ON LIN'S ARITHMETIC–GEOMETRIC MEANS CONJECTURE FOR SINGULAR VALUES

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ABSTRACT. Let A and B be positive semidefinite, and let $0 \le t \le 1$. Lin (2017) conjectured that

$$\lambda_j((1-t)A + tB) \ge \sigma_j^{1/2}(A^{2(1-t)}B^{2t}), \qquad j = 1, \dots, n.$$

The case $t = \frac{1}{2}$ is Drury's theorem, which answered a long-standing question of Bhatia and Kittaneh. Lin remarked that Drury's method does not seem to reach other values of t. We prove his conjecture for every $t \in [0, 1]$. The proof builds on Drury's argument, but replaces his block-matrix step (unique to the midpoint $t = \frac{1}{2}$) by an eigenvalue count based on an inertia identity and a one-variable inequality.

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1. INTRODUCTION

Finding matrix analogues of scalar inequalities has long been a theme of matrix analysis. Many works study analogues of the arithmetic–geometric mean inequality; see, e.g., the books (Bhatia, 1997, 2007; Zhan, 2002). Some of the most useful of these inequalities compare singular values $\sigma(\cdot)$ and eigenvalues $\lambda(\cdot)$ of matrices. A fundamental example of such an inequality is

$$\sigma_j^{1/2}(AB) \le \tfrac{1}{2}\lambda_j(A+B), \qquad A, B \succeq 0, \qquad j = 1, \dots, n, \tag{1.1}$$

which was raised as a question by Bhatia and Kittaneh (2000) and proved twelve years later by Drury (2012).

Bhatia and Kittaneh (1990) proved the weaker inequality $\sigma_j(A^{1/2}B^{1/2}) \le \frac{1}{2}\lambda_j(A+B)$. This inequality was generalized by Ando (1995) to $\sigma_j(A^{1-t}B^t) \le \lambda_j((1-t)A+tB)$ for $t \in [0,1]$. Revisiting Drury's proof, Lin (2017, Conjecture 5.2) conjectured the corresponding weighted form of (1.1), and wrote that Drury's method *"does not seem to lead to a solution."* To our knowledge no proof has appeared since, and Theorem 1.1 supplies one.

Theorem 1.1 (Lin's conjecture)**.** *Let $A, B \succeq 0$ and let $0 \le t \le 1$. Then, for every $j = 1, \dots, n$,*

$$\sigma_j^{1/2}(A^{2(1-t)}B^{2t}) \le \lambda_j((1-t)A + tB). \tag{1.2}$$

More broadly, both Ando's inequality and Theorem 1.1 are members of the family

$$\sigma_j^{q}(A^{(1-t)/q}B^{t/q}) \le \lambda_j((1-t)A + tB), \qquad j = 1, \dots, n, \tag{1.3}$$

indexed by $q > 0$. A brief calculation shows that if inequality (1.3) holds for an exponent q, then it also holds for every larger exponent, so the statements become stronger as q decreases. The cases $q \ge 1$ therefore follow from Ando's inequality, while Lin's conjecture is the case $q = 1/2$, outside that range. Theorem 1.1 gives (1.3) for every $q \ge 1/2$, and another brief calculation shows that at $t = 1/2$ the exponent $1/2$ is the best.

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2. Proof of Theorem 1.1

Proof summary. The proof has four steps. The first two follow Drury's argument (Drury, 2012) exposited in Lin (2017, Sections 2–4). Write $a = 1 - t$ and $b = t$. In Step 1 we use homogeneity of (1.2) to reduce the task to proving $\lambda_j(aA + bB) \geq 1$. Step 2 reduces this task to an eigenvalue inequality for $aA + bB_0$, where $B_0 \preceq B$ is a rank-j matrix. Drury's proof of that inequality works only at $t = 1/2$, so we introduce Steps 3 and 4 to replace it. Step 3 counts eigenvalues using an inertia identity, which reduces the inequality to the sign of a quadratic form on a subspace; Step 4 settles this sign using an elementary scalar inequality.

2.1. **Rescaling.** Using a and b, inequality (1.2) reads

$$\lambda_j(aA + bB) \geq \sigma_j^{1/2}(A^{2a}B^{2b}), \qquad j = 1, \ldots, n. \tag{2.1}$$

If $t = 0$ or $t = 1$ the inequality is trivial; so we assume $t \in (0, 1)$. Also, it suffices to consider $A, B \succ 0$ (the $\succeq 0$ cases follow by continuity). We can also assume $a \leq b$ (otherwise flip the roles of A and B and note $\sigma(AB) = \sigma(BA)$), so that $b \geq 1/2$. Fix $j \in \{1, \ldots, n\}$. If $\sigma_j(A^{2a}B^{2b}) = 0$, then (2.1) is trivial; otherwise by homogeneity we may scale (2.1) so that $\sigma_j(A^{2a}B^{2b}) = 1$, and reduce the task to proving that $\lambda_j(aA + bB) \geq 1$.

2.2. **Rank reduction.** By Weyl's monotonicity principle, lowering B in the Löwner order can only lower $\lambda_j(aA + bB)$. Following Drury, we lower B to a matrix B_0 of rank j by keeping only the part of B that produces the j largest singular values of $A^{2a}B^{2b}$, and resetting those singular values to 1.

More precisely, let $T = A^{2a}B^{4b}A^{2a} = (A^{2a}B^{2b})(A^{2a}B^{2b})^*$. Since $\lambda(T) = \sigma^2(A^{2a}B^{2b})$, the rescaling step tells us $\lambda_j(T) = 1$. Choose an orthonormal basis $v_1, \ldots, v_n$ with $Tv_k = \lambda_k(T)v_k$, and let E be the orthogonal projection onto $\operatorname{span}(v_1, \ldots, v_j)$. Since $T \succeq 0$ and $\lambda_k(T) \geq \lambda_j(T) = 1$ for $k \leq j$,

$$T - E = \sum_{k \leq j} (\lambda_k(T) - 1) v_k v_k^* + \sum_{k > j} \lambda_k(T) v_k v_k^* \succeq 0.$$

Thus, $T \succeq E$. Congruence with A^{-2a} then yields

$$B^{4b} = A^{-2a}TA^{-2a} \quad \succeq \quad B_0^{4b} := A^{-2a}EA^{-2a}. \tag{2.2}$$

Because $b \geq 1/2$, the exponent $1/(4b)$ is at most 1, so by Löwner–Heinz on (2.2) we obtain $B \succeq B_0$. Hence by Weyl $\lambda_j(aA + bB) \geq \lambda_j(aA + bB_0)$.

Why such a choice of B_0? The point of this choice is that B_0 is determined by A and by its own support projection P. Let $V = A^{2a}B_0^{2b}$. By (2.2), $VV^* = A^{2a}B_0^{4b}A^{2a} = E$, so V is a partial isometry with final projection E. Since A^{2a} is invertible, $\operatorname{Ker} V = \operatorname{Ker} B_0^{2b} = \operatorname{Ran}(I - P)$, so the initial projection of V is P. In other words,

$$B_0^{2b}A^{4a}B_0^{2b} = P. \tag{2.3}$$

The rank of P is j, because $B_0^{4b} = A^{-2a}EA^{-2a}$ has rank j. Let $S = PA^{4a}P|_{\operatorname{Ran} P}$ be the compression of A^{4a} to $\operatorname{Ran} P$, which is positive definite because A is. Write $B_0 = PXP$ with $X \succ 0$ on $\operatorname{Ran} P$, and let $Y = X^{2b}$. Compressing (2.3) to $\operatorname{Ran} P$ gives $YSY = I_{\operatorname{Ran} P}$, and multiplying on the left and on the right by Y^{-1} gives $S = Y^{-2}$. Since $Y \succ 0$, uniqueness of positive square roots gives $Y = S^{-1/2}$, and therefore $X = S^{-1/(4b)}$. Thus

$$B_0 = PS^{-1/(4b)}P. \tag{2.4}$$

It remains to prove

$$\lambda_j(aA + bB_0) \geq 1. \tag{2.5}$$

Since (2.4) expresses B_0 through A and P alone, we prove (2.5) for every $A \succ 0$ and every orthogonal projection P of rank j, with B_0 defined by (2.4). At $t = 1/2$ the matrix $B_0 = P(PA^2P)^{-1/2}P$ is Drury's matrix B_1 (Lin, 2017, Section 2), and (2.5) is the inequality $\lambda_j(A + B_1) \geq 2$ that Drury

proves by an eigenvalue estimate for 2×2 block matrices. That estimate uses that B_0 is the inverse square root of PA^2P on $\operatorname{Ran} P$, and we see no analogue of it for the fractional power $S^{-1/(4b)}$.

2.3. **Eigenvalue count.** Inequality (2.5) says that $aA + bB_0 - I$ has at least j nonnegative eigenvalues. This matrix is $H + bB_0$ with $H = aA - I$: a Hermitian matrix, in general indefinite, plus a positive semidefinite perturbation carried by the j-dimensional space $\operatorname{Ran} P$. The following inertia identity transfers such an eigenvalue count to a matrix that acts only on the space carrying the perturbation. It is the finite-dimensional form of the Birman–Schwinger principle (Schwinger, 1961).

Lemma 2.1 (Inertia identity)**.** *Let $H = H^*$ be invertible on a finite-dimensional Hilbert space $\mathcal{H}$. Let $J\colon \mathcal{P} \to \mathcal{H}$ be an isometric inclusion, and let $W \succ 0$ on $\mathcal{P}$. Set*

$$M = H + JWJ^*, \qquad G = W^{-1} + J^*H^{-1}J.$$

Then

$$n_{\geq 0}(M) = n_+(H) + n_{\leq 0}(G),$$

where $n_{\geq 0}$ counts eigenvalues ≥ 0, n_+ counts eigenvalues > 0, and $n_{\leq 0}$ counts eigenvalues ≤ 0, each with multiplicity.

Proof. Consider the Hermitian matrix

$$\mathcal{L} = \begin{pmatrix} H & J \\ J^* & -W^{-1} \end{pmatrix} \quad \text{on } \mathcal{H} \oplus \mathcal{P}.$$

The Schur complements in $\mathcal{L}$ of its blocks $-W^{-1}$ and H are

$$H + JWJ^* = M \qquad \text{and} \qquad -W^{-1} - J^*H^{-1}J = -G.$$

By inertia additivity for Schur complements (Haynsworth, 1968), $\mathcal{L}$ therefore has the inertia of both $M \oplus (-W^{-1})$ and $H \oplus (-G)$. As $-W^{-1} \prec 0$ and H is invertible, counting nonnegative eigenvalues gives $n_{\geq 0}(M) = n_{\geq 0}(\mathcal{L}) = n_+(H) + n_{\leq 0}(G)$. □

To apply Lemma 2.1, $H = aA - I$ must be invertible, that is, $1/a$ must not be an eigenvalue of A. By continuity we may assume so: for fixed P, the left side of (2.5) depends continuously on $A \succ 0$, and $1/a$ is an eigenvalue of $A + \varepsilon I$ for only finitely many $\varepsilon > 0$.

Let m be the number of eigenvalues of A greater than $1/a$, so that $n_+(H) = m$. Apply Lemma 2.1 with $\mathcal{H} = \mathbb{C}^n$, $\mathcal{P} = \operatorname{Ran} P$, J the inclusion map, and $W = bS^{-1/(4b)} \succ 0$. Then $JWJ^* = bB_0$, so $M = aA + bB_0 - I$, and since $(aA - I)^{-1} = -(I - aA)^{-1}$,

$$n_{\geq 0}(aA + bB_0 - I) = m + n_{\leq 0}(G), \qquad G = b^{-1}S^{1/(4b)} - P(I - aA)^{-1}P\big|_{\operatorname{Ran} P}. \tag{2.6}$$

Thus (2.5) holds as soon as G has at least $j - m$ nonpositive eigenvalues. It is enough to find a subspace $U \subseteq \operatorname{Ran} P$ with $\dim U \geq j - m$ on which the quadratic form of G is nonpositive. Indeed, for such a subspace $n_{\leq 0}(G) \geq \dim U$: otherwise the span N of the eigenvectors of G for eigenvalues ≤ 0 would have $\dim N < \dim U$, so U would contain a nonzero vector $v \perp N$, and $\langle Gv, v\rangle > 0$.

The two terms of G indicate where to look. The first is positive definite. The second, $-P(I - aA)^{-1}P$, is negative on vectors built from eigenvectors of A with eigenvalues $x < 1/a$, on which $(I - aA)^{-1}$ acts by the positive numbers $1/(1 - ax)$. Let $\mathcal{F}$ be the span of the eigenvectors of A with eigenvalues in $[0, 1/a)$, and let $U = \operatorname{Ran} P \cap \mathcal{F}$. Because $1/a$ is not an eigenvalue of A, $\mathcal{F}$ has dimension $n - m$, and since $\operatorname{Ran} P$ has dimension j,

$$\dim U \geq j + (n - m) - n = j - m.$$

It therefore remains to prove

$$\langle Gu, u\rangle \leq 0 \qquad \text{for every } u \in U. \tag{2.7}$$

2.4. **Scalar inequality.** By homogeneity it suffices to prove (2.7) for a unit vector $u \in U$. Let μ be the spectral measure of A associated with u, that is, the probability measure on $\mathbb{R}$ with $\langle f(A)u,\, u\rangle = \int f\, d\mu$ for every function f on the spectrum of A. Since $u \in \mathcal{F}$, the measure μ is supported in $[0, 1/a)$. Since $u \in \operatorname{Ran} P$,

$$\langle Su,\, u\rangle = \langle A^{4a}u,\, u\rangle = \int x^{4a}\, d\mu(x) \qquad \text{and} \qquad \langle P(I-aA)^{-1}Pu,\, u\rangle = \int \tfrac{d\mu(x)}{1-ax}.$$

Put $\gamma = 1/(4b)$; because $b \geq 1/2$, we have $0 < \gamma \leq 1$. Concavity of $x \mapsto x^{\gamma}$ and the spectral measure of S associated with u give $\langle S^{\gamma}u,\, u\rangle \leq \langle Su,\, u\rangle^{\gamma}$. By (2.6), therefore,

$$\langle Gu,\, u\rangle \leq b^{-1}\left(\int x^{4a}\, d\mu(x)\right)^{\gamma} - \int \tfrac{d\mu(x)}{1-ax}. \tag{2.8}$$

So (2.7) follows once the right side of (2.8) is shown to be nonpositive for every probability measure μ on $[0, 1/a)$. Lemma 2.2 proves this bound by integrating an inequality in one real variable; its proof is in Appendix A.

Lemma 2.2 (Scalar inequality)**.** *Let $0 < a \leq b$, $a + b = 1$. For every $0 \leq x < 1/a$,*

$$x^{4a} \leq 1 + 4b^2\left(\frac{1}{1-ax} - \frac{1}{b}\right). \tag{2.9}$$

Consequently, if μ is a probability measure on $[0, 1/a)$, then

$$\left(\int x^{4a}\, d\mu(x)\right)^{1/(4b)} \leq b\int \frac{1}{1-ax}\, d\mu(x). \tag{2.10}$$

By (2.10), the right side of (2.8) is nonpositive. This proves (2.7), and with it Theorem 1.1. □

AI USAGE

The results in this paper were developed with an AI assistant (OpenAI's GPT Pro). None of the claims were obtained zero-shot: they emerged through sustained back-and-forth discussion, including failed attempts, counterexamples, audits, and repairs.

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Appendix A. Proof of the scalar inequality

Equality holds in (2.9) at $x = 1$, where both sides equal 1 and both have derivative $4a$, so the two sides are tangent there.

Proof. Put $y = (1 - ax)^{-1}$, so that $y \geq 1$ and $ax = 1 - y^{-1}$. Then (2.9) says $g(y) \geq 0$ for $y \geq 1$, where $g(y) = 1 + 4b^2(y - b^{-1}) - a^{-4a}(1 - y^{-1})^{4a}$. Since $1 - b = a$, we have $g(b^{-1}) = 0$, and $g'(y) = 4b^2 - 4a^{1-4a}(1 - y^{-1})^{4a-1}y^{-2}$ also vanishes at b^{-1}; moreover $g(1) = (1 - 2a)^2 \geq 0$. For $y > 1$,

$$g''(y) = 4a^{1-4a}\,\frac{(1 - y^{-1})^{4a}(2y - 1 - 4a)}{y^2(y - 1)^2},$$

so g is concave on $[1, y_*]$ and convex on $[y_*, \infty)$, where $y_* = (1 + 4a)/2$ (convex throughout if $y_* \leq 1$). Since $2 - (1 + 4a)(1 - a) = 4a^2 - 3a + 1 > 0$, we have $y_* < b^{-1}$. On the convex part g has a double zero at b^{-1}, so $g \geq 0$ there, in particular at y_*; on the concave part g lies above the chord joining $g(1) \geq 0$ and $g(y_*) \geq 0$. This proves (2.9).

Integrating (2.9) against μ gives $\int x^{4a}\,d\mu \leq 1 + 4b^2(\bar{y} - b^{-1})$ with $\bar{y} = \int (1 - ax)^{-1}\,d\mu$. The right side is the tangent line at b^{-1} of $y \mapsto (by)^{4b}$, which is convex since $4b \geq 1$. Hence $\int x^{4a}\,d\mu \leq (b\bar{y})^{4b}$, and taking the power $1/(4b)$ gives (2.10). □