hexagon:2610.00618v1

FROM DÖBLIN TO ALEKSANDROV

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Abstract. We prove Aleksandrov maximum principles for linear second-order kinetic equations in non-divergence form with measurable, uniformly elliptic diffusion coefficients. For coefficients $A(t,v)$ we obtain every source exponent $p > 2d+1$. For autonomous coefficients $a(x,v)$ in one space dimension, we obtain every $p > p_*(\frac{\Lambda}{\lambda})$, where $p_*(\frac{\Lambda}{\lambda}) < 4$ depends only on the ellipticity ratio. The first step in both proofs is Döblin–Fourier cancellation: two positive parts of the kernel have nearly opposite phases, and a common lower bound for their velocity marginals allows us to cancel fixed portions of them. For the autonomous problem, this argument applies on velocity intervals away from zero. A preliminary Aleksandrov estimate gives Hölder continuity and then a Harnack comparison. A homogeneous Bellman barrier controls concentration in position and velocity; together with Harnack, it makes the local density estimates summable across zero velocity, including at $p = 4$. We give the full proofs and the passage from the maximum principle to local Hölder continuity.

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1. Introduction

Diffusion changes velocity; transport turns those changes into motion in position. This interaction is the source of regularity for the kinetic operator $\partial_t + v \cdot \nabla_x - A : D_v^2$. Kolmogorov computed its fundamental solution for $A = I_d$ [15]. For smooth coefficients, Hörmander's theorem explains why diffusion in velocity also smooths position: commutators with transport supply the missing derivatives [13, Theorem 1.1].

Date: 2 October 2026.

Here the coefficient is only measurable. We ask which estimates follow from ellipticity alone, without differentiating the coefficient.

For a uniformly elliptic operator $a^{ij}\partial_{ij}$ on a bounded domain $\Omega \subset \mathbb{R}^N$ the Aleksandrov–Bakelman–Pucci estimate [2, 3, 22] answers this question in the form

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u_+ + C\|f_+\|_{L^N(\Omega)}\,, \tag{1.1}$$

whenever $a^{ij}\partial_{ij}u \ge -f$; see [8, Chapter 9]. The constant depends only on the dimension, ellipticity and domain. Krylov's parabolic estimate uses L^{N+1} in space and time, with a factor $r^{\frac{N}{N+1}}$ on a cylinder of spatial radius r [16, Theorem 3]; see also [23, 7]. These bounds control the effect of a source concentrated on a small set. That control is a starting point for the Krylov–Safonov Harnack theory [17].

The classical proof does not carry over directly to the kinetic operator. It measures the source where the graph of u is touched from above by an affine function, using the determinant of the Hessian to estimate the volume of the resulting slopes. The kinetic equation controls $D_v^2 u$ but contains no second derivative in position, so this argument leaves the position directions uncontrolled. We instead use transport to turn diffusion in velocity into cancellation of spatial waves. This gives two estimates of the form (1.1), with the power of the radius fixed by kinetic scaling. In each coefficient class, uniform ellipticity and Borel measurability suffice: no continuity or derivative bound is assumed. The solution is continuous on the closed cylinder, of class $C^{1,1,2}$ inside it, with no integrability assumption on its derivatives at the boundary.

1.1. **Main results.** Let $d \ge 1$, $P_0 = (t_0, x_0, v_0) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$ and $R > 0$. The kinetic cylinder is

$$Q_R^-(P_0) = \left\{(t,x,v):\ t_0 - R^2 < t < t_0,\quad |v-v_0| < R,\quad |x - x_0 - (t-t_0)v_0| < R^3\right\}. \tag{1.2}$$

We write $V = v - v_0$ and $Y = x - x_0 - (t-t_0)v_0$ for the relative velocity and the relative position. The *kinetic boundary* of $Q_R^-(P_0)$ is

$$\partial_k Q_R^-(P_0) = \overline{Q_R^-(P_0)} \cap \Big(\{t = t_0 - R^2\} \cup \{|V| = R\} \cup \{|Y| = R^3,\ V\cdot Y \le 0\}\Big). \tag{1.3}$$

It consists of the bottom face, the velocity faces, and the part of the position faces through which the relative transport $\partial_t + V\cdot\nabla_Y$ enters the cylinder. It does not contain the relative interior of the top face $t = t_0$, nor the relative interior of the part of the position faces where $V\cdot Y > 0$; on those two sets the estimates below prescribe nothing. Figure 1 shows these faces, and their reflection used in Section 6.

Let $A : \mathbb{R}\times\mathbb{R}^d \to \mathrm{Sym}_d$ be Borel measurable and let $0 < \lambda \le \Lambda < \infty$ with

$$\lambda I_d \le A(t,v) \le \Lambda I_d \quad \text{for almost every } (t,v) \in \mathbb{R}\times\mathbb{R}^d. \tag{1.4}$$

For $u \in C^{1,1,2}$ (see the notation fixed at the end of this introduction) set

$$\mathcal{L}_A u = \partial_t u + v\cdot\nabla_x u - A(t,v) : D_v^2 u\,. \tag{1.5}$$

Theorem 1.1 (Coefficients depending on time and velocity)**.** *Let A satisfy (1.4), let $\mathcal{L}_A$ be the operator (1.5), let $2d+1 < p < \infty$, and let f be a real-valued Borel function on $Q_R^-(P_0)$. Suppose that*

$$u \in C\big(\overline{Q_R^-(P_0)}\big) \cap C^{1,1,2}\big(Q_R^-(P_0)\big)\,,$$

and

$$\mathcal{L}_A u \le f \quad \textit{almost everywhere in } Q_R^-(P_0), \qquad f_+ \in L^p\big(Q_R^-(P_0)\big)\,.$$

Then

$$\sup_{\overline{Q_R^-(P_0)}} u \le \sup_{\partial_k Q_R^-(P_0)} u_+ + C_{d,\lambda,\Lambda,p}\, R^{2-\frac{4d+2}{p}} \|f_+\|_{L^p(Q_R^-(P_0))}\,. \tag{1.6}$$

Let now $d = 1$, let $a : \mathbb{R}\times\mathbb{R} \to [\lambda,\Lambda]$ be Borel measurable, and set

$$\mathcal{L}_a u = \partial_t u + v\partial_x u - a(x,v)\partial_{vv} u\,. \tag{1.7}$$

Theorem 1.2 (Autonomous coefficients in one space dimension)**.** *Let $p_* = p_*(\frac{\Lambda}{\lambda}) := 1 + \beta_*(\frac{\Lambda}{\lambda}) \in [3, 4)$, where β_* is the homogeneous adjoint exponent of Appendix B. For $d = 1$, $p_* < p < \infty$ and a real-valued Borel function f on $Q_R^-(P_0)$, we suppose that*

$$u \in C\big(\overline{Q_R^-(P_0)}\big) \cap C^{1,1,2}\big(Q_R^-(P_0)\big)\,,$$

and

$$\mathcal{L}_a u \le f \quad \text{almost everywhere in } Q_R^-(P_0), \qquad f_+ \in L^p\big(Q_R^-(P_0)\big)\,.$$

Then

$$\sup_{\overline{Q_R^-(P_0)}} u \le \sup_{\partial_k Q_R^-(P_0)} u_+ + C_{\lambda,\Lambda,p}\, R^{2-\frac{6}{p}} \|f_+\|_{L^p(Q_R^-(P_0))}\,. \tag{1.8}$$

For the later Hölder argument, we need to count the source only where the sub-solution is positive. Indeed, we will compare a solution with a smooth barrier and use the smallness of the set where the barrier lies above it. A source norm over the whole cylinder would lose that information. The proofs give the following form, used in Section 9.

Corollary 1.3 (Localisation of the source)**.** *In Theorems 1.1 and 1.2 the norm $\|f_+\|_{L^p(Q_R^-(P_0))}$ may be replaced by $\|f_+ \mathbf{1}_{\{u>0\}}\|_{L^p(Q_R^-(P_0))}$.*

The powers of R follow from kinetic scaling. The map $(t, x, v) \mapsto (R^2 t, R^3 x, Rv)$ carries $Q_1^-(0)$ onto $Q_R^-(0)$ and multiplies Lebesgue measure by R^{4d+2}, the homogeneous dimension being $2 + 3d + d$; a source rescaled from Q_R^- to Q_1^- acquires a factor R^2 and its L^p norm a factor $R^{-\frac{4d+2}{p}}$. The resulting power of R vanishes at $p = 2d+1$, where the estimate would be scale invariant; Theorem 1.1 covers every larger exponent. The strong L^{2d+1} endpoint fails even for constant coefficients, by the nested-bump construction in [20, Section 8.5]. For $d = 1$ the scaling threshold is $p = 3$. Theorem 1.2 reaches every $p > p_*(\frac{\Lambda}{\lambda})$ with $p_* < 4$, and in particular includes $p = 4$. The improvement below four depends on the ellipticity ratio. The threshold p_* is the one supplied by the Bellman argument of Section 8.5; neither its optimality nor an estimate at $p = p_*$ is claimed.

Appendix C proves a stronger obstruction for autonomous coefficients. For every $p < 4d$, a suitable fixed ellipticity ratio admits smooth examples whose height stays positive while their source norm tends to zero. The construction starts with a positive stationary profile H of degree $0 < \alpha < 1$, singular only at the origin. Flattening $1 - H$ there keeps its height and confines its source to a small kinetic annulus. A decaying velocity barrier removes the initial data without increasing the positive source. The source norm is bounded by $Cr^{\alpha-2+\frac{4d}{p}}$, which tends to zero when α is sufficiently close to one.

The quantifiers matter: the coefficient bounds can depend on p, but stay fixed along each sequence. In dimension one, this rules out any threshold below four valid for all ellipticity ratios. It is consistent with Theorem 1.2, whose improvement is for each fixed ratio and may shrink as the ratio grows. In higher dimensions the construction concerns $A(x, v)$, not the restricted class $A(t, v)$ of Theorem 1.1.

For smooth $A(t, v)$, the argument also gives the Lorentz-space endpoint with source in $L^{2d+1,1}$; Remark 5.4 records the short additional argument.

1.2. **Strategy.** Both proofs start from the same Green-density estimate. We reverse time to use terminal-value kernels and reflect position to preserve the sign of transport. This changes neither the estimates nor their exponents; we return to the original variables at the end. Each operator then becomes an instance of

$$\mathcal{K} = \partial_\sigma + B(\sigma, v) : D_v^2 + b(v) \cdot \nabla_z \qquad \text{on } \mathbb{R}_\sigma \times D_v \times \mathbb{R}_z^d\,, \tag{1.9}$$

Diffusion acts in v, transport acts in z, and the coefficients are independent of z. For $A = A(t, v)$, we take z to be the reflected position, $b(v) = v$ and $D = \mathbb{R}^d$. For $a = a(x, v)$ in one dimension, we work on a velocity interval separated from zero. Position then moves in one direction and can serve as time.

After normalisation, D is a fixed bounded interval. The old time becomes the transported variable; autonomy is what makes the new diffusion coefficient independent of it.

Fourier transformation in z separates the waves. The transformed kernel $\widehat{K}^{\xi}$ is a complex measure in terminal velocity. Its variation is bounded by the positive velocity marginal P, obtained at frequency zero. With v denoting the starting velocity, our key estimate is

$$\|\widehat{K}^{\xi}_{\sigma,\tau}(v,\cdot)\|_{\mathrm{TV}} \le C\exp\bigl(-c\,(\tau-\sigma)|\xi|^{\frac{2}{3}}\bigr)\,, \tag{1.10}$$

To obtain this decay, we choose two curves in velocity space whose accumulated displacements in z give phases differing by π. The curves end at the same centre, and a barrier retains a fixed amount of the kernel in a small tube around each curve. The Lipschitz bound on b keeps the phases close to those of the curves. The resulting terminal velocity distributions need not agree. We continue both in one fixed ball, where the parabolic Harnack inequality [17, 7] gives a common minorant: a positive measure bounded above by both velocity marginals. The portions supplied by this lower bound cancel up to the phase error.

This removes a fixed fraction of total variation in a fixed time at frequency one. At frequency ξ, kinetic scaling gives the same contraction over a time proportional to $|\xi|^{-\frac{2}{3}}$: a velocity change of size $r = |\xi|^{-\frac{1}{3}}$ produces a displacement of size r^3 over time r^2. Composition repeats the contraction and gives (1.10). The comparison of two measures with a common positive part is the feature shared with Döblin's argument [5]; here the loss comes from their opposite Fourier phases (Lemma 3.7).

The positive marginal in (σ, v) is controlled by the parabolic Aleksandrov estimate [7, Theorem 2.4]: its occupation density belongs to L^{γ} for $1 \le \gamma \le \frac{d+1}{d}$. Fourier inversion combines this bound with (1.10) to give a full Green density G for unit initial mass. With τ denoting elapsed time, it satisfies

$$\int_T^{2T}\int_{D\times\mathbb{R}^d} G(\tau,w,z)^q \,\mathrm{d}w\,\mathrm{d}z\,\mathrm{d}\tau \le CT^{1-2d(q-1)}.$$

Integrating this inequality against $\frac{\mathrm{d}T}{T}$ is finite near zero precisely when $q < 1 + \frac{1}{2d}$. This gives the finite-horizon Green estimate in the whole space. On a bounded velocity interval, exponential loss at the endpoints also makes the integral finite at infinity; for $d = 1$ the same small-time condition is $q < \frac{3}{2}$.

Integrating the source against this Green density gives a non-negative potential. Subtracting it from the reflected sub-solution leaves a function bounded by its exit-boundary values. Hölder's inequality then gives the Aleksandrov estimate with $p = \frac{q}{q-1}$. For $A(t,v)$, the available time is at most R^2, and the density bound gives exactly the factor $R^{2-\frac{4d+2}{p}}$. We first prove this for smooth coefficients. For measurable coefficients, we keep the classical sub-solution fixed and approximate the coefficient on compact inner cylinders, where its velocity Hessian is bounded. Exhaustion then gives the estimate up to the kinetic boundary.

For $a = a(x,v)$, the difficulty is to combine the estimates on velocity intervals as they approach zero. A first count of repeated visits already gives the Aleksandrov estimate at $p = 6$. This implies Hölder continuity, from which propagation along kinetic paths gives a Harnack comparison for the full-space autonomous evolution. We then use this regularity to improve the estimate from which it came.

The improvement concerns the timing of returns. A convex function of velocity bounds the time-integrated mass in a small velocity interval. Harnack prevents that mass from concentrating in a short time interval: a value at elapsed time t is bounded by an average over $[2t, 3t]$. Together these give the bound $C\frac{r}{\sqrt{t}}$ for mass in an interval of size r, when the starting velocity has size r and $t \ge Cr^2$. Consequently, the mass beginning new visits during successive intervals of length r^2 is bounded by $C(1+j)^{-\frac{1}{2}}$, with time measured from the first arrival in the inner interval. Summing its q-th powers gives the density bound $CR^{\frac{2}{q}-1}r^{\frac{4}{q}-3}$. Its integral against $\frac{\mathrm{d}r}{r}$ is finite for $q < \frac{4}{3}$, giving $p > 4$. This recovers the range proved in the short paper.

To reach below four, we also record the position at which each visit begins. A homogeneous barrier bounds occupation of boxes of width r^3 in position and r in velocity. Harnack converts this integrated bound into faster decay for the mass in each box. In the density sum, this contributes an additional power $r^{\varepsilon(q-1)}$, making the sum converge even at $q = \frac{4}{3}$ and slightly beyond. The preliminary estimate therefore improves through its own regularity consequences, together with the new barrier; the final theorem is used only at the end.

For general $A(t, x, v)$, neither choice of time leaves a transported variable absent from the coefficient. The Fourier waves then interact, and the argument above does not apply. In the autonomous case, changing the clock removes this difficulty only away from zero velocity; the estimates on returns are what let us cross that remaining region.

1.3. **Literature.** For kinetic equations in divergence form, Golse, Imbert, Mouhot and Vasseur [9] proved Hölder continuity and a Harnack inequality with measurable uniformly elliptic diffusion and bounded drift and source. Their cylinders are those of (1.2). Guerand and Mouhot [11] gave short quantitative De Giorgi proofs of the weak Harnack and Harnack inequalities, and Guerand and Imbert [10] gave a constructive proof using a logarithmic transform. In [9] and [10], the source is controlled by its supremum.

In non-divergence form, Abedin and Tralli [1] prove invariant Harnack inequalities under a Cordes–Landis condition on the eigenvalues of the coefficient or under a uniform modulus of continuity. Dong and Yastrzhembskiy [6] prove global L^p estimates and solvability when the coefficient is measurable in time but of vanishing mean oscillation in position and velocity. An Aleksandrov maximum principle for general measurable $A(t, x, v)$ is announced in the forthcoming work [21]. Hao and Zhang [12] prove such an estimate through a probabilistic occupation bound and derive Harnack and Hölder estimates for strong solutions. The present argument instead uses Fourier cancellation in the two restricted coefficient classes above.

Zhu [24] proves interior Hölder estimates for classical stationary solutions in one space dimension with a merely measurable coefficient $a(x, v)$. The autonomous coefficient in Theorem 1.2 is the more general, since our solutions may depend on time, and we obtain a sharper threshold on the Lebesgue exponent.

The short paper [20] presents the cancellation argument, its Aleksandrov estimates, and a mixing theorem for diffusion on the sphere. This companion gives the detailed proofs of the Aleksandrov estimates and the passage to Hölder continuity. It also improves the one-dimensional autonomous range from $p > 4$ to $p > p_*(\frac{\Lambda}{\lambda})$ with $p_* < 4$. The spherical mixing theorem is proved in the short paper. We use its kernel notation and denote the space dimension by d here; the relation between the two conventions for solution operators is explained below.

1.4. **Outline.** Sections 2–5 concern the operator (1.9) alone: Section 2 fixes the setting and the kernels, defines the Green measure and states the density theorem; Section 3 proves the decay (1.10); Section 4 the occupation estimate at frequency zero; Section 5 combines them. Section 6 reflects the variables and proves what the two applications share; Sections 7 and 8 prove Theorems 1.1 and 1.2. Section 7 also obtains local spatial smoothing and a direct Hölder estimate for $A(t, v)$. Section 9 shows that the Aleksandrov principle, in the localised form of Corollary 1.3, implies local Hölder continuity in the kinetic quasi-distance: Theorem 9.1 states this for an arbitrary uniformly elliptic Borel coefficient $A(t, x, v)$, Corollary 9.10 uses this implication for the autonomous class, while Corollary 9.9 follows from the direct local argument. Appendix A constructs the solution operators, the Duhamel potentials and the Green and exit measures of a velocity strip. Appendix B proves the homogeneous adjoint degree gap and constructs the Bellman barrier used for the improvement below four. Appendix C constructs the autonomous counterexamples.

1.5. **Notation.** Smooth means C^∞; Sym_d denotes the real symmetric $d \times d$ matrices, acting by $A : D_v^2 u = \sum_{i,j} A_{ij} \partial_{v_i v_j} u$. For an open set $O \subseteq \mathbb{R}^{1+2d}$ we write $C^{1,1,2}(O)$ for the set of functions on O whose derivatives $\partial_t u$, $\nabla_x u$, $\nabla_v u$ and $D_v^2 u$ exist and are continuous in O; no condition is imposed

at the boundary of O. We write $g_+ = \max\{g, 0\}$, and every L^p norm is taken in all the displayed variables with respect to Lebesgue measure. Constants C and c depend only on the parameters listed as subscripts or named in the statement in which they occur.

Acknowledgements. The first author acknowledges support of the Fondation Sciences Mathématiques de Paris.

AI disclosure. The mathematics is a product of more than a year of work, discussions, and eventually assistance of AI. This manuscript here is written by Codex. It accompanies the shorter human-written file [20].

2. A parabolic equation with a transported variable

Both coefficient classes reduce to an equation with diffusion in v and transport in z, whose coefficients do not depend on z. This independence lets each spatial frequency evolve separately. We introduce its positive evolution kernel first, then its Fourier transform and its time integral, the Green measure. The density estimate for the last of these will give the Aleksandrov principle.

2.1. **Setting.** We fix $d \geq 1$. Throughout Sections 2–5 we consider a domain $D \subseteq \mathbb{R}^d$ and a smooth map $b : \mathbb{R}^d \to \mathbb{R}^d$, in exactly one of the following two situations.

(W) $$D = \mathbb{R}^d, \qquad b(v) = v, \qquad m = L_b = 1\,;$$

(I) $$d = 1, \qquad D = J \text{ a bounded open interval}, \qquad m \leq b' \leq L_b \text{ on } \mathbb{R}, \qquad 0 < m \leq L_b < \infty\,.$$

In both cases, for all $v, w \in \mathbb{R}^d$ and every unit vector $\hat{\xi} \in \mathbb{R}^d$,

$$|b(v) - b(w)| \leq L_b|v - w|, \qquad \hat{\xi} \cdot Db(v)\hat{\xi} \geq m\,. \tag{2.1}$$

Let B be a smooth map from $\mathbb{R} \times \mathbb{R}^d$ to Sym_d with

$$\lambda I_d \leq B(\sigma, v) \leq \Lambda I_d\,, \qquad 0 < \lambda \leq \Lambda < \infty\,. \tag{2.2}$$

In case (I) only the values of B on $\mathbb{R} \times J$ enter any statement below; B is defined on all of $\mathbb{R} \times \mathbb{R}^d$ for convenience. The operator is

$$\mathcal{K} = \partial_\sigma + B(\sigma, v) : D_v^2 + b(v) \cdot \nabla_z \qquad \text{on } \mathbb{R}_\sigma \times D_v \times \mathbb{R}^d_z\,, \tag{2.3}$$

with zero boundary values imposed at $\partial D \times \mathbb{R}^d$ in case (I). We call $d, \lambda, \Lambda, m, L_b$, and in case (I) also $|J|$, the *structural constants*. Every constant below depends only on the structural constants and on the exponents named in the statement; none depends on derivatives of B or of b, on σ, or on the position of J.

2.2. **The terminal solution operator.** For $\sigma < \tau$, we denote by $\mathsf{S}_{\sigma,\tau}F$ the solution at time σ with terminal value F at time τ and zero values at the velocity boundary. The boundary may move: we need this to retain mass in a tube around a prescribed velocity curve. The construction also allows $B = B(\sigma, v, z)$, as required for the original autonomous operator. Independence of z is needed only for the Fourier estimates. Appendix A proves the following properties.

Proposition 2.1 (Terminal solution operator)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be $\mathbb{R}^d$, an open ball, or, when $d = 1$, a bounded open interval; let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be continuous and piecewise C^1 with $\Omega_\sigma := \gamma(\sigma) + \Omega$ (the constant curve $\gamma \equiv 0$ is allowed). Let $B : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathrm{Sym}_d$ be smooth with $\lambda I_d \leq B(\sigma, v, z) \leq \Lambda I_d$, and let $b : \mathbb{R}^d \to \mathbb{R}^d$ be smooth and satisfy (2.1). We consider*

$$\mathcal{K} = \partial_\sigma + B(\sigma, v, z) : D_v^2 + b(v) \cdot \nabla_z$$

on $\{(\sigma, v, z) : v \in \Omega_\sigma\}$ with zero boundary values on $\{v \in \partial\Omega_\sigma\}$. For $\sigma \leq \tau$ there is a linear operator $\mathsf{S}_{\sigma,\tau}$, acting on bounded Borel functions F on $\Omega_\tau \times \mathbb{R}^d$, with the following properties.

(i) *For $F \in C_c^\infty(\Omega_\tau \times \mathbb{R}^d)$, the function $u(\sigma, v, z) = \mathsf{S}_{\sigma,\tau}F(v, z)$ is the unique bounded solution of $\mathcal{K}u = 0$ for $\sigma < \tau$, smooth inside the moving domain and continuous up to its terminal and lateral boundary, with terminal datum F and zero lateral datum.*

(ii) *$\mathsf{S}_{\sigma,\tau}$ is positive, $\mathsf{S}_{\sigma,\tau}\mathbf{1} \le \mathbf{1}$, and $\mathsf{S}_{\sigma,\tau} = \mathsf{S}_{\sigma,r}\mathsf{S}_{r,\tau}$ for $\sigma \le r \le \tau$.*

(iii) *The operators have positive kernels $K_{\sigma,\tau}((v, z), \cdot)$ on $\Omega_\tau \times \mathbb{R}^d$, of mass at most one and jointly Borel in (σ, τ, v, z):*

$$\mathsf{S}_{\sigma,\tau}F(v, z) = \int F \,\mathrm{d}K_{\sigma,\tau}((v, z), \cdot).$$

(iv) *If B does not depend on z, then $K_{\sigma,\tau}((v, z + h), E + (0, h)) = K_{\sigma,\tau}((v, z), E)$ for all $h \in \mathbb{R}^d$.*

(v) *If Ω' and γ' are another admissible domain and curve with $\Omega'_\sigma = \gamma'(\sigma) + \Omega' \subseteq \Omega_\sigma$ for all σ, then the kernels of the two problems satisfy $K'_{\sigma,\tau}((v, z), \cdot) \le K_{\sigma,\tau}((v, z), \cdot)$ for $v \in \Omega'_\sigma$, where the left side is extended by zero.*

(vi) *If B does not depend on z, the velocity marginal $P_{\sigma,\tau}(v, E) \coloneqq K_{\sigma,\tau}((v, z), E \times \mathbb{R}^d)$ is independent of z, and $\phi \mapsto \int \phi(w) P_{\sigma,\tau}(v, \mathrm{d}w)$ is the terminal solution operator, in the sense of (i)–(iii), of the parabolic equation $\partial_\sigma V + B(\sigma, v) : D_v^2 V = 0$ on $\{v \in \Omega_\sigma\}$ with zero boundary values.*

We use v for the starting velocity and w for the terminal velocity; v' denotes an intermediate velocity when we compose two kernels. Time indices are chronological. The kernel acts on terminal functions through S. Its dual action on initial measures is denoted by U:

$$\int F \,\mathrm{d}(\mathsf{U}_{\sigma,\tau}\mu) = \int \mathsf{S}_{\sigma,\tau}F \,\mathrm{d}\mu.$$

These are two actions of the same kernel. For fixed starting variables (v, z) at time σ, the measure $K_{\sigma,\tau}((v, z), \cdot)$ is $\mathsf{U}_{\sigma,\tau}\delta_{(v,z)}$. It solves the adjoint equation in the terminal variables, whereas $\mathsf{S}F$ solves the terminal-value equation.

In the spherical section of the short paper, $S_{s,t}$ denotes the evolution of densities relative to reference probabilities π_s, π_t satisfying $\mathsf{U}_{s,t}\pi_s = \pi_t$. Thus $\mathsf{U}_{s,t}(f\pi_s) = (S_{s,t}f)\pi_t$, and the terminal action denoted here by $\mathsf{S}_{s,t}$ is its Hilbert-space adjoint, from $L^2(\pi_t)$ to $L^2(\pi_s)$. In the autonomous spherical case the reference probability is invariant. This change of representation introduces no additional kernel.

When B is independent of z, we may use displacement $x = z' - z$ in place of terminal position. The image of $K_{s,t}((v, z), \mathrm{d}w\,\mathrm{d}z')$ under this change is denoted by $K_{s,t}(v, \mathrm{d}x, \mathrm{d}w)$. Whenever the density exists, we write

$$K_{s,t}(v, \mathrm{d}x, \mathrm{d}w) = K_{s,t}(x, v, w)\,\mathrm{d}x\,\mathrm{d}w, \qquad P_{s,t}(v, \mathrm{d}w) = P_{s,t}(v, w)\,\mathrm{d}w.$$

The density notation places displacement first, followed by starting and terminal velocity, as in the short paper. If only a density in position is available, $K_{s,t}(x, v, \mathrm{d}w)$ remains a measure in terminal velocity. For coefficients depending on position, the starting position remains an argument of K. We write K^D, P^D for a fixed velocity domain and $K^{\gamma,\rho}, P^{\gamma,\rho}$ for the moving ball $B_\rho(\gamma(\sigma))$, omitting superscripts only when the domain is fixed throughout the argument. The same convention will apply to the boundary measure ω^D, which represents terminal and velocity-boundary data.

2.3. **Fourier kernels.** Since the coefficients are independent of z, we Fourier transform the displacement to obtain complex measures on D. For $\xi \in \mathbb{R}^d$, $v \in D$ and Borel $E \subseteq D$, we put

$$\widehat{K}^{\xi}_{\sigma,\tau}(v, E) = \int_{E \times \mathbb{R}^d} e^{-i\xi\cdot(z'-z)}\, K_{\sigma,\tau}\big((v, z), \mathrm{d}w\,\mathrm{d}z'\big)\,. \tag{2.4}$$

At $\xi = 0$ this is the zero mode $P_{\sigma,\tau}(v, \cdot)$ of Proposition 2.1(vi), the kernel of the parabolic equation in v alone. The lemma records the total-variation domination and temporal composition needed below.

Lemma 2.2 (Fourier kernels)**.** *Under (W) or (I), $\widehat{K}^{\xi}_{\sigma,\tau}(v,\cdot)$ is a complex Borel measure on D which does not depend on z, the map $v \mapsto \widehat{K}^{\xi}_{\sigma,\tau}(v,E)$ is Borel for every Borel $E \subseteq D$, and:*

(i) *For every Borel $E \subseteq D$,*

$$|\widehat{K}^{\xi}_{\sigma,\tau}|(v,E) \le P_{\sigma,\tau}(v,E), \qquad \|\widehat{K}^{\xi}_{\sigma,\tau}(v,\cdot)\|_{\mathrm{TV}} \le P_{\sigma,\tau}(v,D) \le 1.$$

(ii) $\widehat{K}^{\xi}_{\sigma,\tau}(v,E) = \int_D \widehat{K}^{\xi}_{r,\tau}(v',E)\,\widehat{K}^{\xi}_{\sigma,r}(v,\mathrm{d}v')$ *for $\sigma \le r \le \tau$; consequently*

$$\|\widehat{K}^{\xi}_{\sigma,\tau}(v,\cdot)\|_{\mathrm{TV}} \le \|\widehat{K}^{\xi}_{\sigma,r}(v,\cdot)\|_{\mathrm{TV}} \sup_{v'\in D} \|\widehat{K}^{\xi}_{r,\tau}(v',\cdot)\|_{\mathrm{TV}}\,. \tag{2.5}$$

Proof. Translation invariance of K makes (2.4) independent of z, and integration of bounded Borel functions against a Borel kernel gives a Borel kernel. Since the phase has modulus one, its Fourier projection is dominated in total variation by the positive velocity marginal, proving (i). For (ii), we apply the composition law for K and Fubini's theorem, using

$$e^{-i\xi\cdot(z''-z)} = e^{-i\xi\cdot(z'-z)}e^{-i\xi\cdot(z''-z')}.$$

The inner integral is $\widehat{K}^{\xi}_{r,\tau}(v',E)$ and is independent of z'. The outer integral is therefore its integral against $\widehat{K}^{\xi}_{\sigma,r}(v,\cdot)$. Taking total variations gives (2.5). □

2.4. **Green measures.** The evolution kernel records mass at one terminal time. Its integral in time is the Green measure: integrating a source against it gives the solution with zero terminal and velocity-boundary data. We denote the time integral by Γ and its density, when established, by G. Thus P, $\widehat{K}$ and Γ are obtained from K by integration in position, Fourier transformation in position, and integration in time, respectively.

Let μ be a finite positive Borel measure on $D\times\mathbb{R}^d$, prescribed at time σ_0, with mass $M = \mu(D\times\mathbb{R}^d)$. For $0 < S \le \infty$ define the positive measure Γ^S_μ on $(0,S)\times D\times\mathbb{R}^d$ by

$$\int F\,\mathrm{d}\Gamma^S_\mu = \int_{D\times\mathbb{R}^d} \mu(\,\mathrm{d}v\,\mathrm{d}z) \int_0^S \int_{D\times\mathbb{R}^d} F(\tau,w,z')\,K_{\sigma_0,\sigma_0+\tau}\big((v,z),\,\mathrm{d}w\,\mathrm{d}z'\big)\,\mathrm{d}\tau \tag{2.6}$$

for non-negative Borel F, where τ is elapsed time. Kernel contraction gives mass at most MS for finite S, hence local finiteness when $S=\infty$. The next lemma identifies its action on sources; its proof is in Appendix A.

Lemma 2.3 (Duhamel potentials)**.** *In the setting of Proposition 2.1, let $\tau_+ \in \mathbb{R}$, let $\mathcal{U} = \{(\sigma,v,z) : \sigma < \tau_+,\ v\in\Omega_\sigma\}$, and let $0 \le g \in C^\infty_c(\mathcal{U})$. Then*

$$W_g(\sigma,v,z) = \int_\sigma^{\tau_+} \mathsf{S}_{\sigma,r}\big(g(r,\cdot,\cdot)\big)(v,z)\,\mathrm{d}r, \qquad (\sigma,v,z)\in\mathcal{U}, \tag{2.7}$$

is smooth in $\mathcal{U}$, satisfies $0 \le W_g \le (\tau_+-\sigma)\sup g$ and $\mathcal{K}W_g = -g$ in $\mathcal{U}$, and extends continuously by zero to $\{\sigma=\tau_+\}$ and to $\{v\in\partial\Omega_\sigma\}$. It is the integral of g against the Green measure (2.6) from (σ,v,z), written in absolute time. The same assertions hold for the parabolic operator $\partial_\sigma + B(\sigma,v) : D_v^2$, with z omitted and S replaced by its marginal kernel P.

Theorem 2.4 (Green density)**.** *Let μ be a finite positive Borel measure on $D\times\mathbb{R}^d$ of mass M, prescribed at time σ_0.*

(i) *In case (W), for every $S\in(0,\infty)$ and every $1 < q < 1+\frac{1}{2d}$, the measure Γ^S_μ has a density G^S_μ with respect to Lebesgue measure on $(0,S)\times\mathbb{R}^d\times\mathbb{R}^d$, and*

$$\|G^S_\mu\|_{L^q((0,S)\times\mathbb{R}^d\times\mathbb{R}^d)} \le C_{d,\lambda,\Lambda,q}\,M\,S^{\frac{1-2d(q-1)}{q}}\,. \tag{2.8}$$

(ii) *In case* (I)*, for every* $1 < q < \frac{3}{2}$*, the measure* Γ_μ^∞ *is finite, has a density* G_μ^∞ *with respect to Lebesgue measure on* $(0,\infty)\times J\times\mathbb{R}$*, and*

$$\|G_\mu^\infty\|_{L^q((0,\infty)\times J\times\mathbb{R})} \le C_{\lambda,\Lambda,m,L_b,|J|,q}\, M\,. \tag{2.9}$$

Theorem 2.4 is proved in Section 5; its two alternatives are the geometries used below, and its kinetic scaling is recorded in Lemma 2.5.

2.5. **Kinetic scaling.**

Lemma 2.5 (Kinetic scaling)**.** *We fix* $\sigma_*\in\mathbb{R}$*,* $v_*\in D$*,* $z_*\in\mathbb{R}^d$ *and* $r>0$*, and put*

$$\sigma=\sigma_*+r^2\tau,\qquad v=v_*+rY,\qquad z=z_*+(\sigma-\sigma_*)b(v_*)+r^3Z\,. \tag{2.10}$$

Let

$$D_r=r^{-1}(D-v_*),\quad B_r(\tau,Y)=B(\sigma_*+r^2\tau,v_*+rY),\quad b_r(Y)=r^{-1}\big(b(v_*+rY)-b(v_*)\big).$$

Then (D_r,b_r) *is again of type* (W)*, respectively* (I)*, with the same constants* m, L_b *(and* $|J_r|=r^{-1}|J|$*),* B_r *satisfies* (2.2)*, and if* $u(\sigma,v,z)=U(\tau,Y,Z)$ *then*

$$\mathcal{K}u=r^{-2}\big(\partial_\tau+B_r:D_Y^2+b_r\cdot\nabla_Z\big)U\,. \tag{2.11}$$

Moreover, if $\widehat{K}^\xi$ *and* $\widehat{K}^\xi_{(r)}$ *denote the Fourier kernels of the two operators, then for all* $\sigma\le\sigma'$*,* $v\in D$ *and* $\xi\in\mathbb{R}^d$*,*

$$\|\widehat{K}^\xi_{\sigma,\sigma'}(v,\cdot)\|_{\mathrm{TV}}=\|\widehat{K}^{r^3\xi}_{(r),\tau,\tau'}(Y,\cdot)\|_{\mathrm{TV}}\,,\qquad \sigma=\sigma_*+r^2\tau,\ \sigma'=\sigma_*+r^2\tau',\ v=v_*+rY\,. \tag{2.12}$$

Proof. The chain rule gives (2.11); the shift in z subtracts the constant transport $b(v_*)$. Also $Db_r(Y)=Db(v_*+rY)$, so the structural bounds are unchanged. Uniqueness of the terminal problem identifies the original kernel with the affine image of the rescaled kernel. Since

$$z'-z=(\sigma'-\sigma)b(v_*)+r^3(Z'-Z),$$

their Fourier projections satisfy

$$\widehat{K}^\xi_{\sigma,\sigma'}(v,E)=e^{-i(\sigma'-\sigma)\xi\cdot b(v_*)}\,\widehat{K}^{r^3\xi}_{(r),\tau,\tau'}\big(Y,r^{-1}(E-v_*)\big).$$

The affine pushforward and the constant phase preserve total variation, proving (2.12). □

3. Decay of the non-zero Fourier modes

We choose two curves in velocity space that carry a spatial wave into opposite phases. A fixed amount of the kernel stays near each curve. After a short further evolution, the two terminal velocity distributions have a common positive lower bound by parabolic Harnack. The corresponding portions of the kernel cancel up to a small phase error. The resulting contraction is uniform in the starting point, so it can be repeated. Higher frequencies require less time for the same phase separation, giving the following decay.

Proposition 3.1 (Decay of the Fourier kernels)**.** *There are constants* $C,c>0$ *depending only on the structural constants such that for all* $\sigma<\tau$*,* $v\in D$ *and* $\xi\in\mathbb{R}^d$*:*

$$\|\widehat{K}^\xi_{\sigma,\tau}(v,\cdot)\|_{\mathrm{TV}}\le C\exp\big(-c\,(\tau-\sigma)|\xi|^{\frac{2}{3}}\big)\ \textit{in case (W)}, \tag{3.1}$$

$$\|\widehat{K}^\xi_{\sigma,\tau}(v,\cdot)\|_{\mathrm{TV}}\le C\exp\big(-c\,(\tau-\sigma)\big(1+|\xi|^{\frac{2}{3}}\big)\big)\ \textit{in case (I)}. \tag{3.2}$$

In case (W), the dependence on ellipticity can be written in the form used in the short paper:

$$\|\widehat{K}^\xi_{\sigma,\tau}(v,\cdot)\|_{\mathrm{TV}}\le C_{d,\kappa}\exp\Big(-c_{d,\kappa}\lambda^{\frac{1}{3}}(\tau-\sigma)|\xi|^{\frac{2}{3}}\Big)\,,\qquad \kappa=\frac{\Lambda}{\lambda}.$$

Indeed, the change $\widehat{\sigma} = \lambda^{\frac{1}{3}}\sigma$, $\widehat{v} = \lambda^{-\frac{1}{3}}v$, $\widehat{z} = z$ leaves transport in the form $\widehat{v}\cdot\nabla_{\widehat{z}}$ after division by $\lambda^{\frac{1}{3}}$, and normalises the diffusion bounds to $[1,\kappa]$. Position and its frequency are unchanged, while pushing forward the velocity measure preserves total variation.

A velocity change of size r over time r^2 produces a relative displacement of size r^3. Taking $r = |\xi|^{-\frac{1}{3}}$ therefore separates phases in time $|\xi|^{-\frac{2}{3}}$. We prove a fixed contraction at unit frequency in Proposition 3.9, using moving-ball barriers and the common minorant. Near the endpoints of a bounded interval, where the balls may not fit, boundary loss replaces cancellation. Section 3.5 then scales and iterates the contraction; boundary loss also supplies the decay at frequency zero.

3.1. **Two barriers in a moving ball.** The first barrier gives a lower bound on the mass remaining in $B_\rho(\gamma(r))$. The second bounds its displacement in z, and hence its Fourier phase. Both use the following comparison principle. The velocity domain may move along a piecewise C^1 curve. Since position ranges over all of $\mathbb{R}^d$, comparison also requires control at spatial infinity.

Lemma 3.2 (Maximum principle on a moving domain)**.** *Let $\Omega \subseteq \mathbb{R}^d$ be an open ball or, when $d = 1$, a bounded open interval, let $\gamma : [\sigma,\tau] \to \mathbb{R}^d$ be continuous and C^1 on each interval of a subdivision $\sigma = r_0 < r_1 < \cdots < r_k = \tau$, put $\Omega_r = \gamma(r) + \Omega$ and*

$$\mathcal{T} = \{(r,v,z) : \sigma \le r \le \tau,\ v \in \overline{\Omega_r},\ z \in \mathbb{R}^d\}.$$

Let Ψ be continuous on $\mathcal{T}$, with $\partial_r\Psi$, $\nabla_v\Psi$, $\nabla_z\Psi$ and $D_v^2\Psi$ continuous at every interior point whose time coordinate is not one of $r_1,\dots,r_{k-1}$, and suppose that $\mathcal{K}\Psi \ge 0$ at those points, that $\Psi \le 0$ on $\{r = \tau\}$ and on $\{v \in \partial\Omega_r\}$, and that either Ψ does not depend on z or $\Psi(r,v,z) \to -\infty$ as $|z| \to \infty$ uniformly in (r,v). Then $\Psi \le 0$ on $\mathcal{T}$.

Proof. We first suppose that γ is C^1. We replace Ψ by $\Psi_\varepsilon = \Psi - \varepsilon(1+\tau-r)$, so that $\mathcal{K}\Psi_\varepsilon > 0$ and the boundary values remain non-positive. We fix $\sigma' > \sigma$. If Ψ_ε has a positive value on $\{r \ge \sigma'\}$, it attains a positive maximum there: in the first alternative omit z, and in the second use the uniform decay at infinity to restrict z to a compact set.

At a maximiser (r_*, v_*, z_*), the terminal and velocity boundary conditions give $r_* < \tau$ and $v_* \in \Omega_{r_*}$. Hence

$$\nabla_v\Psi_\varepsilon = 0, \qquad D_v^2\Psi_\varepsilon \le 0, \qquad \nabla_z\Psi_\varepsilon = 0.$$

The curve $r \mapsto (r, v_* + \gamma(r) - \gamma(r_*), z_*)$ stays in the moving domain for $r > r_*$ close to r_*. The right derivative at its maximum is non-positive, giving

$$\partial_r\Psi_\varepsilon + \gamma'(r_*)\cdot\nabla_v\Psi_\varepsilon = \partial_r\Psi_\varepsilon \le 0.$$

This also covers a maximiser at $r_* = \sigma'$. Positive semidefiniteness of B now gives $\mathcal{K}\Psi_\varepsilon \le 0$, a contradiction. We let $\sigma' \downarrow \sigma$ and then $\varepsilon \downarrow 0$.

For piecewise C^1 curves, we apply this argument successively on the subdivision intervals, starting with the last. Continuity supplies the terminal inequality for each preceding interval. □

Lemma 3.3 (Retention in a moving ball)**.** *Let $\gamma : [\sigma,\tau] \to \mathbb{R}^d$ be continuous and piecewise C^1 with $|\gamma'| \le H$, and let $\rho > 0$ be such that $B_\rho(\gamma(r)) \subseteq D$ for $\sigma \le r \le \tau$. Let $K^{\gamma,\rho}$ be the kernel of $\mathcal{K}$ with zero boundary values on $\partial B_\rho(\gamma(r)) \times \mathbb{R}^d$. Then for every $z \in \mathbb{R}^d$,*

$$K^{\gamma,\rho}_{\sigma,\tau}\big((\gamma(\sigma),z), B_\rho(\gamma(\tau)) \times \mathbb{R}^d\big) \ge \exp\big(-C_{d,\lambda,\Lambda}\,(\rho^{-2} + H^2)(\tau-\sigma)\big). \tag{3.3}$$

Proof. We extend γ constantly outside $[\sigma,\tau]$. By Proposition 2.1(vi), the mass in (3.3) is the mass of the parabolic kernel $P^{\gamma,\rho}$ on the moving ball, independently of the starting position.

We choose an integer $\mathsf{N} \ge 3$ with $2(\mathsf{N}-1)\lambda > d\Lambda$, and set $c_{\mathsf{N}} = 4\mathsf{N}(\mathsf{N}-1)\lambda - 2\mathsf{N}d\Lambda > 0$. For $0 < s < \rho$, put

$$\mathsf{w}(Y) = 1 - \frac{|Y|^2}{s^2}, \qquad \psi_s(Y) = (\mathsf{w}(Y)_+)^{\mathsf{N}}.$$

Then $\psi_s \in C^2(\mathbb{R}^d)$, $0 \le \psi_s \le 1 = \psi_s(0)$, and its derivatives vanish outside B_s. Inside that ball,

$$\nabla\psi_s = -\frac{2\mathsf{N}}{s^2}\mathsf{w}^{\mathsf{N}-1}Y, \qquad D^2\psi_s = -\frac{2\mathsf{N}}{s^2}\mathsf{w}^{\mathsf{N}-1}I_d + \frac{4\mathsf{N}(\mathsf{N}-1)}{s^4}\mathsf{w}^{\mathsf{N}-2}Y\otimes Y.$$

We use $\operatorname{tr} B \le d\Lambda$, $Y\cdot BY \ge \lambda|Y|^2$, $\mathsf{w}^{\mathsf{N}-1} \le \mathsf{w}^{\mathsf{N}-2}$ for $0 \le \mathsf{w} \le 1$, and $\mathsf{w}^{\mathsf{N}-1} = \mathsf{w}^{\mathsf{N}} + \mathsf{w}^{\mathsf{N}-1}\frac{|Y|^2}{s^2}$ to obtain

$$B : D^2\psi_s \ge -\frac{2\mathsf{N}d\Lambda}{s^2}\psi_s + \frac{c_{\mathsf{N}}}{s^4}\mathsf{w}^{\mathsf{N}-2}|Y|^2\,. \tag{3.4}$$

For $|a| \le H$, Young's inequality gives

$$|a\cdot\nabla\psi_s| \le \frac{c_{\mathsf{N}}}{s^4}\mathsf{w}^{\mathsf{N}-2}|Y|^2 + \frac{\mathsf{N}^2H^2}{c_{\mathsf{N}}}\psi_s.$$

The first term is absorbed by (3.4), leaving

$$B : D^2\psi_s - a\cdot\nabla\psi_s \ge -C_s\psi_s, \qquad C_s = \frac{2\mathsf{N}d\Lambda}{s^2} + \frac{\mathsf{N}^2H^2}{c_{\mathsf{N}}}\,, \tag{3.5}$$

This holds on all of $\mathbb{R}^d$, since both sides vanish outside B_s.

We choose $0 \le \phi \le 1$ in $C_c^\infty(B_\rho(\gamma(\tau)))$, equal to one on $\overline{B_s(\gamma(\tau))}$, and let $V(r,v) = \int \phi\,\mathrm{d}P^{\gamma;\rho}_{r,\tau}(v,\cdot)$. It solves the homogeneous parabolic equation with terminal value ϕ and zero lateral values. By (3.5),

$$\Psi(r,v) = e^{-C_s(\tau-r)}\psi_s(v-\gamma(r)) - V(r,v)$$

has $\mathcal{K}\Psi \ge 0$ on each smooth time interval. Its terminal values are non-positive because $\psi_s \le \phi$, and its lateral values vanish because $s < \rho$. Lemma 3.2 gives $\Psi \le 0$. At the centre,

$$P^{\gamma;\rho}_{\sigma,\tau}(\gamma(\sigma), B_\rho(\gamma(\tau))) \ge V(\sigma,\gamma(\sigma)) \ge e^{-C_s(\tau-\sigma)}.$$

We let $s \uparrow \rho$ and use $C_\rho \le C_{d,\lambda,\Lambda}(\rho^{-2} + H^2)$. □

The second barrier concerns the transported variable. Along the evolution, z is transported at the rate $b(v)$; if v is confined to a ball of radius ρ about a known centre, the accumulated displacement of z in a given direction is determined by that centre, with an error proportional to ρ.

Lemma 3.4 (Phase localisation)**.** *In the setting of Lemma 3.3, let $\hat\xi \in \mathbb{R}^d$ be a unit vector, $z_0 \in \mathbb{R}^d$ and $v \in B_\rho(\gamma(\sigma))$. The measure $K^{\gamma;\rho}_{\sigma,\tau}((v,z_0),\cdot)$ is supported in*

$$\left\{(w,z') :\ \left|\hat\xi\cdot(z'-z_0) - \int_\sigma^\tau \hat\xi\cdot b(\gamma(r))\,\mathrm{d}r\right| \le L_b\,\rho\,(\tau-\sigma)\right\}. \tag{3.6}$$

Proof. For $\sigma \le r \le \tau$, put

$$\varkappa_\pm(r,z) = \pm\left(\hat\xi\cdot(z-z_0) - \int_\sigma^r \hat\xi\cdot b(\gamma(s))\,\mathrm{d}s\right) - L_b\rho(r-\sigma).$$

These functions are independent of v and affine in z. The Lipschitz bound on b gives, for $v' \in B_\rho(\gamma(r))$,

$$\mathcal{K}\varkappa_\pm = \partial_r\varkappa_\pm + b(v')\cdot\nabla_z\varkappa_\pm = \pm\hat\xi\cdot\big(b(v') - b(\gamma(r))\big) - L_b\rho \le L_b|v'-\gamma(r)| - L_b\rho \le 0\,, \tag{3.7}$$

We take $0 \le \phi \le 1$ smooth and compactly supported in the terminal set $\{\varkappa_+(\tau,z) > 0\}$. We choose a smooth increasing $\chi : \mathbb{R} \to [0,1]$, zero on $(-\infty,0]$ and equal to one above the positive minimum of $\varkappa_+$ on $\operatorname{supp}\phi$. Then $F = \chi(\varkappa_+)$ satisfies $\mathcal{K}F \le 0$ and dominates ϕ at time τ.

The terminal solution $u = \mathsf{S}^{\gamma;\rho}_{r,\tau}\phi$ is bounded between zero and one and vanishes on the velocity boundary. To compare it with F on the unbounded position domain, put

$$h_\eta(r,z) = \eta(1+|z-z_0|^2)e^{(M_b+1)(\tau-r)}, \qquad M_b = \sup_{\sigma\le r\le\tau,\ v'\in\overline{B_\rho(\gamma(r))}} |b(v')|.$$

The supremum is finite, and $2|z-z_0| \le 1+|z-z_0|^2$ gives $\mathcal{K}h_\eta \le -h_\eta$. Thus $u-F-h_\eta$ has non-positive terminal and lateral values, tends uniformly to $-\infty$ as $|z|\to\infty$, and has non-negative $\mathcal{K}$. Lemma 3.2 gives $u \le F + h_\eta$. At (σ, v, z_0), $F = 0$, so

$$\int \phi \,\mathrm{d}K^{\gamma;\rho}_{\sigma,\tau}((v,z_0),\cdot) \le \eta e^{(M_b+1)(\tau-\sigma)}.$$

We let $\eta \downarrow 0$. Inner regularity shows that the kernel gives no mass to $\{\varkappa_+(\tau,z) > 0\}$. Repeating the argument with $\varkappa_-$ excludes the other side and proves (3.6). □

3.2. **A common minorant at the terminal time.** The barriers retain a positive amount of mass in each tube. A lower bound on two total masses does not yet give a common positive measure below both distributions. We obtain this common minorant by letting both evolve in one fixed ball and applying parabolic Harnack.

For each non-negative terminal function of velocity, its integral against the parabolic kernel is a non-negative solution of the parabolic equation in the starting time and velocity. We apply Harnack to this solution after reversing time. Because the comparison holds for every such terminal function, it is an inequality between measures. This is the lower bound used in § 3.3; the Fourier kernels themselves are complex and play no part in this application of Harnack.

Theorem 3.5 (Parabolic Harnack inequality [17]; see [7, Theorem 1.5])**.** *Let $N \ge 1$ and let $B : (0,1)\times B_1(0) \to \mathrm{Sym}_N$ be continuous with $\lambda I_N \le B \le \Lambda I_N$. There is $\mathfrak{h} = \mathfrak{h}(N,\lambda,\Lambda) > 0$ such that every non-negative $\mathfrak{u} \in C^{1,2}((0,1)\times B_1(0))$ solving $\partial_\theta \mathfrak{u} = B(\theta, v) : D_v^2 \mathfrak{u}$ in $(0,1)\times B_1(0)$ satisfies*

$$\sup_{(\frac14,\frac12)\times B_{\frac12}(0)} \mathfrak{u} \;\le\; \mathfrak{h}^{-1} \inf_{(\frac34,1)\times B_{\frac12}(0)} \mathfrak{u}\,. \tag{3.8}$$

The two cylinders in (3.8) are concentric and separated in time, so one application compares two times only at points that are close together; a chain of applications along a segment removes that restriction and gives the two-point comparison used below.

Lemma 3.6 (A common terminal minorant)**.** *Let $v_0 \in D$ and $\rho > 0$ with $D_0 = B_{4\rho}(v_0) \subseteq D$, let $T \in \mathbb{R}$, and let $P^{D_0}_{\theta,T}(v,\cdot)$ be the kernel of $\partial_\sigma + B : D_v^2$ with zero boundary values on $\partial B_{4\rho}(v_0)$. There are $h, q_0 > 0$ depending only on d, λ, Λ such that*

$$\begin{aligned} P^{D_0}_{T-4\rho^2,T}(v,\cdot) &\ge h\, P^{D_0}_{T-\rho^2,T}(v_0,\cdot) \quad \text{for every } v \in B_\rho(v_0),\\ P^{D_0}_{T-\rho^2,T}\big(v_0, B_{4\rho}(v_0)\big) &\ge q_0\,. \end{aligned} \tag{3.9}$$

Proof. By parabolic scaling we may take $\rho = 1$ and $v_0 = 0$, retaining B for the rescaled coefficient with the same ellipticity bounds. For $0 \le \phi \in C_c^\infty(B_4)$, the function

$$\mathfrak{u}(\vartheta, v) = \int \phi \,\mathrm{d}P^{B_4}_{T-\vartheta,T}(v,\cdot)$$

is bounded, non-negative and smooth for $\vartheta > 0$ by Proposition 2.1, and solves $\partial_\vartheta \mathfrak{u} = B(T-\vartheta, v) : D_v^2 \mathfrak{u}$. We join $(1,0)$ to $(4,v)$, $|v| < 1$, in six Harnack steps. We set $\vartheta_j = 1 + \frac{j}{2}$, $v_j = \frac{j}{6} v$ and $\bar v_j = \frac12(v_j + v_{j+1})$. The cylinders

$$(\vartheta_j - \frac38, \vartheta_j + \frac58) \times B_1(\bar v_j), \qquad j = 0,\dots,5,$$

lie in $(0,\infty)\times B_4$. The successive points have time offsets $\frac38$ and $\frac78$ and lie within $\frac{1}{12}$ of $\bar v_j$. Theorem 3.5 therefore gives $\mathfrak{u}(1,0) \le \mathfrak{h}^{-6}\mathfrak{u}(4,v)$. Testing with every such ϕ proves the first measure inequality in (3.9), with $h = \mathfrak{h}^6$.

For the second inequality, Lemma 3.3 on B_3, followed by domain monotonicity, bounds the mass from $(T-1,0)$ below by $q_0 = e^{-\frac{C_{d,\lambda,\Lambda}}{9}}$. Scaling back proves both claims. □

3.3. **Cancellation of two phases.** The lemma below applies to the portions selected using the common minorant. Their velocity marginal is Θ, and their phases lie near two values differing by π. Replacing each phase by its central value gives opposite Fourier contributions. The error is small, so the decrease in total variation is proportional to $\Theta(D)$.

Lemma 3.7 (Döblin–Fourier cancellation)**.** *Let M, F_1, F_2 be finite positive Borel measures on $D \times \mathbb{R}^d$ with $F_1 + F_2 \le M$, and suppose that each F_i has velocity marginal Θ. Let $\xi \in \mathbb{R}^d$, $z_0 \in \mathbb{R}^d$, $0 \le \varepsilon < 1$ and $\Phi_1, \Phi_2 \in \mathbb{R}$ with $e^{-i\Phi_1} + e^{-i\Phi_2} = 0$, and suppose that F_i is supported in $\{(w, z') : |\xi \cdot (z' - z_0) - \Phi_i| \le \varepsilon\}$ for $i = 1, 2$. Then the complex measure $E \mapsto \int_{E \times \mathbb{R}^d} e^{-i\xi\cdot(z'-z_0)}\, \mathrm{d}M$ satisfies*

$$\Big\| \int_{\cdot \times \mathbb{R}^d} e^{-i\xi\cdot(z'-z_0)}\, \mathrm{d}M \Big\|_{\mathrm{TV}} \le M(D \times \mathbb{R}^d) - 2(1-\varepsilon)\,\Theta(D)\,. \tag{3.10}$$

Proof. For a finite positive measure N on $D \times \mathbb{R}^d$, denote its Fourier projection on D by $\widehat{N}$. The phase has modulus one, so

$$\|\widehat{N}\|_{\mathrm{TV}} \le N(D \times \mathbb{R}^d). \tag{3.11}$$

If h is a bounded complex Borel function on D with $|h| \le 1$, the common marginal and the phase support give

$$\left| \int_D h \, \mathrm{d}\big(\widehat{F}_i - e^{-i\Phi_i}\Theta\big) \right| \le \int_{D \times \mathbb{R}^d} |e^{-i\xi\cdot(z'-z_0)} - e^{-i\Phi_i}|\, \mathrm{d}F_i \le \varepsilon\Theta(D).$$

Taking the supremum over h yields $\|\widehat{F}_i - e^{-i\Phi_i}\Theta\|_{\mathrm{TV}} \le \varepsilon\Theta(D)$. Since $e^{-i\Phi_1} + e^{-i\Phi_2} = 0$, it follows that $\|\widehat{F}_1 + \widehat{F}_2\|_{\mathrm{TV}} \le 2\varepsilon\Theta(D)$. The remainder $M - F_1 - F_2$ is positive and has mass $M(D \times \mathbb{R}^d) - 2\Theta(D)$. Thus (3.11) and the triangle inequality give

$$\|\widehat{M}\|_{\mathrm{TV}} \le M(D \times \mathbb{R}^d) - 2\Theta(D) + 2\varepsilon\Theta(D),$$

which is (3.10). □

3.4. **A contraction at unit frequency.** We now choose the two paths and apply the preceding estimates. In the whole space there is room for the moving tubes at every starting velocity. In a bounded interval they may not fit near an endpoint, but there the zero boundary condition already removes a fixed amount of mass. The next lemma quantifies this loss by comparison with the one-dimensional heat equation on a half-line.

Lemma 3.8 (Killing near an endpoint)**.** *Let $d = 1$, let J' be a bounded open interval, and let P be the kernel of $\partial_\sigma + B\partial_{vv}$ with zero boundary values at $\partial J'$. For $\ell > 0$ and $v \in J'$,*

$$P_{\sigma,\sigma+\ell}(v, J') \le \operatorname{erf}\Big(\frac{\operatorname{dist}(v, \partial J')}{2\sqrt{\lambda\ell}}\Big), \qquad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, \mathrm{d}t\,. \tag{3.12}$$

In particular $\sup_{v \in J'} P_{\sigma,\sigma+|J'|^2}(v, J') \le \operatorname{erf}\big(\frac{1}{4\sqrt{\lambda}}\big) < 1$.

Proof. We fix $\tau = \sigma + \ell$ and an endpoint $\mathfrak{e}$ nearest to v. The distance $\mathfrak{d}(v') = |v' - \mathfrak{e}|$ is affine on J', with $|\mathfrak{d}'| = 1$. The half-line heat solution

$$W(\vartheta, x) = \operatorname{erf}\left(\frac{x}{2\sqrt{\lambda\vartheta}}\right)$$

satisfies $W_\vartheta = \lambda W_{xx}$ and $W_{xx} \le 0$ for $x \ge 0$. Consequently

$$W_\vartheta - BW_{xx} = (\lambda - B)W_{xx} \ge 0. \tag{3.13}$$

For $\varepsilon > 0$, the function $\mathcal{W}(r, v') = W(\tau - r + \varepsilon, \mathfrak{d}(v'))$ is a supersolution for the terminal problem. It is non-negative at both endpoints and has terminal value $\varphi_\varepsilon(v') = W(\varepsilon, \mathfrak{d}(v'))$. Comparison with smooth terminal data $0 \le \phi \le \varphi_\varepsilon$, compactly supported in J', gives

$$\int \phi \, \mathrm{d}P_{\sigma,\tau}(v, \cdot) \le W(\ell + \varepsilon, \mathfrak{d}(v)).$$

We first approximate φ_ε from below by such data. We then let $\varepsilon \downarrow 0$; since $\varphi_\varepsilon \uparrow 1$ in J', monotone convergence gives (3.12). Finally, $\mathfrak{d}(v) \le \frac{1}{2}|J'|$ and $\ell = |J'|^2$ give the stated strict contraction. □

The four ingredients now combine, at the unit frequency and over a single elapsed time whose length is fixed by the structural constants alone.

Proposition 3.9 (A contraction at unit frequency)**.** *We assume* (W)*, or* (I) *with J replaced by an arbitrary bounded open interval J' on which* (I) *holds. There are $L > 0$ and $\delta \in (0,1)$, depending only on $d, \lambda, \Lambda, m, L_b$, such that for every unit vector $\xi \in \mathbb{R}^d$, every $\sigma \in \mathbb{R}$ and every $v \in D$,*

$$\|\widehat{K}^{\xi}_{\sigma,\sigma+L}(v,\cdot)\|_{\mathrm{TV}} \le 1 - \delta\,. \tag{3.14}$$

The constants do not depend on the length or position of the interval. This is important because the change of scale in Lemma 2.5 changes that interval.

Proof. We denote by $C_* = C_{d,\lambda,\Lambda}$ the constant of Lemma 3.3 and by h, q_0 the constants of Lemma 3.6. We put

$$L_0 = 2 + \frac{2\pi}{m}, \qquad \rho = \min\Big\{1,\ (128 L_b)^{-\frac{1}{3}}\Big\}, \qquad \eta = \min\Big\{\frac{\rho}{4},\ \frac{1}{8 L_b L_0}\Big\}, \tag{3.15}$$

and

$$L = L_0 + 4\rho^2, \qquad R_* = 1 + 4\rho, \qquad m_0 = \exp\Big(-C_*\Big(\eta^{-2} + \tfrac{1}{4}\Big) L_0\Big)\,. \tag{3.16}$$

All of these depend only on $d, \lambda, \Lambda, m, L_b$. We fix a unit vector $\xi \in \mathbb{R}^d$, a time $\sigma \in \mathbb{R}$, a point $z_0 \in \mathbb{R}^d$ and a point $v_0 \in D$, and assume until Step 5 that $B_{R_*}(v_0) \subseteq D$; in case (W) this holds for every v_0. We must bound the total variation of the complex measure $\widehat{K}^{\xi}_{\sigma,\sigma+L}(v_0,\cdot)$, which by (2.4) is the Fourier transform, at the frequency ξ and with base point z_0, of the positive measure

$$M = K_{\sigma,\sigma+L}\big((v_0,z_0),\cdot\big), \qquad M\big(D \times \mathbb{R}^d\big) \le 1\,,$$

the bound on the mass being part of Proposition 2.1 (iii).

Step 1: two paths with opposite phases. For $t \in [0, L_0]$ we set $\mathsf{f}(t) = \min\{t, 1, L_0 - t\}$. Since $L_0 \ge 2$, the function f is continuous and piecewise affine with $\mathsf{f}(0) = \mathsf{f}(L_0) = 0$, $0 \le \mathsf{f} \le 1$, $|\mathsf{f}'| \le 1$ and

$$\int_0^{L_0} \mathsf{f}(t)\,\mathrm{d}t = L_0 - 1 = 1 + \frac{2\pi}{m}\,.$$

For $\alpha \in [0, \frac{1}{2}]$ define curves on $[\sigma, \sigma + L_0]$ by

$$\gamma_1(r) = v_0, \qquad \gamma_2^{\alpha}(r) = v_0 + \alpha\,\mathsf{f}(r-\sigma)\,\xi\,.$$

Both curves begin and end at v_0, have speed at most $\frac{1}{2}$, and stay within distance $\frac{1}{2}$ of v_0. We extend them constantly by v_0 outside the interval. Since $\eta \le \frac{1}{4}$, their tubes fit inside D:

$$B_\eta\big(\gamma_i(r)\big) \subseteq B_{\frac{1}{2}+\eta}(v_0) \subseteq B_1(v_0) \subseteq B_{R_*}(v_0) \subseteq D, \qquad i = 1, 2. \tag{3.17}$$

We define

$$\Delta(\alpha) = \int_0^{L_0} \xi \cdot \big(b(v_0 + \alpha \mathsf{f}(t)\xi) - b(v_0)\big)\,\mathrm{d}t.$$

The bound (2.1) gives

$$\Delta'(\alpha) \ge m \int_0^{L_0} \mathsf{f}(t)\,\mathrm{d}t = m + 2\pi.$$

Since $\Delta(0) = 0$ and $\Delta(\frac{1}{2}) > \pi$, we can choose $\alpha \in (0, \frac{1}{2})$ with $\Delta(\alpha) = \pi$. We fix this value, write $\gamma_2 = \gamma_2^{\alpha}$, and set

$$a_i = \int_{\sigma}^{\sigma+L_0} \xi \cdot b(\gamma_i(r))\,\mathrm{d}r, \qquad a_2 - a_1 = \pi.$$

Step 2: restricting the kernel to the two moving tubes. For $i = 1, 2$, we denote by $K^{\gamma_i,\eta}$ the kernel of $\mathcal{K}$ on $[\sigma, \sigma + L_0]$ with zero boundary values on $\partial B_\eta(\gamma_i(r)) \times \mathbb{R}^d$. It is available by (3.17). From (v_0, z_0), its terminal measure is carried by $B_\eta(v_0) \times \mathbb{R}^d$, since $\gamma_i(\sigma + L_0) = v_0$, and has mass at most one by Proposition 2.1 (iii). Lemma 3.3, applied with the curve γ_i, $H = \frac{1}{2}$ and radius η, gives the lower bound

$$m_0 \le m_i := K^{\gamma_i,\eta}_{\sigma,\sigma+L_0}\big((v_0, z_0), D \times \mathbb{R}^d\big) \le 1\,. \tag{3.18}$$

We retain mass $\frac{m_0}{2}$ from each tube by setting

$$R_i := \frac{m_0}{2m_i} K^{\gamma_i,\eta}_{\sigma,\sigma+L_0}\big((v_0, z_0), \cdot\big), \qquad R_i(D \times \mathbb{R}^d) = \frac{m_0}{2}.$$

Domain monotonicity bounds each tube kernel by the full kernel. Since $\frac{m_0}{2m_i} \le \frac{1}{2}$, each R_i is at most half of that kernel. Hence

$$R_1 + R_2 \le K_{\sigma,\sigma+L_0}((v_0, z_0), \cdot), \tag{3.19}$$

even if the two tubes overlap.

Step 3: selecting portions using a common minorant. We put $T = \sigma + L$ and $D_0 = B_{4\rho}(v_0) \subset D$. We continue R_i with the kernel killed at ∂D_0:

$$E_i(E) = \int K^{D_0}_{\sigma+L_0,T}((v', z'), E) R_i(\,\mathrm{d}v'\,\mathrm{d}z').$$

The starting velocities lie in $B_\eta(v_0) \subset D_0$. Domain monotonicity, (3.19) and composition give

$$E_1 + E_2 \le K_{\sigma,T}((v_0, z_0), \cdot) = M. \tag{3.20}$$

The velocity marginals are

$$\nu_i(E) = \int P^{D_0}_{\sigma+L_0,T}(v', E) R_i(\,\mathrm{d}v'\,\mathrm{d}z').$$

Since $\eta < \rho$, Lemma 3.6 applies throughout this integral. It gives

$$\nu_i \ge \Theta := \frac{m_0 h}{2} P^{D_0}_{T-\rho^2,T}(v_0, \cdot), \qquad \Theta(D) \ge \frac{m_0 h q_0}{2} > 0.$$

We set $g_i = \frac{\mathrm{d}\Theta}{\mathrm{d}\nu_i}$, so $0 \le g_i \le 1$, and define

$$F_i(\,\mathrm{d}w\,\mathrm{d}z') = g_i(w) E_i(\,\mathrm{d}w\,\mathrm{d}z').$$

At each terminal velocity w, this keeps the fraction $g_i(w)$ of E_i. This operation is sometimes called trimming; it is a weighted restriction, which need not be a restriction to a set. It preserves every retained phase, gives $F_1 + F_2 \le M$, and gives each F_i the velocity marginal Θ. The original marginals ν_1 and ν_2 need not agree.

Step 4: the phase error and the cancellation. We set $\Phi_i = a_i + 4\rho^2 \xi \cdot b(v_0)$, so $\Phi_2 - \Phi_1 = \pi$. Lemma 3.4 bounds the phase error during the tube by $L_b \eta L_0$. During the common continuation, the velocity stays within 4ρ of v_0 for time $4\rho^2$, adding at most $16 L_b \rho^3$. Thus E_i, and hence F_i, is supported where

$$|\xi \cdot (z' - z_0) - \Phi_i| \le L_b \eta L_0 + 16 L_b \rho^3 \le \frac{1}{4}.$$

Lemma 3.7 now gives

$$\|\widehat{K}^{\xi}_{\sigma,\sigma+L}(v_0, \cdot)\|_{\mathrm{TV}} \le 1 - \frac{3}{2}\Theta(D) \le 1 - \delta_1, \qquad \delta_1 = \frac{3}{4} m_0 h q_0. \tag{3.21}$$

This proves the contraction whenever the two tubes fit, in particular at every starting velocity in the whole space.

Step 5: points near an endpoint. In case (I), a point not covered above satisfies $\operatorname{dist}(v, \partial J') \le R_*$. Domination by the velocity marginal and Lemma 3.8 give

$$\|\widehat{K}^{\xi}_{\sigma,\sigma+L}(v, \cdot)\|_{\mathrm{TV}} \le P_{\sigma,\sigma+L}(v, J') \le \operatorname{erf}\left(\frac{R_*}{2\sqrt{\lambda L}}\right) =: 1 - \delta_2 < 1. \tag{3.22}$$

This bound is independent of the interval's length and location.

We take $\delta = \min\{\delta_1, \frac{1}{2}\}$ in case (W) and $\delta = \min\{\delta_1, \delta_2, \frac{1}{2}\}$ in case (I). The interior and boundary estimates cover every starting velocity. All constants depend only on $d, \lambda, \Lambda, m, L_b$, as required. □

3.5. **Proof of the decay.** At frequency $\xi \neq 0$, the choice $r = |\xi|^{-\frac{1}{3}}$ turns Proposition 3.9 into a fixed contraction over time Lr^2. Composition repeats this contraction. On a bounded interval, Lemma 3.8 also gives decay at frequency zero; we combine the two bounds by splitting the time interval in half.

Proof of Proposition 3.1. For $\xi \neq 0$, Lemma 2.5 with $r = |\xi|^{-\frac{1}{3}}$ reduces the frequency to $r^3\xi = \frac{\xi}{|\xi|}$. The ellipticity and transport bounds are unchanged, so Proposition 3.9 and (2.12) give

$$\|\widehat{K}^{\xi}_{s,s+Lr^2}(v,\cdot)\|_{\mathrm{TV}} \leq 1-\delta \qquad \text{for all } s \in \mathbb{R},\ v \in D\,. \tag{3.23}$$

We compose $k = \lfloor \frac{\tau-\sigma}{Lr^2} \rfloor$ such blocks. The remaining factor has norm at most one by Lemma 2.2 (i). Thus (2.5) gives a bound $(1-\delta)^k$, and, with $C_1 = (1-\delta)^{-1}$ and $c_1 = \frac{-\log(1-\delta)}{L}$,

$$\|\widehat{K}^{\xi}_{\sigma,\tau}(v,\cdot)\|_{\mathrm{TV}} \leq (1-\delta)^{\frac{\tau-\sigma}{Lr^2}-1} = C_1 \exp\bigl(-c_1(\tau-\sigma)|\xi|^{\frac{2}{3}}\bigr)\,. \tag{3.24}$$

The mass bound covers $\xi = 0$, proving (3.1).

In case (I), Lemma 3.8 gives

$$\sup_{v\in J} P_{s,s+|J|^2}(v,J) \leq 1-\delta_0, \qquad \delta_0 = 1-\operatorname{erf}\Bigl(\frac{1}{4\sqrt{\lambda}}\Bigr) > 0.$$

The same iteration, now in blocks of length $|J|^2$, and $|\widehat{K}^{\xi}| \leq P$ imply

$$\|\widehat{K}^{\xi}_{\sigma,\tau}(v,\cdot)\|_{\mathrm{TV}} \leq P_{\sigma,\tau}(v,J) \leq C_2 \exp\bigl(-c_2(\tau-\sigma)\bigr) \tag{3.25}$$

where $C_2 = (1-\delta_0)^{-1}$ and $c_2 = \frac{-\log(1-\delta_0)}{|J|^2}$. Finally, we split $[\sigma,\tau]$ at its midpoint and apply (3.24) to the first factor and (3.25) to the second. This proves (3.2) with $C = C_1C_2$ and $c = \frac{1}{2}\min\{c_1,c_2\}$. □

4. The velocity marginal

Integrating the positive kernel in z removes transport and leaves the parabolic kernel $P_{\sigma,\tau}(v,\cdot)$. This is also its Fourier transform at frequency zero. The parabolic Aleksandrov estimate, by duality, bounds the density of the time integral of P in $L^{\frac{d+1}{d}}$. Unlike the mass bound alone, this controls concentration in time and velocity. We also check that the estimate is uniform over the whole velocity space, as needed in Section 5.

Theorem 4.1 (Parabolic Aleksandrov estimate [16]; see [7, Theorem 2.4])**.** *Let $N \geq 1$, $0 < h \leq 1$ and $Q = B_1(v_0) \times (t_0-h, t_0)$. We denote its bottom and lateral boundary by*

$$\partial_p Q = \bigl(\overline{B_1(v_0)} \times \{t_0-h\}\bigr) \cup \bigl(\partial B_1(v_0) \times [t_0-h, t_0]\bigr).$$

Let $a : \overline{Q} \to \mathrm{Sym}_N$ be continuous with $\lambda I_N \leq a \leq \Lambda I_N$, let $f \in L^{N+1}(Q)$, and let $u \in C^{2,1}(\overline{Q})$ satisfy

$$\partial_t u - a(v,t) : D_v^2 u \leq f \quad \text{in } Q, \qquad u \leq 0 \quad \text{on } \partial_p Q\,.$$

Then

$$\sup_Q u \leq C_{N,\lambda,\Lambda}\, \|f\|_{L^{N+1}(Q)}\,. \tag{4.1}$$

Proposition 4.2 (Parabolic Green-density estimate)**.** *Let D be $\mathbb{R}^d$ or, when $d = 1$, a bounded open interval, and let P be the kernel of $\partial_\sigma + B(\sigma,v) : D_v^2$ on D with zero boundary values, where B is smooth and satisfies (2.2). Let ρ be a finite positive Borel measure on D of mass M, prescribed at time σ_0, and $S > 0$. There is a non-negative density g^S_ρ on $(0,S) \times D$ characterised, for bounded Borel ϕ, by*

$$\int_0^S \int_D \phi(\tau,w) g^S_\rho(\tau,w)\,\mathrm{d}w\,\mathrm{d}\tau = \int_D \rho(\,\mathrm{d}v) \int_0^S \int_D \phi(\tau,w)\, P_{\sigma_0,\sigma_0+\tau}(v,\,\mathrm{d}w)\,\mathrm{d}\tau \tag{4.2}$$

and, for $1 \le \gamma \le \gamma_0 := \frac{d+1}{d}$,

$$\|g_\rho^S\|_{L^\gamma((0,S)\times D)} \le C_{d,\lambda,\Lambda,\gamma}\, M\, S^{\beta_\gamma}, \qquad \beta_\gamma = \frac{d+2}{2\gamma} - \frac{d}{2}. \tag{4.3}$$

Proposition 4.2 is the dual form of the parabolic Aleksandrov estimate; the short localisation below makes its constant independent of the spatial extent of the source.

Proof. We first take $\sigma_0 = 0$, $S = 1$ and $D = \mathbb{R}^d$. For $0 \le F \in C_c^\infty((0,1)\times\mathbb{R}^d)$, let

$$W(r,v) = \int_r^1 \int_{\mathbb{R}^d} F(t,w)\, P_{r,t}(v,\, \mathrm{d}w)\, \mathrm{d}t, \qquad H = \sup_{[0,1]\times\mathbb{R}^d} W.$$

Then $W \ge 0$, $H \le \|F\|_\infty$, $W(1,\cdot) = 0$ and $\partial_r W + B : D_v^2 W = -F$.

We choose $R^2 = \max\{1, 4d\Lambda\}$. For any $0 \le \alpha < 1$ and v_*, let V solve the same source equation on $(\alpha,1)\times B_R(v_*)$, with zero terminal and lateral values. Parabolic solvability [19, Theorem 5.14] gives a classical solution $V \in C^{1,2}$ in the open cylinder, continuous on its closure; no regularity up to the boundary is used. After reflection $s = 1 - r$ and scaling, put $S = 1 - \alpha$, $\rho = (1 - 2\delta/S)^{\frac12}$ and $T_\delta(s,v) = (\delta + \rho^2 s,\ v_* + \rho(v - v_*))$. Then T_δ maps the closed cylinder into the open one, so $V_\delta = V \circ T_\delta$ is $C^{1,2}$ on a neighbourhood of the closed cylinder and solves the same type of equation, with ellipticity bounds unchanged and source $\rho^2 F \circ T_\delta$, whose L^{d+1}-norm is at most $\|F\|_{L^{d+1}}$. Let $\eta_\delta = \sup|V_\delta|$ over the parabolic boundary; $\eta_\delta \to 0$ by the uniform continuity of V and $V = 0$ there. Theorem 4.1 applied to $V_\delta - \eta_\delta$ gives $V_\delta \le \eta_\delta + CR^{\frac{d}{d+1}}\|F\|_{L^{d+1}}$, and $\delta \downarrow 0$ yields

$$0 \le V \le CR^{\frac{d}{d+1}}\|F\|_{L^{d+1}}.$$

The quadratic function

$$\psi(r,v) = \frac{|v - v_*|^2 + 2d\Lambda(1-r)}{R^2}$$

is non-negative, at least one on the lateral boundary, and satisfies $(\partial_r + B : D_v^2)\psi \le 0$. Thus comparison gives $W \le V + H\psi$ on this cylinder. At (α, v_*), $\psi \le \frac12$. Taking the supremum over α, v_* and absorbing yields

$$H \le CR^{\frac{d}{d+1}}\|F\|_{L^{d+1}} + \frac12 H, \qquad H \le C_{d,\lambda,\Lambda}\|F\|_{L^{d+1}}.$$

This proves the whole-space bound without a time iteration.

The right side of (4.2), with $S = 1$ and $\phi = F$, equals $\int W(0,v)\rho(\,\mathrm{d}v)$. Duality therefore gives the density g_ρ^1 with $\|g_\rho^1\|_{L^{\frac{d+1}{d}}} \le CM$. Its L^1 norm is at most M by contraction. Interpolation gives every $1 \le \gamma \le \gamma_0$.

For general S, we use $(\tau, v) = (S\hat\tau, \sqrt{S}\,\hat v)$ and push forward the initial measure, retaining its mass M. Uniqueness identifies the rescaled parabolic kernels, and the time integral contributes a factor S. Consequently

$$g_\rho^S(\tau,v) = S^{-\frac{d}{2}}\widetilde{g}\left(\frac{\tau}{S}, \frac{v}{\sqrt{S}}\right), \qquad \|g_\rho^S\|_{L^\gamma} = S^{\frac{d+2}{2\gamma} - \frac{d}{2}}\|\widetilde{g}\|_{L^\gamma},$$

which proves (4.3). Finally, the killed interval kernel is dominated by the whole-line kernel with the same coefficient, so the same bound holds there. □

5. Reconstruction of the Green density

The two preceding sections control different parts of the same positive measure. The parabolic estimate bounds its integral in position; Fourier decay bounds its height in position. Their combination gives an L^q bound in all variables. We first work away from the starting time, on $T < \tau < 2T$, and then integrate the resulting power of T down to zero.

Throughout this section, μ has mass M and is prescribed at time σ_0. On each interval $T < \tau < 2T$, the Green measure Γ_μ^∞ has mass at most MT. We retain $\gamma_0 = \frac{d+1}{d}$ and $\beta_\gamma = \frac{d+2}{2\gamma} - \frac{d}{2}$ from Proposition 4.2.

Lemma 5.1 (Fourier marginals on one elapsed-time interval)**.** *Let μ be as in Theorem 2.4 and $T > 0$. On $T < \tau < 2T$, the (τ, w)-marginal of Γ_μ^∞ has a density $g_{\mu,T}$ with*

$$\|g_{\mu,T}\|_{L^\gamma((T,2T)\times D)} \le C_\gamma\, M\, T^{\beta_\gamma}, \qquad 1 \le \gamma \le \gamma_0\,. \tag{5.1}$$

For $\xi \in \mathbb{R}^d$, the complex measure $\Phi \mapsto \int_{\{T<\tau<2T\}} e^{-i\xi\cdot z}\Phi(\tau,w)\,\mathrm{d}\Gamma_\mu^\infty$ on $(T,2T)\times D$ has a density $k_{\mu,T}^\xi$, and

$$\begin{aligned} \|k_{\mu,T}^\xi\|_{L^\gamma((T,2T)\times D)} &\le C_\gamma\, M\, T^{\beta_\gamma}\, \epsilon_T(\xi), \\ \epsilon_T(\xi) &= \begin{cases} \exp\bigl(-cT|\xi|^{\frac{2}{3}}\bigr) & \text{in case (W)}, \\ \exp\bigl(-cT(1+|\xi|^{\frac{2}{3}})\bigr) & \text{in case (I)}, \end{cases} \end{aligned} \tag{5.2}$$

for $1 \le \gamma \le \gamma_0$. Consequently

$$\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \|k_{\mu,T}^\xi\|_{L^\gamma((T,2T)\times D)}\,\mathrm{d}\xi \le C_\gamma\, M\, T^{\beta_\gamma - \frac{3d}{2}} \times \begin{cases} 1 & \text{in case (W)}, \\ e^{-cT} & \text{in case (I)}. \end{cases} \tag{5.3}$$

Proof. The (τ, w)-marginal is the restriction to $(T, 2T)$ of the parabolic occupation measure issued from $\mu_v(E) = \mu(E \times \mathbb{R}^d)$. Proposition 4.2, with $S = 2T$, gives (5.1).

For the Fourier marginal, we evolve first to elapsed time $\frac{T}{2}$ and define the complex velocity measure

$$\eta_T^\xi(E) = \int_{D\times\mathbb{R}^d} e^{-i\xi\cdot z}\, \widehat{K}^\xi_{\sigma_0,\sigma_0+\frac{T}{2}}(v,E)\,\mu(\,\mathrm{d}v\,\mathrm{d}z).$$

Proposition 3.1 gives $\|\eta_T^\xi\|_{\mathrm{TV}} \le CM\epsilon_T(\xi)$, after decreasing the rate c. We continue its positive total variation $|\eta_T^\xi|$ by the parabolic kernel P, and let g be the resulting occupation density over elapsed times $(0, \frac{3T}{2})$. Composition of the Fourier kernels and their domination by P give, for bounded Borel Φ,

$$\left|\int_{\{T<\tau<2T\}} e^{-i\xi\cdot z}\Phi(\tau,w)\,\mathrm{d}\Gamma_\mu^\infty\right| \le \int_T^{2T}\int_D |\Phi(\tau,w)|\, g\left(\tau - \frac{T}{2}, w\right)\,\mathrm{d}w\,\mathrm{d}\tau.$$

Thus the Fourier marginal has a density bounded in modulus by this translate of g. Proposition 4.2 gives (5.2). This uses the parabolic estimate only on a positive measure, not on a complex solution.

Finally, $\zeta = T^{\frac{3}{2}}\xi$ gives

$$\int_{\mathbb{R}^d} e^{-cT|\xi|^{\frac{2}{3}}}\,\mathrm{d}\xi = C_{d,c}T^{-\frac{3d}{2}},$$

which proves (5.3). Measurability of the integrand follows by expressing the L^γ norm as a supremum over a countable norming family of bounded test functions; each associated Fourier pairing is continuous in ξ. □

The next lemma combines the marginal and Fourier bounds to construct the full density G. The input densities g and k^ξ are then its position integral and Fourier transform:

$$g = \int_{\mathbb{R}^d} G(\cdot,z)\,\mathrm{d}z, \qquad k^\xi = \int_{\mathbb{R}^d} e^{-i\xi\cdot z}G(\cdot,z)\,\mathrm{d}z.$$

Lemma 5.2 (Reconstruction from the marginal and the Fourier bounds)**.** *Let Y be a Borel subset of a Euclidean space with a σ-finite Borel measure m, and let Γ be a finite positive Borel measure on $Y \times \mathbb{R}^d$ whose Y-marginal is $g\,\mathrm{d}m$. Let $1 < q < \gamma < \infty$, define q_1 by $\frac{q-1}{\gamma} + \frac{1}{q_1} = 1$, and suppose that $g \in L^{q_1}(Y)$ and that for every $\xi \in \mathbb{R}^d$ the Fourier marginal $E \mapsto \int_{E\times\mathbb{R}^d} e^{-i\xi\cdot z}\,\mathrm{d}\Gamma$ has a density $k^\xi \in L^\gamma(Y)$ with respect to m, where*

$$\mathsf{H} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \|k^\xi\|_{L^\gamma(Y)}\,\mathrm{d}\xi < \infty\,.$$

Then Γ has a density G with respect to $m \otimes \mathrm{Leb}$ and

$$\|G\|_{L^q(Y\times\mathbb{R}^d)}^q \le \mathsf{H}^{q-1}\|g\|_{L^{q_1}(Y)}\,. \tag{5.4}$$

Proof. We disintegrate $\Gamma(\,\mathrm{d}y\,\mathrm{d}z) = g(y)\pi_y(\,\mathrm{d}z)m(\,\mathrm{d}y)$, where π_y has mass one for $g\,\mathrm{d}m$-almost every y. We choose the jointly measurable Fourier densities

$$k^\xi(y) = g(y)\int e^{-i\xi\cdot z}\pi_y(\,\mathrm{d}z), \qquad H(y) = \frac{1}{(2\pi)^d}\int |k^\xi(y)|\,\mathrm{d}\xi.$$

Joint measurability follows from measurability in y and continuity in ξ. For each ξ these representatives agree almost everywhere with the densities in the hypothesis, so their norms and H are unchanged. Minkowski's inequality gives $\|H\|_{L^\gamma} \le \mathsf{H}$, so the Fourier transform of $g(y)\pi_y$ is integrable for almost every y. Fourier inversion gives this measure a non-negative density $G(y,\cdot)$ bounded by $H(y)$, with integral $g(y)$. One may justify this version of inversion by convolution with a Gaussian: the inverse transforms converge uniformly by dominated convergence, and the measures converge weakly. The inverse integral also makes G jointly measurable. Hence

$$\int G(y,z)^q\,\mathrm{d}z \le H(y)^{q-1}g(y).$$

Integrating in y and applying Hölder with $\frac{q-1}{\gamma} + \frac{1}{q_1} = 1$ proves (5.4). □

Proposition 5.3 (Density on one elapsed-time interval)**.** *Let μ and T be as in Lemma 5.1 and $1 < q < 1 + \frac{1}{d}$. Then Γ_μ^∞ has a density $G_{\mu,T}$ on $(T,2T)\times D\times\mathbb{R}^d$ with respect to Lebesgue measure, and*

$$\|G_{\mu,T}\|_{L^q((T,2T)\times D\times\mathbb{R}^d)}^q \le C_q\,M^q\,T^{1-2d(q-1)} \times \begin{cases} 1 & \text{in case (W),} \\ e^{-cT} & \text{in case (I).} \end{cases} \tag{5.5}$$

Proof. We choose $q < \gamma < \gamma_0$ and define q_1 by $\frac{q-1}{\gamma} + \frac{1}{q_1} = 1$. Since $\gamma > q > (d+1)(q-1)$, both γ and q_1 lie in $[1,\gamma_0]$. We apply Lemma 5.2 to the marginal bounds (5.1) and (5.3). The resulting power of T is

$$(q-1)\left(\beta_\gamma - \frac{3d}{2}\right) + \beta_{q_1} = \frac{d+2}{2}\left(\frac{q-1}{\gamma} + \frac{1}{q_1}\right) - 2d(q-1) - \frac{d}{2} = 1 - 2d(q-1).$$

In case (I), the exponential factor is raised to $q-1$; decreasing c gives (5.5). □

Proof of Theorem 2.4. The local densities from Proposition 5.3 agree on overlaps, giving a density G for Γ_μ^∞ on positive elapsed times. We put $\delta_q = 1 - 2d(q-1) > 0$. We integrate the estimates over all interval lengths: for every non-negative function F,

$$\int_0^S F(\tau)\,\mathrm{d}\tau \le \frac{1}{\log 2}\int_0^S \frac{\mathrm{d}T}{T}\int_T^{2T} F(\tau)\,\mathrm{d}\tau.$$

Indeed, each $\tau \in (0,S)$ is counted for $\frac{\tau}{2} < T < \tau$, whose logarithmic length is $\log 2$. We apply this with $F(\tau) = \int |G(\tau,w,z)|^q\,\mathrm{d}w\,\mathrm{d}z$. In case (W), Proposition 5.3 gives

$$\|G_\mu^S\|_{L^q}^q \le C_q M^q \int_0^S T^{\delta_q - 1}\,\mathrm{d}T = \frac{C_q M^q}{\delta_q} S^{\delta_q},$$

which is (2.8). In case (I), the same argument over all positive times gives

$$\|G_\mu^\infty\|_{L^q}^q \le C_q M^q \int_0^\infty T^{\delta_q - 1}e^{-cT}\,\mathrm{d}T < \infty,$$

proving (2.9). Finally, the zero-frequency bound (3.2) gives

$$\Gamma_\mu^\infty((0,\infty)\times J\times\mathbb{R}) \le CM\int_0^\infty e^{-c\tau}\,\mathrm{d}\tau < \infty.$$

□

Remark 5.4 (The Lorentz endpoint). In case (W), the Green density belongs to weak L^{q_*} at $q_* = 1 + \frac{1}{2d}$. To see this, we first take initial mass one and fix $q_* < q < 1 + \frac{1}{d}$. The mass before elapsed time h is at most h, while summing (5.5) over the intervals $(2^j h, 2^{j+1}h)$ gives

$$\int_h^\infty \int G^q \,\mathrm{d}w \,\mathrm{d}z \,\mathrm{d}\tau \le C h^{1-2d(q-1)}.$$

For a level $a > 0$, Chebyshev's inequality on the two time ranges therefore gives

$$|\{G > a\}| \le \frac{h}{a} + C a^{-q} h^{1-2d(q-1)}.$$

Choosing $h = a^{-\frac{1}{2d}}$ bounds this by Ca^{-q_*}. Linearity gives $\|G_\mu^\infty\|_{L^{q_*,\infty}} \le CM$ for initial mass M. Lorentz-space Hölder duality then bounds the source potential by $C\|f\|_{L^{2d+1,1}}$. The comparison argument of the next section gives the corresponding endpoint maximum principle for smooth $A(t,v)$.

6. From Green measures to maximum principles

A positive source produces a non-negative potential. Subtracting that potential from a sub-solution leaves a function controlled by its boundary values; Hölder's inequality then turns a Green-density bound into the Aleksandrov estimate. This section makes that comparison on the kinetic boundary, where data are prescribed only on the initial, velocity and incoming position faces. We first reflect time and position to use the terminal kernels constructed above.

We set $s = -t$, $X = -x$ and $U(s, X, v) = u(-s, -X, v)$, leaving the velocity unreflected. Given $Z_0 = (s_0, X_0, v_0) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ and $R > 0$, the forward cylinder is

$$\mathcal{Q}_R(Z_0) = \left\{(s, X, v) :\ s_0 < s < s_0 + R^2, \quad |v - v_0| < R, \quad |X - X_0 - (s - s_0)v_0| < R^3\right\}, \tag{6.1}$$

with relative variables $V = v - v_0$ and $Y = X - X_0 - (s - s_0)v_0$, and its exit boundary is

$$\partial^+ \mathcal{Q}_R(Z_0) = \overline{\mathcal{Q}_R(Z_0)} \cap \Big(\{s = s_0 + R^2\} \cup \{|V| = R\} \cup \{|Y| = R^3,\ V \cdot Y \ge 0\}\Big). \tag{6.2}$$

For a matrix field A on the cylinder we write

$$\mathcal{K}_A U = \partial_s U + v \cdot \nabla_X U + A : D_v^2 U\,. \tag{6.3}$$

The reflection $\mathscr{R}(t, x, v) = (-t, -x, v)$ preserves Lebesgue measure and maps $Q_R^-(P_0)$ onto $\mathcal{Q}_R(-t_0, -x_0, v_0)$. It fixes V, reverses Y, and therefore carries $\partial_k Q_R^-(P_0)$ onto $\partial^+ \mathcal{Q}_R(-t_0, -x_0, v_0)$. With $\widetilde{A}(s, v) = A(-s, v)$, the chain rule gives

$$\mathcal{K}_{\widetilde{A}} U = -(\mathcal{L}_A u) \circ \mathscr{R}. \tag{6.4}$$

For the autonomous class we use $\widetilde{a}(X, v) = a(-X, v)$. Both reflections preserve ellipticity, measurability and the permitted coefficient dependence.

For the boundary passage, we take the inner cylinders $\mathcal{Q}_\delta$, $0 < \delta < \frac{1}{2}R^2$, given in relative coordinates by

$$\mathcal{Q}_\delta = \{s_0 + \delta < s < s_0 + R^2 - \delta,\ |V| < \rho_\delta R,\ |Y| < \rho_\delta^3 R^3\}, \quad \rho_\delta = \left(1 - \frac{2\delta}{R^2}\right)^{\frac{1}{2}}. \tag{6.5}$$

They exhaust $\mathcal{Q}$, with closures contained in $\mathcal{Q}$. The affine retraction

$$T_\delta(s, Y, V) = \big(s_0 + \delta + \rho_\delta^2 (s - s_0),\, \rho_\delta^3 Y,\, \rho_\delta V\big) \tag{6.6}$$

maps the closure and exit boundary of $\mathcal{Q}$ onto those of $\mathcal{Q}_\delta$ and converges uniformly to the identity on $\overline{\mathcal{Q}}$. The boundary assertion follows because $V \cdot Y$ is multiplied by the positive factor ρ_δ^4.

The maximum principle below carries data only on the exit boundary (6.2), which omits the initial face and the part of the position faces where the relative transport points inwards. Its operator has no second-order part in X, so no uniformly parabolic statement applies to it.

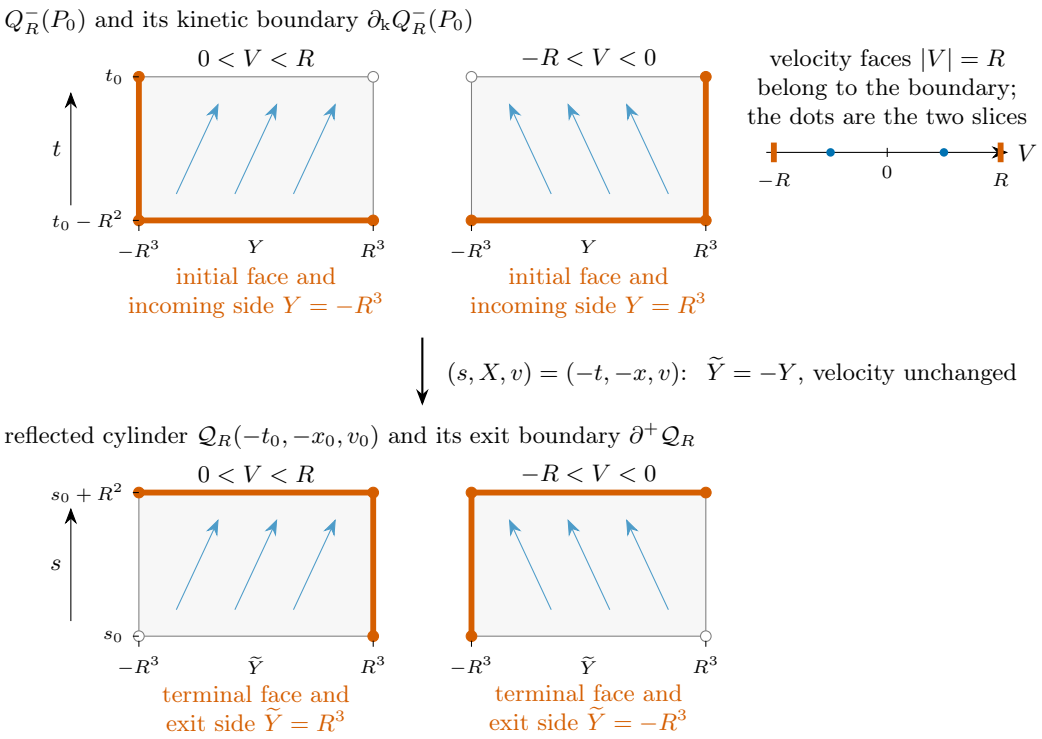

FIGURE 1. The kinetic boundary (1.3), in one-dimensional slices at fixed relative velocity $V = v - v_0$ and in the relative coordinates (Y, t), $Y = x - x_0 - (t - t_0)v_0$, where the cylinder is a rectangle and transport follows $\frac{\mathrm{d}Y}{\mathrm{d}t} = V$. The incoming position face changes with the sign of V, not of v. The highlighted initial, velocity and incoming faces form the kinetic boundary, where boundary control is required; filled corners belong to it, while the relative interiors of the top face and of the outgoing side do not. Under $(t, x, v) \mapsto (-t, -x, v)$, see (6.4), these become the terminal, velocity and outgoing faces of $\partial^+ \mathcal{Q}_R$ in (6.2). The non-strict inequalities also include grazing points: at $V = 0$ both position faces belong to the boundary.

Lemma 6.1 (Kinetic comparison principle)**.** *Let $\mathcal{Q} = \mathcal{Q}_R(Z_0)$ and let $A : \mathcal{Q} \to \mathrm{Sym}_d$ be positive semidefinite at every point. If $Z \in C(\overline{\mathcal{Q}}) \cap C^{1,1,2}(\mathcal{Q})$ satisfies $\mathcal{K}_A Z \geq 0$ at every point of $\mathcal{Q}$ at which $Z > 0$, then*

$$\sup_{\overline{\mathcal{Q}}} Z \leq \sup_{\partial^+ \mathcal{Q}} Z_+ \,. \tag{6.7}$$

Proof. We set $Z_\varepsilon = Z + \varepsilon(s - s_0 - R^2)$. Then $Z_\varepsilon \leq Z$ and $\mathcal{K}_A Z_\varepsilon \geq \varepsilon$ wherever $Z_\varepsilon > 0$. In relative coordinates the operator is $\partial_s + V \cdot \nabla_Y + A : D_V^2$.

We suppose that a positive maximum of Z_ε on $\overline{\mathcal{Q}_\delta}$ occurs outside its exit boundary. The velocity is interior, so $D_V^2 Z_\varepsilon \leq 0$ and $A : D_V^2 Z_\varepsilon \leq 0$. The time is either interior or at the initial face; in the latter case its right derivative is non-positive. Hence $\partial_s Z_\varepsilon \leq 0$. Finally, either the position is interior, or it lies on $|Y| = \rho_\delta^3 R^3$ with $V \cdot Y < 0$. In the second case $Y + hV$ lies inside the ball for small $h > 0$, so the one-sided derivative gives $V \cdot \nabla_Y Z_\varepsilon \leq 0$; in the first this term is zero. Thus $\mathcal{K}_A Z_\varepsilon \leq 0$, a contradiction.

It follows that $\sup_{\overline{\mathcal{Q}_\delta}} Z_\varepsilon \leq \sup_{\partial^+ \mathcal{Q}_\delta} (Z_\varepsilon)_+$. We use (6.6) and continuity to send $\delta \downarrow 0$, then use $Z \leq Z_\varepsilon + \varepsilon R^2$ and send $\varepsilon \downarrow 0$. □

The comparison uses only pointwise positive semidefiniteness of A. The following proposition adds a source potential with a uniform density bound, as constructed in Sections 7 and 8.

Proposition 6.2 (From a Green density to the Aleksandrov estimate)**.** *Let $\mathcal{Q} = \mathcal{Q}_R(Z_0)$, let A be smooth on a neighbourhood of $\overline{\mathcal{Q}}$ with $\lambda I_d \le A \le \Lambda I_d$, let $1 < q < \infty$, $p = \frac{q}{q-1}$, and $\mathsf{K} > 0$. Suppose that to every $P \in \mathcal{Q}$ is associated a positive Borel measure Γ_P on $\mathcal{Q}$ such that:*

(i) *for every $0 \le g \in C_c^\infty(\mathcal{Q})$, the function $W_g(P) = \int_{\mathcal{Q}} g \, \mathrm{d}\Gamma_P$ belongs to $C^{1,1,2}(\mathcal{Q})$, is non-negative, and satisfies $\mathcal{K}_A W_g = -g$ in $\mathcal{Q}$;*

(ii) *Γ_P has a density G_P with $\|G_P\|_{L^q(\mathcal{Q})} \le \mathsf{K}$ for every $P \in \mathcal{Q}$.*

Let $U \in C(\overline{\mathcal{Q}}) \cap C^{1,1,2}(\mathcal{Q})$ and let $g_0 \ge 0$ be a Borel function with $g_0 \in L^p(\mathcal{Q})$ and $\mathcal{K}_A U \ge -g_0$ almost everywhere in $\mathcal{Q}$. Then

$$\sup_{\overline{\mathcal{Q}}} U \le \sup_{\partial^+ \mathcal{Q}} U_+ + \mathsf{K} \|g_0 \mathbf{1}_{\{U>0\}}\|_{L^p(\mathcal{Q})} \, . \tag{6.8}$$

Proof. The defect $g = (-\mathcal{K}_A U)_+$ is continuous, satisfies $g \le g_0$ almost everywhere, and makes $\mathcal{K}_A U + g \ge 0$ pointwise. We fix $\varepsilon > 0$ and choose $0 \le \theta \le 1$ smooth, with $\theta = 0$ below $\frac{\varepsilon}{2}$ and $\theta = 1$ above ε. Then

$$0 \le \theta(U) g \le g_0 \mathbf{1}_{\{U>0\}} \quad \text{almost everywhere.}$$

We choose $\delta_j \downarrow 0$. On each inner cylinder $\mathcal{Q}_j = \mathcal{Q}_{\delta_j}$ choose $0 \le g_j \in C_c^\infty(\mathcal{Q})$ majorising $\theta(U) g$ on $\overline{\mathcal{Q}_j}$, with

$$\|g_j\|_{L^p} \le \|g_0 \mathbf{1}_{\{U>0\}}\|_{L^p} + \frac{1}{j}.$$

Such majorants are obtained using a cutoff between zero and one on a neighbourhood of $\overline{\mathcal{Q}_j}$, mollifying, and adding its vanishing uniform approximation error times a fixed cut-off equal to one there.

We set $W_j(P) = \int g_j \, \mathrm{d}\Gamma_P$. Wherever $U - W_j - \varepsilon > 0$, non-negativity of W_j gives $U > \varepsilon$, hence $g_j \ge g$ and $\mathcal{K}_A(U - W_j - \varepsilon) = \mathcal{K}_A U + g_j \ge 0$. Lemma 6.1 and Hölder's inequality yield, for $P \in \mathcal{Q}_j$,

$$U(P) \le \sup_{\partial^+ \mathcal{Q}_j} U_+ + \mathsf{K} \left(\|g_0 \mathbf{1}_{\{U>0\}}\|_{L^p} + \frac{1}{j} \right) + \varepsilon.$$

The retractions (6.6) and continuity pass the boundary term to $\partial^+ \mathcal{Q}$. We let $j \to \infty$ and then $\varepsilon \downarrow 0$. □

Remark 6.3 (Viscosity subsolutions for smooth coefficients). For smooth A and continuous $g_0 \ge 0$, Proposition 6.2 also holds for continuous viscosity subsolutions of $-\mathcal{K}_A U \le g_0$, without classical derivatives. Consequently Theorems 1.1 and 1.2, including Corollary 1.3, hold for smooth coefficients, after reflection, for continuous viscosity subsolutions with continuous source and with the same constants.

Only comparison needs checking. In the relative coordinates of Lemma 6.1, put $s_1 = s_0 + R^2$ and $M = \sup_{\partial^+ \mathcal{Q}} Z_+$. If $\sup Z > M$, choose $\varepsilon > 0$ so that $Z_\varepsilon = Z + \varepsilon(s - s_1)$ retains a positive excess at some point, and maximise

$$Z_\varepsilon - M - \frac{\delta}{s - s_0} - \frac{\delta}{R^6 - |Y|^2}.$$

For small δ the maximum retains a fixed positive excess and lies in the interior. Since $Z \ge Z_\varepsilon > 0$ there, the smooth upper test gives

$$\varepsilon \le -\frac{\delta}{(s - s_0)^2} + \frac{2\delta \, V \cdot Y}{(R^6 - |Y|^2)^2}.$$

As $\delta \downarrow 0$, a limit of these maximisers cannot lie on the exit boundary, by continuity and the fixed positive excess. If its position is interior, the positive part of the spatial term tends to zero; if it lies on a position face, that face is incoming, so $V \cdot Y < 0$. Both cases contradict the display. We let $\varepsilon \downarrow 0$ to obtain comparison. The preceding proof now applies with smooth majorants of $\theta(U) g_0$ in place of $\theta(U)(-\mathcal{K}_A U)_+$.

This argument does not establish a viscosity extension for Borel coefficients. The approximation below uses the locally bounded classical velocity Hessian and cannot simply be transferred to that solution class.

Our Green measures are constructed for smooth coefficients. The following lemma passes their uniform Aleksandrov estimates to Borel coefficients by approximation on compact inner cylinders, keeping the solution fixed.

Lemma 6.4 (From smooth to Borel coefficients)**.** *Let $1 \le p < \infty$ and $\alpha \in \mathbb{R}$, and let $\mathcal{C}$ be one of the two coefficient classes*

(a) Borel maps $A : \mathbb{R} \times \mathbb{R}^d \to \mathrm{Sym}_d$ with $\lambda I_d \le A(t,v) \le \Lambda I_d$ almost everywhere, regarded as coefficients of $\mathcal{L}_A$;
(b) $d = 1$ and Borel maps $a : \mathbb{R} \times \mathbb{R} \to [\lambda, \Lambda]$, regarded as coefficients of $\mathcal{L}_a$.

We write $\mathcal{L}$ for the operator $\mathcal{L}_A$, respectively $\mathcal{L}_a$, with the coefficient under consideration. Suppose that for every smooth member of $\mathcal{C}$ satisfying its bounds everywhere, every $P_0 \in \mathbb{R}^{1+2d}$ and $R > 0$, and every $u \in C(\overline{Q_R^-(P_0)}) \cap C^{1,1,2}(Q_R^-(P_0))$ and Borel f with $\mathcal{L}u \le f$ almost everywhere and $f_+ \in L^p$, one has

$$\sup_{\overline{Q_R^-(P_0)}} u \le \sup_{\partial_k Q_R^-(P_0)} u_+ + C_0 R^\alpha \|f_+\|_{L^p(Q_R^-(P_0))} \tag{6.9}$$

with C_0 independent of the coefficient. Then (6.9) holds, with the same C_0, for every member of $\mathcal{C}$ and every such u. The same implication holds when $\|f_+\|_{L^p(Q_R^-(P_0))}$ is replaced by $\|f_+ \mathbf{1}_{\{u>0\}}\|_{L^p(Q_R^-(P_0))}$ in hypothesis and conclusion.

Proof. In case (a), we redefine $A = \lambda I_d$ on the null set where ellipticity fails. We mollify the coefficient in (t,v) in case (a), and in (x,v) in case (b), obtaining smooth coefficients C_j in the same class, with the same ellipticity bounds and $C_j \to C$ almost everywhere. Tonelli's theorem lifts this convergence to phase space. We keep u fixed and put

$$e_j = |(C_j - C) : D_v^2 u|.$$

Then $\mathcal{L}_{C_j} u \le f + e_j$ almost everywhere and $e_j \le 2\sqrt{d}\,\Lambda |D_v^2 u|$, with $d = 1$ in case (b). We take Q_δ to be the reflection of (6.5); it is a kinetic cylinder of radius $\rho_\delta R$ with closure inside $Q_R^-(P_0)$. For fixed δ, the classical Hessian is bounded there, so dominated convergence gives $\|e_j\|_{L^p(Q_\delta)} \to 0$. We apply (6.9) on Q_δ and first let $j \to \infty$, using $(f + e_j)_+ \le f_+ + e_j$. This gives

$$\sup_{\overline{Q_\delta}} u \le \sup_{\partial_k Q_\delta} u_+ + C_0 (\rho_\delta R)^\alpha \|f_+\|_{L^p(Q_\delta)}.$$

We now let $\delta \downarrow 0$. The reflected retractions (6.6) carry the original kinetic boundary onto $\partial_k Q_\delta$ and converge uniformly to the identity. Continuity of u therefore passes the boundary supremum to the limit; exhaustion passes the left-hand side and source norm, and $\rho_\delta R \to R$. No uniform Hessian bound as $\delta \downarrow 0$ is needed. For the localised form, use $(f + e_j)_+ \mathbf{1}_{\{u>0\}} \le f_+ \mathbf{1}_{\{u>0\}} + e_j$ on each inner cylinder and take the same two limits. □

7. Coefficients depending on time and velocity

Here the coefficient is independent of position on the whole space. Reflection gives the operator (W), with $z = X$, $B(s,v) = A(-s,v)$ and $b(v) = v$. The Green-density estimate therefore applies directly: Hölder's inequality bounds the source potential, and comparison gives the maximum principle. The second part of this section uses the same Fourier decay to obtain local smoothness in position and then Hölder continuity.

Proof of Theorem 1.1. For smooth A, reflection (6.4) gives $U(s,X,v)=u(-s,-X,v)$ with $\mathcal{K}_{\widetilde{A}}U\geq -g_0$, where $g_0(s,X,v)=f_+(-s,-X,v)$. It preserves the source norm and maps the kinetic boundary onto the exit boundary of $\mathcal{Q}=\mathcal{Q}_R(-t_0,-x_0,v_0)$.

For $P\in\mathcal{Q}$, we denote by Γ_P the whole-space Green measure from P, restricted to $\mathcal{Q}$ and written in absolute time. Lemma 2.3 supplies the potential identity in Proposition 6.2. With $q=\frac{p}{p-1}<1+\frac{1}{2d}$ and remaining time $S_P<R^2$, Theorem 2.4(i) gives

$$\|G_P\|_{L^q(\mathcal{Q})}\leq C_{d,\lambda,\Lambda,q}S_P^{\frac{1-2d(q-1)}{q}}\leq C_{d,\lambda,\Lambda,p}R^{2-\frac{4d+2}{p}}.$$

Proposition 6.2 and reflection now prove (1.6), including its localised-source form. The constants are independent of coefficient derivatives, so Lemma 6.4 gives the Borel case. □

7.1. **Local spatial smoothing and a direct Hölder estimate.** For $A(t,v)$, the Fourier estimate also gives local spatial smoothing. The point is to retain lateral velocity data: a killed terminal kernel alone does not represent a solution on a local cylinder. We first include those data, then estimate how much can arrive through the velocity boundary in a short time.

Proposition 7.1 (Local regularity for $A(t,v)$)**.** *Let A satisfy (1.4) and let $u\in C(\overline{Q_R^-(P_0)})\cap C^{1,1,2}(Q_R^-(P_0))$ satisfy $\mathcal{L}_Au=0$ almost everywhere. Then u is smooth in x in the interior and, for every integer $m\geq 1$,*

$$R^{3m}\|D_x^mu\|_{L^\infty(Q^-_{\frac{3R}{4}}(P_0))}\leq C_m\operatorname{osc}_{Q_R^-(P_0)}u.$$

There are $C>0$ and $\alpha\in(0,1)$, depending only on d,λ,Λ, such that, for $P=(t,x,v)$ and $P'=(s,y,w)$ in $Q^-_{\frac{R}{2}}(P_0)$,

$$|u(P)-u(P')|\leq C\operatorname{osc}_{Q_R^-(P_0)}u\left(\frac{|t-s|^{\frac12}+|v-w|+|x-y-(t-s)v_0|^{\frac13}}{R}\right)^\alpha.$$

Here C_m depends only on d,λ,Λ,m.

Proof. We first suppose that A is smooth and work with the reflected operator $\mathcal{K}_A=\partial_s+v\cdot\nabla_X+A(s,v):D_v^2$. For a starting point (s,X,v) in the image of $Q^-_{\frac{3R}{4}}(P_0)$ under reflection, put $D=B_R(v_0)$ and $T=s+\frac18R^2$. This terminal time is still inside the reflected outer cylinder; the spatial containment is checked below.

The local representation. Proposition 2.1 gives the kernel K^D killed at ∂D. For a source F, we write W_F^D for the Duhamel potential obtained from this kernel. It also defines the solution with prescribed terminal and lateral data: for a smooth compactly supported extension ϕ of those data, set

$$\mathsf{E}\phi=\phi+W^D_{\mathcal{K}_A\phi}.$$

Here signed potentials are defined by subtracting the potentials of the positive and negative parts. We use the bounded-source distributional identity proved in Appendix A; it gives $\mathcal{K}_A\mathsf{E}\phi=0$, and Theorem A.1 gives interior smoothness. The quadratic barrier gives

$$|W_F^D(r,X,v)|\leq\|F\|_\infty\min\left\{T-r,\frac{R^2-|v-v_0|^2}{2d\lambda}\right\};$$

indeed $\mathcal{K}_A(R^2-|v-v_0|^2)=-2\operatorname{tr}A\leq-2d\lambda$. Approximation extends this bound from compactly supported smooth sources to bounded sources. Thus $\mathsf{E}\phi$ has the prescribed traces. Bounded comparison, using the growth barrier of Appendix A, shows that it depends only on those traces and is a positive contraction. Smooth ambient traces are uniformly dense in continuous compactly supported boundary data: extend such data continuously with compact support and mollify. Uniform limits of their solutions

remain distributionally homogeneous, hence smooth inside. Riesz representation therefore gives a positive boundary measure $\omega^D_{s,v}$ on

$$\big(\{T\}\times\mathbb{R}^d\times\overline{D}\big)\cup\big([s,T]\times\mathbb{R}^d\times\partial D\big).$$

We write its position variable as $y = X' - X - (r-s)v_0$; translation invariance removes the starting position X.

For a unit vector e, we put $\varkappa_\pm = \pm e\cdot y - R(r-s)$. Then $\mathcal{K}_A\varkappa_\pm \le 0$. If non-negative boundary data $g \le 1$ have compact support where $\varkappa_+ > 0$, choose a smooth increasing χ with $\chi(0)=0$ and $\chi(\varkappa_+)\ge g$ on the boundary. This is a supersolution, so bounded comparison gives $\mathsf{E}g \le \chi(\varkappa_+)$ and $\mathsf{E}g(s,X,v)=0$. The growth barrier justifies comparison on the unbounded position domain exactly as in Lemma 3.4. Inner regularity, the same argument for $\varkappa_-$, and a countable dense set of directions give

$$|y| \le R(r-s) \le \frac{1}{8}R^3 \quad \text{on the support of } \omega^D_{s,v}.$$

It gives the same support for the killed kernels. We choose $\eta\in C^\infty_c(\mathbb{R}^d)$ equal to one on a neighbourhood of $\{|y|\le R(T-s)\}$ and use $\phi(r,X',w)=\eta(X'-X-(r-s)v_0)$, with time and velocity cutoffs equal to one near the closed strip. Then $\mathcal{K}_A\phi = 0$ on the killed Green support, so

$$1 = \mathsf{E}\phi(s,X,v) = \int \phi\,\mathrm{d}\omega^D_{s,v} = \int 1\,\mathrm{d}\omega^D_{s,v}.$$

Thus the boundary measure has mass one. Bounded classical homogeneous solutions are represented by this measure: cut off their boundary values outside the cone and use comparison. There is no initial-velocity-face mass for an interior starting velocity, as also follows from the short-time estimate below.

Decay before the boundary is reached. For $v\in B_{\frac{3R}{4}}(v_0)$ and $0<h\le\frac{1}{8}R^2$, we fix $t=s+h$ and write $q(r,v') = 1 - P^D_{t-r,t}(v',D)$. At $(r,v')=(h,v)$ this is the mass lost at the velocity boundary between s and $s+h$. We claim

$$q(h,v) \le Ce^{-\frac{cR^2}{h}}, \qquad 0<h\le\frac{1}{8}R^2.$$

We scale to $R=1$. The parabolic equation for q is $\partial_r q - A(t-r,v'):D^2_{v'}q = 0$, with initial value zero and lateral value one. We set $\delta = \frac{1}{4\sqrt{d}}$. The supersolution

$$\Psi(r,v') = \sum_{j=1}^d \sum_{\varepsilon=\pm1} \exp\big(\alpha\varepsilon(v'_j - v_j) - \alpha\delta + \Lambda\alpha^2 r\big)$$

dominates the lateral data, since $|v'-v|\ge\frac{1}{4}$ on ∂D. With $\alpha = \frac{\delta}{2\Lambda h}$, comparison gives $q(h,v)\le 2de^{-\frac{\delta^2}{4\Lambda h}}$. Scaling proves the claim.

The positive difference between the full and killed kernels has mass at most $q(h,v)$. Proposition 3.1 therefore gives

$$\|\widehat{K}^{D,\xi}_{s,s+h}(v,\cdot)\|_{\mathrm{TV}} \le C\big(e^{-\frac{cR^2}{h}} + e^{-ch|\xi|^{\frac{2}{3}}}\big).$$

Here the hat denotes Fourier transformation in position, leaving a measure in terminal velocity. To split the exit measure, write $\mathsf{E}^T g$ for the boundary solution with terminal time T, and let β_h be the lateral part of the boundary measure on the shorter strip ending at $s+h$, including its terminal corner. Its terminal interior part is $K^D_{s,s+h}$: test against smooth data supported inside D and use uniqueness. The short-strip measure has mass one, so β_h has mass $q(h,v)$. Restricting $\mathsf{E}^T g$ to that shorter strip and using uniqueness gives

$$\begin{aligned}\mathsf{E}^T g(s,X,v) = &\int g\,\mathrm{d}\beta_h \\ &+ \int \mathsf{E}^T g(s+h,X',w)\,K^D_{s,s+h}((v,X),\,\mathrm{d}w\,\mathrm{d}X').\end{aligned}$$

This proves the required decomposition for continuous data, and then for bounded Borel data by a monotone class argument. The later boundary operator is a positive contraction. Fourier transformation preserves composition because the coefficients do not depend on position; centring by $(r-s)v_0$ changes only a phase of modulus one. Consequently

$$\|\widehat{\omega}^{D,\xi}_{s,v}\|_{\mathrm{TV}} \le C\big(e^{-\frac{cR^2}{h}} + e^{-ch|\xi|^{\frac{2}{3}}}\big) \le Ce^{-c(R^3|\xi|)^{\frac{1}{3}}}.$$

For the last inequality we take $h = \frac{1}{8}R|\xi|^{-\frac{1}{3}}$ when $R^3|\xi| \ge 1$, and use the mass bound otherwise. The measure on the left retains both exit time and velocity.

Spatial derivatives and parabolic regularity. The preceding bound is integrable in ξ after multiplication by any power of $|\xi|$. The compact y-support also makes $\xi \mapsto \widehat{\omega}^{D,\xi}_{s,v}$ continuous in total variation. Fourier inversion therefore defines a density $\omega^D_{s,v}(y,\cdot)$ in y, taking values in measures of exit time and velocity. Differentiation under the integral gives

$$\sup_y \|D^m_y \omega^D_{s,v}(y,\cdot)\|_{\mathrm{TV}} \le C_m R^{-3(d+m)}, \qquad \int \|D^m_y \omega^D_{s,v}(y,\cdot)\|_{\mathrm{TV}}\,\mathrm{d}y \le C_m R^{-3m}.$$

The second estimate uses the support volume CR^{3d}. From any point in the reflected $Q^-_{\frac{3R}{4}}(P_0)$, the representation stays in the reflected $Q^-_R(P_0)$: the time and position bounds are respectively $(\frac{9}{16}+\frac{1}{8})R^2 < R^2$ and $(\frac{27}{64}+\frac{1}{8})R^3 < R^3$. To justify using a solution defined only on that cylinder, multiply it by a smooth position cutoff $\chi(s,X)$ equal to one on a neighbourhood of the reached cone and supported inside the outer position face. Its defect is $(\partial_s\chi + v\cdot\nabla_X\chi)U$, which is bounded and supported outside the cone, so its Green integral is zero. The boundary representation therefore remains valid at the starting point. We differentiate its convolution against the boundary data, subtract a constant, and use the last estimate. Reflection gives the asserted spatial derivative bounds.

For Hölder continuity, use $(t,x,v) = (t_0+R^2\tau, x_0+R^2\tau v_0+R^3X, v_0+RV)$ and normalise the oscillation. The transformed coefficient still depends only on time and velocity and has the same ellipticity bounds. We relabel the variables, so $P_0 = 0$, $R = 1$ and $|\nabla_x u| \le C$ in $Q^-_{\frac{3}{4}}(0)$. For each fixed $x \in B_{\frac{1}{8}}$,

$$\partial_t u - A(t,v) : D^2_v u = -v\cdot\nabla_x u \quad \text{in } \left(-\frac{9}{16},0\right)\times B_{\frac{3}{4}},$$

with bounded right-hand side. The parabolic Krylov–Safonov estimate [17, Lemma 4.3] gives Hölder continuity in (t,v) on $(-\frac{1}{4},0)\times B_{\frac{1}{2}}$. For the bounded right-hand side this follows from Theorem 3.5 by oscillation decay: on each cylinder Q_r, compare with the homogeneous solution with the same boundary values, which differs by at most Mr^2 when M bounds the right-hand side, so $\operatorname{osc}_{Q_{r/2}} u \le \theta \operatorname{osc}_{Q_r} u + 4Mr^2$ with $\theta < 1$, and iterate. Combining it with the spatial Lipschitz bound gives the stated estimate.

Measurable coefficients. We work first on compact interior cylinders and mollify A in (t,v), obtaining A_j. In reflected coordinates we choose a smooth compactly supported cutoff χ equal to one on the cylinder where the estimates will be applied. We put

$$F_j = \chi\,\mathcal{K}_{A_j}U = \chi(A_j - A) : D^2_v U \quad \text{almost everywhere}, \qquad U_j = U + W_{F_j},$$

where $W_{F_j} = W_{(F_j)_+} - W_{(F_j)_-}$ is the whole-space Duhamel potential for A_j, with a fixed terminal time beyond $\operatorname{supp}\chi$. On this compact set $D^2_v U$ is bounded. We apply Theorem 2.4(i) at the Green-density exponent $q = \frac{2d+2}{2d+1}$ and Hölder's inequality at the conjugate source exponent $2d+2$. On the finite time interval in question this gives

$$\|W_{F_j}\|_\infty \le C\|F_j\|_{L^{2d+2}} \longrightarrow 0.$$

The distributional Duhamel identity, proved in Appendix A also for bounded sources, gives $\mathcal{K}_{A_j}U_j = 0$ where $\chi = 1$. Hörmander's theorem makes U_j smooth there.

We apply the smooth-coefficient bounds to U_j and pass to the locally uniform limit U. All spatial finite-difference bounds pass to the limit. On each fixed (s,v) slice, compactness of the derivatives on smaller position balls, using the bound of the next order, gives spatial smoothness and the same

estimates. The Hölder bound passes directly. Finally, we exhaust the original cylinder, first keeping the top time below t_0 and then letting it increase to t_0. The fixed interior margins keep all constants uniform. □

8. Autonomous coefficients in one space dimension

The coefficient now depends on position, so Fourier transformation in position no longer separates the frequencies. On a velocity interval of one sign, however, position moves monotonically and can serve as time. Autonomy makes the transformed equation independent of the old time, allowing us to apply the Fourier argument in that variable.

The remaining difficulty is to combine these estimates near zero velocity, where the intervals must become small. A single evolution may leave and return to the same interval many times. We first count these visits with velocity barriers, obtaining the preliminary Aleksandrov estimate at $p = 6$. Its regularity consequences give a kinetic Harnack comparison, which improves the count and yields $p > 4$. We then retain the position of each visit. A homogeneous Bellman barrier supplies the additional concentration bound needed for every $p > p_*(\frac{\Lambda}{\lambda})$.

Classical parabolic Harnack is used for the Fourier minorant after changing the clock; the kinetic comparison controls returns in the original variables.

Throughout this section $d = 1$ and $a \in C^\infty(\mathbb{R}^2)$ satisfies $\lambda \le a \le \Lambda$ everywhere; the passage to a Borel coefficient is made at the very end, by Lemma 6.4. By the reflection convention of Section 6, the forward coefficient is $(X, v) \mapsto a(-X, v)$, which is again smooth, independent of s, and $[\lambda, \Lambda]$-valued; we write a for it as well, so that the operator (6.3) reads

$$\mathcal{K}_a U = \partial_s U + v\partial_X U + a(X, v)\partial_{vv} U \,.$$

Its coefficient depends on the transported variable X, so $\mathcal{K}_a$ is not an operator of the form (2.3) and Theorem 2.4 does not apply to it directly.

We work on velocity strips unbounded in position, so that no position boundary interrupts the change of time variable. The Green measure records the evolution inside a strip, and the exit measure records the data needed when it reaches the strip boundary. Composing these measures will account for repeated returns to smaller velocity intervals.

For a bounded open interval H and $-\infty < s_- < s_+ \le \infty$ let $\mathscr{S}_H = (s_-, s_+) \times \mathbb{R}_X \times H$ be the corresponding velocity strip. Its exit boundary consists of the terminal face and of the two velocity faces:

$$\partial^\dagger \mathscr{S}_H = \big(\{s_+\} \times \mathbb{R} \times \overline{H}\big) \cup \big((s_-, s_+) \times \mathbb{R} \times \partial H\big) \qquad \text{(the first set is empty when } s_+ = \infty\text{)} \,. \tag{8.1}$$

The Green and exit measures of $\mathcal{K}_a$ on such a strip, and the properties of them used below, are the following.

Proposition 8.1 (Green and exit measures on a velocity strip)**.** *Let H be a bounded open interval, $-\infty < s_- < s_+ < \infty$, and $\mathscr{S} = \mathscr{S}_H$. For every $e = (s_e, X_e, v_e)$ with $s_- \le s_e < s_+$, $X_e \in \mathbb{R}$ and $v_e \in H$ there are a positive Borel measure Γ_e^H on $\mathscr{S}$ and a positive Borel measure ω_e^H on $\partial^\dagger \mathscr{S}$, both supported in $\{s \ge s_e\}$, depending Borel measurably on e, and depending on s_- only through the requirement $s_- \le s_e$, with the following properties.*

(i) *For every $\phi \in C^{1,1,2}(\mathbb{R}\times\mathbb{R}\times H) \cap C(\mathbb{R}\times\mathbb{R}\times\overline{H})$ such that ϕ and $\mathcal{K}_a\phi$ are bounded on $\mathbb{R}\times\mathbb{R}\times H$,*

$$\phi(e) = \int_{\partial^\dagger \mathscr{S}} \phi \, \mathrm{d}\omega_e^H - \int_{\mathscr{S}} \mathcal{K}_a \phi \, \mathrm{d}\Gamma_e^H \,. \tag{8.2}$$

(ii) $\omega_e^H(\partial^\dagger\mathscr{S}) = 1$ *and, writing* $H = (h_-, h_+)$,

$$\Gamma_e^H(\mathscr{S}) \le \min\Big\{s_+ - s_e,\ \frac{(v_e - h_-)(h_+ - v_e)}{2\lambda}\Big\}. \tag{8.3}$$

(iii) *Both measures are supported in* $\{|X - X_e| \le V_H(s - s_e)\}$, *where* $V_H = \sup_H|v|$. *If* v *has a fixed sign* $\varsigma \in \{-1, 1\}$ *on* H *and* $v_H = \inf_H|v|$, *both measures are supported in* $\{v_H(s - s_e) \le \varsigma(X - X_e) \le V_H(s - s_e)\}$.

(iv) Γ_e^H *increases when* s_+ *increases and, after extension by zero, when* H *is enlarged. For a finite positive Borel measure* ν *on* $\mathscr{S}$ *put* $\Gamma_\nu^H = \int \Gamma_e^H \,\mathrm{d}\nu(e)$ *and* $\omega_\nu^H = \int \omega_e^H \,\mathrm{d}\nu(e)$; *then* (8.2) *holds with* $\phi(e)$ *replaced by* $\int \phi \,\mathrm{d}\nu$.

(v) *Let* $H_1 \subseteq H_2$ *be bounded open intervals with the same* $s_- < s_+$, *let* $\Sigma = (s_-, s_+)\times\mathbb{R}\times(\partial H_1 \cap H_2)$, *and for a finite positive Borel measure* ν *on* $\mathscr{S}_{H_1}$ *let* $\tau\nu = \omega_\nu^{H_1}|_\Sigma$. *Then*

$$\Gamma_\nu^{H_2} = \Gamma_\nu^{H_1} + \Gamma_{\tau\nu}^{H_2}, \qquad \omega_\nu^{H_2} = \omega_\nu^{H_1}|_{\partial^\dagger\mathscr{S}_{H_2}} + \omega_{\tau\nu}^{H_2}, \tag{8.4}$$

where $\Gamma_\nu^{H_1}$ *is extended by zero. The same identities hold when* H_1 *is a finite disjoint union of open intervals contained in* H_2, *the measures* $\Gamma_\nu^{H_1}$, $\omega_\nu^{H_1}$ *being defined component by component.*

(vi) *For every* $0 \le F \in C_c^\infty(\mathscr{S})$, *the function* $e \mapsto \int_{\mathscr{S}} F \,\mathrm{d}\Gamma_e^H$ *is non-negative and smooth in* $\mathscr{S}$, *satisfies* $\mathcal{K}_a \int F \,\mathrm{d}\Gamma_\cdot^H = -F$ *in* $\mathscr{S}$, *and extends continuously by zero to* $\partial^\dagger\mathscr{S}$.

For $s_+ = \infty$, *we take the increasing limit of* Γ_e^H. *The quadratic bound in* (8.3), *items* (iii) *and* (iv), *and the Green identity for tests vanishing for large* s *remain valid.*

The proof is given in Appendix A; since Γ_e^H and ω_e^H do not depend on s_-, the initial time of the strip is not displayed in the notation. The second bound in (8.3) is used below in Lemma 8.6 and Proposition 8.9.

8.1. **Position as the time variable on a velocity interval of one sign.** In physical time, the velocity marginal depends on the starting position and does not satisfy a closed parabolic equation: integrating out position leaves the position-dependent coefficient $a(X, v)$. We obtain a parabolic marginal only after changing the clock. The change is easiest to see on a bounded interval of positive velocities separated from zero. Dividing by v gives

$$\frac{1}{v}\mathcal{K}_a = \partial_X + \frac{1}{v}\partial_s + \frac{a(X, v)}{v}\partial_{vv}.$$

We prescribe terminal data depending only on velocity at a fixed position, and zero data at the velocity endpoints. Since the coefficient and these data are independent of s, so is the solution. Its equation is therefore

$$\partial_X V + \frac{a(X, v)}{v}\partial_{vv}V = 0.$$

After reversing the X direction, this is an ordinary uniformly parabolic equation. Thus the parabolic Harnack inequality in Lemma 3.6 acts here in position and velocity. The marginal integrates out the transported old time. In the centred coordinates below, that variable becomes z, an affine function of s at each fixed new time σ. The lemma treats both signs of velocity and normalises the coefficient bounds uniformly over the intervals we use.

We fix a non-zero reference velocity $\bar v$. We use three nested intervals: the estimate is wanted on the smallest one, visits begin at the middle one, and the equation is solved with zero values at the boundary of the largest one. Their separation leaves room for diffusion before a visit ends at the outer boundary. In the new coordinates below, σ is rescaled position, used as the new time, and z measures displacement from transport at velocity $\bar v$. Figure 2 shows the change of variables for one visit.

Lemma 8.2 (Position as time)**.** *Let* $\bar v \ne 0$ *and* $0 < r \le \frac{2}{3}|\bar v|$, *and put*

$$I_{\mathrm{c}} = \bar v + r\big[-\tfrac14, \tfrac14\big], \qquad I = \bar v + r\big(-\tfrac12, \tfrac12\big), \qquad I^* = \bar v + r\big(-\tfrac34, \tfrac34\big), \qquad J^* = \big(-\tfrac34, \tfrac34\big). \tag{8.5}$$

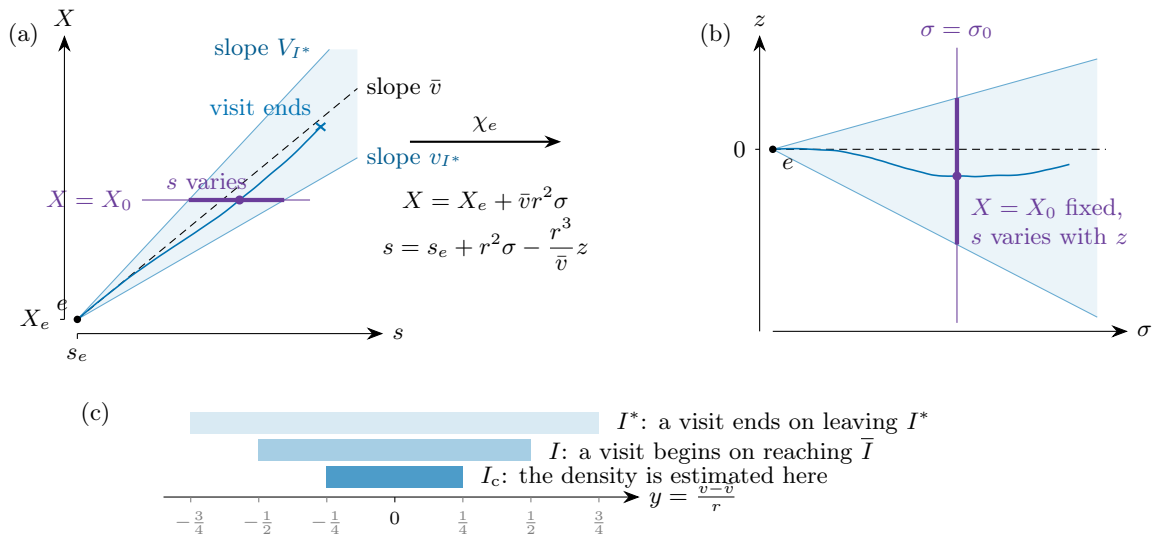

FIGURE 2. Position as time (Lemma 8.2). **(a)** A visit to $I^* \subset (0,\infty)$ in the (s,X) plane. While $v \in I^*$, $v_{I^*}(s-s_e) \le X - X_e \le V_{I^*}(s-s_e)$. Position increases throughout the visit, so each position level reached before exit is crossed exactly once. **(b)** The same visit in the coordinates (8.6). The level $X = X_0$ becomes the slice $\sigma = \sigma_0$; by (8.11), position is fixed along it while the old time $s = s_e + r^2\sigma_0 - \frac{r^3}{\bar v} z$ varies. The autonomous coefficient therefore does not depend on z, see (8.9), and $\mathcal{K}_e$ is translation invariant in z. **(c)** The intervals (8.5) in the normalised velocity $y = \frac{v-\bar v}{r}$, and their roles.

Every $v \in I^*$ has the sign of $\bar v$ and satisfies $\frac{1}{2}|\bar v| \le |v| \le \frac{3}{2}|\bar v|$. Let $e = (s_e, X_e, v_e)$ with $v_e \in I^*$, and define

$$\chi_e(s,X,v) = (\sigma, z, y), \qquad \sigma = \frac{X - X_e}{\bar v r^2}, \qquad z = \frac{X - X_e - \bar v(s-s_e)}{r^3}, \qquad y = \frac{v-\bar v}{r}\,. \tag{8.6}$$

Then χ_e is a diffeomorphism of $\mathbb{R}\times\mathbb{R}\times I^*$ onto $\mathbb{R}\times\mathbb{R}\times J^*$ with

$$\mathrm{d}s\,\mathrm{d}X\,\mathrm{d}v = r^6\,\mathrm{d}\sigma\,\mathrm{d}z\,\mathrm{d}y\,, \tag{8.7}$$

and for every smooth h on $\mathbb{R}\times\mathbb{R}\times J^*$,

$$\mathcal{K}_a(h\circ\chi_e) = \frac{|v|}{|\bar v| r^2}\,(\mathcal{K}_e h)\circ\chi_e, \qquad \mathcal{K}_e = \partial_\sigma + B_e(\sigma,y)\partial_{yy} + b_e(y)\partial_z\,, \tag{8.8}$$

where

$$B_e(\sigma,y) = \frac{|\bar v|\,a(X_e + \bar v r^2\sigma, \bar v + ry)}{|\bar v + ry|}, \qquad b_e(y) = \frac{|\bar v|\,y}{|\bar v + ry|} \qquad \text{for } |y|<1\,. \tag{8.9}$$

The functions B_e and b_e extend to smooth functions on $\mathbb{R}\times\mathbb{R}$ and on $\mathbb{R}$ with

$$\tfrac{3}{5}\lambda \le B_e \le 3\Lambda, \qquad \tfrac{9}{25} \le b_e' \le 9\,, \tag{8.10}$$

so that (J^*, b_e, B_e) is of type (I) with structural constants depending only on λ and Λ. Here the normalised velocity y plays the role of v in Section 2.

Proof. The inverse change of variables is

$$X = X_e + \bar v r^2\sigma, \qquad s = s_e + r^2\sigma - \frac{r^3}{\bar v} z, \qquad v = \bar v + ry. \tag{8.11}$$

Its Jacobian has absolute value r^6. Since $r \le \frac{2}{3}|\bar v|$, the velocity $\bar v + ry$ has the sign of $\bar v$ for $|y| < \frac{3}{2}$, and the asserted velocity bounds hold on I^*. The chain rule gives

$$\mathcal{K}_a(h \circ \chi_e) = \frac{v}{\bar v r^2}(\partial_\sigma h) \circ \chi_e + \frac{v - \bar v}{r^3}(\partial_z h) \circ \chi_e + \frac{a}{r^2}(\partial_{yy} h) \circ \chi_e.$$

Factoring out $\frac{|v|}{|\bar v| r^2}$ gives (8.8). The crucial point is visible in (8.11): X depends only on σ, whereas s also depends on z. Autonomy therefore makes B_e independent of z.

On $|y| < 1$,

$$\frac{3}{5}\lambda \le B_e \le 3\Lambda, \qquad b_e'(y) = \frac{\bar v^2}{(\bar v + ry)^2} \in \left[\frac{9}{25}, 9\right].$$

We choose $0 \le \theta \in C_c^\infty((-1,1))$, equal to one on $[-\frac{3}{4}, \frac{3}{4}]$. The extensions

$$\theta B_e + (1-\theta)\lambda, \qquad y \longmapsto \int_0^y \big(\theta(\zeta) b_e'(\zeta) + 1 - \theta(\zeta)\big)\, \mathrm{d}\zeta$$

agree with the original coefficients on J^*, since $b_e(0) = 0$, and preserve (8.10) by convexity. □

Proposition 8.3 (The Green measure after the change of variables)**.** *In the setting of Lemma 8.2, let Γ_e be the Green measure of $\mathcal{K}_a$ on $\mathscr{S}_{I^*} = (s_e, \infty) \times \mathbb{R} \times I^*$ from e, given by Proposition 8.1 with $s_- = s_e$ and $s_+ = \infty$, and let $\widetilde\Gamma_e$ be the Green measure (2.6) of $\mathcal{K}_e$ on $\mathbb{R}_\sigma \times J^* \times \mathbb{R}_z$ for the point mass at $(y,z) = \big(\frac{v_e - \bar v}{r}, 0\big)$ prescribed at time $\sigma = 0$, up to elapsed time ∞, regarded as a measure on $(0,\infty)_\sigma \times \mathbb{R}_z \times J_y^*$. Then Γ_e is concentrated on $\{\operatorname{sgn}(\bar v)(X - X_e) > 0\}$ and*

$$(\chi_e)_\#\big(|v|\,\Gamma_e\big) = |\bar v|\, r^2\, \widetilde\Gamma_e\,. \tag{8.12}$$

Consequently Γ_e has a density G_e with respect to $\mathrm{d}s\,\mathrm{d}X\,\mathrm{d}v$, and for $1 < q < \frac{3}{2}$,

$$\|G_e\|_{L^q((s_e,\infty)\times\mathbb{R}\times I^*)} \le C_{\lambda,\Lambda,q}\, r^{\frac{6}{q}-4}. \tag{8.13}$$

Proof. The cone bound of Proposition 8.1 gives $\operatorname{sgn}(\bar v)(X - X_e) \ge \frac{1}{2}|\bar v|(s - s_e)$ on the support of Γ_e. Thus χ_e carries it into $\{\sigma > 0\}$.

We take $0 \le F \in C_c^\infty((0,\infty) \times \mathbb{R} \times J^*)$, vanishing for $\sigma \ge \sigma_1$. We write h for its Duhamel potential for $\mathcal{K}_e$. By Lemma 2.3, h is smooth, non-negative, solves $\mathcal{K}_e h = -F$, vanishes for $\sigma \ge \sigma_1$, and has zero velocity-boundary values. It is bounded uniformly over all starting points by Theorem 2.4(ii) and Hölder's inequality, using any fixed exponent in $(1, \frac{3}{2})$.

We put $\phi = h \circ \chi_e$. It belongs to the test class of Proposition 8.1, and

$$\mathcal{K}_a \phi = -\frac{|v|}{|\bar v| r^2}\, F \circ \chi_e.$$

We apply the Green identity on the strip with terminal time $s_+ \ge s_e + 2r^2\sigma_1$, and denote its Green measure by Γ_{e,s_+}. Its velocity interval is I^*. The exit term is zero: ϕ vanishes on the velocity faces, and on the support of the terminal exit measure the cone gives $\sigma \ge \sigma_1$. Hence, with $y_e = \frac{v_e - \bar v}{r}$,

$$|\bar v| r^2 \int F\, \mathrm{d}\widetilde\Gamma_e = |\bar v| r^2 h(0,0,y_e) = \int |v|\,(F \circ \chi_e)\, \mathrm{d}\Gamma_{e,s_+}.$$

We let $s_+ \uparrow \infty$. Both limiting measures are finite by (8.3) and Theorem 2.4(ii); equality against all such F therefore proves (8.12).

If $\widetilde G_e$ is the density of $\widetilde\Gamma_e$, the pushforward identity and the Jacobian (8.7) give

$$G_e = \frac{|\bar v|}{|v|} r^{-4} \widetilde G_e \circ \chi_e, \qquad \int |G_e|^q \le 2^q r^{6-4q} \int |\widetilde G_e|^q.$$

Theorem 2.4(ii) now gives (8.13). □

The estimate for one starting point does not distinguish visits beginning at different times. The next corollary keeps track of these times. The quadratic velocity barrier makes the mass remaining in I^* decay exponentially on the time scale r^2. Thus visits beginning far apart in time contribute little to the same part of the Green density.

Corollary 8.4 (Starting at different times)**.** *In the setting of Lemma 8.2, let ν be a finite positive Borel measure supported in $\mathbb{R}\times\mathbb{R}\times I^*$, and let $\Gamma_\nu = \int \Gamma_e \,\mathrm{d}\nu(e)$, where Γ_e is as in Proposition 8.3. We put $m_j = \nu((jr^2,(j+1)r^2]\times\mathbb{R}\times I^*)$. Then Γ_ν has a density G_ν, and for $1<q<\frac{3}{2}$,*

$$\|G_\nu\|_{L^q}^q \le C_{\lambda,\Lambda,q}\, r^{6-4q} \sum_{j\in\mathbb{Z}} m_j^q \,. \tag{8.14}$$

In particular,

$$\|G_\nu\|_{L^q}^q \le C_{\lambda,\Lambda,q}\, r^{6-4q} \nu(\mathbb{R}^3) \sup_{\tau\in\mathbb{R}} \nu\big((\tau,\tau+r^2]\times\mathbb{R}\times I^*\big)^{q-1} \,. \tag{8.15}$$

The norm is over $\mathbb{R}\times\mathbb{R}\times I^*$.

Proof. By Proposition 8.1 and monotone convergence in the terminal time, $e\mapsto\Gamma_e(E)$ is Borel for every Borel set E. Integrating (8.13) against ν and using duality first gives a density with $\|G_\nu\|_{L^q}\le Cr^{\frac{6}{q}-4}\nu(\mathbb{R}^3)$. We improve this bound by separating the starting times.

We denote by $M_e(t)$ the mass of the kernel killed at ∂I^* at time s_e+t, issued from e. It is non-increasing in t, and the quadratic bound (8.3) gives $\int_0^\infty M_e(t)\,\mathrm{d}t\le Cr^2$. Thus $M_e(2Cr^2)\le\frac{1}{2}$ uniformly over all starting points. Composition of the killed kernels gives $M_e(t)\le C\exp(-c\frac{t}{r^2})$.

We write $\mu_{e,t}$ for the remaining terminal measure, placed on the time slice $s=s_e+t$; its mass is $M_e(t)$. Integrating the kernel composition identity in the later time gives

$$\mathbf{1}_{\{s\ge s_e+t\}}\Gamma_e = \int \Gamma_{e'}\,\mathrm{d}\mu_{e,t}(e') \,.$$

We apply (8.13) to each starting point e' and integrate against $\mu_{e,t}$ to obtain

$$\|\mathbf{1}_{\{s\ge s_e+t\}}G_e\|_{L^q} \le C\exp\left(-c\frac{t}{r^2}\right) r^{\frac{6}{q}-4} \,.$$

The velocity at the restarted point may lie anywhere in I^*, which is why Proposition 8.3 was stated on that whole interval.

We partition time into $J_j=(jr^2,(j+1)r^2]$, $j\in\mathbb{Z}$, and put $m_j=\nu(J_j\times\mathbb{R}\times I^*)$. On J_i, only starting points in J_j with $j\le i$ contribute; those from much earlier intervals have exponentially small remaining mass. The preceding bound therefore gives

$$\|\mathbf{1}_{J_i\times\mathbb{R}\times I^*}G_\nu\|_{L^q} \le Cr^{\frac{6}{q}-4}\sum_{j\le i} e^{-c(i-j-1)_+}m_j \,.$$

The time intervals are disjoint and the geometric sequence is summable. Young's inequality for sequences therefore gives

$$\|G_\nu\|_{L^q}^q \le Cr^{6-4q}\sum_j m_j^q \le Cr^{6-4q}(\sup_j m_j)^{q-1}\nu(\mathbb{R}^3) \,,$$

which proves (8.15). □

8.2. **The Green measure of the velocity strip.** We fix $Z_0 = (s_0, X_0, v_0)$ and $R > 0$, and write $\mathcal{Q} = \mathcal{Q}_R(Z_0)$ for the forward cylinder (6.1), $J_Q = (v_0 - R, v_0 + R)$ for its velocity interval, and $\mathscr{S} = \mathscr{S}_{J_Q}$ for the strip with $s_- = s_0$ and $s_+ = s_0 + R^2$, so that $\mathcal{Q} \subset \mathscr{S}$. We retain the domain superscript in $\Gamma_P^{J_Q}$ and $\omega_P^{J_Q}$. We use $P \in \mathcal{Q}$ for a starting point of these two measures, and e for a starting point of the all-time measures of Proposition 8.3.

At every time the killed kernel has mass at most one. Its Green measure therefore satisfies $\Gamma_P^{J_Q}(J \times \mathbb{R} \times J_Q) \le |J|$ for every time interval J. We will use this bound to count visits beginning during a short interval, as well as over the whole time interval.

We also need to control how much time is spent in a narrow velocity interval. A convex function whose second derivative is concentrated on that interval gives the following bound. In particular, the Green measure cannot put mass on the slice $v = 0$, where position cannot serve as time.

Lemma 8.5 (Velocity occupation of the strip measure)**.** *For every $P \in \mathcal{Q}$ and every bounded interval $E \subset \mathbb{R}$,*

$$\Gamma_P^{J_Q}\big(\mathscr{S} \cap \{v \in E\}\big) \le \frac{2R}{\lambda}|E|\,. \tag{8.16}$$

In particular $\Gamma_P^{J_Q}(\{v = v_1\}) = 0$ for every $v_1 \in \mathbb{R}$.

Proof. For $|E| > 0$, we choose $0 \le \chi \in C_c^\infty(\mathbb{R})$ with $\chi \ge \mathbf{1}_E$ and $\int \chi \le 2|E|$. The smooth function $\Phi(v) = \int |v - \zeta|\chi(\zeta)\,\mathrm{d}\zeta$ satisfies $\Phi'' = 2\chi$ and $\|\Phi'\|_\infty \le 2|E|$. Both Φ and $\mathcal{K}_a\Phi = 2a\chi$ are bounded on the strip, so Proposition 8.1(i) gives

$$2\lambda\Gamma_P^{J_Q}(\mathscr{S} \cap \{v \in E\}) \le \int 2a\chi\,\mathrm{d}\Gamma_P^{J_Q} = \int \big(\Phi(v) - \Phi(v_P)\big)\,\mathrm{d}\omega_P^{J_Q} \le 4R|E|,$$

since the exit measure has mass one and its velocities lie in the interval $\overline{J_Q}$ of length $2R$. This proves (8.16); shrinking E to a point gives the final assertion. □

The starting velocity may be outside I, and mass may leave and return repeatedly. We alternate evolution inside I^* with evolution outside $\overline{I}$. The measure below records where and when each visit to $\overline{I}$ begins, including the initial point if it already lies in $\overline{I}$. A new visit is counted only after the mass has left I^* and returned to $\overline{I}$. The gap between the two boundaries prevents an exit from being counted immediately as a return. Figure 3 illustrates the decomposition.

Lemma 8.6 (Counting visits to a velocity interval)**.** *Let $P \in \mathcal{Q}$ and let $I_c \subset I \subset I^*$ be as in Lemma 8.2 with $I_c \cap J_Q \neq \emptyset$. We put $H_1 = I^* \cap J_Q$ and $H_2 = J_Q \setminus \overline{I}$, and let*

$$\Sigma_{\mathrm{in}} = (s_0, s_0 + R^2) \times \mathbb{R} \times (\partial I \cap J_Q), \qquad \Sigma_{\mathrm{out}} = (s_0, s_0 + R^2) \times \mathbb{R} \times (\partial I^* \cap J_Q)\,.$$

For an empty strip, the integrated Green and exit measures below are understood to be zero. We define finite positive measures γ_k, β_k ($k \ge 1$) by

$$\gamma_1 = \begin{cases} \delta_P & \text{if } v_P \in \overline{I}, \\ \omega_{\delta_P}^{H_2}|_{\Sigma_{\mathrm{in}}} & \text{otherwise,} \end{cases} \qquad \beta_k = \omega_{\gamma_k}^{H_1}|_{\Sigma_{\mathrm{out}}}, \qquad \gamma_{k+1} = \omega_{\beta_k}^{H_2}|_{\Sigma_{\mathrm{in}}}\,. \tag{8.17}$$

Then $\gamma_I = \sum_{k \ge 1} \gamma_k$, which counts all visits, is a finite measure on $\{v \in \overline{I}\}$ with

$$\gamma_I(\mathscr{S}) \le C_{\lambda,\Lambda}\Big(1 + \frac{R}{r}\Big)\,. \tag{8.18}$$

For every time interval J of length r^2 it also satisfies

$$\gamma_I\big(J \times \mathbb{R} \times J_Q\big) \le C_{\lambda,\Lambda}\,. \tag{8.19}$$

Moreover,

$$\Gamma_P^{J_Q}|_{\{v \in I_c\}} \le \Gamma_{\gamma_I}|_{\{v \in I_c\}}\,, \tag{8.20}$$

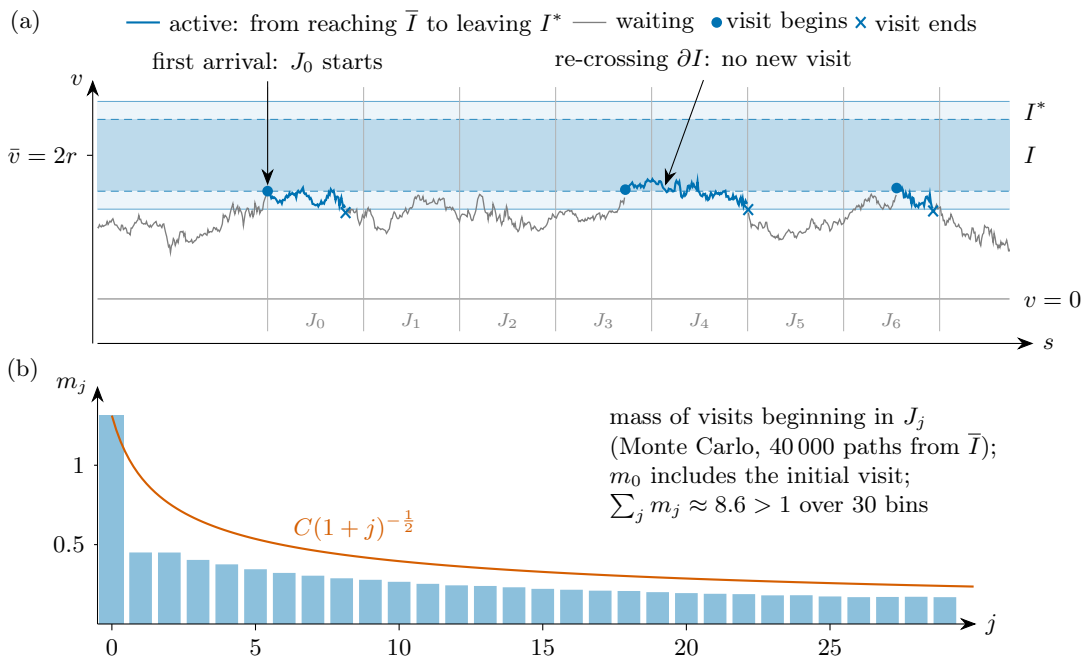

FIGURE 3. Counting visits (Lemmas 8.6 and 8.8), with $\bar{v} = 2r$. **(a)** One velocity path, drawn with a slower diffusion for legibility. A visit begins when the velocity reaches $\overline{I}$ and ends when it leaves I^*; re-crossing ∂I during a visit does not begin another. The time bins J_j, of length r^2, are counted from the first arrival in $\overline{I}$. **(b)** The mass m_j of visits beginning in J_j, estimated from 40 000 paths started in $\overline{I}$, for one rough autonomous coefficient. The orange reference curve has C chosen to bound the plotted Monte Carlo estimates. The masses count repeated visits, so their sum exceeds one. The bound $m_j \le C(1+j)^{-\frac{1}{2}}$ uses the kinetic Harnack comparison of Lemma 8.7, which rests on the preliminary $p = 6$ estimate; this time-only count gives Proposition 8.9, hence $p > 4$.

where Γ_{γ_I} *is the measure of Corollary 8.4. Consequently* $\Gamma_P^{J_Q}|_{\{v\in I_c\}}$ *has a density* $G_{P,I}$ *with*

$$\|G_{P,I}\|_{L^q(\mathscr{S})} \le C_{\lambda,\Lambda,q}\, r^{\frac{6}{q}-4}\Big(1+\frac{R}{r}\Big)^{\frac{1}{q}}, \qquad 1 < q < \frac{3}{2}. \tag{8.21}$$

Proof. The internal boundaries of H_1 and H_2 in J_Q are respectively Σ_{out} and Σ_{in}. The measures γ_k are supported in H_1, and β_k in H_2. We apply Proposition 8.1(v) alternately with inner domain H_1 or H_2 and outer domain J_Q; the two inner domains need not be nested in each other.

If $v_P \notin \overline{I}$, put $\mu_0 = \Gamma_{\delta_P}^{H_2}$ and $\omega_0 = \omega_{\delta_P}^{H_2}|_{\partial^\dagger\mathscr{S}}$; otherwise put both equal to zero. Applying the nested identity alternately on H_1 and H_2 gives

$$\Gamma_P^{J_Q} = \mu_0 + \sum_{k=1}^{N}(\Gamma_{\gamma_k}^{H_1} + \Gamma_{\beta_k}^{H_2}) + \Gamma_{\gamma_{N+1}}^{J_Q}, \tag{8.22}$$

$$\omega_P^{J_Q} = \omega_0 + \sum_{k=1}^{N}(\omega_{\gamma_k}^{H_1} + \omega_{\beta_k}^{H_2})|_{\partial^\dagger\mathscr{S}} + \omega_{\gamma_{N+1}}^{J_Q}. \tag{8.23}$$

Positivity and $\omega_P^{J_Q}(\partial^\dagger\mathscr{S}) = 1$ imply

$$\sum_{k=1}^{N}\omega_{\gamma_k}^{H_1}(\partial^\dagger\mathscr{S}) \le 1, \qquad \sum_{k=1}^{N}\Gamma_{\gamma_k}^{H_1} \le \Gamma_P^{J_Q}|_{\{v\in I^*\}}. \tag{8.24}$$

To bound the total mass of these visits, we write $I^* = (v_-, v_+)$ and test each H_1-identity with $\phi_I(v) = (v - v_-)(v_+ - v)$. This function vanishes on the internal exit boundary, satisfies $\mathcal{K}_a\phi_I = -2a$, is at least $\frac{5}{16}r^2$ on $\overline{I}$, and is at most $\frac{9}{16}r^2$ on $\overline{I^*}$. Summing the identities and using (8.24) gives

$$\frac{5}{16}r^2\sum_{k=1}^{N}\gamma_k(\mathscr{S}) \le \frac{9}{16}r^2 + 2\Lambda\,\Gamma_P^{J_Q}(\mathscr{S}\cap\{v\in I^*\}) \le \frac{9}{16}r^2 + \frac{6\Lambda}{\lambda}Rr,$$

where the last inequality is Lemma 8.5. Monotone convergence proves (8.18).

To count visits beginning in a time interval J of length r^2, we choose $0 \le \eta \le 1$ smooth, equal to one on J, supported in an interval of length $3r^2$, and with $|\eta'| \le Cr^{-2}$. We test the same identities with $\eta(s)\phi_I(v)$. Internal exit terms vanish because $\phi_I = 0$ on ∂I^*; the sum of the outer exit terms, including any terminal face or truncation by J_Q, is at most Cr^2 by (8.24). Hence

$$\frac{5}{16}r^2\sum_{k=1}^{N}\gamma_k(J\times\mathbb{R}\times J_Q) \le Cr^2 + \int_{\mathscr{S}}\big(|\eta'|\phi_I + 2a\eta\big)\,\mathrm{d}\Big(\sum_{k=1}^{N}\Gamma_{\gamma_k}^{H_1}\Big) \le Cr^2\,.$$

For the last bound we use $|\eta'|\phi_I + 2a\eta \le C\mathbf{1}_{\operatorname{supp}\eta}$ on the active strips, their domination by $\Gamma_P^{J_Q}$, and the time-marginal bound above. Letting $N\to\infty$ proves (8.19); a possible initial atom $\gamma_1 = \delta_P$ is included on the left.

Finally, the remainder in (8.22) has mass at most $R^2\gamma_{N+1}(\mathscr{S}) \to 0$. The terms μ_0 and $\Gamma_{\beta_k}^{H_2}$ give no mass to I_{c}, so

$$\Gamma_P^{J_Q}|_{\{v\in I_{\mathrm{c}}\}} = \sum_{k\ge1}\Gamma_{\gamma_k}^{H_1}|_{\{v\in I_{\mathrm{c}}\}}.$$

We enlarge H_1 to I^* and the terminal time to infinity. Monotonicity gives (8.20). We apply Corollary 8.4 with $\nu = \gamma_I$: the total mass is bounded by (8.18), while its mass on each time interval of length r^2 is bounded by (8.19). This gives (8.21). □

8.3. **Harnack improves the Aleksandrov estimate.** We now use the regularity implied by the preliminary estimate to improve the estimate itself. The order of the argument is

$$\begin{aligned}\text{Aleksandrov at } p=6 &\implies \text{Hölder continuity and a kinetic Harnack comparison}\\ &\implies \text{decay of returns in elapsed time}\\ &\implies \text{Aleksandrov for every } p>4.\end{aligned}$$

The preceding estimates already give the starting point, a preliminary Aleksandrov principle at $p = 6$. Indeed, at $q = \frac{6}{5}$, Lemma 8.6 bounds the density on a core of scale $r \le CR$ by $CR^{\frac{5}{6}}r^{\frac{1}{6}}$. The continuous covering (8.32) below therefore gives

$$CR^{\frac{5}{6}}\int_0^{CR} r^{\frac{1}{6}}\,\frac{\mathrm{d}r}{r} \le CR.$$

There is no mass at $v = 0$ by Lemma 8.5; cylinders with $|v_0| > 8R$ are covered by a single interval in Proposition 8.3. Proposition 6.2 thus gives the localised maximum principle at $p = 6$. Applied to $\psi - u$ and $\psi + u$ for a homogeneous solution, it verifies (9.6). We may consequently use the conditional Hölder theorem and propagation lemma of Section 9, with constants depending only on λ, Λ. This uses only the preliminary estimate, before either improvement.

We write S_t for the positive full-space evolution with generator $v\partial_X + a(X,v)\partial_{vv}$. The terminal kernels on $(-M, M)$ from Proposition 8.1 increase with M. For a starting point with velocity v_e, testing with $(v - v_e)^2$ bounds the mass lost before time T by $\frac{2\Lambda T}{(M-|v_e|)^2}$. Their limit therefore has mass one. The bounded-limit and comparison argument of Appendix A gives its classical solutions and composition. Autonomy then gives $\partial_t(\mathsf{S}_tF) = v\partial_X(\mathsf{S}_tF) + a\partial_{vv}(\mathsf{S}_tF)$ for bounded smooth F. Position reflection, $u(t,x,v) = \mathsf{S}_tF(-x,v)$, gives $\partial_t u + v\partial_x u - a(-x,v)\partial_{vv}u = 0$, the convention of Section 9.

Here we write functions in the physical coordinate order (X, v). In terms of the full-space kernel constructed above,

$$\mathsf{S}_t F(X,v) = \int F(X',w)\, K_{0,t}\big((v,X),\, \mathrm{d}w\, \mathrm{d}X'\big).$$

For a non-negative function F, this is its integral against the kernel at elapsed time t. If F is the indicator of a velocity interval, it is the mass in that interval. The next lemma bounds it by a fixed multiple of its value at the same point at any later time in $[2t, 3t]$. Averaging will then turn a time-integrated occupation estimate into a bound at time t.

Lemma 8.7 (A return-time comparison)**.** *For every bounded non-negative Borel function F on $\mathbb{R}^2$,*

$$\mathsf{S}_t F(X,v) \le C_{\lambda,\Lambda} \mathsf{S}_s F(X,v), \qquad 2t \le s \le 3t, \quad |v| \le \sqrt{t}. \tag{8.25}$$

Proof. The argument has three parts. Propagation carries a positive lower bound from a small neighbourhood to a later point. Hölder continuity supplies that neighbourhood from a positive value. An iteration removes the larger supremum that occurs in the Hölder estimate.

A small positive neighbourhood reaches a later point. We work in the reflected position coordinate, with a non-negative solution u of $\mathcal{L}_a u = 0$. We suppose $u \ge k$ on $Q_\ell^-(z)$, where $z = (\tau, x_c, v_c)$ lies in a fixed bounded region. After scaling by ℓ and following the transport of its centre, a cubic position path joins this patch to any point of

$$Q_{2\ell}^-(\tau + 8\ell^2, x_c + 8\ell^2 v_c, v_c).$$

The travel times lie between $4\ell^2$ and $8\ell^2$, and the normalised endpoints and accelerations are bounded. On the preceding interval of length $\frac{1}{4}\ell^2$, we take the path to follow the centre of $Q_\ell^-(z)$, with tube widths $\frac{1}{2}\ell^3$ and $\frac{1}{2}\ell$. The initial tube is then contained in that cylinder. Lemma 9.3, with $T_0 = \frac{1}{4}\ell^2$ and $T_1 = 8\ell^2$, gives a fixed factor $c_* > 0$ throughout the later cylinder.

We repeat with radii $\ell, 2\ell, 4\ell, \dots$ until reaching a fixed small radius. The elapsed time is bounded by a constant times the square of that final radius, while the number of steps is at most $C + \log_2 \frac{1}{\ell}$. One last path of fixed width reaches any prescribed point z_* in a bounded future region separated by a fixed positive time. Choosing the final radius small enough leaves time for this last path. Thus, for some $\beta > 0$ depending only on ellipticity and the fixed regions,

$$u(z_*) \ge c\ell^\beta k. \tag{8.26}$$

All tubes lie in full space. Each doubling loses only a fixed factor in height, and the number of doublings grows like $\log \frac{1}{\ell}$, so the total loss is a power of ℓ. This is why we enlarge the neighbourhood as we propagate it; keeping its width equal to ℓ for the whole time would give a much weaker bound.

Hölder continuity supplies the neighbourhood. We first take F smooth, scale t to one and translate X to zero, without shifting velocity. For the reflected solution u, fix the later point $z_* = (s, 0, v)$ with $2 \le s \le 3$ and $|v| \le 1$. We choose nested open boxes U_R, $\frac{1}{2} \le R \le 1$, around $(1, 0, v)$, whose times lie in $(\frac{15}{16}, \frac{17}{16})$ and whose position and velocity radii are small fixed multiples of R. They can be chosen so that $Q_{2h}^-(z) \subset U_b$ for $z \in U_a$, where $h = c(b-a)$ and $a < b$. We set $M_R = \sup_{U_R} u$.

The size of a neighbourhood on which u stays above half its value at z depends on that value relative to the nearby supremum M_b. Precisely, Theorem 9.1 gives $u \ge \frac{1}{2}u(z)$ on $Q_\ell^-(z)$ for

$$\ell = ch \left(\frac{u(z)}{M_b} \right)^{\frac{1}{\alpha}},$$

provided $M_b > 0$. Taking c small makes the required surrounding cylinders lie in U_b. Applying (8.26) to this neighbourhood and taking points with $u(z)$ approaching M_a gives

$$u(z_*) \ge ch^\beta M_a^{1+\frac{\beta}{\alpha}} M_b^{-\frac{\beta}{\alpha}}.$$

Removing the larger supremum. The last inequality still contains M_b, because Hölder continuity was measured relative to that supremum. We rearrange it and use Young's inequality, with $\theta = \frac{\alpha}{\alpha+\beta}$, to obtain

$$M_a \le Ch^{-\beta\theta}u(z_*)^\theta M_b^{1-\theta} \le \varepsilon M_b + C_\varepsilon h^{-\beta}u(z_*).$$

We set $a_j = 1 - 2^{-j-1}$, so $a_0 = \frac{1}{2}$ and $h_j = c(a_{j+1} - a_j) = c2^{-j-2}$. Repeated substitution gives

$$M_{\frac{1}{2}} \le \varepsilon^N M_{a_N} + C_\varepsilon u(z_*) \sum_{j=0}^{N-1} (\varepsilon 2^\beta)^j.$$

We choose $\varepsilon 2^\beta < 1$ and let $N \to \infty$. Boundedness of u removes the first term, and the sum stays bounded. This proves $u(1,0,v) \le Cu(z_*)$. Scaling back proves (8.25). Approximation of the finite terminal measures extends the inequality from smooth non-negative F to bounded non-negative Borel F. □

We use this comparison only for the full-space evolution, not for a kernel killed at the boundary of a one-sign velocity interval. The latter cannot in general return to the same position at a later time.

Lemma 8.8 (The improved density bound)**.** *In Lemma 8.6, suppose in addition that $|\bar v| = 2r$ and $r \le 6R$. Then*

$$\|G_{P,I}\|_{L^q(\mathscr{S})} \le C_{\lambda,\Lambda,q} R^{\frac{2}{q}-1} r^{\frac{4}{q}-3}, \qquad 1 < q < \frac{3}{2}. \tag{8.27}$$

Proof. By autonomy, we shift the starting time of P to zero. The remaining observation interval has length at most R^2.

The barrier gives a bound integrated in time. We put $\psi_r(v) = \sqrt{r^2 + v^2}$. This is a smooth convex approximation to $|v|$, and $a\psi_r'' \ge \frac{c\lambda}{r}$ on $|v| \le 3r$. The Green identity bounds occupation of this interval by the increase of ψ_r. Since ψ_r has Lipschitz constant one, the quadratic velocity estimate gives, for every starting point $z_e = (X_e, v_e)$,

$$\int_0^T \mathsf{S}_t \mathbf{1}_{\{|v|\le 3r\}}(z_e)\,\mathrm{d}t \le Cr\big(\mathsf{S}_T\psi_r(z_e) - \psi_r(v_e)\big) \le Cr\sqrt{\mathsf{S}_T|v - v_e|^2(z_e)} \le Cr\sqrt{T}.$$

We apply the tests first on $(-M, M)$. The quadratic test gives $\mathsf{S}_T|v - v_e|^2(z_e) \le 2\Lambda T$ in the limit. For ψ_r, the contribution of the distant exit boundary tends to zero: its value is $O(M)$ and its mass is $O(M^{-2})$. This justifies the whole-space Green identity used above.

Harnack gives a bound at each elapsed time. The integrated estimate alone allows the mass to be large on a short time interval. The comparison (8.25) gives the extra control we need. For $|v_e| \le 3r$ and $t \ge 9r^2$, we have $|v_e| \le \sqrt{t}$ and may average that comparison over $s \in [2t, 3t]$. Since $I^* \subset \{|v| \le 3r\}$,

$$\begin{aligned}\mathsf{S}_t\mathbf{1}_{I^*}(z_e) &\le \frac{C}{t}\int_{2t}^{3t} \mathsf{S}_s\mathbf{1}_{I^*}(z_e)\,\mathrm{d}s \\ &\le \frac{C}{t}\int_0^{3t} \mathsf{S}_s\mathbf{1}_{\{|v|\le 3r\}}(z_e)\,\mathrm{d}s \le C\frac{r}{\sqrt{t}}.\end{aligned}$$

Together with the mass bound for $t < 9r^2$, this proves

$$\mathsf{S}_t\mathbf{1}_{I^*}(z_e) \le C\left(1 + \frac{t}{r^2}\right)^{-\frac{1}{2}}. \tag{8.28}$$

From mass present to visits beginning. We choose a bounded velocity interval J containing $\overline{J_Q} \cup \overline{I^*}$ in its interior. We apply the alternating construction of Lemma 8.6 with outer interval J, active interval I^* and waiting set $J \setminus \overline{I}$. All measures are supplied by Proposition 8.1; no unbounded waiting domain is needed. As in that construction, γ_k records where and when the kth visit begins; we put $\nu = \sum_{k\ge1}\gamma_k$ for the measure counting all visits in this enlarged strip. For the first N active pieces, let $\mu_b^{(N)}$ be

the sum of their terminal measures at time b, and let $\Gamma^{(N)}_{\mathrm{act}}$ be the sum of their occupation measures. Repeated use of Proposition 8.1 (v) expresses the evolution on J as these active pieces, the waiting pieces and a final continuation. Every measure is non-negative. Hence, for $f \geq 0$ and times measured from the starting point e,

$$\int f \,\mathrm{d}\mu^{(N)}_b \leq \mathsf{S}_b f(e), \qquad \Gamma^{(N)}_{\mathrm{act}} \leq \Gamma^J_e.$$

Integrating the first inequality in b also bounds active occupation by the full-space evolution. These inequalities hold before taking any infinite sum. Only a waiting piece can reach ∂J, because an active velocity exit lies on ∂I^*.

We now keep track of when each visit begins. We write $I^* = (v_-, v_+)$ and again use $\phi_I(v) = (v - v_-)(v_+ - v)$. We sum its Green identities over the active pieces, restricted to a time interval $[a, b]$ of length r^2. The test vanishes on every velocity exit; it is at least $\frac{5}{16}r^2$ at the beginning of a visit and at most $\frac{9}{16}r^2$ at time b. The active mass already present at time a has a favourable sign and may be discarded. Since $-\mathcal{K}_a\phi_I = 2a(X, v)$, this gives, with times measured from a starting point z_e,

$$\nu((a, b] \times \mathbb{R}^2) \leq C\left[\mathsf{S}_b \mathbf{1}_{I^*}(z_e) + r^{-2}\int_a^b \mathsf{S}_t \mathbf{1}_{I^*}(z_e)\,\mathrm{d}t\right]. \tag{8.29}$$

One can first sum finitely many pieces. The same test over the whole observation interval, and the integrated velocity bound above, bound the total mass beginning these visits by $C(1+\frac{R}{r})$, independently of J. Monotone convergence justifies the full sum, and this finiteness makes the remainder in the alternating Green identity vanish, as in Lemma 8.6. The initial visit is counted separately. Crucially, no constant is added on each later time interval in (8.29).

We use $J_0 = [0, r^2]$ and $J_j = (jr^2, (j+1)r^2]$ for $j \geq 1$, and put $m_j = \nu(J_j \times \mathbb{R} \times \overline{I})$. We first start in $\overline{I}$ at time zero. Then $|v_e| \leq 3r$, and (8.28)–(8.29) bound the mass beginning visits during the j-th interval of length r^2 by $C(1+j)^{-\frac{1}{2}}$. For the initial interval, we use $\mathsf{S}_t 1 \leq 1$ in (8.29) and add the unit mass of the initial visit.

Allowing any first-arrival time. If the starting velocity lies outside $\overline{I}$, the first visit may begin much later. The first-arrival measure γ_1 records where and when it begins and has total mass at most one. If the starting velocity already lies in $\overline{I}$, then $\gamma_1 = \delta_P$. We apply the preceding bound from each first-arrival point. We write $a_l = \gamma_1(J_l \times \mathbb{R} \times \overline{I})$, so $\sum_l a_l \leq 1$. An arrival in J_l shifts any later interval J_j across at most two elapsed-time bins. Consequently,

$$m_j \leq C\sum_{l \leq j} a_l (1 + j - l)^{-\frac{1}{2}}.$$

Young's inequality for this discrete convolution gives, on an observation interval of length at most R^2,

$$\sum_j m_j^q \leq C_q \sum_{0 \leq i \leq 2+\frac{R^2}{r^2}} (1+i)^{-\frac{q}{2}} \leq C_q\left(1 + \frac{R}{r}\right)^{2-q}, \qquad 1 < q < 2\,. \tag{8.30}$$

We now return to the Green density. The improvement is that we control the sum of the qth powers of the masses in successive time intervals, rather than use only their total mass. The Green measure on J equals the sum of its active pieces on I_{c}, since the waiting pieces give no mass there. It dominates the original measure on J_Q. We extend each active piece to the all-time Green measure on I^* from the start of the visit to obtain $\Gamma_P^{J_Q}|_{\{v \in I_{\mathrm{c}}\}} \leq \Gamma_\nu|_{\{v \in I_{\mathrm{c}}\}}$. Grouping the possible atom at time zero with the first interval only increases the sum of the qth powers. We apply (8.14) and (8.30) to obtain

$$\|G_{P,I}\|^q_{L^q} \leq Cr^{6-4q}\left(1 + \frac{R}{r}\right)^{2-q} \leq CR^{2-q}r^{4-3q},$$

where the last inequality uses $r \leq 6R$. Taking the q-th root proves the lemma. □

8.4. **The density on the kinetic cylinder.** Near $v = 0$, using position as time requires smaller and smaller velocity intervals. Lemma 8.8 bounds the density on a core of scale $r \le 6R$ centred at $\bar v = \pm 2r$ by $CR^{\frac{2}{q}-1} r^{\frac{4}{q}-3}$. We now integrate over these cores. The factor $r^{\frac{4}{q}-3}$ is integrable against $\frac{\mathrm{d}r}{r}$ at zero when $q < \frac{4}{3}$.

Proposition 8.9 (Green density on the kinetic cylinder)**.** *For every $P \in \mathcal{Q}$ and every $1 < q < \frac{4}{3}$, the measure $\Gamma_P^{J_Q}|_{\mathcal{Q}}$ has a density G_P with respect to $\mathrm{d}s\,\mathrm{d}X\,\mathrm{d}v$ and*

$$\|G_P\|_{L^q(\mathcal{Q})} \le C_{\lambda,\Lambda,q}\, R^{\frac{6}{q}-4}. \tag{8.31}$$

Proof. We first suppose $|v_0| \le 8R$, so $|v| < 9R$ on $\mathcal{Q}$. For $\rho > 0$ and $\varsigma = \pm 1$, we use (8.5) with $\bar v = \varsigma\rho$ and $r = \frac{\rho}{2}$. Its core is $I_{\mathrm{c}}^{\varsigma}(\rho) = \{v : \frac{7\rho}{8} \le \varsigma v \le \frac{9\rho}{8}\}$. For $\varsigma v > 0$,

$$\int_0^\infty \mathbf{1}_{I_{\mathrm{c}}^{\varsigma}(\rho)}(v)\frac{\mathrm{d}\rho}{\rho} = \int_{\frac{8\varsigma v}{9}}^{\frac{8\varsigma v}{7}} \frac{\mathrm{d}\rho}{\rho} = \log\frac{9}{7}, \tag{8.32}$$

When $|v| \le 9R$, the integral is unchanged if truncated at $L_R = \frac{72R}{7}$. For $\rho \le L_R$, Lemma 8.8 applies since $r \le \frac{36}{7}R < 6R$. For a non-negative Borel function F on $\mathcal{Q}$, extended by zero to $\mathscr{S}$, Tonelli's theorem and Hölder's inequality give, with $q' = \frac{q}{q-1}$,

$$\int_{\mathcal{Q}\cap\{\varsigma v>0\}} F\,\mathrm{d}\Gamma_P^{J_Q} \le C_{\lambda,\Lambda,q} R^{\frac{2}{q}-1}\|F\|_{L^{q'}(\mathcal{Q})} \int_0^{L_R} \rho^{\frac{4}{q}-3}\frac{\mathrm{d}\rho}{\rho} \le C_{\lambda,\Lambda,q} R^{\frac{6}{q}-4}\|F\|_{L^{q'}(\mathcal{Q})}.$$

The integral converges precisely for $q < \frac{4}{3}$. Lemma 8.5 gives zero mass to $\{v = 0\}$; summing the two signs and using duality proves (8.31).

If $|v_0| > 8R$, we instead use one triple with $\bar v = v_0$ and $r = 4R \le \frac{2}{3}|v_0|$. Its core contains J_Q and I contains $\overline{J_Q}$. Domain and time monotonicity bound $\Gamma_P^{J_Q}$ by the all-time measure on I^*, so Proposition 8.3 gives the result for every $1 < q < \frac{3}{2}$. □

8.5. **A Bellman barrier controls position as well.** The preceding argument counts when a visit begins, but integrates over all starting positions. This loses information at $q = \frac{4}{3}$: the power $4 - 3q$ in the velocity-scale sum becomes zero. We now retain the position of each visit. The extra information comes from a barrier that controls occupation of a box of width r^3 in position and r in velocity. Figure 4 compares the two counts.

We continue to use the positive full-space evolution S_t introduced before Lemma 8.7. Its generator is $L = v\partial_X + a(X,v)\partial_{vv}$, so $\mathcal{K}_a = \partial_s + L$. The following barrier works for every coefficient value in the ellipticity interval. It need not be positive.

Proposition 8.10 (A homogeneous Bellman barrier)**.** *There is $\beta_* = \beta_*(\frac{\Lambda}{\lambda}) \in [2,3)$ such that, for every $\beta_* - 2 < \alpha < 1$, there are a function $\Phi \in C^2(\mathbb{R}^2 \setminus \{0\})$ and a constant $c > 0$ satisfying*

$$\begin{aligned} \Phi(r^3X, rv) &= r^\alpha \Phi(X,v), \qquad \rho(X,v) = (X^2 + v^6)^{\frac{1}{6}}, \\ v\Phi_X + b\Phi_{vv} &\ge c\rho^{\alpha-2} \qquad \text{for every } b \in [\lambda,\Lambda]. \end{aligned} \tag{8.33}$$

The function and constants depend only on λ, Λ, α.

We seek one barrier that works for every diffusion coefficient $b \in [\lambda,\Lambda]$. Taking the least favourable coefficient gives the stationary Bellman inequality

$$v\Phi_X + \min_{\lambda\le b\le\Lambda} b\Phi_{vv} \ge c\rho^{\alpha-2}.$$

Appendix B proves existence by separation: if no such homogeneous barrier existed, there would be a positive stationary adjoint measure of degree $2+\alpha$. The definition of β_* excludes this when $2+\alpha > \beta_*$, and the strict gap $\beta_* < 3$ allows us to choose $\alpha < 1$. Testing the evolution against this barrier bounds the time spent in a small box in position and velocity. Together with the Harnack comparison and

the quadratic velocity test, this controls the mass of visits beginning in each time and position cell. Retaining where visits begin, as well as when, supplies the additional estimate needed below $p = 4$.

For the rest of the argument we fix one admissible α and set

$$\gamma = 2 - \alpha \in (1, 2), \qquad \varepsilon = 1 - \alpha = \gamma - 1 > 0. \tag{8.34}$$

Until the end of this section, constants may depend on λ, Λ, α and on the displayed integrability exponent.

The barrier extends continuously to the origin by zero. Homogeneity bounds its derivatives, and integration along the coordinate lines gives

$$|\Phi(X, v) - \Phi(X', v')| \le C\big(|X - X'|^{\frac{\alpha}{3}} + |v - v'|^{\alpha}\big). \tag{8.35}$$

Indeed, $|\Phi_X| \le C|X|^{\frac{\alpha}{3}-1}$ and $|\Phi_v| \le C|v|^{\alpha-1}$ away from the respective zero coordinates, and both singularities are integrable.

Lemma 8.11 (Occupation and concentration in a phase-space box)**.** *For fixed $A_0, M > 0$, we write*

$$\mathcal{B}_r(Y) = \{(X, v) : |X - Y| \le A_0 r^3,\ |v| \le A_0 r\}.$$

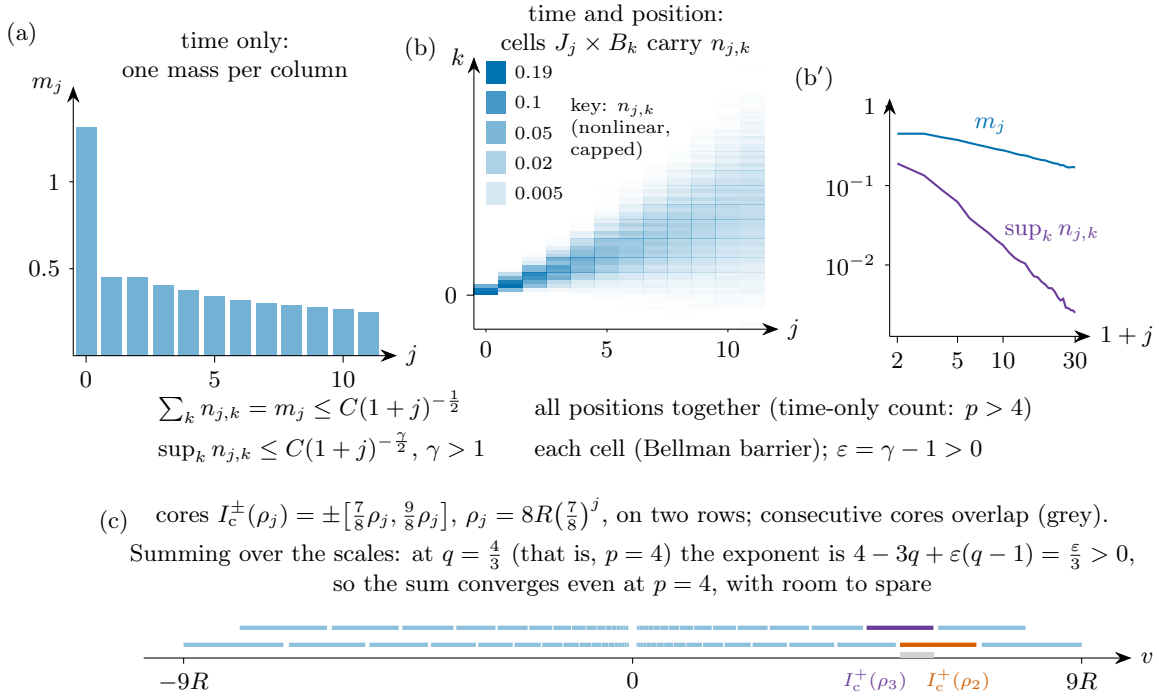

FIGURE 4. What retaining position gains (Section 8.5; Monte Carlo as in Figure 3). The preliminary $p = 6$ estimate supplies Hölder continuity and the kinetic Harnack comparison; this improves the time counts, shown in (a), and the Bellman barrier supplies the additional position bound, shown in (b). **(a)** One mass m_j per time bin. **(b)** The same bins divided into position cells B_k of width r^3; the key is nonlinear and capped. **(b′)** The column totals decay slowly, the largest single cell faster. Lemma 8.12 proves $m_j \le C(1+j)^{-\frac{1}{2}}$ and, through Proposition 8.10, $\sup_k n_{j,k} \le C(1+j)^{-\frac{\gamma}{2}}$ with $\gamma > 1$ and $\varepsilon = \gamma - 1$; these are uniform upper bounds, not the rates of this example. **(c)** The overlapping cores of Proposition 8.14. The core estimates (8.43) sum for $q < \frac{4-\varepsilon}{3-\varepsilon}$, which gives $p > 4 - \varepsilon = 3 + \alpha$; the admissible α give every $p > p_*(\kappa) = 1 + \beta_*(\kappa) < 4$. Neither the endpoint nor optimality is claimed.

Uniformly in X, Y, r, T, we have

$$\int_0^T \mathsf{S}_s \mathbf{1}_{\mathcal{B}_r(Y)}(X,v)\,\mathrm{d}s \le C r^\gamma T^{\frac{\alpha}{2}}, \qquad |v| \le M\sqrt{T}. \tag{8.36}$$

Consequently, if $|v| \le 3r$, then

$$\mathsf{S}_t \mathbf{1}_{\mathcal{B}_r(Y)}(X,v) \le C\left(1 + \frac{t}{r^2}\right)^{-\frac{\gamma}{2}}. \tag{8.37}$$

Constants may also depend on $A_0, M, \lambda, \Lambda, \alpha$.

Proof. We justify the use of the singular barrier first. We convolve only in position, with a non-negative smooth approximate identity η_δ, and put $\Phi_\delta = \eta_\delta *_X \Phi$. Appendix B verifies that the derivatives of Φ have no distribution supported at the origin. The bound

$$|\Phi_{vv}(X,v)| \le C|X|^{\frac{\alpha-2}{3}}$$

is locally integrable in X. Thus Φ_δ is smooth in position and twice continuously differentiable in velocity, including at $v = 0$. Since (8.33) holds for every scalar b, we may hold $b = a(X,v)$ fixed while convolving the shifted barrier. This gives

$$L[\Phi_\delta(X - Y, v)] \ge c(\eta_\delta *_X \rho^{-\gamma})(X - Y, v).$$

Neither a derivative of a nor a transport error appears.

For clarity, the moment bounds used to apply the Green identity can be obtained from the smooth diffusion representing the same kernel:

$$\mathrm{d}X_s = V_s\,\mathrm{d}s, \qquad \mathrm{d}V_s = \sqrt{2a(X_s, V_s)}\,\mathrm{d}B_s, \qquad (X_0, V_0) = (X, v),$$

where B_s is one-dimensional Brownian motion. Smoothness gives local existence and uniqueness, and bounded diffusion prevents explosion. Its terminal solutions agree with those of Proposition 2.1 by bounded comparison. In particular,

$$\mathbb{E}\sup_{s\le T}|V_s - v|^2 \le 8\Lambda T, \qquad \sup_{s\le T}|X_s - X| \le |v|T + T\sup_{s\le T}|V_s - v|.$$

These estimates also justify stopping on large compact sets and then removing the stops in the Green identity. The modulus (8.35), unchanged by position convolution, makes the endpoint differences uniformly integrable. We next let $\delta \downarrow 0$ and use Fatou's lemma to obtain

$$\begin{aligned} c\,\mathbb{E}\int_0^T \rho(X_s - Y, V_s)^{-\gamma}\,\mathrm{d}s &\le \mathbb{E}\Phi(X_T - Y, V_T) - \Phi(X - Y, v) \\ &\le C_M T^{\frac{\alpha}{2}}. \end{aligned}$$

At the origin the regularised integrand tends to $+\infty$; elsewhere it tends to $\rho^{-\gamma}$. On $\mathcal{B}_r(Y)$ we have $\rho \le C_{A_0} r$, which proves (8.36).

For $t \ge 9r^2$, Lemma 8.7 and (8.36) give

$$\begin{aligned} \mathsf{S}_t \mathbf{1}_{\mathcal{B}_r(Y)}(X,v) &\le \frac{C}{t}\int_{2t}^{3t} \mathsf{S}_s \mathbf{1}_{\mathcal{B}_r(Y)}(X,v)\,\mathrm{d}s \\ &\le C r^\gamma t^{\frac{\alpha}{2}-1} = C\left(\frac{r}{\sqrt{t}}\right)^\gamma. \end{aligned}$$

For $t < 9r^2$, the mass bound gives (8.37). Harnack is applied to the positive solution $\mathsf{S}_t F$ in its starting variables, not to an adjoint density. □

8.6. **Counting visits by time and position.** We divide time into intervals of length r^2 and position into cells of width r^3. The quantity $n_{j,k}$ below counts the mass of visits beginning in one such cell; its sum over k is the earlier time count m_j. We will bound the largest cell using the Bellman barrier and the sum using the return-time comparison.

We use the intervals (8.5) with $\bar v = \pm 2r$, so $I^* \subset \{|v| < 3r\}$. We first start at $e = (0, X_e, v_e)$ with $v_e \in \overline{I}$, and observe up to $T \le R^2$. As in Lemma 8.8, we work in a bounded interval J containing $\overline{I^*}$ in its interior. All constants are independent of J.

A visit begins when the velocity reaches $\overline{I}$ and ends when it leaves I^*. Another visit is counted only after a return to $\overline{I}$. The measure ν is the sum of the distributions of these starting times, positions and velocities, as in Lemma 8.6. Its total mass counts visits and can exceed one. The existing quadratic test gives

$$\nu(\mathbb{R}^3) \le C\left(1 + \frac{R}{r}\right).$$

The sum of the occupations during active visits, and of their terminal measures at any fixed time, is dominated by the full-space evolution. This follows from the alternating Green and exit identities, or from the fact that at most one visit is active along each path.

We divide time into $J_0 = [0, r^2]$ and $J_j = (jr^2, (j+1)r^2]$ for $j \ge 1$, truncating the last interval at T. We divide position into $B_k = [kr^3, (k+1)r^3)$, $k \in \mathbb{Z}$, and put

$$n_{j,k} = \nu(J_j \times B_k \times \overline{I}), \qquad m_j = \sum_k n_{j,k}. \tag{8.38}$$

Thus m_j counts all starting positions in one time interval, whereas $n_{j,k}$ retains a single position interval.

Lemma 8.12 (Visits beginning in one time and position interval)**.** *The masses in* (8.38) *satisfy*

$$m_j \le C(1+j)^{-\frac{1}{2}}, \qquad \sup_k n_{j,k} \le C(1+j)^{-\frac{\gamma}{2}}. \tag{8.39}$$

In particular, for $1 < q < \frac{3}{2}$,

$$\sum_{j,k} n_{j,k}^q \le C_q \left(1 + \frac{R}{r}\right)^{1-\gamma(q-1)}. \tag{8.40}$$

Proof. The estimate for m_j was proved in Lemma 8.8. To retain position, we write $I^* = (v_-, v_+)$ and use

$$\phi(v) = (v - v_-)(v_+ - v), \qquad h_k(X, v) = \chi_k(X)\phi(v),$$

where $0 \le \chi_k \le 1$ equals one on B_k, is supported in an interval of length $3r^3$ centred at $Y_k = (k + \frac{1}{2})r^3$, and satisfies $|\chi_k'| \le Cr^{-3}$. Since $|v| \le 3r$ and $\phi \le Cr^2$ on I^*,

$$Lh_k = -2a\chi_k + v\phi\chi_k', \qquad |Lh_k| \le C\mathbf{1}_{\mathcal{B}_r(Y_k)} \quad \text{on } I^*,$$

with a fixed sufficiently large A_0 in the definition of $\mathcal{B}_r$.

We sum the Green identities over active visits, each restricted to $[a, b] \subset [0, T]$, where $b - a \le r^2$. If $\mu_s^{(N)}$ is the sum of the terminal measures of the first N active pieces at time s, and ν_N counts their starts, the exact identity is

$$\int_{(a,b]\times\mathbb{R}^2} h_k \,\mathrm{d}\nu_N = \int h_k \,\mathrm{d}\mu_b^{(N)} - \int h_k \,\mathrm{d}\mu_a^{(N)} - \int_a^b \int Lh_k \,\mathrm{d}\mu_s^{(N)}\,\mathrm{d}s.$$

The internal velocity exits contribute zero because $\phi = 0$ on ∂I^*. The term at a is non-positive. We retain the term at b and use domination by the full-space evolution. At a new start in B_k, $h_k \ge \frac{5}{16}r^2$. Monotone convergence, justified by the finite total count, therefore gives

$$\nu((a,b] \times B_k \times \overline{I}) \le C\left[\mathsf{S}_b\mathbf{1}_{\mathcal{B}_r(Y_k)}(X_e, v_e) + r^{-2}\int_a^b \mathsf{S}_s\mathbf{1}_{\mathcal{B}_r(Y_k)}(X_e, v_e)\,\mathrm{d}s\right]. \tag{8.41}$$

Lemma 8.11 bounds this by $C(1+j)^{-\frac{\gamma}{2}}$ on J_j, for $j \geq 1$, including a truncated final interval. On J_0 the mass bound and the unit initial atom suffice. This proves (8.39). Finally,

$$\begin{aligned}\sum_{j,k} n_{j,k}^q &\leq \sum_j (\sup_k n_{j,k})^{q-1} m_j \\ &\leq C_q \sum_{0 \leq j \leq 2+\frac{R^2}{r^2}} (1+j)^{-\frac{1+\gamma(q-1)}{2}} \leq C_q \left(1+\frac{R}{r}\right)^{1-\gamma(q-1)}.\end{aligned}$$

The exponent in the last expression is positive because $\gamma < 2$ and $q < \frac{3}{2}$. □

8.7. **Summing the densities below the exponent four.** Lemma 8.12 bounds the sum of the qth powers of the starting masses. To pass this bound to densities, we control overlap between cells. A visit killed at ∂I^* survives for time h with mass at most $Ce^{-c\frac{h}{r^2}}$, by Corollary 8.4, and moves by at most $3rh$. The number of overlapping cells grows polynomially, while their contributions decay exponentially.

Lemma 8.13 (Starting at different times and positions)**.** *For any finite positive starting measure ν on $[0,\infty) \times \mathbb{R} \times I^*$, with $\bar{v} = \pm 2r$, we continue the time grid to infinity and put*

$$n_{j,k} = \nu(J_j \times B_k \times I^*), \qquad j \geq 0, \quad k \in \mathbb{Z}.$$

For the visit measure these are the masses in (8.38), since it is supported in $\overline{I}$. The all-time Green measure $\Gamma_\nu^{I^*}$ has a density satisfying

$$\|G_\nu^{I^*}\|_{L^q}^q \leq C_q r^{6-4q} \sum_{j,k} n_{j,k}^q, \qquad 1 < q < \frac{3}{2}. \tag{8.42}$$

The same estimate holds after translating the time origin and the position grid.

Proof. Restarting the surviving measure at time $s_e + h$ and applying Proposition 8.3 gives

$$\|\mathbf{1}_{\{s \geq s_e+h\}} G_e^{I^*}\|_{L^q} \leq C_q e^{-c\frac{h}{r^2}} r^{\frac{6}{q}-4}.$$

The estimate is valid for every restarted velocity in I^*. On an output cell $J_i \times B_\ell \times I^*$, starting mass in $J_j \times B_k \times I^*$ contributes only if $j \leq i$ and $|\ell - k| \leq C(i-j+1)$, by the position support bound. Minkowski's inequality therefore gives

$$\|\mathbf{1}_{J_i \times B_\ell \times I^*} G_\nu^{I^*}\|_{L^q} \leq C_q r^{\frac{6}{q}-4} \sum_{j \leq i} e^{-c(i-j-1)_+} \sum_{|k-\ell| \leq C(i-j+1)} n_{j,k}.$$

The sequence defining this convolution on the two indices has finite ℓ^1 norm, since $\sum_{h \geq 0}(h+1)e^{-c(h-1)_+} < \infty$. Young's inequality and disjointness of the output cells prove (8.42). Absolute continuity follows first by integrating the one-point estimate of Proposition 8.3 against the finite measure ν. □

We take J large enough to contain both $\overline{J_Q}$ and $\overline{I^*}$. For a start in $\overline{I}$, the Green measure on J, restricted to I_c and to the observation horizon, is dominated by $\Gamma_\nu^{I^*}$. The original measure on J_Q is smaller by domain monotonicity. Combining Lemmas 8.12 and 8.13, for $r \leq 6R$, gives

$$\begin{aligned}\|G_{e,I}\|_{L^q}^q &\leq C_q r^{6-4q} \left(1+\frac{R}{r}\right)^{1-\gamma(q-1)} \\ &\leq C_q R^{6-4q} \left(\frac{r}{R}\right)^{4-3q+\varepsilon(q-1)}.\end{aligned} \tag{8.43}$$

Here $G_{e,I}$ denotes the density restricted to the core, with the same convention as $G_{P,I}$ above.

If the starting velocity is outside $\overline{I}$, we first use the distribution of the first arrival in $\overline{I}$. Its mass is at most one, and no occupation of I_c occurs before that arrival. From each arrival point we apply (8.43) with time measured from that point and remaining horizon at most R^2. Minkowski's inequality gives

the same bound after integration against the first-arrival measure. The position grid can stay fixed: the estimates are uniform in the starting position and box centre, and require no translation invariance of a. The different arrival times introduce no extra factor.

Proposition 8.14 (The improved cylinder density)**.** *For every $P \in \mathcal{Q}$ and*

$$1 < q < \frac{4-\varepsilon}{3-\varepsilon},$$

the Green measure $\Gamma_P^{J_Q}|_{\mathcal{Q}}$ has a density G_P satisfying

$$\|G_P\|_{L^q(\mathcal{Q})} \le C_{\lambda,\Lambda,q,\alpha} R^{\frac{6}{q}-4}. \tag{8.44}$$

Proof. We first suppose $|v_0| \le 8R$. The cores with centres $\bar v = \pm\rho_j$ and scales $r_j = \frac{\rho_j}{2}$, where

$$\rho_j = 8R\left(\frac{7}{8}\right)^j, \qquad j \ge 0,$$

cover $0 < |v| \le 9R$. We enlarge J_Q to a bounded interval containing both $\overline{J_Q}$ and the corresponding $\overline{I^*}$, and use monotonicity and (8.43). The countable cover and Lemma 8.5 first give absolute continuity on $\mathcal{Q}$, including across $v = 0$. Summing restrictions of that density gives

$$\int_{\mathcal{Q}} G_P^q \le C_q R^{6-4q} \sum_{j\ge 0} \left(4\left(\frac{7}{8}\right)^j\right)^{4-3q+\varepsilon(q-1)}.$$

This geometric series converges precisely when $4 - 3q + \varepsilon(q-1) > 0$, which is the stated range. Its upper endpoint is less than $\frac{3}{2}$, as required in the local estimates. For $|v_0| > 8R$, the single-interval argument at the end of Proposition 8.9 already gives the result for every $q < \frac{3}{2}$. □

At $q = \frac{4}{3}$, the extra power in (8.43) is $\frac{\varepsilon}{3} > 0$. Thus the density belongs to strong $L^{\frac{4}{3}}$, and the maximum principle includes strong L^4 sources. Nearby values of q give $p > 4 - \varepsilon = 3 + \alpha$. The barrier provides the additional position bound; the Döblin argument still supplies the local density on each one-sign interval. Neither bound alone gives the conclusion.

Proof of Theorem 1.2. We fix $p > p_* = 1 + \beta_*$ and choose

$$\beta_* - 2 < \alpha < \min\{1, p-3\}$$

in Proposition 8.10; this is possible since $\beta_* < 3$ and $p > 1 + \beta_*$. We set $\varepsilon = 1 - \alpha \in (0,1)$. These exponents depend only on $\frac{\Lambda}{\lambda}$ and p. Since $p > 3 + \alpha = 4 - \varepsilon$, the conjugate $q = \frac{p}{p-1}$ is in the range of Proposition 8.14.

We first assume a smooth and reflect time and position as in Section 6. The function $U(s,X,v) = u(-s,-X,v)$ satisfies $\mathcal{K}_{\widetilde a} U \ge -g_0$, where $\widetilde a(X,v) = a(-X,v)$ and $g_0(s,X,v) = f_+(-s,-X,v)$. The coefficient remains autonomous with the same bounds. For each $P \in \mathcal{Q}$, we use $\Gamma_P^{J_Q}|_{\mathcal{Q}}$. Proposition 8.1(vi) gives the source potential, and Proposition 8.14 bounds its density. Proposition 6.2 therefore gives

$$\sup_{\overline{\mathcal{Q}}} U \le \sup_{\partial^+\mathcal{Q}} U_+ + C_{\lambda,\Lambda,p} R^{\frac{6}{q}-4} \|g_0 \mathbf{1}_{\{U>0\}}\|_{L^p(\mathcal{Q})}.$$

Since $\frac{6}{q} - 4 = 2 - \frac{6}{p}$, reflection gives (1.8), including the localised source. Finally, Lemma 6.4 applies: we mollify a only in position and velocity, keep u fixed, pass to the limit on compact inner cylinders, and then exhaust the original cylinder. All constants are independent of coefficient derivatives. □

The threshold of Theorem 1.2, supplied by the Bellman construction, is

$$p_*(\kappa) = 1 + \beta_*(\kappa) \in [3,4), \qquad \kappa = \frac{\Lambda}{\lambda}. \tag{8.45}$$

This definition is nonquantitative: the proof gives neither a numerical formula for p_* nor the estimate at $p = p_*$. Whether p_* is the optimal Aleksandrov threshold remains open. Since $\beta_* < 3$, every $p \geq 4$ is admissible; with $\alpha = \frac{\beta_* - 1}{2}$ one obtains in particular every $p > 4 - \varepsilon$ with $\varepsilon = \frac{3-\beta_*}{2} \in (0, \frac{1}{2}]$.

Proof of Corollary 1.3. For smooth coefficients, both theorem proofs use Proposition 6.2, whose source is already restricted to $\{U > 0\}$. Reflection preserves this set and its source norm. The localised form of Lemma 6.4 passes the same estimate to measurable coefficients. □

9. From the Aleksandrov principle to Hölder continuity

The Aleksandrov estimate first turns positivity on most of a cylinder into a lower bound on a smaller region. A moving barrier carries that lower bound to later points. A covering argument then shows that positivity on any fixed fraction of a cylinder suffices. Applying this to a solution and its negative reduces the oscillation on a smaller cylinder, and iteration gives Hölder continuity. This is the growth argument of Landis and Krylov–Safonov [18, Chapter 3, Sections 6 and 8][17], adapted to kinetic transport.

In this section the localised Aleksandrov estimate is an assumption. The implication is valid for coefficients depending on all three variables; we have proved its hypothesis only for the two coefficient classes of Theorems 1.1 and 1.2.

9.1. **Setting.** Throughout this section $A : \mathbb{R}^{1+2d} \to \mathrm{Sym}_d$ is Borel measurable and satisfies

$$\lambda I_d \leq A(t,x,v) \leq \Lambda I_d \quad \text{for almost every } (t,x,v) \in \mathbb{R}^{1+2d}, \tag{9.1}$$

and, extending (1.5), we write

$$\mathcal{L}_A u = \partial_t u + v \cdot \nabla_x u - A(t,x,v) : D_v^2 u. \tag{9.2}$$

The cylinder $Q_R^-(P_0)$ and its kinetic boundary $\partial_k Q_R^-(P_0)$ are those of (1.2) and (1.3). For $P_0 = (t_0, x_0, v_0)$ and $R > 0$ let

$$\Phi_{P_0,R}(t,x,v) = \big(t_0 + R^2 t,\ x_0 + R^3 x + R^2 t v_0,\ v_0 + Rv\big), \tag{9.3}$$

the composition of the kinetic dilation of ratio R with the Galilean translation by P_0. It satisfies $\Phi_{P_0,R}\big(Q_1^-(0)\big) = Q_R^-(P_0)$, $\Phi_{P_0,R} \circ \Phi_{0,\varrho} = \Phi_{P_0,R\varrho}$ and $|\Phi_{P_0,R}(E)| = R^{4d+2}|E|$ for every Borel set E; the exponent $4d+2$ is the homogeneous dimension already met in (1.6). The kinetic quasi-distance between $P = (t,x,v)$ and $P' = (t',x',v')$ is

$$d_k(P,P') = |t-t'|^{\frac{1}{2}} + |v-v'| + \max\big\{|x'-x-(t'-t)v|,\ |x-x'-(t-t')v'|\big\}^{\frac{1}{3}}. \tag{9.4}$$

For $m \in \mathbb{N}$ the *forward stack* of $Q_r^-(P_0)$ is

$$Q_r^{m,+}(P_0) = \big\{(t,x,v) :\ 0 < t - t_0 \leq m r^2,\ |v - v_0| < r,\ |x - x_0 - (t-t_0)v_0| < (m+2)r^3\big\}, \tag{9.5}$$

so that $Q_r^{m,+}(P_0) = \Phi_{P_0,r}\big(Q_1^{m,+}(0)\big)$. It occupies the m cylinder heights that follow the top of $Q_r^-(P_0)$, and is wider in the position variable by the amount the transport moves in that time.

9.2. **The hypothesis.** We use the equation only through comparison with smooth barriers. The assumption below bounds the excess of a barrier ψ over u by the source of ψ on the set where $\psi > u$. Let $O \subseteq \mathbb{R}^{1+2d}$ be open, let $1 \leq p < \infty$ and $C_{\mathrm{A}} < \infty$. A function $u \in C(O)$ is an *admissible supersolution in O with Aleksandrov data (p, C_{A})* if, for every $P_0 \in \mathbb{R}^{1+2d}$ and $R > 0$ with $\overline{Q_R^-(P_0)} \subset O$ and every ψ smooth on a neighbourhood of $\overline{Q_R^-(P_0)}$,

$$\sup_{\overline{Q_R^-(P_0)}} (\psi - u) \leq \sup_{\partial_k Q_R^-(P_0)} (\psi - u)_+ + C_{\mathrm{A}} R^{2 - \frac{4d+2}{p}} \big\| (\mathcal{L}_A \psi)_+ \mathbf{1}_{\{\psi > u\}} \big\|_{L^p(Q_R^-(P_0))}. \tag{9.6}$$

A function u is an *admissible solution in O* if both u and $-u$ are admissible supersolutions in O. If u is an admissible supersolution, so is $\theta u + c$ for every $\theta > 0$ and $c \in \mathbb{R}$, with the same data; taking $\psi = 0$ in (9.6) gives the minimum principle for u on every cylinder. Condition (9.6) is precisely the conclusion of Theorem 1.1, in the localised form of Corollary 1.3, applied to $\psi - u$ with source $\mathcal{L}_A \psi$.

Theorem 9.1 (The Aleksandrov principle implies Hölder continuity)**.** *Let A satisfy (9.1), let $1 \le p < \infty$ and $C_{\mathrm{A}} < \infty$. There are $\alpha \in (0,1]$ and $C < \infty$, depending only on d, λ, Λ, p and C_{A}, with the following property. Let $\Omega \subseteq \mathbb{R}^{1+2d}$ be open, let u be a bounded admissible solution in Ω with Aleksandrov data (p, C_{A}), let $\mathcal{C} \subset \Omega$ be compact, and let $R_0 > 0$ be such that $\overline{Q^-_{2R_0}(P)} \subset \Omega$ for every $P \in \mathcal{C}$. Then*

$$|u(P) - u(P')| \le C\Big(\operatorname{osc}_{\Omega_{R_0,\mathcal{C}}} u\Big)\Big(\frac{d_k(P,P')}{R_0}\Big)^{\alpha} \qquad \text{for all } P, P' \in \mathcal{C}\,, \tag{9.7}$$

where $\Omega_{R_0,\mathcal{C}} = \bigcup_{P\in\mathcal{C}} Q^-_{2R_0}(P)$.

The rest of the section proves Theorem 9.1 and applies it to the autonomous class. For $A(t,v)$, the final corollary uses the direct local estimate of Proposition 7.1.

9.3. **Positivity from a nearly full superlevel set.** We suppose that $u \ge 1$ on most of a cylinder. We choose a smooth cutoff ψ between zero and one, equal to one on a smaller interior region. The source in (9.6) is then counted only where $\psi > u$, a subset of the small set $\{u < 1\}$. Its norm is small, so u stays close to one wherever $\psi = 1$. We use the following fixed interior region, called the *reference cap*:

$$\mathcal{B} = \big\{(t,x,v) :\ -\tfrac14 < t < -\tfrac18,\ |x| < 8^{-3},\ |v| < \tfrac18\big\}\,, \qquad \overline{\mathcal{B}} \subset Q_1^-(0)\,. \tag{9.8}$$

Lemma 9.2 (Positivity from a nearly full superlevel set)**.** *There is $C_1 < \infty$, depending only on d, Λ, p and C_{A}, with the following property. Let $P_0 \in \mathbb{R}^{1+2d}$, $r > 0$, $\ell \ge 0$ and $\eta \in (0,1)$, and let $u \ge 0$ be an admissible supersolution with Aleksandrov data (p, C_{A}) in an open set containing $\overline{Q_r^-(P_0)}$. If*

$$\big|\{u \ge \ell\} \cap Q_r^-(P_0)\big| \ge (1-\eta)\big|Q_r^-(P_0)\big|\,, \tag{9.9}$$

then

$$\inf_{\Phi_{P_0,r}(\mathcal{B})} u \ge \big(1 - C_1\eta^{\frac1p}\big)\ell\,. \tag{9.10}$$

Proof. We choose $\psi_0 \in C_c^\infty(Q_1^-(0))$, $0 \le \psi_0 \le 1$, with $\psi_0 = 1$ on $\mathcal{B}$, and put $\psi = \ell\,\psi_0 \circ \Phi^{-1}_{P_0,r}$, extended by zero. The kinetic chain rule gives $|\mathcal{L}_A\psi| \le C_0\ell r^{-2}$, where C_0 depends only on d, Λ and the fixed cutoff. The boundary term in (9.6) vanishes, and $\{\psi > u\} \subseteq \{u < \ell\}$ has measure at most $\eta r^{4d+2}|Q_1^-(0)|$. Therefore

$$\sup_{\overline{Q_r^-(P_0)}} (\psi - u) \le C_{\mathrm{A}} C_0 |Q_1^-(0)|^{\frac1p}\eta^{\frac1p}\ell.$$

Since $\psi = \ell$ on $\Phi_{P_0,r}(\mathcal{B})$, this is (9.10). □

Only one consequence of Lemma 9.2 is used below, at one fixed density. We fix $\eta_* \in (0,1)$ with

$$C_1\eta_*^{\frac1p} \le \tfrac14\,, \tag{9.11}$$

depending only on $d, \Lambda, p, C_{\mathrm{A}}$. Lemma 9.2 then gives $\inf u \ge \frac34\ell$ on the cap at density $1 - \eta_*$.

9.4. **Propagation along kinetic paths.** We carry a positive lower bound along a position path with bounded acceleration, using its derivative as the velocity path. A Gaussian barrier centred on this pair expands with velocity width proportional to $h^{\frac12}$ and position width proportional to $h^{\frac32}$. Its differential inequality holds for every admissible coefficient. On each short time interval, we reduce its height so that it starts below the preceding barrier; comparison follows from (9.6), whose boundary and source terms then vanish. Figure 5 shows the construction.

Each comparison cylinder reaches slightly before its time interval. We therefore need the initial lower bound on a tube of positive duration. The ratios in (9.12) describe its widths, the travel time and the acceleration in kinetic units. The propagation constant depends on these ratios and ellipticity, but not on p or C_{A}.

Lemma 9.3 (Propagation along kinetic paths)**.** *Let $H, \kappa_x, \kappa_v > 0$ and $0 < T_0 \le T_1 < \infty$. There is $c_{\mathrm{prop}} \in (0,1]$, depending only on d, λ, Λ and*

$$\frac{T_0}{T_1}, \qquad HT_1^{\frac{1}{2}}, \qquad \frac{\kappa_v}{T_1^{\frac{1}{2}}}, \qquad \frac{\kappa_x}{T_1^{\frac{3}{2}}}, \tag{9.12}$$

with the following property. Let $\mathcal{T} \subseteq \mathbb{R}^{1+2d}$ be open, let $t_- \in \mathbb{R}$, and let $P = (t_P, x_P, v_P) \in \mathcal{T}$ with $T_P := t_P - t_- \in [T_0, T_1]$. We assume that there is $\bar{x} \in C^{1,1}\big([-T_0, T_P]; \mathbb{R}^d\big)$ with

$$\big|\ddot{\bar{x}}\big| \le H \quad \text{almost everywhere on } [-T_0, T_P], \qquad \big(\bar{x}(T_P), \dot{\bar{x}}(T_P)\big) = (x_P, v_P)\,, \tag{9.13}$$

such that the tube

$$\mathcal{V} = \big\{\big(t_- + s,\ \bar{x}(s) + y,\ \dot{\bar{x}}(s) + V\big) :\ -T_0 \le s \le T_P,\ |y| \le \kappa_x,\ |V| \le \kappa_v\big\} \tag{9.14}$$

is contained in $\mathcal{T}$. Then every $\ell \ge 0$ and every non-negative admissible supersolution u in $\mathcal{T}$ satisfy

$$u \ge \ell \ \text{ on } \mathcal{V} \cap \{t \le t_-\} \qquad \Longrightarrow \qquad u(P) \ge c_{\mathrm{prop}}\, \ell\,. \tag{9.15}$$

Proof. We construct the Gaussian in coordinates centred on the path. The matrix J describes transport from velocity to position, and V inserts diffusion in velocity. Their covariance matrix, computed in Step 1, determines the Gaussian. At each step we start with the same rescaled profile and a smaller amplitude, using a time cutoff to make the comparison smooth. Throughout, $Z = (y, V) \in \mathbb{R}^{2d}$,

$$\mathsf{J} = \begin{pmatrix} 0 & I_d \\ 0 & 0 \end{pmatrix}, \qquad \mathsf{V} = \begin{pmatrix} 0 \\ I_d \end{pmatrix}, \qquad \mathsf{D}_h = \begin{pmatrix} h^{\frac{3}{2}} I_d & 0 \\ 0 & h^{\frac{1}{2}} I_d \end{pmatrix} \quad (h > 0)\,,$$

$\varepsilon = \frac{\lambda}{64}$, and $\chi \in C^\infty(\mathbb{R})$ is a fixed function with $0 \le \chi \le 1$, $\chi = 0$ on $\big(-\infty, -\frac{1}{128}\big]$ and $\chi = 1$ on $[0, \infty)$; no constant below depends on the choice of χ.

To obtain smooth barriers, we extend $\bar{x}$ by its tangent lines at the endpoints and mollify. The acceleration bound H is preserved, and convergence in C^1 keeps the tube with halved radii inside (9.14). It retains the initial lower bound. Step 5 allows a fixed positive endpoint error. We choose one mollification small enough to preserve the tube inclusion and satisfy both endpoint bounds. We use the halved radii below.

Step 1: the covariance matrix. For $s \in [-T_0, T_P]$ we use the coordinates

$$Z = \big(x - \bar{x}(s),\ v - \dot{\bar{x}}(s)\big)\,, \tag{9.16}$$

and, for $\Psi(t, x, v) = \Upsilon(s, Z)$ with $s = t - t_-$, the chain rule gives

$$\mathcal{L}_A \Psi = \partial_s \Upsilon + (\mathsf{J}Z) \cdot \nabla_Z \Upsilon - \ddot{\bar{x}}(s) \cdot \nabla_V \Upsilon - A : D_V^2 \Upsilon\,. \tag{9.17}$$

We define G_σ by $\dot{\mathsf{G}}_\sigma = \mathsf{J}\mathsf{G}_\sigma + \mathsf{G}_\sigma \mathsf{J}^{\mathsf{T}} + \lambda \mathsf{V}\mathsf{V}^{\mathsf{T}}$ with $\mathsf{G}_0 = \varepsilon \mathsf{D}_h^2$, that is

$$\mathsf{G}_\sigma = e^{\sigma \mathsf{J}} \big(\varepsilon \mathsf{D}_h^2\big) e^{\sigma \mathsf{J}^{\mathsf{T}}} + \lambda\, \mathsf{G}_\star(\sigma), \qquad \mathsf{G}_\star(\theta) = \int_0^\theta e^{\varsigma \mathsf{J}} \mathsf{V}\mathsf{V}^{\mathsf{T}} e^{\varsigma \mathsf{J}^{\mathsf{T}}}\, \mathrm{d}\varsigma = \begin{pmatrix} \frac{\theta^3}{3} I_d & \frac{\theta^2}{2} I_d \\ \frac{\theta^2}{2} I_d & \theta I_d \end{pmatrix}, \tag{9.18}$$

valid for every real σ. Since $e^{\sigma \mathsf{J}} = I_{2d} + \sigma \mathsf{J}$, rescaling gives

$$\mathsf{G}_\sigma = \mathsf{D}_h \widehat{\mathsf{G}}_{\frac{\sigma}{h}} \mathsf{D}_h, \qquad \widehat{\mathsf{G}}_\theta = \varepsilon\, e^{\theta \mathsf{J}} e^{\theta \mathsf{J}^{\mathsf{T}}} + \lambda\, \mathsf{G}_\star(\theta)\,; \tag{9.19}$$

every bound below is therefore a bound on the fixed function $\theta \mapsto \widehat{\mathsf{G}}_\theta$, uniform in h and in the block. We write $\mathfrak{q}_\sigma(Z) = Z^{\mathsf{T}} \mathsf{G}_\sigma^{-1} Z$, $\mathfrak{p}_\sigma(Z) = \mathsf{V}^{\mathsf{T}} \mathsf{G}_\sigma^{-1} Z$ and $\mathsf{R}_\sigma = \mathsf{V}^{\mathsf{T}} \mathsf{G}_\sigma^{-1} \mathsf{V}$. Differentiating G_σ^{-1} and using the equation for G_σ,

$$\big(\partial_\sigma + (\mathsf{J}Z) \cdot \nabla_Z\big) \mathfrak{q}_\sigma = -\lambda |\mathfrak{p}_\sigma|^2, \qquad \nabla_V \mathfrak{q}_\sigma = 2\mathfrak{p}_\sigma, \qquad D_V^2 \mathfrak{q}_\sigma = 2\mathsf{R}_\sigma\,, \tag{9.20}$$

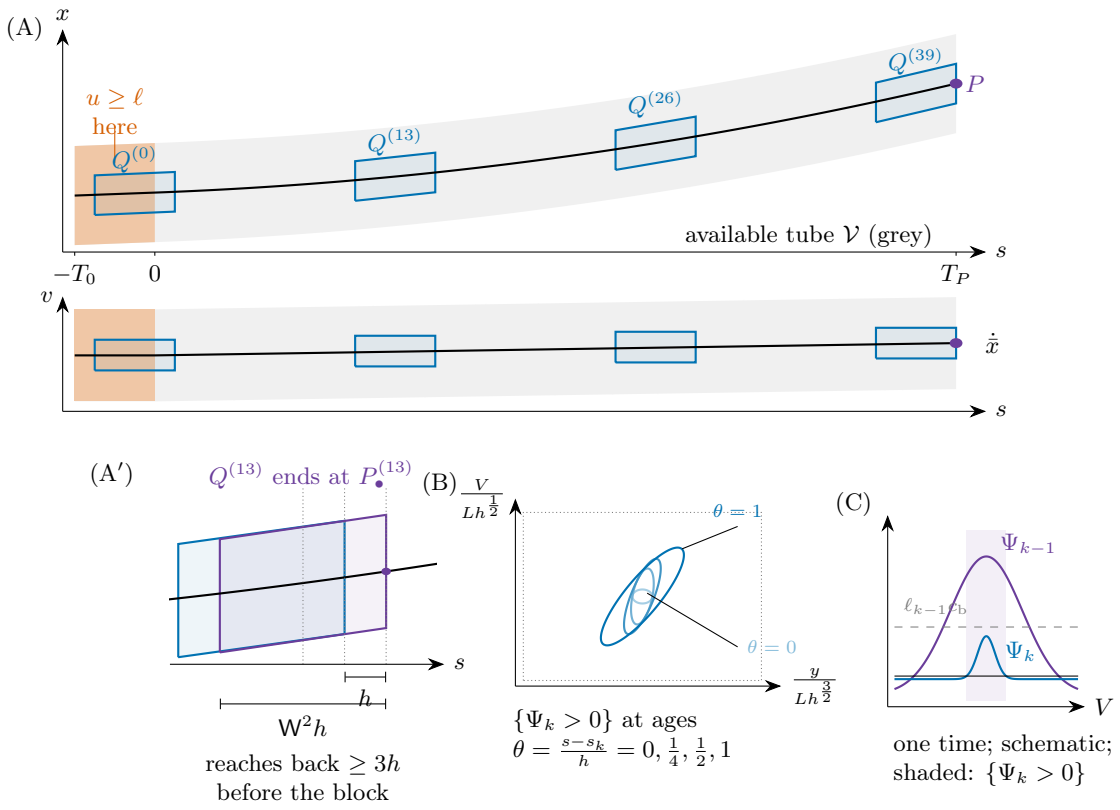

FIGURE 5. Propagation of a non-negative admissible supersolution, $d = 1$ (Lemma 9.3). **(A)** The bound $u \geq \ell$ on an initial tube segment of duration $T_0 > 0$ propagates to P; velocity is the derivative of position. The cylinders $Q^{(k)}$ have radius $R = \mathsf{W}h^{\frac{1}{2}}$, end at the next path point, and are sheared by the velocity there; only some are drawn. The cylinder proportions are schematic: they are drawn with $\mathsf{W} = 2$, whereas (9.24) gives $\mathsf{W} = 2 + \frac{3}{2}L\lambda^{\frac{1}{2}} + 2HT_1^{\frac{1}{2}}$. **(A′)** Each cylinder extends W^2h into the past, beyond its block of length h. **(B)** The exact positive region $\{\Psi_k > 0\}$ for $\lambda = \Lambda = 1$ and $H = 0$, where $\Xi_h h = \frac{L^2}{4}$, in path-centred coordinates $\frac{\mathsf{D}_h^{-1}Z}{L}$, within the bounds (9.26). **(C)** Schematic heights as the next barrier starts, $s_k - \frac{h}{128} < s \leq s_k$. On $\{\Psi_k > 0\}$, $\Psi_k \leq \ell_k e^{\frac{L^2}{512}} = \ell_{k-1}c_{\mathrm{b}} \leq \Psi_{k-1}$; both barriers are non-positive outside their positive regions. Repetition gives $u(P) \geq c_{\mathrm{prop}}\ell$.

where the transport terms cancel in the first identity. Four properties of $\widehat{\mathsf{G}}$ are used: positive definiteness for $\theta \geq -\frac{1}{128}$, and

$$\begin{aligned} &\widehat{\mathsf{G}}_\theta \leq 2\lambda I_{2d} && (|\theta| \leq 1), \\ &\operatorname{tr}\bigl(\mathsf{V}^{\mathsf{T}}\widehat{\mathsf{G}}_\theta^{-1}\mathsf{V}\bigr) \leq C_{\mathsf{G}} := \frac{128\,d}{\lambda} && (0 \leq \theta \leq 1), \\ &\widehat{\mathsf{G}}_\theta \leq \tfrac{13}{50}\,\widehat{\mathsf{G}}_{\theta+1} && \bigl(-\tfrac{1}{128} \leq \theta \leq 0\bigr). \end{aligned} \tag{9.21}$$

These are scalar 2×2 calculations. For $|\theta| \leq 1$, row sums give $\|\widehat{\mathsf{G}}_\theta\| \leq 3\varepsilon + \frac{11}{6}\lambda < 2\lambda$. For $\theta \geq 0$, the non-negative covariance term gives $\mathsf{V}^{\mathsf{T}}\widehat{\mathsf{G}}_\theta^{-1}\mathsf{V} \leq \varepsilon^{-1}(1+\theta^2)I_d$. For $-\frac{1}{128} \leq \theta \leq 0$, the diagonal entries are at least $\frac{\varepsilon}{2}$ and the off-diagonal entry has modulus at most $\frac{\varepsilon}{128}$, so the matrix is positive definite. Finally, $\mathsf{G}_\star(a) = \mathsf{D}_a\mathsf{G}_\star(1)\mathsf{D}_a$ and $\mathsf{G}_\star(1)^{-1} \leq 16I_{2d}$ give, in this last range,

$$\widehat{\mathsf{G}}_{\theta+1} \geq \frac{125\lambda}{2048}I_{2d}, \qquad \widehat{\mathsf{G}}_\theta \leq \frac{65\lambda}{4096}I_{2d}.$$

Their quotient is $\frac{13}{50}$.

Step 2: the barrier and its differential inequality. Let $\ell' \geq 0$, $h > 0$, $L > 0$ and let $s_\bullet$ be the initial time of a block. Writing $\sigma = s - s_\bullet$ and $\mathcal{G} = e^{-\Xi_h \sigma - \mathfrak{q}_\sigma(Z)}$, with Z as in (9.16), put

$$\Psi(t,x,v) = \ell'\,\chi\Big(\frac{\sigma}{h}\Big)\big(\mathcal{G} - e^{-L^2}\big), \qquad \Xi_h = \frac{2\Lambda C_{\mathsf{G}}}{h} + \frac{2H^2}{3\lambda}\,. \tag{9.22}$$

Positive definiteness holds on a neighbourhood of $[-\frac{1}{128}, \infty)$, so the cutoff makes Ψ, extended by zero, smooth everywhere. For $0 < \sigma \leq h$, the cutoff is one. Equations (9.17) and (9.20) give

$$\begin{aligned} \mathcal{L}_A \Psi &= \ell'\,\mathcal{G}\Big(-\Xi_h + \lambda|\mathsf{p}_\sigma|^2 + 2\,\ddot{\bar{x}}(s)\cdot \mathsf{p}_\sigma + 2A : \mathsf{R}_\sigma - 4\,\mathsf{p}_\sigma \cdot A\mathsf{p}_\sigma\Big) \\ &\leq \ell'\,\mathcal{G}\Big(-\Xi_h - 3\lambda|\mathsf{p}_\sigma|^2 + 2H|\mathsf{p}_\sigma| + \frac{2\Lambda C_{\mathsf{G}}}{h}\Big) \\ &\leq \ell'\,\mathcal{G}\Big(-\Xi_h - \frac{3\lambda}{2}|\mathsf{p}_\sigma|^2 + \frac{2H^2}{3\lambda} + \frac{2\Lambda C_{\mathsf{G}}}{h}\Big) = -\frac{3\lambda}{2}\,\ell'\,\mathcal{G}\,|\mathsf{p}_\sigma|^2 \leq 0\,, \end{aligned} \tag{9.23}$$

almost everywhere on $\{0 < \sigma \leq h\}$. We used $\ell'\mathcal{G} \geq 0$, ellipticity, the trace bound (9.21), $|\ddot{\bar{x}}| \leq H$ and Young's inequality.

Step 3: the constants. We fix, in this order,

$$\begin{aligned} &\varepsilon = \frac{\lambda}{64}, \quad C_{\mathsf{G}} = \frac{128\,d}{\lambda}, \quad \Xi_* = 2\Lambda C_{\mathsf{G}} + \frac{2H^2 T_1}{3\lambda}, \quad L = 2\,\Xi_*^{\frac12}, \quad \mathsf{W} = 2 + \tfrac32 L\lambda^{\frac12} + 2HT_1^{\frac12}, \\ &c_1 = \mathsf{W} + H\mathsf{W}^2 T_1^{\frac12}, \quad c_2 = \mathsf{W}^3 + \tfrac12 H\mathsf{W}^4 T_1^{\frac12}, \quad h_* = \min\Big\{\frac{T_0}{\mathsf{W}^2},\ \Big(\frac{\kappa_v}{2c_1}\Big)^2,\ \Big(\frac{\kappa_x}{2c_2}\Big)^{\frac23}\Big\}, \end{aligned} \tag{9.24}$$

and put

$$c_{\mathrm{b}} = e^{-\frac34 L^2} - e^{-L^2} > 0, \qquad \gamma = c_{\mathrm{b}}\,e^{-\frac{L^2}{512}} \in (0,1)\,. \tag{9.25}$$

Thus $\mathsf{W} \geq 2$ and $\mathsf{W}^2 h_* \leq T_0$; and for $0 < h \leq h_*$ one has $h \leq \frac14 T_0 \leq T_1$, so that $\Xi_h h = 2\Lambda C_{\mathsf{G}} + \frac{2H^2 h}{3\lambda} \leq \Xi_* = \frac14 L^2$, $c_1 h^{\frac12} \leq \frac12 \kappa_v$ and $c_2 h^{\frac32} \leq \frac12 \kappa_x$.

Step 4: comparison over one time interval. Let $0 < h \leq h_*$, let $s_\bullet \in [0, T_P - h]$, let Ψ be as in (9.22), and let

$$Q = Q_R^-(P_\bullet), \qquad P_\bullet = \big(t_- + s_\bullet + h,\ \bar{x}(s_\bullet + h),\ \dot{\bar{x}}(s_\bullet + h)\big), \qquad R = \mathsf{W}h^{\frac12}\,,$$

so that Q has its top face at the final time of the block and reaches back to $\sigma = -(\mathsf{W}^2 - 1)h$, well before the block starts. We claim that if $\Psi \leq u$ on $\overline{Q} \cap \{\sigma \leq 0\}$, then $\Psi \leq u$ on $\overline{Q}$. On $\{\Psi > 0\}$ one has $\chi > 0$ and $\mathcal{G} > e^{-L^2}$, hence $\sigma > -\frac{1}{128}h$ and $\mathfrak{q}_\sigma(Z) < L^2 + \Xi_h \frac{h}{128} \leq \frac{513}{512}L^2$; as $\sigma \leq h$ on $\overline{Q}$, the first bound of (9.21) applies and, with (9.19), gives $|\mathsf{D}_h^{-1} Z| \leq \big(2\lambda \cdot \frac{513}{512}\big)^{\frac12} L \leq \frac32 L\lambda^{\frac12}$, that is

$$|v - \dot{\bar{x}}(s)| \leq \tfrac32 L\lambda^{\frac12} h^{\frac12}, \qquad |x - \bar{x}(s)| \leq \tfrac32 L\lambda^{\frac12} h^{\frac32} \qquad \text{on } \{\Psi > 0\} \cap \overline{Q}\,. \tag{9.26}$$

The acceleration bound places every point of $\overline{Q}$ within $R + H\mathsf{W}^2 h \leq c_1 h^{\frac12}$ in velocity and $R^3 + \frac12 H\mathsf{W}^4 h^2 \leq c_2 h^{\frac32}$ in position of the path at the same time. Since $\mathsf{W}^2 h \leq T_0$, the whole cylinder lies in the tube. On $\{\Psi > 0\}$, moreover, $|\sigma - h| \leq 2h$. Thus (9.26) gives

$$|v - \dot{\bar{x}}(s_\bullet + h)| \leq \left(\frac32 L\lambda^{\frac12} + 2HT_1^{\frac12}\right) h^{\frac12} < R$$

and

$$|x - \bar{x}(s_\bullet + h) - (\sigma - h)\dot{\bar{x}}(s_\bullet + h)| \leq \left(\frac32 L\lambda^{\frac12} + 2HT_1^{\frac12}\right) h^{\frac32} < R^3,$$

by (9.24). Thus $\Psi \leq 0$ on the position and velocity faces. It also vanishes on the bottom face, where $\sigma \leq -3h$. The assumed comparison for $\sigma \leq 0$ gives $\{\Psi > u\} \subseteq \{0 < \sigma \leq h\}$, where (9.23) holds. Both terms on the right of (9.6) vanish, proving $\Psi \leq u$ on $\overline{Q}$.

Step 5: iteration along the path. We set $n = \lceil \frac{T_P}{h_*} \rceil$, $h = \frac{T_P}{n}$, $s_k = kh$, $\ell_0 = e^{-\frac{L^2}{512}}\ell$ and $\ell_k = \gamma \ell_{k-1}$. We denote by Ψ_k and $Q^{(k)}$ the barrier and cylinder starting at s_k with amplitude ℓ_k, for $0 \le k < n$. By Step 4, the induction $\Psi_k \le u$ requires comparison only where $\Psi_k > 0$ and $\sigma = s - s_k \le 0$. There $-\frac{h}{128} < \sigma \le 0$, so $\mathfrak{q}_\sigma \ge 0$ and $\Xi_h h \le \frac{1}{4}L^2$ give

$$\Psi_k \le \ell_k \, \mathcal{G} \le \ell_k \, e^{\Xi_h \frac{h}{128}} \le \ell_k \, e^{\frac{L^2}{512}} . \tag{9.27}$$

For $k = 0$, this set lies in the initial tube by (9.26) and (9.24). There $u \ge \ell = \ell_0 e^{\frac{L^2}{512}}$, so (9.27) gives the comparison. For $k \ge 1$ that set lies in $\overline{Q^{(k-1)}}$: its times $s \in [s_k - \frac{1}{128}h, s_k]$ lie in $[s_k - \mathsf{W}^2 h, s_k]$, and $|s - s_k| \le h$ turns (9.26) into

$$|v - \dot{\bar{x}}(s_k)| \le \left(\frac{3}{2} L \lambda^{\frac{1}{2}} + H T_1^{\frac{1}{2}} \right) h^{\frac{1}{2}} < R$$

and

$$|x - \bar{x}(s_k) - (s - s_k)\dot{\bar{x}}(s_k)| \le \left(\frac{3}{2} L \lambda^{\frac{1}{2}} + \frac{1}{2} H T_1^{\frac{1}{2}} \right) h^{\frac{3}{2}} < \mathsf{W}^3 h^{\frac{3}{2}} = R^3 .$$

There the inductive hypothesis gives $u \ge \Psi_{k-1}$, and Ψ_{k-1} is large: its own time variable is $\sigma + h \in [h - \frac{1}{128}h, h]$, so its cutoff equals one, and the third bound of (9.21), applied at $\theta = \frac{\sigma}{h} \in [-\frac{1}{128}, 0]$, gives

$$\mathfrak{q}_{\sigma+h}(Z) \le \tfrac{13}{50}\, \mathfrak{q}_\sigma(Z) \le \tfrac{13}{50} \cdot \tfrac{513}{512} L^2 < \tfrac{1}{2} L^2 ,$$

whence

$$\Psi_{k-1} \ge \ell_{k-1}\big(e^{-\frac{1}{4}L^2 - \frac{1}{2}L^2} - e^{-L^2} \big) = \ell_{k-1} c_{\mathrm{b}} = \ell_k e^{\frac{L^2}{512}}$$

by (9.25), which by (9.27) is at least Ψ_k. For the original path, P is the centre of the final top face. For the smoothed path, we choose the endpoint error at most $\frac{1}{8}L\lambda^{\frac{1}{2}}h^{\frac{3}{2}}$ in position and $\frac{1}{8}L\lambda^{\frac{1}{2}}h^{\frac{1}{2}}$ in velocity. This is a fixed positive allowance because $h \ge \frac{1}{2}h_*$. It keeps P in $\overline{Q^{(n-1)}}$, and $\widehat{\mathsf{G}}_1 \ge \frac{\lambda}{16} I_{2d}$ gives $\mathfrak{q}_h(Z_P) \le \frac{1}{2}L^2$. Hence

$$u(P) \ge \Psi_{n-1}(P) = \ell_{n-1}\big(e^{-\Xi_h h - \mathfrak{q}_h(Z_P)} - e^{-L^2} \big) \ge \ell_{n-1} c_{\mathrm{b}} = \gamma^n \ell .$$

Since $n \le n_* := \lceil \frac{T_1}{h_*} \rceil$, we may take $c_{\mathrm{prop}} = \gamma^{n_*}$. Formula (9.24) shows that $\frac{T_1}{h_*}$, and hence this constant, depends only on d, λ, Λ and the four ratios (9.12). □

The hypothesis is met by the cubic Hermite path joining $(\bar{x}(0), \dot{\bar{x}}(0))$ to a prescribed (x_P, v_P) in time T_P, continued before time 0 with zero acceleration: its acceleration obeys

$$|\ddot{\bar{x}}| \le \frac{6\big|x_P - \bar{x}(0) - T_P \dot{\bar{x}}(0)\big|}{T_P^2} + \frac{4\big|v_P - \dot{\bar{x}}(0)\big|}{T_P} \qquad \text{on } [0, T_P] . \tag{9.28}$$

The constant c_{prop} degenerates as $H T_1^{\frac{1}{2}}$ grows and as $\kappa_v T_1^{-\frac{1}{2}}$ and $\kappa_x T_1^{-\frac{3}{2}}$ decrease; below this makes the constant of Lemma 9.5 degenerate as m grows, which is harmless because m is fixed before it.

9.5. **Ink spots and the forward stack.** The covering theorem uses a lower bound on the forward stack (9.5), not just on the interior cap. Lemma 9.2 gives positivity on $\Phi_{P_0,r}(\mathcal{B})$, whose times lie between $t_0 - \frac{1}{4}r^2$ and $t_0 - \frac{1}{8}r^2$. This leaves a fixed time gap before the stack begins. We will join the cap to every point of that stack by the paths of Lemma 9.3.

The covering theorem itself is quoted; it is the only external input of this section, and it is the kinetic form of the crawling ink-spots lemma of Krylov and Safonov [17]; Figure 6 illustrates it, together with the positivity it is combined with below.

Theorem 9.4 (Crawling ink spots, kinetic form [14, Section 10], [10, Theorem A.1 and Remark 12])**.** *There are $c_d \in (0,1)$ and $C_d > 0$, depending only on d, with the following property. Let $m \in \mathbb{N}$ with $m \ge 1$, let $\eta, r_0 \in (0,1)$, and let E and F be bounded measurable sets with $E \subseteq F \cap Q_1^-(0)$. We assume that every kinetic cylinder $Q_r^-(P) \subseteq Q_1^-(0)$ with*

$$\big|E \cap Q_r^-(P)\big| \ge (1 - \eta)\big|Q_r^-(P)\big|$$

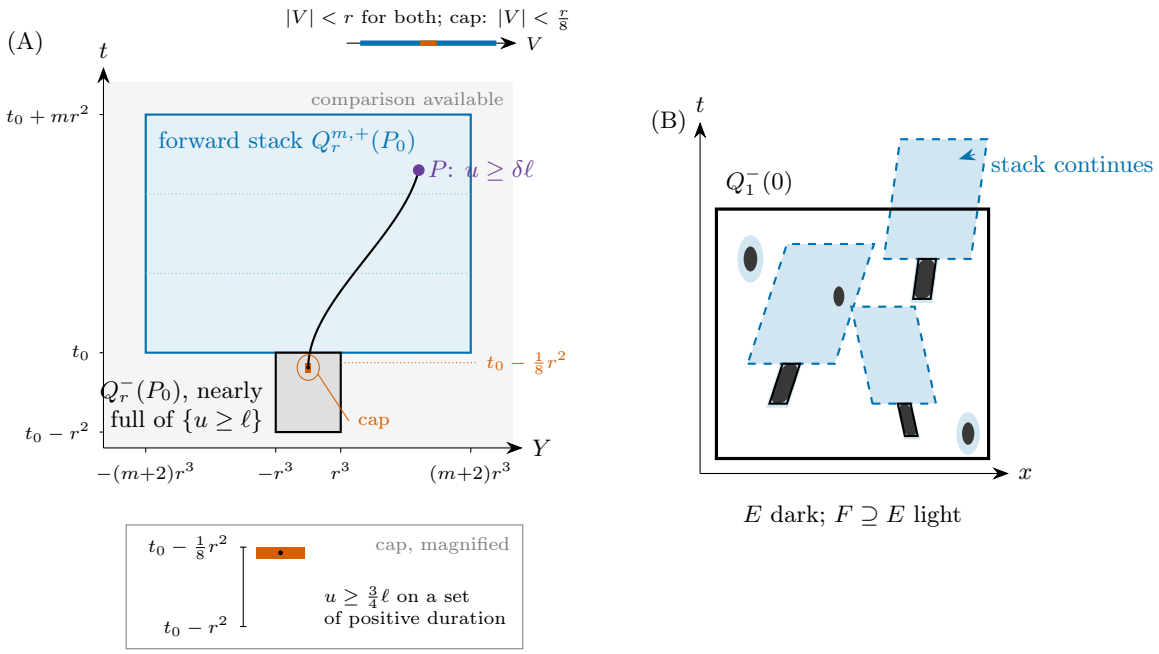

FIGURE 6. From a nearly full cylinder to its forward stack. **(A)** In relative coordinates. If $Q_r^-(P_0)$ is nearly full of $\{u \geq \ell\}$, Lemma 9.2 gives $u \geq \frac{3}{4}\ell$ on the cap $\Phi_{P_0,r}(\mathcal{B})$, whose times lie between $t_0 - \frac{1}{4}r^2$ and $t_0 - \frac{1}{8}r^2$. Kinetic paths from the cap then give $u \geq \delta\ell$ throughout the forward stack (9.5), of height mr^2 and half-width $(m+2)r^3$ (Lemma 9.5); the paths need only stay in the larger region where comparison is available. **(B)** The covering statement, Theorem 9.4: every cylinder nearly full of E has its forward stack in $F \supseteq E$, and stacks may leave $Q_1^-(0)$. Both panels are position–time projections (display value $m = 3$); density is measured in the full kinetic variables.

satisfies $r < r_0$ *and* $Q_r^{m,+}(P) \subseteq F$. *Then*

$$|E| \leq \frac{m+1}{m}\,(1 - c_d\eta)\left(\left|F \cap Q_1^-(0)\right| + C_d\, m\, r_0^2\right). \tag{9.29}$$

We use the unit-cylinder covering statement with an error for stacks that leave the cylinder [14, Section 10], at their parameter $s = 1$. Their Q_1 then differs from $Q_1^-(0)$ only by a null top face, their delayed cylinder $\overline{Q}^m$ is exactly (9.5), and their density parameter μ is our η. The restriction $r < r_0$ gives the error $C_d m r_0^2$. Guérand and Imbert record the same form on their earlier cylinder Q^- in Theorem A.1 and explain the relation in Remark 12. We now fix $m \in \mathbb{N}$ so large that

$$a := \frac{m+1}{m}\left(1 - c_d\eta_*\right) < 1\,. \tag{9.30}$$

Then m and a depend only on d, Λ, p and C_{A}.

Lemma 9.5 (Positivity on the forward stack)**.** *There are a bounded open set* $\mathcal{T}_* \subset \mathbb{R}^{1+2d}$ *containing* $\overline{Q_1^-(0)} \cup \overline{Q_1^{m,+}(0)}$ *and a constant* $\delta \in (0,1)$*, both depending only on* d*,* λ*,* Λ*,* p *and* C_{A}*, with the following property. Let* $P_0 \in \mathbb{R}^{1+2d}$*,* $r > 0$ *and* $\ell \geq 0$*, and let* $u \geq 0$ *be an admissible supersolution with Aleksandrov data* (p, C_{A}) *in an open set containing* $\overline{\Phi_{P_0,r}(\mathcal{T}_*)}$*. If*

$$\left|\{u \geq \ell\} \cap Q_r^-(P_0)\right| \geq (1 - \eta_*)\left|Q_r^-(P_0)\right|,$$

then $u \geq \delta\ell$ *on* $Q_r^{m,+}(P_0)$.

Proof. We assume $\ell > 0$. Applying Lemma 9.2 directly on $Q_r^-(P_0)$, with the choice (9.11), gives

$$u \geq \tfrac{3}{4}\ell \qquad \text{on } \Phi_{P_0,r}(\mathcal{B})\,. \tag{9.31}$$

We start every path at the image $P_- = (t_-, x_-, v_-)$ of $(-\frac{3}{16}, 0, 0)$ under $\Phi_{P_0,r}$:

$$t_- = t_0 - \tfrac{3}{16}r^2, \qquad v_- = v_0, \qquad x_- = x_0 - (t_0 - t_-)v_0\,, \tag{9.32}$$

with zero acceleration before t_-. We use

$$T_0 = \tfrac{1}{32}r^2, \qquad \kappa_x = \tfrac{1}{2}\,8^{-3}r^3, \qquad \kappa_v = \tfrac{1}{16}r\,. \tag{9.33}$$

In the coordinates of the cap, this initial tube has times in $[-\frac{7}{32}, -\frac{3}{16}]$, position radius $\frac{1}{2}8^{-3}$ and velocity radius $\frac{1}{16}$. It lies inside (9.8), so (9.31) supplies its lower bound.

For $P \in Q_r^{m,+}(P_0)$, the travel time $T_P = t_P - t_-$ lies between $\frac{1}{8}r^2$ and $T_1 = (m+1)r^2$, and exceeds T_0. We use the cubic Hermite path (9.28) from (x_-, v_-) to (x_P, v_P). Its position mismatch is $x_P - x_- - T_P v_- = x_P - x_0 - (t_P - t_0)v_0$, bounded by $(m+2)r^3$ by (9.5); the velocity mismatch is at most r. Hence its acceleration is bounded by

$$H = \big(384(m+2) + 32\big)r^{-1}.$$

Integrating this bound, including the zero-acceleration extension before t_-, gives a constant $c_m \geq m+3$ such that

$$|\dot{\bar{x}}(s) - v_0| \leq c_m r, \qquad \big|\bar{x}(s) - x_0 - (t_- + s - t_0)v_0\big| \leq c_m r^3, \qquad -T_0 \leq s \leq T_P\,, \tag{9.34}$$

with c_m depending only on m. We take $\mathcal{T}_* = \{(t,x,v) : -2 < t < m+1,\ |x|, |v| < c_m + 1\}$. It contains the two required cylinders. The path bounds, together with $\kappa_v < r$, $\kappa_x < r^3$ and $t_- - T_0 > t_0 - 2r^2$, place every tube inside $\Phi_{P_0,r}(\mathcal{T}_*)$.

Lemma 9.3 now gives $u(P) \geq \frac{3}{4}c_{\mathrm{prop}}\ell$. Its four ratios (9.12) depend only on m, uniformly in P, P_0, r. Thus $\delta = \frac{3}{4}c_{\mathrm{prop}}$ has the asserted dependence. □

9.6. **Positivity from every positive density.** We no longer assume that the positive set fills most of a cylinder. Starting from any fixed positive fraction, we lower the level from 1 to δ, then to δ^2, and so on. At each level, a cylinder on which the set is nearly full gives positivity on its forward stack by Lemma 9.5. The covering theorem then forces the measure of the next superlevel set to grow, up to an error controlled by a small radius threshold. After finitely many steps there must be a nearly full cylinder of radius above that threshold. Its cap can be joined to every point of the target cylinder by Lemma 9.3.

In the proof we choose the number of levels from the initial fraction β, then choose the radius threshold small enough to control the accumulated covering error. This order is important: propagation from a smaller cap gives a weaker lower bound, but the chosen cap still has a positive lower bound on its radius. The sampling and target cylinders are fixed inside the original cylinder, with the target sharing its top time, so the final estimate can be iterated on smaller cylinders.

Theorem 9.6 (Growth lemma)**.** *There are $\vartheta \in (0,1)$ and a kinetic cylinder Σ with $\overline{\Sigma} \subset Q_1^-(0)$, depending only on d, λ, Λ, p and C_{A}, and, for every $\beta \in (0,1]$, a constant $\kappa_\beta > 0$ depending only on β and the same parameters, with the following property. Let $P_0 \in \mathbb{R}^{1+2d}$ and $R > 0$, and let $u \geq 0$ be an admissible supersolution with Aleksandrov data (p, C_{A}) in an open set containing $\overline{Q_R^-(P_0)}$. If*

$$\big|\{u \geq 1\} \cap \Phi_{P_0,R}(\Sigma)\big| \geq \beta\big|\Phi_{P_0,R}(\Sigma)\big|\,, \tag{9.35}$$

then

$$u \geq \kappa_\beta \quad \text{on } Q_{\vartheta R}^-(P_0)\,. \tag{9.36}$$

Proof. Step 1: the reference geometry. Relative time, velocity and transported position scale under $\Phi_{P_0,s}$ as s^2, s, s^3. Thus

$$\Phi_{P_0,s}\big(Q_1^-(P')\big) = Q_s^-\big(\Phi_{P_0,s}(P')\big), \qquad \Phi_{P_0,s}\circ\Phi_{P',r} = \Phi_{\Phi_{P_0,s}(P'),\,sr}\,, \tag{9.37}$$

and the same scaling carries forward stacks to forward stacks.

We put $P^\sharp = (-(m+2),0,0)$ and $\Sigma^\sharp = Q_1^-(P^\sharp)$, whose times fill $\big(-(m+3),-(m+2)\big)$. A kinetic cylinder $Q_r^-(P)\subseteq\Sigma^\sharp$ has $r\le 1$ and top time at most $-(m+2)$, so its forward stack lies in $\{t\le -2\}$, whereas $Q_1^-(0)$ lies in $\{-1<t<0\}$. We choose a bounded open set $\mathcal{G}\subseteq\{t<0\}$ containing

(i) $\overline{\Sigma^\sharp}$ and $Q_1^-(0)$;
(ii) $\Phi_{P,r}(\mathcal{T}_*)$ for every kinetic cylinder $Q_r^-(P)\subseteq\Sigma^\sharp$, with $\mathcal{T}_*$ as in Lemma 9.5;
(iii) the points $\big(t'+s,\bar x(s)+y,\dot{\bar x}(s)+V\big)$ with $|y|\le 1$, $|V|\le 1$ and $-1\le s\le T$, where $\bar x$ runs over the paths that join a point $\big(t',\bar x(0),\dot{\bar x}(0)\big)$ of $\overline{\Sigma^\sharp}$ to a point of $Q_1^-(0)$ in a time $T\in[m+1,m+3]$, being the cubic Hermite path of (9.28) on $[0,T]$ and of zero acceleration on $[-1,0]$.

These sets are bounded and lie in $\{t<0\}$. Indeed, the images in (ii) have top time at most -1. The paths in (iii) have travel time in $[m+1,m+3]$, endpoints in fixed bounded sets and acceleration at most $6(m+5)+8$ by (9.28).

We choose $\varepsilon_0>0$ small enough that $\Phi_{0,\varepsilon_0}(\mathcal{G})\subseteq Q_1^-(0)$ and $\overline{\Phi_{0,\varepsilon_0}(\Sigma^\sharp)}\subset Q_1^-(0)$. We set $\Sigma=\Phi_{0,\varepsilon_0}(\Sigma^\sharp)$, $\vartheta=\varepsilon_0$ and $\Phi_\star=\Phi_{P_0,R\varepsilon_0}$. These choices depend only on the stated parameters, and (9.37) gives

$$\begin{aligned}
&\Phi_\star(\mathcal{G})\subseteq Q_R^-(P_0),\\
&\Phi_\star\big(\Sigma^\sharp\big)=\Phi_{P_0,R}(\Sigma)=:\Sigma_R=Q_{R\varepsilon_0}^-\big(\Phi_\star(P^\sharp)\big),\\
&\Phi_\star\big(Q_1^-(0)\big)=Q_{\vartheta R}^-(P_0)\,.
\end{aligned} \tag{9.38}$$

In particular $u\ge 0$ is an admissible supersolution in an open set containing $\overline{\Phi_\star(\mathcal{G})}$.

Step 2: a dense cylinder whose radius is bounded below. Let $r_0\in(0,1)$ and $j\ge 0$, and suppose that some kinetic cylinder $Q_r^-(P)\subseteq\Sigma_R$ satisfies

$$r\ge r_0R\varepsilon_0, \qquad \big|\{u\ge\delta^j\}\cap Q_r^-(P)\big|\ge(1-\eta_*)\big|Q_r^-(P)\big|\,. \tag{9.39}$$

The cap construction in Lemma 9.5 gives $P_-\in\Sigma_R$ and $u\ge\frac34\delta^j$ on an initial tube with the data (9.33). We join it to any $P'\in Q_{\vartheta R}^-(P_0)$ by the cubic Hermite path, continued with zero acceleration before its start.

In the coordinates of $\Phi_\star$, the travel time lies in $[m+1,m+3]$ and the acceleration is at most $6(m+5)+8$. The radius becomes $\frac{r}{R\varepsilon_0}\in[r_0,1]$, so the initial duration and the two tube widths in (9.33) have positive lower bounds depending only on r_0. These tubes lie in $\Phi_\star(\mathcal{G})$ by (iii); the affine map preserves the path class and the ratios (9.12). Using the largest travel time and acceleration and the smallest duration and widths, Lemma 9.3 gives one constant $c_{\mathrm{mac}}>0$, depending only on d,λ,Λ,m,r_0, such that

$$u\ge\tfrac34 c_{\mathrm{mac}}\delta^j \qquad \text{on } Q_{\vartheta R}^-(P_0)\,. \tag{9.40}$$

Step 3: iteration over the levels. Let $\beta\in(0,1]$ and let u satisfy (9.35). We choose $N\in\mathbb{N}$ with $a^{-N}\beta>1-\eta_*$, which is possible because $a<1$ by (9.30), and then $r_0\in(0,1)$ so small that

$$L_0\sum_{k=0}^{N-1}a^{-k}<a^{-N}\beta-(1-\eta_*), \qquad L_0:=\frac{C_d m\,r_0^2}{|Q_1^-(0)|}\,. \tag{9.41}$$

The order is important: N is chosen from β, then r_0 from (9.41), and finally c_{mac} from Step 2. All earlier constants are already fixed. We put $E_j=\{u\ge\delta^j\}\cap\Sigma_R$, $e_j=\frac{|E_j|}{|\Sigma_R|}$ and $F_j=\{u\ge\delta^{j+1}\}\cap\Phi_\star(\mathcal{G})$ for $0\le j\le N$, so that $e_0\ge\beta$ by (9.35). If (9.39) holds for some $j\le N$ then (9.40) gives $u\ge\frac34 c_{\mathrm{mac}}\delta^N$ on $Q_{\vartheta R}^-(P_0)$ and we are done; assume therefore that (9.39) fails for every $j\le N-1$, and fix such a j.

We pull E_j, F_j back by $\Phi_\Sigma = \Phi_{\Phi_\star(P^\sharp),R\varepsilon_0}$ to sets E, F on the unit-cylinder geometry. Then $E \subseteq F \cap Q_1^-(0)$ and $F \cap Q_1^-(0) = \Phi_\Sigma^{-1}(E_{j+1})$. Every cylinder contained in $Q_1^-(0)$ on which E has density at least $1-\eta_*$ has radius less than r_0, since otherwise its image would satisfy (9.39). For each such cylinder $Q_r^-(P)$, the comparison region satisfies

$$\Phi_{\Phi_\Sigma(P),rR\varepsilon_0}(\mathcal{T}_*) = \Phi_\star\big(\Phi_{\Phi_{P^\sharp,1}(P),r}(\mathcal{T}_*)\big) \subseteq \Phi_\star(\mathcal{G})$$

by (ii) and (9.37); its closure lies in the open set of admissibility. The image of $Q_r^{m,+}(P)$ is $Q_{rR\varepsilon_0}^{m,+}(\Phi_\Sigma(P))$, so Lemma 9.5 gives $Q_r^{m,+}(P) \subseteq F$. Theorem 9.4 therefore applies. After division by $|Q_1^-(0)|$, it gives

$$e_j \le a\big(e_{j+1} + L_0\big), \qquad \text{that is} \qquad e_{j+1} \ge a^{-1}e_j - L_0\,. \tag{9.42}$$

Iterating (9.42) over $j = 0, \dots, N-1$ and using $e_0 \ge \beta$ and (9.41),

$$e_N \ge a^{-N}e_0 - L_0 \sum_{k=0}^{N-1} a^{-k} > 1 - \eta_*\,.$$

The cylinder Σ_R itself, of radius $R\varepsilon_0 \ge r_0 R\varepsilon_0$, therefore satisfies (9.39) at the level N, and (9.40) gives $u \ge \frac{3}{4}c_{\mathrm{mac}}\delta^N$ on $Q_{\vartheta R}^-(P_0)$. Both alternatives thus yield (9.36) with $\kappa_\beta = \frac{3}{4}c_{\mathrm{mac}}\delta^N > 0$, a constant depending only on β, d, λ, Λ, p and C_{A}. □

9.7. **Oscillation contraction.** We normalise the solution between -1 and 1. At least one of the sets $\{u \le 0\}$ and $\{u \ge 0\}$ occupies half the sampling cylinder. In the first case the growth lemma applied to $1-u$ lowers the upper bound on the target; in the second, applied to $1+u$, it raises the lower bound. Either way the oscillation decreases by a fixed factor. Only affine changes of the solution are used, so no truncation or extension by zero is needed.

Proposition 9.7 (Oscillation contraction)**.** *Let ϑ be as in Theorem 9.6. There is $q \in (0,1)$, depending only on d, λ, Λ, p and C_{A}, such that, for every $P_0 \in \mathbb{R}^{1+2d}$ and $R > 0$, every admissible solution u with Aleksandrov data (p, C_{A}) in an open set containing $\overline{Q_R^-(P_0)}$ satisfies*

$$\operatorname{osc}_{Q_{\vartheta R}^-(P_0)} u \le q \operatorname{osc}_{Q_R^-(P_0)} u\,. \tag{9.43}$$

Proof. We may assume $\kappa_{\frac12} \le 1$ and normalise u so that $-1 \le u \le 1$ and its oscillation on $Q_R^-(P_0)$ is two; the constant case is immediate. At least one of $\{u \le 0\}$ and $\{u \ge 0\}$ fills half the sampling cylinder. By replacing u with $-u$, we may assume it is $\{u \le 0\}$.

If O is the open set of admissibility, then for every $\epsilon > 0$ the function $1 - \frac{u}{1+\epsilon}$ is a positive admissible supersolution on $O \cap \{u < 1+\epsilon\}$, which contains $\overline{Q_R^-(P_0)}$. It is at least one on $\{u \le 0\}$. Theorem 9.6 gives

$$u \le (1+\epsilon)(1-\kappa_{\frac12}) \quad \text{on } Q_{\vartheta R}^-(P_0).$$

Letting $\epsilon \downarrow 0$ reduces the oscillation to at most $2 - \kappa_{\frac12}$. Undoing the normalisation proves (9.43) with $q = 1 - \frac12\kappa_{\frac12}$. □

Proof of Theorem 9.1. We use ϑ, q from Proposition 9.7 and put $\alpha = \min\{1, \frac{\log\frac1q}{\log\frac1\vartheta}\}$. Iterating at the same centre gives $\operatorname{osc}_{Q_{\vartheta^k R}^-(P_1)} u \le q^k \operatorname{osc}_{Q_R^-(P_1)} u$. For $\vartheta^{k+1}R < \rho \le \vartheta^k R$, inclusion of the cylinders and $q \le \vartheta^\alpha$ therefore imply

$$\operatorname{osc}_{Q_\rho^-(P_1)} u \le \vartheta^{-\alpha}\left(\frac{\rho}{R}\right)^{\alpha} \operatorname{osc}_{Q_R^-(P_1)} u. \tag{9.44}$$

For $P, P' \in \mathcal{C}$, we order the times so that P is earlier and put $\rho_* = d_k(P,P')$. If $\rho_* > 0$, both points lie in $\overline{Q_{\rho_*}^-(P')}$ by (9.4). The set

$$\overline{\Omega_{R_0,\mathcal{C}}} = \bigcup_{P'' \in \mathcal{C}} \overline{Q_{2R_0}^-(P'')}$$

is compact as the image of $\mathcal{C} \times \overline{Q_1^-(0)}$ under the kinetic affine map, and is contained in Ω. For $\rho_* < R_0$, we apply (9.44) with centre P' and radius R_0; continuity includes the two points in the closure. For $\rho_* \geq R_0$, we use the oscillation on $\Omega_{R_0,\mathcal{C}}$ directly. Both cases give (9.7); $\rho_* = 0$ means $P = P'$. □

Remark 9.8 (What the exponent is used for). The exponent p enters only through Lemma 9.2 and the choice (9.11). All that is required there of the right-hand side of (9.6) is that it tend to zero with the measure of the set $\{\psi > u\}$ on which the source is counted, so that every $p < \infty$ serves and Theorem 9.1 needs no lower bound on p. A bound by the supremum of the source alone, having no gain from the measure of that set, does not start the argument.

9.8. **The two coefficient classes.** For the classical solution class used here, the maximum principles apply on every compact inner cylinder.

Corollary 9.9 (Hölder continuity for coefficients depending on time and velocity)**.** *Let A satisfy (1.4) and let $\Omega \subseteq \mathbb{R}^{1+2d}$ be open. Every bounded $u \in C^{1,1,2}(\Omega)$ solving $\mathcal{L}_A u = 0$ almost everywhere in Ω satisfies (9.7), with C and α depending only on d, λ and Λ.*

Proof. We apply Proposition 7.1. For $P, P' \in \mathcal{C}$ with $d_k(P, P') < \frac{1}{4}R_0$, centre the cylinder at whichever point has the later time, say P. Both points lie in the closure of its inner half-cylinder. The spatial increment in Proposition 7.1 is one of those defining d_k, and continuity permits the point on the top face. Its outer cylinder lies in $\Omega_{R_0,\mathcal{C}}$, so the proposition gives (9.7). For pairs at distance at least $\frac{1}{4}R_0$, the oscillation bound gives the same conclusion after increasing C. □

Corollary 9.10 (Hölder continuity for autonomous coefficients in one space dimension)**.** *Let $d = 1$, let $a : \mathbb{R} \times \mathbb{R} \to [\lambda, \Lambda]$ be Borel measurable and let $\Omega \subseteq \mathbb{R}^3$ be open. Every bounded $u \in C^{1,1,2}(\Omega)$ solving $\mathcal{L}_a u = 0$ almost everywhere in Ω satisfies (9.7), with C and α depending only on λ and Λ.*

Proof. For every smooth test ψ, apply Theorem 1.2 and Corollary 1.3, with $p = 6$, to $\psi - u$ and $\psi + u$ on each compactly contained kinetic cylinder. Their sources are $\mathcal{L}_a\psi$, which is bounded and Borel. Thus u and $-u$ satisfy (9.6), and Theorem 9.1 applies. □

For a Borel coefficient these remain a priori estimates for solutions of the stated classical regularity; no existence assertion is made.

Appendix A. Construction of the solution operators

This appendix proves the three statements that were used but not proved in the body: the terminal solution operator and its kernels (Proposition 2.1), the Duhamel potential (Lemma 2.3), and the Green and exit measures on a velocity strip (Proposition 8.1).

Theorem A.1 (Hörmander [13, Theorem 1.1])**.** *Let $X_0, X_1, \dots, X_k$ be smooth real vector fields on an open set $\mathcal{O} \subseteq \mathbb{R}^N$ such that the Lie algebra generated by $X_0, \dots, X_k$ spans $\mathbb{R}^N$ at every point of $\mathcal{O}$. Then every distribution u on $\mathcal{O}$ with $(X_0 + \sum_{i=1}^k X_i^2 + c)u \in C^\infty(\mathcal{O})$, $c \in C^\infty(\mathcal{O})$, belongs to $C^\infty(\mathcal{O})$; here c is real-valued.*

For our operator, the positive square root $\beta = B^{\frac{1}{2}}$ is smooth. We put

$$X_i = \sum_j \beta_{ij}\partial_{v_j}, \qquad X_0 = \partial_\sigma + b(v)\cdot\nabla_z - \sum_k c_k\partial_{v_k}, \qquad c_k = \sum_{i,j} \beta_{ij}\partial_{v_j}\beta_{ik}.$$

Then $\mathcal{K} = X_0 + \sum_i X_i^2$. The X_i span the velocity directions, and the z-components of $[X_i, X_0]$ are $Db(v)\beta_i$, where β_i is the ith column of β. These span the z-directions by (2.1). Modulo these directions, X_0 gives ∂_σ. Thus Theorem A.1 applies, also when B depends on z.

Proof of Proposition 2.1. We fix a finite time interval $[\sigma_-, \tau]$ and $F \in C_c^\infty(\Omega_\tau \times \mathbb{R}^d)$. We write $\mathcal{K}_\varepsilon = \mathcal{K} + \varepsilon\Delta_z$, $0 \le \varepsilon \le 1$. We construct its terminal solution and then let $\varepsilon \downarrow 0$.

Comparison and boundary barriers. Straightening the moving domain by $Y = v - \gamma(\sigma)$ adds the drift $-\gamma'(\sigma)\cdot\nabla_Y$. On a bounded cylinder the usual maximum argument, applied to $u + \delta\sigma$, gives comparison for $\mathcal{K}_\varepsilon$, including $\varepsilon = 0$. We apply it successively across the finitely many smooth pieces of γ. Since b has at most linear growth, for C sufficiently large

$$\Phi = e^{C(\tau-\sigma)}(1 + |Y|^2 + |z|^2), \qquad \mathcal{K}_\varepsilon\Phi \le -\Phi \quad (0 \le \varepsilon \le 1).$$

Here C depends only on bounds for γ, γ', $b(0)$, L_b, d, Λ on the fixed interval. Comparing $u - \delta\Phi$ on bounded truncations and then letting $\delta \downarrow 0$ proves comparison for bounded functions on the full moving domain.

We suppose $\Omega = B_{r_0}$ and put $\mathsf{d} = r_0 - |v - \gamma(\sigma)|$. With L_γ a Lipschitz constant for γ, we set

$$\kappa = \frac{1 + L_\gamma}{\lambda}, \qquad w(t) = \frac{1 - e^{-\kappa t}}{\kappa}.$$

In a boundary collar, with $e = \frac{v-\gamma}{|v-\gamma|}$,

$$\mathcal{K}_\varepsilon w(\mathsf{d}) = w'' e\cdot Be - \frac{w'}{|v-\gamma|}\big(\operatorname{tr} B - e\cdot Be\big) + w'\gamma'\cdot e \le w'(L_\gamma - \lambda\kappa) \le 0.$$

We choose $\mathsf{d}_* > 0$ smaller than $\frac14 r_0$ and half the distance of $\operatorname{supp} F$ from the terminal moving velocity boundary. The functions $\|F\|_\infty \frac{w(\mathsf{d})}{w(\mathsf{d}_*)}$ and $\pm F + (\tau-\sigma)M_F$, where $M_F = \sup_{0\le\varepsilon\le1}\|\mathcal{K}_\varepsilon F\|_\infty$, are lateral and terminal barriers. Comparison gives

$$|u_\varepsilon| \le \|F\|_\infty \min\left\{1, \frac{w(\mathsf{d})}{w(\mathsf{d}_*)}\right\}, \qquad |u_\varepsilon - F| \le (\tau-\sigma)M_F \quad \text{near } \sigma = \tau. \tag{A.1}$$

Compact support of F, its positive terminal-boundary distance and the common Lipschitz bound on the approximating curves give a terminal neighbourhood uniform in the exhaustion and in ε. For $\Omega = \mathbb{R}^d$ there is no lateral barrier and the terminal bound holds throughout the interval.

Construction. For $\varepsilon > 0$, we approximate γ uniformly by smooth curves γ_k with the same Lipschitz bound. We choose $\alpha_k \downarrow 0$ with $\alpha_{k+1} \le \frac14\alpha_k$ and $\|\gamma_k - \gamma\|_\infty \le \frac12\alpha_k$. The balls $B_{r_0-\alpha_k}(\gamma_k(\sigma))$ then increase to Ω_σ. In the straightened variables $V = v - \gamma_k(\sigma)$, we use the ellipsoids

$$\frac{|V|^2}{(r_0-\alpha_k)^2} + \frac{|z|^2}{L^2} < 1$$

and let $L \to \infty$. For $\Omega = \mathbb{R}^d$, we use balls in $\mathbb{R}^{2d}$. These are fixed smooth bounded spatial domains. We take zero lateral data and the transformed terminal datum F, choosing k, L large enough that its compact support lies strictly inside the terminal domain. The datum and all its derivatives then vanish near the terminal–lateral corner. After time reflection, the coefficients are smooth and uniformly parabolic for each fixed $\varepsilon > 0$, and the data satisfy the boundary compatibility conditions. Classical Dirichlet solvability [19, Theorem 5.14] therefore applies. Its regularity constants may depend on the truncation and on ε; the uniform bounds used in the limit come from our comparison barriers.

The barriers are uniform along the exhaustion. For fixed $\delta > 0$, we choose m with $\frac32\alpha_m < \delta$ and compare on $B_{r_0-\delta}(\gamma(\sigma)) \times \{|z| < R\}$, contained in the approximating domains for large enough ellipsoidal truncations. Its velocity face is within $\delta + \frac32\alpha_m$ of each approximating ball boundary. The terminal data agree, and the difference at the artificial position boundary is at most $2\|F\|_\infty$. Growth comparison gives

$$|u_\varepsilon^{(k)} - u_\varepsilon^{(l)}| \le C\left(\frac{\Phi}{1+R^2} + w\left(\delta + \frac32\alpha_m\right)\right) \quad (k, l \ge m)$$

on each common inner cylinder, once its ellipsoidal truncations are large enough. We let $R \to \infty$, $m \to \infty$ and $\delta \downarrow 0$. This proves local uniform convergence. In the whole space the same argument uses only Φ. The limit solves $\mathcal{K}_\varepsilon u_\varepsilon = 0$ distributionally, is smooth by Theorem A.1, and satisfies (A.1). For

$\varepsilon > 0$, put $Y_j = \sqrt{\varepsilon}\,\partial_{z_j}$. Then $\mathcal{K}_\varepsilon = X_0 + \sum_i X_i^2 + \sum_j Y_j^2$, and $X_0, X_1, \dots, X_d, Y_1, \dots, Y_d$ already span $\mathbb{R}^{1+2d}$ at every point, so Theorem A.1 applies to u_ε without brackets.

We now take a weak-$*$ limit in L^∞ as $\varepsilon \downarrow 0$. For every compactly supported smooth test function,

$$\int u_\varepsilon(\mathcal{K}^*\psi + \varepsilon\Delta_z\psi) = 0, \qquad \mathcal{K}^*\psi = -\partial_\sigma\psi + \sum_{i,j}\partial_{v_iv_j}(B_{ij}\psi) - b(v)\cdot\nabla_z\psi. \tag{A.2}$$

The last term carrying ε tends to zero, so the limit solves $\mathcal{K}u = 0$ and is smooth by Theorem A.1. The bounds (A.1) pass first almost everywhere, then everywhere by continuity. They give the required terminal and lateral traces, including the corners. Positivity and contraction pass from the approximations. Growth comparison gives uniqueness, independence of the subsequence, and independence of σ_-. This proves (i).

Kernels and composition. At a fixed starting point, $F \mapsto u(\sigma, v, z)$ is a positive functional on $C_c(\Omega_\tau \times \mathbb{R}^d)$ of norm at most one. Riesz representation gives $K_{\sigma,\tau}$; we use its integral to define $\mathsf{S}_{\sigma,\tau}$ on bounded Borel data.

For $\sigma < r < \tau$ and smooth $F \geq 0$, choose smooth compactly supported $\Phi_j \uparrow \mathsf{S}_{r,\tau}F$. Comparison gives

$$V := \sup_j \mathsf{S}_{\sigma,r}\Phi_j \leq \mathsf{S}_{\sigma,\tau}F.$$

The bounded monotone limit V is a distributional solution, hence smooth. It has zero lateral trace by domination. Its terminal trace at r is $\mathsf{S}_{r,\tau}F$: approximate that trace by Φ_j uniformly on compact sets, use the terminal bound in (A.1), and control the complement by $\delta\Phi$. Growth comparison now gives equality. Uniform approximation and the monotone class theorem extend composition to bounded Borel data.

For joint measurability, we choose continuous cutoffs $\chi_j(\tau, w) \uparrow \mathbf{1}_{\Omega_\tau}(w)$, smooth in w with locally bounded derivatives, whose supports stay a positive distance from $\partial\Omega_\tau$ locally uniformly in τ. We fix $\psi \in C_c^\infty(\mathbb{R}^{2d})$ and write $F_\tau = \chi_j(\tau, \cdot)\psi$. Near any fixed terminal time, these data have a common compact support away from the moving boundary. Comparison with $F_\tau \pm C_j(\tau - r)$ gives

$$\|\mathsf{S}_{r,\tau}F_\tau - F_\tau\|_\infty \leq C_j(\tau - r)$$

for nearby $r \leq \tau$, with C_j locally uniform. Composition and contraction therefore give

$$\|\mathsf{S}_{\sigma,\tau}F_\tau - \mathsf{S}_{\sigma,\tau_0}F_{\tau_0}\|_\infty \leq \|F_\tau - F_{\tau_0}\|_\infty + C_j|\tau - \tau_0|, \qquad \sigma < \min\{\tau, \tau_0\}.$$

For fixed τ_0 the solution is smooth in (σ, v, z), and the terminal bound also handles $\sigma = \tau$. Thus these cutoff-test evaluations are jointly continuous. Monotone convergence for $\psi \geq 0$, followed by a monotone class argument, gives Borel evaluation of every Borel set. This proves (ii) and (iii).

Domain monotonicity follows by comparing the two terminal solutions on the smaller domain and then using Riesz uniqueness. When B is independent of z, uniqueness also gives translation covariance. Consequently the velocity marginal is independent of the starting z. For $\varphi \in C_c^\infty(\Omega_\tau)$, approximate the terminal datum $\varphi(w)$ by $\varphi(w)\chi(\frac{z'}{N})$, with increasing cutoffs in z'. The limit V is the bounded monotone limit of the classical solutions $\mathsf{S}F_N$; by dominated convergence it solves $\mathcal{K}V = 0$ distributionally, so it is smooth by Theorem A.1. It is independent of z, and hence solves $\partial_\sigma V + B : D_v^2 V = 0$. The two boundary estimates are uniform in N: the lateral distance is unchanged, and the extra derivatives of $\chi(\frac{z'}{N})$ are bounded by $\frac{C}{N} + \frac{C}{N^2}$ on the velocity support of φ. Thus the limit has datum φ and zero lateral trace. Parabolic comparison identifies it with the stated parabolic solution. This proves the remaining items. □

Proof of Lemma 2.3. Kernel measurability and contraction make (2.7) a Borel integral with $0 \leq W_g \leq (\tau_+ - \sigma)\|g\|_\infty$. Its identification with the Green integral is (2.6). For $\psi \in C_c^\infty(\mathcal{U})$, put $u_r(\sigma) = \mathsf{S}_{\sigma,r}g(r)$.

Fubini, integration by parts up to $\sigma = r - h$, and the terminal trace give

$$\int_{\mathcal{U}} W_g \mathcal{K}^* \psi = -\int_{-\infty}^{\tau_+} \lim_{h\downarrow 0} \int_{\Omega_{r-h}\times\mathbb{R}^d} u_r(r-h)\psi(r-h)\,\mathrm{d}v\,\mathrm{d}z\,\mathrm{d}r = -\int_{\mathcal{U}} g\psi.$$

Contraction justifies dominated convergence; the only boundary term is at $r - h$, since ψ is compactly supported in $\mathcal{U}$. Thus $\mathcal{K}W_g = -g$ distributionally, and Theorem A.1 gives smoothness.

The velocity distance of $\operatorname{supp} g$ from the moving boundary is positive. Hence the lateral estimate (A.1) holds uniformly for the data $g(r,\cdot)$ and, after integration, gives

$$W_g \le (\tau_+ - \sigma)\|g\|_\infty \min\left\{1, \frac{w(\mathsf{d})}{w(\mathsf{d}_*)}\right\}.$$

This proves the zero lateral trace; the terminal trace follows from the first bound. Omitting z gives the parabolic statement. Finally, the distributional identity extends to bounded Borel sources by a monotone class argument and dominated convergence. Smoothness is asserted only when the source is smooth. □

Proof of Proposition 8.1. We apply Proposition 2.1 with $z = X$, $b(v) = v$, $B = a$ and $\Omega = H$. We define

$$\int F\,\mathrm{d}\Gamma_e^H = \int_{s_e}^{s_+} \int_{H\times\mathbb{R}} F(r, X', w)\, K^H_{s_e,r}((v_e, X_e),\, \mathrm{d}w\,\mathrm{d}X')\,\mathrm{d}r. \tag{A.3}$$

This is a positive Borel measure, Borel in e and independent of s_-. Contraction gives its mass bound $s_+ - s_e$. For the other bound, put $\mathfrak{q}(v) = (v - h_-)(h_+ - v)$. Since $\mathcal{K}_a\mathfrak{q} = -2a$, comparison with $\frac{\mathfrak{q}}{2\lambda}$ gives $W_{g_j} \le \frac{\mathfrak{q}}{2\lambda}$ for smooth compactly supported $0 \le g_j \uparrow 1$. Monotone convergence proves (8.3). In particular, for every bounded Borel source,

$$|W_g| \le \|g\|_\infty \min\left\{s_+ - s, \frac{\mathfrak{q}(v)}{2\lambda}\right\}.$$

Thus W_g has zero terminal and velocity-boundary limits.

The exit measure. Let $\mathcal{E} = (\{s_+\}\times\mathbb{R}\times\overline{H}) \cup ([s_-, s_+]\times\mathbb{R}\times\partial H)$. For $\phi \in C_c^\infty(\mathbb{R}^3)$ put

$$u_\phi = \phi - W_{-\mathcal{K}_a\phi}.$$

The distributional Duhamel identity makes u_ϕ a bounded smooth solution of $\mathcal{K}_a u_\phi = 0$, with boundary value ϕ on $\mathcal{E}$. Growth comparison gives positivity and $\|u_{\phi_1} - u_{\phi_2}\|_\infty \le \|\phi_1 - \phi_2\|_{L^\infty(\mathcal{E})}$. Uniform approximation therefore defines a positive boundary solution operator E on $C_c(\mathcal{E})$. Riesz representation gives $\mathsf{E}h(e) = \int h\,\mathrm{d}\omega_e^H$, with mass at most one. For $h \in C_c(\mathcal{E})$, $\mathsf{E}h$ is continuous as a uniform limit of the smooth u_ϕ; the monotone class theorem therefore gives Borel dependence of ω_e^H on e. By construction,

$$\phi(e) = \int_{\mathcal{E}} \phi\,\mathrm{d}\omega_e^H - \int_{\mathscr{S}} \mathcal{K}_a\phi\,\mathrm{d}\Gamma_e^H. \tag{A.4}$$

This identity also holds for smooth ϕ bounded on $\overline{\mathscr{S}}$ with $\mathcal{K}_a\phi$ bounded on $\mathscr{S}$. Indeed, we insert a position cutoff η_R and a time–velocity cutoff equal to one near $[s_-, s_+]\times\overline{H}$. The only remaining error satisfies

$$\left|\int v\eta_R'(X)\phi\,\mathrm{d}\Gamma_e^H\right| \le \frac{2V_H}{R}\|\phi\|_\infty \Gamma_e^H(\mathscr{S}) \longrightarrow 0.$$

Taking $\phi = 1$ gives $\omega_e^H(\mathcal{E}) = 1$; time cutoffs supported before s_e show that ω_e^H is carried by $\{s \ge s_e\}$.

Support and the full Green identity. We choose a smooth bounded increasing function χ with bounded derivatives, zero on $(-\infty, 0]$ and with $\chi, \chi' > 0$ on $(0,\infty)$. For $\delta > 0$ put $\rho_\pm = \pm(X - X_e) - (V_H + \delta)(s - s_e)$. Then $\mathcal{K}_a\chi(\rho_\pm) \le -\delta\chi'(\rho_\pm)$ and $\chi(\rho_\pm(e)) = 0$. The Green identity is a sum of two non-negative integrals, so both measures vanish on $\{\rho_\pm > 0\}$. We let $\delta \downarrow 0$ to obtain the cone. If the velocity has sign ς, use also $\rho = -\varsigma(X - X_e) + (v_H - \delta)(s - s_e)$ to obtain the lower cone bound.

There is no exit mass on the initial velocity face. To see this, let m_H and ℓ be the midpoint and half-length of H, and $d_0 = \ell - |v_e - m_H| > 0$. The function

$$\Upsilon_{\theta,\kappa}(s,v) = e^{\Lambda\kappa^2(s_- + \theta - s)} \frac{\cosh(\kappa(v - m_H))}{\cosh(\kappa\ell)}$$

satisfies $\mathcal{K}_a \Upsilon_{\theta,\kappa} \le 0$ and is at least one on $\{v \in \partial H,\ s \le s_- + \theta\}$. The Green identity therefore gives

$$\omega_e^H\{v \in \partial H,\ s \le s_- + \theta\} \le 2e^{\Lambda\kappa^2\theta - \kappa d_0} = 2e^{-\frac{d_0^2}{4\Lambda\theta}}, \qquad \kappa = \frac{d_0}{2\Lambda\theta}.$$

Letting $\theta \downarrow 0$ shows that ω_e^H is carried by $\partial^\dagger \mathscr{S}$. Uniqueness of bounded homogeneous solutions with prescribed exit values now also proves its independence of s_-.

For later use we extend (A.4) to every bounded $\phi \in C^{1,1,2}(\mathscr{S}) \cap C(\mathscr{S} \cup \partial^\dagger \mathscr{S})$ with bounded $\mathcal{K}_a\phi$, at an interior starting point. The function $u = \phi - W_{-\mathcal{K}_a\phi}$ is a bounded smooth homogeneous solution with exit value ϕ. To approximate its boundary values, take a smooth increasing χ_1, zero below 0 and one above 1, and put

$$\Theta_j = \sum_{\pm} \chi_1\big(\pm(X - X_e) - (V_H + 1)(s - s_e) - j\big).$$

Then $\mathcal{K}_a\Theta_j \le 0$, $\Theta_j(e) = 0$, and $\Theta_j \ge 1$ when $|X - X_e| \ge j + 1 + (V_H + 1)(s_+ - s_-)$. We choose a smooth compactly supported position cutoff $0 \le \rho_j \le 1$, equal to one when $|X - X_e| \le j + 1 + (V_H + 1)(s_+ - s_-)$, and a smooth time cutoff $0 \le \eta_j \le 1$, zero for $s \le s_- + \frac{1}{2j}$ and one for $s \ge s_- + \frac{1}{j}$. For large j, the boundary datum $h_j = \rho_j\eta_j\phi$, extended by zero at the initial velocity face, belongs to $C_c(\mathcal{E})$ and converges to ϕ on $\partial^\dagger\mathscr{S}$. On $\partial^\dagger\mathscr{S}$, its discarded part is bounded by $\|\phi\|_\infty(\Theta_j + \Upsilon_{\frac{1}{j},\kappa_j})$, where $\kappa_j = \frac{jd_0}{2\Lambda}$: the first barrier controls the position cutoff, and the second is at least one on the early velocity face. Growth comparison yields

$$|u(e) - \mathsf{E}h_j(e)| \le 2\|\phi\|_\infty e^{-\frac{jd_0^2}{4\Lambda}} \longrightarrow 0.$$

Dominated convergence against ω_e^H proves the Green identity for this class. The class in (i) also permits a starting point with $s_e = s_-$: we first lower s_- and use the independence just proved.

Monotonicity and decomposition. Formula (A.3) and domain comparison give monotonicity in s_+ and H. Integrating in the starting measure gives (iv), and Lemma 2.3 gives (vi).

For $H_1 \subseteq H_2$, we take $F \in C_c^\infty(\mathscr{S}_{H_2})$ and h continuous and compactly supported on its closed exit boundary. The function

$$W(e') = \int F \,\mathrm{d}\Gamma_{e'}^{H_2} + \int h \,\mathrm{d}\omega_{e'}^{H_2}$$

is bounded and smooth, solves $\mathcal{K}_a W = -F$, and has exit value h. Its restriction to the smaller strip belongs to the enlarged test class just proved, including across the internal boundary Σ. We apply the Green identity there, split the boundary integral along Σ, and integrate against ν. This gives

$$\int F \,\mathrm{d}\Gamma_\nu^{H_2} + \int h \,\mathrm{d}\omega_\nu^{H_2} = \int F \,\mathrm{d}(\Gamma_\nu^{H_1} + \Gamma_{\tau\nu}^{H_2}) + \int h \,\mathrm{d}(\omega_\nu^{H_1}|_{\partial^\dagger\mathscr{S}_{H_2}} + \omega_{\tau\nu}^{H_2}).$$

Taking $h = 0$ and $F = 0$ separately proves (v) by Riesz uniqueness. For a finite union of intervals, we add the component identities.

Finally, we let $s_+ \uparrow \infty$. The Green measures increase, with total mass bounded by $\frac{\mathfrak{q}(v_e)}{2\lambda}$. Their limit inherits the cone and monotonicity. For tests vanishing after a finite time T, the Green integral is already independent of $s_+ > T$. The identity then shows, by Riesz uniqueness, that the exit measures restricted to $\{s < T,\ v \in \partial H\}$ are also independent of $s_+ > T$. These consistent restrictions define the infinite-horizon exit measure and give the claimed limiting identity. □

Appendix B. Construction of the Bellman barrier

We prove Proposition 8.10. The first step excludes positive stationary adjoint measures with a certain homogeneity. The second turns that exclusion into a barrier by separation. We work with measures throughout; no smooth adjoint density or extremising coefficient is assumed.

B.1. **The homogeneous adjoint exponent.** We write $\delta_r(X,v)=(r^3X,rv)$ and $\rho(X,v)=(X^2+v^6)^{\frac16}$. We consider nonzero pairs of positive Radon measures (μ,η) on the punctured plane satisfying

$$\begin{aligned} -\partial_X(v\mu)+\partial_{vv}\eta=0, \qquad & \lambda\mu\le\eta\le\Lambda\mu,\\ \mu(\delta_rE)=r^{4-\beta}\mu(E), \qquad & \eta(\delta_rE)=r^{4-\beta}\eta(E). \end{aligned} \tag{B.1}$$

The exponent 4 is the homogeneous dimension of position and velocity, without time. We call β the density degree, meaning the scaling in this display; a density need not exist. We define β_* as the supremum of the degrees for which such a pair exists. The time integral of the constant-coefficient Gaussian kernel gives a positive stationary fundamental measure of degree two. Its source is at the origin, which is excluded here, so $\beta_*\ge2$.

Lemma B.1 (A strict gap below degree three)**.** *We have $2\le\beta_*<3$, and β_* depends only on $\frac{\Lambda}{\lambda}$.*

Proof. We first exclude every $\beta\ge3$. On $X>0$ we write $X=s^3$ and $v=sy$. Homogeneity gives

$$\mathrm{d}\mu=s^{3-\beta}\,\mathrm{d}s\,F(\,\mathrm{d}y), \qquad \mathrm{d}\eta=s^{3-\beta}\,\mathrm{d}s\,H(\,\mathrm{d}y), \qquad \lambda F\le H\le\Lambda F.$$

Testing (B.1) with separated functions of s and y yields

$$\int\varphi''\,\mathrm{d}H-\frac13\int y^2\varphi'\,\mathrm{d}F+\frac{\beta-2}{3}\int y\varphi\,\mathrm{d}F=0. \tag{B.2}$$

Indeed, the transport derivative of $\zeta(s)\varphi(y)$ is $\frac{y}{3s}\zeta'\varphi-\frac{y^2}{3s^2}\zeta\varphi'$; integration by parts in s gives the last term in (B.2).

The distribution $J=H'+\frac13y^2F$ satisfies $J'=-\frac{\beta-2}{3}yF$, so it has a locally bounded-variation representative. Thus H' is a Radon measure. A measure whose derivative is a measure on an interval has a bounded-variation density: subtracting a primitive of the derivative leaves a constant distribution. We obtain $H=h\,\mathrm{d}y$, and comparison gives $F=f\,\mathrm{d}y$ with $\lambda f\le h\le\Lambda f$. The identities then imply $h,J\in W^{1,1}_{\mathrm{loc}}$.

We set

$$c_\beta=\frac{\beta-2}{3}, \qquad d_\beta=\frac{\beta-3}{3}, \qquad I(y)=\int_0^y zf(z)\,\mathrm{d}z, \qquad M(y)=\int_0^y z^2f(z)\,\mathrm{d}z.$$

Two integrations give

$$h(y)=h_0+j_0y-c_\beta yI(y)+d_\beta M(y), \qquad h_0=h(0), \quad j_0=J(0). \tag{B.3}$$

Local finiteness on rectangles meeting $X=0$ away from the origin gives the tail condition

$$\int_{|y|>2}f(y)|y|^{\beta-4}\,\mathrm{d}y<\infty. \tag{B.4}$$

For example, on a rectangle with $1\le v\le2$, substituting $s=\frac{v}{y}$ in $s^{3-\beta}\,\mathrm{d}s\,f(y)\,\mathrm{d}y$ gives the factor $y^{\beta-4}$ after integration in v. Negative velocities give the other tail.

For $y>0$, we have $I(y)\ge0$, $0\le M(y)\le yI(y)$ and $c_\beta-d_\beta=\frac13$. Non-negativity in (B.3) implies

$$0\le\frac{h(y)}{y}\le\frac{h_0}{y}+j_0-\frac13I(y).$$

Hence $I(\infty)<\infty$. Dominated convergence gives $\frac{M(y)}{y}\to0$, and therefore

$$\lim_{y\to\infty}\frac{h(y)}{y}=j_0-c_\beta I(\infty)\ge0.$$

A positive limit would force $f(y) \geq cy$ for large y, contradicting (B.4). Thus $j_0 = c_\beta I(\infty)$, and

$$h(y) = h_0 + c_\beta y\big(I(\infty) - I(y)\big) + d_\beta M(y) \geq h_0.$$

Since $\beta \geq 3$, the weight in (B.4) has divergent integral at infinity. Comparison forces $h_0 = 0$. Near zero, (B.3) now gives $h(y) = j_0 y + o(|y|)$. Non-negativity on both sides of zero forces $j_0 = 0$. Consequently $I(\infty) = 0$, and $f = h = 0$ on the positive half-line.

For $y = -z < 0$, we have $0 \leq -M(-z) \leq zI(-z)$. With $h_0 = j_0 = 0$, (B.3) gives

$$h(-z) \geq (c_\beta - d_\beta) z I(-z) = \frac{z}{3} I(-z).$$

Any positive mass of f on the negative half-line would make $I(-z)$ bounded below for large z, again contradicting (B.4). Both angular measures vanish. Reflection $(X, v) \mapsto (-X, -v)$ gives the same conclusion on $X < 0$.

The remaining pair, if any, is supported on $X = 0$, $v \neq 0$. Writing $\mu = \delta_0(\,\mathrm{d}X) m(\,\mathrm{d}v)$ and $\eta = \delta_0(\,\mathrm{d}X) n(\,\mathrm{d}v)$, the equation becomes

$$-\delta_0'(X) \otimes (vm) + \delta_0(X) \otimes n'' = 0.$$

The independent normal distributions force $vm = 0$. Since the origin is excluded, $m = 0$, and comparison gives $n = 0$. This contradicts nontriviality.

Excluding degree three alone does not yet give a strict gap. We suppose, for a contradiction, that admissible degrees $\beta_j \uparrow 3$ exist. We choose $0 \leq \chi = \chi_0(\rho) \leq 1$, equal to one on $1 \leq \rho \leq 2$ and supported in $\frac{1}{2} < \rho < 4$, and normalise by $\int \chi \,\mathrm{d}\mu_j = 1$. The normalising integral is positive because a dilation of this annulus meets every nonzero homogeneous measure. On $A_0 = \{1 \leq \rho \leq 2\}$ we have $\mu_j(A_0) \leq 1$, and homogeneity gives

$$\mu_j(\delta_{2^k} A_0) = 2^{k(4-\beta_j)} \mu_j(A_0) \qquad (k \in \mathbb{Z}).$$

Every compact subset of the punctured plane is covered by finitely many of these annuli. The degrees are bounded, so their masses are uniformly bounded; the same holds for $\eta_j \leq \Lambda \mu_j$. A subsequence therefore converges vaguely to a pair (μ, η).

The normalisation passes to the limit, giving $\int \chi \,\mathrm{d}\mu = 1$. Testing with compactly supported smooth functions passes the adjoint equation and ellipticity inequalities to the limit. For a fixed $r > 0$ and compactly supported continuous F, we pass to the limit in

$$\int F \circ \delta_r \,\mathrm{d}\mu_j = r^{\beta_j - 4} \int F \,\mathrm{d}\mu_j,$$

and likewise for η_j. Thus the limit is a nonzero pair of degree three, which we have just excluded. This proves $\beta_* < 3$.

Finally, the map $T(X, v) = (\lambda X, v)$ commutes with δ_r. The transformed measures

$$\widetilde{\mu} = T_\# \mu, \qquad \widetilde{\eta} = \lambda^{-1} T_\# \eta$$

satisfy the same stationary equation, have the same degree, and obey $\widetilde{\mu} \leq \widetilde{\eta} \leq \frac{\Lambda}{\lambda} \widetilde{\mu}$. This invertible correspondence proves the claimed dependence on the ratio alone. □

B.2. **From the degree gap to a universal barrier.** We seek one homogeneous function Φ for which $v\Phi_X + b\Phi_{vv} > 0$ for every $b \in [\lambda, \Lambda]$. If no such function existed, separation would supply a positive measure that annihilates all these expressions. Extending that measure by homogeneity produces a stationary adjoint pair of degree $2 + \alpha$. Choosing $2 + \alpha > \beta_*$ rules out this alternative and yields the barrier.

Proof of Proposition 8.10. We fix $\beta_* - 2 < \alpha < 1$ and put $\Sigma = \{\rho = 1\}$. In the space $C(\Sigma \times [\lambda, \Lambda])$, we consider the linear subspace of functions

$$(z, b) \longmapsto v\Phi_X(z) + b\Phi_{vv}(z), \qquad z = (X, v),$$

where Φ ranges over the C^2, α-homogeneous functions on the punctured plane. If this subspace contained no strictly positive function, neither would its closure: strict positivity on a compact set is open in the uniform norm. Hahn–Banach separation from that open cone then gives a nonzero continuous functional which vanishes on the subspace and is non-negative on the cone. By approximation with strictly positive functions it is positive on all non-negative functions. Riesz representation and normalisation give a probability measure Π on $\Sigma \times [\lambda, \Lambda]$.

We extend it radially to two positive measures:

$$\overline{\mu}(F) = \int \int_0^\infty r^{1-\alpha} F(\delta_r z)\, \mathrm{d}r\, \mathrm{d}\Pi(z,b),$$
$$\overline{\eta}(F) = \int \int_0^\infty b r^{1-\alpha} F(\delta_r z)\, \mathrm{d}r\, \mathrm{d}\Pi(z,b).$$

They are nonzero and locally finite on the punctured plane, obey the ellipticity comparison, and have density degree $2+\alpha$. For $\zeta \in C_c^2(\mathbb{R}^2 \setminus \{0\})$, the function

$$\Phi_\zeta(z) = \int_0^\infty r^{-1-\alpha} \zeta(\delta_r z)\, \mathrm{d}r$$

is C^2 and α-homogeneous. The identity

$$(v\partial_X + b\partial_{vv})[\zeta(\delta_r z)] = r^2 \big[(rv)\zeta_X(\delta_r z) + b\zeta_{vv}(\delta_r z)\big]$$

shows that its annihilation by Π is exactly

$$\int v\zeta_X\, \mathrm{d}\overline{\mu} + \int \zeta_{vv}\, \mathrm{d}\overline{\eta} = 0.$$

Thus $(\overline{\mu}, \overline{\eta})$ is admissible in (B.1) with degree $2+\alpha > \beta_*$, a contradiction. The subspace therefore contains a strictly positive function. Its positive minimum on the compact set, followed by homogeneity, gives (8.33). □

We finally check the origin, as required in Lemma 8.11. The derivatives Φ_X, Φ_v and Φ_{vv} have homogeneous degrees $\alpha-3$, $\alpha-1$ and $\alpha-2$. All are greater than -4, so the derivatives are locally integrable. On $\partial\{\rho < \delta\}$, the normal surface elements in the X and v directions have sizes $O(\delta)$ and $O(\delta^3)$. Integration by parts gives an inner boundary term $O(\delta^{\alpha+1})$ for one position derivative, and terms $O(\delta^{\alpha+3})$ and $O(\delta^{\alpha+2})$ for two velocity derivatives. They tend to zero. No distribution supported at the origin is omitted, and (8.33) holds distributionally on the whole plane.

Appendix C. The autonomous counterexample

A stationary singularity gives a stronger obstruction than the shrinking smooth bump. We construct a positive homogeneous function H which solves the homogeneous equation away from the origin and vanishes there. Flattening $1-H$ near the origin preserves its height and confines its source to a small set. A decaying velocity barrier then removes the initial data without increasing the positive source.

Proposition C.1 (No smaller threshold uniform over ellipticity ratios)**.** *For every $d \geq 1$ and $1 \leq p < 4d$, there are fixed $0 < \lambda \leq \Lambda$, a fixed kinetic cylinder Q, an autonomous measurable matrix $A(x,v)$ with $\lambda I_d \leq A \leq \Lambda I_d$, and smooth non-negative functions U_j such that*

$$U_j = 0 \text{ on the initial and full lateral boundary of } Q, \qquad \sup_Q U_j \geq \frac{1}{4}, \qquad \|(\mathcal{L}_A U_j)_+\|_{L^p(Q)} \longrightarrow 0.$$

The same conclusion holds with smooth autonomous matrices A_j in place of A, with the same ellipticity bounds. Here $\mathcal{L}_A = \partial_t + v\cdot\nabla_x - A : D_v^2$. The bounds may depend on d, p, but do not change along the sequence.

C.1. **A positive homogeneous profile.** We fix $0 < \alpha < 1$ and seek a coefficient A and a function H with

$$H(r^3 x, rv) = r^\alpha H(x,v), \qquad H \asymp \rho^\alpha, \qquad A : D_v^2 H = v \cdot \nabla_x H, \qquad \rho = (|x|^2 + |v|^6)^{\frac{1}{6}}. \tag{C.1}$$

The equation holds away from the origin. We construct H separately for $d \geq 2$ and $d = 1$.

Dimensions at least two. For $x \neq 0$, we put $e = \frac{x}{|x|}$, $y = \frac{v}{|x|^{\frac{1}{3}}}$ and

$$H(x,v) = |x|^{\frac{\alpha}{3}} \phi(y,e).$$

Then $D_v^2 H = |x|^{\frac{\alpha-2}{3}} D_y^2 \phi$, while $v \cdot \nabla_x H = |x|^{\frac{\alpha-2}{3}} T_\alpha \phi$, where

$$T_\alpha \phi = (y \cdot e) \left(\frac{\alpha}{3} \phi - \frac{1}{3} y \cdot D_y \phi \right) + (y - (y \cdot e)e) \cdot D_e \phi.$$

Here D_e is the tangential gradient on the unit sphere. We arrange for $D_y^2 \phi$ to have a positive eigenvalue everywhere, and for $T_\alpha \phi > 0$ wherever that Hessian has no negative eigenvalue. A positive matrix can then weight its eigendirections to give the required trace.

We choose $\sigma > 0$ with $\frac{\sigma}{\sqrt{1-\alpha}} < \frac{1}{4}$ and start with

$$\psi(y,e) = C_0 + (\sigma^2 + |y-e|^2)^{\frac{\alpha}{2}}.$$

Writing $s = |y - e|$, its tangential and radial Hessian eigenvalues are

$$\alpha(\sigma^2 + s^2)^{\frac{\alpha}{2}-1}, \qquad \alpha(\sigma^2 + s^2)^{\frac{\alpha}{2}-2}\big(\sigma^2 - (1-\alpha)s^2\big).$$

Thus the Hessian has both signs outside $s \leq \frac{\sigma}{\sqrt{1-\alpha}}$. On that small ball, $y \cdot e \geq \frac{3}{4}$. Increasing C_0 adds $\frac{\alpha}{3}(y \cdot e) C_0$ to the transport term, so we choose it to make $T_\alpha \psi \geq 1$ there.

To extend through $x = 0$, we remove the dependence on e for large y. We choose a smooth radial cutoff χ_R, equal to one on $|y| \leq R$ and zero on $|y| \geq 2R$, with $|D^k \chi_R| \leq C_k R^{-k}$, and set

$$\phi(y,e) = |y|^\alpha + \chi_R(y)(\psi(y,e) - |y|^\alpha).$$

Near zero this means $\phi = \psi$, so the apparent singular terms cancel. On the transition annulus,

$$\|D_y^2 \phi - D_y^2 |y|^\alpha\| \leq C\big(C_0 R^{-2} + R^{\alpha-3} + \sigma^2 R^{\alpha-4}\big).$$

The eigenvalues of $D_y^2 |y|^\alpha$ are $\alpha |y|^{\alpha-2}$ in the tangential directions and $-\alpha(1-\alpha)|y|^{\alpha-2}$ radially. Since $\alpha > 0$, choosing R large preserves both signs on the annulus. Hence $M = D_y^2 \phi$ always has a positive eigenvalue, and $b = T_\alpha \phi > 0$ wherever $M \geq 0$. Outside $|y| \geq 2R$, $b = 0$.

We write $m_+ = \operatorname{tr} M_+$, $m_- = \operatorname{tr} M_-$ and Π_+, Π_-, Π_0 for the spectral projections of M. Thus $m_+ > 0$ and $m_- = 0$ implies $b > 0$. On the compact set $\{|y| \leq 2R\} \times \mathbb{S}^{d-1}$, we can choose $c_- \geq 1$ so large that $b + c_- m_- > 0$: on the closed set $\{b \leq 0\}$, m_- has a positive minimum. The matrix

$$A = \frac{b + c_- m_-}{m_+} \Pi_+ + c_- \Pi_- + \Pi_0$$

is measurable, positive definite and satisfies $A : M = b$. Its eigenvalues have fixed positive lower and finite upper bounds on that compact set. Outside it, $\frac{b + c_- m_-}{m_+} = \frac{c_-(1-\alpha)}{d-1}$, so the bounds are global.

We use this matrix at (y,e) when $x \neq 0$. Near $(0,v)$ with $v \neq 0$, the profile equals $|v|^\alpha$. It extends smoothly there. At $x = 0$, $v \neq 0$, we set

$$A(0,v) = c_- \widehat{v} \otimes \widehat{v} + \frac{c_-(1-\alpha)}{d-1}(I_d - \widehat{v} \otimes \widehat{v}), \qquad \widehat{v} = \frac{v}{|v|},$$

so the equation continues to hold. We set $A(0,0) = I_d$ and $H(0,0) = 0$, including the value 1 in the ellipticity bounds. Positivity and compactness on $\{\rho = 1\}$ prove $H \asymp \rho^\alpha$; homogeneity also gives $|D_v H| \leq C \rho^{\alpha-1}$.

Dimension one. We put $\gamma = \frac{\alpha}{3}$ and choose

$$a_\Lambda(x,v) = \begin{cases} \Lambda, & xv < 0, \\ 1, & xv \geq 0. \end{cases}$$

Here $\lambda = 1$, and the ratio $\Lambda > 1$ will be fixed below. The values on the axes do not affect the almost-everywhere equation. For $x > 0$, the ansatz $H(x,v) = x^\gamma F(s)$, $s = -\frac{v}{x^{\frac{1}{3}}}$, reduces the equation to

$$a(s)F'' - \frac{s^2}{3}F' + \gamma s F = 0, \qquad a(s) = \Lambda \ (s > 0), \quad a(s) = 1 \ (s < 0).$$

With $z = \frac{s^3}{9a}$ this is Kummer's equation. We use its standard solutions $M(a,b,z)$ and $U(a,b,z)$ [4, Sections 13.2, 13.4 and 13.7], and set

$$\begin{aligned} F_+(s) &= U\left(-\gamma, \frac{2}{3}, \frac{s^3}{9\Lambda}\right), \qquad s > 0, \\ F_-(s) &= A_\gamma M\left(-\gamma, \frac{2}{3}, \frac{s^3}{9}\right) + \frac{B_\gamma s}{(9\Lambda)^{\frac{1}{3}}} M\left(\frac{1}{3} - \gamma, \frac{4}{3}, \frac{s^3}{9}\right), \qquad s < 0, \\ A_\gamma &= \frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{1}{3} - \gamma)}, \qquad B_\gamma = \frac{\Gamma(-\frac{1}{3})}{\Gamma(-\gamma)}. \end{aligned}$$

The connection formula gives the same value and first derivative at $s = 0$; the equation gives $F''(0) = 0$. Both constants A_γ, B_γ are positive.

We extend by $H(-x,-v) = H(x,v)$. The large-argument expansions give

$$\begin{aligned} F_+(s) &= (9\Lambda)^{-\gamma} s^\alpha (1 + O(s^{-3})), \qquad s \to +\infty, \\ F_-(s) &= 9^{-\gamma}(P_\gamma - Q_\gamma \Lambda^{-\frac{1}{3}})|s|^\alpha + O(|s|^{\alpha-3}), \qquad s \to -\infty, \\ P_\gamma &= \cos(\pi\gamma) - \frac{\sin(\pi\gamma)}{\sqrt{3}} > 0, \qquad Q_\gamma = \frac{2\sin(\pi\gamma)}{\sqrt{3}} > 0. \end{aligned}$$

Continuity across $x = 0$ therefore requires

$$P_\gamma - Q_\gamma \Lambda^{-\frac{1}{3}} = \Lambda^{-\gamma}.$$

The left side minus the right side is negative at $\Lambda = 1$ and tends to $P_\gamma > 0$ as $\Lambda \to \infty$. We fix a finite root. The common trace is $H(0,v) = (9\Lambda)^{-\gamma}|v|^\alpha$.

The transformation $U(-\gamma, \frac{2}{3}, z) = z^{\frac{1}{3}} U(\frac{1}{3} - \gamma, \frac{4}{3}, z)$ and the positive integral representation of the latter give $F_+ > 0$. The function F_- is positive at zero and for large negative s. A negative interior minimum would satisfy $F' = 0$, $F'' \geq 0$, whereas its equation gives $F'' = -\gamma s F < 0$. A zero minimum is excluded by uniqueness of the ordinary differential equation. Hence $H > 0$ away from the origin.

The differentiated expansions follow from $M' = \frac{a}{b} M(a+1, b+1, \cdot)$ and $U' = -aU(a+1, b+1, \cdot)$ [4, 13.3.15 and 13.3.22], applying the same expansions with shifted parameters and the chain rule for the arguments $\frac{s^3}{9\Lambda}$ and $\frac{s^3}{9}$. They give, with $C_\alpha = (9\Lambda)^{-\gamma}$,

$$\frac{\mathrm{d}^j}{\mathrm{d}s^j}\big(F_\pm(s) - C_\alpha |s|^\alpha\big) = O(|s|^{\alpha-3-j}), \qquad j = 0, 1, 2,$$

as $s \to \pm\infty$. In $H_x = x^{\gamma-1}(\gamma F - \frac{s}{3}F')$, the leading terms cancel. The remainder is bounded by $C|v|^{\alpha-3}$, locally bounded away from $v = 0$. The same expansions give the common traces of H_v, H_{vv} across $x = 0$. Across $v = 0$, F, F', F'' agree. Thus H_x, H_v, H_{vv} are bounded weak derivatives on compact sets away from the origin, with no singular measures on the axes. The equation holds almost everywhere and the weak chain rule applies there. Positivity and homogeneity again give (C.1) and $|H_v| \leq C\rho^{\alpha-1}$.

C.2. **Flattening the singularity and removing the initial data.** We use either profile and put $\Omega = \{H < 1\}$. We choose a smooth convex Ψ on $[0,\infty)$, zero on $[0,1]$, with $0 \le \Psi' \le 1$ and $\Psi(s) = s - c_\Psi$ for $s \ge 2$, where $1 < c_\Psi < 2$. For small $r > 0$, we set

$$u_r = 1 - c_\Psi r^\alpha - r^\alpha \Psi\left(\frac{H}{r^\alpha}\right).$$

Then $0 \le u_r \le 1$ on Ω, $u_r = 0$ on its boundary and $u_r(0,0) \to 1$. The singularity has been flattened. The chain rule gives the non-negative stationary source

$$f_r = v \cdot \nabla_x u_r - A : D_v^2 u_r = r^{-\alpha} \Psi''\left(\frac{H}{r^\alpha}\right) \langle A D_v H, D_v H\rangle.$$

Its support lies in $\{r^\alpha \le H \le 2r^\alpha\}$, a set of volume at most Cr^{4d}. There $|D_v H| \le Cr^{\alpha-1}$, so

$$\|f_r\|_{L^p(\Omega)} \le C r^{\alpha - 2 + \frac{4d}{p}}. \tag{C.2}$$

For the given $p < 4d$, we choose $\max\{0, 2 - \frac{4d}{p}\} < \alpha < 1$. All coefficient and geometric choices are now fixed; only r tends to zero.

To obtain zero initial data, we choose $R > \sup_{\overline{\Omega}} |v|$ and set

$$B(t,v) = e^{-\mu t}\left(2 - \frac{|v|^2}{R^2}\right), \qquad \mu = \frac{d\lambda}{R^2}, \qquad T = \frac{\log 8}{\mu}.$$

We have $B(0,\cdot) > 1$, $0 < B(T,\cdot) \le \frac{1}{4}$, and

$$\mathcal{L}_A B = e^{-\mu t}\left[\frac{2 \operatorname{tr} A}{R^2} - \mu\left(2 - \frac{|v|^2}{R^2}\right)\right] \ge 0.$$

We choose a smooth convex θ, zero on $(-\infty, 0]$, with $0 \le \theta' \le 1$ and $\theta(s) \ge s - \frac{1}{8}$ for $s \ge 0$. Then $U_r = \theta(u_r - B)$ has zero initial and lateral values and

$$\mathcal{L}_A U_r \le \theta'(u_r - B)(f_r - \mathcal{L}_A B) \le f_r.$$

For small r, $U_r(T,0,0) \ge \frac{3}{8}$, while the positive source norm is at most $T^{\frac{1}{p}} \|f_r\|_{L^p(\Omega)} \to 0$. The time T does not depend on r.

We choose S with $S^2 > T$ and $\overline{\Omega} \subset B_{S^3} \times B_S$, and use the fixed cylinder

$$Q = Q_S^-(T,0,0) = (T - S^2, T) \times B_{S^3} \times B_S.$$

We extend U_r by zero for $t \le 0$ and outside Ω. This extension has a fixed zero neighbourhood of the initial and lateral faces: $\inf_{\overline{\Omega}} B(0,\cdot) > 1$, and $B \ge \frac{1}{8}$ on $[0,T] \times \overline{\Omega}$, whereas $u_r = 1 - H$ near $\partial\Omega$. For $d \ge 2$ the extension is smooth. For $d = 1$, we first fix r and convolve in (x,v) at scale h. We also extend f_r by zero outside $(0,T) \times \Omega$. The bounded weak derivatives give

$$\mathcal{L}_A(U_r * \eta_h) = (\mathcal{L}_A U_r) * \eta_h + e_{r,h}, \qquad \|e_{r,h}\|_{L^p} \longrightarrow 0.$$

Indeed, the transport error is bounded by $Ch\|\nabla_x U_r\|_{L^p}$. For diffusion, both $(AD_v^2 U_r) * \eta_h$ and $A(D_v^2 U_r * \eta_h)$ tend to $AD_v^2 U_r$ in L^p. Since the mollifier is non-negative,

$$(\mathcal{L}_A(U_r * \eta_h))_+ \le f_r * \eta_h + |e_{r,h}|.$$

We choose $h = h(r)$ small enough to keep the error below r, retain height at least $\frac{1}{4}$ and preserve the zero boundary neighbourhood. This gives smooth functions with vanishing positive source. Finally, mollifying A preserves ellipticity. For each fixed smooth function, the additional error tends to zero in L^p because its velocity Hessian is bounded. Choosing this second mollification scale after $h(r)$ proves Proposition C.1 with smooth A_j as well.

This construction excludes every $p < 4d$ for a suitable fixed ratio. It says nothing at $p = 4d$, and does not exclude all $p < 4d$ at each fixed ratio. In particular, it is consistent with the improvement below four in Theorem 1.2.

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