

POSITIVE DEFINITE FUNCTIONS OF NONCOMMUTING CONTRACTIONS, HUA–BELLMAN MATRICES, AND A NEW DISTANCE METRIC

SUVRIT SRA*

ABSTRACT. We study positive definiteness of Hua–Bellman matrices (i.e., matrices of the form $[\det(I - A_i^* A_j)^{-\alpha}]_{ij}$, where A_i and A_j are strict contractions and $\alpha \in \mathbb{R}$). We start by revisiting a 1959 work of Bellman (*Representation theorems and inequalities for Hermitian matrices*; Duke Mathematical J., 26(3), 1959) that studies positive definiteness of Hua–Bellman matrices and claims a strengthening of Hua’s representation-theoretic results (*Inequalities involving determinants*; Acta Mathematica Sinica, 5 (1955)). We uncover a critical error in Bellman’s proof that has surprisingly escaped notice to date, and we show by an explicit example that his claim itself is false. We then provide conditions under which $\det(I - A^* B)^{-\alpha}$ is a positive definite function. For complex contractions our condition is the known sharp one, with the integer exponents below Hua’s range reached by an elementary argument. For real symmetric contractions we recover the part of Bellman’s claims that survives. Building on our result, we introduce a new Poincaré-like distance metric on noncommuting strict contractions, and yet another ‘log-det’ based distance in the appendix.

1. INTRODUCTION

We study a basic function on strictly contractive matrices, namely, $\det(I - Z)^{-\alpha}$, where $\|Z\| < 1$ and $\alpha \in \mathbb{R}$. This function arises in a variety of contexts, including complex analysis (Hua, 1955), non-Euclidean geometry (e.g., Riemannian, Finsler, and Kähler geometry) (Tian, 2000; Sra, 2016b; Nielsen, 2020), optimization (Sra and Hosseini, 2015), combinatorics (Goulden and Jackson, 1992; Brändén, 2012; Vere-Jones, 1988), probability theory (Shirai, 2007), matrix analysis (Zhang, 2009; Xu et al., 2009; Ando, 2008), and mathematical physics (Orlov, 2004), among others.

Specifically, we study conditions under which $(A, B) \mapsto \det(I - A^* B)^{-\alpha}$ is a *positive definite function*, i.e., it can be written as an inner product $\langle \Phi_\alpha(A), \Phi_\alpha(B) \rangle$ for a suitable map Φ_α . We warn the reader that we use the term “positive” in its weak sense, e.g., we say a matrix A is *positive*, denoted $A \geq 0$, (*strictly positive*, denoted $A > 0$) if it is Hermitian and all its eigenvalues lie in $\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{> 0}$). The term “positive definite” should also thus be interpreted in the weak sense. We make this slight abuse of terminology to avoid focusing on merely technical conditions that ensure strict positivity but to rather focus on the more important task of ensuring positivity.

The study of positive definite functions has a long history in mathematics, dating back at least to Szegő (1933), who studied the closely related question about positivity of Taylor series coefficients of $P^{-\alpha}$ for the polynomial $P(x) = \sum_{i=1}^n \prod_{j \neq i} (1 - x_j)$. Scott and Sokal (2014) revisit this question for other polynomials such as the determinantal polynomial $P(x) = \det(\sum_i x_i A_i)$, and they study it via complete monotonicity; their results (which circulated as a preprint from 2011) form the basis of the work (Brändén, 2012) that we will need later in this paper.

There is an additional, separate motivation behind our work. It lies in the following elegant identity of Hua (1955) for *strict contractions* A and B (i.e., matrices whose largest singular value is strictly smaller than one):

$$(1.1) \quad I - B^* B + (A - B)^*(I - A^* A)^{-1}(A - B) = (I - A^* B)(I - A A^*)^{-1}(I - B^* A).$$

*TU Munich, Dept. of Mathematics, School of CIT; Garching, Germany. Email: s.sra@tum.de.
Also on arXiv: <https://arxiv.org/abs/2112.00056>.

Hua's identity (1.1) immediately implies positivity of the block matrix

$$\begin{bmatrix} (I - A^*A)^{-1} & (I - A^*B)^{-1} \\ (I - B^*A)^{-1} & (I - B^*B)^{-1} \end{bmatrix},$$

a property that is otherwise not obvious. Subsequent to Hua's discovery, various other works, e.g., (Marcus, 1958), (Bellman, 1959), (Ando, 1980), (Xu et al., 2009, 2011), and (Zhang, 2009), among others, continued the study of Hua-like block matrices and associated operator inequalities. Beyond Hua's seminal work (Hua, 1955), a common denominator of research on this topic has been Bellman's (1959) work that studies Hua matrices through the lens of positive definite functions and integral representations.

Specifically, Bellman claims that for strictly contractive real matrices $A_1, \dots, A_M$ of size $n \times n$, the *Hua–Bellman* matrix

$$(1.2) \quad H_\alpha := [\det(I - A_i^T A_j)^{-\alpha}]_{i,j=1}^M,$$

is positive for α in the set $\{\frac{j}{2} \mid j = 1, 2, \dots\} \cup (\frac{1}{2}(n-1), \infty)$ (his Theorems 3 and 5). The positivity of H_α is tantamount to $\det(I - A^*B)^{-\alpha}$ being a positive definite function for the noted range of α . Bellman claims a significant generalization to Hua's work that had already established $H_\alpha \geq 0$ for $\alpha > n-1$; and seeks to prove his claim in a more elementary manner than Hua, whose results use representation theory and multivariable complex analysis.

While Bellman's approach is foundational, his proof contains an error that impacts subsequent works that rely on his claims. We uncover this old error, and study the positive definiteness of $\det(I - A^*B)^{-\alpha}$ afresh. Our main result on this topic is Theorem 2.1. As we will show, Bellman's claim is false, not merely his proof of it, and only its restriction to symmetric matrices survives. Subsequently, in Theorem 3.1 we present a closely related Poincaré-like geometry on noncommuting strict contractions that may be of independent interest.

1.1. Uncovering an old error.

Claim 1.1 (Theorem 3 in (Bellman, 1959)). *Let $A_1, \dots, A_M$ be strictly contractive real matrices. Then, the Hua-matrix*

$$(1.3) \quad H_{k/2} = \left[\frac{1}{\det(I - A_i^T A_j)^{k/2}} \right]_{i,j=1}^M,$$

is positive semidefinite for all integers $k \geq 1$.

Bellman begins with the well-known Gaussian integral $\frac{1}{\det(A)^{1/2}} \propto \int_{\mathbb{R}^m} e^{-x^T A x} dx$, for a real symmetric positive definite matrix A . Then, on pg. 488, (7.6), he invokes the following inequality of Ostrowski and Taussky (1951):

$$(1.4) \quad \det(I - A_i^T A_j) \geq \det(I - (A_i^T A_j)_s) > 0,$$

where $X_s := \frac{1}{2}(X + X^T)$ denotes the symmetric part of a matrix X . Subsequently, he states that (*sic*. Theorem) Claim 1.1 “... will be demonstrated if we prove that $[\det(I - (A_i^T A_j)_s)^{-k/2}]$ is positive semidefinite.”

This statement is where the error lies: just because inequality (1.4) holds, from positivity of $H_{k/2,s} := [\det(I - (A_i^T A_j)_s)^{-k/2}]$ we cannot conclude that $H_{k/2}$ is also positive definite, since $H_{k/2,s}$ only dominates $H_{k/2}$ entrywise. The claim itself is false, as the following example shows.

Counterexample. Let $n = 2$, $k = 1$, and $M = 6$, and let

$$(1.5) \quad (A_j)_{j=1}^6 = \left(\begin{bmatrix} -2 & -9 \\ -5 & -10 \end{bmatrix}, \begin{bmatrix} 9 & -5 \\ 9 & 6 \end{bmatrix}, \begin{bmatrix} -10 & -3 \\ -6 & 3 \end{bmatrix}, \begin{bmatrix} -8 & -8 \\ 1 & -10 \end{bmatrix}, \begin{bmatrix} -2 & 1 \\ -6 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ 10 & -6 \end{bmatrix} \right),$$

$$A_j \leftarrow \frac{1}{2} \frac{A_j}{\|A_j\|} \quad \text{for } 1 \leq j \leq 6,$$

so that each A_j is a real strict contraction of operator norm $\frac{1}{2}$. The 6×6 matrix $H_{1/2} = [\det(I - A_i^T A_j)^{-1/2}]_{i,j=1}^6$ has smallest eigenvalue

$$\lambda_{\min}(H_{1/2}) \approx -1.2066 \times 10^{-3},$$

so $H_{1/2}$ is not positive.¹ Bellman’s claim thus fails already for 2×2 real matrices and $k = 1$.

What, then, survives of Bellman’s claim? We show that $H_\alpha \geq 0$ for every integer $\alpha \geq 0$ and for complex contractions of any size (Theorem 2.1-(i)); together with Hua’s range $\alpha > n - 1$ this is the complete set of admissible exponents for complex contractions (Faraut and Korányi, 1990), and our proof of the integer part, via α -permanents, is elementary. We also show that half-integer exponents are admissible when the A_i are real and symmetric (Theorem 2.1-(ii)), which is the symmetric case of Bellman’s claims.

2. POSITIVE DEFINITE FUNCTIONS ON NONCOMMUTING CONTRACTIONS

In this section we present results characterizing the positive definiteness of Hua–Bellman matrices. To that end, we study the corresponding function $\det(I - A^* B)^{-\alpha}$ on $A, B \in \mathcal{C}_n$, the class of $n \times n$ strictly contractive (in operator norm) complex matrices.² We provide sufficient conditions on α to ensure the desired positive definiteness.

We are now ready to state the first main result of this paper.

Theorem 2.1. *Let $A, B \in \mathcal{C}_n$. Then, $\det(I - A^* B)^{-\alpha}$ is a positive definite function:*

(i) *for $\alpha \in \{0, 1, 2, \dots\} \cup (n - 1, \infty)$; and*

(ii) *for $\alpha \in \{\frac{j}{2} \mid j = 0, 1, 2, \dots\} \cup (\frac{n-1}{2}, \infty)$ if A, B are real and symmetric.*

Part (i) contains Hua’s result ($\alpha > n - 1$) and adds the integers $0, 1, \dots, n - 1$. The set in part (i) is the Wallach set of the matrix ball, and by Faraut and Korányi (1990) (see also (Faraut and Korányi, 1994, Sec. XIII.2)) it is exactly the set of α for which $\det(I - A^* B)^{-\alpha}$ is positive definite on $\mathcal{C}_n$; so part (i) is sharp, and the point of the proof below is that the integers $0, 1, \dots, n - 1$ are reached by an elementary argument. Part (ii) covers Theorem 5 of Bellman’s paper, albeit with an additional requirement of symmetry.³ We note in passing that the set of valid α noted in part-(ii) is precisely the Gindikin–Wallach set—see e.g., (Scott and Sokal, 2014) and (Belton et al., 2022, pg. 42) and references therein for additional connections of this set. Bellman closes his Theorem 5 with the words “what the precise exponent should be is not clear”. For real symmetric contractions the Gindikin set is the answer: by Gindikin’s theorem (Gindikin, 1964) (see also (Faraut and Korányi, 1994, Ch. VII)) it is exactly the set of α for which $\det(X + Y)^{-\alpha}$ is a positive definite kernel on the cone of positive definite real symmetric matrices, and the Cayley transform in the proof of part (ii) below shows that $\det(I - A^* B)^{-\alpha}$ on real symmetric contractions is positive definite for exactly the same α ; so part (ii) is sharp as well.

The key ingredient in our proof of Theorem 2.1 is the α -permanent, which we now recall.

¹The value was confirmed in 60-digit arithmetic. Every 5×5 principal submatrix of this $H_{1/2}$ is positive.

²To reduce clutter, we limit ourselves to square matrices; similar arguments can be made when A, B are $n \times k$ strict contractions.

³We present this result since the error in the proof of Claim 1.1 also impacts Theorem 5 of Bellman, wherein he claims that “the proof proceeds along the same lines...”.

2.1. The α -permanent. The α -permanent generalizes the matrix permanent and determinant, and was introduced by Vere-Jones (1988). Let $\alpha \in \mathbb{C}$ and $A = (a_{ij})$ be an $n \times n$ matrix. Then, the α -permanent is defined as

$$(2.1) \quad \text{per}_\alpha(A) := \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\#\sigma} \prod_{i=1}^n a_{i,\sigma(i)},$$

where $\#\sigma$ denotes the number of disjoint cycles in the permutation σ . This object interpolates between the determinant and permanent (indeed, $\text{per}_1(A) = \text{per}(A)$, while $\text{per}_{-1}(A) = (-1)^n \det(A)$), and it enjoys a variety of applications and connections; see e.g., (Shirai, 2007; Crane, 2013; Frenkel, 2010). The key property of α -permanents that we will need is an “inner-product” representation derived in Lemma 2.2.

Lemma 2.2. *Let $A, B \in \mathbb{C}^{n \times m}$, $\alpha \in \mathbb{R}$, and $n, m \geq 1$. Then, there exist scalars $\{c_\lambda^\alpha\}_{\lambda \vdash m}$ and nonlinear maps $\{\psi_\lambda\}_{\lambda \vdash m} : \mathbb{C}^{n \times m} \rightarrow \mathbb{C}^{n_\lambda \times m_\lambda}$ for suitable integers $\{n_\lambda, m_\lambda\}_{\lambda \vdash m}$ ($\lambda \vdash m$ signifies that λ is a partition of m), such that the identity*

$$\text{per}_\alpha(A^*B) = \sum_{\lambda \vdash m} c_\lambda^\alpha \langle \psi_\lambda(A), \psi_\lambda(B) \rangle,$$

holds, where for each λ , $\langle \cdot, \cdot \rangle$ is the associated trace inner product on $\mathbb{C}^{n_\lambda \times m_\lambda}$.

Proof. The proof relies on the (known) observation that the α -permanent can be written in terms of immanants. The *immanant* of $X \in \mathbb{C}^{m \times m}$ indexed by a partition λ of m is defined as (see (Merris, 1997) for details)

$$(2.2) \quad d_\lambda(X) := \sum_{\sigma \in \mathfrak{S}_m} \chi_\lambda(\sigma) \prod_{i=1}^m x_{i,\sigma(i)},$$

where $\chi_\lambda(\sigma)$ is the character associated to the irreducible representation S^λ , the Specht module corresponding to $\lambda \vdash m$ —see e.g., (Fulton and Harris, 1991). Then, Crane (2013, Eq. (12)) shows that for each λ , there exist constants c_λ^α such that:⁴

$$(2.3) \quad \text{per}_\alpha(X) = \sum_{\lambda \vdash m} c_\lambda^\alpha d_\lambda(X).$$

From classic results, e.g., (Merris, 1997, Thm. 7.26), it follows that for each $\lambda \vdash m$, there exists a matrix P_λ with m^m rows and m_λ columns such that $d_\lambda(X) = \text{tr } P_\lambda^* X^{\otimes m} P_\lambda$. Thus (with $n_\lambda = m^m$),

$$(2.4) \quad d_\lambda(A^*B) = \text{tr } P_\lambda^* (A^*B)^{\otimes m} P_\lambda = \text{tr } P_\lambda^* (A^{\otimes m})^* (B^{\otimes m}) P_\lambda =: \langle \psi_\lambda(A), \psi_\lambda(B) \rangle,$$

where $\psi_\lambda : A \mapsto A^{\otimes m} P_\lambda$. Using representation (2.4) in (2.3) we obtain the identity

$$\text{per}_\alpha(A^*B) = \sum_{\lambda \vdash m} c_\lambda^\alpha \langle \psi_\lambda(A), \psi_\lambda(B) \rangle,$$

which shows that $\text{per}_\alpha(A^*B)$ is indeed a weighted sum of inner products. $\square$

While Lemma 2.2 shows that $\text{per}_\alpha(A^*B)$ can be written as a weighted sum of inner products, the coefficients c_λ^α can be negative, in which case the sum need not be a positive definite kernel. For nonnegative integers α they are not.

Proposition 2.3. *Let $k \in \mathbb{Z}_{\geq 0}$ and $m \geq 1$. Then $c_\lambda^k \geq 0$ for every $\lambda \vdash m$; indeed, c_λ^k is the multiplicity of the Specht module S^λ in the representation of $\mathfrak{S}_m$ on $(\mathbb{C}^k)^{\otimes m}$ that permutes the tensor factors.*

Proof. A permutation σ fixes the basis tensor $e_{i_1} \otimes \cdots \otimes e_{i_m}$ iff $(i_1, \dots, i_m)$ is constant on the cycles of σ , so the character of this representation is $\sigma \mapsto k^{\#\sigma}$. Hence $c_\lambda^k = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} k^{\#\sigma} \chi_\lambda(\sigma)$ is the inner product of this character with χ_λ , i.e., the multiplicity of S^λ , a nonnegative integer. (By Schur–Weyl duality it equals the number of semistandard Young tableaux of shape λ with entries in $\{1, \dots, k\}$; see (Fulton and Harris, 1991).) $\square$

⁴(Crane, 2013, Theorem 2.4) notes that $c_\lambda^\alpha = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \alpha^{\#\sigma} \chi_\lambda(\sigma)$.

For non-integer α the coefficients can be negative: for $m = 2$ one has $\text{per}_\alpha = \frac{\alpha^2 + \alpha}{2} \text{per} + \frac{\alpha^2 - \alpha}{2} \det$, so $c_{(1,1)}^{1/2} = -\frac{1}{8}$. Brändén (2012), building on the complete monotonicity results of Scott and Sokal (2014) (then a preprint), shows that nevertheless $\text{per}_\alpha(C) \geq 0$ for every real positive matrix C when α is a half-integer (his Theorem 2.3 is stated for the α -determinant $\det_\alpha(C) := \alpha^n \text{per}_{1/\alpha}(C)$). For the kernel $(A, B) \mapsto \text{per}_\alpha(A^T B)$ this is only the one-point condition $\text{per}_\alpha(A^T A) \geq 0$, which is weaker than positive definiteness; the counterexample of Section 1.1 shows that the difference is real.

In addition to Lemma 2.2 and Proposition 2.3, our proof of Theorem 2.1 uses the following generalization of MacMahon’s Master Theorem.

Theorem 2.4 ((Vere-Jones, 1988; Foata and Zeilberger, 1988)). *Let $A = [a_{ij}]_{i,j=1}^n$ and let $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{N}^n$, where $\mathbb{N} = \{0, 1, 2, \dots\}$. Let $A[\mathbf{m}]$ be the $|\mathbf{m}| \times |\mathbf{m}|$ matrix with $|\mathbf{m}| = \sum_{j=1}^n m_j$, obtained by replacing the (i, j) -entry of A by the $m_i \times m_j$ matrix $a_{ij} \mathbf{1}_{m_i} \mathbf{1}_{m_j}^T$. Let $X = \text{Diag}(x_1, \dots, x_n)$ and $\alpha \in \mathbb{C}$. Then,*

$$(2.5) \quad \det(I - XA)^{-\alpha} = \sum_{\mathbf{m} \in \mathbb{N}^n} \frac{\mathbf{x}^{\mathbf{m}}}{\mathbf{m}!} \text{per}_\alpha(A[\mathbf{m}]),$$

where $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \cdots x_n^{m_n}$ and $\mathbf{m}! = m_1! \cdots m_n!$.

2.2. Proof of Theorem 2.1, part (i). First, recall from (Hua, 1955) that if $\alpha > n - 1$, then $\det(I - A^* B)^{-\alpha}$ is a positive definite function⁵, a result that holds for complex contractions (and thus *a fortiori* also for real ones). What remains to prove is that the integers $\alpha = 0, 1, \dots, n - 1$ are admissible (the case $\alpha = 0$ being trivial). The main idea is to invoke Theorem 2.4 and use it to represent $\det(I - A^* B)^{-\alpha}$ as an inner product $\langle \Phi_\alpha(A), \Phi_\alpha(B) \rangle$, where Φ_α maps into a suitable Hilbert space. As a consequence, we obtain that the associated Hua–Bellman matrix H_α is a Gram matrix, and hence positive.

The first step is to realize that $A[\mathbf{m}] = Q_{\mathbf{m}}^*(A \otimes \mathbf{1}\mathbf{1}^T)Q_{\mathbf{m}}$ for matrices $Q_{\mathbf{m}}$ and $\mathbf{1}\mathbf{1}^T$ of appropriate sizes induced by Theorem 2.4. Let E_{ij} denote the $m_i \times m_j$ matrix of all ones, and $\mathbf{1}\mathbf{1}^T = E = [E_{ij}]_{i,j=1}^n$ the corresponding block matrix of size $|\mathbf{m}| \times |\mathbf{m}|$.⁶ Then, consider the Khatri–Rao product between conformally partitioned matrices A and E :

$$(2.6) \quad A * E = \begin{bmatrix} a_{11} \otimes E_{11} & \cdots & a_{1n} \otimes E_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} \otimes E_{n1} & \cdots & a_{nn} \otimes E_{nn} \end{bmatrix},$$

which is nothing but $A[\mathbf{m}]$ since $a_{ij} \otimes E_{ij} = a_{ij} E_{ij}$. From basic properties of Khatri–Rao products (Liu and Trenkler, 2008) it follows that there is a matrix $Q_{\mathbf{m}}$ such that $A[\mathbf{m}] = Q_{\mathbf{m}}^*(A \otimes E)Q_{\mathbf{m}}$. More explicitly, let $Q_{\mathbf{m}}$ be the diagonal matrix $\text{Diag}(U_1^{\mathbf{m}}, \dots, U_n^{\mathbf{m}})$, where each $U_i^{\mathbf{m}}$ is an $|\mathbf{m}| \times m_i$ matrix formed by suitable columns of the identity matrix $I_{|\mathbf{m}|}$, such that $(U_i^{\mathbf{m}})^* E U_j^{\mathbf{m}} = E_{ij}$.

⁵Bellman (1959) recalls this result of Hua as Theorem 2 of his paper.

⁶To reduce notational burden, we let the dependency of blocks of E on $\mathbf{m}$ to remain implicit.

Therewith, we have the factorization

$$\begin{aligned}
 Q_m^*(A \otimes E)Q_m &= \begin{bmatrix} (U_1^m)^* & & \\ & \ddots & \\ & & (U_n^m)^* \end{bmatrix} \begin{bmatrix} a_{11}E & \cdots & a_{1n}E \\ \vdots & \ddots & \vdots \\ a_{n1}E & \cdots & a_{nn}E \end{bmatrix} \begin{bmatrix} U_1^m & & \\ & \ddots & \\ & & U_n^m \end{bmatrix} \\
 (2.7) \quad &= \begin{bmatrix} a_{11}(U_1^m)^*EU_1^m & a_{12}(U_1^m)^*EU_2^m & \cdots & a_{1n}(U_1^m)^*EU_n^m \\ \vdots & \vdots & & \vdots \\ a_{n1}(U_n^m)^*EU_1^m & a_{n2}(U_n^m)^*EU_2^m & \cdots & a_{nn}(U_n^m)^*EU_n^m \end{bmatrix} \\
 &= \begin{bmatrix} a_{11}E_{11} & \cdots & a_{1n}E_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1}E_{n1} & \cdots & a_{nn}E_{nn} \end{bmatrix} = A[m].
 \end{aligned}$$

From identity (2.5) we have $\det(I - XA^*B)^{-\alpha} = \sum_{m \in \mathbb{N}^n} \frac{x^m}{m!} \text{per}_\alpha((A^*B)[m])$, so that using factorization (2.7) we can write

$$(2.8) \quad (A^*B)[m] = Q_m^*(A^*B \otimes \mathbf{1}\mathbf{1}^T)Q_m = Q_m^*((A \otimes \mathbf{1}^T)^*(B \otimes \mathbf{1}^T))Q_m =: \tilde{A}_m^* \tilde{B}_m.$$

Finally, identity (2.8) combined with Lemma 2.2 immediately implies that

$$(2.9) \quad \text{per}_\alpha((A^*B)[m]) = \sum_{\lambda \vdash |m|} c_\lambda^\alpha \langle \psi_\lambda(\tilde{A}_m), \psi_\lambda(\tilde{B}_m) \rangle,$$

for suitable nonlinear maps $\{\psi_\lambda\}_{\lambda \vdash |m|}$. Each summand $\langle \psi_\lambda(\tilde{A}_m), \psi_\lambda(\tilde{B}_m) \rangle$ is a Gram kernel, hence positive definite, and by Proposition 2.3 the coefficients c_λ^α are nonnegative for integer α . Thus (2.9) is a positive definite kernel in (A, B) .

Now invoking Theorem 2.4, we see that $\det(I - A^*B)^{-\alpha}$ is a nonnegative sum of positive definite kernels. Consequently, there exists a suitable nonlinear map Φ_α (that maps its argument to a suitable Hilbert space) so that $\det(I - A^*B)^{-\alpha} = \langle \Phi_\alpha(A), \Phi_\alpha(B) \rangle$, concluding the proof. $\square$

Remark. For integer α there is also a direct route that avoids α -permanents: since $\det(I - Z)^{-1} = \sum_{k \geq 0} \text{tr Sym}^k(Z)$ for $\|Z\| < 1$ and $\text{Sym}^k(A^*B) = \text{Sym}^k(A)^*\text{Sym}^k(B)$, the kernel $\det(I - A^*B)^{-1} = \sum_k \langle \text{Sym}^k(A), \text{Sym}^k(B) \rangle$ is positive definite, and its integer powers are positive definite by the Schur product theorem. The α -permanent expansion is kept because it exhibits the structure of the coefficients (Proposition 2.3) and shows exactly where the half-integer case breaks down.

It remains to prove Theorem 2.1-(ii). To that, we first recall a closely related result.

Theorem 2.5 (Gindikin (Gindikin, 1964); stated in this form in (Sra, 2016b, Theorem 3.10)). *Let $A_1, \dots, A_M \in \mathbb{R}^{n \times n}$ be symmetric and strictly positive. Then, the matrix*

$$(2.10) \quad [\det(A_i + A_j)^{-\alpha}]_{i,j=1}^M,$$

is positive whenever $\alpha \in \{\frac{j}{2} \mid j = 0, 1, 2, \dots\} \cup (\frac{n-1}{2}, \infty)$.

2.3. Proof of Theorem 2.1, part (ii). The key idea is to introduce a change of variables that turns $\det(I - A^*B)^{-\alpha}$ essentially into the form (2.10). Specifically, consider the *Cayley transformation* $X = (I - A)(I + A)^{-1}$ (similarly, $Y = (I - B)(I + B)^{-1}$). Matrix X (similarly, Y) is well-defined because $I + A$ is invertible due to A being a strict contraction. The reverse transformation is $X \mapsto (I - X)(I + X)^{-1}$, which is also well-defined because $I + X = 2I(I + A)^{-1}$ is also invertible. Using this transformation on A and B we obtain

$$\begin{aligned}
 I - A^*B &= I - (I + X^*)^{-1}(I - X^*)(I - Y)(Y + I)^{-1} \\
 &= I - (I + X^*)^{-1}[I + X^*Y - X^* - Y](Y + I)^{-1} \\
 &= (I + X^*)^{-1}[(I + X^*)(Y + I) - X^*Y + X^* + Y - I](Y + I)^{-1} \\
 &= (I + X^*)^{-1}[2X^* + 2Y](Y + I)^{-1}.
 \end{aligned}$$

Next, recall the notation $\operatorname{Re}(Z) = \frac{Z+Z^*}{2}$ and $\operatorname{Im}(Z) = \frac{Z-Z^*}{2i}$ (where $i = \sqrt{-1}$), so that any complex matrix Z can be written as $Z = \operatorname{Re}(Z) + i\operatorname{Im}(Z)$. Since $A \in \mathcal{C}_n$, $I - A^*A = (I + X^*)^{-1}[2X^* + 2X](I + X)^{-1}$ is strictly positive. Thus,

$$\operatorname{Re}(X) = \frac{1}{4}(I + X^*)(I - A^*A)(I + X),$$

which is clearly strictly positive. If A and B are Hermitian, then $\operatorname{Re}(X) = X = X^* > 0$ (similarly, $Y > 0$), whereby

$$(2.11) \quad \det(I - A^*B)^{-\alpha} = \frac{\det(I + X)^\alpha \det(I + Y)^\alpha}{\det(2(X + Y))^\alpha},$$

whose positivity for the range of values claimed in Theorem 2.1-(ii) immediately follows from Theorem 2.5, once we additionally assume A and B to be real.

3. A POINCARÉ-LIKE GEOMETRY ON NONCOMMUTING CONTRACTIONS

In this section, we introduce a new Poincaré-like geometry on the space of strict contractions. The distance function introduced is suggested by Hua–Bellman matrices, whose positivity yields the positivity of the proposed distance. The second main result of this paper is Theorem 3.1, which formally introduces the said geometry. We state the results for the set of $n \times k$ strict contractions, denoted $\mathcal{C}_{n,k}$.

Theorem 3.1. *Define $\delta : \mathcal{C}_{n,k} \times \mathcal{C}_{n,k} \rightarrow \mathbb{R}_{\geq 0}$ as*

$$(3.1) \quad \delta^2(A, B) := \log \frac{|\det(I - A^*B)|}{\sqrt{\det(I - A^*A)}\sqrt{\det(I - B^*B)}}, \quad \text{for } A, B \in \mathcal{C}_{n,k}.$$

Then, $(\mathcal{C}_{n,k}, \delta)$ is a metric space.

It is worth noting that positivity of an $M \times M$ Hua–Bellman matrix depends only on the dimensionality n of its constituent matrices and not on M , and thus, Theorem 2.1 holds for arbitrary M . In contrast, the distance function of Theorem 3.1 corresponds to an absolute value version of H_α that holds for $M = 3$, and for which a much wider range of values of α is admissible. More precisely, we have the following corollary:

Corollary 3.2. *Let A , B , and $C \in \mathcal{C}_{n,k}$. Let $\alpha \geq 0$. Then, the following matrix*

$$(3.2) \quad H_{\frac{\alpha}{2}} \circ \overline{H}_{\frac{\alpha}{2}} = \begin{bmatrix} \frac{1}{\det(I - A^*A)^\alpha} & \frac{1}{|\det(I - A^*B)|^\alpha} & \frac{1}{|\det(I - A^*C)|^\alpha} \\ \frac{1}{|\det(I - B^*A)|^\alpha} & \frac{1}{\det(I - B^*B)^\alpha} & \frac{1}{|\det(I - B^*C)|^\alpha} \\ \frac{1}{|\det(I - C^*A)|^\alpha} & \frac{1}{|\det(I - C^*B)|^\alpha} & \frac{1}{\det(I - C^*C)^\alpha} \end{bmatrix},$$

is positive, where H_α denotes a Hua–Bellman matrix as before.

Proof. Since δ^2 is the square of a distance, we know from classical results (see e.g., the exposition in (Sra, 2021, App. A)) that the matrix

$$M := \begin{bmatrix} 0 & \delta^2(A, B) & \delta^2(A, C) \\ \delta^2(A, B) & 0 & \delta^2(B, C) \\ \delta^2(A, C) & \delta^2(B, C) & 0 \end{bmatrix}$$

is conditionally negative definite. Hence, its elementwise exponentiation $M' := [e^{-\alpha m_{ij}}]_{i,j=1}^3$ is positive (i.e., $M' \geq 0$) for all $\alpha \geq 0$ —see e.g., (Berg et al., 1984, Thm. 3.2.2). We conclude the proof by introducing the diagonal matrix

$$D = \begin{bmatrix} \frac{1}{\det(I - A^*A)^{\frac{\alpha}{2}}} & & \\ & \frac{1}{\det(I - B^*B)^{\frac{\alpha}{2}}} & \\ & & \frac{1}{\det(I - C^*C)^{\frac{\alpha}{2}}} \end{bmatrix}$$

and observing that $H_{\frac{\alpha}{2}} \circ \overline{H}_{\frac{\alpha}{2}} = DM'D \geq 0$. $\square$

Remark. The range $\alpha \geq 0$ proved in Corollary 3.2 is non-trivial. Indeed, from Theorem 2.1 it follows that $H_1 \geq 0$. Since here H_1 is 3×3 , an observation of Marcus and Watkins (1971) shows that its elementwise absolute value $|H_1|$ is again positive (this does not follow from Theorem 2.1, as that would require $H_{1/2} \geq 0$). Consequently, from a well-known result of FitzGerald and Horn (1977) on Hadamard powers, since $M = 3$, we can only conclude positive definiteness of the Hadamard power $|H_1|^{\circ\alpha}$ for $\alpha \geq M - 2 = 1$, whereas Corollary 3.2 permits $\alpha \geq 0$.

Corollary 3.3 explains why we call the geometry induced by distance (3.1) Poincaré-like.

Corollary 3.3. *Let $A, B \in \mathcal{C}_{n,1}$, i.e., vectors of Euclidean norm less than one. Then*

$$\delta^2(A, B) = \log \frac{|1 - A^*B|}{\sqrt{(1 - \|A\|^2)}\sqrt{(1 - \|B\|^2)}}.$$

For real vectors this equals $\log \cosh d(A, B)$, where d is the Cayley–Klein–Hilbert (Beltrami–Klein) distance of the unit ball (Deza and Deza, 2013, pg. 120); for complex vectors it equals $\log \cosh(\frac{1}{2}d_P(A, B))$, where d_P is the Poincaré–Bergman distance of the complex unit ball, normalized so that for $n = 1$ it is the Poincaré distance of the disk.

Before proving Theorem 3.1, we first recall a closely related distance function.

Theorem 3.4 (S-Divergence (Sra, 2016b)). *Let X, Y be strictly positive definite. Then,*

$$(3.3) \quad \delta_S^2(X, Y) := \log \frac{\det\left(\frac{X+Y}{2}\right)}{\sqrt{\det(X)}\sqrt{\det(Y)}},$$

is the square of a distance, i.e., δ_S is a distance.

At this point, one may wonder whether we can use the Cayley transformation used in the proof of Theorem 2.1-(ii) to approach Theorem 3.1. Directly applying this transformation to $\delta^2(A, B)$ leads to the following object:

$$(3.4) \quad \log \frac{|\det(X^* + Y)|}{\sqrt{\det(X^* + X)}\sqrt{\det(Y^* + Y)}}.$$

This quantity bears a close resemblance to the S-Divergence (3.3), yet it does not seem amenable to the proof technique developed in (Sra, 2016b) for proving the triangle inequality for the S-Divergence.⁷ On the other hand, if A and B were Hermitian, then identity (2.11) would apply, and the quantity (3.4) would reduce to the S-Divergence (note $\text{Re}(X), \text{Re}(Y)$ are strictly positive definite). This observation was crucial to the discovery of our proof of Theorem 3.1 below.

3.1. Proof of Theorem 3.1. The key idea is to consider the *Hermitian dilations* of A and B , i.e.,

$$(3.5) \quad P := \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}, \quad Q := \begin{bmatrix} 0 & B^* \\ B & 0 \end{bmatrix}.$$

Then, computing $\det(I - P^*Q)$ we see that

$$(3.6) \quad \det(I - P^*Q) = \det\left(\begin{bmatrix} I - A^*B & 0 \\ 0 & I - AB^* \end{bmatrix}\right) = \det(I - A^*B) \det(I - AB^*).$$

But $\det(I - AB^*) = \det(I - B^*A)$, so that $\det(I - P^*Q) = |\det(I - A^*B)|^2$. Consequently,

$$(3.7) \quad \frac{|\det(I - A^*B)|}{\sqrt{\det(I - A^*A)}\sqrt{\det(I - B^*B)}} = \frac{\sqrt{\det(I - P^*Q)}}{\sqrt{\det(I - P^*P)}\sqrt{\det(I - Q^*Q)}}.$$

⁷Indeed, in the initial version of this article, there was an error that arose from following that approach.

From identity (3.7) upon taking logarithms it follows that

$$(3.8) \quad \delta^2(A, B) = \frac{1}{2} \delta^2(P, Q).$$

Since P and Q are Hermitian strict contractions, as argued above, we can reduce $\delta^2(P, Q)$ to a corresponding S-Divergence, whereby upon invoking Theorem 3.4, we immediately conclude that δ is a distance metric.

Remarks: We believe that the distance function (3.1) may find several applications. Our belief is based on the diverse body of applications the S-Divergence (3.3) enjoys, e.g., in computer vision (Cherian et al., 2013), brain-computer interfaces and imaging (Yger et al., 2017), matrix means (Sra, 2016b; Chebbi and Moakher, 2012), geometric optimization (Sra and Hosseini, 2015), numerical linear algebra (Sra, 2016a), signal processing (Bouchard et al., 2018), machine learning (Zern et al., 2019; Tiomoko et al., 2019), quantum information theory (Virosztek, 2021), among many others.

ACKNOWLEDGEMENTS

We thank the referee for their exceptional review of an initial version of this paper, including uncovering an error in our initial proof (which we have now fixed). We also acknowledge support from an NSF CAREER grant (IIS-1846088), and NSF CCF-2112665 (TILOS AI Research Institute). The author (SS) thanks the Alexander von Humboldt (AvH) Foundation for their generous support via a Humboldt Professorship in AI.

APPENDIX A. A RELATED DISTANCE ON ARBITRARY COMPLEX MATRICES

An earlier version of this article proved the triangle inequality for δ by first applying the Cayley transformation and then invoking the theorem below. That route turned out to be flawed (see the footnote following (3.4)), and the Hermitian dilation argument of Section 3 supersedes it. Theorem A.1 is nevertheless valid on its own, and since it seems to be of independent interest, we record it here; it does not play any role in the results of Sections 1–3.

Theorem A.1. *Let X, Y be arbitrary complex matrices, and $0 < p \leq 2$, then*

$$(A.1) \quad \rho_p^2(X, Y) := \log \det(I + |X - Y|^p),$$

*is the square of a distance, i.e., ρ_p is a distance (here $|X| := (X^*X)^{1/2}$).*

We will need the following simple observation in our proof of Theorem A.1:

Lemma A.2. *Let $0 < p \leq 2$. The function $f(t) = \sqrt{\log(1 + t^p)}$ is concave on $(0, \infty)$.*

Proof. We prove that $f''(t) \leq 0$. Since $f''(t) = -\frac{pt^{p-2}(pt^p + 2(t^p - p + 1)\log(t^p + 1))}{4(t^p + 1)^2 \log^{\frac{3}{2}}(t^p + 1)}$, it suffices to verify that $(pt^p + 2(t^p - p + 1)\log(t^p + 1)) \geq 0$. Writing $t^p = x$, this inequality is equivalent to $px + 2(1 + x)\log(1 + x) \geq 2p\log(1 + x)$. Since $(1 + x)\log(1 + x) \geq x$, the left-hand side exceeds $px + 2x$ which combined with the inequality $\log(1 + x) \leq x$ yields $px + 2x \geq (p + 2)\log(1 + x) \geq 2p\log(1 + x)$ as desired since $p \leq 2$. $\square$

Proof of Theorem A.1. Nonnegativity of ρ_p and $\rho_p(X, Y) = 0$ iff $X = Y$ are immediate from (A.1), while symmetry follows since $|X - Y|$ and $|Y - X|$ have the same singular values. Thus, only the triangle inequality requires an argument.

The key idea is to reduce the question to a setting where we can apply a subadditivity theorem for singular values. For complex matrices A, B, C of size $n \times n$ such that $C = A + B$, and any concave function $f : \mathbb{R}_+ \rightarrow [0, \infty)$ with $f(0) = 0$, we have

$$(A.2) \quad \{f(\sigma_j(C))\}_{j=1}^n \prec_w \{f(\sigma_j(A)) + f(\sigma_j(B))\}_{j=1}^n,$$

where $\prec_w$ denotes the weak-majorization partial order (see e.g., (Bhatia, 1997, Chapter 2)), and $\sigma_j(\cdot)$ denotes the j -th singular value (in decreasing order). The case of the full sum ($k = n$ in the partial sums defining $\prec_w$) is the classical inequality of Rotfel'd (1969); the weak-majorization form (A.2), equivalently the same inequality in every unitarily invariant norm, is due to Uchiyama (2006), and it is this form that we need below.

Let $A = X - Z$, $B = Z - Y$, and $C = X - Y$; let a_j , b_j , and c_j be their singular values. Since $f(t) = \sqrt{\log(1 + t^p)}$ is concave (see Lemma A.2), from (A.2) we thus obtain

$$(A.3) \quad \{f(c_j)\}_j \prec_w \{f(a_j) + f(b_j)\}_j.$$

We further know that if $x \prec_w y$ for $x, y \in \mathbb{R}_+^n$, then $x^s \prec_w y^s$ for $s \geq 1$ (see e.g., (Bhatia, 1997, Example II.3.5)). Consequently, for $s = 2$ from inequality (A.3) we obtain

$$(A.4) \quad \{f(c_j)^2\}_j \prec_w (\{f(a_j) + f(b_j)\}_j)^2.$$

Thus, in particular it follows from the majorization inequality (A.4) that

$$(A.5) \quad \sum_{j=1}^n \log(1 + c_j^p) \leq \sum_j \left(\sqrt{\log(1 + a_j^p)} + \sqrt{\log(1 + b_j^p)} \right)^2,$$

so that upon taking square roots on both sides we obtain

$$\sqrt{\sum_{j=1}^n \log(1 + c_j^p)} \leq \sqrt{\sum_j \left(\sqrt{\log(1 + a_j^p)} + \sqrt{\log(1 + b_j^p)} \right)^2}.$$

Applying Minkowski's inequality on the right hand side we get

$$(A.6) \quad \sqrt{\sum_{j=1}^n \log(1 + c_j^p)} \leq \sqrt{\sum_j \log(1 + a_j^p)} + \sqrt{\sum_j \log(1 + b_j^p)}.$$

But $\sum_j \log(1 + c_j^p) = \log \det(I + |C|^p) = \log \det(I + |X - Y|^p) = \rho_p^2(X, Y)$. Thus, inequality (A.6) is nothing but the triangle inequality

$$\rho_p(X, Y) \leq \rho_p(X, Z) + \rho_p(Y, Z),$$

which concludes the proof. $\square$

REFERENCES

- T. Ando. Hua–Marcus inequalities. *Linear and Multilinear Algebra*, 8(4):347–352, 1980. doi: 10.1080/03081088008817340.
- T. Ando. Positivity of operator-matrices of Hua-type. *Banach Journal of Mathematical Analysis*, 2(2):1–8, 2008. doi: 10.15352/bjma/1240336286.
- Richard Bellman. Representation theorems and inequalities for Hermitian matrices. *Duke Mathematical Journal*, 26(3):485–490, 1959. doi: 10.1215/s0012-7094-59-02645-6.
- Alexander Belton, Dominique Guillot, Apoorva Khare, and Mihai Putinar. Preservers of totally positive kernels and Pólya frequency functions. *Mathematics Research Reports*, 3:35–56, 2022. doi: 10.5802/mrr.12.
- Christian Berg, Jens Peter Reus Christensen, and Paul Ressel. *Harmonic analysis on semigroups: theory of positive definite and related functions*, volume 100 of *Graduate Texts in Mathematics*. Springer, New York, 1984. doi: 10.1007/978-1-4612-1128-0.
- Rajendra Bhatia. *Matrix Analysis*. Graduate Texts in Mathematics 169. Springer, New York, 1997. doi: 10.1007/978-1-4612-0653-8.
- Florent Bouchard, Jérôme Malick, and Marco Congedo. Riemannian optimization and approximate joint diagonalization for blind source separation. *IEEE Transactions on Signal Processing*, 66(8): 2041–2054, 2018. doi: 10.1109/TSP.2018.2795539.
- Petter Brändén. Solutions to two problems on permanents. *Linear Algebra and its Applications*, 436(1):53–58, 2012. doi: 10.1016/j.laa.2011.06.022.

- Zeineb Chebbi and Maher Moakher. Means of Hermitian positive-definite matrices based on the log-determinant α -divergence function. *Linear Algebra and its Applications*, 436(7):1872–1889, 2012. doi: 10.1016/j.laa.2011.12.003.
- Anoop Cherian, Suvrit Sra, Arindam Banerjee, and Nikolaos Papanikolopoulos. Jensen–Bregman logdet divergence with application to efficient similarity search for covariance matrices. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 35(9):2161–2174, 2013. doi: 10.1109/TPAMI.2012.259.
- Harry Crane. Some algebraic identities for the α -permanent. *Linear Algebra and its Applications*, 439(11):3445–3459, 2013. doi: 10.1016/j.laa.2013.09.028.
- Michel Marie Deza and Elena Deza. *Encyclopedia of Distances*. Springer, Berlin, 2nd edition, 2013. doi: 10.1007/978-3-642-30958-8.
- Jacques Faraut and Adam Korányi. Function spaces and reproducing kernels on bounded symmetric domains. *Journal of Functional Analysis*, 88(1):64–89, 1990. doi: 10.1016/0022-1236(90)90119-6.
- Jacques Faraut and Adam Korányi. *Analysis on Symmetric Cones*. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1994. ISBN 978-0-19-853477-8.
- Carl H. FitzGerald and Roger A. Horn. On fractional Hadamard powers of positive definite matrices. *Journal of Mathematical Analysis and Applications*, 61(3):633–642, 1977. doi: 10.1016/0022-247X(77)90167-6.
- Dominique Foata and Doron Zeilberger. Laguerre polynomials, weighted derangements, and positivity. *SIAM Journal on Discrete Mathematics*, 1(4):425–433, 1988. doi: 10.1137/0401043.
- Péter E. Frenkel. Remarks on the α -permanent. *Mathematical Research Letters*, 17(4):795–802, 2010. doi: 10.4310/MRL.2010.v17.n4.a17.
- William Fulton and Joe Harris. *Representation Theory: A First Course*. Graduate Texts in Mathematics 129. Springer, New York, 1991.
- Simon G. Gindikin. Analysis in homogeneous domains. *Russian Mathematical Surveys*, 19(4):1–89, 1964. doi: 10.1070/RM1964v019n04ABEH001153.
- I. P. Goulden and D. M. Jackson. Immanants, Schur functions, and the MacMahon master theorem. *Proceedings of the American Mathematical Society*, 115(3):605–612, 1992. doi: 10.2307/2159206.
- Loo-Keng Hua. Inequalities involving determinants. *Acta Mathematica Sinica*, 5:463–470, 1955.
- Shuangzhe Liu and Götz Trenkler. Hadamard, Khatri–Rao, Kronecker and other matrix products. *International Journal of Information and Systems Sciences*, 4(1):160–177, 2008.
- Marvin Marcus. On a determinantal inequality. *The American Mathematical Monthly*, 65:266–268, 1958. doi: 10.2307/2310245.
- Marvin Marcus and William Watkins. Partitioned Hermitian matrices. *Duke Mathematical Journal*, 38(2):237–249, 1971. doi: 10.1215/S0012-7094-71-03830-0.
- Russell Merris. *Multilinear Algebra*. Gordon and Breach, Amsterdam, 1997.
- Frank Nielsen. The Siegel–Klein disk: Hilbert geometry of the Siegel disk domain. *Entropy*, 22(9):1019, 2020. doi: 10.3390/e22091019.
- A. Yu. Orlov. New solvable matrix integrals. *International Journal of Modern Physics A*, 19(supp02):276–293, 2004. doi: 10.1142/S0217751X04020476.
- A. M. Ostrowski and Olga Taussky. On the variation of the determinant of a positive definite matrix. *Indagationes Mathematicae (Proceedings)*, 54:383–385, 1951. doi: 10.1016/s1385-7258(51)50055-0.
- S. Yu. Rotfel’d. The singular values of the sum of completely continuous operators. In M. Sh. Birman, editor, *Topics in Mathematical Physics, Volume 3: Spectral Theory*, pages 73–78. Consultants Bureau, New York, 1969.
- Alexander D Scott and Alan D Sokal. Complete monotonicity for inverse powers of some combinatorially defined polynomials. *Acta Mathematica*, 213:323–392, 2014.
- Tomoyuki Shirai. Remarks on the positivity of α -determinants. *Kyushu Journal of Mathematics*, 61(1):169–189, 2007. doi: 10.2206/kyushujm.61.169.

- Suvrit Sra. On the matrix square root via geometric optimization. *Electronic Journal of Linear Algebra*, 31:433–443, 2016a. doi: 10.13001/1081-3810.3196.
- Suvrit Sra. Positive definite matrices and the S-divergence. *Proceedings of the American Mathematical Society*, 144(7):2787–2797, 2016b. doi: 10.1090/proc/12953.
- Suvrit Sra. Metrics induced by Jensen–Shannon and related divergences on positive definite matrices. *Linear Algebra and its Applications*, 616:125–138, 2021. doi: 10.1016/j.laa.2020.12.023.
- Suvrit Sra and Reshad Hosseini. Conic geometric optimization on the manifold of positive definite matrices. *SIAM Journal on Optimization*, 25(1):713–739, 2015. doi: 10.1137/140978168.
- Gabor Szegő. Über gewisse Potenzreihen mit lauter positiven Koeffizienten. *Mathematische Zeitschrift*, 37(1):674–688, 1933. doi: 10.1007/bf01474608.
- Gang Tian. *Canonical Metrics in Kähler Geometry*. Lectures in Mathematics ETH Zürich. Birkhäuser, Basel, 2000.
- Malik Tiomoko, Romain Couillet, Florent Bouchard, and Guillaume Ginolhac. Random matrix improved covariance estimation for a large class of metrics. In *Proceedings of the 36th International Conference on Machine Learning*, Proceedings of Machine Learning Research 97, pages 6254–6263, 2019.
- Mitsuru Uchiyama. Subadditivity of eigenvalue sums. *Proceedings of the American Mathematical Society*, 134(5):1405–1412, 2006. doi: 10.1090/s0002-9939-05-08116-5.
- David Vere-Jones. A generalization of permanents and determinants. *Linear Algebra and its Applications*, 111:119–124, 1988. doi: 10.1016/0024-3795(88)90053-5.
- Dániel Virosztek. The metric property of the quantum Jensen–Shannon divergence. *Advances in Mathematics*, 380:107595, 2021. doi: 10.1016/j.aim.2021.107595.
- Changqing Xu, Zhaodi Xu, and Fuzhen Zhang. Revisiting Hua–Marcus–Bellman–Ando inequalities on contractive matrices. *Linear Algebra and its Applications*, 430(5–6):1499–1508, 2009. doi: 10.1016/j.laa.2007.11.011.
- Guanghui Xu, Changqing Xu, and Fuzhen Zhang. Contractive matrices of Hua type. *Linear and Multilinear Algebra*, 59(2):159–172, 2011. doi: 10.1080/03081080903266888.
- Florian Yger, Maxime Berar, and Fabien Lotte. Riemannian approaches in brain-computer interfaces: a review. *IEEE Transactions on Neural Systems and Rehabilitation Engineering*, 25(10):1753–1762, 2017. doi: 10.1109/TNSRE.2016.2627016.
- Artjom Zern, Matthias Zisler, Freddie Åström, Stefania Petra, and Christoph Schnörr. Unsupervised label learning on manifolds by spatially regularized geometric assignment. In *Pattern Recognition (GCPR 2018)*, Lecture Notes in Computer Science 11269, pages 698–713. Springer, 2019. doi: 10.1007/978-3-030-12939-2_48.
- Fuzhen Zhang. Recent results and research problems on Hua matrices. NIU Linear Algebra Meeting, presentation slides, August 2009.