

A PROOF OF MCLEOD'S 1959 CONJECTURE ON COMPLETE HOMOGENEOUS SYMMETRIC RATIOS

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ABSTRACT. We prove McLeod's (1959) conjecture that $(h_{\ell+r}/h_\ell)^{1/r}$ is subadditive for all $\ell \geq 0$ and $r \geq 1$, where h_k is the complete homogeneous symmetric polynomial of degree k (J. B. McLeod. "*On four inequalities in symmetric functions.*" Proc. Edinburgh Math. Soc., 1959). We first prove that the ratio h_k/h_{k-1} is convex, and our proof takes an unexpected detour through queuing theory. The geometric mean of r consecutive such ratios equals McLeod's conjectured function, but geometric means of convex functions need not be convex. Our second and decisive step is a tensorization trick that couples neighboring ratios tightly enough to make their geometric mean convex.

1. INTRODUCTION

For $x = (x_1, \dots, x_n) \in \mathbb{R}_+^n$, the complete homogeneous symmetric polynomial of degree k is

$$(1.1) \quad h_k(x) := \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}, \quad k \geq 1, \quad h_0 = 1.$$

McLeod (1959, §1, p. 211) proved that $h_r^{1/r}$ is subadditive on $\mathbb{R}_+^n$ and asked whether $(h_{\ell+r}/h_\ell)^{1/r}$ is also subadditive. Our main result proves McLeod's conjecture in the equivalent form of convexity.

Theorem 1.1 (McLeod's inequality). *For every $n \geq 1$ and all integers $\ell \geq 0$ and $r \geq 1$, the function*

$$(1.2) \quad G_{\ell,r}(x) = \left(\frac{h_{\ell+r}(x)}{h_\ell(x)} \right)^{1/r}, \quad G_{\ell,r}(0) = 0,$$

is convex on $\mathbb{R}_+^n$. Equivalently (since $G_{\ell,r}$ is positively homogeneous of degree one),

$$(1.3) \quad G_{\ell,r}(x+y) \leq G_{\ell,r}(x) + G_{\ell,r}(y), \quad x, y \in \mathbb{R}_+^n.$$

Beyond McLeod's $\ell = 0$ result, the case $\ell = 1$ was proved by Baston (1978). Sra (2020, Theorems 14–15) extended both cases to coordinatewise powers (see also Section 4). Theorem 1.1 finally resolves the general case. Our analysis also yields a quantitative lower bound for the directional curvature of $G_{\ell,r}$ —see Appendix A.4.

1.1. High-level proof strategy. Taking cue from the proof of concavity of an analogous ratio of elementary symmetric polynomials (Marcus and Lopes, 1957; Sra, 2020), we could consider

$$(1.4) \quad \frac{h_{\ell+r}}{h_\ell} = \frac{h_{\ell+r}}{h_{\ell+r-1}} \times \cdots \times \frac{h_{\ell+1}}{h_\ell}.$$

However, we are pursuing convexity *not* concavity—could we still hope that convexity of the individual ratios $\rho_k = h_k/h_{k-1}$ helps us toward proving convexity of the product (1.4)? Curiously, even proving the convexity of the ratio ρ_k turns out to be far from trivial; it is the content of Theorem 2.1, a result whose proof relies on an unexpected connection to queuing.

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Convexity of ρ_k by itself is insufficient: $G_{\ell,r}$ is the geometric mean of these ratios, and geometric means of convex functions need not be convex. This step is precisely where the elementary symmetric argument succeeds and ours stalls. This obstacle forces us to revisit geometric means more closely. While geometric means do not preserve convexity in general, perhaps for the ratios ρ_k they still do, because neighboring ratios are tightly linked. We use two ingredients to turn this hope into a proof:

- (i) work with the *reciprocal ratios* h_k/h_{k+1} , and exploit their concavity in k ;
- (ii) *tensorize an auxiliary variable*, splitting it equally among many auxiliary variables; importantly, it yields a bound coupling the curvatures and logarithmic slopes of neighboring ratios.

Concavity in k is already known in queuing theory; our tensorization trick also recovers an independent proof. Tensorization rests on the relation $\partial_z h_k(x, z)|_{z=0} = h_{k-1}(x)$ for an auxiliary variable z ; importantly, such auxiliary variables let neighboring degrees interact. We may use as many auxiliaries as needed, and in the limit only cross terms survive. A discrete Poincaré inequality then combines concavity in k with a bound on curvature to control the geometric mean (Section 3).

2. CONVEXITY OF CONSECUTIVE RATIOS

The polynomial $h_k(x)$ normalizes the stationary law of a closed network of n exponential single-server stations with service means x_i and k circulating customers (Gordon and Newell, 1967). Its reciprocal ratio h_{k-1}/h_k is the *throughput* at each station (the long-run rate of service completions). For a fixed customer count, $\rho_k = h_k/h_{k-1}$ is therefore the *time per completion*: the average time between successive service completions at a station. We use this interpretation to prove convexity of consecutive ratios in the service means.

Theorem 2.1 (Consecutive-ratio convexity). *For every $n \geq 1$ and $k \geq 1$, the function*

$$(2.1) \quad \rho_k(x) = \frac{h_k(x)}{h_{k-1}(x)}, \quad \rho_k(0) = 0,$$

is convex and coordinatewise nondecreasing on $\mathbb{R}_+^n$.

The monotonicity is known: since $h_k = s_{(k)}$, it is a special case of the monotonicity of Schur ratios s_λ/s_μ for $\mu \subseteq \lambda$ (Khare and Tao, 2021, Proposition 8.1). Our proof also recovers it directly.

Note. We first tried a direct Hessian computation to establish convexity. For $k \leq 4$ with some algebra and a Cauchy–Schwarz argument this route succeeds—see Appendix B.1. But for $k \geq 5$, the resulting identity does not seem to admit any direct proof of Hessian positivity.

2.1. The queuing theory proof. The reason queuing theory unlocks the proof is almost coincidental: there exists a queuing system in which ρ_k is the average time between successive service completions, and in this system the completion times are convex almost by construction. More specifically, we must find a way to compare completion times as the service means vary, and to quantify the throughput (i.e., the long-run rate of completions) to identify their long-run behavior. The max-plus recursions of Baccelli et al. (1992) supply the comparison, while the product-form stationary law of Gordon and Newell (1967) supplies the throughput.

Proof of Theorem 2.1. For $n = 1$ the ratio is x_1 , so assume $n \geq 2$ and first take $x \in \mathbb{R}_{++}^n$. Put k customers at station 1 of a cyclic network $1 \rightarrow 2 \rightarrow \dots \rightarrow n \rightarrow 1$. Each station serves one customer at a time, in arrival order, with independent exponential service times of mean x_i .

Step 1. We begin by proving convexity of completions in the means x . To compare networks with different service means, fix independent standard exponentials $E_{i,j}$ and use $x_i E_{i,j}$ as the j th service time at station i for every x . Only the deterministic scales x_i vary between networks; each service time is therefore linear in x on this common probability space.

Service begins when both the server and the next customer are available, so its starting time is the maximum of these two availability times. Because all customers follow the same route and each

station serves them in arrival order, customer order is preserved. The customer completing the j th service at station $i - 1$ is the j th arrival to station i for $2 \leq i \leq n$. At station 1, the first k required arrivals are present at time zero; thereafter the j th required arrival is the $(j - k)$ th completion at station n . Writing $D_i(j; x)$ for the time of the j th completion at station i , we therefore obtain

$$(2.2) \quad \begin{aligned} D_1(j; x) &= x_1 E_{1,j} + \max\{D_1(j - 1; x), D_n(j - k; x)\}, \\ D_i(j; x) &= x_i E_{i,j} + \max\{D_i(j - 1; x), D_{i-1}(j; x)\}, \quad 2 \leq i \leq n, \end{aligned}$$

with $D_i(j; x) = 0$ for $j \leq 0$. For each realization of the exponentials, these recursions use only addition of nonnegative linear functions of x and taking a maximum. Both operations preserve convexity and coordinatewise monotonicity. The return to station 1 uses the earlier index $j - k < j$, so the cyclic route permits induction in the order $(j, 1), \dots, (j, n)$. Starting from the zero initial times, this induction shows that every $D_i(j; \cdot)$ is convex and coordinatewise nondecreasing.

Step 2. Next, we show that the normalized completion times $D_i(j; x)/j$ converge to $\rho_k(x)$ as $j \rightarrow \infty$. To identify the long-run behavior of the completion times, consider the *queue-length process*. Let $e_i \in \mathbb{Z}_+^n$ denote the i th standard basis vector, with indices read cyclically ($e_{n+1} = e_1$). The process has states $q \in \mathbb{Z}_+^n$ with $\sum_i q_i = k$ and transitions $q \rightarrow q - e_i + e_{i+1}$ at rate $1/x_i$ when $q_i > 0$. Global balance gives the stationary law $\pi_x(q) = x^q/h_k(x)$, where $x^q := x_1^{q_1} x_2^{q_2} \dots x_n^{q_n}$. Incoming and outgoing fluxes both equal $\pi_x(q) \sum_{i: q_i > 0} x_i^{-1}$ (see Appendix A.1). Consequently, the throughput at any station is

$$(2.3) \quad \sum_{q: q_i > 0} \frac{\pi_x(q)}{x_i} = \frac{1}{h_k(x)} \sum_{q: q_i > 0} x^{q - e_i} = \frac{h_{k-1}(x)}{h_k(x)} = \frac{1}{\rho_k(x)}.$$

By ergodicity, this throughput (2.3) is the long-run number of completions per unit time at each station; inverting the completion count therefore yields

$$(2.4) \quad \frac{D_i(j; x)}{j} \longrightarrow \rho_k(x) \quad \text{almost surely as } j \rightarrow \infty.$$

Appendix A.1 proves this limit from the bounded-rate counting-process martingale.

Finally, since convexity and coordinatewise monotonicity each compare ρ_k at finitely many points, both pass from the completion times to the almost-sure limit (2.4) on $\mathbb{R}_{++}^n$. Since ρ_k is continuous on $\mathbb{R}_+^n$, both properties extend to $\mathbb{R}_+^n$. $\square$

3. THE PROOF OF THEOREM 1.1: RECIPROCAL RATIOS AND TENSORIZATION

The queuing argument proves convexity of every ρ_k . To prove convexity of the geometric mean $G_{\ell,r} = (\prod_{k=\ell+1}^{\ell+r} \rho_k)^{1/r}$ we develop new ideas below. It suffices to establish convexity of $t \mapsto G_{\ell,r}(x + tv)$ for $x \in \mathbb{R}_+^n$ and an arbitrary direction v (ensuring that $x + tv \in \mathbb{R}_+^n$ too). As before we may work with $x > 0$. We write D_v for the directional derivative along v , and seek to prove that $D_v^2 G_{\ell,r}(x) \geq 0$. Because $\log G_{\ell,r}$ is the average of the $\log \rho_k$, we describe each factor by its logarithmic slope and relative curvature,

$$(3.1) \quad d_k = D_v \log \rho_k, \quad \kappa_k = \frac{D_v^2 \rho_k}{\rho_k}, \quad \bar{d} = \frac{1}{r} \sum_{k=\ell+1}^{\ell+r} d_k.$$

Twice differentiating the logarithm of $G_{\ell,r}$ (in the geometric mean form above) gives

$$(3.2) \quad \frac{D_v^2 G_{\ell,r}}{G_{\ell,r}} = \frac{1}{r} \left\{ \sum_{k=\ell+1}^{\ell+r} \kappa_k - \sum_{k=\ell+1}^{\ell+r} (d_k - \bar{d})^2 \right\}.$$

Indeed, $D_v^2 \log \rho_k = \kappa_k - d_k^2$, and the first derivative of $\log G_{\ell,r}$ is $\bar{d}$. Theorem 2.1 makes each κ_k nonnegative, but (3.2) subtracts the variance of the logarithmic slopes. Convexity of $G_{\ell,r}$, equivalently nonnegativity of (3.2) therefore requires enough curvature to control this variance.

3.1. Reciprocal ratios. To control the variance in (3.2), we need to know how the ratios change from one degree to the next. Since (3.2) is written in logarithms, the first candidate is convexity of $k \mapsto \log \rho_k$. We work instead with a stronger property: concavity of the reciprocal ratios

$$(3.3) \quad \sigma_k := \frac{h_{k-1}}{h_k} = \rho_k^{-1} \quad (k \geq 1), \quad \sigma_0 = 0,$$

that is, nonnegativity of their second differences

$$(3.4) \quad \alpha_k := 2\sigma_k - \sigma_{k-1} - \sigma_{k+1}.$$

Concavity of σ_k implies the logarithmic statement, since $\log \sigma_k$ is then concave and $\log \rho_k = -\log \sigma_k$. Because σ_k is the station throughput (Section 2), $\alpha_k \geq 0$ says that throughput is concave in the customer count. This concavity is known (Dowdy et al., 1984), and Shanthikumar and Yao (1988, Theorem 1) extended it to networks whose service rates are nondecreasing and concave in the local queue length, a class that contains ours (Appendix B.2).

But concavity of σ_k alone does not control the logarithmic slopes in (3.2). We also need to bound their neighboring differences by curvature in the service means. The tensorization argument below proves this crucial coupled estimate (it also recovers an independent proof of concavity).

3.2. Tensorization. The coupled estimate comes from augmenting x to the tuple (x, z) , where z is an auxiliary, and then computing the Hessian of $\rho_k(x, z)$. Derivatives $\partial_z|_{z=0}$ lower the degree, e.g., $\partial_z h_j(x, z)|_{z=0} = h_{j-1}(x)$, so the Hessian involves both ρ_{k-1} and ρ_k . Convexity of ρ_k therefore ties the curvature κ_k (see (3.1)) to the slope difference $d_k - d_{k-1}$.

A single auxiliary, however, ties them too loosely. With $\tau_k = \rho_k / \rho_{k-1} > 0$ ($k \geq 2$), its diagonal entry is $\partial_z^2 \rho_k = \tau_k(2\sigma_{k-1} - 2\sigma_{k-2})$, a first difference of σ , whereas the variance bound of Section 3.3 needs the second differences α_k . This limitation is what tensorization addresses. More precisely, we split z into M auxiliaries that share its value equally. We keep track of what this split changes using the generating function $H_x(t) = \sum_{m \geq 0} h_m(x)t^m = \prod_i (1 - x_i t)^{-1}$; each auxiliary multiplies it by one factor, and the coefficient of t^s in that factor multiplies h_{j-s} . A single auxiliary with value u contributes $(1 - ut)^{-1} = 1 + ut + u^2 t^2 + \dots$, so that $h_j(x, u) = h_j(x) + u h_{j-1}(x) + u^2 h_{j-2}(x) + \dots$. Splitting u equally among M auxiliaries gives instead

$$(3.5) \quad H_{(x, (u/M)\mathbf{1}_M)}(t) = H_x(t) \left(1 - \frac{ut}{M}\right)^{-M} \xrightarrow{M \rightarrow \infty} H_x(t) e^{ut}.$$

As $M \rightarrow \infty$, the split keeps the first-order term ut but halves the second-order term, from $u^2 t^2$ to $u^2 t^2 / 2$; this halving turns $\tau_k(2\sigma_{k-1} - 2\sigma_{k-2})$ into $\tau_k(2\sigma_{k-1} - \sigma_{k-2} - \sigma_k) = \tau_k \alpha_{k-1}$.

Proposition 3.1 (Coupled curvature). *For every $k \geq 1$, $\alpha_k \geq 0$. For $k \geq 2$,*

$$(3.6) \quad \kappa_k \geq \frac{\sigma_{k-1}}{\alpha_{k-1}} (d_k - d_{k-1})^2 \quad \text{if } \alpha_{k-1} > 0; \quad \text{and if } \alpha_{k-1} = 0, \text{ then } d_k = d_{k-1}.$$

Proof. Let $z = (z_1, \dots, z_M)$ and write $\tilde{\rho}_k(x, z) = \rho_k(x_1, \dots, x_n, z_1, \dots, z_M)$. By Theorem 2.1:

$$(3.7) \quad \nabla^2 \tilde{\rho}_k(x, 0) \succeq 0.$$

At $(x, 0)$, the auxiliary derivatives of h_j are h_{j-1} at first order and h_{j-2} at mixed second order. The quotient rule then gives, for $a \neq b$,

$$(3.8) \quad \partial_{z_a} \tilde{\rho}_k = 1 - \tau_k, \quad \partial_{z_a} \partial_{z_b} \tilde{\rho}_k = \tau_k \alpha_{k-1},$$

$$(3.9) \quad D_v \partial_{z_a} \tilde{\rho}_k = -\tau_k (d_k - d_{k-1}).$$

Here $h_{-1} = 0$; Appendix A.2 gives the full calculation. In particular, the second formula is

$$\partial_{z_a} \partial_{z_b} \tilde{\rho}_k = \tau_k \left(2 \frac{h_{k-2}}{h_{k-1}} - \frac{h_{k-1}}{h_k} - \frac{h_{k-3}}{h_{k-2}} \right),$$

which explains the choice of reciprocal ratios.

Concavity in k comes from the auxiliary direction $\mathbf{1}_M$ alone. By symmetry, the auxiliary Hessian block has one common diagonal entry a_k and one common off-diagonal entry $b_k = \tau_k \alpha_{k-1}$. Neither depends on M . Positivity in the direction $\mathbf{1}_M$ gives $a_k + (M-1)b_k \geq 0$ for every M , and hence $b_k \geq 0$. Applying this bound at degree $k+1$, where $b_{k+1} = \tau_{k+1} \alpha_k$, proves $\alpha_k \geq 0$.

To bring in the service means, test the Hessian (3.7) against v together with the auxiliary direction, i.e., against $(v, s/M, \dots, s/M)$. Writing $u_k = -\tau_k(d_k - d_{k-1})$ for the mixed entry (3.9), the result is

$$(3.10) \quad D_v^2 \rho_k + 2su_k + s^2 \left(b_k + \frac{a_k - b_k}{M} \right) \geq 0.$$

The mixed term $2su_k$ does not depend on M , while the diagonal correction tends to zero. Sending $M \rightarrow \infty$ leaves

$$(3.11) \quad D_v^2 \rho_k + 2su_k + b_k s^2 \geq 0 \quad (s \in \mathbb{R}).$$

If $b_k > 0$, minimize over s and divide by ρ_k to obtain

$$(3.12) \quad \kappa_k \geq \frac{u_k^2}{\rho_k b_k} = \frac{\sigma_{k-1}}{\alpha_{k-1}} (d_k - d_{k-1})^2.$$

If $b_k = 0$, nonnegativity of the affine expression for all s forces $u_k = 0$, so $d_k = d_{k-1}$. The same tensorization therefore proves both reciprocal concavity and the curvature estimate. $\square$

3.3. A finite variance bound. We must now combine the lower-bound (3.12) with another inequality that helps us prove non-negativity of (3.2). Lemma 3.2 performs this task.

Lemma 3.2 (Poincaré inequality). *Fix $\ell \geq 0$ and $r \geq 2$, and let $J = \{1 \leq p < r : \alpha_{\ell+p} > 0\}$. Let $w_0, \dots, w_{r-1} \in \mathbb{R}$ have mean $\bar{w}$ and increments $\eta_p = w_p - w_{p-1}$, with $\eta_p = 0$ for $p \notin J$. Then*

$$(3.13) \quad \sum_{j=0}^{r-1} (w_j - \bar{w})^2 \leq \sum_{p \in J} \frac{R_p}{\alpha_{\ell+p}} \eta_p^2 \leq \sum_{p \in J} \frac{\sigma_{\ell+p}}{\alpha_{\ell+p}} \eta_p^2,$$

where $R_p \geq 0$ is the gap between σ and its chord over $[\ell, \ell+r]$, equalling

$$(3.14) \quad R_p = \sigma_{\ell+p} - \frac{r-p}{r} \sigma_\ell - \frac{p}{r} \sigma_{\ell+r}.$$

Proof. A quick calculation (see Lemma A.6) allows us to write the left hand side of (3.13) as

$$(3.15) \quad \sum_{j=0}^{r-1} (w_j - \bar{w})^2 = \sum_{p,q \in J} \Gamma_{pq} \eta_p \eta_q, \quad \text{where} \quad \Gamma_{pq} = \min(p, q) - pq/r \geq 0,$$

where the sum runs over J because $\eta_p = 0$ for $p \notin J$. Since $\Gamma \geq 0$, we can bound (3.15) as follows: for any $\beta_p > 0$, multiply $\eta_p \eta_q \leq \frac{1}{2} (\eta_p^2 \beta_q / \beta_p + \eta_q^2 \beta_p / \beta_q)$ by Γ_{pq} and sum over $p, q \in J$ to obtain

$$\sum_{p,q \in J} \Gamma_{pq} \eta_p \eta_q \leq \sum_{p \in J} c_p \eta_p^2, \quad \text{where} \quad c_p = \frac{1}{\beta_p} \sum_{q \in J} \Gamma_{pq} \beta_q.$$

Set $\beta_p = \alpha_{\ell+p}$ (positive on J and by Proposition 3.1, zero outside). Then $\alpha_{\ell+p} c_p = \sum_{q=1}^{r-1} \Gamma_{pq} \alpha_{\ell+q}$. The matrix Γ inverts the second difference with zero boundary values: every f with $f_0 = f_r = 0$ satisfies $f_p = \sum_q \Gamma_{pq} (2f_q - f_{q-1} - f_{q+1})$. Extended by (3.14) to $p = 0, r$, the sequence R has $R_0 = R_r = 0$ and, since the chord is linear in p , $2R_p - R_{p-1} - R_{p+1} = \alpha_{\ell+p}$. Hence $\alpha_{\ell+p} c_p = R_p$, which gives the first inequality in (3.13); it also shows $R_p \geq 0$, since Γ and α are nonnegative. The second inequality follows from $R_p \leq \sigma_{\ell+p}$, which holds because $\sigma_\ell, \sigma_{\ell+r} \geq 0$. $\square$

Proof of Theorem 1.1. For $r = 1$ it is Theorem 2.1. For $r \geq 2$, take $w_j = d_{\ell+1+j}$, so that $\eta_p = d_{\ell+p+1} - d_{\ell+p}$. Proposition 3.1 gives $\eta_p = 0$ for $p \notin J$, which is the hypothesis of Lemma 3.2, and it bounds $\kappa_{\ell+p+1}$ for $p \in J$. Since every $\kappa_k \geq 0$, dropping $\kappa_{\ell+1}$ and the terms with $p \notin J$ gives

$$(3.16) \quad \sum_{k=\ell+1}^{\ell+r} \kappa_k \stackrel{(3.12)}{\geq} \sum_{p \in J} \frac{\sigma_{\ell+p}}{\alpha_{\ell+p}} \eta_p^2 \stackrel{(3.13)}{\geq} \sum_{k=\ell+1}^{\ell+r} (d_k - \bar{d})^2.$$

Using (3.16), non-negativity of the right side of (3.2) is immediate, which proves Theorem 1.1. $\square$

4. MINOR EXTENSION TO POWERS

Sra (2020, Theorems 14–15) extended the cases $\ell = 0$ and $\ell = 1$ of Theorem 1.1 to coordinatewise powers $x^{\circ p} = (x_1^p, \dots, x_n^p)$. Theorem 1.1 already implies the same extension for all ℓ and r .

Corollary 4.1 (Power extension). *For every $n \geq 1$, all integers $\ell \geq 0$, $r \geq 1$, and every real $p \geq 1$,*

$$(4.1) \quad G_{\ell,r}^{[p]}(x) = \left(\frac{h_{\ell+r}(x^{\circ p})}{h_{\ell}(x^{\circ p})} \right)^{1/(pr)}, \quad G_{\ell,r}^{[p]}(0) = 0,$$

is convex on $\mathbb{R}_+^n$, or equivalently subadditive. No $0 < p < 1$ has this property for all n, ℓ, r .

The same product formula that led to the Hessian argument also preserves coordinatewise monotonicity: each factor ρ_k is nondecreasing by Theorem 2.1, so $G_{\ell,r}$ is nondecreasing as well. Together with convexity, this monotonicity allows coordinatewise powers to be inserted into the triangle inequality. The following lemma isolates that last step.

Lemma 4.2 (Monotone power convexification). *Let $F : \mathbb{R}_+^n \rightarrow [0, \infty)$ be convex, positively homogeneous of degree one, coordinatewise nondecreasing, and positive off the origin. Then $x \mapsto F(x^{\circ p})^{1/p}$ is convex and one-homogeneous for every $p \geq 1$.*

Proof. The transformed function is one-homogeneous, so it suffices to prove subadditivity. Normalize each vector to lie on its unit level set: for nonzero x, y , put $a = F(x^{\circ p})^{1/p}$, $b = F(y^{\circ p})^{1/p}$, $u = x/a$, $v = y/b$, and $\lambda = a/(a + b)$. Thus $F(u^{\circ p}) = F(v^{\circ p}) = 1$. Coordinatewise Jensen, followed by monotonicity and convexity of F , gives

$$F((\lambda u + (1 - \lambda)v)^{\circ p}) \leq F(\lambda u^{\circ p} + (1 - \lambda)v^{\circ p}) \leq 1.$$

Since $\lambda u + (1 - \lambda)v = (x + y)/(a + b)$, homogeneity gives $F((x + y)^{\circ p})^{1/p} \leq a + b$. If either vector is zero the inequality is immediate. Subadditivity and homogeneity imply convexity. $\square$

Proof of Corollary 4.1. Apply Lemma 4.2 to $F = G_{\ell,r}$. For sharpness, take $\ell = 0$, $r = 1$, $n = 2$ and $x = e_1$, $y = e_2$. For every $0 < p < 1$,

$$G_{0,1}^{[p]}(e_1 + e_2) = 2^{1/p} > 2 = G_{0,1}^{[p]}(e_1) + G_{0,1}^{[p]}(e_2).$$

Thus no $p < 1$ works uniformly over the family. $\square$

AI USAGE

The results in this paper were developed through extensive use of an AI assistant (OpenAI’s GPT Pro). None of the claims were obtained zero-shot: they emerged through sustained back-and-forth discussion, including failed attempts, counterexamples, audits, and repairs. For example, our first approach to Theorem 2.1 used the exponential-moment representation $\mathbb{E}S_x^k = k! h_k(x)$, where $S_x = \sum_i x_i \xi_i$ with independent standard exponential ξ_i ; this representation was key to the h_k/h_1 case in the author’s earlier work (Sra, 2020). Here it ran into rough weather, so we cast a wider net: we asked the assistant to survey where else h_k or the ratio h_k/h_{k-1} arise as probabilities or rates in the literature, and among the candidates it returned, the queuing interpretation proved the most fruitful. The tensorization idea was also our own contribution. We used GPT and Claude for assistance with drafting and writing, though the main text is almost entirely human written.

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APPENDIX A. DEFERRED PROOFS

This appendix supplies the proofs that Sections 2 and 3 defer, in the order in which they are used, and ends with the quantitative remainder mentioned in the introduction.

A.1. The queuing network. For the queuing proof of Theorem 2.1, this subsection derives the stationary law of the network and its completion-time limit. Fix throughout integers $n \geq 2$ and $k \geq 1$, and let $x \in \mathbb{R}_{++}^n$. The network of Section 2.1 has state space $\mathcal{S}_k = \{q \in \mathbb{Z}_+^n : \sum_{i=1}^n q_i = k\}$, with station indices understood cyclically.

Proposition A.1 (Stationary law). *The queue-length chain of Section 2.1 is irreducible and has stationary distribution $\pi_x(q) = x^q / h_k(x)$ for $q \in \mathcal{S}_k$, where $x^q = \prod_{i=1}^n x_i^{q_i}$.*

Proof. Irreducibility is immediate: by repeated service completions, every customer can be moved around the cycle, so every composition of k is reachable from every other one.

The normalization $\sum_{q \in \mathcal{S}_k} x^q = h_k(x)$ is exactly (1.1). It remains to check global balance. The total outgoing flux from a state q is $\pi_x(q) \sum_{i: q_i > 0} 1/x_i$. To enter q through a completion at station i , the preceding state must be $q + e_i - e_{i+1}$, which is admissible exactly when $q_{i+1} > 0$. Since $x^{q+e_i-e_{i+1}} = x^q x_i / x_{i+1}$, the associated incoming flux is $\pi_x(q + e_i - e_{i+1}) / x_i = \pi_x(q) / x_{i+1}$. Summing over the admissible incoming transitions gives $\pi_x(q) \sum_{i: q_{i+1} > 0} 1/x_{i+1} = \pi_x(q) \sum_{j: q_j > 0} 1/x_j$, which is the outgoing flux. Hence π_x is stationary. Uniqueness follows from finite-state irreducibility. $\square$

To justify the completion-time limit in (2.4), we first count service completions. For a fixed station i , let $N_i(t; x)$ count the completions by time t , and let $\Theta_k(x)$ be the throughput (2.3).

Lemma A.2 (Long-run throughput). *Fix $x \in \mathbb{R}_{++}^n$. From any initial state of the closed network, every station i satisfies*

$$(A.1) \quad \frac{N_i(t; x)}{t} \longrightarrow \Theta_k(x) = \frac{h_{k-1}(x)}{h_k(x)} \quad \text{almost surely.}$$

Proof. The queue-length process $Q(t)$ is an irreducible continuous-time Markov chain on the finite state space $\mathcal{S}_k$, hence is positive recurrent and satisfies the ergodic theorem from every initial state.

The counting process at station i has the compensator decomposition

$$(A.2) \quad N_i(t; x) = M_i(t) + \int_0^t \frac{\mathbf{1}_{\{Q_i(s) > 0\}}}{x_i} ds,$$

where M_i is a square-integrable martingale. Its predictable quadratic variation satisfies $\langle M_i \rangle_t \leq t/x_i$. To see that $M_i(t)/t \rightarrow 0$ almost surely, Doob's inequality gives $\Pr(\sup_{0 \leq s \leq 2^m} |M_i(s)| > \varepsilon 2^m) \leq 1/(\varepsilon^2 x_i 2^m)$, which is summable in m . Borel–Cantelli, followed by interpolation between consecutive dyadic times, proves the claim. The Markov-chain ergodic theorem gives

$$\frac{1}{t} \int_0^t \frac{\mathbf{1}_{\{Q_i(s) > 0\}}}{x_i} ds \longrightarrow \sum_{q: q_i > 0} \pi_x(q) \frac{1}{x_i} = \Theta_k(x),$$

where the last identity is (2.3). Substitution into (A.2) proves (A.1). The Markov-chain facts used here are standard; see, for example, (Norris, 1997). $\square$

Lemma A.3 (Inversion of counting processes). *Let $N : [0, \infty) \rightarrow \mathbb{Z}_+$ be nondecreasing and right-continuous, with $N(t)/t \rightarrow \theta \in (0, \infty)$. Then $D(r) = \inf\{t : N(t) \geq r\}$ satisfies $D(r)/r \rightarrow 1/\theta$.*

Proof. Fix $0 < \varepsilon < \theta$, and choose $T < \infty$ with $(\theta - \varepsilon)t \leq N(t) \leq (\theta + \varepsilon)t$ for $t \geq T$. For all sufficiently large r , the times $t_- = (r - 1)/(\theta + \varepsilon)$ and $t_+ = r/(\theta - \varepsilon)$ are at least T . The upper estimate on N gives $N(t_-) \leq r - 1$, so the r th completion has not occurred by t_- and $D(r) > t_-$. The lower estimate gives $N(t_+) \geq r$, so $D(r) \leq t_+$. Therefore $(r - 1)/(r(\theta + \varepsilon)) < D(r)/r \leq 1/(\theta - \varepsilon)$; let $r \rightarrow \infty$ and then $\varepsilon \downarrow 0$. $\square$

Combining the two lemmas gives $D_i(r; x)/r \rightarrow 1/\Theta_k(x) = \rho_k(x)$ almost surely, which is (2.4).

A.2. Derivatives at auxiliary zero coordinates. For Proposition 3.1, the two lemmas below verify the derivative formulas (3.8) and (3.9).

Lemma A.4 (Auxiliary derivatives). *Let $k \geq 2$, and let $a \neq b$ be auxiliary indices. Then*

$$\frac{\partial \tilde{\rho}_k}{\partial z_a}(x, 0) = 1 - \tau_k, \quad \frac{\partial^2 \tilde{\rho}_k}{\partial z_a \partial z_b}(x, 0) = \tau_k \alpha_{k-1}.$$

Proof. For a single auxiliary variable z one has $h_j(x, z) = \sum_{s=0}^j z^s h_{j-s}(x)$, so $\partial_z h_j(x, 0) = h_{j-1}(x)$; for two auxiliary variables, the coefficient of $z_a z_b$ in $h_j(x, z_a, z_b)$ is $h_{j-2}(x)$, so $\partial_{z_a z_b}^2 h_j(x, 0) = h_{j-2}(x)$. Here $h_{-1} = 0$. The quotient rule for $\tilde{\rho}_k = h_k/h_{k-1}$ at $(x, 0)$ gives the first derivative $(h_{k-1}^2 - h_k h_{k-2})/h_{k-1}^2 = 1 - \tau_k$ and, by (3.4), the mixed second derivative

$$\begin{aligned} \frac{\partial^2 \tilde{\rho}_k}{\partial z_a \partial z_b}(x, 0) &= -\frac{h_{k-2}}{h_{k-1}} - \frac{h_k h_{k-3}}{h_{k-1}^2} + 2 \frac{h_k h_{k-2}^2}{h_{k-1}^3} \\ &= \tau_k \left(2 \frac{h_{k-2}}{h_{k-1}} - \frac{h_{k-1}}{h_k} - \frac{h_{k-3}}{h_{k-2}} \right) = \tau_k (2\sigma_{k-1} - \sigma_k - \sigma_{k-2}) = \tau_k \alpha_{k-1}. \end{aligned} \quad \square$$

Lemma A.5 (Mixed derivative). *Let $k \geq 2$, let a be an auxiliary index, and let D_v act on the x variables. Then $D_v \partial_{z_a} \tilde{\rho}_k(x, 0) = -\tau_k (d_k - d_{k-1})$.*

Proof. Mixed partial derivatives of the smooth function $\tilde{\rho}_k$ commute, and $\partial_{z_a} \tilde{\rho}_k(\cdot, 0) = 1 - \tau_k$ on $\mathbb{R}_{++}^n$ by Lemma A.4. Since $\tau_k = \rho_k/\rho_{k-1}$, we have $D_v \tau_k = \tau_k (D_v \log \rho_k - D_v \log \rho_{k-1}) = \tau_k (d_k - d_{k-1})$. $\square$

A.3. The finite variance identity. Lemma 3.2 uses the following identity, which writes the variance of a finite sequence as a quadratic form in its increments.

Lemma A.6 (Finite-path Green kernel). *Let $r \geq 1$, and let $w_0, \dots, w_{r-1} \in \mathbb{R}$ have mean $\bar{w}$ and increments $\eta_p = w_p - w_{p-1}$ for $1 \leq p \leq r-1$. Then*

$$(A.3) \quad \sum_{j=0}^{r-1} (w_j - \bar{w})^2 = \sum_{p,q=1}^{r-1} \Gamma_{pq} \eta_p \eta_q, \quad \Gamma_{pq} = \min(p, q) - \frac{pq}{r} = \frac{\min(p, q)(r - \max(p, q))}{r},$$

and the matrix Γ is positive semidefinite with nonnegative entries.

Proof. The left side of (3.15) is unchanged when a constant is added to every w_j , so assume $w_0 = 0$; then $w_j = \sum_{p=1}^{j-1} \mathbf{1}_{\{p \leq j\}} \eta_p$. Let B be the $r \times (r-1)$ matrix $B_{jp} = \mathbf{1}_{\{p \leq j\}}$, and let $P = I - r^{-1} \mathbf{1} \mathbf{1}^\top$ be the orthogonal projection onto mean-zero vectors, so that the left side equals $\eta^\top B^\top P B \eta$. Direct counting gives $(B^\top B)_{pq} = r - \max(p, q)$ and $(B^\top \mathbf{1})_p = r - p$, hence for $p \leq q$, $(B^\top P B)_{pq} = r - q - (r - p)(r - q)/r = p(r - q)/r = \min(p, q) - pq/r$. The Gram form $\Gamma = B^\top P B$ gives positive semidefiniteness, and the product formula gives nonnegative entries. $\square$

A.4. A quantitative remainder. The first inequality in (3.13) uses the weights $R_p \leq \sigma_{\ell+p}$; keeping the difference gives a lower bound on the directional curvature of $G_{\ell,r}$. For $r \geq 2$, with $\eta_p = d_{\ell+p+1} - d_{\ell+p}$ and J as in Lemma 3.2, equations (3.13) and (3.2) give

$$(A.4) \quad \frac{D_v^2 G_{\ell,r}}{G_{\ell,r}} \geq \frac{\kappa_{\ell+1}}{r} + \frac{1}{r^2} \sum_{p \in J} \frac{(r-p)\sigma_\ell + p\sigma_{\ell+r}}{\alpha_{\ell+p}} \eta_p^2.$$

Indeed, the curvature bound (3.12) contributes $\sigma_{\ell+p} \eta_p^2 / \alpha_{\ell+p}$, whereas the variance bound subtracts only $R_p \eta_p^2 / \alpha_{\ell+p}$, and $\sigma_{\ell+p} - R_p$ is the chord term in (3.14). We do not claim optimality.

APPENDIX B. DIRECT HESSIAN PROOFS (UP TO $k = 4$) AND OTHER MINOR RESULTS

The results of this appendix are not needed for Theorem 1.1 and Corollary 4.1. They give a direct Hessian proof of convexity of ρ_k in low degrees and record how ρ_k and σ_k depend on the degree k . Both parts use the elementary bounds

$$(B.1) \quad \max_i x_i \leq \rho_k(x) \leq \sum_i x_i \quad (x \in \mathbb{R}_+^n \setminus \{0\}).$$

Coefficient comparison proves both. For each j the polynomial $x_j h_{k-1}(x)$ consists precisely of the monomials of $h_k(x)$ that contain x_j , each with coefficient one, so $h_k(x) - x_j h_{k-1}(x) = h_k(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_n) \geq 0$, which is the lower bound. For the upper bound, the coefficient of a degree- k monomial $x_{i_1} \cdots x_{i_k}$ in $h_1 h_{k-1}$ is the number of distinct indices among $i_1, \dots, i_k$, hence at least one, so $h_k \leq h_1 h_{k-1}$ coefficientwise.

B.1. Direct Hessian proofs in low degrees. We compute the Hessian of a consecutive ratio directly. The resulting identity holds in every degree, and in low degrees it proves convexity.

For this calculation, it is convenient to write $F_k = h_{k+1}/h_k = \rho_{k+1}$, with $k \geq 1$. Fix $x \in \mathbb{R}_{++}^n$. Since F_k is homogeneous of degree one, Euler's identity gives $\nabla^2 F_k(x) x = 0$. Thus subtracting a multiple of x from a direction v does not change $D_v^2 F_k$. We choose that multiple to make $D_v h_k = 0$: explicitly, replace v by $v - (D_v h_k)/(k h_k) x$. This centering removes the first-derivative terms from the quotient rule, leaving

$$(B.2) \quad h_k^2 D_v^2 F_k = h_k D_v^2 h_{k+1} - h_{k+1} D_v^2 h_k.$$

To expand the difference on the right, define

$$(B.3) \quad a_j = \sum_i v_i x_i^j, \quad c_j = \sum_i v_i^2 x_i^j, \quad H_x(t) = \sum_{m \geq 0} h_m(x) t^m = \prod_i (1 - x_i t)^{-1}.$$

Differentiating the generating function gives $D_v H_x = H_x \sum_{j \geq 0} a_j t^{j+1}$. Coefficient comparison therefore yields $D_v h_m = \sum_{j=0}^{m-1} a_j h_{m-j-1}$, so the centering condition is

$$(B.4) \quad \sum_{j=0}^{k-1} a_j h_{k-j-1} = 0.$$

A second differentiation, followed by coefficient comparison, gives

$$(B.5) \quad D_v^2 H_x = H_x \left[\left(\sum_{j \geq 0} a_j t^{j+1} \right)^2 + \sum_{j \geq 0} (j+1) c_j t^{j+2} \right],$$

$$(B.6) \quad D_v^2 h_m = \sum_{i+j \leq m-2} a_i a_j h_{m-i-j-2} + \sum_{j=0}^{m-2} (j+1) c_j h_{m-j-2},$$

where $i, j \geq 0$ in the first sum. The same quadratic expressions occur at degrees k and $k+1$. Their coefficients in (B.2) are $\Delta_q = h_k h_{k-q-1} - h_{k+1} h_{k-q-2}$ for $0 \leq q < k$, with $h_{-1} = 0$, and Jacobi–Trudi identifies each coefficient as a Schur polynomial (Macdonald, 1995):

$$(B.7) \quad \Delta_q = s_{(k, k-q-1)}(x) \geq 0.$$

Here s_λ denotes the Schur polynomial indexed by λ ; its nonnegative coefficients give the inequality. Grouping terms by $q = i + j$ now gives the Hessian formula.

Proposition B.1 (Centered Hessian identity). *Under (B.4),*

$$(B.8) \quad h_k^2 D_v^2 F_k = \sum_{q=0}^{k-1} \Delta_q \left(\sum_{j=0}^q a_j a_{q-j} + (q+1) c_q \right).$$

The coefficients Δ_q are nonnegative, but the quadratic expressions they multiply contain signed cross terms. The centering relation can control these cross terms in low degrees, giving the following direct proofs without using the queuing representation.

Proposition B.2 (Direct algebraic convexity). *The functions $F_1 = h_2/h_1$, $F_2 = h_3/h_2$, and $F_3 = h_4/h_3$ are convex on $\mathbb{R}_+^n$, with value 0 at the origin.*

Proof. For $k = 1$, centering gives $a_0 = 0$, and (B.8) reduces to $h_1^2 D_v^2 F_1 = h_1 c_0 \geq 0$.

For $k = 2$, centering eliminates a_1 : the relation $h_1 a_0 + a_1 = 0$ gives $a_1 = -h_1 a_0$. With $\Delta_0 = h_1 h_2 - h_3 \geq 0$ and $\Delta_1 = h_2$, identity (B.8) becomes

$$h_2^2 D^2 F_2[v, v] = \Delta_0(a_0^2 + c_0) + h_2(2a_0 a_1 + 2c_1) = \Delta_0 c_0 + 2h_2 c_1 - (h_1 h_2 + h_3) a_0^2.$$

Cauchy–Schwarz gives $a_1^2 = (\sum_i x_i v_i)^2 \leq h_1 c_1$, hence $c_1 \geq h_1 a_0^2$ and $h_2^2 D^2 F_2[v, v] \geq \Delta_0(c_0 + a_0^2) \geq 0$.

At $k = 3$, the same strategy leaves a quadratic form in a_0, a_1 . Write $s = h_1$ and $p_2 = \sum_i x_i^2 = 2h_2 - s^2$; then $\Delta_0 = h_3 h_2 - h_4 s$, $\Delta_1 = h_3 s - h_4$ and $\Delta_2 = h_3$ are nonnegative by (B.7). Centering now eliminates $a_2 = -h_2 a_0 - s a_1$, and (B.8) expands to

$$h_3^2 D^2 F_3[v, v] = \Delta_0 c_0 + 2\Delta_1 c_1 + 3h_3 c_2 + h_3 a_1^2 - 2h_4 a_0 a_1 - (h_3 h_2 + h_4 s) a_0^2.$$

Three Cauchy–Schwarz inequalities give $c_0 \geq a_0^2/n$, $c_1 \geq a_1^2/s$ and $c_2 \geq a_2^2/p_2 = (h_2 a_0 + s a_1)^2/p_2$. Substitution leaves a quadratic form that completes to a square. Using $p_2 + s^2 = 2h_2$ and the definitions of Δ_0, Δ_1 gives

$$h_3^2 D^2 F_3[v, v] \geq \left(h_3 + \frac{2\Delta_1}{s} + \frac{3h_3 s^2}{p_2} \right) \left(a_1 + \frac{s}{2} a_0 \right)^2 + \frac{n+2}{2n} \Delta_0 a_0^2 \geq 0,$$

which proves the claim for F_3 .

For each of the three ratios, undoing the radial centering leaves the Hessian quadratic form unchanged. The bound $0 \leq F_k \leq h_1$ from (B.1) gives continuity at the origin; elsewhere the denominator is positive. Convexity therefore extends from the positive orthant to its boundary. $\square$

At higher degrees, the identity still describes the full Hessian, but it does not by itself establish its sign. The queuing proof of Section 2.1 proves the required positivity for all k simultaneously.

B.2. Dependence on the degree. The queuing interpretation makes the degree index natural: k is the number of circulating customers. This subsection shows that σ_k is nondecreasing and concave in k , and that ρ_k decreases to $\max_i x_i$.

Lemma B.3 (Degree log-concavity). *For every $x \in \mathbb{R}_+^n \setminus \{0\}$ and every $k \geq 1$,*

$$(B.9) \quad h_k(x)^2 - h_{k-1}(x)h_{k+1}(x) = s_{(k,k)}(x) \geq 0.$$

Consequently $\sigma_k(x)$ is nondecreasing in k , and $0 \leq \sigma_k(x) \leq (\max_i x_i)^{-1}$ by (B.1).

Proof. The Jacobi–Trudi identity (Macdonald, 1995) expresses the left side of (B.9) as the Schur polynomial $s_{(k,k)}$, which has nonnegative coefficients. Dividing by $h_k h_{k+1} > 0$ gives $\sigma_{k+1} \geq \sigma_k$ for $k \geq 1$, and $\sigma_1 \geq 0 = \sigma_0$ is trivial. The upper bound is the lower bound in (B.1). $\square$

Equilibrium-rate representation. For $x \in \mathbb{R}_{++}^n$, choose $0 < a < (\max_i x_i)^{-1}$ and let $Y_1, \dots, Y_n$ be independent geometric random variables with $\Pr(Y_i = q) = (1 - ax_i)(ax_i)^q$ for $q \geq 0$. Writing $Y = \sum_i Y_i$, convolution gives $\Pr(Y = k) = a^k h_k(x) / H_x(a)$ for $k \geq 0$. The equilibrium rate of a random variable Y with full support on $\mathbb{Z}_+$ is $r_Y(0) = 0$ and $r_Y(k) = \Pr(Y = k - 1) / \Pr(Y = k)$ for $k \geq 1$ (Shanthikumar and Yao, 1988, §2, equation (2)). Thus $r_Y(k) = \sigma_k / a$ for $k \geq 0$, while $r_{Y_i}(0) = 0$ and $r_{Y_i}(k) = 1 / (ax_i)$ for $k \geq 1$, so each r_{Y_i} is nondecreasing and concave. Shanthikumar and Yao (1988, Proposition 1(A)) prove that these properties of equilibrium rates are preserved under convolution. Iteration therefore gives the nondecreasing concavity of σ_k on $\mathbb{Z}_+$. In their cyclic network representation, choosing the common visit weight a gives exactly the rate $r_Y = \sigma / a$; this choice accounts for the throughput normalization.

Proposition B.4. *For fixed $x \in \mathbb{R}_+^n \setminus \{0\}$, $\rho_k(x)$ is nonincreasing in k and tends to $\max_i x_i$.*

Proof. Lemma B.3 gives $h_k^2 \geq h_{k-1}h_{k+1}$, so $\rho_{k+1} = h_{k+1}/h_k \leq h_k/h_{k-1} = \rho_k$. Hence $\rho_k(x)$ decreases to some limit $L \geq \max_i x_i$ by (B.1). Let $M = \max_i x_i$. Since h_k has $\binom{n+k-1}{k}$ monomials, $M^k \leq h_k(x) \leq \binom{n+k-1}{k} M^k$, and therefore $h_k(x)^{1/k} \rightarrow M$. But $h_k(x)^{1/k} = (\prod_{j=1}^k \rho_j(x))^{1/k}$ is a geometric mean of a positive sequence converging to L , so it also converges to L . Hence $L = M$. $\square$

In queuing language, increasing the number of circulating customers decreases the time per completion and drives the system toward the bottleneck service time.