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COUNTEREXAMPLES TO HIRONAKA'S CHARACTERISTIC-p RESOLUTION: A REPORT ON QUERIES TO FRONTIER AGENTS

Abstract. Created on 23 October 2026, this manuscript draws on a `frontier-dump` investigation of Hironaka's proposed positive-characteristic resolution method. Taylor Dupuy initiated the queries during his talk *How to Use AI in Mathematical Research Effectively* at the University of Rochester Number Theory Seminar on that date. The findings were synthesized with `harvest-dumps` and developed into this manuscript with `using-latexpowers`.

We examine the proposed positive-characteristic resolution mechanism of [Hir17]. Its negative-degree pieces are the unit ideal at a closed singular point where, for a common multiple m of the generator degrees, $\wp(E, m)$ contains an element of order m. For the family $x^p + y^n$, with $n > 2p$ and $p \nmid n$, the asserted transformation rule fails under the algebraic description of the characteristic algebra. With core focusing read by (44), the fixed local chain and the $\check{E}$ or $\hat{E}$ reading of the invariant, the length-one string agrees before and after the first blowup. With renewed chains and corresponding transported choices, the corresponding strings agree along the resolution sequence under either core-focusing reading. The two formulas for core focusing respectively retain the base-hiked exponent and, when its Inv-max stratum is smooth and closed, forget it. We give characteristic-2 limits on tail enclosure and a shade increase at a kangaroo point. These calculations do not supply a decreasing invariant or settle resolution. The mathematical draft of this manuscript was AI-generated by Claude Code (model `claude-opus-5-5`) from an AI-generated research record; its proofs received only AI review, and no human has checked them.

AI Generated PRINTED BY: Taylor Dupuy

Origin. This manuscript began with a prompt during Taylor Dupuy's talk *How to Use AI in Mathematical Research Effectively* at the University of Rochester Number Theory Seminar on 23 October 2026. The demonstration used `frontier-dump`, followed by `harvest-dumps` and `using-latexpowers`, to investigate Hironaka's preprint and present the counterexamples found.

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1. Introduction

Does the mechanism in Hironaka's proposed resolution of singularities in positive characteristic [Hir17, §1, pp. 2–4] supply an invariant that decreases after its prescribed blowups? The preprint builds a characteristic algebra $\wp(E)$, adds negative-degree pieces, chooses tail hypersurfaces, and compares invariant strings after ambient reduction [Hir17, §§5, 9–10, 15–16]. We test those steps at the points where they are used. This matters because the termination argument requires strict descent; the order of the coefficient ideal in the ambient reduction to $\{x = 0\}$ decreases in the examples below, but that order is absent from the proposed invariant.

Results in every characteristic. The first obstruction occurs before any blowup. It is natural to expect the new negative-degree pieces to constrain an LL-chain. In fact the differential operators in their definition already produce a unit.

Theorem 1.1 (Negative degrees; Theorem 7.1)**.** *Let $\xi \in \operatorname{Sing}(E)$ be a closed point. Let $m > 0$ be divisible by the degrees of a finite generating system of $\wp(E)$, and suppose there is $g \in \wp(E, m)_\xi$ with $\operatorname{ord}_\xi g = m$, as in the preprint's condition (37). For every $a > 0$, $\wp^{\mathrm{nega}}(E, -a)_\xi = \mathcal{O}_\xi$, and $\tilde{\wp}(E, 0)_\xi = \mathcal{O}_\xi$.*

The proof of Theorem 7.1 applies a Hasse derivative of order equal to the order of a suitable power of the element supplied by (37). It works in every characteristic. Consequently, the condition in Hironaka's "master key" [Hir17, Thm. 9.2, p. 47] is vacuous for an LL-head in this setting (Corollary 7.3); its version for a permissible centre also uses the stated condition that the head parameter vanish along that centre (Corollary 7.3). The expectation that the negative-degree pieces transform by Eq. (59) likewise fails: Theorem 7.4 gives $\wp^{\mathrm{nega}}(E', -a)_{\xi'} = \mathcal{O}_{\xi'}$ while its claimed transform is the proper ideal (y^a) for E_n below.

For the second obstruction let K be perfect of characteristic p, and set $E_n = (x^p + y^n, p)$ on $\mathbb{A}^2_K$, where $n > 2p$ and $p \nmid n$. All calculations of $\wp$ and of the edge invariant here use the algebraic description of $\wp$ in [Hir17, Thm. 4.4, p. 18]. Hironaka attributes its asserted agreement with the geometric description to his earlier work [Hir17, p. 17]; the earlier paper was not checked here (Remark 6.6). With this scope, the edge invariant at the origin is $(2, 1, p)$, not $(2, 1, 1)$: $x \notin \wp(E_n, 1)$ (Proposition 6.3).

Theorem 1.2 (The local chain; Theorems 9.3, 10.2, and 10.3)**.** *Let $n > 2p$ with $p \nmid n$. Use the algebraic description of $\wp$, read core focusing by formula (44), and use the admissible local chain and representatives of Proposition 9.1 under Readings 4.5 and 4.11. Take the first stabilisation of its singular loci, and interpret $\operatorname{Inv}(\mathcal{Y}(i))$ as the invariant of $\check{E}(i)$ or of $\hat{E}(i)$. Then $\operatorname{Sing}\mathcal{Y}(s) = \{0\}$ and $\dim \mathcal{Y}(s) = 1$ for all s; the Step 2 power containment and strict dimension drop asserted in* [Hir17, Thm. 15.10, pp. 79–80] *fail. Blowing up the terminal plat $\{0\}$ leaves the transform E_{n-p} singular at the origin ξ' of the chart $x = x'y$, contrary to* [Hir17, Thm. 16.6(4), p. 84]*. The length-one invariant string is $((2, 1, p))$ both at 0 and at ξ'.*

For the chosen chain, these assertions hold for every permitted covering and cotangential generators (Proposition 9.1, Theorem 9.3); the claims about the terminal plat and invariant string retain the readings stated in Theorems 10.2 and 10.3. Under either core-focusing reading, renewed chains with corresponding transported choices give the same string at each point blown up in the resolution of E_n, in readings (I) and (II) (Corollary 10.7). This refutes strict descent for those strings; it does not address the other two readings of 10.1. The example itself resolves after

$\lfloor n/p \rfloor$ point blowups: the y-exponent drops by p (Proposition 6.1), and the coefficient-ideal order for the ambient reduction to $Y = \{x = 0\}$ drops by $p!$ (Proposition 6.8). Neither quantity is a component of the string used here. The proposed strings therefore cannot bound this resolution sequence as required by [Hir17, Thm. 16.13, p. 87].

The two printed prescriptions for core focusing do not agree. The following statements identify the obstruction independently of the family E_n.

Theorem 1.3 (Core focusing; Propositions 8.1 and 8.2)**.** *For every maximum base-hiked ideal exponent $\hat{E}$, formula (44) of* [Hir17, p. 30] *gives $\check{E} \sim \hat{E}$ and $\mathrm{Sing}(\check{E}) = \mathrm{Sing}(\hat{E})$. If its Inv-max stratum Σ is smooth and closed, formula (43) instead gives $\check{E} \sim (I(\Sigma), 1)$.*

Formula (44) retains the old singular locus because each term in its intersection already contains $S(\hat{E})$ (Proposition 8.1). Formula (43) takes the largest characteristic algebra allowed by the smooth stratum, namely the Rees algebra of $I(\Sigma)$ (Proposition 8.2). It can therefore forget the exponent that took several blowups to resolve; intersecting $\hat{E}$ with $(I(\Sigma), 1)$ does the same (Remark 8.3). Under the alternative reading (43), transformed members may improve, but renewed strings with corresponding choices still agree at 0 and ξ' (Proposition 10.5).

Characteristic-two limits and open questions. The characteristic-2 examples locate further failures without claiming the same conclusion in all characteristics. It is natural to expect the nonzero degree-one part of $\wp(\hat{E})$ to have order one at every singular point, as asserted in [Hir17, Thm. 6.1, p. 29]; Proposition 8.5 refutes its second sentence at the origin of a monomial-curve example. Under formula (44), the same example has singular locus lying in no smooth hypersurface germ at the origin, contrary to the claimed enclosure by a tail (Proposition 8.7 and [Hir17, Thms. 10.3–10.4, p. 60]). Under formula (43) this example does not settle enclosure; the corresponding assertion is Question 12.2. The preprint's proof gives a tail through each point but does not establish one tail through the whole singular locus (Remark 12.1).

Hauser's kangaroo example [Hau08, §G, p. 16] supplies a different limit: at an equiconstant point in characteristic 2, the shade rises from 2 to 3 (Example 11.2), so shade by itself cannot replace the failed descent measure. It is natural to expect the edge data to jump along its cusp; under the algebraic description of $\wp$, no element of $\wp(E, 1)$ has order one at any closed point of that cusp (Proposition 11.4). Whether the full invariant is constant there remains Question 11.5. The example does not compute the full invariant string at the kangaroo points (Remark 11.3).

Some limits concern the available text and the questions it leaves. It is natural to expect a proofread preprint to have complete proofs; in fact two proofs end incomplete (Observation 13.1). Observation 13.2 also locates the false value identity in H-Cor. 7.15, the reversed inclusion in the proof of H-Lem. 5.6, fragments in H-Lemmas 3.14 and 5.9(4), ill-typed H-Definitions 4.14 and 4.18, the mismatched types in H-Thm. 14.6, the power/iteration ambiguity in H-Remark 9.11, the definition of s in H-Lemma 16.5, and an unverified coefficient extraction in H-Theorems 8.6–8.7. These are located defects, with the last item recorded without verification; they do not replace the counterexamples above.

The corrected E_5 computation does not settle whether edge data are always determined by a tangent cone (Question 14.3); non-increase of the $\wp$-based edge invariant at exceptional points and smoothness of every terminal plat also remain Questions 14.2 and 14.1. In particular, neither repairability of the full method nor impossibility of every refinement follows from these calculations (Remark 14.6). The needed canonical focusing construction and a strictly decreasing invariant through kangaroo points remain Questions 14.4 and 14.5.

Organisation. Section 2 gives the Software and AI assistance disclosure. Section 3 fixes conventions and readings; Section 4 records the objects used from the preprint. Section 5 proves a valuative test, and Section 6 computes E_n. Section 7 examines negative degrees; Section 8

compares the two core-focusing formulas. Section 9 constructs the local chain, and Section 10 tests its invariant after blowup. Section 11 tests residual order and edge data at a kangaroo point; Section 12 examines containment in tail hypersurfaces. Section 13 locates textual defects; Section 14 states open questions; Section 15 records local repairs and surviving foundational parts. Appendix A records token accounting.

2. Software and AI assistance

2.1. **Methods: typesetting.** This article was assembled as LaTeX source, a bibliography and a PDF by Claude Code agents on 23–24 October 2026. A Codex desktop agent revised the source and rebuilt the PDF on 26 October 2026. The source map, staged reviews, and computation outputs were retained with the project. Compilation checks typesetting and cross-references; it does not verify an argument.

An OpenAI Codex agent (GPT-6 family; exact runtime model identifier unavailable) in Codex desktop revised the title, date, and opening abstract paragraph on 27 October 2026 [Ope26].

2.2. **Methods: agent management.** During Taylor Dupuy's University of Rochester Number Theory Seminar talk *How to Use AI in Mathematical Research Effectively* on 23 October 2026, a prompt initiated the investigation of [Hir17]. The `frontier-dump` workflow [Dup26a, skill source] produced an adversarial research record with a literature-search subagent. A follow-up request to explain the counterexamples used `harvest-dumps` [Dup26a, skill source] and `using-latexpowers` [Dup26b, skill source] to make this manuscript. Claude Code [Ant26, product page] ran the original work: `claude-fable-5-1` for the research record and `claude-opus-5-5` for the manuscript and its AI reviewers. On 26 October a Codex desktop agent [Ope26, product page] based on GPT-6 revised the manuscript; its exact serving model identifier was not exposed to this task. No repository-specific `AI-POLICY.md` was in force for this project.

2.3. **Methods: record keeping.** The project retains an internal source-to-claim map, frozen manuscript versions, agent review reports, computation scripts and outputs, and AI usage and token records. The research record is cited as [Cla26a]; the computational checks are cited as [Cla26b]. Those repository materials may require access permission. The material facts about methods, findings and limits are stated here. The available token accounting is summarized in Appendix A.

2.4. **Methods: natural language proofs.** The `frontier-dump` record identified possible counterexamples and defects in [Hir17]; it was an AI research record, not a source of mathematical authority. The `harvest-dumps` agent read the relevant passages of [Hir17] and [Hau08], reconstructed the arguments in this article, and generalized the main family to $E_n = (x^p + y^n, p)$ under the hypotheses stated in the body. The algebraic description of $\wp$ and the chosen readings of ambiguous preprint definitions remain explicit dependencies (Remark 6.6 and Readings 4.5–10.1). The 26 October Codex pass repaired the induction in Proposition 9.1, narrowed Theorem 9.3 to an admissible representative chain, and narrowed Remark 11.3. These are AI-drafted mathematical changes, not human verification. The abstract and introduction were drafted on 26 October by a fresh Codex agent from a frozen copy of the settled body using `write-introduction` [Dup26b, skill source]. The earlier introduction was a placeholder; no earlier prose was used as the new draft's input.

2.5. **Methods: adversarial reviews.** Separate Claude Code reviewer agents, each reporting `claude-opus-5-5`, reviewed the outline, filled presentation and integrated manuscript. The outline was accepted in round 3; the presentation in round 4 before restructuring and round 2 after it; and the integrated version in round 3. Those verdicts apply to the frozen versions

reviewed, not automatically to the 26 October changes. A separate Codex agent reviewed the affected proof chain on 26 October; its findings led to the repairs just described. All these reviews were by AI agents. A separate Codex agent reviewed the new introduction and abstract against the frozen body in three rounds. Its final verdict found no remaining issue in that bounded comparison. This was an AI review, not a human check of the mathematics.

2.6. **Methods: AI assisted computations.** The original research agent used SageMath 10.8 [The25, project page] and Python 3.13.7 [Pyt26, documentation] to check examples [Cla26a]. The manuscript agent ran three further SageMath 10.8 scripts [Cla26b]: an integral-closure check in characteristic 2, numerical checks for $p = 2, 3, 5, 7$, and checks of the kangaroo and Narasimhan examples. The computations corroborate specified finite cases; they do not prove the universal statements. The body supplies hand derivations for computations it uses as proof, except Example 8.4(c), which is used only in Remark 8.8.

2.7. **Methods: AI statement.** This Software and AI assistance section was drafted by a Claude Code agent using `claude-opus-5-5` on 23 October 2026 and revised by a Codex desktop agent based on GPT-6 on 26 October 2026 under the `using-latexpowers` disclosure workflow. Taylor Dupuy chose its placement after the introduction; that choice is not a mathematical or citation check. No human check of this section or of the mathematics has been recorded. Agents are not authors or mathematical warrants; responsibility for every argument, result and citation rests with the human who prints the manuscript. The 26 October revision used the `write-materials-and-methods` layout and disclosure procedure [Dup26b, skill source]. Its account is AI-drafted, and Taylor's placement choice did not constitute human review or acceptance of its content.

2.8. **Proofreading and adversarial review.** The AI review corrected the research record's claim that $x \in \wp((x^2+y^5, 2), 1)$ (Proposition 6.3), found a second tail $x+y^2$ for E_5 (Example 9.2), and separated the two readings of core focusing (Propositions 8.1 and 8.2). The 26 October check found that containment in (c) did not imply order p along $c = 0$; Theorem 9.3 now tests the power containment $(c)^p$ actually used in [Hir17, Thm. 15.10, Step 2, p. 80]. It also found that the former induction in Proposition 9.1 had not established its claim for every stage, and that a claim about appending shade to the full invariant was unsupported. The revised proposition gives an induction for an explicit admissible chain, and Remark 11.3 states the remaining limit.

2.9. **Literature review.** The manuscript agent opened the local copy of [Hir17] and read the passages on which the counterexamples depend. It opened [Hau08], version 2, for the kangaroo example and related context. The original Moh paper [Moh87, Stability Thm., p. 966] was opened and its bound checked. The Narasimhan papers [Nar83b, Nar83a] were not opened. [Hir03] is cited as [Hir17] cites it, and was not opened. The polynomial of Example 8.4 was not located in an opened Narasimhan source. Reception facts in the research record's bounded search were not reopened by the manuscript agent. No human source audit is recorded.

2.10. **AI findings and procedure.** The starting material was Hironaka's human-authored preprint [Hir17]. The 23 October `claude-fable-5-1` Claude Code `frontier-dump` produced thirteen negative findings [Cla26a, report §6]; the `claude-opus-5-5` Claude Code `harvest-dumps` checked, corrected and presented all thirteen below. Two inherited claims were corrected rather than repeated. The numbered destinations state the mathematical scope and proofs.

(1) The negative-degree algebra was expected to carry new information; it is the unit ideal at singular points (Theorem 7.1).
(2) Its claimed transformation rule fails for E_n under the algebraic description of $\wp$ (Theorem 7.4).

(3) The "master key" was expected to constrain tails; its condition is vacuous (Corollary 7.3).
(4) The proposed invariant was expected to drop after blowup; it remains constant along the specified chain under the stated readings (Theorem 10.3 and Corollary 10.7).
(5) Blowing up the terminal plat was expected to remove singular points over it; a transformed member remains singular (Theorem 10.2(a)).
(6) Stabilisation was expected to force the Step 2 divisibility and a strict dimension drop; the specified chain violates both (Theorem 9.3).
(7) The two definitions of core focusing were expected to agree; (44) returns the base-hike and (43) forgets it (Propositions 8.1 and 8.2).
(8) The second sentence of H-Theorem 6.1 is false in the characteristic-2 example (Proposition 8.5).
(9) Narasimhan's example does not directly settle the enclosure claim under (43), but under (44) gives the stated incompatibility (Proposition 8.7 and Question 12.2).
(10) The research record's proposed edge elements on the kangaroo cusp were false; the corrected edge statement is Proposition 11.4, while the shade increase is Example 11.2.
(11) The research record expected $x \in \wp(E_5, 1)$ and an invariant $(2, 1, 1)$; the valuative test gives $(2, 1, 2)$ instead (Proposition 6.3). The tangent-cone question remains open (Question 14.3).
(12) Repairable versus demonstrably irreparable was a false dichotomy; the surviving problem is Question 14.5 (Remark 14.6).
(13) Acknowledged proofreading was expected to imply a complete preprint text; two proofs end mid-sentence and other located defects remain (Observations 13.1 and 13.2).

All thirteen findings are carried; none is omitted. The 26 October Codex agent did not create an additional counterexample: it corrected the scope and proof of the chain results.

2.11. **Provenance.** For the failures of specific statements of [Hir17], the comparison is with the definitions and claims at the pinpoint locators attached to each body statement; the research record's bounded literature search found no published analysis of those precise steps [Cla26a, report §7]. This is not a claim of novelty or an exhaustive search. The characteristic-2 shade jump and Moh's bound are compared directly with [Hau08, pp. 4, 6–8, 16–17]; the present calculation states what is needed from that source. The alleged geometric/algebraic agreement for $\wp$ comes from [Hir17, Thm. 4.4, p. 18], which cites the unopened [Hir03]; Remark 6.6 records the resulting conditional scope. The primary Narasimhan texts have not been independently checked.

2.12. **Due diligence.** The related `frontier-dump` for this project is the 23 October 2026 adversarial review [Cla26a]; its thirteen negative findings were mapped into this article, with the corrections above. No other research dump is used as evidence for the mathematical claims here. The source map and AI review reports are retained internally; they are not substitutes for the arguments in the paper.

2.13. **Formalization.** No result in this paper was formalized in Lean or another proof assistant. The SageMath checks are finite computations, and the proof reviews were natural-language AI reviews. No human has checked the proofs, computations or citations.

3. Scope, conventions and readings

The object of study is the preprint [Hir17]. It defines a characteristic algebra $\wp(E)$ of an ideal exponent E, extends it by parts $\wp^{\text{nega}}$ of negative degree, builds from these "LL-chains" whose last members ("tails") are meant to replace hypersurfaces of maximal contact, and uses the tails to define a descending chain of ambient reductions ("CotAR-schemes") whose last singular locus, the "terminal plat" ∇, is the next centre of blowup. Main Theorem I [Hir17, Thm. 16.6, p. 84] asserts

that blowing up ∇ lowers an invariant, and Main Theorems II–III [Hir17, Thms. 16.13–16.14, p. 87] conclude resolution by repeating this. We show by explicit examples that the steps listed in Table 1 fail.

Conventions. K is a perfect field of characteristic $p > 0$, Z is a smooth irreducible K-scheme, written $\dim Z$ for its dimension (the preprint's n; we reserve n for the exponent in the family of §6), and $\mathcal{O} = \mathcal{O}_{Z,\xi}$ is the local ring at a closed point ξ, with maximal ideal $\mathfrak{m}$. A locator "[Hir17, p. N]" refers to the printed page N of [Hir17]; in the copy we used, printed and PDF page numbers agree. For a multi-index β we write $\partial^{(\beta)}$ for the Hasse derivative with respect to a regular system of parameters, so $\partial^{(\beta)} x^\gamma = \binom{\gamma}{\beta} x^{\gamma-\beta}$ [Hir17, Eq. (8), p. 9]. For $h \in \mathcal{O}$, $\operatorname{ord} h = \operatorname{ord}_\xi h$ is its order and $\operatorname{in} h$ its initial form in $\operatorname{gr}_{\mathfrak{m}} \mathcal{O}$. A theorem-type or section number with the prefix "H-", such as H-Thm. 9.2 or H-§10, refers to [Hir17]; unprefixed such numbers refer to this paper. Equation numbers between (35) and (129), such as (43), (44) and (59), are the preprint's; our own displays are numbered (1)–(3). Throughout, a refuted expectation is introduced in the form "it is natural to expect X; in fact Y".

Readings. Some definitions of [Hir17] do not typecheck as printed, and some admit more than one reading. We fix readings in Readings 4.5, 4.7, 4.11 and 10.1. Each counterexample below says which readings it uses, and where a result holds under every reading we say so.

Statement of [Hir17]	Status here	Where
$\wp^{\mathrm{nega}}$ (H-Def. 5.1, p. 25) carries information	$\wp^{\mathrm{nega}} = \mathcal{O}$ at singular points, any p	Thm. 7.1
Master key H-Thm. 9.2 (p. 47), H-Thm. 9.3	vacuous, any p	Cor. 7.3
H-Thm. 7.8, Eq. (59) (p. 37)	false for E_n, any p	Thm. 7.4
Core focusing (43), (44) (p. 30)	(44) returns $\hat{E}$; (43) returns $(I(\Sigma), 1)$; any p	Props. 8.1, 8.2
H-Thm. 6.1, second sentence (p. 29)	false (char. 2)	Prop. 8.5
H-Thm. 15.10(2) and Step 2 (pp. 79–80)	power containment fails for the stated chain; dimension does not drop under the first-stabilisation convention	Thm. 9.3
Main Thm. I, H-Thm. 16.6(2),(4); H-Cor. 16.9 (pp. 84–85)	false for E_n under (44), any p; under (43), (4) may hold but the renewed string does not drop	Thms. 10.2, 10.3, Rem. 10.6
Main Thm. II, H-Thm. 16.13 (p. 87): proposed descent	the stated invariant does not decrease for E_n under the readings fixed here	Cor. 10.7
Residual order alone as a descent measure	increases (kangaroo, char. 2)	Ex. 11.2
H-§10: $\operatorname{Sing}(\check{E}) \subset$ tail (H-Thms. 10.3–10.6)	false under (44) (char. 2); open under (43)	Prop. 8.7

TABLE 1. Summary of the statements of [Hir17] examined here.

4. The objects of the preprint

We state the definitions of [Hir17] that we use, in its notation, with locators. Where we simplify, we say so.

4.1. **Ideal exponents and infinitely near singularities.**

Definition 4.1 ([Hir17, §2.1, p. 4; Def. 2.1, p. 5; Defs. 2.4–2.5, p. 6])**.**

(a) An *ideal exponent* on Z is a pair $E = (J, b)$ of a nonzero coherent ideal $J \subset \mathcal{O}_Z$ and an integer $b > 0$.
(b) Its *singular locus* is $\operatorname{Sing}(E) = \{\xi \in Z : \operatorname{ord}_\xi J \geq b\}$.
(c) A blowup $\pi\colon Z' \to Z$ is *permissible* for E if its centre D is a closed smooth irreducible subscheme of $\operatorname{Sing}(E)$.
(d) The *transform* of E by a permissible π is $E' = (J', b)$, where $J\mathcal{O}_{Z'} = I(D)^b J'$ and $I(D)$ is the ideal of D.
(e) An *LSB* over Z is a finite sequence of blowups $Z_r \to U_{r-1} \subset Z_{r-1} \to \cdots \to U_0 \subset Z_0 = Z$, each U_i open and each centre $D_i \subset U_i$ smooth irreducible closed.
(f) $S(E)$ is the collection, over all finite systems t of indeterminates, of the LSBs over $Z[t] = Z \times_K \operatorname{Spec} K[t]$ that are permissible (at every step) for $E[t] = (J\mathcal{O}_{Z[t]}, b)$.

The preprint writes $E_1 \subset E_2$ for $S(E_1) \subset S(E_2)$ [Hir17, §4.3.1, p. 24], and asserts

$$S(E_1) = S(E_2) \iff \wp(E_1) = \wp(E_2) \qquad \text{[Hir17, Eq. (35), p. 25]}. \tag{1}$$

We write $E_1 \sim E_2$ when $S(E_1) = S(E_2)$.

Definition 4.2 ([Hir17, §4.3.1, p. 24; Eq. (13), p. 11])**.**

(a) The *intersection* of $E_1 = (J_1, b_1)$ and $E_2 = (J_2, b_2)$ is $E_1 \cap E_2 = (J_1^{b_2} + J_2^{b_1}, b_1 b_2)$.
(b) For a smooth closed $Y \subset Z$ with ideal $I(Y)$, the ideal exponent $E\{Y\}$ is characterised by $S(E\{Y\}) = S(E) \cap S(I(Y), 1)$.

We use $S(E_1 \cap E_2) = S(E_1) \cap S(E_2)$; this holds because $\operatorname{ord}(J_1^{b_2} + J_2^{b_1}) \geq b_1 b_2$ exactly when $\operatorname{ord} J_1 \geq b_1$ and $\operatorname{ord} J_2 \geq b_2$, at every point of every transform. In particular $E\{Y\} \sim E \cap (I(Y), 1)$.

4.2. **Differential operators.** $\operatorname{Diff}_Z^{(m)}$ is the $\mathcal{O}_Z$-module of differential operators of order $\leq m$ [Hir17, §3.1.1, p. 7]. At ξ it is free on the Hasse derivatives $\partial^{(\beta)}$ with $|\beta| \leq m$ [Hir17, Rem. 3.5, p. 9]. For an ideal J, $\operatorname{Diff}^{(m)} J$ is the ideal generated by the $\partial(h)$ with $\partial \in \operatorname{Diff}^{(m)}$ and $h \in J$; by the Leibniz rule it is generated by the $\partial^{(\beta)} h$ with $|\beta| \leq m$ and h running over generators of J. An operator of order $\leq m$ lowers orders by at most m.

4.3. **The characteristic algebra.**

Definition 4.3 ([Hir17, p. 17; Thm. 4.4, p. 18])**.** Let $E = (J, b)$ and let $\operatorname{Bl}(Z)^* = \bigoplus_{d \geq 0} \mathcal{O}_Z T^d$.

(a) (*Geometric definition.*) $\wp(E, a)$, $a > 0$, is the union of the ideals I with $S(I, a) \supset S(E)$.
(b) (*Algebraic definition.*) $\wp(E) = \bigoplus_d \wp(E, d) T^d$ is the integral closure in $\operatorname{Bl}(Z)^*$ of the $\mathcal{O}_Z$-subalgebra $A(E)$ generated by the ideals $(\operatorname{Diff}^{(j)} J)\, T^{b-j}$, $0 \leq j < b$.

The preprint states that the two definitions agree when K is perfect, citing [Hir03] [Hir17, p. 17]; we have not checked this. All our computations of $\wp$ use the algebraic definition (b), which the preprint restates as H-Theorem 4.4. The geometric definition is used only in Proposition 8.1, where the preprint's own formula (44) is phrased in terms of S. We also use two statements of [Hir17]: $\operatorname{Diff}^{(\mu)} \wp(E, a) \subset \wp(E, a - \mu)$ for $0 \leq \mu < a$ [Hir17, Thm. 4.1(2), p. 17], and

$$\wp(E, a) \subset \wp(E, c) \quad \text{for integers } a > c \geq 0 \qquad \text{[Hir17, Thm. 4.5, p. 18]}. \tag{2}$$

With Definition 4.3(b), H-Theorem 4.5 follows because the degree-e part of $A(E)$ lies in its degree-$(e-1)$ part: we have $\operatorname{Diff}^{(j)} J \subset \operatorname{Diff}^{(j+1)} J$, and $\operatorname{Diff}^{(j+1)} J$ sits in degree $b - j - 1$. So an integral equation for hT^a over $A(E)$ is also one for hT^{a-1}. (With Definition 4.3(a) it follows from $I'_c = I(D)^{a-c} I'_a$ for the transforms of (I, c) and (I, a).)

4.4. **Edge data and the invariant.**

Definition 4.4 ([Hir17, Eq. (21), p. 18; items (6)–(8), p. 19; Def. 4.9, p. 20; Eq. (34), p. 24])**.** Let $\xi \in \mathrm{Sing}(E)$ be a closed point and choose a regular system of parameters z of $\mathcal{O}$, with images $\bar{z}$ in $\mathrm{gr}_{\mathfrak{m}} \mathcal{O} \cong \kappa(\xi)[\bar{z}]$.

(a) The *edge algebra* $\bar{\wp}(E)(\xi)$ is the image of $\wp(E)_\xi$ in $\kappa(\xi)[\bar{z}]$ under $hT^d \mapsto$ (class of h in $\mathfrak{m}^d/\mathfrak{m}^{d+1}$). Its degree-$d$ part consists of the initial forms of the elements of $\wp(E,d)_\xi$ of order exactly d, together with 0.

(b) The edge algebra generates a homogeneous ideal G of $\kappa(\xi)[\bar{z}]$. Its minimal homogeneous generators $\bar{g}_1, \dots, \bar{g}_r$, chosen from the lowest degree up, lie in $\rho^{e_j}(\kappa(\xi)[\bar{z}])$ and have degrees $q_j = p^{e_j}$ [Hir17, item (8), p. 19]. Since $\kappa(\xi)$ is perfect, each $\bar{g}_j$ is the p^{e_j}-th power of a linear form. Homogeneous lifts $g_j \in \wp(E, q_j)$ are the *edge data* of E at ξ.

(c) The *edge invariant* is $\mathrm{Inv}_\xi(E) = (\dim Z,\ \dim Z - r,\ q_1, \dots, q_r)$, ordered lexicographically. E is *edge-trivial* at ξ if $r = 0$, and then $\mathrm{Inv}_\xi(E) = (\dim Z, \dim Z)$.

The preprint gives the edge algebra both as a graded algebra and as an ideal of $\kappa(\xi)[\bar{z}]$ [Hir17, pp. 18–20]. We use only the minimal generators of the ideal G that it generates, which is how r and the q_j are defined [Hir17, items (6)–(7), p. 19]. Every element of $\wp(E,d)_\xi$ has order at least d (Lemma 5.2 below).

Reading 4.5 (cotangent module)**.** As printed [Hir17, Def. 4.14, Eq. (29), p. 22], $\mathrm{cot}(E,\ell)_\xi$ is the set of p^ℓ-th roots of elements of $\rho^\ell(\mathrm{Bl}(Z,1)) \cap \wp(E,p^\ell) + \mathfrak{m}\,\mathrm{Bl}(Z,p^\ell)$, where ρ is Frobenius. Since $\mathrm{Bl}(Z,p^\ell) = \mathcal{O}$ as a module [Hir17, Def. 2.2, p. 5], this makes $\mathrm{cot}(E)_\xi = \mathfrak{m}$. The global version [Hir17, Def. 4.18, Eq. (31), p. 23] has the same defect, with $I(\mathrm{Sing}\,E)$ in place of $\mathfrak{m}$. We read $\mathfrak{m}^{p^\ell+1}$ for $\mathfrak{m}\,\mathrm{Bl}(Z,p^\ell)$ and $I(\mathrm{Sing}\,E)^{(p^\ell+1)}$ for $I(\mathrm{Sing}\,E)\,\mathrm{Bl}(Z,p^\ell)$. Then $y \in \mathrm{cot}(E)_\xi$ with $\mathrm{ord}\,y = 1$ means $\bar{y}^{p^\ell} \in G$ for $\ell \gg 0$. The *cotangent vector space* $v(E)_\xi = \mathrm{cot}(E)_\xi \bmod \mathfrak{m}^2$ [Hir17, Def. 4.15, p. 22] is then spanned by the $\bar{y}$ with $\bar{y}^{p^\ell} \in G$.

4.5. **Base-hike and core focusing.**

Definition 4.6 ([Hir17, §6.1, p. 29; Eqs. (43)–(44), p. 30])**.** Let $E = (J, b)$.

(a) The *maximum base-hike* is $\hat{E} = (J, d)$, where d is the maximum of $\mathrm{ord}_\xi J$ over the closed points $\xi \in \mathrm{Sing}(E)$.

(b) Let $\mathrm{Inv}_{\max}$ be the lexicographic maximum of $\mathrm{Inv}_\xi(\hat{E})$ over the closed points of $\mathrm{Sing}(\hat{E})$, and let Σ be the set of closed points where it is attained (the *Inv-max stratum*).

(c) (Eq. (43).) The *core-edge focusing* $\check{E}$ is an ideal exponent with $\wp(\check{E}) \supset \wp(\hat{E})$ and $\mathrm{Sing}(\check{E}) = \Sigma$ whose $\wp(\check{E})$ is maximal under these conditions.

(d) (Eq. (44).) Alternatively, with $\mathrm{Inv}_{\max} = (\dim Z, \dim Z - r, q_1, \dots, q_r)$, "pick any $\check{E}$ such that" $S(\check{E}) = \bigcap_{i \le r} S(\wp(\hat{E}, q_i), q_i) \cap S(\hat{E})$.

Reading 4.7 (core focusing)**.** The preprint presents (c) and (d) as two definitions of the same object. They are not (Propositions 8.1 and 8.2). Unless we say otherwise we read $\check{E}$ by (d); for the examples of §§9–10 this reading also satisfies the condition $\mathrm{Sing}(\check{E}) = \Sigma$ of (c). We treat (c) separately in Proposition 8.2 and Proposition 10.5.

4.6. **The negative-degree part.**

Definition 4.8 ([Hir17, Def. 5.1, Eq. (36), p. 25; Rem. 5.5, Eq. (37), p. 26])**.** Let E be an ideal exponent and let $m > 0$ be divisible by the degrees of a finite system of generators of $\wp(E)$. For $a \in \mathbb{Z}$ put

$$\tilde{\wp}(E, -a) = \sum_{d \in \mathbb{Z},\ dm \ge |a|} \mathrm{Diff}^{(dm+a)}\ \wp^{\mathrm{posi}}(E, dm), \qquad \wp^{\mathrm{posi}}(E,c) = \wp(E,c)\ (c > 0),$$

and $\wp^{\mathrm{nega}}(E,-a)=\tilde{\wp}(E,-a)$ for $a>0$. The preprint assumes, "from now on throughout this section" and in its later uses, that at every closed point $\xi\in\mathrm{Sing}(E)$

$$\text{there is } g\in\wp(E,m)_\xi \text{ with } \operatorname{ord}_\xi g=m. \tag{37}$$

It states that (37) holds for $E=\hat{E}$ and $E=\check{E}$ [Hir17, p. 26; Rem. 7.7, p. 36].

The value of $\wp^{\mathrm{nega}}(E,-a)$ does not depend on the choice of m [Hir17, p. 28]. We also record the preprint's criterion [Hir17, Thm. 6.3, p. 30]: if $g\in\wp(E,q)_\xi$ satisfies (37) with $m=q$, then $0\neq f\in\mathcal{O}_\xi$ lies in $\wp^{\mathrm{nega}}(E,-a)_\xi$ if and only if $g^{p^\ell}f\in\wp(E,qp^\ell-a)$ for all $\ell\gg 1$.

4.7. **LL-chains and tails.** We need only the following special case. An *LL-head* of order $q=p^e$, $e>0$, at $\xi\in\mathrm{Sing}(\check{E})$ is an element $g=y^q+\varepsilon\in\wp(\check{E},q)$ with $\operatorname{ord} y=1$ and $\operatorname{ord}\varepsilon>q$ [Hir17, Def. 6.12, Eq. (52), p. 33]. An *LL-chain* headed by g is a sequence $g=g(0),g(1),\dots,g(e)$ with

$$g(i)=g(i+1)^p+h(i),\qquad \operatorname{ord} g(i)=p^{e-i},\qquad \operatorname{ord} h(i)>p^{e-i},$$

where each $h(i)$ is a "differentiation-product" of $\varepsilon(i)$ taken from a set $\mathcal{E}(g(i),p^{e-i})\subset\wp^{\mathrm{nega}}(\check{E})$, and $\operatorname{ord} g(e)=1$ [Hir17, Eqs. (83)–(84), pp. 54–55]. The *tail* is $g(e)$, and the *tail hypersurface* is $\{g(e)=0\}$ [Hir17, Defs. 9.20–9.21, p. 56]. The operators producing $h(i)$ are listed case by case in [Hir17, Lem. 9.6, Rems. 9.9–9.11, pp. 50–51]; for instance, in "case (I)" of [Hir17, Rem. 9.9, p. 51] a monomial $\varepsilon=u_0x^{\alpha+p\beta+q\gamma_0}$ with $|q\gamma_0|\geq 2q$ gives $H^\flat(\varepsilon)=u_0^{-1}\partial^{(\alpha+p\beta)}(\varepsilon)\,\partial^{(q\gamma_0)}(\varepsilon)$.

4.8. **Ambient reduction, CotAR-schemes and the terminal plat.**

Definition 4.9 ([Hir17, Thm. 3.10, p. 10])**.** For $E=(J,b)$ put $b^\sharp=b!$ and

$$J^\sharp=\sum_{j=0}^{b-1}(\mathrm{Diff}^{(j)}\,J)^{b^\sharp/(b-j)}.$$

For a smooth $W\subset Z$ the *ambient reduction* of E to W is $F=E(W)=(J^\sharp\mathcal{O}_W,b^\sharp)$. By [Hir17, Thm. 3.10] an LSB with all centres in the strict transforms of $W[t]$ lies in $S(E)$ if and only if the induced LSB of $W[t]$ lies in $S(F)$.

Definition 4.10 ([Hir17, Defs. 15.1, 15.5, 15.7, 15.8, 15.12, pp. 74–80])**.** Let $E=E(0)$.

(a) An *AR-scheme* $\mathcal{Y}=\{(V_j,F_j,Y_{ji})\}$ consists of open sets V_j, ideal exponents F_j on V_j, and smooth closed $Y_{ji}\subset V_j$ of a common dimension, with ambient reductions $F_j(Y_{ji})$. Its singular locus is $\mathrm{Sing}(\mathcal{Y})=\bigcap_{j,i}\mathrm{Sing}(F_j(Y_{ji}))$ (Eq. (124), p. 78).

(b) The *CotAR-scheme* $\mathcal{Y}(s)=\mathcal{Y}(E(s))$ takes $F_j=\check{E}(s)|_{V_j}$ and $Y_{ji}=\{c(ji)=0\}$ for a finite system of generators $c(ji)$ of $\mathrm{Cot}(\check{E}(s))$ on V_j inducing a basis of the cotangent space. Here $\mathrm{Cot}(\check{E})$ is the module generated by the tails of the LL-chains of $\check{E}$ [Hir17, Def. 13.2, p. 67; Cor. 14.3, p. 69]. Its dimension is $\dim\mathcal{Y}(s)=\dim Z-\operatorname{rank} v(\check{E}(s))$.

(c) $E(s+1)$ is defined by $S(E(s+1))=S(\check{E}(s))\cap S(\mathrm{Cot}(\check{E}(s)),1)$, and then $E(s+1)\to\hat{E}(s+1)\to\check{E}(s+1)\to\mathcal{Y}(s+1)$. This is the chain $\mathcal{Y}(0)\rightsquigarrow\mathcal{Y}(1)\rightsquigarrow\cdots$ of Eq. (122).

(d) If t is the index with $\mathrm{Sing}\,\mathcal{Y}(t-1)\supsetneq\mathrm{Sing}\,\mathcal{Y}(t)=\mathrm{Sing}\,\mathcal{Y}(t+k)$ for all $k\geq 1$, the *terminal plat* is $\nabla=\mathrm{Sing}\,\mathcal{Y}(t)$ [Hir17, Thm. 15.10, Def. 15.12, p. 80].

The preprint lets one choose the covering and the cotangential generators $c(ji)$ ("choose any", [Hir17, Def. 15.8(3), p. 78]). The module $\mathrm{Cot}(\check{E})$ itself is determined by $\check{E}$ [Hir17, Def. 13.2, p. 67; Thm. 9.29, p. 59], but its definition passes through the GLUED operators of H-§§11–13, which we do not reproduce. We use only the following two properties.

Reading 4.11 (cotangent generators)**.** $\mathrm{Cot}(\check{E})_\xi$ contains the tail of every LL-chain of $\check{E}$ at ξ. This is our reading of [Hir17, Eq. (106), p. 67], together with [Hir17, Eq. (108), p. 69], which places tails in $T(\check{E})$, and [Hir17, Thm. 9.29, p. 59]. The class mod $\mathfrak{m}^2$ of every element of $\mathrm{Cot}(\check{E})_\xi$ lies in $v(\check{E})_\xi$ [Hir17, Cor. 14.3, Eq. (109)]. Hence, if $v(\check{E})_\xi = K\bar{c}$, the ideal generated by $\mathrm{Cot}(\check{E})_\xi$ lies in $(c) + \mathfrak{m}^2$.

The chain counterexamples use the representatives fixed in Proposition 9.1; within that chain they hold for every admissible covering and choice of cotangential generators.

5. Two tools

The following valuative test is standard for integral closures; we state it for $\wp$.

Lemma 5.1 (valuative test)**.** *Let $E = (J, b)$, let $\zeta \in Z$, and let $v\colon \mathcal{O}_{Z,\zeta} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ satisfy $v(1) = 0$, $v(gh) = v(g) + v(h)$ and $v(g + h) \geq \min(v(g), v(h))$. Put*

$$\mu_v = \min_{0 \leq j < b} \frac{v(\mathrm{Diff}^{(j)} J)}{b - j}, \qquad v(I) = \min_{h \in I} v(h).$$

Then $v(h) \geq d\mu_v$ for every $d \geq 0$ and every $h \in \wp(E, d)_\zeta$.

Proof. The degree-e part A_e of the algebra $A(E)$ of Definition 4.3(b) is the $\mathcal{O}$-span of products $h_1 \cdots h_k$ with $h_i \in \mathrm{Diff}^{(j_i)} J$ and $\sum(b - j_i) = e$, so $v(A_e) \geq e\mu_v$. Let hT^d be integral over $A(E)$. Comparing the coefficients of T^{sd} in an integral equation, we get $h^s + \sum_{i=1}^{s} c_i h^{s-i} = 0$ with $c_i \in A_{di}$. If $v(h) < d\mu_v$, then for $i \geq 1$ $v(c_i h^{s-i}) \geq di\mu_v + (s - i)v(h) > s\, v(h) = v(h^s)$, which is impossible. Integral closure commutes with localisation, so this covers $\wp(E, d)_\zeta$. □

We apply Lemma 5.1 to three kinds of v:

(i) the order ord_η at a point η;

(ii) a *monomial valuation* on $\hat{\mathcal{O}}_{Z,\zeta} = \kappa[[z_1, \ldots, z_{\dim Z}]]$, where ζ is a closed point and κ the algebraic closure of K in the completion [Hir17, Def. 3.3, p. 8], $v(\sum c_\alpha z^\alpha) = \min\{\langle w, \alpha\rangle : c_\alpha \neq 0\}$ for a weight vector $w > 0$;

(iii) an *arc* $v = \mathrm{ord}_s \circ \varphi$ for a local homomorphism $\varphi\colon \hat{\mathcal{O}} \to \overline{K}[[s]]$.

Lemma 5.2. *For every ideal exponent E, every $d \geq 0$ and every point $\eta \in \mathrm{Sing}(E)$, every $h \in \wp(E, d)_\eta$ has $\mathrm{ord}_\eta h \geq d$. If $\Sigma \subset \mathrm{Sing}(E)$ is smooth, then $\wp(E, d) \subset I(\Sigma)^d$ near Σ.*

Proof. Take $v = \mathrm{ord}_\eta$ in Lemma 5.1. Since $\mathrm{ord}_\eta J \geq b$ and operators of order $\leq j$ lower orders by at most j, we get $\mu_v \geq 1$. For smooth Σ, an element of order $\geq d$ at every point of Σ lies in $I(\Sigma)^d$, because locally $I(\Sigma)$ is generated by part of a regular system of parameters. □

6. The basic example

Throughout this section p is a prime, K is perfect of characteristic p, $Z = \mathbb{A}^2_K = \mathrm{Spec}\, K[x, y]$ and $\xi = 0$. For $n \geq 1$ with $p \nmid n$ put

$$f_n = x^p + y^n, \qquad E_n = (f_n, p).$$

The main example of this paper is E_n with $n > 2p$. Its smallest instance is $E_5 = (x^2 + y^5, 2)$ in characteristic 2, which we carry along as a worked example. Note that $p \nmid n$ excludes $n = 2p$.

For $j < p$ we have $\partial_x^{(i)} x^p = \binom{p}{i} x^{p-i} = 0$ when $0 < i < p$, and $\partial_x^{(p)} x^p = 1$. Hence for $j < p$

$$\mathrm{Diff}^{(j)}(f_n) = \big(f_n,\ y^{n-i} : 1 \leq i \leq j,\ \tbinom{n}{i} \not\equiv 0 \bmod p\big), \tag{3}$$

and $\binom{n}{1} = n \not\equiv 0$.

Proposition 6.1. *Let $n > p$ with $p \nmid n$.*

(a) $\mathrm{Sing}(E_n) = \{0\}$ *and* $\hat{E}_n = E_n$.
(b) *Let* π *be the blowup of* 0. *In the chart* $x = x'y$ *the transform is* $(x'^p + y^{n-p}, p)$, *that is,* E_{n-p} *at the origin* ξ' *of the chart. In the chart* $y = y'x$ *it is* $(1 + x^{n-p}y'^n, p)$, *which has no singular point. On the first chart no point other than* ξ' *has order* $\geq p$, *and* ξ' *is singular if and only if* $n > 2p$.
(c) E_n *is resolved by* $\lfloor n/p \rfloor$ *successive blowups of the unique singular point. The points blown up carry* $E_n, E_{n-p}, \ldots, E_{n-(\lfloor n/p \rfloor - 1)p}$, *all with exponent* $> p$. *The exponent of the* y*-term drops by* p *at each step, from* n *to* $n \bmod p$. *Equivalently, the order of the coefficient ideal (Proposition 6.8) drops by* $p!$ *at each step.*

This assertion is germwise at 0. In Definition 4.10, use these representatives on a neighborhood of 0; complementary opens have empty singular locus for these exponents.

Proof. (a) At a point of order $\geq p$ the derivative $\partial_y f_n = n y^{n-1}$ vanishes, so $y = 0$ and then $x = 0$. The order is at most p everywhere, because $\partial_x^{(p)} f_n = 1$. (b) We have $f_n(x'y, y) = y^p(x'^p + y^{n-p})$ and $f_n(x, y'x) = x^p(1 + x^{n-p}y'^n)$. Away from the exceptional divisor π is an isomorphism, so by (a) there are no singular points there. At a point $(c, 0)$ of the first chart with $c \neq 0$, the transform takes the value $c^p \neq 0$. At ξ' the order is $\min(p, n-p)$. (c) follows from (b) by induction, since $p \nmid n - jp$. □

Example 6.2. For $p = 2$ and $n = 5$: $f_5(x'y, y) = y^2(x'^2 + y^3)$, so the blowup of 0 gives $E_3 = (x'^2 + y^3, 2)$ at ξ'. A second blowup gives $(x''^2 + y, 2)$, which is resolved. The exponent of the y-term goes $5 \to 3 \to 1$, in $\lfloor 5/2 \rfloor = 2$ blowups.

It is natural to expect, as in characteristic zero, that $\{x = 0\}$ has maximal contact with E_n. By Proposition 6.1, every point of order p of every transform under point blowups lies on the strict transform of $\{x = 0\}$. So one expects $x \in \wp(E_n, 1)$ and $\mathrm{Inv}_0(E_n) = (2, 1, 1)$. In fact $x \notin \wp(E_n, 1)$.

Proposition 6.3. *Let* $n > p$ *with* $p \nmid n$. *The ideal* G *generated by the edge algebra of* E_n *at* 0 *is* $(\bar{x}^p)$, *and* $\mathrm{Inv}_0(E_n) = (2, 1, p)$. *In particular, for* $1 \leq d < p$ *no element of* $\wp(E_n, d)$ *has order* d, *and* $x \notin \wp(E_n, 1)$.

Proof. Step 1: initial forms. Let v be the monomial valuation with $v(x) = n$ and $v(y) = p$. By (3), the generator f_n in degree p gives the ratio $v(f_n)/p = n$. A generator y^{n-i} in degree $p - j$ $(i \leq j)$ gives $p(n-i)/(p-j) \geq p(n-j)/(p-j) \geq n$, because $n \geq p$. So $\mu_v = n$, and by Lemma 5.1 every $h \in \wp(E_n, d)$ has $v(h) \geq dn$. If $\operatorname{ord} h = d$, each monomial $x^a y^{d-a}$ of $\operatorname{in} h$ has v-value $dn - (n-p)(d-a)$, which is $< dn$ unless $a = d$. Hence $\operatorname{in} h = c\,\bar{x}^d$.

Step 2: no element of order $d < p$. Let $N \gg n$ and let $\varphi\colon K[[x, y]] \to K[[s]]$ be the arc $x \mapsto -s^n + s^N$, $y \mapsto s^p$, with $w = \mathrm{ord}_s \circ \varphi$. Then $\varphi(f_n) = -s^{np} + s^{Np} + s^{np} = s^{Np}$. The ratios of the generators y^{n-i} become $p(n-i)/(p-j) \geq p(n-1)/(p-1)$, with equality for $i = j = 1$, so $\mu_w = \min(N, p(n-1)/(p-1))$. Suppose $h \in \wp(E_n, d)$, $1 \leq d < p$, has order d. By Step 1, $\operatorname{in} h = c\,x^d$ with $c \neq 0$. A monomial $x^a y^b$ has leading exponent $na + pb$ in φ. Since $p \nmid n$ and $d < p$, the value nd is attained only by $(a, b) = (d, 0)$, so the coefficient of s^{nd} in $\varphi(h)$ is $\pm c$. Hence $w(h) \leq nd < d\,p(n-1)/(p-1) \leq d\mu_w$, where the middle inequality holds because $n > p$. This contradicts Lemma 5.1.

Step 3. By Steps 1–2, the degree-d part of the edge algebra lies in $K\bar{x}^d$ and vanishes for $d < p$. It contains $\bar{x}^p = \operatorname{in} f_n$. Hence $G = (\bar{x}^p)$, $r = 1$ and $q_1 = p$. □

Example 6.4. For E_5 in characteristic 2 the proof reads as follows. The generators are f_5 in degree 2 and $\mathrm{Diff}^{(1)}(f_5) = (f_5, y^4)$ in degree 1. The monomial valuation $v(x) = 5$, $v(y) = 2$ has $\mu_v = 5$, so initial forms of order-d elements are $c\,\bar{x}^d$. The arc $x = s^5 + s^N$, $y = s^2$ has $\varphi(f_5) = s^{2N}$ and $\mu_w = 8$. An element $cx + h_2$ of $\wp(E_5, 1)$ would have $w \leq 5 < 8$. So $G = (\bar{x}^2)$ and $\mathrm{Inv}_0(E_5) = (2, 1, 2)$.

Remark 6.5. In characteristic 2 a SageMath computation of integral closures confirms Proposition 6.3 for $n = 3, 5, 7$ [Cla26b]. The degree-2 Veronese subalgebra of $A(E_n)$ is the Rees algebra of $I = (f_n, y^{2n-2})$, so $x \in \wp(E_n, 1)$ if and only if $x^2 \in \bar{I}$. The integral closure is $(f_n,\ y^{2n-2},\ xy^{3(n-1)/2})$ for these n, and it contains neither x^2 nor y^n. For $p > 2$ we did not compute integral closures; the proof above does not use them. The numerical inputs of the proof (the singular loci, the blowup charts, μ_v and μ_w) were checked in SageMath for $p = 2, 3, 5, 7$ and four values of n each [Cla26b]. If the two definitions of $\wp$ agree, as [Hir17, p. 17] asserts, then $S(x, 1) \not\supset S(E_n)$: some LSB over some $Z[t]$ is permissible for E_n but not for $(x, 1)$. We do not exhibit one. If so, the point blowups over Z alone do not detect $\wp$.

Remark 6.6 (dependence on the algebraic description). Proposition 6.3 uses the algebraic description of $\wp$, which is H-Theorem 4.4, imported there from [Hir03]. Strictly, what is proved is a disjunction: either $\mathrm{Inv}_0(E_n) = (2, 1, p)$, or H-Theorem 4.4 fails for E_n. The counterexamples of §§7–10 are proved under the algebraic description. Their essential input is that $\mathrm{Inv}_0(E_n)$ is the same for all $n > p$ with $p \nmid n$ (Proposition 6.3). Theorem 10.3(b) also uses Proposition 6.7. If x lay in the geometrically defined $\wp(E_n, 1)$ for *every* such n, then $\mathrm{Inv}_0(E_n) = (2, 1, 1)$ for all n, and the comparison of Theorem 10.3(a) would go through unchanged. If it held for some n only, the invariant could change along the resolution of E_n, and we make no claim in that case. A witness LSB, if one exists, might use the "smooth (or only regular)" centres that [Hir17, Def. 2.4, p. 6] allows over localisations of $Z[t]$ with imperfect residue fields.

Proposition 6.7. *Let $n > p$ with $p \nmid n$, and let I be an ideal with $x \in I \subset (x, y^2)$. Put $E_n^I = E_n \cap (I, 1) = (f_n + I^p, p)$. Then $\mathrm{Sing}(E_n^I) = \{0\}$, $\widehat{E_n^I} = E_n^I$, $x \in \wp(E_n^I, 1)$, the ideal generated by the edge algebra at 0 is $(\bar{x})$, and $\mathrm{Inv}_0(E_n^I) = (2, 1, 1)$.*

Proof. $\mathrm{Sing}(E_n^I) \subset \mathrm{Sing}(E_n) = \{0\}$ by Proposition 6.1(a). Since $I \subset \mathfrak{m}$ and $x^p \in J = f_n + I^p$, the ideal J has order exactly p at 0. So $\mathrm{Sing}(E_n^I) = \{0\}$ and $\widehat{E_n^I} = E_n^I$. Since $x^p \in J$, $(xT)^p \in A(E_n^I)$, so $x \in \wp(E_n^I, 1)$. Let v be the monomial valuation with $v(x) = \lambda \geq n$ and $v(y) = 1$; then $v(f_n) = n$. Every element of $I^p \subset (x, y^2)^p$ is a combination of the $x^a y^{2(p-a)}$, and a derivative of order j of such a term has $v \geq 2p - 2j$, since $\lambda \geq 2$. By (3), $\mathrm{Diff}^{(j)}(f_n)$ has $v \geq n - j$. So for $0 \leq j < p$,

$$\frac{v(\mathrm{Diff}^{(j)} J)}{p - j} \geq \min\Big(\frac{2p - 2j}{p - j}, \frac{n - j}{p - j}\Big) > 1,$$

because $n > p$, and hence $\mu_v > 1$. Let $g \in \wp(E_n^I, d)$ have order d. By Lemma 5.1, $v(g) \geq d\mu_v > d = v(y^d)$, so $\mathrm{in}\, g$ contains no monomial $\bar{y}^d$. Every monomial of $\mathrm{in}\, g$ is therefore divisible by $\bar{x}$. Hence the edge algebra lies in $(\bar{x})$ and contains $\bar{x}$. □

Proposition 6.8. *Let $n > p$ with $p \nmid n$, let $Y = \{x = 0\} \cong \mathbb{A}^1$, and put $c_n = n(p - 1)!$.*

- *(a) The ambient reduction of E_n to Y is $F_n = ((y^{c_n}), p!)$.*
- *(b) F_n is edge-trivial at 0, so $\mathrm{Inv}_0(F_n) = (1, 1)$. Its maximum base-hike $\hat{F}_n = (y^{c_n}, c_n)$ has $\mathrm{Inv}_0(\hat{F}_n) = (1, 0, 1)$.*
- *(c) The transform of F_n under the blowup of $0 \in Y$ is $((y^{c_n - p!}), p!)$, which is F_{n-p} when $n - p > p$.*

Proof. (a) By (3), $\mathrm{Diff}^{(j)}(f_n)$ restricts on Y to $(y^{n - i_j})$, where $i_j \leq j$ is the largest index with $\binom{n}{i_j} \not\equiv 0$. So $J^\sharp|_Y$ is generated by y^e with $e = \min_j (n - i_j)\, p!/(p - j)$. Since $(n - j)/(p - j) \geq n/p$, the minimum is $n(p - 1)!$, attained at $j = 0$. (b) Since $c_n > p!$, every generator $y^{c_n - i}$ $(i \leq j)$ of $A(F_n)$ in degree $p! - j$ has ratio $(c_n - i)/(p! - j) \geq (c_n - j)/(p! - j) > 1$. So $\mu_{\mathrm{ord}} > 1$, and no element of $\wp(F_n, d)$ has order d. For $\hat{F}_n$, let $p^v \parallel c_n$. The element $\partial^{(c_n - p^v)} y^{c_n} = \binom{c_n}{c_n - p^v} y^{p^v}$ is nonzero by

Lucas's theorem and lies in degree p^v. So $(yT)^{p^v}$ lies in the algebra, $y \in \wp(\hat{F}_n, 1)$, and the ideal generated by the edge algebra is $(\bar{y})$. (c) $y^{c_n} = y^{p!} \cdot y^{c_n - p!}$, and $c_n - p! = (n-p)(p-1)!$. □

Example 6.9. For $p = 2$: $J^\sharp = (f_n) + (f_n, y^{n-1})^2$ restricts on Y to (y^n), so $F_n = ((y^n), 2)$, and the transform of F_5 is $F_3 = ((y^3), 2)$. On the other hypersurface $Y^\flat = \{x + y^2 = 0\}$ (relevant in Example 9.2), the reduction of E_5 in the coordinate y is $((y^4), 2)$. Indeed $f_5 = y^4 + y^5$ there, and $(f_5, y^4)^2$ restricts into (y^8). This reduction is edge-trivial, since $\mathrm{Diff}^{(1)}(y^4) = (y^4)$ gives $\mu_{\mathrm{ord}} = 2$. Its transform $((y^2), 2)$ under the blowup of 0 has $\mathrm{Inv}_0 = (1, 0, 1)$, because $(yT)^2 = y^2T^2$ lies in its algebra.

7. The negative-degree part is trivial

The preprint introduces $\tilde{\wp}$ as the first of its new tools ([Hir17, §1, item (1), p. 3]; [Hir17, §5]). It is natural to expect that $\wp^{\mathrm{nega}}(E, -a)$ carries information about E. In fact it is the unit ideal.

Theorem 7.1. *Let E satisfy (37) at a closed point $\xi \in \mathrm{Sing}(E)$. Then $\wp^{\mathrm{nega}}(E, -a)_\xi = \mathcal{O}_\xi$ for every $a > 0$, and $\tilde{\wp}(E, 0)_\xi = \mathcal{O}_\xi$.*

First proof (from Definition 4.8). Let $a \geq 0$ and choose $d \geq 1$ with $dm \geq a$. Then $g^d \in \wp^{\mathrm{posi}}(E, dm)$ has order dm, so some monomial z^β with $|\beta| = dm$ occurs in g^d with a coefficient $c \neq 0$. Then $\partial^{(\beta)}(g^d) \equiv c \pmod{\mathfrak{m}}$ is a unit. Since $\mathrm{Diff}^{(dm+a)}$ consists of the operators of order $\leq dm + a$, it contains $\partial^{(\beta)}$. Hence the summand $\mathrm{Diff}^{(dm+a)}\, \wp^{\mathrm{posi}}(E, dm)$ of $\tilde{\wp}(E, -a)$ contains a unit. □

Second proof (from Theorems 4.5 and 6.3 of [Hir17]). Take $f = 1$ and $q = m$ in the criterion of [Hir17, Thm. 6.3], with g as in (37). We have $g^{p^\ell} \in \wp(E, mp^\ell)$, which lies in $\wp(E, mp^\ell - a)$ by (2). Hence $1 \in \wp^{\mathrm{nega}}(E, -a)_\xi$. □

Remark 7.2. The key step appears in the preprint itself. The proof of [Hir17, Lem. 5.6, p. 27] uses "$\mathrm{Diff}^{(md'-md)}\, g^{d'-d} = \mathcal{O}_{Z,\xi}$ by the assumption on g". The proof of [Hir17, Lem. 7.10, pp. 37–38] constructs an operator $\partial = u^{-1}\delta \in \mathrm{Diff}^{(m)}$ with $\partial(g) = 1$. The characteristic of K plays no role in Theorem 7.1.

The preprint calls H-Theorem 9.2 "the master key to the conceptual development and applications of LLUED" [Hir17, p. 47]. It is natural to expect it to be the technical heart of the argument. In fact it is vacuous.

Corollary 7.3. *Let $g = y^q + \varepsilon$ be an LL-head of $\check{E}$ at ξ, where $\check{E}$ satisfies (37). Then $h = \varepsilon$ satisfies the conclusion of* [Hir17, Thm. 9.2, p. 47]*: $h \in \wp^{\mathrm{nega}}(\check{E})_\xi$, $\mathrm{ord}\, h \geq \mathrm{ord}\, \varepsilon$, and $(g - h)^{1/p} \in \mathcal{O}_\xi$. If D is permissible for (g, q) and $y \in I(D)$, then $h = \varepsilon$ also satisfies the conclusion of* [Hir17, Thm. 9.3].

Proof. $\varepsilon \in \mathcal{O}_\xi = \wp^{\mathrm{nega}}(\check{E}, -a)_\xi$ by Theorem 7.1, and $g - \varepsilon = (y^{q/p})^p$. H-Theorem 9.3 asks for $h \in \wp^{\mathrm{nega}}((g, q))$. The ideal exponent (g, q) satisfies (37) at ξ, since $\mathrm{ord}\, g = q$, so $\wp^{\mathrm{nega}}((g, q))_\xi = \mathcal{O}_\xi$ by Theorem 7.1. Finally, permissibility gives $\mathrm{ord}_D\, g \geq q$, and $y \in I(D)$ gives $\mathrm{ord}_D\, y^q \geq q$. Therefore $\mathrm{ord}_D\, \varepsilon = \mathrm{ord}_D(g - y^q) \geq q$. □

The hypothesis $y \in I(D)$ can always be arranged, by [Hir17, Lem. 7.3].

The preprint asserts that elements of $\wp^{\mathrm{nega}}(E, -a)$ transform by "adding a-times the exceptional divisor" [Hir17, Rem. 7.7, p. 36]. It is natural to expect the resulting rule, Eq. (59), to hold. In fact it fails for E_n whenever $n > 2p$.

Theorem 7.4. [Hir17, Thm. 7.8, Eq. (59), p. 37] *is false. Let $n > 2p$ with $p \nmid n$, let $E = E_n$, $\xi = 0$, let π be the blowup of $D = \{0\}$, let ξ' be the origin of the chart $x = x'y$, and let $a \geq 1$. The hypotheses of H-Theorem 7.8 hold: E satisfies (37) on* $\mathrm{Sing}(E)$, $\xi' \in \pi^{-1}(\xi) \cap \mathrm{Sing}(E')$, *and* $\mathrm{Inv}_{\xi'}(E') = \mathrm{Inv}_{\xi}(E)$. *But*

$$\wp^{\mathrm{nega}}(E', -a)_{\xi'} = \mathcal{O}_{\xi'} \neq (y^a) = I(D)^a\, \wp^{\mathrm{nega}}(E, -a)\, \mathcal{O}_{Z',\xi'}.$$

Proof. By Proposition 6.1, E and $E' = E_{n-p}$ (at ξ') are maximum base-hiked, with $\mathrm{Sing}(E) = \{0\}$ and $\mathrm{Sing}(E') = \{\xi'\}$. So they satisfy (37) at every point of their singular loci, with g and g' suitable powers of f_n and f_{n-p}. Since $n > p$ and $n - p > p$, Proposition 6.3 gives $\mathrm{Inv}_{\xi}(E) = (2, 1, p) = \mathrm{Inv}_{\xi'}(E')$. By Theorem 7.1 the left side is $\mathcal{O}_{\xi'}$ and $\wp^{\mathrm{nega}}(E, -a) = \mathcal{O}$, so the right side is $I(D)^a \mathcal{O}_{\xi'} = (y^a)$. □

For E_5 in characteristic 2 this reads $\wp^{\mathrm{nega}}(E_3, -a)_{\xi'} = \mathcal{O} \neq (y^a)$.

Remark 7.5. The derivation in [Hir17, Rem. 7.9, Eq. (60), p. 37] fails at the step labelled "degree down by division". That step uses $\wp^{\mathrm{posi}}(E', mp^\ell - a)_{\xi'} = (g')^{p^\ell} \wp^{\mathrm{nega}}(E', -a)_{\xi'}$. H-Theorem 6.3 gives only one inclusion, and here the right side is $(g')^{p^\ell}\mathcal{O}$, which is strictly smaller. Take $E' = E_{n'}$ with $n' = n - p > p$, and any admissible m with g' as in (37). By (3), $y^{n'-1} \in \mathrm{Diff}^{(1)}(f_{n'}) \subset \wp(E', p-1)$. For $N(p-1) \geq mp^\ell$ we get $y^{N(n'-1)} \in \wp(E', N(p-1)) \subset \wp^{\mathrm{posi}}(E', mp^\ell - a)$ by (2). But $y^{N(n'-1)} \in (g')^{p^\ell}\mathcal{O}$ would force $g' = uy^m$ with u a unit. Then $\bar{y}^m$ would lie in the edge algebra, which contradicts Proposition 6.3. For E_3 in characteristic 2 the element is $y^2 \in \wp(E_3, 1)$.

Remark 7.6 (the domino fall). The "domino fall" [Hir17, Lem. 16.8(1), p. 85] argues that $g'(e-1) = g'(e)^p + h'(e-1)$ does not vanish at ξ' when $g'(e)$ does not, "because $h'(e-1)$ vanishes there by the rule of transformation of $\wp^{\mathrm{nega}}$". Let z generate the exceptional ideal at ξ'. The identity $g(e-1) = g(e)^p + h(e-1)$ in $\mathcal{O}$ pulls back to an identity in $\mathcal{O}_{Z',\xi'}$ only if all three terms are divided by the same power of z. With $g'(e-1) = z^{-p}\pi^* g(e-1)$ and $g'(e) = z^{-1}\pi^* g(e)$, this forces $h'(e-1) = z^{-p}\pi^* h(e-1)$. That element vanishes on the exceptional divisor only if $\mathrm{ord}_D\, h(e-1) > p$, whereas [Hir17, Lem. 16.7, p. 85] provides only $\mathrm{ord}_D\, h(e-1) \geq p$. The rule invoked, Eq. (59), is false (Theorem 7.4), and membership in $\wp^{\mathrm{nega}}$ imposes no condition at all (Theorem 7.1). The lemma is therefore unsupported. Theorem 10.2 shows that its intended consequence fails.

8. Core focusing

The preprint offers (43) and (44) as two definitions of the same $\check{E}$ [Hir17, p. 30]. It is natural to expect them to agree, and to expect each to "focus" $\hat{E}$ onto its Inv-max stratum while keeping the information in $\hat{E}$. In fact (44) does not focus at all, and (43) keeps no information about $\hat{E}$.

Proposition 8.1. *For every $\hat{E}$, the right side of (44) equals $S(\hat{E})$. Hence (44) gives $\check{E} \sim \hat{E}$ and* $\mathrm{Sing}(\check{E}) = \mathrm{Sing}(\hat{E})$.

Proof. We use the geometric definition, in which (44) is phrased. If $S(I_1, q)$ and $S(I_2, q)$ both contain $S(\hat{E})$, then so does $S(I_1 + I_2, q) = S(I_1, q) \cap S(I_2, q)$ [Hir17, p. 24]. So the ideals I with $S(I, q) \supset S(\hat{E})$ form a directed family, and their union $\wp(\hat{E}, q)$ is one of them, because the family is directed and $\mathcal{O}_Z$ is noetherian. Hence $S(\wp(\hat{E}, q_i), q_i) \supset S(\hat{E})$ for each i, and the intersection in (44) is $S(\hat{E})$. □

Proposition 8.2. *Suppose the Inv-max stratum Σ of $\hat{E}$ is smooth (and closed). Then the maximum in (43) exists and equals $\wp((I(\Sigma), 1)) = \bigoplus_d I(\Sigma)^d T^d$. Hence (43) gives $\check{E} \sim (I(\Sigma), 1)$, and* $\mathrm{Inv}_{\xi}(\check{E}) = (\dim Z, \dim Z - c, 1, \ldots, 1)$ *at every $\xi \in \Sigma$, where $c = \mathrm{codim}\, \Sigma$ and 1 is repeated c times.*

Proof. $(I(\Sigma),1)$ has singular locus Σ. Its algebra A is the Rees algebra of $I(\Sigma)$, which is integrally closed because $I(\Sigma)$ is locally generated by part of a regular system of parameters. So $\wp((I(\Sigma),1)) = \bigoplus_d I(\Sigma)^d T^d$. If E is any ideal exponent with $\mathrm{Sing}(E) = \Sigma$, then $\wp(E) \subset \bigoplus_d I(\Sigma)^d T^d$ by Lemma 5.2. The same lemma, applied to $\hat{E}$ with $\Sigma \subset \mathrm{Sing}(\hat{E})$, gives $\wp(\hat{E}) \subset \bigoplus_d I(\Sigma)^d T^d$. So $\wp((I(\Sigma),1))$ is the largest $\wp$ allowed by (43), and it is attained. By (1), $\check{E} \sim (I(\Sigma),1)$. At $\xi \in \Sigma$ the edge algebra is generated by the linear forms $\bar{z}_1, \dots, \bar{z}_c$ of a regular system of parameters with $I(\Sigma) = (z_1, \dots, z_c)$. □

Remark 8.3. For $E = E_n$ with $n > 2p$ (§6), $\Sigma = \{0\}$, and (43) gives $\check{E} \sim ((x,y),1)$. This is resolved by one blowup, while E_n is not; for instance E_5 in characteristic 2. A natural repair is $\check{E} := \hat{E} \cap (I(\Sigma),1)$ for smooth Σ. It has singular locus Σ and $\wp \supset \wp((I(\Sigma),1))$, so by the proof of Proposition 8.2 it is equivalent to $(I(\Sigma),1)$ as well.

We now turn to an example of the type studied by Narasimhan [Nar83b, Nar83a]. Every property of it that we use is checked below.

Example 8.4. Let $p = 2$, $Z = \mathbb{A}^4$ with coordinates (x,y,z,w), $f = x^2 + yz^3 + zw^3 + y^7 w$, $E = (f, 2)$, and let C be the monomial curve $t \mapsto (t^{32}, t^7, t^{19}, t^{15})$. Then:

(a) $C \subset \mathrm{Sing}(E)$ and $\hat{E} = E$;
(b) the embedding dimension of C at 0 is 4, so no smooth hypersurface germ at 0 contains C;
(c) $\mathrm{Sing}(E) = C$ as sets.

Proof. (a) Along C the four terms of f all equal t^{64}. The derivatives are $f_x = 0$, $f_y = z^3 + y^6 w$, $f_z = yz^2 + w^3$ and $f_w = zw^2 + y^7$. Along C the two terms of each derivative are equal (t^{57}, t^{45} and t^{49} respectively), so the derivatives vanish on C. Since $\partial_x^{(2)} f = 1$, $\operatorname{ord} f \le 2$ everywhere, so $\hat{E} = E$. (b) The coordinate ring of C is the semigroup ring of $\langle 7, 15, 19, 32\rangle$. None of $7, 15, 19, 32$ is a sum of the others: 15 is not a multiple of 7; $19 - 15$ and $19 - 7a$ are not in $\langle 7\rangle$; and 32 minus any element of $\langle 7, 15, 19\rangle$ below 32 is again outside $\langle 7, 15, 19\rangle$. So the maximal ideal of $\mathcal{O}_{C,0}$ needs four generators. (c) is a computation: the radical of (f, f_x, f_y, f_z, f_w) is $I(C)$ [Cla26a, V2]. It is used only in Remark 8.8. □

The preprint asserts [Hir17, Thm. 6.1, p. 29] that "for every $d > 0$ and every $\xi \in \mathrm{Sing}(\hat{E})_{cl}$ we have $d = \operatorname{ord}_\xi \wp(\hat{E}, d)$ provided $\wp(\hat{E}, d) \neq (0)$". It is natural to expect this, since $\hat{E}$ has an element of degree equal to its order. In fact it fails for $d = 1$.

Proposition 8.5. *In Example 8.4, $\wp(\hat{E},1) \ni f_y \neq 0$, but every element of $\wp(\hat{E},1)_0$ has order ≥ 2 at 0. So* [Hir17, Thm. 6.1, second sentence] *is false.*

Proof. $f_y \in \mathrm{Diff}^{(1)}(f) \subset \wp(E,1)$. Let $h \in \wp(E,1)_0$. By Lemma 5.2, h vanishes on $\mathrm{Sing}(E) \supset C$. If $\operatorname{ord}_0 h = 1$, then near 0 the curve C lies in the smooth hypersurface $\{h = 0\}$, which contradicts Example 8.4(b). □

The restriction of H-Theorem 6.1 to d divisible by the hike degree is what Definition 4.8 uses, and that restricted form holds for the element of (37).

Proposition 8.6. *In Example 8.4, $\mathrm{Inv}_\eta(\hat{E}) \neq \mathrm{Inv}_0(\hat{E})$ for every closed point $\eta \in C \setminus \{0\}$. Hence $\Sigma \neq \mathrm{Sing}(\hat{E})$, and the singular-locus prescriptions in (43) and (44) differ wherever (43) defines an ideal exponent.*

Proof. Let η be the image of $t_0 \in \overline{K}^\times$. Then $f_z \in \wp(E,1)$ vanishes at η, and $\partial_y(f_z) = z^2$ takes the value $t_0^{38} \neq 0$ there, so $\operatorname{ord}_\eta f_z = 1$. So the edge algebra at η has a nonzero element of degree 1, and the generators of G at η include one of degree $q = 1$. At 0 the edge algebra contains $\bar{x}^2 = \mathrm{in}_0 f$, so $r(0) \ge 1$, and it has no element of degree 1 (Proposition 8.5), so all $q_j(0) \ge 2$. The

two invariants differ in the entry $\dim Z - r$ if $r(0) \neq r(\eta)$, and otherwise in the q-entries. So Inv is not constant on $C \subset \mathrm{Sing}(\hat{E})$, the stratum Σ is a proper subset of $\mathrm{Sing}(\hat{E})$. Where (43) defines an ideal exponent, it prescribes $\mathrm{Sing}(\check{E}) = \Sigma$; (44) gives $\mathrm{Sing}(\check{E}) = \mathrm{Sing}(\hat{E})$ (Proposition 8.1). □

§10 of [Hir17] asserts that $\mathrm{Sing}(\check{E})$ lies in every tail hypersurface [Hir17, Thms. 10.3–10.4, p. 60], and the construction there uses "the constancy of $\mathrm{Inv}_\eta(\check{E})$ throughout $\mathrm{Sing}(\check{E})$" [Hir17, p. 60; Def. 10.2]. It is natural to expect core focusing to guarantee both. Under reading (44) it guarantees neither.

Proposition 8.7. *In Example 8.4, read $\check{E}$ by (44). Then:*

(a) $\mathrm{Sing}(\check{E}) \supset C$, *and* $\mathrm{Inv}_\eta(\check{E})$ *is not constant on* $\mathrm{Sing}(\check{E})$;

(b) no smooth hypersurface germ at 0 *contains* $\mathrm{Sing}(\check{E})$, *so the conclusion of* [Hir17, Thm. 10.4] *fails for every LL-chain of $\check{E}$ at* 0;

(c) [Hir17, Cor. 9.19, p. 56] *and* [Hir17, Thm. 10.4] *cannot both hold at* 0.

Proof. (a) By Proposition 8.1, $\check{E} \sim \hat{E}$. So $\mathrm{Sing}(\check{E}) = \mathrm{Sing}(\hat{E}) \supset C$ and, by (1), $\wp(\check{E}) = \wp(\hat{E})$. The second claim is Proposition 8.6. (b) The first claim is Example 8.4(b). The tail $g(e)$ of an LL-chain at 0 has order 1, so the tail hypersurface is smooth at 0 and cannot contain C. (c) Since $\bar{x}^2 = \mathrm{in}_0 f$ lies in the edge algebra at 0, Reading 4.5 gives $\bar{x} \in v(\check{E})_0 \neq 0$. H-Cor. 9.19 asserts that $v(\check{E})_0$ is generated by the images of tails of LL-chains at 0, so at least one such chain would exist, and by (b) it violates H-Thm. 10.4. □

Remark 8.8. Under reading (43), Example 8.4 does not refute H-§10 directly. By Proposition 8.6, Σ is a proper subset of $\mathrm{Sing}(\hat{E})$, which equals C by Example 8.4(c) (a computation). So if Σ is closed, as (43) requires, it is finite, and every finite set of points lies locally in a smooth hypersurface. One may expect that $\mathrm{Inv}_0(\hat{E}) = (4, 3, 2)$ and $\Sigma = \{0\}$; we have not verified this. Refining the top locus by edge data is the device that Hauser names as the way around Narasimhan's example [Hau08, p. 7].

Example 8.4 and Propositions 8.5, 8.6 and 8.7 are proved only in characteristic 2. The general statements they rest on, Lemma 5.2 and Propositions 8.1 and 8.2, hold in every characteristic.

Question 8.9. For $p > 2$, is there a maximum base-hiked ideal exponent $\hat{E}$ with $\wp(\hat{E}, 1) \neq 0$ whose singular locus contains a curve lying in no smooth hypersurface germ at a point ξ where $v(\hat{E})_\xi \neq 0$?

For such an $\hat{E}$ the arguments of Propositions 8.5 and 8.7 would refute H-Thm. 6.1 (second sentence) and, under reading (44), H-Thm. 10.4 in characteristic p. We have not checked any such example. Narasimhan's paper [Nar83b], which we have not opened, concerns, by its title, equimultiple curves in positive characteristic and may already provide one.

9. The ambient-reduction chain of E_n

We compute the chain of Definition 4.10 for $E = E_n$, $n > p$, $p \nmid n$, under Readings 4.5, 4.7 and 4.11. We use the coordinates (x, y) of §6; the preprint's edge parameter, called y there, is our x.

Proposition 9.1. *Let $n > p$ with $p \nmid n$ and $E(0) = E_n$. Work near the origin, let I_s be the ideal generated by* $\mathrm{Cot}(\check{E}(s))$, *and choose the representatives recursively by $A_0 = (0)$, $A_{s+1} = A_s + I_s$, $E(s+1) = E_n^{A_{s+1}}$, and $\check{E}(s) = \hat{E}(s) = E(s)$. These are admissible under reading (44). For every admissible choice of covering and cotangential generators with these representatives:*

(a) $\check{E}(0) = E_n$. The LL-chain $f_n = x^p + h(0)$ with $h(0) = y^n$ is produced by the case analysis of [Hir17, §9.7], *and its tail is x. Hence $x \in I_0 \subset (x, y^2)$, and $E(1) \sim E_n^{I_0}$;*

(b) *For $s \geq 1$, put $A_s = I_0 + \cdots + I_{s-1}$. Then $E(s) = (f_n + A_s^p, p)$, $x \in A_s \subset (x, y^2)$, $\mathrm{Inv}_0(\check{E}(s)) = (2, 1, 1)$, and $I_s \subset (x, y^2)$;*
(c) $\operatorname{Sing} \mathcal{Y}(s) = \{0\}$ *for every $s \geq 0$. The singular loci therefore stabilise from $s = 0$, and the terminal plat is $\nabla = \{0\}$ under the first-stabilisation convention;*
(d) $\operatorname{rank} v(\check{E}(s)) = 1$ *and* $\dim \mathcal{Y}(s) = 1$ *for every $s \geq 0$.*

Proof. (a) We have $\operatorname{Sing}(E_n) = \{0\}$ and $\operatorname{ord}_0(f_n) = p$, so $\hat{E}(0) = E_n$ and (44) permits the chosen representative $\check{E}(0) = E_n$. The head is $x^p + \varepsilon$ with $\varepsilon = y^n$ and $q = p$, so $e = 1$. In the decomposition $y^n = y^{\alpha + p\beta + q\gamma_0}$ of [Hir17, §9.3–9.4], we have $0 \leq \alpha < p$ and $0 \leq \beta < p^{e-1} = 1$, so $\beta = 0$; the unit coefficient u_0 is 1. Thus $n = \alpha + p\gamma_0$ with $0 < \alpha < p$.

- If $n > 2p$, then $\gamma_0 \geq 2$ and $|p\gamma_0| \geq 2p$. This is case (I) of [Hir17, Lem. 9.6, Rem. 9.9], and $H^\flat(y^n) = \partial^{(\alpha)}(y^n)\, \partial^{(p\gamma_0)}(y^n) = \binom{n}{\alpha}\binom{n}{p\gamma_0} y^n = y^n$ by Lucas's theorem.
- If $p < n < 2p$, then $\gamma_0 = 1$, which is case (III) of [Hir17, Lem. 9.6, Rem. 9.11]. There $h^\flat(y^n) = \binom{n}{\alpha}\binom{n}{p}\, y^n = y^n$ by Lucas's theorem, since the base-p digits of n are $(1, \alpha)$. This is a fixed point, so $H^\flat(y^n) = y^n$.

In both cases $g(1) = (f_n - y^n)^{1/p} = x$. By Reading 4.11, $x \in I_0 \subset (x) + \mathfrak{m}^2 = (x, y^2)$, because $v(\check{E}(0)) = K\bar{x}$ (Proposition 6.3 and Reading 4.5). Then Definition 4.2 gives $E(1) \sim (f_n + I_0^p, p) = E_n^{I_0}$. (b) Put $A_0 = (0)$ and $A_s = I_0 + \cdots + I_{s-1}$ for $s \geq 1$. The first part gives $x \in A_1 \subset (x, y^2)$ and $E(1) = (f_n + A_1^p, p)$. Suppose inductively that $E(s) = (f_n + A_s^p, p)$ and $x \in A_s \subset (x, y^2)$. Proposition 6.7 gives $\operatorname{Sing} E(s) = \{0\}$, $\hat{E}(s) = E(s)$, $\mathrm{Inv}_0(E(s)) = (2, 1, 1)$, and $v(E(s))_0 = K\bar{x}$. Thus (44) permits our choice $\check{E}(s) = E(s)$, and Reading 4.11 gives $I_s \subset (x, y^2)$. Hence $x \in A_{s+1} = A_s + I_s \subset (x, y^2)$. By Definition 4.2, the defining relation for the chain is $S(E(s+1)) = S(E(s)) \cap S(I_s, 1)$. The chosen representative satisfies this relation, since $S(E_n^{A_{s+1}}) = S(E_n) \cap S(A_s + I_s, 1)$ and $S(A_s + I_s, 1) = S(A_s, 1) \cap S(I_s, 1)$. Thus $E(s+1) = E_n^{A_{s+1}} = (f_n + A_{s+1}^p, p)$ is admissible, closing the induction. (c) For $s = 0$, Proposition 6.1 gives $\operatorname{Sing} E(0) = \{0\}$; for $s \geq 1$ this is the induction above. Every reduction $F_j(Y_{ji})$ has singular locus contained in $\operatorname{Sing} E(s) \cap Y_{ji}$ by [Hir17, Thm. 3.10, p. 10]. Conversely, the generators of $E(s)$ have order at least p at 0, and all Y_{ji} pass through 0. In the formula for ambient reduction (Definition 4.9), each derivative of order j has order at least $p - j$, so its contribution to $J^\sharp|_{Y_{ji}}$ has order at least $p!$. Thus 0 is singular for every reduction, giving $\operatorname{Sing} \mathcal{Y}(s) = \{0\}$. (d) Proposition 6.3 gives $v(\check{E}(0)) = K\bar{x}$. The induction and Proposition 6.7 give the same for all $s \geq 1$. Definition 4.10 therefore gives $\dim \mathcal{Y}(s) = 2 - 1 = 1$ for every s. □

Example 9.2. For E_5 in characteristic 2 the ideal I_0 can be computed exactly. With the edge parameter $X = x + y^2$ the head is $X^2 + \varepsilon$ with $\varepsilon = y^4 + y^5$. This has no X-terms, so it is its own g-cleaning [Hir17, Def. 7.18, Eq. (64), p. 41]. Its top monomial y^5 is in case (I) of [Hir17, Lem. 9.6], and $H^\flat(\varepsilon) = \partial_y(\varepsilon)\, \partial_y^{(4)}(\varepsilon) = y^4(1 + y) = \varepsilon$. So $X = x + y^2$ is a second tail. Hence $I_0 = (x, y^2)$, and

$$E(1) \sim E_5^\natural = E_5 \cap ((x, y^2), 1) = ((x^2, xy^2, y^4), 2).$$

The preprint asserts that once the singular loci stabilise, $\operatorname{Sing} \mathcal{Y}(s) = \operatorname{Sing} \mathcal{Y}(s+1)$, the chain produces a smooth centre $\{f = 0\}$ defined by cotangent generators, and that the dimensions of the $\mathcal{Y}(s)$ strictly decrease [Hir17, Thm. 15.9, p. 79; Thm. 15.10, Step 2, p. 80]. It is natural to expect stabilisation to force this. In fact it does not.

Theorem 9.3. *Let $n > p$ with $p \nmid n$ and use the admissible chain of Proposition 9.1, with any covering and cotangential generators permitted there.*

(a) *The claims of* [Hir17, Thm. 15.10, Step 2] *for $s=0$ are false for every cotangential generator c of order* 1. *First, $S((c),1)\neq S(\check{E}(0))\cap S(\mathrm{Cot},1)=S(E(1))$. Second, no ideal J' with $(J',p)\sim\check{E}(0)$ lies in $(c)^p$. Third,* $\mathrm{Sing}(\check{E}(1))=\{0\}\neq\{c=0\}$.

(b) *The singular loci stabilise already at $s=0$, but* $\dim\mathcal{Y}(0)=\dim\mathcal{Y}(1)=1$. *Thus the strict dimension drop of* [Hir17, Thm. 15.10(2)] *under the first-stabilisation convention, and the chain* $\dim Z>\dim\mathcal{Y}(0)>\dim\mathcal{Y}(1)$ *of* [Hir17, Thm. 15.9], *fail.*

Proof. (a) The blowup of the smooth curve $\{c=0\}$ is permissible for $((c),1)$. It is not permissible for $E(1)$, whose singular locus is $\{0\}$. If $J'\subset(c)^p$, then $\mathrm{Sing}(J',p)\supset\{c=0\}$, whereas $\mathrm{Sing}\,\check{E}(0)=\{0\}$. The third claim is Proposition 9.1(b). (b) By Proposition 9.1(c) and (d), the singular loci stabilise at $s=0$ while $\dim\mathcal{Y}(0)=\dim\mathcal{Y}(1)=1$. □

Remark 9.4. Here $\nabla=\{0\}$ happens to be smooth, so the conclusion of [Hir17, Thm. 15.10(3)] holds in this example although its proof fails. Whether ∇ is always smooth is open (Question 14.1). [Hir17, Thm. 15.9] also asserts that "$Y(t+1)$ is smooth, i.e. for all i, $F(t+1)_j$ is the zero ideal". By [Hir17, Def. 15.11] this means $\mathrm{Sing}\,\mathcal{Y}(t+1)=\emptyset$. That contradicts [Hir17, Thm. 15.10(3)] in general, and it fails here, where $\mathrm{Sing}\,\mathcal{Y}(1)=\{0\}$.

10. Main Theorem I

Throughout this section let $n>2p$ with $p\nmid n$, and let $E=E_n$ with the admissible chain of Proposition 9.1, so that $\nabla=\{0\}$ (Proposition 9.1(c)). Take $D=\nabla$. The blowup π of 0 is permissible, $\nabla'=\emptyset$, and $D'=\emptyset$. So every point $\xi'\in\pi^{-1}(0)$ is a point of $\pi^{-1}(D)\setminus D'$, as in [Hir17, Thm. 16.6(2)], and of $\pi^{-1}(\nabla)\setminus\nabla'$, as in [Hir17, Thm. 16.6(4), Cor. 16.9]. Let ξ' be the origin of the chart $x=x'y$. By Proposition 6.1, the transform at ξ' is E_{n-p}, and it is singular there.

Reading 10.1 (the invariant of an AR-scheme)**.** Main Theorem I compares strings

$$(\mathrm{Inv}_\xi(\mathcal{Y}(0)),\mathrm{Inv}_\xi(\mathcal{Y}(1)),\dots,\mathrm{Inv}_\xi(\mathcal{Y}(m-1))).$$

Here $\mathcal{Y}(m-1)$ is the terminal level, as in [Hir17, Cor. 16.9] and the proof of H-Thm. 16.6 (p. 86), so $m-1=t$. But $\mathrm{Inv}_\xi(\mathcal{Y}(i))$ is not defined in [Hir17]. We consider four readings:

(I) $\mathrm{Inv}_\xi(\check{E}(i))$;
(II) $\mathrm{Inv}_\xi(\hat{E}(i))$;
(III) Inv_ξ of the reduced members $F_j(Y_{ji})$ on Y_{ji};
(IV) Inv_ξ of their maximum base-hikes.

At ξ' the members $\mathcal{Y}'(i)$ may be either the transforms of the members of $\mathcal{Y}(i)$ or the members of the chain computed afresh from the transform E'. The preprint uses both: [Hir17, Thm. 16.6] speaks of transforms, and [Hir17, §16.3, p. 87] says that "core-edge focusing is renewed … after each sequence of blowups".

It is natural to expect that once the whole terminal plat is blown up, the transforms have no singular point over it, as [Hir17, Thm. 16.6(4), Cor. 16.9] assert. In fact they do.

Theorem 10.2. *Under Reading 4.7:*

(a) [Hir17, Thm. 16.6(4), p. 84] *is false: the member $\check{E}(0)=E_n$ of $\mathcal{Y}(0)$ has transform E_{n-p}, which is singular at ξ';*

(b) *in readings* (I) *and* (II) *of Reading 10.1, the conclusion of* [Hir17, Cor. 16.9, p. 85] *is false at ξ'.*

Proof. (a) This is Proposition 6.1(b). (b) The conclusion of H-Cor. 16.9 is a strict drop of $\mathrm{Inv}(\mathcal{Y}(m-1))$ at ξ'. It fails by Theorem 10.3(a). □

The proof of H-Cor. 16.9 fails in every reading. It asserts that $g(0)'$ does not vanish at $\xi' \in \pi^{-1}(\nabla) \setminus \nabla'$. But the transform of $g(0) = f_n$ is $y^{-p}\pi^* f_n = x'^p + y^{n-p}$, which vanishes at ξ'. The identification "$\nabla = \{g(e) = 0\}$" of [Hir17, Eq. (129), p. 84] also fails for every tail $g(e)$: a tail has order 1, so $\{g(e) = 0\}$ is a curve, while $\nabla = \{0\}$. Part (a) does not depend on the covering or cotangential generators within this chosen chain.

It is natural to expect that blowing up ∇ strictly lowers the string at the points of $\pi^{-1}(D) \setminus D'$, as [Hir17, Thm. 16.6(2)] asserts, so that the induction of [Hir17, Thm. 16.13] terminates. In fact the string does not change.

Theorem 10.3. *Under Reading 4.7, in readings* (I) *and* (II)*, and for the admissible chain fixed above:*

- *(a) the string has length $m = t + 1 = 1$ and is $((2, 1, p))$ both at 0 and at ξ', whether $\mathcal{Y}'(0)$ is transformed or renewed. Hence* [Hir17, Thm. 16.6(2)] *is false;*
- *(b) with renewed chains, even the length-2 strings agree: both are $((2, 1, p), (2, 1, 1))$.*

Proof. (a) By Proposition 9.1, $t = 0$ at 0, and also for the renewed chain at ξ', since $n - p > p$. At 0, $\check{E}(0) = \hat{E}(0) = E_n$, and at ξ' both the transform and the renewed $\check{E}'(0) = \hat{E}'$ are E_{n-p}. Since $n > p$ and $n - p > p$, Proposition 6.3 gives $\mathrm{Inv} = (2, 1, p)$ for both. (b) By Proposition 9.1(b), applied to E_n and to E_{n-p}, the second entry is $(2, 1, 1)$ at both points. □

For E_5 in characteristic 2 the strings are $((2, 1, 2))$, and with renewed chains $((2, 1, 2), (2, 1, 1))$, at the origin and at ξ'.

Remark 10.4. Two variants behave differently. Part (i) and the second half of part (ii) are known only in the example E_5.

- (i) *Transformed $i = 1$ entry.* If the string had length 2, the transformed entry for $i = 1$ would depend on I_0. For E_5 (Example 9.2) it drops: the transform of $E_5^\natural = ((x^2, xy^2, y^4), 2)$ at ξ' is $((x', y)^2, 2)$, with $\mathrm{Inv} = (2, 0, 1, 1)$.
- (ii) *Readings* (III) *and* (IV). These depend on the members. For the member on $\{x = 0\}$, Proposition 6.8 gives the same values $(1, 1)$ and $(1, 0, 1)$ at 0 and at ξ', transformed or renewed, because $n - p > p$. For E_5 a system of generators of $\mathrm{Cot}(\check{E}(0))$ may also contain $x + y^2$. The member on $\{x + y^2 = 0\}$ has reduction $((y^4), 2)$, and its transform has $\mathrm{Inv} = (1, 0, 1) \neq (1, 1)$ (Example 6.9). Since [Hir17] does not say how the members' invariants combine into $\mathrm{Inv}(\mathcal{Y}(i))$, we draw no conclusion in these readings.

Proposition 10.5. *Under the alternative reading (43) of $\check{E}$, the renewed string at ξ' equals the string at 0 in every reading* (I)–(IV)*, for corresponding choices.*

Proof. By Proposition 8.2, (43) gives $\check{E}(0) \sim ((x, y), 1)$ at 0 and $\check{E}'(0) \sim ((x', y), 1)$ at ξ'. Every later term of the chain is built from $\check{E}(0)$ alone (Definition 4.10). The germs $(\mathbb{A}^2, 0, ((x, y), 1))$ and $(Z', \xi', ((x', y), 1))$ are isomorphic by a change of coordinates. Transporting an admissible choice at 0 along this isomorphism gives an admissible choice at ξ' with the same invariants. This covers all four readings for every $i \geq 1$, and readings (I), (III) and (IV) for $i = 0$. The remaining entry, reading (II) at $i = 0$, is $\mathrm{Inv}_0(\hat{E}) = (2, 1, p) = \mathrm{Inv}_{\xi'}(\hat{E}')$ by Proposition 6.3. □

Remark 10.6. Under reading (43) with *transformed* members, H-Theorem 16.6(2),(4) may hold for E_n: the transform of $((x, y), 1)$ is $(\mathcal{O}, 1)$, which has empty singular locus. But then $\check{E}$ has forgotten E (Proposition 8.2). $E' = E_{n-p}$ is still singular at ξ', with $\mathrm{Inv}_{\xi'}(\hat{E}') = \mathrm{Inv}_0(\hat{E})$, and the renewed string does not drop (Proposition 10.5). So this reading gives no route to H-Theorem 16.13 either.

Corollary 10.7. *Let $n > 2p$ with $p \nmid n$. Under either reading of $\check{E}$, with renewed chains and corresponding transported choices, the string in readings* (I) *and* (II) *is the same at every one of the $\lfloor n/p \rfloor$ points blown up in resolving E_n (Proposition 6.1(c)).*

Proof. The points blown up carry E_m with $m = n - jp > p$. Under (44), Propositions 6.3 and 9.1 give the same values for each such m as at 0. Under (43), Proposition 8.2 gives $\check{E} \sim ((x, y), 1)$ at each point, and the proof of Proposition 10.5 gives the same renewed string. □

So these strings cannot bound the number of applications of H-Theorem 16.6. The termination claim of [Hir17, Thm. 16.13, p. 87], "by applying Th. (16.6) ... repeatedly but finitely many times", is therefore not justified by these proposed strings. In the family E_n, the order $c_n = n(p-1)!$ of the coefficient ideal F_n decreases by $p!$ at each step (Proposition 6.1(c)), and that quantity is not a component of Inv. For E_5 in characteristic 2 it goes $5 \to 3 \to 1$.

11. Residual order can increase

Definition 11.1 (after [Hau08, pp. 6–7])**.** Let $g = x^{p^e} + \varepsilon$ with ε free of x, and let M be the largest monomial in the exceptional coordinates dividing ε. The *residual order* of g with respect to $V = \{x = 0\}$ is $\operatorname{ord}(\varepsilon/M)$. Hauser's *shade* is the maximum of the residual order over hypersurfaces V transversal to the exceptional divisor; a V attaining it has *weak maximal contact*.

At the origin, where there is no exceptional divisor, the residual order of f_n with respect to $\{x = 0\}$ is n. After a blowup the exceptional monomial absorbs the y-term, so for E_n the quantity that drops along the resolution is the order of the coefficient ideal F_n, not the residual order. The shade is a natural secondary quantity to test, but it can increase. Whether it can be combined with the full string of H-Theorem 16.6 to give a decreasing invariant remains open here.

Example 11.2 (Hauser's kangaroo point [Hau08, p. 4, §G p. 16])**.** Let $p = 2$ and $f_0 = x^2+y^7+yz^4$ on $\mathbb{A}^3$. Three point blowups in the charts $(x, y, z) \mapsto (xy, y, zy)$, (xz, yz, z) and $(xz, yz + z, z)$ give

$$f_1 = x^2 + y^5 + y^3z^4, \qquad f_2 = x^2 + y^3z^3(y^2 + z^2), \qquad f_3 = x^2 + z^6(y^5 + y^4 + y^3 + y^2),$$

each of order 2 at the origin a_i of its chart. At a_2 the exceptional divisors are $\{y = 0\}$ and $\{z = 0\}$; at a_3 the only one is $\{z = 0\}$. With respect to $\{x = 0\}$ the residual parts are $y^2 + z^2$ at a_2 (order 2) and $y^5+y^4+y^3+y^2$ at a_3 (order 2). With respect to $\{x+\phi = 0\}$, $\phi = y^2z^3+yz^3 = yz^3(1+y)$, we have $f_3 = (x + \phi)^2 + z^6(y^5 + y^3)$ and $y^5 + y^3 = y^3(1 + y)^2$, so the residual order is 3. The residual order at the equiconstant point a_3 is thus larger than at a_2: $2 \to 3$.

Proof. Direct substitution: $f_0(xy, y, zy) = y^2 f_1$, $f_1(xz, yz, z) = z^2 f_2$, and $f_2(xz, yz+z, z) = z^2 f_3$, using $(y + 1)^5 + (y + 1)^3 = y^5 + y^4 + y^3 + y^2$ in characteristic 2. Then $\phi^2 = y^4z^6 + y^2z^6$. The exceptional divisors are read off from the charts. The computation is reproduced in [Cla26a, V3] and [Cla26b]. □

Hauser shows that at a_2 the residual order with respect to every hypersurface of weak maximal contact is 2, so the increase is genuine [Hau08, §G, p. 16]. He uses $x \mapsto x + yz^3$, which gives $f_3 = x^2 + z^6(y^5 + y^4 + y^3)$. The hypersurface $\{x + \phi = 0\}$ is the tail obtained by removing the whole square part of $z^6(y^5 + y^4 + y^3 + y^2)$; we have not checked whether the case analysis of [Hir17, §9.7] produces exactly this tail. Moh's bound [Moh87, Stability Thm., p. 966], expressed in Hauser's shade notation [Hau08, §B, p. 8], limits the increase: $\operatorname{shade}_{a'} f' \le \operatorname{shade}_a f + p^{e-1}$ at equiconstant points for $f = x^{p^e} + y^r g(y)$. It does not prevent the increase.

Remark 11.3. The increase shows that shade alone is not a decreasing invariant along this sequence. We have not computed the full invariant string of H-Theorem 16.6 at a_2 and a_3, so this example does not decide whether a lexicographic refinement using shade decreases there. The preprint does not discuss kangaroo points, Moh's bound, or any mechanism for a decrease at them. Hauser reports that Hironaka's notes, including those of the 2008 lectures, assert that such points "can be resolved directly", and that "there is no written proof of this available yet" [Hau08, §H, p. 17].

Every infinitely near singular point of $(f_0, 2)$ near a point $\eta \neq 0$ of the cusp $\mathrm{Sing}(f_0, 2) = \{x = 0,\ u = 0\}$, $u = y^3 + z^2$, lies on $\{x = 0\}$ and on $\{u = 0\}$. By the maximal-contact reasoning that preceded Proposition 6.3, it is natural to expect that x and u lie in $\wp((f_0, 2), 1)$ near η, so that the edge data jump at 0 and $\Sigma = \{0\}$. In fact $\wp(E, 1)$ has no element of order 1 anywhere on the cusp.

Proposition 11.4. *Let $E = (f_0, 2)$ as in Example 11.2, and let $C = \{x = u = 0\}$, $u = y^3 + z^2$. For every closed point $\eta \in C$, no element of $\wp(E, 1)_\eta$ has order 1 at η.*

Proof. We have $f_0 = x^2 + yu^2$, and $\mathrm{Diff}^{(1)}(f_0) = (f_0, u^2)$, because $\partial_y f_0 = u^2$ and $\partial_x f_0 = \partial_z f_0 = 0$.

At $\eta = 0$. For the monomial valuation $v(x) = \lambda \gg 0$, $v(y) = 2$, $v(z) = 3$ we have $v(f_0) \geq \min(2\lambda, 14, 14)$ and $v(u^2) = 12$, so $\mu_v \geq 7$. An element $h \in \wp(E, 1)$ of order 1 can therefore have no linear term in y or z, so $h = cx + h_2$ with $c \neq 0$ and $h_2 \in \mathfrak{m}^2$. Take the arc $y = s^2$, $z = s^4$, $x = s\,u(s) + s^N$ with $u(s) = s^6 + s^8$ and $N \gg 0$. Then $\varphi(f_0) = s^{2N}$, $w(u^2) = 12$, and $\mu_w = 12$. Odd powers of s below s^N come only from monomials with an odd power of x, and $x \cdot y^b z^c$ has leading exponent $7 + 2b + 4c$. So the coefficient of s^7 is c, and $w(h) \leq 7 < 12$, a contradiction.

At $\eta \neq 0$. Here $\eta = (0, y_0, z_0)$ with $y_0^3 = z_0^2$ and $y_0 \neq 0$. Let $c_0 = y_0^{1/2}$ (residue fields are perfect) and put $X = x + c_0 u$, $U = u$, $W = z - z_0$. These form a regular system of parameters at η, since $\partial_y u = y^2 \neq 0$ and $\partial_z u = 0$. Moreover $f_0 = X^2 + (y - y_0)U^2$. From $y^3 - y_0^3 = U + W^2$ we get $y - y_0 = e\,(U + W^2)$ with e a unit, so $f_0 = X^2 + eU^2(U + W^2)$. For the monomial valuation $v(X) = \lambda \gg 0$, $v(U) = 1$, $v(W) = \gamma$ with $0 < \gamma < 1$, $\mu_v \geq \min(\lambda, \frac{3}{2}, 1 + \gamma, 2) > \max(1, \gamma)$. So an order-1 element $h \in \wp(E, 1)_\eta$ has the form $h = cX + h_2$ with $c \neq 0$ and $h_2 \in \mathfrak{m}_\eta^2$. Take the arc $y = (c_0 + s)^2$, $z = z_0 + s^2$, $x = (c_0 + s)u(s) + s^N$. Then $\varphi(f_0) = s^{2N}$, and $u(s) = c_0^4 s^2 + (c_0^2 + 1)s^4 + s^6$ has $w(u) = 2$. So $\varphi(X) = s\,u(s) + s^N$ has $w(X) = 3$ with leading coefficient c_0^4, while $w(U) = w(W) = 2$, $w(h_2) \geq 4$ and $\mu_w = \min(N, 4) = 4$. Hence $w(h) = 3 < 4$, a contradiction. □

Under reading (44), $\mathrm{Sing}(\check{E}) = \mathrm{Sing}(\hat{E})$ is the cusp C, which is singular at 0, whatever the edge data are (Proposition 8.1).

Question 11.5. Is $\mathrm{Inv}_\eta(\hat{E})$ constant along the cusp C of Example 11.2?

Proposition 11.4 shows only that all $q_j \geq 2$ along C. If Inv is constant, then under reading (43) the Inv-max stratum of $\hat{E}$ is the singular curve C, so Proposition 8.2 does not apply, and the example becomes a test case for the smoothness assertion [Hir17, Thm. 15.10(3)].

Under reading (44) the example bears on [Hir17, Thm. 15.10(3)] whatever the edge data are, since $\mathrm{Sing}(\check{E}) = C$ is singular. Suppose $\{x = 0\}$ is taken as the tail hypersurface. (We have not checked that some LL-chain along a neighbourhood of 0 produces it: $f_0 - x^2 = yu^2$ has order 2 at $\eta \neq 0$, so f_0 is not an LL-head along a neighbourhood.) Then the reduction of $(f_0, 2)$ to $\{x = 0\}$ is $J^\sharp|_{x=0} = (yu^2, u^4)$, whose singular locus is C. The open step is the next term of the chain at 0. Near $\eta \neq 0$, where y is a unit and $E(1) \sim ((x^2, u^2), 2)$, we have $u \in \wp(E(1), 1)$; but u has order 2 at 0.

Example 11.2 and Proposition 11.4 are in characteristic 2. Moh's bound and Hauser's classification of kangaroo points concern every characteristic [Hau08, §§B–C]. We have not checked a kangaroo example for $p > 2$ against the invariants of [Hir17].

12. Enclosure of $\mathrm{Sing}(\check{E})$ by tails

§10 of [Hir17] asserts that $\mathrm{Sing}(\check{E}) \cap V$ lies in every tail hypersurface [Hir17, Thm. 10.3, Thm. 10.4, p. 60], and that the tail vanishes along every smooth $D \subset \mathrm{Sing}(\check{E})$ [Hir17, Thm. 10.6, p. 61]. Permissibility of the reduced centres depends on these assertions.

Remark 12.1 (the proofs). [Hir17, Lem. 10.1, p. 59] produces, for each $\eta \in \mathrm{Sing}(\check{E}) \cap V$, a modified chain with $y(i,\eta) = y(i) - u(i,\eta)$ whose tail vanishes at η. The modification depends on η. The proof of H-Thm. 10.4, "Lem. (10.1) applies to every η", therefore yields one hypersurface through each η, not one hypersurface through all of them. The proof of H-Thm. 10.6 replaces $y(j)$ by $y(j) - c(j)$ and asserts that the tail $g(e) = y(e)$ is unchanged, although by Eq. (84)(7)–(8) the tail is $y(e)$ itself. For a single smooth D a correct argument is available when every $h(j)$ is chosen with $\mathrm{ord}_D\, h(j) \geq p^{e-j}$, as [Hir17, Lem. 16.7, p. 85] asserts is possible. Indeed $\mathrm{ord}_D\, g(0) \geq q$ by Lemma 5.2. Inductively, $g(j+1)^p = g(j) - h(j) \in I(D)^{p^{e-j}}$ gives $g(j+1) \in I(D)^{p^{e-j-1}}$, so in particular $g(e) \in I(D)$. This handles one D at a time, not all of $\mathrm{Sing}(\check{E})$ at once.

Question 12.2. Is the Inv-max stratum of a maximum base-hiked ideal exponent, with the $\wp$-based edge data of Definition 4.4, always contained locally in a smooth hypersurface? (This is the statement of [Hir17, Thms. 10.3–10.4].)

The following lemma excludes the simplest kind of counterexample: hypersurfaces $x^2 + \varepsilon(y,z,w)$ whose order-2 locus contains a monomial curve along which the tangent cone is the square of a linear form. If $\mathrm{Inv} = (4,3,2)$ along such a curve then the tangent cone must have this shape at every point, since $\mathrm{in}_\eta f$ lies in the ideal $(\bar{\ell}^2)$ generated by the edge algebra.

Lemma 12.3. *Let $p = 2$, $f = x^2 + \varepsilon$ with $\varepsilon \in K[[y,z,w]]$, and let $C\colon t \mapsto (t^A, t^a, t^b, t^c)$ with $A, a, b, c \geq 1$ and $\gcd(A,a,b,c) = 1$. Suppose that along C the following vanish: f; the first derivatives $\partial_y\varepsilon, \partial_z\varepsilon, \partial_w\varepsilon$; and the mixed derivatives $\partial_y\partial_z\varepsilon, \partial_y\partial_w\varepsilon, \partial_z\partial_w\varepsilon$. Then $C \subset \{x = \phi(y,z,w)\}$ for some $\phi \in K[[y,z,w]]$. In particular C lies in a smooth hypersurface.*

Proof. For a monomial $m = y^i z^j w^k$ let $D(m) = ia + jb + kc$ be its t-weight and $\sigma(m) = \{\ell :$ the ℓ-th exponent is odd$\} \subset \{1,2,3\}$ its parity class. For each weight D and class σ let $S_D(\sigma)$ be the sum of the coefficients of the monomials of ε with weight D and class σ. There are finitely many such monomials, since $a, b, c \geq 1$. In characteristic 2, $\partial_y(y^i z^j w^k) = i y^{i-1} z^j w^k$ is nonzero exactly when i is odd, and similarly for the mixed derivatives. So the hypotheses say that, for each D, $G_D(T) = \sum_{\sigma \supset T} S_D(\sigma)$ vanishes for $1 \leq |T| \leq 2$, and $G_D(\emptyset) = [D = 2A]$, because $\varepsilon(C(t)) = t^{2A}$. Möbius inversion in characteristic 2 gives $S_D(\sigma) = \sum_{T \supset \sigma} G_D(T)$. Hence all seven classes $\sigma \neq \emptyset$ have the common sum $s_D = G_D(\{1,2,3\})$, and $S_D(\emptyset) = [D = 2A] + s_D$.

Case $s_D = 0$ for all D. Let ε_0 be the part of ε with all exponents even. Since K is perfect, $\varepsilon_0 = \phi^2$. The rest of ε vanishes on C weight by weight, so $\phi(C(t))^2 = \varepsilon_0(C(t)) = t^{2A}$. Hence $\phi(C(t)) = t^A$, and $C \subset \{x = \phi\}$.

Case $s_D \neq 0$ for some D. Then every class has a monomial of weight D. The classes $\{1\}$, $\{2\}$, $\{3\}$ and $\{1,2\}$ give $D \equiv a \equiv b \equiv c$ and $D \equiv a + b \pmod 2$, so a, b, c are even, and A is odd since the gcd is 1. Apply the second Hasse derivative $D_t^{(2)}$ to $\varepsilon(C(t)) = t^{2A}$. By the chain rule,

$$D_t^{(2)}(\varepsilon \circ C) = \sum_j (\partial_j\varepsilon)(C)\, D_t^{(2)} C_j + \sum_{i<j} (\partial_i\partial_j\varepsilon)(C)\, C_i' C_j' + \sum_j (\partial_j^{(2)}\varepsilon)(C)\, (C_j')^2.$$

This vanishes, because the first and mixed derivatives vanish on C and $C_j' = a_j t^{a_j - 1} = 0$ for the even exponents $a_j \in \{a,b,c\}$. But $D_t^{(2)} t^{2A} = \binom{2A}{2} t^{2A-2} = A(2A-1) t^{2A-2} \neq 0$, since A is odd. This is a contradiction. □

Lemma 12.3 and its proof use $p = 2$: they rest on the parity classes and on $\binom{2A}{2} = A(2A-1)$. We do not know an analogue for $p > 2$.

Computational evidence 12.4. Exhaustive $\mathbb{F}_2$-linear searches for $f = x^2 + \varepsilon$ with monomials of degree 3 to 10, and monomial curves of embedding dimension 4 with $A, a, b, c < 24$, found no solution [Cla26a, search scripts], as Lemma 12.3 predicts.

Remark 12.5. For a non-monomial curve C with tangent cone $\mathrm{in}_\eta f = \ell_\eta^2$ at each point $\eta \in C$, the same Hasse computation gives the tangency condition $x'(t)^2 = \sum_j (\partial_j^{(2)}\varepsilon)(C(t))\, C_j'(t)^2$. It says that the tangent line of C lies in $\ker \ell_\eta$. This condition alone does not force C into a smooth hypersurface: in characteristic 2 the curve (t^2, t^3) has tangent in $\ker \bar{x}$ everywhere but does not lie in $\{x = 0\}$. So the shape of a possible counterexample to Question 12.2 beyond Lemma 12.3 remains unclear.

13. Further located defects

The acknowledgments mention "multiple proof-reading and misprints-checking" [Hir17, §1.1, p. 4]. It is natural to expect the text to be complete. In fact it is not.

Observation 13.1. *Two proofs in* [Hir17] *end mid-sentence: the proof of* [Hir17, Thm. 14.4, p. 69] *ends "Infer to Lem. (5.9) of §5. It is important to note that the", and the proof of* [Hir17, Lem. 14.5, p. 70] *ends with the induction hypothesis stated and nothing proved.*

Observation 13.2. *The following statements of* [Hir17] *are misstated, garbled, or have proofs with gaps. Item (8) is stated without verification (see §2).*

(1) [Hir17, Cor. 7.15, p. 40]*: "$\partial_j(x - \rho^N(a_N))^\alpha = \partial_j(x^\alpha)$" is false for values, since in characteristic 2 $\partial_x(x+c)^3 = (x+c)^2 \neq x^2$. It is true for the operators with respect to x and $x - a$.*

(2) [Hir17, Lem. 5.6, p. 27]*: the displayed proof uses the Leibniz inclusion in the wrong direction. The conclusion is trivially true by Theorem 7.1.*

(3) [Hir17, Lem. 3.14, p. 12] *and* [Hir17, Lem. 5.9(4), p. 28]*: the proofs or statements are textual fragments. H-Lemma 3.14 follows from the construction of* [Hir17, Lem. 15.3, p. 75].

(4) [Hir17, Defs. 4.14, 4.18, pp. 22–23]*: ill-typed as printed (Reading 4.5).*

(5) [Hir17, Thm. 14.6, p. 70]*, second line, "$S(T(\check{E}), 1) \subset I(\mathrm{Sing}(\check{E}))\mathrm{Bl}(Z)$", compares a set of LSBs with a module. Its proof invokes "the rule of transformation of the negative degree parts", which is Eq. (59) (Theorem 7.4).*

(6) [Hir17, Rem. 9.11(5)–(7), p. 51]*: "$(h^\flat)^k$" must mean iteration, not power. As a power it does not remove the top monomial: for $g = x^2 + y^3z + z^4$ one has $h^\flat(\varepsilon) = y^3z$, and $\varepsilon - (y^3z)^2$ still contains y^3z.*

(7) [Hir17, Lem. 16.5, p. 83]*: s is defined as "the shortest length of those subsystems of H having non-empty intersection", which is 1 whenever some $\Gamma_i \neq \emptyset$. The intended definition uses subsystems C with $a(C) \geq b$.*

(8) [Hir17, Thms. 8.6–8.7, pp. 44–45]*: the "lexicographical induction" extracting coefficients by $\partial^{(\theta)}h$ ignores the contributions of monomials $x^{\theta'}$ with $\theta' \in \theta + \mathbb{Z}_{\geq 0}^{\dim Z}$.*

14. Open questions

Question 14.1. Does the chain $\mathcal{Y}(0) \rightsquigarrow \mathcal{Y}(1) \rightsquigarrow \cdots$ always stabilise with a smooth terminal plat ([Hir17, Thm. 15.10(3)]), under some reading of $\check{E}$ that does not degenerate in the sense of Propositions 8.1–8.2?

Question 14.2. Is [Hir17, Thm. 7.1] (non-increase of Inv at points of $\pi^{-1}(\xi)$ for the $\wp$-based edge data) true?

A test of the analogous statement for the tangent cone is described in §2.

It is natural to expect that Inv is computable from the tangent cone. The candidate counterexample, $x \in \wp(E_5, 1)$, is false (Proposition 6.3), and for E_n the edge data and the tangent cone agree. So the expectation remains open.

Question 14.3. Is there a maximum base-hiked ideal exponent whose edge data G at a point is not determined by its tangent cone there?

Question 14.4. Is there a canonical ideal exponent $\check{E}$, with finitely generated $\wp(\check{E})$, that retains $\hat{E}$ and has singular locus the Inv-max stratum?

By Proposition 8.2 the two obvious choices fail for smooth Σ: the maximal one forgets $\hat{E}$, and $\hat{E} \cap (I(\Sigma), 1)$ coincides with it.

Question 14.5. Is there an invariant that decreases strictly through kangaroo points in all dimensions?

This is the open problem itself; nothing in [Hir17] bears on it.

Remark 14.6 (on repairability). It is natural to ask whether the technique of [Hir17] is either repairable or demonstrably irreparable. This is a false dichotomy. By Theorem 7.1 the LL-chain reduces to writing $g = y^q + \varepsilon$ and removing the p-th-power part of ε; the tails are therefore hypersurfaces of weak maximal contact chosen in coordinates. Theorem 10.3 and Corollary 10.7 show that the invariant built from them does not decrease. Example 11.2 shows that the natural secondary invariant can increase. What remains after the false steps are removed is Question 14.5. No proof is known that every refinement of the edge data fails, so "demonstrably irreparable" is not established either.

15. Context: local repairs and correct parts

This section lists, without proof, local repairs and parts of [Hir17] that appear correct. It is context only, and none of it repairs the failures of §§7–10. Its provenance is recorded in §2.

Local repairs.

- read H-Defs. 4.14 and 4.18 as in Reading 4.5;
- restrict the second sentence of H-Thm. 6.1 to d divisible by the hike degree;
- prove H-Lem. 3.14 by the construction of H-Lem. 15.3, and state H-Cor. 7.15 for operators;
- define s in H-Lem. 16.5 through subsystems with $a(C) \geq b$;
- prove H-Thm. 10.6 for one smooth D as in Remark 12.1.

The natural repair of H-§6.2, $\check{E} := \hat{E} \cap (I(\Sigma), 1)$, is not a repair (Remark 8.3).

Parts that appear correct. These are: the numerical exponent theorem (H-Thm. 3.15 with H-Prop. 3.26); H-Thms. 4.5 and 6.3, which we used in §7; H-Lems. 7.3–7.5; H-Thm. 7.1, whose proof is only sketched (Question 14.2); H-Prop. 9.15; H-Thm. 9.18 and H-Cor. 9.19; H-Thm. 14.8; H-Lems. 15.2–15.3; and, as strategies, H-Thm. 16.4 ("Theorem Δ") and H-Lem. 16.5. Of these, only H-Thms. 4.5 and 6.3 are used here. H-Theorem 4.5 is proved in §4. H-Theorem 6.3 is used only in the second proof of Theorem 7.1, and the first proof does not need it.

What survives. The parts of [Hir17] that survive the counterexamples are its foundational ones, and none of them reaches resolution.

(i) The algebraic theory of $\wp(E)$: the numerical exponent theorem, H-Thms. 4.5 and 6.3, and the transformation lemmas H-Lems. 7.3–7.5, with H-Thm. 7.1 pending Question 14.2. All of it rests on H-Thm. 4.4, the agreement of the geometric and algebraic definitions of $\wp$, which is imported from [Hir03] and which we have not checked (Remark 6.6).

(ii) The reduction of resolution to a decreasing invariant: H-Thm. 16.4 and H-Lem. 16.5 are correct as strategies, but they assume the termination step that §10 refutes.

The new parts do not survive. By Theorem 7.1 the LL-chain removes the p-th-power part of ε in $g = y^q + \varepsilon$, so its tails are hypersurfaces of weak maximal contact chosen in coordinates. The invariant built from them does not drop (Corollary 10.7), the natural secondary invariant can increase (Example 11.2), and the natural repair of core focusing is degenerate (Remark 8.3). What remains is Question 14.5, together with Question 14.4.

Appendix A. Token accounting

The master token ledger records 1,801,828 tokens for the priced `harvest-dumps` rows available on 26 October 2026. This is a lower bound on that workflow's usage: its first coordinator segment, some final review rounds, and the caller were not recorded. The priced rows were produced with `claude-opus-5-5` in Claude Code. The recorded aggregates do not separate input, output, and cache-read tokens, so those classes below are estimated rather than measured. The breakdown is 721,704 for the resumed coordinator segment, 248,350 for the outline reviewer, 387,996 and 215,008 for the presentation reviewers before and after restructuring, and 228,770 for the integrated reviewer through round 2. The ledger estimated a \$1.6216 token charge for these rows using an assumed 90/8/2 split among cache-read, input, and output tokens and rates checked on 23 October 2026. This is an estimate, not a bill. The `frontier-dump` workflow, later review rounds, and the 26 October Codex revisions are not included in that subtotal because comparable usage measurements are unavailable.

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