UNIVERSAL TEST RINGS FOR RATIONAL GERSTEN INJECTIVITY

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Abstract. For each prime p, rational Gersten injectivity in a fixed degree for mixed-characteristic henselian discrete valuation rings of residue characteristic p is equivalent to injectivity for $W(\overline{\mathbf{F}_p(t_1, t_2, \ldots)})$. For a finite bound d on residue transcendence degree, the test ring is $W(\overline{\mathbf{F}_p(t_1, \ldots, t_d)})$. For these henselian DVRs, rational localization boundary images depend only on the residue field and are reflected by arbitrary residue-field extensions. A normalization by the absolute ramification index makes the resulting comparisons of kernels canonical and functorial. The proof uses regular approximation, finite transfers, and perfection. An appendix gives an alternative comparison based on the existential embedding theorem of Anscombe–Dittmann–Jahnke, including an example with nonnoetherian valuation rings. We also describe the additional boundary condition that induction between consecutive test rings would require.

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1. Introduction

We use $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $\mathbb{Z}$ and $\mathbb{Q}$ denote the integers and rational numbers. Rings are commutative with identity, and ring homomorphisms preserve the identity. For a local ring R, its residue field is denoted by $\kappa(R)$. For a discrete valuation ring (DVR) R, the valuation v_R is normalized by $v_R(\pi) = 1$ for a uniformizer π, and $\widehat{R}$ denotes completion for the maximal-ideal topology. If R has mixed characteristic $(0, p)$, its *absolute ramification index* is $v_R(p)$, the ramification index of $\mathbb{Z}_{(p)} \subset R$ [Sta26, Tag 09EK].

Let R be a DVR and $n \in \mathbb{Z}$. Rational Gersten injectivity in degree n is the assertion that

$$K_n(R)_{\mathbb{Q}} \longrightarrow K_n(\operatorname{Frac} R)_{\mathbb{Q}} \quad \text{is injective.} \tag{1.1}$$

Here K denotes nonconnective algebraic K-theory and $G_{\mathbb{Q}} = G \otimes_{\mathbb{Z}} \mathbb{Q}$ for an abelian group G. For DVRs, the rings in (1.1) are regular and noetherian. Their negative K-groups vanish

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[TT90, Proposition 6.8]. We allow $n \in \mathbb{Z}$ in the statements involving arbitrary spectrum coefficients. For a discrete valuation ring, injectivity in every degree is equivalent to the rational Gersten exact sequences; this follows directly from localization, as recalled in section 2.

Fix a prime p. For every integer $d \geq 0$, define

$$\Omega_{p,d} = \overline{\mathbf{F}_p(t_1, \dots, t_d)}, \qquad \Omega_{p,\infty} = \overline{\mathbf{F}_p(t_1, t_2, \dots)}, \tag{1.2}$$

where the variables are algebraically independent. The algebraic closures and the embeddings between them are chosen compatibly. For example, inside a fixed algebraic closure of $\mathbf{F}_p(t_1, t_2, \dots)$, one can take the algebraic closure of $\mathbf{F}_p(t_1, \dots, t_d)$. In a bound on residue transcendence degree, the symbol ∞ will mean that no bound is imposed. Thus a bound by ∞ permits arbitrary cardinalities, whereas $\Omega_{p,\infty}$ has countable transcendence degree. For a perfect field k of characteristic p, $W(k)$ denotes its ring of p-typical Witt vectors. A strict p-ring is a p-adically separated complete ring in which p is a nonzerodivisor and whose quotient by p is perfect of characteristic p. The ring $W(k)$ is the unique strict p-ring with prescribed residue field k; it is a complete DVR with uniformizer p. Existence and uniqueness are proved in [Ser62, Chapitre II, §6]; functoriality is [Ser62, Chapitre II, §6, Proposition 10, p. 48].

For a field k of characteristic p, its perfection is $k^{\mathrm{perf}} = \bigcup_{m \in \mathbb{N}} k^{1/p^m}$ inside an algebraic closure. Restriction and transfer in K-theory are denoted by res and Tr, respectively, with the conventions of section 2. For a DVR R with residue field k and $n \in \mathbb{Z}$, the maps

$$\partial_R : K_{n+1}(\operatorname{Frac} R)_{\mathbb{Q}} \longrightarrow K_n(k)_{\mathbb{Q}}, \qquad \mathrm{Tr}_{k/R} : K_n(k)_{\mathbb{Q}} \longrightarrow K_n(R)_{\mathbb{Q}}$$

are the localization boundary and the closed-point Gysin map. The latter is induced by restriction of scalars on perfect complexes.

The principal comparison is the following theorem, proved in section 6.

Theorem 1.1 (Residue fields and rational boundary images)**.** *Let R be a henselian DVR of mixed characteristic $(0, p)$ with residue field k, and let $n \in \mathbb{Z}$.*

(i) *In $K_n(k)_{\mathbb{Q}}$ one has*

$$\operatorname{im} \partial_R = \mathrm{res}^{-1}_{k^{\mathrm{perf}}/k}\big(\operatorname{im} \partial_{W(k^{\mathrm{perf}})}\big). \tag{1.3}$$

In particular this image depends only on k.

(ii) *If S is another henselian DVR of mixed characteristic $(0, p)$ with residue field ℓ, every field embedding $k \subseteq \ell$ makes the square*

$$\begin{array}{ccc} \operatorname{im} \partial_R & \hookrightarrow & K_n(k)_{\mathbb{Q}} \\ \downarrow{\scriptstyle \mathrm{res}_{\ell/k}} & & \downarrow{\scriptstyle \mathrm{res}_{\ell/k}} \\ \operatorname{im} \partial_S & \hookrightarrow & K_n(\ell)_{\mathbb{Q}} \end{array}$$

cartesian.

(iii) *There is a canonical isomorphism*

$$\begin{aligned} &\ker\big(K_n(R)_{\mathbb{Q}} \to K_n(\operatorname{Frac} R)_{\mathbb{Q}}\big) \\ &\qquad \xrightarrow{\sim} \ker\big(K_n(W(k^{\mathrm{perf}}))_{\mathbb{Q}} \to K_n(\operatorname{Frac} W(k^{\mathrm{perf}}))_{\mathbb{Q}}\big), \end{aligned} \tag{1.4}$$

characterized, for every $a \in K_n(k)_{\mathbb{Q}}$, by

$$\mathrm{Tr}_{k/R}(a) \longmapsto \frac{1}{v_R(p)} \mathrm{Tr}_{k^{\mathrm{perf}}/W(k^{\mathrm{perf}})} \mathrm{res}_{k^{\mathrm{perf}}/k}(a). \tag{1.5}$$

These isomorphisms are natural for ring homomorphisms between henselian DVRs of mixed characteristic $(0, p)$. Every such homomorphism induces an injection on rational kernels.

Definition 1.2. An *arithmetic henselian trait* of residue characteristic p is a ring

$$U = (\widetilde{A}_{p\widetilde{A}})^h, \tag{1.6}$$

where $\widetilde{A}$ is a smooth finitely presented $\mathbb{Z}_{(p)}$-algebra and $\widetilde{A}/p\widetilde{A}$ is a nonzero integral domain. The superscript h denotes henselization.

Such a ring is a henselian DVR with uniformizer p and residue field $\mathrm{Frac}(\widetilde{A}/p\widetilde{A})$. Indeed, smoothness implies flatness and regularity over the regular base $\mathbb{Z}_{(p)}$ [Sta26, Tag 00TA, Tag 07NF]. The prime $p\widetilde{A}$ has height one: its height is at most one by the principal ideal theorem [Sta26, Tag 00KV, (1)] and is not zero because p is a nonzerodivisor in the regular localization. That localization is therefore a one-dimensional regular local ring. Henselization preserves its dimension, regularity, residue field, and uniformizer [Sta26, Tag 0AP3, Tag 04GN].

Corollary 1.3 (Universal Witt test)**.** *Fix a prime p, an integer n, and $d \in \mathbb{N} \cup \{\infty\}$. The following assertions are equivalent.*

(i) *The map $K_n(W(\Omega_{p,d}))_{\mathbb{Q}} \to K_n(\mathrm{Frac}\, W(\Omega_{p,d}))_{\mathbb{Q}}$ is injective.*

(ii) *For every mixed-characteristic henselian DVR R of residue characteristic p with $\mathrm{trdeg}_{\mathbf{F}_p} \kappa(R) \le d$, the map $K_n(R)_{\mathbb{Q}} \to K_n(\mathrm{Frac}\, R)_{\mathbb{Q}}$ is injective.*

(iii) *For every arithmetic henselian trait U of residue characteristic p with $\mathrm{trdeg}_{\mathbf{F}_p} \kappa(U) \le d$, the map $K_n(U)_{\mathbb{Q}} \to K_n(\mathrm{Frac}\, U)_{\mathbb{Q}}$ is injective.*

For $d = \infty$, there is no residue transcendence-degree bound in (ii) *or* (iii).

More precisely, a nonzero rational kernel class over any such R produces a nonzero kernel class over $W(\Omega_{p,s})$ for a finite integer s, over an arithmetic henselian trait of residue transcendence degree s, and over $W(\Omega_{p,\infty})$. One can require $s \le \mathrm{trdeg}_{\mathbf{F}_p} \kappa(R)$ whenever the latter number is finite. The equivalences also hold when injectivity is required in every degree simultaneously.

Corollary 1.3 detects every possible failure on a prescribed family of complete unramified DVRs. It does not supply the vanishing needed on that family: proving injectivity for the test ring is equivalent to the original rational henselian DVR problem.

The proof. The comparison in Theorem 1.1 has three steps. Completion preserves the kernel by Theorem 3.6. A Cohen subring of the completion has the same rational boundary image by Corollary 4.3 and Proposition 4.4. Adjoining residue p-th roots and completing then compares that subring with $W(k^{\mathrm{perf}})$ by Theorem 5.3. The compatibility with arbitrary residue-field embeddings follows from Corollary 6.3. The factor $v_R(p)^{-1}$ in (1.5) compensates for the ramification multiplicity in (2.3).

To deduce Corollary 1.3, localization represents a kernel class by $a \in K_n(k)_{\mathbb{Q}}$. By Lemma 2.4, $a = \mathrm{res}_{k/k_0}(a_0)$ for a finitely generated subfield $k_0/\mathbf{F}_p$ of k. If $s = \mathrm{trdeg}_{\mathbf{F}_p} k_0$, choose an embedding $k_0 \hookrightarrow \Omega_{p,s}$. The field diagram is

$$k \longleftarrow k_0 \longhookrightarrow \Omega_{p,s} \longhookrightarrow \Omega_{p,\infty}.$$

Part (ii) of Theorem 1.1 reflects membership in the boundary image along every arrow. A nonzero kernel class over R therefore yields one over each indicated Witt ring. An arithmetic trait with residue field k_0 gives the third equivalent condition. Its construction and the complete deduction are in section 7.

Approximation and existential embeddings. There are two routes to the comparison needed in this argument. In the main text, Néron–Popescu desingularization factors a regular map of DVRs through smooth algebras. A point on the closed fibre lifts by henselianity; a rational point gives a retraction, and a point over a finite separable extension gives a transfer. This is the approximation method of [Pop86, Theorem 2.5 and §3], implemented in Proposition 3.5. For completion it specializes to the classical approximation theorems

of Greenberg and Artin. The passage from finite polynomial equations to injectivity on K-groups already appears in Banaszak's work [Ban07].

The second route uses the existential embedding theorem of Anscombe–Dittmann–Jahnke [ADJ24, Theorem 5.14]. It transfers existential closedness of the residue-field and value-group embeddings to the valued-field embedding, under equal finite initial ramification. Finite equations over the valuation ring can then be reflected to the smaller ring. The finite-presentation argument in Theorem 3.3 turns this into pure monomorphisms on K-groups, with arbitrary spectrum coefficients. Section A develops it with all the required logical definitions. Example A.5 exhibits its application to nonnoetherian valuation rings of rank two. For the DVR comparisons in Corollary 1.3, the classical approximation argument already suffices; the model-theoretic proof is an alternative with the broader valued-field scope stated in Corollary A.4.

Relation to the Gersten problem. The classical obstruction already factors through the part of Quillen K-theory not generated by Milnor symbols. For every DVR R with residue field k, Lemma 2.2 recalls the surjection

$$\operatorname{coker}\bigl(K_n^M(k)_{\mathbb{Q}} \to K_n(k)_{\mathbb{Q}}\bigr) \twoheadrightarrow \ker\bigl(K_n(R)_{\mathbb{Q}} \to K_n(\operatorname{Frac} R)_{\mathbb{Q}}\bigr) \qquad (n \in \mathbb{N}).$$

Parshin's conjecture asserts the vanishing of $K_i(X)_{\mathbb{Q}}$ for $i > 0$ and X smooth projective over a finite field. Geisser [Gei98, Theorem 1.3(ii)] proves that it implies $K_n^M(k)_{\mathbb{Q}} = K_n(k)_{\mathbb{Q}}$ for every field k of positive characteristic. Consequently it implies rational Gersten injectivity for every DVR of positive residue characteristic. Geisser–Levine [GL00, following Theorem 8.2, p. 490] explain that this rational generation by Milnor symbols, usually called Beilinson's conjecture for fields of positive characteristic, implies rational Gersten injectivity and, together with their finite-coefficient theorem, integral Gersten injectivity. The unconditional cases of residue transcendence degree at most one in Proposition 8.1 come from the classical results of Quillen and Harder.

Skalit reduces the integral Gersten conjecture for unramified regular local rings to DVRs essentially smooth over $\mathbb{Z}$ [Ska17, Theorem 5.4]. The arithmetic traits used here are their henselizations. A passage back from a henselization to the original DVR would require an additional comparison, which is not proved here. In equicharacteristic, integral Gersten injectivity is known by Panin [Pan03, Theorem A]. Hence the rings $W(\Omega_{p,\infty})$, as p varies, test rational Gersten injectivity for all henselian DVRs.

For DVRs, Gersten injectivity with finite coefficients is classical: the prime-to-residue-characteristic case is due to Gillet [Gil86], and the remaining case to Geisser–Levine; see [GL00, Theorem 8.2 and its proof]. Two companion preprints concern broader henselian settings. For coefficients invertible in the ring, [Fel26b, Theorem 6.1 and Corollary 6.13] treats henselian valuation rings and noetherian henselian regular local rings. [Fel26a, Theorem 1.1] gives a failure with coefficients $\mathbb{Z}/3$ for a complete ramified regular local surface of residue characteristic 3. Neither preprint is used in the proofs below. Integral Gersten injectivity requires control of torsion in addition to the rational problem; the higher-dimensional case requires further arguments.

Integral Gersten injectivity for DVRs is announced in [Moc07, Theorem 0.3] and [Moc20, Corollary 2]; [Dru25, Theorem 4.5] includes henselian DVRs. The author does not regard the cited versions as complete proofs. In Mochizuki's arguments, the retraction asserted in [Moc07, Theorem 3.13] and the functoriality asserted in [Moc20, Lemma 13(III)] require correction. In Druzhinin's version 1, Theorems 2.1–2.2, Theorems 2.3–2.4, and the factorization in the proof of Theorem 3.1 invoke, respectively, references [9], [8], and [10], all listed there as work in preparation.[1] Proofs of these inputs are not supplied in that version. We therefore do not treat these announcements as established resolutions of the mixed-characteristic Gersten problem.

[1] All three are attributed there to A. E. Druzhinin: [9], *ϕ-invariance and lifting properties*; [8], *$(\phi, \underline{\overline{\mathbb{A}}}, \mathrm{Bl}_\beta)$-localisation and blow-ups and loops functors*; [10], *Blow-ups and ρ-filtrations.*

Organization of the paper. We include details of several standard arguments to make the use of the hypotheses and the compatibility of the comparison maps explicit. The aim is to facilitate verification and a possible future formalization in Lean.

Expert readers interested in the universal test rings may skip directly to section 7, where Corollary 1.3 is deduced from Theorem 1.1. Sections 2 to 6 establish the latter theorem. Section 9 compares consecutive test rings and states the additional condition required for induction.

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Use of AI-assisted tools. The main mathematical ideas were found during discussions with chatgpt (GPT-5.5 and then Sol 5.6). The paper was mostly written by Sol 5.6 and GPT-6 Astra, under strict supervision. Separate AI-assisted (GPT-6 Astra and Opus 5.5) referee passes were also used to search for proof gaps and incorrect hypotheses. The author assumes full responsibility for the mathematical statements, proofs, and references.

2. Localization and closed-point support

For a ring R, a spectrum M, and an integer n, the notation

$$K_n(R;M) = \pi_n(K(R) \wedge M)$$

denotes algebraic K-theory with spectrum coefficients. Here $\mathbb{S}$ is the sphere spectrum and $H\mathbb{Q}$ is the Eilenberg–Mac Lane spectrum of $\mathbb{Q}$. For a positive integer m, $\mathbb{S}[1/m]$ is the localization of $\mathbb{S}$ obtained by inverting m. Thus

$$K_n(R;\mathbb{S}) = K_n(R), \qquad K_n(R;H\mathbb{Q}) = K_n(R)_{\mathbb{Q}}, \qquad K_n(R;\mathbb{S}[1/m]) = K_n(R)[1/m].$$

For a ring homomorphism $R \to S$, derived extension of scalars $P \mapsto S \otimes^{\mathbf{L}}_R P$ on perfect complexes induces

$$\mathrm{res}_{S/R} : K_n(R;M) \longrightarrow K_n(S;M).$$

If S is perfect as an R-module, restriction of scalars preserves perfect complexes and induces

$$\mathrm{Tr}_{S/R} : K_n(S;M) \longrightarrow K_n(R;M).$$

These are the restriction and transfer conventions used throughout; see [Wei13, V, 3.3.2]. The notation applies to quotient maps as well as to inclusions. For a DVR R with residue field k, the resolution $0 \to R \xrightarrow{\pi} R \to k \to 0$ makes k a perfect R-module. Thus $\mathrm{Tr}_{k/R}$ is defined; it is the closed-point Gysin map. Localization boundaries are denoted by ∂_R.

The next proposition is the DVR case of Quillen localization and dévissage [Wei13, V, Theorems 4.1 and 5.1, and (6.6)]. We recall the argument, including the ramification multiplicity.

Proposition 2.1 (Localization and base change)**.** *Let R be a DVR with residue field k. For every spectrum M and every integer n, localization and dévissage give an exact sequence*

$$\begin{aligned} K_{n+1}(\mathrm{Frac}\, R;M) \xrightarrow{\partial_R} K_n(k;M) \xrightarrow{\mathrm{Tr}_{k/R}} K_n(R;M) \\ \longrightarrow K_n(\mathrm{Frac}\, R;M). \end{aligned} \tag{2.1}$$

In particular,

$$\ker\big(K_n(R;M) \to K_n(\mathrm{Frac}\, R;M)\big) = \mathrm{im}\big(\mathrm{Tr}_{k/R} : K_n(k;M) \to K_n(R;M)\big). \tag{2.2}$$

If $R \to S$ is a local map of DVRs with residue extension $k \to \ell$, π_R is a uniformizer of R, and $0 < v_S(\pi_R) = e < \infty$, then it is flat and

$$\mathrm{res}_{S/R}\, \mathrm{Tr}_{k/R} = e\ \mathrm{Tr}_{\ell/S}\, \mathrm{res}_{\ell/k}\,. \tag{2.3}$$

Proof. Nonconnective localization [TT90, Theorem 7.4] identifies the fibre of $K(R) \to K(\operatorname{Frac} R)$ with the K-theory of perfect complexes whose cohomology is supported at the closed point. Every finitely generated R-module supported there has finite length and admits a filtration whose quotients are finite-dimensional k-vector spaces. Since R is regular, these modules are perfect. The exact functor sending a finite-dimensional k-vector space to its underlying perfect R-complex induces the map to this fibre. Quillen localization and dévissage, together with the resolution theorem for the regular ring R, say that this map is an isomorphism on homotopy groups in degrees at least zero [Wei13, V, Theorems 3.1, 4.1, and 5.1]. The negative groups of R and $\operatorname{Frac} R$ vanish by regularity, and $K_0(R) \to K_0(\operatorname{Frac} R)$ is the identity on ranks $\mathbb{Z}$. The homotopy exact sequence gives zero in degree -1 for the fibre because this rank map is surjective. In every smaller degree, both adjacent groups in that sequence vanish, so the fibre group vanishes as well. The field k also has vanishing negative K-groups. Hence the dévissage map is an isomorphism on every homotopy group and is an equivalence of spectra. This gives the cofiber sequence

$$K(k) \xrightarrow{\operatorname{Tr}_{k/R}} K(R) \longrightarrow K(\operatorname{Frac} R).$$

Smash product with M preserves cofiber sequences. Its homotopy long exact sequence is (2.1), which gives (2.2).

For base change, the kernel of $R \to S$ is a prime ideal. It is not the maximal ideal, since the image of π_R is nonzero. It is therefore zero. The R-module S is torsion-free, hence flat over the DVR R [Sta26, Tag 0539]. If π_S is a uniformizer of S, then $\pi_R S = \pi_S^e S$. The (S,k)-bimodule $S/\pi_R S$ has the filtration

$$S/\pi_S^e S \supset \pi_S S/\pi_S^e S \supset \cdots \supset \pi_S^{e-1} S/\pi_S^e S \supset 0.$$

For every integer j with $0 \le j < e$, multiplication by π_S^j induces the (S,k)-bimodule isomorphism

$$\ell = S/\pi_S S \xrightarrow{\sim} \pi_S^j S/\pi_S^{j+1} S, \qquad \overline{s} \longmapsto \pi_S^j s \bmod \pi_S^{j+1} S.$$

For a finite-dimensional k-vector space V, flat pullback is $(S/\pi_R S) \otimes_k V$. Tensoring the filtration with V is exact; each quotient is $\ell \otimes_k V$. Every term has finite length over S and is perfect. Additivity [Wei13, V, Theorem 1.2] therefore identifies the induced map with e copies of $\operatorname{Tr}_{\ell/S} \operatorname{res}_{\ell/k}$. This is an equality of homotopy classes of maps of K-theory spectra [TT90, Corollary 1.7.3], hence remains valid after smashing with M. For $e = 1$ it gives the commutative square

$$\begin{array}{ccc} K_n(k;M) & \xrightarrow{\operatorname{Tr}_{k/R}} & K_n(R;M) \\ \downarrow{\scriptstyle \operatorname{res}_{\ell/k}} & & \downarrow{\scriptstyle \operatorname{res}_{S/R}} \\ K_n(\ell;M) & \xrightarrow[\operatorname{Tr}_{\ell/S}]{} & K_n(S;M). \end{array}$$

□

It follows that injectivity of (1.1) is equivalent to surjectivity of $\partial_R : K_{n+1}(\operatorname{Frac} R)_{\mathbb{Q}} \to K_n(k)_{\mathbb{Q}}$. If injectivity holds in all degrees, localization gives

$$0 \longrightarrow K_n(R)_{\mathbb{Q}} \longrightarrow K_n(\operatorname{Frac} R)_{\mathbb{Q}} \xrightarrow{\partial_R} K_{n-1}(k)_{\mathbb{Q}} \longrightarrow 0$$

for every n. Conversely these exact sequences include the asserted injectivity.

For a ring A and $u \in A^\times$, denote by $\{u\} \in K_1(A)$ the class of the automorphism $A \to A$, $x \mapsto ux$. For a field k and $n \in \mathbb{N}$, $K_n^M(k)$ denotes Milnor K-theory: $K_0^M(k) = \mathbb{Z}$, and for $n \ge 1$ it is the degree-n part of the tensor algebra on $k^\times$ modulo the two-sided ideal generated by $a \otimes (1-a)$ for $a \in k \setminus \{0,1\}$ [Wei13, III, Definition 7.1]. Its symbol $\{a_1, \dots, a_n\}$ maps to the product of the classes of $a_1, \dots, a_n$ in Quillen K-theory [Wei13, IV, Example 1.10.1]. The following is the classical symbol-lifting argument; compare [GL00, proof of Theorem 8.2 and the following paragraph].

Lemma 2.2 (Milnor symbols)**.** *For every DVR R with residue field k and every $n \in \mathbb{N}$, the transfer $\mathrm{Tr}_{k/R} : K_n(k) \to K_n(R)$ vanishes on the image of $K_n^M(k)$. It induces a surjection*

$$\operatorname{coker}\bigl(K_n^M(k) \to K_n(k)\bigr) \twoheadrightarrow \ker\bigl(K_n(R) \to K_n(\operatorname{Frac} R)\bigr).$$

Proof. For a uniformizer π, the resolution $0 \to R \xrightarrow{\pi} R \to k \to 0$ gives $\mathrm{Tr}_{k/R}(1) = [R] - [R] = 0$ in $K_0(R)$. For $n \geq 1$ and $a_1, \dots, a_n \in k^\times$, choose lifts $u_1, \dots, u_n \in R^\times$. Their product in $K_n(R)$ restricts to the image of $\{a_1, \dots, a_n\}$. The projection formula [Wei13, V, Example 3.5.3] gives

$$\mathrm{Tr}_{k/R}(\mathrm{res}_{k/R}(\{u_1\} \cdots \{u_n\})) = \{u_1\} \cdots \{u_n\}\ \mathrm{Tr}_{k/R}(1) = 0.$$

Symbols generate $K_n^M(k)$, and the calculation for 1 treats $n = 0$. Localization (2.2) gives the surjection. □

Lemma 2.3 (Degrees at most two)**.** *For every DVR R and every integer $n \leq 2$, the map $K_n(R) \to K_n(\operatorname{Frac} R)$ is injective.*

Proof. For $n < 0$, both groups vanish by regularity [TT90, Proposition 6.8]. For $n = 0$, rank identifies the map with the identity of $\mathbb{Z}$. For $n = 1$, the determinant identifies it with $R^\times \subseteq (\operatorname{Frac} R)^\times$. For $n = 2$, Matsumoto's theorem identifies $K_2^M(k)$ with $K_2(k)$ [Wei13, III, Theorem 6.1]; apply Lemma 2.2. This is the degree-two theorem of Dennis–Stein [DS75], in the form [Wei13, V, Corollary 6.6.2]. □

Lemma 2.4 (Filtered colimits)**.** *For every spectrum M and every integer n, the functor $K_n(-; M)$ preserves filtered colimits of rings. If $(R_\lambda)_\lambda$ is a filtered system of domains with injective transition maps, then*

$$\varinjlim_\lambda \ker\bigl(K_n(R_\lambda; M) \to K_n(\operatorname{Frac} R_\lambda; M)\bigr) \simeq \ker\bigl(K_n(\varinjlim_\lambda R_\lambda; M) \to K_n(\operatorname{Frac}(\varinjlim_\lambda R_\lambda); M)\bigr). \tag{2.4}$$

Proof. Nonconnective K-theory commutes with filtered colimits of rings [TT90, Theorem 7.2]; see also [BGT13, the proof of Theorem 9.8]. Smash product with M preserves colimits. Homotopy groups of spectra preserve filtered colimits, by commutation of filtered colimits of spaces with finite limits [Lur26, Tag 05XS]. This proves the first assertion. For the second, the injective transition maps identify $\operatorname{Frac}(\varinjlim R_\lambda)$ with $\varinjlim \operatorname{Frac} R_\lambda$: each fraction has its numerator and nonzero denominator in one stage. Filtered colimits of abelian groups are exact [Sta26, Tag 00DB]; applying this to the two systems of K-groups gives (2.4). □

3. Algebraic purity and regular approximation

Definition 3.1. An inclusion of rings $R \subseteq S$ is *algebraically pure* if, for all integers $r, s \in \mathbb{N}$ and all $f_i \in R[X_1, \dots, X_r]$, $1 \leq i \leq s$,

$$\exists b \in S^r, \quad f_i(b) = 0 \quad (1 \leq i \leq s)$$
$$\Longrightarrow$$
$$\exists a \in R^r, \quad f_i(a) = 0 \quad (1 \leq i \leq s).$$

An inclusion of abelian groups $A \subseteq B$ is *pure* if, for all integers $r, s \in \mathbb{N}$, all integers c_{ij}, $1 \leq i \leq s$, $1 \leq j \leq r$, and all $a_i \in A$, $1 \leq i \leq s$,

$$\exists (b_1, \dots, b_r) \in B^r, \quad \sum_{j=1}^{r} c_{ij} b_j = a_i \quad (1 \leq i \leq s)$$
$$\Longrightarrow$$
$$\exists (x_1, \dots, x_r) \in A^r, \quad \sum_{j=1}^{r} c_{ij} x_j = a_i \quad (1 \leq i \leq s).$$

Empty conjunctions are true, and empty sums are zero.

For rings, this is Popescu's definition of algebraic purity [Pop86, §3, p. 91]. The corresponding categorical notion is recalled in [BR07, §2, p. 18]. The following argument expresses $R \to S$ as a filtered colimit of maps $R \to Q$, each admitting an R-algebra retraction. This is the approximation by split monomorphisms in [BR07, §2, property (5)], proved below directly from polynomial equations.

Lemma 3.2 (Finite presentations and retractions)**.** *Let $R \subseteq S$ be an inclusion of rings.*

(i) *The category of finitely presented R-algebras Q equipped with an R-algebra map $Q \to S$ is filtered, and its canonical colimit map*

$$\varinjlim_{(Q \to S)} Q \longrightarrow S$$

is an isomorphism.

(ii) *The inclusion $R \subseteq S$ is algebraically pure if and only if every object $Q \to S$ of this category admits an R-algebra homomorphism $\rho : Q \to R$. Such a homomorphism satisfies $\rho \circ (R \to Q) = \mathrm{id}_R$.*

Proof. Part (i) is [Sta26, Tag 0BUF], applied to R-algebras; the morphisms are the R-algebra maps over S. For (ii), suppose first that $R \subseteq S$ is algebraically pure. A presentation

$$Q = R[X_1, \dots, X_r]/(f_1, \dots, f_s)$$

and a map $Q \to S$ give elements $b_j \in S$, $1 \le j \le r$, with $f_i(b_1, \dots, b_r) = 0$ for $1 \le i \le s$. Choose $a_j \in R$, $1 \le j \le r$, satisfying the same equations. Evaluation $X_j \mapsto a_j$ induces an R-algebra map $Q \to R$. Conversely, a solution in S of a finite system $f_i(X_1, \dots, X_r) = 0$, $1 \le i \le s$, defines a map from the displayed algebra Q to S. An R-algebra map $Q \to R$ sends the classes of the X_j to a solution in R. □

Finite-equation hypotheses were used for injectivity in K-theory by Banaszak [Ban07]. We give a variant that records purity for arbitrary rings and spectrum coefficients.

Theorem 3.3 (Purity on K-groups)**.** *If $R \subseteq S$ is algebraically pure, then*

$$\mathrm{res}_{S/R} : K_n(R; M) \longrightarrow K_n(S; M)$$

is a pure monomorphism for every spectrum M and every integer n.

Proof. By Lemmas 3.2 and 2.4,

$$K_n(S; M) = \varinjlim_{(Q \to S)} K_n(Q; M),$$

and every structural map $R \to Q$ in this diagram has an R-algebra retraction. Fix integers $r, s \in \mathbb{N}$, integers c_{ij} for $1 \le i \le s$, $1 \le j \le r$, elements $a_i \in K_n(R; M)$ for $1 \le i \le s$, and elements $b_j \in K_n(S; M)$ for $1 \le j \le r$, satisfying

$$\sum_{j=1}^{r} c_{ij} b_j = \mathrm{res}_{S/R}(a_i) \qquad (1 \le i \le s).$$

Filteredness gives an object $\sigma : Q \to S$ and elements $\widetilde{b}_j \in K_n(Q; M)$ such that $K_n(\sigma; M)(\widetilde{b}_j) = b_j$ for $1 \le j \le r$. For each $i \in \{1, \dots, s\}$, the description of equality in a filtered colimit [Sta26, Tag 04AX], applied to the i-th equality, gives an arrow $u_i : Q \to Q_i$ over S with

$$K_n(u_i; M) \left(\sum_{j=1}^{r} c_{ij} \widetilde{b}_j - \mathrm{res}_{Q/R}(a_i) \right) = 0.$$

A cocone on the finite diagram consisting of Q, the Q_i, and the arrows u_i consists of an object $T \to S$ and arrows

$$u : Q \to T, \qquad v_i : Q_i \to T, \qquad v_i u_i = u \quad (1 \le i \le s).$$

Such a cocone exists by filteredness. Applying $K_n(v_i; M)$ to the preceding equality gives

$$\sum_{j=1}^{r} c_{ij} K_n(u; M)(\widetilde{b}_j) = \operatorname{res}_{T/R}(a_i) \qquad (1 \le i \le s).$$

For an R-algebra retraction $\rho : T \to R$, the elements

$$x_j = K_n(\rho u; M)(\widetilde{b}_j) \in K_n(R; M) \qquad (1 \le j \le r)$$

therefore satisfy

$$\sum_{j=1}^{r} c_{ij} x_j = K_n(\rho; M) \operatorname{res}_{T/R}(a_i) = a_i \qquad (1 \le i \le s).$$

For $r = 0$ and $s = 1$, this calculation says that $\operatorname{res}_{S/R}(a_1) = 0$ implies $a_1 = 0$, proving injectivity. For arbitrary r, s, it proves purity as in Definition 3.1. □

Remark 3.4. The proof applies to every functor from commutative rings to abelian groups which preserves filtered colimits. If $A \subseteq B$ is pure and $m \ge 1$, then $A/mA \to B/mB$ is injective: for $a \in A \cap mB$, solvability of $mX = a$ in B implies $a \in mA$.

The following comparison is the usual approximation argument: factor through a smooth algebra, lift a point by henselianity, and use a retraction or a finite free transfer. For completion, the approximation input goes back to Greenberg [Gre66, Theorem 1 and Corollary 2]; compare also Artin [Art69, Theorem 1.10]. We use Néron–Popescu desingularization [Pop86, Theorem 2.5], in the form [Sta26, Tag 07GC].

Proposition 3.5 (Comparison by regular approximation)**.** *Let $R \to S$ be a local homomorphism of DVRs with* $\operatorname{char}(\operatorname{Frac} R) = 0$. *Assume that R is henselian, that a uniformizer π of R maps to a uniformizer of S, and that the residue extension $k = R/\pi R \subseteq \ell = S/\pi S$ is separable. For every spectrum M and every $n \in \mathbb{Z}$, the maps*

$$K_n(R; M) \longrightarrow K_n(S; M), \qquad K_n(\operatorname{Frac} R; M) \longrightarrow K_n(\operatorname{Frac} S; M)$$

have torsion kernels. If $k \subseteq \ell$ is algebraically pure, both maps are pure monomorphisms.

Proof. The kernel of $R \to S$ is prime and does not contain π; it is therefore zero. The R-module S is torsion-free, hence flat [Sta26, Tag 0539]. Its fibres over the two primes of R are

$$S \otimes_R \operatorname{Frac} R = \operatorname{Frac} S, \qquad S \otimes_R k = \ell.$$

The generic fibre is geometrically regular since its base field has characteristic zero. The same argument applies to the closed fibre if $\operatorname{char} k = 0$; in positive characteristic, for every finite purely inseparable extension k'/k, the ring $\ell \otimes_k k'$ is reduced by [Sta26, Tag 030W]. It is Artinian, being finite-dimensional over ℓ, and hence is a product of fields. It is therefore regular. The criterion in [Sta26, Tag 0381] proves geometric regularity. Thus $R \to S$ is a regular homomorphism of noetherian rings [Sta26, Tag 07BZ]. Popescu's theorem gives a filtered system of smooth R-algebras B_λ equipped with compatible maps $B_\lambda \to S$ such that

$$S = \varinjlim_{\lambda} B_\lambda, \qquad \operatorname{Frac} S = \varinjlim_{\lambda} B_\lambda[1/\pi]. \tag{3.1}$$

For each λ, the smooth k-algebra $B_\lambda/\pi B_\lambda$ is nonzero, since it maps to the field ℓ. It has a point with values in a finite separable extension k'/k [Sta26, Tag 056U]. The equivalence for finite étale algebras over a henselian local ring supplies a finite étale R-algebra R' with $R'/\pi R' = k'$ [Sta26, Tag 04GK]. Every maximal ideal of R' lies over (π), and the quotient by

π is a field. Hence R' is local. It is henselian [Sta26, Tag 04GH], and it is free as an R-module of rank $[k' : k]$. Smooth lifting over the henselian pair $(R', \pi R')$ [Sta26, Tag 0H74] gives a lift

$$\begin{array}{ccc} B_\lambda & \longrightarrow & k' \\ \uparrow & \nearrow & \uparrow \\ R & \longrightarrow & R'. \end{array}$$

Fix $x \in K_n(R; M)$ whose image in $K_n(S; M)$ is zero. By Lemma 2.4 and (3.1), there is λ with $\operatorname{res}_{B_\lambda/R}(x) = 0$. The displayed lift gives $\operatorname{res}_{R'/R}(x) = 0$. Finite freeness and additivity give

$$[k' : k]x = \operatorname{Tr}_{R'/R} \operatorname{res}_{R'/R}(x) = 0.$$

For $y \in K_n(\operatorname{Frac} R; M)$ with zero image in $K_n(\operatorname{Frac} S; M)$, choose λ with $\operatorname{res}_{B_\lambda[1/\pi]/\operatorname{Frac} R}(y) = 0$ instead. Localizing the displayed lift and applying the transfer for the finite free $\operatorname{Frac} R$-algebra $R'[1/\pi]$ gives $[k' : k]y = 0$. This proves the two torsion assertions.

Suppose now that $k \subseteq \ell$ is algebraically pure. A finite presentation of $B_\lambda/\pi B_\lambda$ and its map to ℓ give a finite polynomial system over k with a solution in ℓ. Purity gives a solution in k, hence a k-algebra map $B_\lambda/\pi B_\lambda \to k$. We may therefore take $k' = k$ and $R' = R$ in the preceding lift. Every map $R \to B_\lambda$ admits an R-algebra retraction. Given a finitely presented R-algebra Q and a map $Q \to S$, finite presentation factors it through some B_λ; composing with its retraction gives $Q \to R$. By Lemma 3.2, $R \subseteq S$ is algebraically pure. Every finitely presented $\operatorname{Frac} R$-algebra mapping to $\operatorname{Frac} S = \varinjlim_\lambda B_\lambda[1/\pi]$ factors through some $B_\lambda[1/\pi]$. Composing with the localized retraction $B_\lambda[1/\pi] \to \operatorname{Frac} R$ proves algebraic purity of $\operatorname{Frac} R \subseteq \operatorname{Frac} S$ in the same way. Apply Theorem 3.3 to both inclusions. □

Theorem 3.6 (Completion)**.** *Let R be a henselian DVR with residue field k and* $\operatorname{char}(\operatorname{Frac} R) = 0$*. For every spectrum M and every integer n, completion induces pure monomorphisms*

$$K_n(R; M) \longrightarrow K_n(\widehat{R}; M), \qquad K_n(\operatorname{Frac} R; M) \longrightarrow K_n(\operatorname{Frac} \widehat{R}; M)$$

and an isomorphism

$$\ker\big(K_n(R; M) \to K_n(\operatorname{Frac} R; M)\big) \xrightarrow{\sim} \ker\big(K_n(\widehat{R}; M) \to K_n(\operatorname{Frac} \widehat{R}; M)\big). \tag{3.2}$$

Proof. The map from R to its completion is injective because $\bigcap_{m \geq 1} \pi^m R = 0$ for a uniformizer π. By [Sta26, Tag 0AP1], $\widehat{R}$ is a complete DVR. [Sta26, Tag 05GG] gives

$$\widehat{R}/\pi\widehat{R} = R/\pi R.$$

Thus π generates the maximal ideal of $\widehat{R}$, and the residue map is the identity under this identification. The residue extension is the identity, hence is separable and algebraically pure. Proposition 3.5 gives the two pure monomorphisms.

The map $R \to \widehat{R}$ preserves the uniformizer and induces the identity on k. By Proposition 2.1 and naturality of restriction, the following diagram commutes and its rows are exact at their middle terms:

$$\begin{array}{ccccc} K_n(k; M) & \xrightarrow{\operatorname{Tr}_{k/R}} & K_n(R; M) & \longrightarrow & K_n(\operatorname{Frac} R; M) \\ \| & & \downarrow{\scriptstyle \operatorname{res}_{\widehat{R}/R}} & & \downarrow \\ K_n(k; M) & \xrightarrow[\operatorname{Tr}_{k/\widehat{R}}]{} & K_n(\widehat{R}; M) & \longrightarrow & K_n(\operatorname{Frac} \widehat{R}; M). \end{array}$$

The middle vertical arrow is injective, so its restriction to the kernel of the top rightward arrow is injective. For $\widehat{x} \in \ker(K_n(\widehat{R}; M) \to K_n(\operatorname{Frac} \widehat{R}; M))$, exactness gives $a \in K_n(k; M)$

with $\mathrm{Tr}_{k/\widehat{R}}(a) = \widehat{x}$. Exactness of the top row and commutativity of the left square give, respectively,

$$\mathrm{Tr}_{k/R}(a) \in \ker(K_n(R;M) \to K_n(\mathrm{Frac}\, R;M)), \qquad \mathrm{res}_{\widehat{R}/R}\,\mathrm{Tr}_{k/R}(a) = \widehat{x}.$$

Thus the map in (3.2) is surjective. □

Remark 3.7. For a positive integer m, the Moore spectrum $\mathbb{S}/m$ is the cofiber of multiplication by m on $\mathbb{S}$. For $M = \mathbb{S}/m$, Theorem 3.6 concerns finite-coefficient K-theory $\pi_n(K(R)\wedge\mathbb{S}/m)$. The coefficient cofiber sequence gives

$$0 \longrightarrow K_n(R)/m \longrightarrow K_n(R;\mathbb{S}/m) \longrightarrow K_{n-1}(R)[m] \longrightarrow 0,$$

where, for any abelian group G and any integer $m \geq 1$, $G[m] = \{g \in G : mg = 0\}$. Separately, purity for $M = \mathbb{S}$ implies that $K_n(R)/m \to K_n(\widehat{R})/m$ is injective: if the image of $a \in K_n(R)$ is mb in the target, the equation $mX = a$ has a solution in $K_n(R)$.

4. Finite extensions and Cohen subrings

The identities below express the compatibility of transfer and restriction with the maps from residue fields [Wei13, V, 3.3.2 and (6.6.3)–(6.6.4)]. We record their effect on kernels of restriction to fraction fields, keeping the degree and ramification factors explicit.

Lemma 4.1 (Transfer and ramification)**.** *Let $A \subseteq B$ be a finite extension of DVRs, with residue extension $k \subseteq \ell$ and ramification index e. The A-module B is free of rank $e[\ell : k]$. For every integer n, the following identities hold on integral K-groups:*

$$\mathrm{Tr}_{B/A}\,\mathrm{Tr}_{\ell/B} = \mathrm{Tr}_{k/A}\,\mathrm{Tr}_{\ell/k}, \qquad \mathrm{res}_{B/A}\,\mathrm{Tr}_{k/A} = e\ \mathrm{Tr}_{\ell/B}\,\mathrm{res}_{\ell/k}\,. \tag{4.1}$$

The maps $\mathrm{res}_{B/A}$ and $\mathrm{Tr}_{B/A}$ preserve the kernels of restriction to the fraction fields.

Proof. The A-module B is torsion-free, hence flat; being finite over the local ring A, it is finite free. The map is local: the maximal ideal of the integral extension B contracts to the unique maximal ideal of A. Choose uniformizers π_A, π_B. By definition of e, $\pi_A = u\pi_B^e$ for some $u \in B^\times$. The quotient $B/\pi_A B$ has the filtration

$$B/\pi_B^e B \supset \pi_B B/\pi_B^e B \supset \cdots \supset \pi_B^{e-1} B/\pi_B^e B \supset 0.$$

Multiplication by π_B^j identifies the j-th successive quotient with $B/\pi_B B = \ell$, for $0 \leq j < e$. Its k-action is the one induced by $A \to B \to \ell$. Taking k-dimensions gives

$$\mathrm{rank}_A\, B = \dim_k(B/\pi_A B) = e[\ell : k].$$

For the first identity in (4.1), an ℓ-vector space regarded successively as a B-module and an A-module is the same A-module as the one obtained by first restricting scalars from ℓ to k and then applying $\mathrm{Tr}_{k/A}$. The natural isomorphism of these exact functors proves the identity.

The second identity is (2.3).

Finally, $B \otimes_A \mathrm{Frac}\, A = B[1/\pi_A] = B[1/\pi_B] = \mathrm{Frac}\, B$. The extension-of-scalars square and the restriction-of-scalars square therefore commute with passage to fraction fields. Consequently both maps preserve the kernels of restriction to fraction fields. □

Proposition 4.2 (Purely inseparable residue extension)**.** *Under the hypotheses of Lemma 4.1, assume that ℓ/k is purely inseparable. For every integer n, restriction induces an isomorphism*

$$\ker\big(K_n(A) \to K_n(\mathrm{Frac}\, A)\big)[1/(e[\ell : k])] \xrightarrow{\mathrm{res}_{B/A}} \ker\big(K_n(B) \to K_n(\mathrm{Frac}\, B)\big)[1/(e[\ell : k])]. \tag{4.2}$$

Its inverse is $(e[\ell : k])^{-1}\,\mathrm{Tr}_{B/A}$.

Proof. On $K_n(A)$ the composite $\operatorname{Tr}_{B/A}\operatorname{res}_{B/A}$ is multiplication by $e[\ell:k]$. Indeed, after a choice of an A-basis of B, the underlying functor is naturally isomorphic to the direct sum of $e[\ell:k]$ copies of the identity functor. Additivity gives the asserted equality of maps on K-groups.

The purely inseparable field identity [Wei13, IV, Exercise 6.13] is

$$\operatorname{res}_{\ell/k}\operatorname{Tr}_{\ell/k} = [\ell:k]\,\mathrm{id}_{K_n(\ell)}. \tag{4.3}$$

We include its bimodule proof. If $\ell = k$, this is the identity. Otherwise the characteristic is a positive prime p. Since the purely inseparable extension is finite, there is an integer $s \geq 0$ with $b^{p^s} \in k$ for every $b \in \ell$. For an element $z = \sum_i a_i \otimes b_i$ in the kernel of multiplication $\ell \otimes_k \ell \to \ell$, one has

$$z^{p^s} = \sum_i a_i^{p^s} \otimes b_i^{p^s} = \left(\sum_i a_i b_i\right)^{p^s} \otimes 1 = 0.$$

Thus this kernel is nil, and the quotient is the field ℓ. The finite-dimensional ℓ-algebra $\ell \otimes_k \ell$ is therefore local Artinian with residue field ℓ. Every factor of a composition series by ideals is that residue field. The number of factors is $[\ell:k]$, since the ℓ-dimension of $\ell \otimes_k \ell$ is $[\ell:k]$ and each factor has dimension one. On each factor, the two ℓ-actions agree through multiplication. The functor inducing the left side of (4.3) is tensor product with this bimodule. Its composition series gives, by additivity, $[\ell:k]$ copies of the identity functor, proving the equality.

Take x in the integral kernel for B. By localization, $x = \operatorname{Tr}_{\ell/B}(a)$ for some $a \in K_n(\ell)$. The two identities of Lemma 4.1 and (4.3) give

$$\begin{aligned}\operatorname{res}_{B/A}\operatorname{Tr}_{B/A} x &= \operatorname{res}_{B/A}\operatorname{Tr}_{k/A}\operatorname{Tr}_{\ell/k}(a)\\ &= e\,\operatorname{Tr}_{\ell/B}\operatorname{res}_{\ell/k}\operatorname{Tr}_{\ell/k}(a)\\ &= e[\ell:k]\,\operatorname{Tr}_{\ell/B}(a) = e[\ell:k]\,x.\end{aligned}$$

Both $\operatorname{res}_{B/A}$ and $\operatorname{Tr}_{B/A}$ preserve the kernels. Their two composites on those kernels are multiplication by $e[\ell:k]$. After this integer is inverted, the displayed inverse follows. □

A *Cohen ring* with residue field k of characteristic p means a complete DVR with uniformizer p and residue field k [Sta26, Tag 0327]. A Cohen subring of a complete mixed-characteristic DVR is such a subring inducing an isomorphism on residue fields.

Corollary 4.3 (Comparison with a Cohen subring)**.** *Let A be a complete DVR of mixed characteristic $(0,p)$ with residue field k and absolute ramification index $e = v_A(p)$. There is a Cohen subring $C \subseteq A$ inducing the identity on k. For every such Cohen subring, A is finite free of rank e over C. For every integer n, restriction induces an isomorphism*

$$\begin{aligned}&\ker\big(K_n(C) \to K_n(\operatorname{Frac} C)\big)[1/e]\\ &\qquad\xrightarrow{\operatorname{res}_{A/C}} \ker\big(K_n(A) \to K_n(\operatorname{Frac} A)\big)[1/e], \qquad (\operatorname{res}_{A/C})^{-1} = e^{-1}\operatorname{Tr}_{A/C}.\end{aligned} \tag{4.4}$$

Integrally, $\operatorname{Tr}_{A/C}$ is surjective on these kernels. If the kernel for C is zero, the kernel for A is killed by e; if the kernel for A is zero, the kernel for C is zero.

Proof. The Cohen structure theorem for a complete noetherian local domain gives a complete regular local subring of A with the same residue field, over which A is finite [Sta26, Tag 032D]. In mixed characteristic this subring is a formal power series ring over a Cohen ring. Finite integral extensions preserve dimension; hence that subring has dimension one. A formal power series ring in r variables over a Cohen DVR has dimension $r+1$, so $r = 0$. This subring is therefore a Cohen ring C.

For any Cohen embedding $C \subseteq A$ inducing the identity on k, choose a uniformizer π of A. Since $pA = \pi^e A$, the filtration by powers of π gives

$$\dim_k(A/pA) = \sum_{i=0}^{e-1} \dim_k(\pi^i A/\pi^{i+1}A) = e.$$

Here the C-action on each quotient factors through $C/pC = k$, and multiplication by π^i identifies that quotient with k. The ring C is p-adically complete and $\bigcap_{m\geq 1} p^m A = 0$. Hence A is a finite C-module by [Sta26, Tag 031D]. It is torsion-free over the DVR C, so it is finite free; its rank is $\dim_k(A/pA) = e$. Apply Proposition 4.2 with $\ell = k$ to obtain (4.4).

For integral surjectivity, every element of the kernel for C is $\operatorname{Tr}_{k/C}(a)$ for some $a \in K_n(k)$. The first identity in (4.1) gives $\operatorname{Tr}_{A/C}\operatorname{Tr}_{k/A}(a) = \operatorname{Tr}_{k/C}(a)$. If the kernel for A is zero, its surjective image is zero. If the kernel for C is zero and x lies in the kernel for A, then $\operatorname{Tr}_{A/C} x = 0$ and the calculation in the preceding proposition gives $ex = \operatorname{res}_{A/C}\operatorname{Tr}_{A/C} x = 0$. □

Proposition 4.4 (Boundary images and total ramification)**.** *Let $A \subseteq B$ be a finite extension of DVRs inducing the identity on their common residue field k, and let e be their ramification index. For every integer n, one has*

$$\begin{aligned} e\,\operatorname{im}(\partial_A : K_{n+1}(\operatorname{Frac} A) \to K_n(k)) &\subseteq \operatorname{im}(\partial_B : K_{n+1}(\operatorname{Frac} B) \to K_n(k)) \\ &\subseteq \operatorname{im}(\partial_A : K_{n+1}(\operatorname{Frac} A) \to K_n(k)). \end{aligned} \tag{4.5}$$

The images are therefore equal after inverting e; in particular,

$$\begin{aligned} &\operatorname{im}(K_{n+1}(\operatorname{Frac} A)_{\mathbb{Q}} \xrightarrow{\partial_A} K_n(k)_{\mathbb{Q}}) \\ &\qquad = \operatorname{im}(K_{n+1}(\operatorname{Frac} B)_{\mathbb{Q}} \xrightarrow{\partial_B} K_n(k)_{\mathbb{Q}}). \end{aligned} \tag{4.6}$$

Proof. By Proposition 2.1,

$$\operatorname{im}\partial_A = \ker \operatorname{Tr}_{k/A}, \qquad \operatorname{im}\partial_B = \ker \operatorname{Tr}_{k/B}\,.$$

The two identities in Lemma 4.1, with residue degree one, are

$$\operatorname{Tr}_{B/A}\operatorname{Tr}_{k/B} = \operatorname{Tr}_{k/A}, \qquad \operatorname{res}_{B/A}\operatorname{Tr}_{k/A} = e\,\operatorname{Tr}_{k/B}\,.$$

For $a \in K_n(k)$, the first identity gives $\operatorname{Tr}_{k/B}(a) = 0 \Rightarrow \operatorname{Tr}_{k/A}(a) = 0$; the second gives $\operatorname{Tr}_{k/A}(a) = 0 \Rightarrow \operatorname{Tr}_{k/B}(ea) = 0$. These are exactly the right and left inclusions in (4.5). Inverting e makes the outer subgroups equal. Tensoring with $\mathbb{Q}$ gives (4.6). □

5. Perfection of the residue field

At limit stages we use the following valuation argument. The preservation of henselianity is [Sta26, Tag 04GI].

Lemma 5.1 (Directed colimits with compatible uniformizers)**.** *Let I be a nonempty directed partially ordered set and $(C_i, \iota_{ij})_{i\leq j}$ a directed system of henselian DVRs. Assume that each $\iota_{ij} : C_i \to C_j$ is an injective local homomorphism. Suppose that uniformizers $\pi_i \in C_i$ have been chosen with*

$$\iota_{ij}(\pi_i) = \pi_j \qquad (i \leq j).$$

For

$$C = \varinjlim_{i\in I} C_i, \qquad \iota_i : C_i \longrightarrow C,$$

every ι_i is injective and local. The ring C is a henselian DVR, and the common image $\pi = \iota_i(\pi_i)$ is a uniformizer. The canonical maps induce isomorphisms

$$\varinjlim_{i\in I} C_i/\pi_i C_i \xrightarrow{\sim} C/\pi C, \qquad \varinjlim_{i\in I} \operatorname{Frac} C_i \xrightarrow{\sim} \operatorname{Frac} C.$$

Its normalized valuation satisfies

$$v_C(\iota_i(x)) = v_{C_i}(x) \qquad (i \in I,\ x \in C_i),$$

where every valuation takes the value ∞ at zero.

Proof. For $i \le j$ and $x \in C_i \setminus \{0\}$, express $x = \pi_i^m u$ with $m \in \mathbb{N}$ and $u \in C_i^\times$. Since $\iota_{ij}(u) \in C_j^\times$, one has

$$\iota_{ij}(x) = \pi_j^m \iota_{ij}(u), \qquad v_{C_j}(\iota_{ij}(x)) = m = v_{C_i}(x). \tag{5.1}$$

In a directed colimit of rings, $\iota_i(x) = \iota_j(y)$ if and only if some $h \ge i, j$ satisfies $\iota_{ih}(x) = \iota_{jh}(y)$. In particular, injectivity of the transition maps implies injectivity of each ι_i. If two nonzero elements of C are represented in one C_i, their product is the image of a nonzero product in that domain. Thus C is a domain.

Equation (5.1) and the equality criterion show that

$$v_C(0) = \infty, \qquad v_C(\iota_i(x)) = v_{C_i}(x) \quad (x \neq 0)$$

defines a function on C. For all $a, b \in C$, computation in a common stage gives

$$v_C(ab) = v_C(a) + v_C(b), \qquad v_C(a+b) \ge \min\{v_C(a), v_C(b)\}.$$

For $a \neq 0$, its expression at one stage gives $a = \pi^{v_C(a)} u$ with $u \in C^\times$. The element π is not invertible: an equation $\pi b = 1$ would hold at a stage and make its uniformizer invertible. It follows that

$$C^\times = \{a \in C : v_C(a) = 0\}, \qquad C \setminus C^\times = \pi C.$$

Consequently C is local with maximal ideal πC. Equation (5.1) also gives $\iota_i^{-1}(\pi C) = \pi_i C_i$, so ι_i is local.

For a nonzero ideal $J \subseteq C$, the nonempty subset $\{v_C(a) : a \in J \setminus \{0\}\}$ of $\mathbb{N}$ has a least element m. Choose $a \in J$ with $v_C(a) = m$. Then $a = \pi^m u$ with $u \in C^\times$, whence $\pi^m C \subseteq J$. Every $b \in J \setminus \{0\}$ has the form $b = \pi^{v_C(b)} v$ with $v_C(b) \ge m$ and $v \in C^\times$. Thus $b \in \pi^m C$ and

$$J = \pi^m C.$$

Hence C is noetherian, is a local domain, and has nonzero principal maximal ideal. It is a DVR [Sta26, Tag 00PD], with the stated valuation.

The maps $C_i/\pi_i C_i \to C_j/\pi_j C_j$ and $C_i/\pi_i C_i \to C/\pi C$ are injective by (5.1). Every element of $C/\pi C$ is represented at one stage; the first asserted isomorphism follows. The injections ι_{ij} and ι_i extend to fraction fields. Every fraction in $\operatorname{Frac} C$ has numerator and nonzero denominator represented in a common C_i. The induced map $\varinjlim_i \operatorname{Frac} C_i \to \operatorname{Frac} C$ is therefore surjective. It is injective because all its maps from the fields $\operatorname{Frac} C_i$ are injective.

Finally, C is henselian by [Sta26, Tag 04GI], since it is a filtered colimit of henselian local rings under local maps. □

Lemma 5.2 (Adjoining one residue p-th root)**.** *Let C be a henselian DVR with uniformizer p and residue field k of characteristic p. For $a \in k \setminus k^p$ and a lift $\widetilde{a} \in C$, consider*

$$D = C[X]/(X^p - \widetilde{a}).$$

The canonical map $C \to D$ is injective and local, and D is a henselian DVR with uniformizer p. It is free of rank p over C, and the residue identification is

$$D/pD \xrightarrow{\sim} k(a^{1/p}), \qquad X \bmod pD \longmapsto a^{1/p}.$$

For every $n \in \mathbb{Z}$, the maps

$$K_n(C)[1/p] \xrightarrow{\operatorname{res}_{D/C}} K_n(D)[1/p],$$

$$K_n(\operatorname{Frac} C)[1/p] \xrightarrow{\operatorname{res}_{\operatorname{Frac} D/\operatorname{Frac} C}} K_n(\operatorname{Frac} D)[1/p]$$

are injective, and restriction induces an isomorphism

$$\ker\bigl(K_n(C) \to K_n(\operatorname{Frac} C)\bigr)[1/p] \xrightarrow{\sim} \ker\bigl(K_n(D) \to K_n(\operatorname{Frac} D)\bigr)[1/p]. \tag{5.2}$$

Its inverse on these kernels is $p^{-1}\operatorname{Tr}_{D/C}$.

Proof. Division by the monic polynomial gives the C-basis $1, X, \dots, X^{p-1}$ of D. The map $C \to D$ is therefore injective, and multiplication by p on D is injective. Reduction modulo p gives

$$D/pD = k[X]/(X^p - a) = k(a^{1/p}).$$

Here $X^p - a$ is irreducible because $a \notin k^p$ [Sta26, Tag 09HF]. Every maximal ideal of the finite integral C-algebra D contracts to the maximal ideal pC. Since D/pD is a field, pD is the unique maximal ideal of D. The ring D is noetherian and $\dim D = \dim C = 1$, by injectivity and integrality of $C \to D$. Its maximal ideal is generated by p, so it is regular of dimension one and hence a DVR [Sta26, Tag 00PD]. Since it is finite and local over C, it is henselian [Sta26, Tag 04GH].

Because p is a uniformizer in both rings,

$$D \otimes_C \operatorname{Frac} C = D[1/p] = \operatorname{Frac} D.$$

Thus $\operatorname{Frac} D$ has degree p over $\operatorname{Frac} C$. The free bases over the rings and fields, together with additivity, give, for every $n \in \mathbb{Z}$,

$$\operatorname{Tr}_{D/C} \operatorname{res}_{D/C} = p \operatorname{id}_{K_n(C)}, \qquad \operatorname{Tr}_{\operatorname{Frac} D/\operatorname{Frac} C} \operatorname{res}_{\operatorname{Frac} D/\operatorname{Frac} C} = p \operatorname{id}_{K_n(\operatorname{Frac} C)}.$$

After inverting p, each restriction consequently has a left inverse and is injective. The residue extension $k(a^{1/p})/k$ is purely inseparable of degree p, and the ramification index is one. Proposition 4.2 gives (5.2). □

Theorem 5.3 (Perfection)**.** *Let C be a henselian DVR of mixed characteristic $(0, p)$ with uniformizer p and residue field k. For a fixed perfection k^{perf} of k, there is an injective local map $C \to W(k^{\mathrm{perf}})$ inducing the prescribed embedding $k \to k^{\mathrm{perf}}$. For every integer n, it induces an isomorphism*

$$\ker\bigl(K_n(C) \to K_n(\operatorname{Frac} C)\bigr)[1/p] \xrightarrow{\sim} \ker\bigl(K_n(W(k^{\mathrm{perf}})) \to K_n(\operatorname{Frac} W(k^{\mathrm{perf}}))\bigr)[1/p]. \tag{5.3}$$

For every integer n, the two restriction maps

$$K_n(C)[1/p] \longrightarrow K_n(W(k^{\mathrm{perf}}))[1/p],$$
$$K_n(\operatorname{Frac} C)[1/p] \longrightarrow K_n(\operatorname{Frac} W(k^{\mathrm{perf}}))[1/p]$$

are injective.

Proof. We construct rings C_α, subfields $k_\alpha \subseteq k^{\mathrm{perf}}$, and injective local maps $\iota_{\alpha\beta} : C_\alpha \to C_\beta$ for $\alpha \le \beta$. The required identities and residue identifications are

$$\iota_{\alpha\alpha} = \mathrm{id}, \qquad \iota_{\beta\gamma}\iota_{\alpha\beta} = \iota_{\alpha\gamma}, \qquad \iota_{\alpha\beta}(p) = p, \tag{5.4}$$

$$C_\alpha/pC_\alpha \simeq k_\alpha, \qquad \overline{\iota_{\alpha\beta}} : k_\alpha \hookrightarrow k_\beta \subseteq k^{\mathrm{perf}}. \tag{5.5}$$

Each C_α will be henselian with uniformizer p. Choose an infinite cardinal $\mu > |k^{\mathrm{perf}}|$, viewed as an ordinal. The recursion is performed for ordinals less than μ and stops at the first stage whose residue field is perfect.

Start with $C_0 = C$ and $k_0 = k$. Suppose that C_α has been constructed and k_α is imperfect. Choose $a_\alpha \in k_\alpha \setminus k_\alpha^p$ and a lift $\widetilde{a}_\alpha \in C_\alpha$. Define

$$C_{\alpha+1} = C_\alpha[X]/(X^p - \widetilde{a}_\alpha), \qquad k_{\alpha+1} = k_\alpha(a_\alpha^{1/p}) \subseteq k^{\mathrm{perf}},$$

where $a_\alpha^{1/p}$ is the unique p-th root in k^{perf}. Take the canonical map $\iota_{\alpha,\alpha+1}$ and define $\iota_{\beta,\alpha+1} = \iota_{\alpha,\alpha+1}\iota_{\beta\alpha}$ for $\beta \le \alpha$. The residue class of X maps to $a_\alpha^{1/p}$. Lemma 5.2 proves all the required ring properties and the identities (5.4)–(5.5) at this successor stage.

For a nonzero limit ordinal $\beta < \mu$, provided the construction has not stopped earlier, define

$$C_\beta = \varinjlim_{\alpha<\beta} C_\alpha, \qquad k_\beta = \bigcup_{\alpha<\beta} k_\alpha \subseteq k^{\mathrm{perf}},$$

and take the canonical colimit maps for $\iota_{\alpha\beta}$. Lemma 5.1 proves that C_β is a henselian DVR with uniformizer p and gives the compatible identifications

$$C_\beta/pC_\beta \simeq \varinjlim_{\alpha<\beta} C_\alpha/pC_\alpha \simeq k_\beta, \qquad \operatorname{Frac} C_\beta \simeq \varinjlim_{\alpha<\beta} \operatorname{Frac} C_\alpha.$$

The colimit identities give (5.4).

If no stage less than μ had perfect residue field, the elements $a_\alpha^{1/p}$, for $\alpha < \mu$, would all be defined and distinct. Indeed, for $\alpha < \beta < \mu$,

$$a_\alpha^{1/p} \in k_{\alpha+1} \subseteq k_\beta, \qquad a_\beta^{1/p} \notin k_\beta.$$

This would give an injection $\mu \to k^{\mathrm{perf}}$, contrary to $\mu > |k^{\mathrm{perf}}|$. Thus a first terminal ordinal $\lambda < \mu$ exists. Since k_λ is perfect and contains k, induction on $r \ge 0$ shows that it contains the unique p^r-th root of every element of k. Every element of k^{perf} is such a root. The inclusion $k_\lambda \subseteq k^{\mathrm{perf}}$ is therefore an equality.

Fix $n \in \mathbb{Z}$. We next prove, by transfinite induction on $\beta \le \lambda$, that for every $\alpha \le \beta$ the maps

$$K_n(C_\alpha)[1/p] \longrightarrow K_n(C_\beta)[1/p],$$
$$K_n(\operatorname{Frac} C_\alpha)[1/p] \longrightarrow K_n(\operatorname{Frac} C_\beta)[1/p]$$

are injective, and that

$$\ker\bigl(K_n(C_\alpha) \to K_n(\operatorname{Frac} C_\alpha)\bigr)[1/p] \longrightarrow \ker\bigl(K_n(C_\beta) \to K_n(\operatorname{Frac} C_\beta)\bigr)[1/p]$$

is an isomorphism. For $\alpha = \beta$ these are identity maps. At a successor stage, compose the induction assertions for the preceding stage with Lemma 5.2.

Suppose that β is a nonzero limit ordinal and $\alpha < \beta$. The subset $\{\gamma : \alpha \le \gamma < \beta\}$ is cofinal in β, since $\max(\alpha, \eta) < \beta$ for every $\eta < \beta$. By Lemma 2.4 and the induction hypothesis,

$$\begin{aligned}\ker\bigl(K_n(C_\alpha)[1/p] \to K_n(C_\beta)[1/p]\bigr) &= \bigcup_{\alpha\le\gamma<\beta} \ker\bigl(K_n(C_\alpha)[1/p] \to K_n(C_\gamma)[1/p]\bigr) = 0.\end{aligned}$$

Replacing every C_γ by $\operatorname{Frac} C_\gamma$ gives the fraction-field assertion by the same equality. Exactness of localization of abelian groups and Lemma 2.4 give

$$\ker\bigl(K_n(C_\beta) \to K_n(\operatorname{Frac} C_\beta)\bigr)[1/p] \simeq \varinjlim_{\alpha\le\gamma<\beta} \ker\bigl(K_n(C_\gamma) \to K_n(\operatorname{Frac} C_\gamma)\bigr)[1/p].$$

On these kernels, for $\alpha \le \gamma \le \delta < \beta$, functoriality and the induction hypothesis give

$$(\mathrm{res}_{C_\delta/C_\alpha})^{-1}\,\mathrm{res}_{C_\delta/C_\gamma} = (\mathrm{res}_{C_\gamma/C_\alpha})^{-1}.$$

Hence the inverses on the right define a map from the displayed colimit to the kernel over C_α. Its composites with the canonical map from that kernel are identities, since this holds at each stage. This completes the induction.

The completion $\widehat{C_\lambda}$ is a complete DVR with uniformizer p and residue field k^{perf}. The map $C_\lambda \to \widehat{C_\lambda}$ is injective because

$$\ker(C_\lambda \to \widehat{C_\lambda}) = \bigcap_{m\geq 1} p^m C_\lambda = 0;$$

the last equality follows from the finite valuation of every nonzero element of C_λ. This completion is a strict p-ring. By [Ser62, Chapitre II, §6, Proposition 10], there is an isomorphism

$$\widehat{C_\lambda} \xrightarrow{\sim} W(k^{\mathrm{perf}})$$

inducing the identity on k^{perf}. The composite

$$C = C_0 \xrightarrow{\iota_{0\lambda}} C_\lambda \longrightarrow \widehat{C_\lambda} \xrightarrow{\sim} W(k^{\mathrm{perf}})$$

is an injective local map with the prescribed residue embedding.

Apply Theorem 3.6 to C_λ with $M = \mathbb{S}[1/p]$. It gives injectivity on the two K-groups of the ring and fraction field, and an isomorphism on the kernels of restriction to fraction fields. For every domain R, flatness of $\mathbb{Z}[1/p]$ over $\mathbb{Z}$ identifies

$$\ker(K_n(R)[1/p] \to K_n(\operatorname{Frac} R)[1/p]) = \ker(K_n(R) \to K_n(\operatorname{Frac} R))[1/p].$$

Combining the completion assertions with the induction assertions for $\alpha = 0$ and $\beta = \lambda$ proves (5.3) and both injectivity statements. The integer n was arbitrary. □

6. Witt vectors and rational Gersten kernels

We first recall the finite extension property of Witt rings. It follows from the equivalence for finite étale algebras over a henselian local ring [Sta26, Tag 04GK] and [Ser62, Chapitre II, §6, Proposition 10].

Lemma 6.1 (Finite extensions of Witt rings)**.** *For every finite extension $k \subseteq \ell$ of perfect fields of characteristic p, the extension $W(k) \to W(\ell)$ is finite étale and free of rank $[\ell : k]$.*

Proof. The extension ℓ/k is separable. By [Sta26, Tag 04GK], it lifts to a finite étale $W(k)$-algebra B with $B/pB = \ell$. A finite étale algebra is finite projective as a module; over the local ring $W(k)$ it is free. Its rank is $\dim_k(B/pB) = [\ell : k]$. Finite freeness also makes B p-adically separated, complete, and p-torsion-free. Thus B is a strict p-ring with residue field ℓ. [Ser62, Chapitre II, §6, Proposition 10] gives a unique isomorphism $B \simeq W(\ell)$ inducing the identity on ℓ. Its composite with $W(k) \to B$ equals $W(k) \to W(\ell)$, since both maps induce the prescribed embedding $k \subseteq \ell$ and their source is a strict p-ring. This proves the assertion for the prescribed Witt homomorphism. □

Theorem 6.2 (Extension with fixed ramification)**.** *Let A be a complete DVR of mixed characteristic $(0, p)$ with perfect residue field k, and fix a Cohen embedding $W(k) \hookrightarrow A$ inducing the identity on k. For an extension $k \subseteq \ell$ of perfect fields, define*

$$A_\ell = A \otimes_{W(k)} W(\ell).$$

Then the following assertions hold.

(i) *The ring A_ℓ is a complete DVR with residue field ℓ. Every uniformizer of A is a uniformizer of A_ℓ, and their absolute ramification indices are equal.*

(ii) *For every integer n, the maps*

$$K_n(A)_{\mathbb{Q}} \longrightarrow K_n(A_\ell)_{\mathbb{Q}}, \qquad K_n(\operatorname{Frac} A)_{\mathbb{Q}} \longrightarrow K_n(\operatorname{Frac} A_\ell)_{\mathbb{Q}}$$

are injective.

(iii) *If $k \subseteq \ell$ is algebraically pure, then, for every integer n and every spectrum M,*

$$K_n(A; M) \longrightarrow K_n(A_\ell; M), \qquad K_n(\operatorname{Frac} A; M) \longrightarrow K_n(\operatorname{Frac} A_\ell; M)$$

are pure monomorphisms. In particular this applies to an embedding with k algebraically closed.

Proof. (i) Choose a uniformizer π of A. For a unit $u \in A^\times$ and the positive integer $e = v_A(p)$, one has $p = u\pi^e$. By Corollary 4.3, A is finite free of rank e over $W(k)$. Functoriality of Witt vectors gives an injection $W(k) \to W(\ell)$: in Witt coordinates it is the coordinatewise injection. The target is a domain and is torsion-free over the DVR $W(k)$, so this map is flat. The ring A_ℓ is finite free of rank e over $W(\ell)$. A $W(k)$-basis of A also shows that $A \to A_\ell$ is injective.

Every maximal ideal of A_ℓ contracts to (p) in $W(\ell)$, since the extension is finite integral. It contains p and hence π, because $p = u\pi^e$. Moreover

$$A_\ell/\pi A_\ell = (A/\pi A) \otimes_{W(k)} W(\ell) = k \otimes_{W(k)} W(\ell) = \ell.$$

Thus A_ℓ is local with maximal ideal (π). Multiplication by π on A is injective and remains so after the flat base change from $W(k)$ to $W(\ell)$. The ring A_ℓ is noetherian and has dimension one, since it is finite integral over $W(\ell)$. It is a one-dimensional regular local ring, hence a DVR. It is p-adically complete because it is a finite free module over the complete ring $W(\ell)$. The equality $p = u\pi^e$ identifies the p-adic and π-adic topologies and gives $v_{A_\ell}(p) = e$.

(ii), (iii). By (i), $A \to A_\ell$ is a local map of DVRs preserving a uniformizer. The field k is perfect, so ℓ/k is separable. Proposition 3.5 gives torsion kernels for the two integral maps, hence the injectivity in (ii) after tensoring with $\mathbb{Q}$. Under the purity hypothesis, it gives (iii) directly. If k is algebraically closed, every nonzero finitely presented k-algebra has a k-point by the weak Nullstellensatz [Sta26, Tag 00FV]. Applying this to the algebras in Lemma 3.2 proves that $k \subseteq \ell$ is algebraically pure. □

Corollary 6.3 (Restriction between Witt rings)**.** *For every extension $k \subseteq \ell$ of perfect fields of characteristic p and every integer n, restriction induces injections*

$$\operatorname{res}_{W(\ell)/W(k)} : K_n(W(k))_\mathbb{Q} \hookrightarrow K_n(W(\ell))_\mathbb{Q},$$
$$\operatorname{res}_{\operatorname{Frac} W(\ell)/\operatorname{Frac} W(k)} : K_n(\operatorname{Frac} W(k))_\mathbb{Q} \hookrightarrow K_n(\operatorname{Frac} W(\ell))_\mathbb{Q}.$$

In particular restriction induces an injection

$$\ker\big(K_n(W(k))_\mathbb{Q} \to K_n(\operatorname{Frac} W(k))_\mathbb{Q}\big) \hookrightarrow \ker\big(K_n(W(\ell))_\mathbb{Q} \to K_n(\operatorname{Frac} W(\ell))_\mathbb{Q}\big). \tag{6.1}$$

If the residue embedding is algebraically pure, then, for every integer n and every spectrum M, the maps

$$K_n(W(k); M) \longrightarrow K_n(W(\ell); M), \qquad K_n(\operatorname{Frac} W(k); M) \longrightarrow K_n(\operatorname{Frac} W(\ell); M)$$

are pure monomorphisms.

Proof. Apply Theorem 6.2 with $A = W(k)$. The square

$$\begin{array}{ccc} K_n(W(k))_\mathbb{Q} & \longrightarrow & K_n(\operatorname{Frac} W(k))_\mathbb{Q} \\ \big\downarrow{\scriptstyle \operatorname{res}} & & \big\downarrow{\scriptstyle \operatorname{res}} \\ K_n(W(\ell))_\mathbb{Q} & \longrightarrow & K_n(\operatorname{Frac} W(\ell))_\mathbb{Q} \end{array}$$

commutes. It sends the upper kernel to the lower kernel, and injectivity of the left vertical arrow proves (6.1). □

Proof of Theorem 1.1. Choose a Cohen subring $C \subseteq \widehat{R}$ inducing the identity on k, and a map $C \to W(k^{\mathrm{perf}})$ furnished by Theorem 5.3. For $a \in K_n(k)_\mathbb{Q}$, localization, completion, total

ramification, and perfection give, respectively,

$$\begin{aligned} a \in \operatorname{im}\partial_R &\iff \operatorname{Tr}_{k/R}(a) = 0 \\ &\iff \operatorname{Tr}_{k/\widehat{R}}(a) = 0 \\ &\iff \operatorname{Tr}_{k/C}(a) = 0 \\ &\iff \operatorname{Tr}_{k^{\mathrm{perf}}/W(k^{\mathrm{perf}})} \operatorname{res}_{k^{\mathrm{perf}}/k}(a) = 0 \\ &\iff \operatorname{res}_{k^{\mathrm{perf}}/k}(a) \in \operatorname{im}\partial_{W(k^{\mathrm{perf}})}. \end{aligned}$$

The second equivalence uses the injectivity in Theorem 3.6; the third uses the finiteness of $\widehat{R}$ over C from Corollary 4.3 and then Proposition 4.4. For the fourth, (2.3) identifies the two transfers after restriction, and Theorem 5.3 gives injectivity on their images. The final expression contains no choice of Cohen subring or lift, which proves (i).

For perfect fields $k^{\mathrm{perf}} \subseteq \ell^{\mathrm{perf}}$, flat base change and Corollary 6.3 give, for every $b \in K_n(k^{\mathrm{perf}})_{\mathbb{Q}}$,

$$\operatorname{Tr}_{k^{\mathrm{perf}}/W(k^{\mathrm{perf}})}(b) = 0 \quad \iff \quad \operatorname{Tr}_{\ell^{\mathrm{perf}}/W(\ell^{\mathrm{perf}})} \operatorname{res}_{\ell^{\mathrm{perf}}/k^{\mathrm{perf}}}(b) = 0.$$

Apply this with $b = \operatorname{res}_{k^{\mathrm{perf}}/k}(a)$ and use (i) for R and S. One obtains $a \in \operatorname{im}\partial_R$ if and only if $\operatorname{res}_{\ell/k}(a) \in \operatorname{im}\partial_S$, which is precisely the cartesian assertion in (ii).

Restriction $K_n(k)_{\mathbb{Q}} \to K_n(k^{\mathrm{perf}})_{\mathbb{Q}}$ is surjective by (4.3) and Lemma 2.4. Equation (1.3) therefore induces an isomorphism of the quotients by the boundary images. Localization identifies these quotients with the kernels in (1.4). Multiplication by $v_R(p)^{-1}$ then gives exactly (1.5). This proves existence, uniqueness, and choice independence.

Every ring map $R \to S$ between DVRs of mixed characteristic $(0, p)$ is injective and local: its kernel is a prime not containing p, hence is zero; the inverse image of the maximal ideal is a prime containing p, hence is the maximal ideal of R. For such a map of ramification index e, one has $v_S(p) = e\, v_R(p)$ and

$$\operatorname{res}_{S/R} \operatorname{Tr}_{k/R}(a) = e\ \operatorname{Tr}_{\ell/S} \operatorname{res}_{\ell/k}(a) \qquad (a \in K_n(k)_{\mathbb{Q}}).$$

Applying (1.5) for S to this equality gives

$$\frac{e}{v_S(p)} \operatorname{Tr}_{\ell^{\mathrm{perf}}/W(\ell^{\mathrm{perf}})} \operatorname{res}_{\ell^{\mathrm{perf}}/k}(a) = \frac{1}{v_R(p)} \operatorname{Tr}_{\ell^{\mathrm{perf}}/W(\ell^{\mathrm{perf}})} \operatorname{res}_{\ell^{\mathrm{perf}}/k}(a).$$

By flat base change this is the image under $\operatorname{res}_{W(\ell^{\mathrm{perf}})/W(k^{\mathrm{perf}})}$ of the right side of (1.5). The classes $\operatorname{Tr}_{k/R}(a)$ exhaust the source kernel, so naturality follows. The Witt restriction is injective by Corollary 6.3; the two canonical isomorphisms give the last assertion. □

7. The universal test rings

Proposition 7.1 (Arithmetic realization of a residue field)**.** *Every finitely generated field $k_0/\mathbf{F}_p$ is the residue field of an arithmetic henselian trait.*

Proof. Since $\mathbf{F}_p$ is perfect, a finitely generated extension admits a separating transcendence basis [Sta26, Tag 030Q, Tag 030Z]. Choose such a basis $t_1, \dots, t_s$ for $k_0/\mathbf{F}_p$. Then $k_0/\mathbf{F}_p(t_1, \dots, t_s)$ is finite separable. Choose a primitive element α [Sta26, Tag 030N], with monic minimal polynomial $\overline{f} \in \mathbf{F}_p(t_1, \dots, t_s)[X]$. This includes the trivial extension, for which a degree-one polynomial may be used.

Choose a nonzero polynomial $\overline{g}$ in the variables $T_1, \dots, T_s$ over $\mathbf{F}_p$ so that all coefficients of $\overline{f}$ belong, under $T_i \mapsto t_i$, to the localization

$$\mathbf{F}_p[T_1, \dots, T_s, 1/\overline{g}].$$

Lift $\overline{g}$ to $g \in \mathbb{Z}_{(p)}[T_1, \dots, T_s]$. Lifting the numerators of the coefficients of $\overline{f}$ and replacing their denominators by the corresponding powers of g gives a monic polynomial of the same degree

$$f \in \mathbb{Z}_{(p)}[T_1, \dots, T_s, 1/g][X].$$

Define

$$\widetilde{A} = (\mathbb{Z}_{(p)}[T_1, \dots, T_s, 1/g, 1/\operatorname{disc}(f)])[X]/(f). \tag{7.1}$$

The discriminant of $\overline{f}$ is nonzero by separability, so the reduction of the base ring in this formula is nonzero. The monic algebra is finite free; its discriminant in the power basis is $\operatorname{disc}(f)$. Since this element has been inverted, the algebra is finite étale [Sta26, Tag 0BJF]. The base is a localization of a polynomial algebra by finitely many elements, and is smooth and finitely presented over $\mathbb{Z}_{(p)}$. These properties are preserved by a finite étale extension. Thus $\widetilde{A}$ is smooth and finitely presented over $\mathbb{Z}_{(p)}$.

We check the special fibre. The quotient by the monic polynomial $\overline{f}$ is free, with basis $1, X, \dots, X^{\deg f - 1}$, over $\mathbf{F}_p[T_1, \dots, T_s, 1/\overline{g}, 1/\operatorname{disc}(\overline{f})]$. It therefore injects into its localization at the nonzero elements of this domain. That localization is

$$\mathbf{F}_p(t_1, \dots, t_s)[X]/(\overline{f}) = k_0.$$

Hence $\widetilde{A}/p\widetilde{A}$ is a domain. Its fraction field is k_0, since its image contains the t_i and α. The ring $U = (\widetilde{A}_{p\widetilde{A}})^h$ is consequently an arithmetic henselian trait with residue field k_0.

□

Proof of Corollary 1.3. Fix R as in (ii), with residue field k, and a nonzero class

$$x \in \ker\bigl(K_n(R)_{\mathbb{Q}} \to K_n(\operatorname{Frac} R)_{\mathbb{Q}}\bigr).$$

By Proposition 2.1 and Lemma 2.4, there are a finitely generated subfield $k_0/\mathbf{F}_p$ of k and $a_0 \in K_n(k_0)_{\mathbb{Q}}$ such that

$$x = \operatorname{Tr}_{k/R} \operatorname{res}_{k/k_0}(a_0), \qquad \operatorname{res}_{k/k_0}(a_0) \notin \operatorname{im} \partial_R.$$

Choose an arithmetic henselian trait U with residue field k_0 by Proposition 7.1. The cartesian square in Theorem 1.1(ii), for U and R, gives $a_0 \notin \operatorname{im} \partial_U$. Thus

$$0 \neq \operatorname{Tr}_{k_0/U}(a_0) \in \ker\bigl(K_n(U)_{\mathbb{Q}} \to K_n(\operatorname{Frac} U)_{\mathbb{Q}}\bigr).$$

The integer $s = \operatorname{trdeg}_{\mathbf{F}_p} k_0$ is finite. A transcendence basis of $k_0/\mathbf{F}_p$ identifies its rational function subfield with $\mathbf{F}_p(t_1, \dots, t_s)$; extending to algebraic closures gives an embedding $k_0 \hookrightarrow \Omega_{p,s}$. For each $b \in \mathbb{N} \cup \{\infty\}$ with $b \geq s$, compose this with the fixed embedding $\Omega_{p,s} \hookrightarrow \Omega_{p,b}$. Applying Theorem 1.1(ii) to U and $W(\Omega_{p,b})$ gives

$$a_0 \notin \operatorname{im} \partial_U \quad \Longleftrightarrow \quad \operatorname{res}_{\Omega_{p,b}/k_0}(a_0) \notin \operatorname{im} \partial_{W(\Omega_{p,b})}.$$

Consequently

$$0 \neq \operatorname{Tr}_{\Omega_{p,b}/W(\Omega_{p,b})} \operatorname{res}_{\Omega_{p,b}/k_0}(a_0) \in \ker\bigl(K_n(W(\Omega_{p,b}))_{\mathbb{Q}} \to K_n(\operatorname{Frac} W(\Omega_{p,b}))_{\mathbb{Q}}\bigr).$$

For finite d, the inclusion $k_0 \subseteq k$ gives $s \leq d$. For $d = \infty$, s is still finite, regardless of the cardinality of k. Taking $b = d$ proves (i)⇒(ii) by contraposition. The class over U proves (iii)⇒(ii) by contraposition. Conversely, (ii) implies (i) and (iii) because all their rings occur in (ii). Taking $b = s$ and $b = \infty$ proves the more precise assertion. Applying the equivalence separately for every n proves the simultaneous assertion. □

Remark 7.2. In condition (iii) of Corollary 1.3, one may choose just one arithmetic henselian trait for each finitely generated residue field: the proof uses only its residue field. Continuity was applied to $K_n(k)$, not to $W(k)$; full Witt vectors need not commute with filtered unions of perfect fields.

8. Known vanishing and the residue field

The low-dimensional vanishing below is classical. For finitely generated residue fields, the Quillen–Harder argument appears in [Ska17, Corollary 5.12(i) and the proof of Proposition 5.13]. We give its rational form and pass to arbitrary residue fields by continuity. The vanishing used here is the finite-field and curve case of the consequence of Parshin's conjecture in [Gei98, Theorem 1.3(i)]. The subsequent comparisons follow from Theorem 1.1 and Corollary 1.3.

Proposition 8.1 (Residue transcendence degree at most one)**.** *Let R be a DVR whose residue field k has characteristic $p > 0$ and satisfies* $\operatorname{trdeg}_{\mathbf{F}_p} k \leq 1$. *Then $K_n(R)_{\mathbb{Q}} \to K_n(\operatorname{Frac} R)_{\mathbb{Q}}$ is injective for every n.*

Proof. If $k/\mathbf{F}_p$ is algebraic, it is the union of its finite subfields. Quillen's finite-field computation [Qui72, Theorem 8(i)] gives $K_m(l)_{\mathbb{Q}} = 0$ for each such field l and each $m \geq 1$. Filtered-colimit continuity gives $K_m(k)_{\mathbb{Q}} = 0$ for $m \geq 1$.

Suppose $\operatorname{trdeg}_{\mathbf{F}_p} k = 1$ and choose an element $t \in k$ transcendental over $\mathbf{F}_p$. Every element of k is algebraic over $\mathbf{F}_p(t)$, and any finite collection of such elements generates a finite extension. Consequently k is the filtered union of the finite extensions $L/\mathbf{F}_p(t)$ contained in it. Each L is the function field of a smooth projective curve X over a finite field. Harder's theorem [Har77], in the K-theoretic form [Wei13, VI, Theorem 6.1], gives $K_m(X)_{\mathbb{Q}} = 0$ for $m \geq 1$. Localization, followed by the filtered colimit over the nonempty open subsets of X, gives the exact sequence

$$K_m(X)_{\mathbb{Q}} \longrightarrow K_m(L)_{\mathbb{Q}} \longrightarrow \bigoplus_{x \in X^{(1)}} K_{m-1}(\kappa(x))_{\mathbb{Q}}.$$

For $m \geq 2$ the first group is zero by Harder's theorem, and each summand on the right is zero by Quillen's computation, since $\kappa(x)$ is finite and $m - 1 \geq 1$. Thus $K_m(L)_{\mathbb{Q}} = 0$ for $m \geq 2$. This is also [Ska17, Corollary 5.12(i)]. Continuity now gives $K_m(k)_{\mathbb{Q}} = 0$ for $m \geq 2$.

In either case, for $n \geq 2$ the source of $\operatorname{Tr}_{k/R} : K_n(k)_{\mathbb{Q}} \to K_n(R)_{\mathbb{Q}}$ is zero, so its image is zero. Equation (2.2) proves injectivity in those degrees. For $n \leq 1$, use Lemma 2.3. □

Remark 8.2. Restriction between $W(\Omega_{p,a})$ and $W(\Omega_{p,b})$ is injective on rational K-groups for $a \leq b$, by Corollary 6.3. Thus a failure of rational Gersten injectivity at finite transcendence degree a persists at every $b \geq a$. By Corollary 1.3 and Proposition 8.1, any failure occurs at some finite transcendence degree at least two.

Remark 8.3. By Theorem 1.1 and Corollary 1.3, $W(\Omega_{p,d})$ in condition (i) may be replaced by any henselian DVR of mixed characteristic $(0, p)$ with residue field $\Omega_{p,d}$. The comparison of its rational kernel with that of $W(\Omega_{p,d})$ is (1.5).

9. Comparing consecutive test rings

Fix a prime p. Corollary 1.3 turns induction on the residue transcendence degree into a comparison of successive Witt rings. For each integer $d \geq 0$, we factor that comparison through the perfection of a rational function field:

$$W(\Omega_{p,d}) \longrightarrow W(\Omega_{p,d}(t)^{\mathrm{perf}}) \longrightarrow W(\Omega_{p,d+1}).$$

Here t is transcendental over $\Omega_{p,d}$, and the last residue field is an algebraic closure of $\Omega_{p,d}(t)$. The first arrow is controlled by localization on the affine line. For the second, Proposition 9.8 gives a criterion involving cyclic extensions of prime degree over all finite intermediate fields. Combining the two statements with Corollary 1.3 gives Corollaries 9.10 and 9.11.

The affine-line calculation, Galois transfer identities, and normalized transfers below use classical localization, homotopy invariance, and additivity [Wei13, IV, Exercise 6.13; V, Theorem 6.3 and Corollary 6.7.1]. We apply these identities to the kernels defined in (9.1). Injectivity under extension of perfect residue fields supplies the descent argument. Corollary 1.3 then gives the statements for henselian DVRs in Corollaries 9.10 and 9.11.

For every perfect field E of characteristic p and every integer $n \geq 0$, define

$$M_n(E) = \ker(K_n(W(E))_{\mathbb{Q}} \longrightarrow K_n(\operatorname{Frac} W(E))_{\mathbb{Q}})\,. \tag{9.1}$$

By Proposition 2.1,

$$M_n(E) = \operatorname{im}\Big(\operatorname{Tr}_{E/W(E)} : K_n(E)_{\mathbb{Q}} \longrightarrow K_n(W(E))_{\mathbb{Q}}\Big)\,. \tag{9.2}$$

Restriction sends $M_n(E)$ injectively to $M_n(E')$ for every extension $E \subseteq E'$ of perfect fields, by Corollary 6.3.

Lemma 9.1 (Unions of perfect residue fields)**.** *Let E be the union of a filtered family $(E_i)_{i\in I}$ of perfect subfields of characteristic p. For every $n \geq 0$, restriction induces an isomorphism*

$$\varinjlim_{i\in I} M_n(E_i) \xrightarrow{\sim} M_n(E).$$

Proof. Let $x \in M_n(E)$. By (9.2), there is $a \in K_n(E)_{\mathbb{Q}}$ with $\operatorname{Tr}_{E/W(E)}(a) = x$. By Lemma 2.4, there are $i \in I$ and $a_i \in K_n(E_i)_{\mathbb{Q}}$ with $\operatorname{res}_{E/E_i}(a_i) = a$. The element $x_i = \operatorname{Tr}_{E_i/W(E_i)}(a_i)$ belongs to $M_n(E_i)$. Both Witt rings have uniformizer p, so (2.3) gives

$$\operatorname{res}_{W(E)/W(E_i)}(x_i) = \operatorname{Tr}_{E/W(E)} \operatorname{res}_{E/E_i}(a_i) = x.$$

This proves surjectivity. If an element of the colimit represented by $y_i \in M_n(E_i)$ maps to zero, then $\operatorname{res}_{W(E)/W(E_i)}(y_i) = 0$. Injectivity in Corollary 6.3 gives $y_i = 0$, which proves injectivity. □

Lemma 9.2 (The rational function field)**.** *Let k be an algebraically closed field of characteristic p, let t be transcendental over k, and let $n \geq 1$. The homomorphism*

$$\begin{aligned} K_n(k)_{\mathbb{Q}} \oplus \bigoplus_{a\in k} K_{n-1}(k)_{\mathbb{Q}} &\longrightarrow K_n(k(t))_{\mathbb{Q}}, \\ (b, (c_a)_{a\in k}) &\longmapsto \operatorname{res}_{k(t)/k}(b) + \sum_{a\in k} \{t-a\} \cdot \operatorname{res}_{k(t)/k}(c_a) \end{aligned} \tag{9.3}$$

is an isomorphism; the sum is finite because $(c_a)_{a\in k}$ has finite support. Moreover, restriction gives an isomorphism $K_n(k(t))_{\mathbb{Q}} \xrightarrow{\sim} K_n(k(t)^{\mathrm{perf}})_{\mathbb{Q}}$.

Proof. Quillen's calculation [Wei13, V, Corollary 6.7.1] gives the split exact sequence

$$0 \longrightarrow K_n(k)_{\mathbb{Q}} \longrightarrow K_n(k(t))_{\mathbb{Q}} \xrightarrow{(\partial_a)_{a\in k}} \bigoplus_{a\in k} K_{n-1}(k)_{\mathbb{Q}} \longrightarrow 0. \tag{9.4}$$

Every closed point has the form $(t-a)$ because k is algebraically closed. With $\partial_a\{t-a\} = 1$, the product formula for boundaries [Wei13, V, (6.1.1) and Corollary 6.3.1] gives, for $a, b \in k$ and $c \in K_{n-1}(k)_{\mathbb{Q}}$,

$$\partial_b(\{t-a\} \cdot \operatorname{res}_{k(t)/k}(c)) = \begin{cases} c & b = a, \\ 0 & b \neq a. \end{cases}$$

These identities identify its splitting with (9.3).

For completeness, the assertion about perfection is the field case of [KM21, Lemma 4.1]. It also follows from the field transfer identity already proved in (4.3). For every finite purely inseparable extension $H/k(t)$, both composites of restriction and transfer are multiplication by $[H : k(t)]$. The degree is a power of p, so restriction is an isomorphism after tensoring with $\mathbb{Q}$. The finite subextensions H of $k(t)^{\mathrm{perf}}/k(t)$ form a filtered system with union $k(t)^{\mathrm{perf}}$. Apply Lemma 2.4 to conclude. □

Proposition 9.3 (Adjoining a variable)**.** *Let k be an algebraically closed field of characteristic p, let t be transcendental over k, and let $F = k(t)^{\mathrm{perf}}$. For $a \in k$, denote the Teichmüller lift*

of $t-a$ by $[t-a]\in W(F)^\times$. For every integer $n\ge 1$, one has the equality of subspaces of $K_n(W(F))_{\mathbb{Q}}$

$$M_n(F)=\operatorname{res}_{W(F)/W(k)}M_n(k)+\sum_{a\in k}\{[t-a]\}\cdot\operatorname{res}_{W(F)/W(k)}M_{n-1}(k). \tag{9.5}$$

In particular,

$$M_n(k)=M_{n-1}(k)=0\quad\Longrightarrow\quad M_n(F)=0. \tag{9.6}$$

More generally, if $M_{n-1}(k)=0$, restriction induces an isomorphism $M_n(k)\xrightarrow{\sim}M_n(F)$. In particular, $M_3(k)\xrightarrow{\sim}M_3(F)$ for every such k.

Proof. Let $x\in M_n(F)$. Choose $a\in K_n(F)_{\mathbb{Q}}$ with $\operatorname{Tr}_{F/W(F)}(a)=x$, by (9.2). By Lemma 9.2, there are $b\in K_n(k)_{\mathbb{Q}}$, a finite subset $S\subset k$, and $c_s\in K_{n-1}(k)_{\mathbb{Q}}$ for each $s\in S$, such that

$$a=\operatorname{res}_{F/k}(b)+\sum_{s\in S}\{t-s\}\cdot\operatorname{res}_{F/k}(c_s).$$

Flat base change and the projection formula give

$$\begin{aligned}\operatorname{Tr}_{F/W(F)}\operatorname{res}_{F/k}(b)&=\operatorname{res}_{W(F)/W(k)}\operatorname{Tr}_{k/W(k)}(b),\\ \operatorname{Tr}_{F/W(F)}(\{t-s\}\cdot\operatorname{res}_{F/k}(c_s))&=\{[t-s]\}\cdot\operatorname{Tr}_{F/W(F)}\operatorname{res}_{F/k}(c_s)\\ &=\{[t-s]\}\cdot\operatorname{res}_{W(F)/W(k)}\operatorname{Tr}_{k/W(k)}(c_s).\end{aligned}$$

The projection formula is [Wei13, V, Example 3.5.3]. These identities put x in the right side of (9.5). Conversely, restriction preserves the kernels in (9.1). For $u\in K_1(W(F))_{\mathbb{Q}}$ and $y\in M_{n-1}(F)$, one has

$$\operatorname{res}_{\operatorname{Frac}W(F)/W(F)}(u\cdot y)=\operatorname{res}_{\operatorname{Frac}W(F)/W(F)}(u)\cdot\operatorname{res}_{\operatorname{Frac}W(F)/W(F)}(y)=0.$$

Every summand on the right therefore belongs to $M_n(F)$. This proves the equality and (9.6). If $M_{n-1}(k)=0$, (9.5) makes restriction $M_n(k)\to M_n(F)$ surjective. Its injectivity is Corollary 6.3. For $n=3$, the hypothesis holds because $M_2(k)=0$ by Lemma 2.3. □

Remark 9.4. Retain k,t,F,n from Proposition 9.3. The sum in (9.5) has not been proved direct. A sufficient additional compatibility is the following: for every $b\in K_n(k(t))_{\mathbb{Q}}$,

$$\operatorname{res}_{F/k(t)}(b)\in\operatorname{im}\partial_{W(F)}\quad\Longrightarrow\quad\partial_a(b)\in\operatorname{im}(\partial_{W(k)}:K_n(\operatorname{Frac}W(k))_{\mathbb{Q}}\to K_{n-1}(k)_{\mathbb{Q}})\quad(a\in k).$$

To check this, fix $b_0\in K_n(k)_{\mathbb{Q}}$, a finite subset $S\subset k$, and $c_a\in K_{n-1}(k)_{\mathbb{Q}}$ for $a\in S$, and suppose

$$\operatorname{res}_{W(F)/W(k)}\operatorname{Tr}_{k/W(k)}(b_0)+\sum_{a\in S}\{[t-a]\}\cdot\operatorname{res}_{W(F)/W(k)}\operatorname{Tr}_{k/W(k)}(c_a)=0.$$

By the projection formula and localization this is equivalent to

$$\operatorname{res}_{F/k(t)}\left(\operatorname{res}_{k(t)/k}(b_0)+\sum_{a\in S}\{t-a\}\cdot\operatorname{res}_{k(t)/k}(c_a)\right)\in\operatorname{im}\partial_{W(F)}.$$

The displayed compatibility and (9.4) give $c_a\in\operatorname{im}\partial_{W(k)}$, and hence $\operatorname{Tr}_{k/W(k)}(c_a)=0$, for every $a\in S$. The original relation reduces to $\operatorname{res}_{W(F)/W(k)}\operatorname{Tr}_{k/W(k)}(b_0)=0$. By Corollary 6.3, $\operatorname{Tr}_{k/W(k)}(b_0)=0$. Thus every component of the relation is zero. Theorem 1.1 concerns field embeddings; the displayed compatibility with tame boundaries is an additional condition.

Remark 9.5 (The degree-three comparison). The invariance in degree three has a classical antecedent stronger than the calculation above. Levine proves

$$K_3^{\mathrm{ind}}(k)\xrightarrow{\sim}K_3^{\mathrm{ind}}(k(t)),\qquad K_3^{\mathrm{ind}}(k)=\operatorname{coker}(K_3^M(k)\to K_3(k)),$$

for every field k [Lev89, §4.2, pp. 327–328]. Thus $M_3(k)\to M_3(k(t)^{\mathrm{perf}})$ is an isomorphism for every perfect field k of characteristic p. Indeed, given $a\in K_3(k(t)^{\mathrm{perf}})_{\mathbb{Q}}$, rational invariance

under perfection gives $a_0 \in K_3(k(t))_{\mathbb{Q}}$ restricting to a. Levine's isomorphism gives $b \in K_3(k)_{\mathbb{Q}}$ and $c \in K_3^M(k(t))_{\mathbb{Q}}$ such that

$$a_0 = \mathrm{res}_{k(t)/k}(b) + \mathrm{image}(c).$$

After restriction to $k(t)^{\mathrm{perf}}$, Lemma 2.2 annihilates the last summand under the closed-point Gysin map. Flat base change therefore gives

$$\mathrm{Tr}_{k(t)^{\mathrm{perf}}/W(k(t)^{\mathrm{perf}})}(a) = \mathrm{res}_{W(k(t)^{\mathrm{perf}})/W(k)}\,\mathrm{Tr}_{k/W(k)}(b).$$

This proves surjectivity on M_3; injectivity is Corollary 6.3.

Lemma 9.6 (Finite extensions and fixed subfields)**.** *Let $K \subseteq L$ be a finite extension of perfect fields of characteristic p, and let $n \geq 0$. Transfer between the Witt rings induces a homomorphism $\mathrm{Tr}_{W(L)/W(K)} : M_n(L) \to M_n(K)$, and*

$$M_n(L) = \mathrm{res}_{W(L)/W(K)}\,M_n(K) \oplus \ker\bigl(\mathrm{Tr}_{W(L)/W(K)} : M_n(L) \to M_n(K)\bigr). \tag{9.7}$$

If L/K is Galois with group G, then every $g \in G$ acts on $M_n(L)$ by the homomorphism induced by $W(g) : W(L) \to W(L)$, and

$$\mathrm{res}_{W(L)/W(K)}\,M_n(K) = M_n(L)^G. \tag{9.8}$$

If $G = \langle\sigma\rangle$ is cyclic of finite order $m \geq 1$, then

$$\ker\bigl(\mathrm{Tr}_{W(L)/W(K)} : M_n(L) \to M_n(K)\bigr) = (\sigma - 1)M_n(L). \tag{9.9}$$

Proof. By Lemma 6.1, $W(L)/W(K)$ is finite étale and free of rank $[L : K]$. Transfer commutes with localization at p, by finite flat base change [Wei13, V, (6.6.3)–(6.6.4)]. Hence it sends $M_n(L)$ to $M_n(K)$. The restriction–transfer identity is

$$\mathrm{Tr}_{W(L)/W(K)}\,\mathrm{res}_{W(L)/W(K)} = [L : K]\,\mathrm{id}_{M_n(K)}. \tag{9.10}$$

Indeed, $W(L)$ is free of rank $[L : K]$ over $W(K)$, so the composite exact functor is a direct sum of $[L : K]$ copies of the identity. For every $x \in M_n(L)$ the decomposition

$$\begin{aligned} x = {} & \frac{1}{[L : K]}\,\mathrm{res}_{W(L)/W(K)}\,\mathrm{Tr}_{W(L)/W(K)}(x) \\ & + \left(x - \frac{1}{[L : K]}\,\mathrm{res}_{W(L)/W(K)}\,\mathrm{Tr}_{W(L)/W(K)}(x)\right) \end{aligned}$$

has its first summand in the restriction image and its second in the transfer kernel. If $\mathrm{res}_{W(L)/W(K)}(y)$ is in both, then $[L : K]y = 0$ by (9.10), and therefore $y = 0$. This proves (9.7).

Suppose L/K is Galois. The homomorphism

$$W(L) \otimes_{W(K)} W(L) \longrightarrow \prod_{g \in G} W(L), \qquad a \otimes b \longmapsto (a\,W(g)(b))_{g \in G},$$

is an isomorphism: modulo p it is the Galois isomorphism $L \otimes_K L \simeq \prod_{g \in G} L$, and both sides are free $W(L)$-modules of rank $[L : K]$. A matrix of this homomorphism has invertible determinant modulo p, hence invertible determinant in $W(L)$. Applying additivity to its bimodule decomposition gives the usual Galois transfer identity [Wei13, IV, Exercise 6.13]:

$$\mathrm{res}_{W(L)/W(K)}\,\mathrm{Tr}_{W(L)/W(K)}(x) = \sum_{g \in G} g(x) \qquad (x \in M_n(L)). \tag{9.11}$$

Every restricted class is fixed by G. Conversely, if $x \in M_n(L)^G$, then

$$x = \mathrm{res}_{W(L)/W(K)}\left(\frac{1}{|G|}\,\mathrm{Tr}_{W(L)/W(K)}(x)\right),$$

which proves (9.8).

In the cyclic case, (9.11) and the injectivity of restriction identify the transfer kernel with the kernel of $1+\sigma+\cdots+\sigma^{m-1}$ on $M_n(L)$. This operator annihilates $(\sigma-1)M_n(L)$. For an element x in its kernel, the element

$$y=\frac{1}{m}\sum_{j=0}^{m-1} j\sigma^j(x)\in M_n(L)$$

satisfies

$$(\sigma-1)y=x-\frac{1}{m}\sum_{j=0}^{m-1}\sigma^j(x)=x.$$

This proves the reverse inclusion in (9.9). □

Corollary 9.7 (Normalized transfer)**.** *Let $F\subseteq E$ be an algebraic extension of perfect fields of characteristic p, and let $n\geq 0$. Restriction $M_n(F)\to M_n(E)$ has a canonical retraction determined by*

$$\mathrm{res}_{W(E)/W(L)}(x)\longmapsto\frac{1}{[L:F]}\,\mathrm{Tr}_{W(L)/W(F)}(x)\quad(F\subseteq L\subseteq E,\ [L:F]<\infty,\ x\in M_n(L)).$$

Proof. For finite intermediate fields $F\subseteq L\subseteq L'\subseteq E$, transitivity of transfer and (9.10) give

$$\begin{aligned}\frac{1}{[L':F]}\,\mathrm{Tr}_{W(L')/W(F)}\,\mathrm{res}_{W(L')/W(L)}&=\frac{1}{[L':F]}\,\mathrm{Tr}_{W(L)/W(F)}\,\mathrm{Tr}_{W(L')/W(L)}\,\mathrm{res}_{W(L')/W(L)}\\&=\frac{1}{[L:F]}\,\mathrm{Tr}_{W(L)/W(F)}\,.\end{aligned}$$

Thus these homomorphisms are compatible with the restriction maps. The finite intermediate fields form a filtered family with union E. By Lemma 9.1, the displayed rule defines a unique homomorphism on $M_n(E)$. Its composite with restriction from $M_n(F)$ is the identity, by taking $L=F$ in the rule. □

Proposition 9.8 (Descent from an algebraic closure)**.** *Let F be a perfect field of characteristic p, choose an algebraic closure $\overline{F}$, and let $n\geq 0$. The following are equivalent.*

(i) *Restriction induces an isomorphism $M_n(F)\xrightarrow{\sim}M_n(\overline{F})$.*

(ii) *For every tower $F\subseteq K\subsetneq L\subseteq\overline{F}$ with $[K:F]<\infty$ and L/K cyclic Galois of prime degree, $\mathrm{Tr}_{W(L)/W(K)}:M_n(L)\to M_n(K)$ is injective.*

(iii) *For every tower in* (ii)*, the group $\mathrm{Gal}(L/K)$ acts trivially on $M_n(L)$.*

(iv) *For every tower in* (ii) *and every generator $\sigma\in\mathrm{Gal}(L/K)$,*

$$(\sigma-1)K_n(L)_{\mathbb{Q}}\subseteq\mathrm{im}\Big(K_{n+1}(\mathrm{Frac}\,W(L))_{\mathbb{Q}}\xrightarrow{\partial_{W(L)}}K_n(L)_{\mathbb{Q}}\Big).\tag{9.12}$$

Here σ acts on $K_n(L)_{\mathbb{Q}}$ by restriction along $\sigma:L\to L$.

If $M_n(F)=0$, these conditions are equivalent to $M_n(\overline{F})=0$.

Proof. The equivalence of (ii) and (iii) is (9.9). For a tower as in (ii), functoriality of the closed-point Gysin map gives

$$(\sigma-1)\,\mathrm{Tr}_{L/W(L)}(a)=\mathrm{Tr}_{L/W(L)}((\sigma-1)a)\qquad(a\in K_n(L)_{\mathbb{Q}}).$$

The map $\mathrm{Tr}_{L/W(L)}$ surjects onto $M_n(L)$, and its kernel is the boundary image in (9.12), by Proposition 2.1. Thus (iii) and (iv) are equivalent.

Assume (i). For $x\in M_n(L)$, choose $y\in M_n(F)$ such that

$$\mathrm{res}_{W(\overline{F})/W(L)}(x)=\mathrm{res}_{W(\overline{F})/W(F)}(y).$$

Injectivity of restriction from $W(L)$ to $W(\overline{F})$ gives $x=\mathrm{res}_{W(L)/W(F)}(y)$. Every $\sigma\in\mathrm{Gal}(L/K)$ fixes F, so $\sigma(x)=x$. This proves (iii).

Assume (iii) and let E/F be a finite Galois extension inside $\overline{F}$, with group G. The action of G on $M_n(E)$ has a normal kernel

$$N = \{g \in G : g(x) = x \text{ for every } x \in M_n(E)\}.$$

Suppose $N \neq G$. Choose an element of G/N of order $r > 1$ and a prime divisor ℓ of r. Its (r/ℓ)-th power has order ℓ; let $C \subseteq G$ be the inverse image of the subgroup it generates. Then $N \subsetneq C$, C/N is cyclic of order ℓ, and

$$F \subseteq E^C \subsetneq E^N \subseteq \overline{F}, \qquad \mathrm{Gal}(E^N/E^C) = C/N,$$

by the finite Galois correspondence [Sta26, Tag 09DW, Tag 0BMH]. By (9.8), restriction identifies $M_n(E^N)$ with $M_n(E)^N = M_n(E)$. This identification respects the C/N-actions: an element of C restricts to its corresponding automorphism of E^N. By (iii), C/N acts trivially on $M_n(E^N)$, hence on $M_n(E)$. This contradicts the definition of N and $C \neq N$. Therefore $N = G$, and (9.8) yields

$$M_n(E) = M_n(E)^G = \mathrm{res}_{W(E)/W(F)}\, M_n(F).$$

Because F is perfect, every finite extension inside $\overline{F}$ is separable. Its normal closure is Galois [Sta26, Tag 0EXM] and finite: it is generated by the finitely many roots of the minimal polynomials of a finite set of field generators. Composita of finite Galois extensions are finite Galois, since the union of the corresponding finite sets of separable polynomials has that compositum as its splitting field. Thus the finite Galois subextensions form a filtered family with union $\overline{F}$. For $x \in M_n(\overline{F})$, Lemma 9.1 gives such an extension E/F and $x_E \in M_n(E)$ satisfying $\mathrm{res}_{W(\overline{F})/W(E)}(x_E) = x$. The preceding equality gives $y \in M_n(F)$ with $x_E = \mathrm{res}_{W(E)/W(F)}(y)$. Therefore

$$x = \mathrm{res}_{W(\overline{F})/W(E)}\, \mathrm{res}_{W(E)/W(F)}(y) = \mathrm{res}_{W(\overline{F})/W(F)}(y),$$

proving surjectivity in (i). Its injectivity is Corollary 6.3. The final assertion follows by substituting $M_n(F) = 0$ in (i). □

Corollary 9.9 (Consecutive test rings)**.** *Fix a prime p and integers $d \geq 0$, $n \geq 1$. Choose t transcendental over $\Omega_{p,d}$ and identify $\Omega_{p,d+1}$ with an algebraic closure of $\Omega_{p,d}(t)$. Assume $M_{n-1}(\Omega_{p,d}) = 0$. The composite*

$$M_n(\Omega_{p,d+1}) \xrightarrow{\textit{normalized transfer}} M_n(\Omega_{p,d}(t)^{\mathrm{perf}}) \xrightarrow{\mathrm{res}^{-1}} M_n(\Omega_{p,d})$$

is a retraction of restriction $M_n(\Omega_{p,d}) \to M_n(\Omega_{p,d+1})$. The first arrow is the map in Corollary 9.7; the second is the inverse of restriction from $W(\Omega_{p,d})$ to $W(\Omega_{p,d}(t)^{\mathrm{perf}})$. Restriction between the two kernels is an isomorphism if and only if the equivalent conditions (ii)–(iv) *of Proposition 9.8 hold with $F = \Omega_{p,d}(t)^{\mathrm{perf}}$ and $\overline{F} = \Omega_{p,d+1}$. In particular, this construction gives a retraction of $M_3(\Omega_{p,d}) \to M_3(\Omega_{p,d+1})$ for every $d \geq 0$.*

Proof. By Proposition 9.3 and Corollary 6.3, the left arrow in the diagram

$$\begin{array}{ccc} M_n(\Omega_{p,d}) & \xhookrightarrow{\ \mathrm{res}\ } & M_n(\Omega_{p,d+1}) \\ {\scriptstyle \mathrm{res}}\big\downarrow{\scriptstyle \sim} & \nearrow_{\text{normalized transfer}} & \\ M_n(\Omega_{p,d}(t)^{\mathrm{perf}}) & & \end{array}$$

is an isomorphism. The diagonal arrow is Corollary 9.7, applied to the algebraic extension $\Omega_{p,d}(t)^{\mathrm{perf}} \subseteq \Omega_{p,d+1}$. The triangle commutes because normalized transfer is a retraction of restriction from its base field. Its diagonal arrow followed by the inverse of the left arrow is therefore a retraction of the top arrow. The top arrow is an isomorphism if and only if restriction

$$M_n(\Omega_{p,d}(t)^{\mathrm{perf}}) \longrightarrow M_n(\Omega_{p,d+1})$$

is an isomorphism. Apply Proposition 9.8 for the asserted criterion. For $n = 3$, the hypothesis is automatic because $M_2(\Omega_{p,d}) = 0$ by Lemma 2.3. □

Corollary 9.10 (One step in residue transcendence degree)**.** *Fix a prime p and integers $d \geq 0$, $n \geq 1$. Assume*

$$M_n(\Omega_{p,d}) = M_{n-1}(\Omega_{p,d}) = 0. \tag{9.13}$$

Let t be transcendental over $\Omega_{p,d}$, and identify $\Omega_{p,d+1}$ with an algebraic closure of $F = \Omega_{p,d}(t)^{\mathrm{perf}}$. The following are equivalent.

(i) *Rational Gersten injectivity in degree n holds for every mixed-characteristic henselian DVR R of residue characteristic p with $\operatorname{trdeg}_{\mathbf{F}_p} \kappa(R) \leq d+1$.*

(ii) *$M_n(\Omega_{p,d+1}) = 0$.*

(iii) *For every tower $F \subseteq K \subsetneq L \subseteq \Omega_{p,d+1}$ with $[K : F] < \infty$ and L/K cyclic Galois of prime degree, $\operatorname{Tr}_{W(L)/W(K)} : M_n(L) \to M_n(K)$ is injective.*

(iv) *For every tower in* (iii) *and every generator $\sigma \in \mathrm{Gal}(L/K)$, the inclusion* (9.12) *holds.*

Proof. By Proposition 9.3 and (9.13), $M_n(F) = 0$. Since $\overline{F} = \Omega_{p,d+1}$, Proposition 9.8 gives the equivalence of (ii), (iii), and (iv). The equivalence of (i) and (ii) is Corollary 1.3. □

Corollary 9.11 (A finite induction criterion)**.** *Fix a prime p and $N, d \in \mathbb{N}$. The following are equivalent.*

(i) *For every mixed-characteristic henselian DVR R with residue characteristic p and $\operatorname{trdeg}_{\mathbf{F}_p} \kappa(R) \leq d$, rational Gersten injectivity holds in every degree $0 \leq n \leq N$.*

(ii) *For every pair of integers $3 \leq n \leq N$ and $1 \leq e < d$, every tower*

$$\Omega_{p,e}(t)^{\mathrm{perf}} \subseteq K \subsetneq L \subseteq \Omega_{p,e+1}, \qquad [K : \Omega_{p,e}(t)^{\mathrm{perf}}] < \infty,$$

in which L/K is cyclic Galois of prime degree, and every generator $\sigma \in \mathrm{Gal}(L/K)$, one has

$$(\sigma - 1)K_n(L)_{\mathbb{Q}} \subseteq \operatorname{im}\Big(K_{n+1}(\operatorname{Frac} W(L))_{\mathbb{Q}} \xrightarrow{\partial_{W(L)}} K_n(L)_{\mathbb{Q}} \Big). \tag{9.14}$$

For each e, t is transcendental over $\Omega_{p,e}$ and $\Omega_{p,e+1}$ is an algebraic closure of $\Omega_{p,e}(t)$.

Requiring (9.14) *for all $n \geq 3$ and all $e \geq 1$ is equivalent to rational Gersten injectivity in every degree for all mixed-characteristic henselian DVRs of residue characteristic p.*

Proof. Assume (i). For a tower in (ii), L is perfect and has transcendence degree $e+1 \leq d$ over $\mathbf{F}_p$. Thus (i), applied to $W(L)$, gives $M_n(L) = 0$. Localization gives $\operatorname{im} \partial_{W(L)} = K_n(L)_{\mathbb{Q}}$, proving (9.14).

Assume (ii). By Lemma 2.3, the assertion is true if $N \leq 2$. By Proposition 8.1, it is true if $d \leq 1$. Suppose $N \geq 3$ and $d \geq 2$. We prove by induction on e, for $1 \leq e \leq d$, that

$$M_n(\Omega_{p,e}) = 0 \qquad (0 \leq n \leq N).$$

The case $e = 1$ is Proposition 8.1. Suppose $1 \leq e < d$ and the displayed assertion holds for e. For each n with $3 \leq n \leq N$, it gives both $M_n(\Omega_{p,e}) = 0$ and $M_{n-1}(\Omega_{p,e}) = 0$. Hypothesis (ii) and Corollary 9.10 therefore give $M_n(\Omega_{p,e+1}) = 0$. The degrees $0, 1, 2$ are Lemma 2.3, which completes the induction. Apply Corollary 1.3 in each degree $n \leq N$ at residue bound d to obtain (i).

For the last assertion, the necessity follows as in the first paragraph. Conversely, assume all the stated inclusions. The finite assertion gives $M_n(\Omega_{p,d}) = 0$ for every $n, d \geq 0$. With the compatible choices in (1.2),

$$\Omega_{p,\infty} = \bigcup_{d \geq 0} \Omega_{p,d}.$$

Indeed, the coefficients of a polynomial over $\mathbf{F}_p(t_1, t_2, \ldots)$ involve only finitely many variables, so every element algebraic over that field is algebraic over $\mathbf{F}_p(t_1, \ldots, t_d)$ for some d. By

Lemma 9.1, $M_n(\Omega_{p,\infty}) = 0$ for every $n \geq 0$. The unbounded form of Corollary 1.3 proves the claim. □

Remark 9.12 (The remaining condition). The initial dimensions $0, 1$ are covered by Proposition 8.1; the first additional step in this induction is $1 \to 2$. For a cyclic extension L/K of prime degree ℓ, the identity

$$\mathrm{res}_{W(L)/W(K)} \mathrm{Tr}_{W(L)/W(K)} = 1 + \sigma + \cdots + \sigma^{\ell-1}$$

is multiplication by ℓ on the invariant subspace and zero on $(\sigma - 1)M_n(L)$. Hence averaging alone does not establish the required transfer injectivity. The required assertion is precisely that each difference $(\sigma - 1)a$, with $a \in K_n(L)_{\mathbb{Q}}$, is a localization boundary as in (9.14).

For a fixed $n \geq 1$, the vanishing step in Corollary 9.10 uses degrees $n - 1$ and n. The retraction in Corollary 9.9 only requires vanishing in degree $n - 1$. Its construction uses the chosen intermediate field $\Omega_{p,d}(t)^{\mathrm{perf}}$. The intermediate base fields K range over all finite extensions of $\Omega_{p,d}(t)^{\mathrm{perf}}$; they are essential in the proof of Proposition 9.8.

Appendix A. Existential embeddings and valuation rings

The argument in this appendix uses the existential embedding theorem of Anscombe–Dittmann–Jahnke. It gives another proof of the pure comparisons for DVRs in the main text and applies more generally to valuation rings which need not be noetherian. The conclusion about K-groups is a formal consequence of that embedding theorem and Theorem 3.3. We recall the logical definitions so that the hypotheses can be checked without prior model-theoretic background. A language specifies sorts, operations, constants, and relations; a structure interprets these symbols on a set for each sort. An embedding is injective on each sort, preserves the operations and constants, and preserves and reflects the relations. A term is a finite expression in operations, variables, and constants. An atomic formula is an equality between terms or a named relation applied to terms. A quantifier-free formula is a finite Boolean combination of atomic formulas. Parameters are elements of the structure used as additional constants. We use the ring language $\mathcal{L}_{\mathrm{ring}} = \{+, -, \cdot, 0, 1\}$ and the ordered-group language $\mathcal{L}_{\mathrm{oag}} = \{0, +, <\}$.

Definition A.1. An embedding $A \subseteq B$ of $\mathcal{L}$-structures is *existentially closed*, denoted $A \preceq_{\exists}^{\mathcal{L}} B$, if, for every finite tuple a of parameters in A, every integer $r \geq 0$, and every quantifier-free $\mathcal{L}$-formula $\varphi(X_1, \ldots, X_r, a)$,

$$(B \models \exists X_1 \cdots \exists X_r \, \varphi(X_1, \ldots, X_r, a)) \implies (A \models \exists X_1 \cdots \exists X_r \, \varphi(X_1, \ldots, X_r, a)).$$

Each variable and parameter has a specified sort. The notation $A \models \psi$ means that ψ is true in A. The superscript $\mathcal{L}$ is omitted when the language is specified.

For an inclusion of fields $k \subseteq \ell$, the condition $k \preceq_{\exists}^{\mathcal{L}_{\mathrm{ring}}} \ell$ is equivalent to algebraic purity. Indeed, after disjunctive normal form, a quantifier-free ring formula is a finite disjunction of systems

$$f_i(X) = 0 \quad (1 \leq i \leq s), \qquad g_j(X) \neq 0 \quad (1 \leq j \leq t),$$

where r, s, t are nonnegative integers and $f_i, g_j \in k[X_1, \ldots, X_r]$. A solution in ℓ satisfies one disjunct. In a field, the inequations in that disjunct are equivalent to the existence of Y satisfying $Y \prod_{j=1}^{t} g_j(X) = 1$. Thus algebraic purity gives a solution in k of that disjunct. The converse follows by applying existential closedness to a conjunction of polynomial equations.

For a valued field (E, v), its valuation ring, maximal ideal, and residue field are

$$\mathcal{O}_v = \{x \in E : v(x) \geq 0\}, \qquad \mathfrak{m}_v = \{x \in E : v(x) > 0\}, \qquad \kappa(v) = \mathcal{O}_v/\mathfrak{m}_v.$$

The language $\mathcal{L}_{\mathrm{val}}$ of [ADJ24, §2] has the three sorts

$$E, \qquad \kappa(v), \qquad v(E^{\times}) \cup \{\infty\}.$$

The first two have the ring language. The last has $0, +, <, \infty$, where $\gamma < \infty$ for $\gamma \in v(E^{\times})$ and $\gamma + \infty = \infty + \gamma = \infty$ for $\gamma \in v(E^{\times}) \cup \{\infty\}$. The connecting functions are the valuation

$v : E \to v(E^\times) \cup \{\infty\}$, with $v(0) = \infty$, and the map $\mathrm{red} : E \to \kappa(v)$ which is the quotient map on $\mathcal{O}_v$ and is zero on $E \setminus \mathcal{O}_v$. For $(E, v) \subseteq (F, w)$, one has

$$\mathcal{O}_w = \{x \in F : w(x) \geq 0\}, \qquad \mathcal{O}_v = E \cap \mathcal{O}_w \subseteq \mathcal{O}_w.$$

It induces the embeddings on the other two sorts. The notation $(E, v) \preceq_\exists (F, w)$ refers to $\mathcal{L}_{\mathrm{val}}$.

Corollary A.2. *If $(E, v) \preceq_\exists (F, w)$, then, for every spectrum M and every integer n, both maps*

$$\mathrm{res}_{F/E} : K_n(E; M) \longrightarrow K_n(F; M), \qquad \mathrm{res}_{\mathcal{O}_w/\mathcal{O}_v} : K_n(\mathcal{O}_v; M) \longrightarrow K_n(\mathcal{O}_w; M)$$

are pure monomorphisms.

Proof. Fix integers $r, s \in \mathbb{N}$ and polynomials $f_i \in E[X_1, \ldots, X_r]$, $1 \leq i \leq s$. If $b \in F^r$ satisfies $f_i(b) = 0$ for $1 \leq i \leq s$, then

$$(F, w) \models \exists X_1 \cdots \exists X_r \bigwedge_{i=1}^{s} f_i(X) = 0.$$

All coefficients are parameters in the valued-field sort E. Existential closedness gives $a \in E^r$ satisfying $f_i(a) = 0$ for $1 \leq i \leq s$. Hence $E \subseteq F$ is algebraically pure.

Now fix polynomials $f_i \in \mathcal{O}_v[X_1, \ldots, X_r]$, $1 \leq i \leq s$, and a solution $b \in \mathcal{O}_w^r$. Then $f_i(b) = 0$ for every $1 \leq i \leq s$ and $w(b_j) \geq 0$ for every $1 \leq j \leq r$. Consequently

$$(F, w) \models \exists X_1 \cdots \exists X_r \left(\bigwedge_{i=1}^{s} f_i(X) = 0 \wedge \bigwedge_{j=1}^{r} v(X_j) \geq 0 \right).$$

In this formula v denotes the valuation symbol, interpreted as w in (F, w). Its coefficients belong to $\mathcal{O}_v \subseteq E$. Existential closedness gives $a \in E^r$ with $f_i(a) = 0$ for $1 \leq i \leq s$ and $v(a_j) \geq 0$ for $1 \leq j \leq r$. Thus $a \in \mathcal{O}_v^r$, proving that $\mathcal{O}_v \subseteq \mathcal{O}_w$ is algebraically pure. Apply Theorem 3.3 to these two inclusions. □

A valued field (E, v) of mixed characteristic $(0, p)$ is *finitely ramified* if

$$0 < \#\{\gamma \in v(E^\times) : 0 < \gamma \leq v(p)\} < \infty.$$

This cardinality is its *initial ramification* [ADJ24, Definition 2.1]. For a normalized discrete valuation it equals $v(p)$.

Theorem A.3 (Anscombe–Dittmann–Jahnke)**.** *Let $(E, v) \subseteq (F, w)$ be finitely ramified henselian valued fields of mixed characteristic $(0, p)$ with the same initial ramification. Then*

$$\left.\begin{aligned} \kappa(v) &\preceq_\exists^{\mathcal{L}_{\mathrm{ring}}} \kappa(w), \\ v(E^\times) &\preceq_\exists^{\mathcal{L}_{\mathrm{oag}}} w(F^\times) \end{aligned}\right\} \quad \Longrightarrow \quad (E, v) \preceq_\exists (F, w).$$

This is [ADJ24, Theorem 5.14]. Adjoining ∞ to the value-group sort gives the same existential-closedness condition: each variable can be separated into the cases where it is finite and where it is ∞, reducing a formula to finitely many formulas in $\mathcal{L}_{\mathrm{oag}}$. We use the corrected arXiv proof. As the authors explain in [ADJ24, Remark A], the published proof of Lemma 4.2 incorrectly treated an arbitrary p-basis as a separating transcendence basis. The revised proof repairs that step; the statement and numbering of Theorem 5.14 are unchanged.

Corollary A.4 (Purity from residue fields and value groups)**.** *Under the hypotheses of Theorem A.3, assume*

$$\kappa(v) \preceq_\exists^{\mathcal{L}_{\mathrm{ring}}} \kappa(w), \qquad v(E^\times) \preceq_\exists^{\mathcal{L}_{\mathrm{oag}}} w(F^\times).$$

For every spectrum M and every $n \in \mathbb{Z}$, restriction induces pure monomorphisms

$$K_n(\mathcal{O}_v; M) \longrightarrow K_n(\mathcal{O}_w; M), \qquad K_n(E; M) \longrightarrow K_n(F; M).$$

Consequently injectivity of $K_n(\mathcal{O}_w; M) \to K_n(F; M)$ implies injectivity of $K_n(\mathcal{O}_v; M) \to K_n(E; M)$.

Proof. Theorem A.3 gives $(E, v) \preceq_\exists (F, w)$, and Corollary A.2 gives the two pure monomorphisms. For the last assertion, the commutative square

$$\begin{array}{ccc} K_n(\mathcal{O}_v; M) & \longrightarrow & K_n(E; M) \\ \downarrow & & \downarrow \\ K_n(\mathcal{O}_w; M) & \longrightarrow & K_n(F; M) \end{array}$$

sends the upper kernel to the lower kernel; injectivity of the left vertical map proves the implication. □

For mixed-characteristic DVRs, the completion map and the maps in Theorem 6.2(iii) preserve the uniformizer and have identical value groups $\mathbb{Z}$. Their residue embeddings are existentially closed, by the field case of Definition A.1. Corollary A.4 therefore recovers the purity assertions proved in the main text by regular approximation.

Example A.5 (A valuation of rank two). Let $k \subseteq \ell$ be perfect fields of characteristic p with $k \preceq_\exists^{\mathcal{L}_{\mathrm{ring}}} \ell$. For $h = k$ or $h = \ell$, consider the subring

$$W(h) + t \operatorname{Frac} W(h)[[t]] \subseteq \operatorname{Frac} W(h)((t)).$$

For a nonzero Laurent series $\sum_{i \geq m} a_i t^i$ with $a_m \neq 0$, define

$$v_h\left(\sum_{i \geq m} a_i t^i\right) = (m, v_{W(h)}(a_m)) \in \mathbb{Z} \times \mathbb{Z}$$

with the lexicographic order, and define $v_h(0) = \infty$. Products of leading coefficients and the valuation inequality for sums show that this is a valuation. Its value group is all of $\mathbb{Z} \times \mathbb{Z}$, since

$$v_h(t^a p^b) = (a, b) \qquad ((a, b) \in \mathbb{Z} \times \mathbb{Z}).$$

Its valuation ring is the displayed subring, and its maximal ideal is

$$\mathfrak{m}_{v_h} = pW(h) + t \operatorname{Frac} W(h)[[t]] = p\mathcal{O}_{v_h}.$$

Its residue field is therefore h. Since $v_h(p) = (0, 1)$, its initial ramification is one.

These valuation rings are henselian. Indeed, let $f(X)$ be monic over the displayed ring with a simple residue root in h. Taking constant terms gives a polynomial $f_0(X) \in W(h)[X]$. Henselianity of $W(h)$ gives $a \in W(h)$ with $f_0(a) = 0$ and $f_0'(a) \in W(h)^\times$, lifting that root. In $\operatorname{Frac} W(h)[[t]]$ one has

$$f(a) \in t \operatorname{Frac} W(h)[[t]], \qquad f'(a) \in \operatorname{Frac} W(h)[[t]]^\times.$$

Hensel's lemma in this complete DVR gives a root $b \in a + t \operatorname{Frac} W(h)[[t]]$. It belongs to the displayed valuation ring and has the required residue, proving henselianity.

The ideals generated by t/p^m, for $m \in \mathbb{N}$, form a strictly increasing chain: equality of two successive ideals would imply $1/p$ belongs to the valuation ring. Thus these rings are not noetherian. The coefficient embedding induces the identity on the value groups $\mathbb{Z} \times \mathbb{Z}$ and the prescribed embedding on residue fields. Corollary A.4 gives, for every $n \in \mathbb{Z}$ and every spectrum M, pure monomorphisms

$$K_n\big(W(k) + t \operatorname{Frac} W(k)[[t]]; M\big) \longrightarrow K_n\big(W(\ell) + t \operatorname{Frac} W(\ell)[[t]]; M\big),$$
$$K_n\big(\operatorname{Frac} W(k)((t)); M\big) \longrightarrow K_n\big(\operatorname{Frac} W(\ell)((t)); M\big).$$

The example illustrates Corollary A.4 for nonnoetherian valuation rings. Its conclusion also follows from regular approximation, as the next remark shows.

Remark A.6 (An alternative proof of Example A.5). The proof of Proposition 3.5 makes $W(k) \subseteq W(\ell)$ algebraically pure. It also reflects finite equations over $\operatorname{Frac} W(k)$ with specified coordinates required to lie in $W(k)$. To verify this, take $r, s \in \mathbb{N}$, polynomials $f_i \in \operatorname{Frac} W(k)[X_1, \dots, X_r]$ for $1 \le i \le s$, and $b \in \operatorname{Frac} W(\ell)^r$ with $f_i(b) = 0$ for $1 \le i \le s$. Take a subset $J \subseteq \{1, \dots, r\}$ such that $b_j \in W(\ell)$ for $j \in J$. Choose $N \in \mathbb{N}$ with $p^N b_j \in W(\ell)$ for $j \notin J$. Substitute $X_j = p^{-N} Y_j$ for $j \notin J$, leave $X_j = Y_j$ for $j \in J$, and clear powers of p in the finitely many coefficients. Algebraic purity solves the resulting equations in $W(k)$. Reversing the substitutions gives a solution over $\operatorname{Frac} W(k)$ whose J-coordinates belong to $W(k)$.

Now take $r, s \in \mathbb{N}$, polynomials

$$f_i \in \big(W(k) + t \operatorname{Frac} W(k)[[t]]\big)[X_1, \dots, X_r] \quad (1 \le i \le s),$$

and a solution $b \in \big(W(\ell) + t \operatorname{Frac} W(\ell)[[t]]\big)^r$. Smooth approximation for $\operatorname{Frac} W(k)[[t]] \subseteq \operatorname{Frac} W(\ell)[[t]]$, as in (3.1), factors the corresponding map

$$\operatorname{Frac} W(k)[[t]][X_1, \dots, X_r]/(f_1, \dots, f_s) \longrightarrow \operatorname{Frac} W(\ell)[[t]]$$

through a smooth $\operatorname{Frac} W(k)[[t]]$-algebra B. Denote the images of X_j in B by x_j. Choose $u, v \in \mathbb{N}$ and a finite presentation

$$B/tB = \operatorname{Frac} W(k)[Y_1, \dots, Y_u]/(g_1, \dots, g_v)$$

and representatives $q_j(Y)$ of $x_j \bmod tB$, for $1 \le j \le r$. The induced point with values in $\operatorname{Frac} W(\ell)$ solves

$$g_i(Y) = 0 \ (1 \le i \le v), \qquad Z_j = q_j(Y) \ (1 \le j \le r), \qquad Z_j \in W(\ell),$$

because the constant coefficient of each b_j belongs to $W(\ell)$. The preceding argument gives a $\operatorname{Frac} W(k)$-point of B/tB at which every $x_j \bmod tB$ belongs to $W(k)$. Smooth henselian lifting [Sta26, Tag 0H74] gives $B \to \operatorname{Frac} W(k)[[t]]$ lifting that point. The images of the x_j have constant coefficients in $W(k)$ and satisfy every $f_i = 0$; they lie in $W(k) + t \operatorname{Frac} W(k)[[t]]$. This proves algebraic purity of the valuation-ring embedding. For the fraction-field embedding, apply Proposition 3.5 to $\operatorname{Frac} W(k)[[t]] \subseteq \operatorname{Frac} W(\ell)[[t]]$; its residue embedding is algebraically pure by the first paragraph. Finally apply Theorem 3.3.

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