $v \in A(G) \cap C_c(G)$ equal to one on $\text{supp}(u)$, by [KaL18, Proposition 2.3.2 pp. 50–51]. Since u is a multiplier of $A(G)$, we have $u = uv \in A(G)$. Thus this definition is also equivalent, with the same constant, to the one using compactly supported Herz–Schur multipliers [Knu16b, Section 1 p. 691].

We use the standard permanence properties [CoH89, Proposition 1.3 p. 512]

$$(2.4) \quad \Lambda_{WA}(H) \leq \Lambda_{WA}(G), \quad \Lambda_{WA}(G/K) = \Lambda_{WA}(G),$$

where H is a closed subgroup and K is a compact normal subgroup, as well as

$$(2.5) \quad \Lambda_{WA}(G_1 \times G_2) = \Lambda_{WA}(G_1) \Lambda_{WA}(G_2),$$

see [CoH89, Corollary 1.5 p. 514]. According to [Ver24, Proposition 4.3], every amenable locally compact group G satisfies $\Lambda_{WA}(G) = 1$. We will also use the invariance [Knu18, (2.4) and the following remark, p. 105], where the removal of the second countability assumption of [Jol15] is explicitly stated, under extensions with amenable quotient:

$$(2.6) \quad \Lambda_{WA}(G) = \Lambda_{WA}(N) \quad \text{if } N \text{ is closed and normal and } G/N \text{ is amenable.}$$

In particular, we have

$$(2.7) \quad \Lambda_{WA}(G) = \Lambda_{WA}(G^0) \quad \text{if } G/G^0 \text{ is amenable.}$$

Lie groups We use the standard structure theory described in the books [Bou89], [Hel01], [HiN12] and [Var84] of real Lie groups.

Recall that in every finite-dimensional Lie algebra $\mathfrak{g}$, there is a largest solvable ideal [HiN12, Definition 5.4.5 p. 104] [Bou89, Definition 2 p. 44] called the radical of $\mathfrak{g}$, and is denoted by $\text{rad}(\mathfrak{g})$. A Levi subalgebra [Bou89, Definition 7 p. 63] of $\mathfrak{g}$ is any subalgebra of $\mathfrak{g}$ supplementary to the radical $\mathfrak{r}$. The nilradical of a finite-dimensional real Lie algebra $\mathfrak{g}$ is its largest nilpotent ideal [HiN12, Definition 5.2.10 p. 94]. This convention differs from that of [Bou89, Chapter I, Section 5.3, Definition 3 p. 44], where the nilpotent radical is defined as the intersection of the kernels of all finite-dimensional irreducible representations, see also [HiN12, Remark 5.2.11 p. 94].

A real Lie algebra is said to be compact [Bou05, Definition 1 p. 284] if it is isomorphic to the Lie algebra of a compact Lie group. The real rank of a real semisimple Lie algebra $\mathfrak{h}$ is the dimension of a maximal abelian subspace of $\mathfrak{p}$, where $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition [HKdL16, Section 2.5 p. 932].

Let G be a connected real Lie group with Lie algebra $\mathfrak{g} = T_e G$. We denote by $\exp_G: \mathfrak{g} \rightarrow G$ its exponential map. For any $X \in \mathfrak{g}$, the map $t \mapsto \exp_G(tX)$ is the unique one-parameter subgroup of G whose derivative at 0 is X [HiN12, Section 9.2 pp. 296–298]. For any $s \in G$, let $c_s: G \rightarrow G$ be the conjugation automorphism defined by $c_s(x) \stackrel{\text{def}}{=} sxs^{-1}$. The adjoint representation of G is the homomorphism $\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g})$ defined by $\text{Ad}(s) \stackrel{\text{def}}{=} d_e c_s$. Its differential at the identity is the adjoint representation ad of $\mathfrak{g}$, given by $\text{ad}(X)Y = [X, Y]$ for any $X, Y \in \mathfrak{g}$. For any $s \in G$ and $X \in \mathfrak{g}$, we have

$$s \exp_G(X) s^{-1} = \exp_G(\text{Ad}(s)X), \quad \text{Ad}(\exp_G(X)) = e^{\text{ad}(X)}.$$

Moreover, since G is connected, we have $\text{Ker Ad} = Z(G)$ [HiN12, Section 9.2.3 pp. 306–307]. If $\rho: G \rightarrow \text{GL}(V)$ is a smooth representation on a finite-dimensional real vector space V , we denote by $d\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ its differential at the identity. For any $X \in \mathfrak{g}$, we have

$$d\rho(X) = \left. \frac{d}{dt} \right|_{t=0} \rho(\exp_G(tX)), \quad \rho(\exp_G(X)) = e^{d\rho(X)}.$$

For any Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, the associated connected analytic subgroup is $H = \langle \exp_G(\mathfrak{h}) \rangle$. We equip H with its intrinsic Lie group topology, which may be finer than the subspace topology inherited from G [HiN12, Theorem 9.4.8 p. 332]. If $\mathfrak{j} \subseteq \mathfrak{h}$ are ideals of $\mathfrak{g}$, the connectedness of G implies that they are invariant under $\text{Ad}(G)$. Hence the adjoint representation induces a representation ρ on $\mathfrak{h}/\mathfrak{j}$ given by

$$\rho(g)(X + \mathfrak{j}) = \text{Ad}(g)X + \mathfrak{j}, \quad d\rho(Y)(X + \mathfrak{j}) = [Y, X] + \mathfrak{j}$$

for any $g \in G$, $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$.

Ozawa's invariant-state theorem We use Ozawa's invariant-state theorem [Oza12, Theorem A p. 333].

Theorem 2.1 *if G is weakly amenable and N is an amenable closed normal subgroup, there exists a state on the algebra $L^\infty(N)$ invariant under left translations by N and conjugation by G .*

Induced representations We refer to the books [BHV08, Appendix E] and [KaT13, Chapter 2] for more information on induced representations. Let H be a closed subgroup of a locally compact group G , and let $\sigma: H \rightarrow \mathcal{B}(\mathcal{H}_\sigma)$ be a strongly continuous unitary representation of H on a Hilbert space $\mathcal{H}_\sigma$. We denote by $\text{Ind}_H^G \sigma$ the unitary representation of G induced by σ [KaT13, Section 2.3 pp. 61–65].

If G/H admits a G -invariant Radon measure μ , the representation $\text{Ind}_H^G \sigma$ can be realized on the completion of the space of continuous functions $F: G \rightarrow \mathcal{H}_\sigma$ with compact support modulo H satisfying $F(xh) = \sigma(h^{-1})F(x)$ for any $x \in G$ and $h \in H$, with respect to the norm

$$\|F\| \stackrel{\text{def}}{=} \left(\int_{G/H} \|F(x)\|_{\mathcal{H}_\sigma}^2 d\mu(xH) \right)^{\frac{1}{2}}.$$

The action is given by $(\text{Ind}_H^G \sigma(s)F)(x) \stackrel{\text{def}}{=} F(s^{-1}x)$ [BHV08, Remark E.1.7 p. 388]. In particular, $\text{Ind}_H^G 1_H$ is the quasi-regular representation. If H is normal and $q: G \rightarrow G/H$ is the quotient map, it is equivalent to $\lambda_{G/H} \circ q$. Induction preserves unitary equivalence, Hilbert direct sums and subrepresentations [BHV08, Propositions E.2.1 and E.2.2 p. 389], see also [KaT13, Proposition 2.42 p. 93]. Moreover, induction in stages gives $\text{Ind}_H^G \lambda_H \simeq \lambda_G$ [KaT13, Example 2.23 p. 66 and Theorem 2.47 p. 96]. For any strongly continuous unitary representation π of G , the tensor product formula gives

$$\pi \otimes \text{Ind}_H^G \sigma \simeq \text{Ind}_H^G ((\pi|_H) \otimes \sigma)$$

[KaT13, Theorem 2.58 p. 106]. Consequently, if Z is a closed central subgroup of G and $q: G \rightarrow G/Z$ is the quotient map, then

$$\pi \otimes (\lambda_{G/Z} \circ q) \simeq \text{Ind}_Z^G (\pi|_Z).$$

Here $\simeq$ denotes unitary equivalence.

For a closed subgroup H of a locally compact group G , the quasi-regular representation is the representation $\text{Ind}_H^G 1_H$ induced by the trivial representation of H [BHV08, Example E.1.8 (ii) p. 388]. In particular, if Z is a closed central subgroup and $q: G \rightarrow G/Z$ is the quotient map, this representation is realized on $L^2(G/Z)$ as $\rho = \lambda_{G/Z} \circ q$. Thus

$$(\rho(s)f)(xZ) = f(s^{-1}xZ)$$

for any $s \in G$, any $f \in L^2(G/Z)$ and almost every $xZ \in G/Z$, where G/Z is equipped with a left Haar measure.

3 The action of the Levi factor on the radical

The proof below applies Theorem 2.1 to the nilradical of G and passes to a vector group quotient. For an action of a group Γ on a set X , we use $(\gamma \cdot f)(x) \stackrel{\text{def}}{=} f(\gamma^{-1}x)$ on $\ell^\infty(X)$. Invariance of a state means invariance under these operators. The same convention is used on L^∞ -spaces.

Lemma 3.1 *Let G be a Lie group and H be a connected Lie subgroup of G whose Lie algebra $\mathfrak{h}$ is semisimple. If the centre of H is finite, then H is closed in G .*

Proof : Let $\mathfrak{g}$ be the Lie algebra of G and let $\iota: H \rightarrow G$ be the inclusion, where H is equipped with its intrinsic Lie group topology. Consider the representation $\theta \stackrel{\text{def}}{=} \text{Ad}_G \circ \iota$ of H on $\mathfrak{g}$. Its image is closed in $\text{GL}(\mathfrak{g})$ by [HiN12, Corollary 14.5.7 p. 556]. Any element of $\text{Ker } \theta$ centralizes

$\exp_G(\mathfrak{h})$, hence H , so that $\text{Ker } \theta \subset Z(H)$ is finite. Since H is σ -compact, [CdlH16, Corollary 2.D.6(2) p. 39] implies that θ is proper. The group G is Hausdorff and $\text{Ad}_G \circ \iota$ is proper, so [Bou66, Chapter I, Section 10.1, Proposition 5(d) p. 99] implies that ι is proper. In particular, ι is closed, and $H = \iota(H)$ is closed in G . ■

Proposition 3.2 *Let G be a connected Lie group and let $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{s}$ be a Levi decomposition of its Lie algebra. Suppose that some simple ideal $\mathfrak{s}_i$ of $\mathfrak{s}$ is non-compact and satisfies $[\mathfrak{s}_i, \mathfrak{r}] \neq 0$. Then G is not weakly amenable.*

Proof : Suppose that the group G is weakly amenable.

Step 1: a closed normal nilpotent subgroup. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$, that is, its largest nilpotent ideal. By [Bou89, Chapter I, Section 5.5, Proposition 6 p. 49], any derivation of $\mathfrak{g}$ maps $\mathfrak{r}$ into $\mathfrak{n}$. Applying this to the inner derivations gives $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n}$. Let N be the connected analytic subgroup of G with Lie algebra $\mathfrak{n}$. By [Bou89, Chapter III, Section 9.7, Proposition 23 p. 354], N is closed, nilpotent and invariant under every continuous automorphism of G . Being nilpotent, N is solvable, hence amenable by [BHV08, Corollary G.2.3 p. 426]. Thus N is a closed amenable normal subgroup of G .

By Weyl's theorem on complete reducibility [HiN12, Theorem 5.5.21 p. 120], the $\mathfrak{s}_i$ -module $\mathfrak{r}$ is a direct sum of irreducible submodules. On each irreducible summand E on which $\mathfrak{s}_i$ acts non-trivially, the subspace $[\mathfrak{s}_i, E]$ is nonzero and invariant, hence equal to E . On each trivial summand, it is zero. Consequently,

$$[\mathfrak{s}_i, [\mathfrak{s}_i, \mathfrak{r}]] = [\mathfrak{s}_i, \mathfrak{r}].$$

Since $[\mathfrak{s}_i, \mathfrak{r}] \subset [\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n}$, we obtain $[\mathfrak{s}_i, \mathfrak{n}] \supset [\mathfrak{s}_i, \mathfrak{r}] \neq 0$.

Step 2: $\mathfrak{s}_i$ acts non-trivially on $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$. The subspaces $\mathfrak{n}$ and $[\mathfrak{n}, \mathfrak{n}]$ are ideals of $\mathfrak{g}$, hence $\mathfrak{s}_i$ -invariant. By Weyl's theorem, there exists an $\mathfrak{s}_i$ -invariant subspace U such that $\mathfrak{n} = U \oplus [\mathfrak{n}, \mathfrak{n}]$. Since $\mathfrak{n}$ is nilpotent and $U + [\mathfrak{n}, \mathfrak{n}] = \mathfrak{n}$, the subspace U generates $\mathfrak{n}$ as a Lie algebra by [Bou89, Chapter I, Section 4, Exercise 4(b) p. 91]. If $\mathfrak{s}_i$ acted trivially on $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$, it would act trivially on U , since the quotient map restricts to an isomorphism of $\mathfrak{s}_i$ -modules from U onto this quotient. Each element of $\text{ad}(\mathfrak{s}_i)$ is a derivation, so it would then vanish on every iterated bracket of elements of U , hence on $\mathfrak{n}$. This contradicts Step 1.

Step 3: a vector group quotient. By [HiN12, Corollary 11.2.7 p. 446 and Theorem 11.2.10 p. 447], the exponential map identifies N with $(\mathfrak{n}, *)/\Gamma$, where $*$ is the Campbell–Hausdorff multiplication and $\Gamma = \text{Ker } \exp_N$ is a discrete subgroup of $\mathfrak{z}(\mathfrak{n})$. For $g \in G$ and $X \in \mathfrak{n}$, equivariance gives $\exp_N(\text{Ad}(g)X) = g \exp_N(X) g^{-1}$, so $\text{Ad}(g)\Gamma = \Gamma$. Since G is connected and Γ is discrete, every element of Γ is fixed by $\text{Ad}(G)$. Thus $\mathfrak{t} \stackrel{\text{def}}{=} \text{span}_{\mathbb{R}} \Gamma$ is pointwise fixed by $\text{Ad}(G)$.

Put $\mathfrak{k} \stackrel{\text{def}}{=} [\mathfrak{n}, \mathfrak{n}] + \mathfrak{t}$ and $\mathfrak{w} \stackrel{\text{def}}{=} \mathfrak{n}/\mathfrak{k}$. The quotient map from $(\mathfrak{n}, *)$ to the additive group $W \stackrel{\text{def}}{=} \mathfrak{w}$ kills Γ , hence descends to a continuous surjective homomorphism

$$q: N \longrightarrow W, \quad q(\exp_N X) = X + \mathfrak{k}.$$

This map is a submersion, so it is open. Its kernel K_N is closed and normal in G , has Lie algebra $\mathfrak{k}$, and $N/K_N \simeq W$. The adjoint action induces a continuous representation $\rho: G \rightarrow \text{GL}(\mathfrak{w})$ satisfying $q(gxg^{-1}) = \rho(g)q(x)$.

By Step 2, the $\mathfrak{s}_i$ -action on $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ is non-trivial. The submodule $\mathfrak{k}/[\mathfrak{n}, \mathfrak{n}]$ is trivial, since it is the image of $\mathfrak{t}$. Weyl's complete reducibility theorem therefore shows that the action on the quotient $\mathfrak{w}$ is non-trivial.

Step 4: an invariant state on $L^\infty(\mathfrak{w})$. By Theorem 2.1, there exists a state ω_N on $L^\infty(N)$ which is invariant under left translations by elements of N and under conjugations by elements of G . We first check that pullback under q preserves Haar null sets. Let B be a Borel Haar null subset of W and let $(F_j)_{j \geq 1}$ be a sequence of compact subsets covering N . The Haar measures of N , K_N and W are σ -finite. Weil's formula [BHV08, Corollary B.1.7 p. 330], extended to non-negative Borel functions by monotone convergence, gives

$$\int_N 1_{F_j}(x) 1_B(q(x)) dx = \int_W 1_B(w) \int_{K_N} 1_{F_j}(x_w k) dk dw = 0,$$

where x_w is any representative of the coset w . Hence $F_j \cap q^{-1}(B)$ is Haar null for any j , and so is $q^{-1}(B)$. The same conclusion holds for subsets of Borel Haar null sets. So $f \mapsto f \circ q$ is a well-defined unital positive map from $L^\infty(\mathfrak{w})$ into $L^\infty(N)$, and $\omega_W(f) \stackrel{\text{def}}{=} \omega_N(f \circ q)$ defines a state on $L^\infty(\mathfrak{w})$. For $a, x \in N$ and $g \in G$, we have $f(q(a) + q(x)) = (f \circ q)(ax)$ and $f(\rho(g)q(x)) = (f \circ q)(gxg^{-1})$. Hence ω_W is invariant under translations by elements of $\mathfrak{w}$ and under the linear action of $\rho(G)$.

Step 5: conclusion. Let $H \stackrel{\text{def}}{=} \rho(S_i)$. By Step 3, the differential of $\rho|_{S_i}$ is nonzero, hence injective since $\mathfrak{s}_i$ is simple. By [HiN12, Corollary 14.5.7 p. 556], the image H of this representation of the connected semisimple Lie group S_i is closed in $\text{GL}(\mathfrak{w})$. Its Lie algebra is isomorphic to the non-compact simple Lie algebra $\mathfrak{s}_i$. Thus H is a non-compact connected simple Lie group and is not amenable by [BHV08, Example G.3.6(ii) p. 431].

For every Borel subset B of $\mathfrak{w}$, put $m(B) \stackrel{\text{def}}{=} \omega_W([1_B])$. This defines a finitely additive probability measure on the Borel subsets of $\mathfrak{w}$. By Step 4 it is invariant under translations and under H . These transformations preserve the Haar null sets. Apply [CoV15, Proposition 2.2, (i) $\Leftrightarrow$ (iii)] to the abstract group H equipped with the discrete topology and its inclusion into $\text{GL}(\mathfrak{w})$. The existence of m implies that the closure of H in $\text{GL}(\mathfrak{w})$ is amenable. This contradicts the preceding paragraph. ■

Corollary 3.3 *Let G be a connected weakly amenable Lie group and let $\mathfrak{g} = \mathfrak{r} \ltimes \mathfrak{s}$ be a Levi decomposition of its Lie algebra. Then every simple ideal $\mathfrak{s}_i$ of $\mathfrak{s}$ either is compact, or has real rank one and satisfies $[\mathfrak{s}_i, \mathfrak{r}] = 0$.*

Proof : A weakly amenable group has the approximation property. The characterization of connected Lie groups with this property in [HKdL16, Introduction and Theorem C, pp. 928–930] therefore implies that every simple ideal of a Levi subalgebra has real rank at most one. A simple ideal of real rank zero is compact, and Proposition 3.2 shows that every non-compact simple ideal commutes with $\mathfrak{r}$. ■

The closed nilradical and the kernel of its exponential covering are handled inside the ambient group in Proposition 3.2. Thus the argument imposes neither simple connectedness nor closedness of a Levi factor.

4 Covariant positive definite functions and central lifting

Throughout this section all locally compact groups are second countable. We fix a group G , a closed central subgroup Z , and the quotient map $q: G \rightarrow G/Z$. We write $\hat{Z}$ for the dual group, with Haar measure $d\chi$, and use the Fourier transform $\hat{h}(z) \stackrel{\text{def}}{=} \int_{\hat{Z}} h(\chi) \chi(z) d\chi$, where $z \in Z$.

The families below are taken from the generalized Haagerup property of [CCJJV01, Definition 4.2.1 p. 47]. We retain the covariance and continuity requirements, which are the ones needed to lift weak amenability.

Definition 4.1 Let G be a locally compact group and Z be a closed central subgroup of G . We say that the pair (G, Z) has the covariant approximation property if for any compact subset K of G and any $\varepsilon > 0$ there exist a compact neighbourhood V of the trivial character in $\hat{Z}$ and a family $(\varphi_\chi)_{\chi \in V}$ of elements of $P_1(G)$ satisfying the following conditions.

1. For any $x \in K$ and any $\chi \in V$, we have $|\varphi_\chi(x) - 1| < \varepsilon$.
2. For any $\chi \in V$, any $x \in G$ and any $z \in Z$, we have $\varphi_\chi(xz) = \varphi_\chi(x)\chi(z)$.
3. The map $V \times G \rightarrow \mathbb{C}$, $(\chi, x) \mapsto \varphi_\chi(x)$ is continuous.

The generalized Haagerup property requires in addition that $|\varphi_\chi|$ vanish at infinity on G/Z . That additional condition will not be used. Recall also that, if $q_N: G \rightarrow G/N$ is a quotient by a closed normal subgroup and K' is a compact subset of G/N , there exists a compact subset K of G such that $q_N(K) = K'$, by [BHV08, Lemma B.1.1 p. 324].

Lemma 4.2 Let G be a locally compact group and Z be a closed central subgroup of G .

1. If Z is compact, then (G, Z) has the covariant approximation property.
2. If G is amenable, then (G, Z) has the covariant approximation property.
3. If G is a connected simple Lie group locally isomorphic to $\mathrm{SO}(n, 1)$ for some $n \geq 2$ or to $\mathrm{SU}(n, 1)$ for some $n \geq 1$, and if Z is the centre of G , then (G, Z) has the covariant approximation property.
4. If (G_1, Z_1) and (G_2, Z_2) have the covariant approximation property, then so does $(G_1 \times G_2, Z_1 \times Z_2)$.
5. If (G, Z) has the covariant approximation property and if Z_0 is a closed subgroup of Z , then $(G/Z_0, Z/Z_0)$ has the covariant approximation property.

Proof : 1. Since Z is compact, its dual group $\hat{Z}$ is discrete by [HeR79, Theorem 23.17 p. 362], so that $V \stackrel{\text{def}}{=} \{1\}$ is a compact neighbourhood of the trivial character. The constant function $\varphi_1 = 1$ satisfies the three conditions of Definition 4.1.

2. By [CCJJV01, Proposition 4.2.12 p. 54], the pair (G, Z) has the generalized Haagerup property, hence the covariant approximation property.

3. If G is the universal covering group of the group $\mathrm{SU}(n, 1)$, the pair (G, Z) has the generalized Haagerup property by [CCJJV01, Proposition 4.2.14 p. 59]. In the other cases, it has this property by [CCJJV01, Theorem 4.2.15 p. 60]. The analytic input of these results is the positive definiteness of some covariant spherical functions of Flensted-Jensen on a central extension of $\mathrm{PU}(n, 1)$ by $\mathbb{R}$ which contains the universal covering group of $\mathrm{SU}(n, 1)$ as a closed subgroup, see [CCJJV01, Section 4.2.5 pp. 57–59], [FJ77, Theorem 2.7 p. 77 and Remark 2.8 p. 78] and [Han90, Proposition 22 p. 462 and Remark 24 pp. 464–465].

4. Let K be a compact subset of $G_1 \times G_2$ and $\varepsilon > 0$. The projections K_1 and K_2 of K are compact and $K \subset K_1 \times K_2$. Let V_i and $(\varphi_{\chi_i}^{(i)})_{\chi_i \in V_i}$ be given by Definition 4.1 for (G_i, Z_i) , K_i and $\frac{\varepsilon}{2}$. We identify the dual group of $Z_1 \times Z_2$ with $\hat{Z}_1 \times \hat{Z}_2$ by [HeR79, Theorem 23.18 pp. 362–363]. Then $V_1 \times V_2$ is a compact neighbourhood of the trivial character, and the functions $\varphi_{(\chi_1, \chi_2)}(x_1, x_2) \stackrel{\text{def}}{=} \varphi_{\chi_1}^{(1)}(x_1)\varphi_{\chi_2}^{(2)}(x_2)$ belong to $P_1(G_1 \times G_2)$, since they are coefficients of tensor products of representations. Conditions 2 and 3 are clear, and condition 1 follows from $|ab - 1| \leq |a||b - 1| + |a - 1| \leq |b - 1| + |a - 1|$ for complex numbers a, b with $|a| \leq 1$.

5. Let $q_0: G \rightarrow G/Z_0$ be the quotient map, K' be a compact subset of G/Z_0 and $\varepsilon > 0$. Let K be a compact subset of G with $q_0(K) = K'$ and let V and $(\varphi_\chi)_{\chi \in V}$ be given by Definition 4.1 for K and ε . The annihilator $Z_0^\perp \stackrel{\text{def}}{=} \{\chi \in \hat{Z} : \chi|_{Z_0} = 1\}$ is a closed subgroup of $\hat{Z}$ by [HeR79, Definition 23.23 and Remark 23.24(c) p. 365]. Pullback along $Z \rightarrow Z/Z_0$ identifies $\widehat{Z/Z_0}$ topologically with $Z_0^\perp$ by [HeR79, Theorem 23.25 p. 365]. Thus $V' \stackrel{\text{def}}{=} V \cap Z_0^\perp$ is a compact neighbourhood of the trivial character in $\widehat{Z/Z_0}$. For $\chi \in V'$, the function φ_χ is constant on the cosets of Z_0 by condition 2, hence of the form $\varphi'_\chi \circ q_0$, where φ'_χ is a positive definite function on G/Z_0 with $\varphi'_\chi(e) = 1$, continuous since q_0 is open. Conditions 1 and 2 for the family $(\varphi'_\chi)_{\chi \in V'}$, the pair $(G/Z_0, Z/Z_0)$ and the compact set K' are clear. Let $Q_0: V' \times G \rightarrow V' \times G/Z_0$ be the map $(\chi, x) \mapsto (\chi, q_0(x))$. The image of a basic open set $U \times O$ is $U \times q_0(O)$, which is open because q_0 is open. Thus Q_0 is a continuous open surjection, hence a quotient map. The function $(\chi, xZ_0) \mapsto \varphi'_\chi(xZ_0)$ is continuous by [Eng89, Proposition 2.4.2 p. 90], since its composition with Q_0 is the continuous function $(\chi, x) \mapsto \varphi_\chi(x)$. This proves condition 3. ■

Averaging these covariant spherical functions over central characters already appears in Hansen's proof for the universal covering group of the group $\text{SU}(n, 1)$ [Han90, Lemma 30 pp. 468–469 and Definition 31 p. 469]. The next two lemmas provide the Fourier algebra assertion needed for arbitrary central extensions satisfying Definition 4.1.

Lemma 4.3 *Let Z be a second countable locally compact abelian group and σ be a unitary representation of Z on $\mathcal{H}$. Suppose that there is a dense subspace $\mathcal{D}$ such that, for any $\xi \in \mathcal{D}$, the function $z \mapsto \langle \sigma(z)\xi, \xi \rangle_{\mathcal{H}}$ is the Fourier transform of an element of $L^1(\hat{Z})$. Then σ is equivalent to a subrepresentation of a direct sum of copies of λ_Z .*

Proof : By the SNAG theorem [BHV08, Theorem D.3.1 p. 375], let E be the projection-valued measure on $\hat{Z}$ associated with σ , and put $\mu_\xi(B) \stackrel{\text{def}}{=} \|E(B)\xi\|_{\mathcal{H}}^2$. For any $\xi \in \mathcal{D}$, uniqueness of the Fourier–Stieltjes transform [BHV08, Section D.1 pp. 370–371] shows that μ_ξ has an L^1 density. If B is Haar null, $\xi \in \mathcal{H}$ and $\eta \in \mathcal{D}$, then

$$\mu_\xi(B) = \|E(B)(\xi - \eta)\|_{\mathcal{H}}^2 \leq \|\xi - \eta\|_{\mathcal{H}}^2.$$

Density gives $\mu_\xi(B) = 0$ for every $\xi \in \mathcal{H}$. Thus $\mu_\xi = h_\xi d\chi$ with $h_\xi \in L^1(\hat{Z})$ non-negative.

The cyclic representation generated by ξ identifies with multiplication by the characters on $L^2(\hat{Z}, \mu_\xi)$. The map $f \mapsto fh_\xi^{\frac{1}{2}}$ embeds it isometrically into the Hilbert space $L^2(\hat{Z})$ and intertwines these multiplication operators. By Plancherel's theorem [BHV08, Theorem D.1.2 and Remark D.1.4 p. 371], the latter representation is equivalent to λ_Z . Finally, [BHV08, Proposition C.4.9 p. 354] decomposes $\mathcal{H}$ into mutually orthogonal cyclic subrepresentations. Applying the preceding embedding to each summand proves the assertion. ■

Lemma 4.4 *Let G be a locally compact group, Z be a closed central subgroup of G and $q: G \rightarrow G/Z$ be the quotient map. Let $\Psi \in P(G)$ and suppose that the restriction to Z of the representation π_Ψ is equivalent to a subrepresentation of a direct sum of copies of λ_Z . Then for any $u \in A(G/Z)$, the function $(u \circ q)\Psi$ belongs to $A(G)$ and*

$$\|(u \circ q)\Psi\|_{A(G)} \leq \|u\|_{A(G/Z)} \Psi(e).$$

Proof : Let $\delta > 0$ and $f, g \in L^2(G/Z)$ with $u(\dot{y}) = \langle \lambda_{\dot{y}} f, g \rangle_{L^2(G/Z)}$ for any $\dot{y} \in G/Z$ and

$$\|f\|_{L^2(G/Z)} \|g\|_{L^2(G/Z)} \leq \|u\|_{A(G/Z)} + \delta.$$

Since the Haar measure of G/Z is G -invariant, the representation $\rho \stackrel{\text{def}}{=} \lambda_{G/Z} \circ q$ is the quasi-regular representation of G on the Hilbert space $L^2(G/Z)$, that is the representation $\text{Ind}_Z^G 1_Z$ induced by the trivial representation of Z [BHV08, Example E.1.8 (ii) p. 388]. For any $s \in G$, we have

$$(u \circ q)(s)\Psi(s) = \langle (\rho \otimes \pi_\Psi)(s)(f \otimes \xi_\Psi), g \otimes \xi_\Psi \rangle.$$

Here $\otimes_2$ denotes the Hilbert space tensor product. By [BHV08, Corollary E.2.6 (i) p. 391], the representation $\pi_\Psi \otimes \rho$, hence $\rho \otimes \pi_\Psi$, is equivalent to $\text{Ind}_Z^G(\pi_\Psi|_Z)$. By assumption, there exist a set J and a representation σ' of Z such that $\pi_\Psi|_Z \oplus \sigma'$ is equivalent to $\oplus_{j \in J} \lambda_Z$. By [BHV08, Propositions E.2.1 and E.2.2 p. 389], the representation $\text{Ind}_Z^G(\pi_\Psi|_Z) \oplus \text{Ind}_Z^G \sigma'$ is equivalent to $\oplus_{j \in J} \text{Ind}_Z^G \lambda_Z$. By [BHV08, Example E.1.8 (i) p. 388 and Theorem E.2.4 p. 390], we have $\text{Ind}_Z^G \lambda_Z = \text{Ind}_Z^G \text{Ind}_{\{e\}}^Z 1 \simeq \text{Ind}_{\{e\}}^G 1 = \lambda_G$. Consequently, there exists an isometry $W: L^2(G/Z) \otimes_2 \mathcal{H}_\Psi \rightarrow \oplus_{j \in J} L^2(G)$ such that $W(\rho \otimes \pi_\Psi)(s) = (\oplus_{j \in J} \lambda_s)W$ for any $s \in G$. Write $W(f \otimes \xi_\Psi) = (a_j)_{j \in J}$ and $W(g \otimes \xi_\Psi) = (b_j)_{j \in J}$. Then

$$(u \circ q)(s)\Psi(s) = \sum_{j \in J} \langle \lambda_s a_j, b_j \rangle_{L^2(G)}, \quad s \in G.$$

Each function $s \mapsto \langle \lambda_s a_j, b_j \rangle_{L^2(G)}$ belongs to the Fourier algebra $A(G)$ with norm at most $\|a_j\|_{L^2(G)} \|b_j\|_{L^2(G)}$ and, by the Cauchy–Schwarz inequality,

$$\sum_{j \in J} \|a_j\|_{L^2(G)} \|b_j\|_{L^2(G)} \leq \|f \otimes \xi_\Psi\| \|g \otimes \xi_\Psi\| = \|f\|_{L^2(G)} \|g\|_{L^2(G)} \Psi(e).$$

Since $A(G)$ is a Banach space and $\|\cdot\|_\infty \leq \|\cdot\|_{A(G)}$, the series converges in $A(G)$ to $(u \circ q)\Psi$, and $\|(u \circ q)\Psi\|_{A(G)} \leq (\|u\|_{A(G/Z)} + \delta)\Psi(e)$. We let $\delta \rightarrow 0$. $\blacksquare$

Remark 4.5 The assumption in Lemma 4.4 cannot be replaced by weak containment. For example, take $G = Z = \mathbb{R}$, $\Psi = 1$ and $u = 1$ on $G/Z = \{e\}$. The trivial representation of $\mathbb{R}$ is weakly contained in $\lambda_{\mathbb{R}}$ by [BHV08, Theorem G.3.2 p. 430], whereas $(u \circ q)\Psi = 1 \notin A(\mathbb{R})$, since $A(\mathbb{R}) \subset C_0(\mathbb{R})$.

Proposition 4.6 *Let G be a second countable locally compact group and Z be a closed central subgroup of G . If (G, Z) has the covariant approximation property, then*

$$\Lambda_{\text{WA}}(G) \leq \Lambda_{\text{WA}}(G/Z).$$

Proof : Fix a compact subset K of G and $\varepsilon > 0$. Take V and $(\varphi_\chi)_{\chi \in V}$ as in Definition 4.1. Choose a non-negative continuous function ω on $\hat{Z}$ with compact support in the interior of V and integral one. Extend $\omega(\chi)\varphi_\chi(x)$ by zero outside $V \times G$ and put

$$(4.1) \quad h_x(\chi) \stackrel{\text{def}}{=} \omega(\chi)\varphi_\chi(x) \quad \text{and} \quad \Psi(x) \stackrel{\text{def}}{=} \int_{\hat{Z}} h_x(\chi) d\chi.$$

Fix $x_0 \in G$. Joint continuity and compactness of V give

$$\sup_{\chi \in V} |\varphi_\chi(x) - \varphi_\chi(x_0)| \rightarrow 0 \quad \text{as } x \rightarrow x_0.$$

Indeed, for any $\eta > 0$, continuity at each (χ, x_0) gives a neighbourhood V_χ of χ in V and a neighbourhood U_χ of x_0 such that $|\varphi_{\chi'}(x) - \varphi_{\chi'}(x_0)| < \eta$ for $\chi' \in V_\chi$ and $x \in U_\chi$. We choose

finitely many V_χ covering V and intersect the corresponding U_χ . Since $\int_{\hat{Z}} \omega(\chi) d\chi = 1$, we obtain

$$\|h_x - h_{x_0}\|_{L^1(\hat{Z})} \leq \sup_{\chi \in V} |\varphi_\chi(x) - \varphi_\chi(x_0)| \rightarrow 0.$$

Hence Ψ is continuous. Integrating the finite positive-definiteness inequalities shows that $\Psi \in P_1(G)$, and

$$(4.2) \quad |\Psi(x) - 1| \leq \int_{\hat{Z}} \omega(\chi) |\varphi_\chi(x) - 1| d\chi < \varepsilon, \quad x \in K.$$

Covariance gives

$$(4.3) \quad \Psi(xz) = \hat{h}_x(z), \quad x \in G, z \in Z.$$

Let E be the spectral measure of $\pi_\Psi|_Z$. Since $\Psi(z) = \hat{\omega}(z)$ for $z \in Z$, uniqueness of the Fourier–Stieltjes transform gives

$$\|E(B)\xi_\Psi\|^2 = \int_B \omega(\chi) d\chi$$

for every Borel set $B \subset \hat{Z}$. By centrality and [BHV08, Theorem D.3.1(i) p. 375], every $E(B)$ commutes with $\pi_\Psi(G)$. If B is Haar null, then $E(B)\xi_\Psi = 0$, and cyclicity gives $E(B) = 0$. Thus all scalar spectral measures of $\pi_\Psi|_Z$ are absolutely continuous with respect to Haar measure. Their densities belong to $L^1(\hat{Z})$, and Lemma 4.3, applied with $\mathcal{D} = \mathcal{H}_\Psi$, embeds $\pi_\Psi|_Z$ in a direct sum of copies of λ_Z .

We may suppose $L \stackrel{\text{def}}{=} \Lambda_{\text{WA}}(G/Z) < \infty$. For $\delta > 0$, choose a function $u \in A(G/Z)$ with $\|u\|_{B_2(G/Z)} \leq L + \delta$ and $|u - 1| < \varepsilon$ on $q(K)$. Lemma 4.4 gives $w \stackrel{\text{def}}{=} (u \circ q)\Psi \in A(G)$. By (2.2), (2.3) and the Banach algebra inequality, we obtain

$$\|w\|_{B_2(G)} = \|(u \circ q)\Psi\|_{B_2(G)} \leq \|u \circ q\|_{B_2(G)} \|\Psi\|_{B_2(G)} \leq L + \delta.$$

For any $x \in K$, we have

$$|w(x) - 1| \leq |u(q(x)) - 1| |\Psi(x)| + |\Psi(x) - 1| \stackrel{(4.2)}{<} 2\varepsilon.$$

Since K, ε, δ are arbitrary, this proves the assertion. $\blacksquare$

5 The product formula for connected Lie groups

We retain the Levi decomposition and the notation S_i from the introduction. Let I_{nc} be the indices of the non-compact simple ideals, let $\mathfrak{s}_c$ be the sum of the compact ones and let $\mathfrak{s}_{nc}$ be the sum of the others. The closedness statement of Lemma 3.1 will be used for the lower bound.

Lemma 5.1 *Let G be a connected Lie group. Suppose that for any $i \in I_{nc}$, the ideal $\mathfrak{s}_i$ has real rank one and satisfies $[\mathfrak{s}_i, \mathfrak{r}] = 0$. Let $\mathfrak{m} \stackrel{\text{def}}{=} \mathfrak{r} \rtimes \mathfrak{s}_c$.*

1. *The subspace $\mathfrak{m}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{s}_{nc}$ is a direct sum of ideals.*
2. *Let $\tilde{M}$ and $\tilde{S}_i$ be the simply connected Lie groups with Lie algebras $\mathfrak{m}$ and $\mathfrak{s}_i$. Then the universal covering group $\tilde{G}$ of G is isomorphic to $\tilde{M} \times \prod_{i \in I_{nc}} \tilde{S}_i$, and $\tilde{M}$ is amenable.*

3. The pair $(G, Z(G))$ has the covariant approximation property.

4. The topological groups $G/Z(G)$ and $\tilde{M}/Z(\tilde{M}) \times \prod_{i \in I_{nc}} \tilde{S}_i/Z(\tilde{S}_i)$ are isomorphic.

Proof : 1. We have $[\mathfrak{r}, \mathfrak{g}] \subset \mathfrak{r}$ and $[\mathfrak{s}_c, \mathfrak{g}] \subset [\mathfrak{s}_c, \mathfrak{r}] + [\mathfrak{s}_c, \mathfrak{s}] \subset \mathfrak{r} + \mathfrak{s}_c$, so that $\mathfrak{m}$ is an ideal. By assumption, $[\mathfrak{s}_{nc}, \mathfrak{r}] = 0$, and $[\mathfrak{s}_{nc}, \mathfrak{s}_c] = 0$, so that $[\mathfrak{s}_{nc}, \mathfrak{m}] = 0$.

2. The group $M \times \prod_{i \in I_{nc}} \tilde{S}_i$ is connected and simply connected, with Lie algebra isomorphic to $\mathfrak{m} \oplus \mathfrak{s}_{nc} = \mathfrak{g}$. By the integrability theorem [HiN12, Theorem 9.5.9 p. 340], applied to an isomorphism of Lie algebras and to its inverse, it is isomorphic to $\tilde{G}$. The group $\tilde{M}$ is the semidirect product of its simply connected solvable radical by the simply connected group with Lie algebra $\mathfrak{s}_c$ [Bou89, Chapter III, Section 9.8, Proposition 31 p. 356]. The latter group is compact by Weyl's theorem [HiN12, Theorem 12.1.17 p. 467]. Thus M is an extension of a compact group by a solvable normal subgroup, and is amenable by [BHV08, Corollary G.2.3 p. 426].

3. By Lemma 4.2, the pair $(\tilde{M}, Z(\tilde{M}))$ has the covariant approximation property. Let $i \in I_{nc}$. If $\mathfrak{s}_i$ is isomorphic to some $\mathfrak{so}(m, 1)$ or $\mathfrak{su}(m, 1)$, the pair $(\tilde{S}_i, Z(\tilde{S}_i))$ has the covariant approximation property by Lemma 4.2. Otherwise, $\mathfrak{s}_i$ is isomorphic to $\mathfrak{sp}(m, 1)$ for some $m \geq 2$ or to $\mathfrak{f}_{4(-20)}$, see [HaK15, proof of Theorem C p. 6962], the centre of $\tilde{S}_i$ is finite by the same reference, and we conclude with the first point of Lemma 4.2. Since $Z(\tilde{G}) = Z(\tilde{M}) \times \prod_{i \in I_{nc}} Z(\tilde{S}_i)$, the pair $(\tilde{G}, Z(\tilde{G}))$ has the covariant approximation property by the fourth point of Lemma 4.2. Let $p: \tilde{G} \rightarrow G$ be the covering homomorphism and $D \stackrel{\text{def}}{=} \text{Ker } p$, which is a discrete central subgroup. By [Bou89, Chapter III, Section 9.8, Lemma 1 pp. 355–356], the centre of a connected group modulo a discrete central subgroup is the quotient of its centre. Applied to $\tilde{G}$ and D , this gives $p^{-1}(Z(G)) = Z(\tilde{G})$. Thus the covering homomorphism identifies G with $\tilde{G}/D$ and $Z(G)$ with $Z(\tilde{G})/D$. We conclude with the fifth point of Lemma 4.2.

4. Since $D \subset Z(\tilde{G})$, transitivity of quotients [Bou66, Chapter III, Section 2.7, Corollary to Proposition 22 p. 235] gives

$$G/Z(G) \simeq (\tilde{G}/D)/(Z(\tilde{G})/D) \simeq \tilde{G}/Z(\tilde{G}).$$

The centre of a product is the product of the centres. Applying this to the decomposition in the second point proves the fourth point. $\blacksquare$

Now, we prove Theorem 2.

Theorem 2 *Let $\mathfrak{g} = \mathfrak{r} \rtimes \mathfrak{s}$ be a Levi decomposition of the Lie algebra $\mathfrak{g}$ of a connected Lie group G , with radical $\mathfrak{r}$. Write $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ as a sum of simple ideals. Let S_i be the connected analytic subgroup of G with Lie algebra $\mathfrak{s}_i$. Then G is weakly amenable if and only if each $\mathfrak{s}_i$ is compact, or has real rank one and satisfies $[\mathfrak{s}_i, \mathfrak{r}] = 0$. In this case, we have*

$$(5.1) \quad \Lambda_{\text{WA}}(G) = \prod_{i=1}^n \Lambda_{\text{WA}}(S_i).$$

Consequently, the constant Λ_{WA} is invariant under local isomorphism of connected Lie groups.

Proof : Assume first that every simple ideal is compact, or has real rank one and commutes with $\mathfrak{r}$.

Upper bound. By Lemma 5.1 and Proposition 4.6, we have $\Lambda_{\text{WA}}(G) \leq \Lambda_{\text{WA}}(G/Z(G))$. By Lemma 5.1 and [CoH89, Corollary 1.5 p. 514], we have

$$\Lambda_{\text{WA}}(G/Z(G)) = \Lambda_{\text{WA}}(\tilde{M}/Z(\tilde{M})) \prod_{i \in I_{nc}} \Lambda_{\text{WA}}(\tilde{S}_i/Z(\tilde{S}_i)).$$

The group $\tilde{M}/Z(\tilde{M})$ is amenable by [BHV08, Proposition G.2.2(i) p. 425], since it is a quotient of the amenable group $\tilde{M}$ by a closed normal subgroup. Hence $\Lambda_{\text{WA}}(\tilde{M}/Z(\tilde{M})) = 1$. For $i \in I_{nc}$, the group $\tilde{S}_i/Z(\tilde{S}_i)$ is a connected simple Lie group with Lie algebra $\mathfrak{s}_i$, so that $\Lambda_{\text{WA}}(\tilde{S}_i/Z(\tilde{S}_i)) = \Lambda_{\text{WA}}(S_i)$ by [Knu18, Theorem 1.1 p. 102]. Since $\Lambda_{\text{WA}}(S_i) = 1$ if $\mathfrak{s}_i$ is compact, we obtain $\Lambda_{\text{WA}}(G) \leq \prod_{i=1}^n \Lambda_{\text{WA}}(S_i)$.

Lower bound. Empty products are understood to be 1. If there are no simple factors with constant greater than 1, the lower bound is simply $\Lambda_{\text{WA}}(G) \geq 1$. Otherwise, let I_F be the set of indices i such that $\mathfrak{s}_i$ is isomorphic to the Lie algebra $\mathfrak{sp}(m, 1)$ for some $m \geq 2$ or to $\mathfrak{f}_{4(-20)}$, and let S_F be the connected Lie subgroup of G with Lie algebra $\mathfrak{s}_F \stackrel{\text{def}}{=} \bigoplus_{i \in I_F} \mathfrak{s}_i$. The simply connected Lie group $\prod_{i \in I_F} \tilde{S}_i$ with Lie algebra $\mathfrak{s}_F$ has finite centre, see [HaK15, proof of Theorem C p. 6962], and S_F is its quotient by a discrete central subgroup. As in the proof of Lemma 5.1, the centre of S_F is the image of the centre of $\prod_{i \in I_F} \tilde{S}_i$, hence is finite, and $S_F/Z(S_F)$ is isomorphic to $\prod_{i \in I_F} \tilde{S}_i/Z(\tilde{S}_i)$. By Lemma 3.1, the subgroup S_F is closed in G . Using [CoH89, Proposition 1.3 p. 512], [CoH89, Corollary 1.5 p. 514] and [Knu18, Theorem 1.1 p. 102], we obtain

$$\Lambda_{\text{WA}}(G) \geq \Lambda_{\text{WA}}(S_F) = \Lambda_{\text{WA}}(S_F/Z(S_F)) = \prod_{i \in I_F} \Lambda_{\text{WA}}(S_i) = \prod_{i=1}^n \Lambda_{\text{WA}}(S_i),$$

since $\Lambda_{\text{WA}}(S_i) = 1$ for $i \notin I_F$.

Conversely, weak amenability implies the stated condition by Corollary 3.3.

The last assertion follows, since the right-hand side of (1.2) and the weak amenability criterion only depend on $\mathfrak{g}$. $\blacksquare$

Example 5.2 Let H be the universal covering group of the group $\text{SL}(2, \mathbb{R})$, let z generate its infinite cyclic centre, and fix $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Following the construction of [HoM07, Example 10.33 pp. 442–443], also used in [Knu18, Example 5.2 p. 112], set

$$D \stackrel{\text{def}}{=} \{(-m - n\theta, z^m, z^n) : m, n \in \mathbb{Z}\} \quad \text{and} \quad G \stackrel{\text{def}}{=} (\mathbb{R} \times H \times H)/D.$$

The subgroup D is central, closed and discrete: it is the graph of a map from the closed discrete subgroup $Z(H) \times Z(H)$ to $\mathbb{R}$. Hence G is a connected Lie group with Lie algebra $\mathbb{R} \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$.

Let $q : \mathbb{R} \times H \times H \rightarrow G$ be the quotient homomorphism. The analytic Levi subgroup is $S \stackrel{\text{def}}{=} q(\{0\} \times H \times H)$. Irrationality of θ gives $D \cap (\{0\} \times H \times H) = \{(0, e, e)\}$, so S , with its intrinsic Lie group topology, is isomorphic to $H \times H$. However, we have

$$q^{-1}(S) = (\mathbb{Z} + \theta\mathbb{Z}) \times H \times H.$$

Since $\mathbb{Z} + \theta\mathbb{Z}$ is dense and proper in $\mathbb{R}$, the subgroup S is dense and proper in G , and is therefore not closed. Thus the restriction theorem for closed subgroups cannot be applied directly to this Levi subgroup.

Theorem 2 gives $\Lambda_{\text{WA}}(G) = 1$, and Theorem 1 gives the Haagerup property. The value of the constant was already covered by [Knu18, Theorem 5.4 p. 113]; the example illustrates why closedness of the Levi subgroup cannot be assumed.

Example 5.3 The same construction can be carried out with H the universal covering group of $\text{SU}(2, 1)$ and z a generator of its infinite cyclic centre [CCJJV01, Section 4.2.4 pp. 56–57]. The resulting group has Lie algebra $\mathbb{R} \oplus \mathfrak{su}(2, 1) \oplus \mathfrak{su}(2, 1)$ and fundamental group $D \simeq \mathbb{Z}^2$,

since $\mathbb{R} \times H \times H$ is simply connected. As previously, $D \cap (\{0\} \times H \times H) = \{e\}$ and the inverse image of the analytic Levi subgroup is $(\mathbb{Z} + \theta\mathbb{Z}) \times H \times H$. Thus this Levi subgroup is dense and proper, and is isomorphic to $H \times H$ in its intrinsic topology. Theorem 2 gives $\Lambda_{\text{WA}}(G) = 1$. This example is outside the assumptions of [CDSW05, Main Theorem], [Knu16b, Theorem 4], and [Knu18, Theorems 3.1 and 5.4].

Corollary 5.4 *The finite values of $\Lambda_{\text{WA}}(G)$, as G ranges over connected Lie groups, are precisely the positive odd integers. In particular, every connected Lie group satisfies $\Lambda_{\text{WA}}(G) = 1$ or $\Lambda_{\text{WA}}(G) \geq 3$.*

Proof : By Theorem 2 and the simple group constants [Knu18, Theorem 1.1 p. 102], every finite value is a product of numbers $1, 2m - 1$ with $m \geq 2$, and 21, hence is a positive odd integer. Conversely, the trivial group has constant one, and $\text{Sp}(m, 1)$ has constant $2m - 1$ for every $m \geq 2$. This realizes every positive odd integer. ■

Remark 5.5 The oddness of the constants in the previously classified cases was already noted in [Ver24, Remark 5.5 p. 16]. The present corollary extends this observation to all connected Lie groups.

6 Cowling's conjecture and amenable component groups

We prove both implications in Theorem 1. The reverse implication uses the known invariance (2.7).

Theorem 1 *Let G be a locally compact group such that G/G^0 is amenable. Then*

$$\Lambda_{\text{WA}}(G) = 1 \iff G \text{ has the Haagerup property.}$$

In particular, Conjecture 1.1 holds for any connected Lie group and for any almost connected locally compact group.

Proof : *Step 1: connected Lie groups.* Suppose G is a connected Lie group and $\Lambda_{\text{WA}}(G) = 1$. By Corollary 3.3, every non-compact simple ideal $\mathfrak{s}_i$ of a Levi subalgebra has real rank one and centralizes $\mathfrak{t}$. If one of these ideals were isomorphic to $\mathfrak{sp}(m, 1)$, $m \geq 2$, or to $\mathfrak{f}_{4(-20)}$, its simply connected group would have finite centre [HaK15, proof of Theorem C p. 6962]. The corresponding analytic subgroup S_i of G , being a quotient of that group by a discrete central subgroup, also has finite centre and is closed by Lemma 3.1. Restriction and Theorem 1.2 would then give $1 = \Lambda_{\text{WA}}(G) \geq \Lambda_{\text{WA}}(S_i) \geq 3$, a contradiction. Thus every non-compact simple Levi ideal is isomorphic to $\mathfrak{so}(m, 1)$ or $\mathfrak{su}(m, 1)$. Hence

$$\mathfrak{g} = (\mathfrak{t} \rtimes \mathfrak{s}_c) \oplus \mathfrak{s}_{nc},$$

where the simply connected group with Lie algebra $\mathfrak{t} \rtimes \mathfrak{s}_c$ is amenable, as in Lemma 5.1. Thus G is locally isomorphic to a product of an amenable Lie group and groups $\text{SO}_0(m, 1)$ and $\text{SU}(m, 1)$. The classification of [CCJJV01, Theorem 4.0.1 p. 41] implies that the group G has the Haagerup property. Conversely, that classification implies that a connected Lie group with the Haagerup property has precisely this Lie algebra form. Theorem 2 and the simple group constants then give $\Lambda_{\text{WA}}(G) = 1$.

Step 2: connected locally compact groups. Let G be connected and locally compact with $\Lambda_{\text{WA}}(G) = 1$. By Gleason–Yamabe Theorem [CdlH16, Theorem 2.E.14 p. 46], there is a compact

normal subgroup K such that G/K is a connected Lie group. Its weak amenability constant is one by (2.4). Step 1 gives the Haagerup property for G/K . Pulling back the normalized positive definite approximants along the proper quotient map gives the Haagerup property for G . Conversely, if G has the Haagerup property, then so does G/K , by the compact quotient argument in Section 2. Step 1 and (2.4) give $\Lambda_{\text{WA}}(G) = \Lambda_{\text{WA}}(G/K) = 1$.

Step 3: second countable groups with amenable component quotient. Now, suppose that the group G is second countable, $\Lambda_{\text{WA}}(G) = 1$ and G/G^0 is amenable. The closed subgroup G^0 has constant one by (2.4) and has the Haagerup property by Step 2. The permanence result [CCJJV01, Proposition 6.1.5 and Example 6.1.6 pp. 90–91] then gives the Haagerup property for G .

Step 4: removal of the countability assumption. Let G be any locally compact group with $\Lambda_{\text{WA}}(G) = 1$ such that G/G^0 is amenable. Fix a compact subset Q of G and $\varepsilon > 0$. The subgroup generated by $Q \cup Q^{-1}$ and a compact symmetric identity neighbourhood is a compactly generated open subgroup H containing Q . The group H is σ -compact. Since it is open and closed, it contains G^0 and $H^0 = G^0$. Therefore H/H^0 is an open, hence closed, subgroup of the amenable locally compact group G/G^0 . It is amenable by [BHV08, Corollary G.3.4 p. 431].

By Kakutani–Kodaira [HeR79, Theorem 8.7 p. 71], choose a compact normal subgroup K of H such that $L \stackrel{\text{def}}{=} H/K$ is second countable. Let $q: H \rightarrow L$ be the quotient map. The connected group $q(H^0)$ is contained in L^0 . Thus q induces the continuous surjective homomorphism

$$H/H^0 \rightarrow L/L^0, \quad xH^0 \mapsto q(x)L^0.$$

This homomorphism is open, since the quotient maps $H \rightarrow H/H^0$, q and $L \rightarrow L/L^0$ are open. Consequently, L/L^0 is a Hausdorff quotient of the amenable group H/H^0 and is amenable by [BHV08, Proposition G.2.2(i) p. 425]. By (2.4), we have $\Lambda_{\text{WA}}(L) = \Lambda_{\text{WA}}(H) = 1$.

Step 3 gives $\varphi \in P_1(L) \cap C_0(L)$ with $|\varphi - 1| < \varepsilon$ on $q(Q)$. Its pullback belongs to $P_1(H) \cap C_0(H)$, since q is proper. Extend this function by zero outside the open subgroup H . The resulting function belongs to $P_1(G) \cap C_0(G)$ and is within ε of one on Q . Varying Q and ε gives the required net.

Step 5: the converse. Suppose that the group G/G^0 is amenable and that G has the Haagerup property. Restriction of the positive definite approximants to the closed subgroup G^0 shows that G^0 has the Haagerup property. Step 2 gives $\Lambda_{\text{WA}}(G^0) = 1$, and (2.7) gives $\Lambda_{\text{WA}}(G) = 1$. Finally, an almost connected group has compact component quotient, which is amenable by [BHV08, Example G.1.5 p. 423]. This proves all assertions. ■

7 Quotients and lattices

We first treat arbitrary quotients in the connected case. Centrality will then give equality, including when weak amenability fails.

Corollary 7.1 *Let G be a connected locally compact group and let N be a closed normal subgroup of G . Then $\Lambda_{\text{WA}}(G/N) \leq \Lambda_{\text{WA}}(G)$.*

Proof : Suppose first that G is a Lie group and that $\Lambda_{\text{WA}}(G) < \infty$. Let $\mathfrak{n}$ be the Lie algebra of N and let $\phi: \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{n}$ be the quotient map. By [Bou89, Chapter I, Section 6.2, Corollary 2 to Proposition 2 p. 53], the radical of $\mathfrak{g}/\mathfrak{n}$ is $\phi(\mathfrak{r})$. Moreover, $\mathfrak{s} \cap \mathfrak{n}$ is an ideal of $\mathfrak{s}$, hence is the sum of the $\mathfrak{s}_i$ for i in some set J , by [Bou89, Chapter I, Section 6.2, Corollary 1 to Proposition 2 p. 53]. Consequently, $\phi(\mathfrak{s})$ is semisimple, with simple ideals isomorphic to the $\mathfrak{s}_i$ for $i \notin J$. Its intersection with $\phi(\mathfrak{r})$ is a solvable ideal of $\phi(\mathfrak{s})$, hence is zero. Since $\phi(\mathfrak{r}) + \phi(\mathfrak{s}) = \mathfrak{g}/\mathfrak{n}$, this

makes $\phi(\mathfrak{s})$ a Levi subalgebra. For $i \notin J$, the isomorphism from $\mathfrak{s}_i$ onto $\phi(\mathfrak{s}_i)$ preserves compact type and real rank, and

$$[\phi(\mathfrak{s}_i), \phi(\mathfrak{t})] = \phi([\mathfrak{s}_i, \mathfrak{t}]).$$

Thus the commutation conditions pass to the quotient. Theorem 2 gives

$$\Lambda_{\text{WA}}(G/N) = \prod_{i \notin J} \Lambda_{\text{WA}}(S_i) \leq \prod_i \Lambda_{\text{WA}}(S_i) \stackrel{(1.2)}{=} \Lambda_{\text{WA}}(G).$$

Here the constants of the simple analytic subgroups depend only on their Lie algebras, by [Knu18, Theorem 1.1 p. 102]. If $\Lambda_{\text{WA}}(G) = \infty$, the assertion is immediate.

For a connected locally compact group G , choose a compact normal subgroup K such that G/K is a connected Lie group, by Gleason–Yamabe Theorem [CdlH16, Theorem 2.E.14 p. 46]. The subgroup NK is normal and is closed by [Bou66, Chapter III, Section 4.1, Corollary 1 to Proposition 1 p. 251], since N is closed and K is compact. Its image NK/N in G/N is the compact image of K . Transitivity of quotients [Bou66, Chapter III, Section 2.7, Corollary to Proposition 22 p. 235] identifies

$$(G/N)/(NK/N) \simeq G/NK \simeq (G/K)/(NK/K).$$

In particular, G/NK is a quotient of the connected Lie group G/K . Hence

$$\Lambda_{\text{WA}}(G/N) \stackrel{(2.4)}{=} \Lambda_{\text{WA}}(G/NK) \leq \Lambda_{\text{WA}}(G/K) \stackrel{(2.4)}{=} \Lambda_{\text{WA}}(G).$$

This proves the assertion. ■

Lemma 7.2 *Let G be a connected locally compact group and Z be a closed central subgroup. Then*

$$(7.1) \quad \Lambda_{\text{WA}}(G/Z) = \Lambda_{\text{WA}}(G).$$

Proof : Corollary 7.1 gives one inequality. Suppose first that G is a Lie group and $\Lambda_{\text{WA}}(G/Z) < \infty$. Let $\mathfrak{z}$ be the Lie algebra of Z . It is a central ideal contained in $\mathfrak{t}$, so the simple Levi ideals of $\mathfrak{g}/\mathfrak{z}$ are isomorphic to the $\mathfrak{s}_i$. By Theorem 2 applied to G/Z , every non-compact $\mathfrak{s}_i$ has real rank one and $[\mathfrak{s}_i, \mathfrak{t}] \subset \mathfrak{z}$. Since $\mathfrak{s}_i = [\mathfrak{s}_i, \mathfrak{s}_i]$, the Jacobi identity and centrality give

$$[\mathfrak{s}_i, \mathfrak{t}] = [[\mathfrak{s}_i, \mathfrak{s}_i], \mathfrak{t}] \subset [\mathfrak{s}_i, [\mathfrak{s}_i, \mathfrak{t}]] \subset [\mathfrak{s}_i, \mathfrak{z}] = \{0\}.$$

Thus G satisfies the same weak amenability criterion as G/Z , with the same simple factors. The product formula proves equality. If $\Lambda_{\text{WA}}(G/Z) = \infty$, Corollary 7.1 already gives equality.

For general connected G , choose a compact normal subgroup K with G/K a Lie group. The subgroup ZK/K is closed and central in G/K , while ZK/Z is compact. The Lie case gives

$$\Lambda_{\text{WA}}(G/Z) \stackrel{(2.4)}{=} \Lambda_{\text{WA}}(G/ZK) = \Lambda_{\text{WA}}(G/K) \stackrel{(2.4)}{=} \Lambda_{\text{WA}}(G).$$

■

We will use one fact about component groups. Let $q: G \rightarrow Q = G/Z$ be a quotient by a closed central subgroup and let $p: G \rightarrow G/G^0$ be the component quotient. By [HeR79, Theorem 7.12 p. 63], we have $Q^0 = \overline{q(G^0)}$. Put $A \stackrel{\text{def}}{=} \overline{p(Z)}$, which is a closed central subgroup of G/G^0 . In particular, A is abelian, hence amenable by [BHV08, Theorem G.2.1 p. 424]. The subsets $G^0 Z = q^{-1}(q(G^0)) = p^{-1}(p(Z))$ are saturated for both quotient maps. By [Bou66, Chapter

I, Section 5.3, Propositions 6 and 7 pp. 54–55], closure commutes with passing to these open quotients. Therefore,

$$q^{-1}(Q^0) = \overline{G^0 Z} = p^{-1}(A).$$

The continuous open surjection $G \rightarrow Q/Q^0$ consequently has kernel $p^{-1}(A)$. Factoring first through G/G^0 gives the topological isomorphism

$$(7.2) \quad Q/Q^0 \simeq (G/G^0)/A.$$

If G/G^0 is amenable, then its quotient Q/Q^0 is amenable by [BHV08, Proposition G.2.2(i) p. 425]. Conversely, if Q/Q^0 is amenable, then G/G^0 is an extension of that group by the closed amenable normal subgroup A , and is amenable by [BHV08, Proposition G.2.2(ii) p. 425]. Thus G/G^0 is amenable if and only if Q/Q^0 is amenable. Notice that $q(G^0)$ need not be closed.

Now, we prove Theorem 3.

Theorem 3 *Let G be a locally compact group such that the group G/G^0 is amenable, and let Z be a closed central subgroup of G . Then*

$$(7.3) \quad \Lambda_{\text{WA}}(G/Z) = \Lambda_{\text{WA}}(G).$$

The conclusion also holds if the group $(G/Z)/(G/Z)^0$ is amenable. In particular, it holds if either G or G/Z is almost connected.

Proof : We first suppose that the group G is almost connected. Choose a compact normal subgroup K such that $L = G/K$ is a Lie group with finitely many components [CdH16, Theorem 2.E.14 p. 46]. Put $B \stackrel{\text{def}}{=} ZK/K$, a closed central subgroup of L . By (2.4), $\Lambda_{\text{WA}}(G) = \Lambda_{\text{WA}}(L)$ and $\Lambda_{\text{WA}}(G/Z) = \Lambda_{\text{WA}}(L/B)$. The image of the open connected subgroup L^0 in L/B is open and connected, hence is $(L/B)^0$. Thus

$$(L/B)^0 \simeq L^0/(L^0 \cap B).$$

Both L and L/B have finitely many components. Lemma 7.2 therefore yields

$$\Lambda_{\text{WA}}(L) \stackrel{(2.7)}{=} \Lambda_{\text{WA}}(L^0) = \Lambda_{\text{WA}}(L^0/(L^0 \cap B)) = \Lambda_{\text{WA}}((L/B)^0) \stackrel{(2.7)}{=} \Lambda_{\text{WA}}(L/B).$$

This proves the almost connected case.

Now, suppose that the group G/G^0 is amenable, with no countability assumption. Choose an open almost connected subgroup H of G containing G^0 [CdH16, Corollary 2.E.7 p. 44]. Then $H^0 = G^0$. The subgroup $q(H)$ is open and almost connected in $Q = G/Z$, and $q(H)^0 = Q^0$, since an open subgroup has the same identity component as its ambient group. By (7.2), Q/Q^0 is amenable. Applying (2.7) to G, H, Q and $q(H)$, and the almost connected case to $(H, H \cap Z)$, gives

$$\Lambda_{\text{WA}}(G) = \Lambda_{\text{WA}}(H) = \Lambda_{\text{WA}}(H/(H \cap Z)) = \Lambda_{\text{WA}}(q(H)) = \Lambda_{\text{WA}}(Q).$$

Here $H/(H \cap Z) \rightarrow q(H)$ is a topological isomorphism because H is open and q is an open quotient map. This proves (1.3). The alternative assumption on Q/Q^0 is equivalent to that on G/G^0 , by (7.2). The last assertion follows. ■

A lattice in a locally compact group is a discrete subgroup whose homogeneous space admits an invariant probability measure. For a lattice Γ in G , we use the equality [CoH89, Proposition 6.2 and the following observation, p. 546]

$$(7.4) \quad \Lambda_{\text{WA}}(\Gamma) = \Lambda_{\text{WA}}(G).$$

The observation following that proposition removes the second countability assumption.

Corollary 7.4 *Let G be a connected locally compact group, let Γ be a lattice of G , and let Z be a subgroup of Γ contained in the centre of G . Then*

$$\Lambda_{\text{WA}}(\Gamma/Z) = \Lambda_{\text{WA}}(\Gamma).$$

If this constant is one, then the groups Γ and Γ/Z have the Haagerup property.

Proof : The subgroup Z is closed and discrete in G . Let $q: G \rightarrow G/Z$ be the quotient map. Choose an open identity neighbourhood U in G with $U \cap \Gamma = \{e\}$. Since $Z \subset \Gamma$, we have $UZ \cap \Gamma = Z$, so $q(U) \cap q(\Gamma) = \{e\}$. Thus $q(\Gamma)$ is discrete and is isomorphic to Γ/Z . The natural map $G/\Gamma \rightarrow (G/Z)/q(\Gamma)$ is an equivariant homeomorphism by transitivity of quotient spaces [Bou66, Chapter I, Section 3.4, Proposition 7 p. 40], applied to the coset relations defined by $Z \subset \Gamma$. The invariant probability measure on the first space gives one on the second. Hence $q(\Gamma)$ is a lattice of the group G/Z . The lattice equality and Lemma 7.2 give

$$\Lambda_{\text{WA}}(\Gamma) \stackrel{(7.4)}{=} \Lambda_{\text{WA}}(G) \stackrel{(7.1)}{=} \Lambda_{\text{WA}}(G/Z) \stackrel{(7.4)}{=} \Lambda_{\text{WA}}(\Gamma/Z).$$

If the common value is one, the connected groups G and G/Z have the Haagerup property by Theorem 1. Restriction to their closed subgroups Γ and $q(\Gamma)$ proves the last assertion. ■

AI statement. The author acknowledges the use of AI tools for language polishing, exploratory mathematical discussions and feedback during the development and preparation of this manuscript.

Competing interests The author declares that he has no competing interests.

Data availability No data sets were generated during this study.

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