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NATURAL COHOMOLOGY OF GENERAL STEINER BUNDLES

VLADO VOJDANOVSKI

Abstract. We prove that a general Steiner bundle on projective space over a field of characteristic zero has natural cohomology. The proof constructs a matrix of linear forms from multiplication of sections on a curve. When the rank exceeds the dimension of the projective space, a finite étale isogeny of elliptic curves splits the bundles controlling the graded kernels and cokernels into line bundles of equal degree. Choosing the auxiliary line bundle outside a finite set gives maximal rank in every degree. A rational curve provides the boundary case. Two open rank conditions then yield the assertion for general matrices and all twists.

1. Introduction

Let k be a field of characteristic zero. A vector bundle F on $\mathbb{P}^n_k$ is a *Steiner bundle* if it admits a presentation

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus t} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{\oplus(t+r)} \longrightarrow F \longrightarrow 0. \tag{1.1}$$

It has *natural cohomology* if, for each integer a, at most one of the groups $H^i(\mathbb{P}^n, F(a))$ is nonzero. Here $t \geq 1$ and $r \geq n \geq 1$; the parameter r is the rank of F.

Coskun, Huizenga, and Smith conjecture that a general Steiner bundle has natural cohomology [4, Conjecture 1.4]. They recall the known case of $\mathbb{P}^2$ and establish the conjecture for rank divisible by n and in an asymptotic range. We prove the assertion for all the parameters above.

Theorem 1.1. *Let k have characteristic zero, and let $n \geq 1$, $t \geq 1$, and $r \geq n$. In the affine space of $t \times (t+r)$ matrices of linear forms in $S = k[x_0, \ldots, x_n]$, there is a nonempty Zariski open subset on which every induced map*

$$\mu_s : S_{s-1}^{\oplus(t+r)} \longrightarrow S_s^{\oplus t}, \qquad s \geq 0, \tag{1.2}$$

has maximal rank, where $S_{-1} = 0$. For every matrix in this open subset, the transpose defines a presentation (1.1), and F has natural cohomology.

The proof combines several established constructions. Arrondo obtains Steiner bundles from multiplication of sections in [2, Example 1.5] and gives elliptic examples in [2, Remark 2.9]. The version using a generating subspace of sections is explicit in Lazarsfeld and Sheridan [6, Lemma 1.1]. Elliptic evaluation kernels obtained by choosing one more generating section than the rank of a globally generated bundle and dualizing appear in Hein's appendix to Brenner [3, Theorem A.1] and in Aprodu, Farkas, and Ortega [1, Proposition 1.3].

The isogeny pushforward and splitting into translates are classical constructions treated by Oda [7, §1]. Symmetric contraction, which we use to describe symmetric powers of evaluation kernels, appears in Kaid and Kasprowitz [5, Proposition 4.1, equation (4.3)].

Our contribution is the uniform deduction of natural cohomology from these constructions. For $r > n$, symmetric contraction identifies the duals of the graded kernels and cokernels with H^1 and H^0 of $B \otimes \operatorname{Sym}^s K$. After isogeny pullback, this bundle splits into line bundles of degree $nt - rs$. Avoiding finitely many choices of B handles degree zero, giving maximal rank in every degree. The boundary $r = n$ uses the classical rational Schwarzenberger construction [2, Example 1.7].

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The passage from two critical rank conditions to all degrees follows [4, Lemma 2.10]. We include proofs of these steps.

All curve constructions through Section 4 are over an algebraically closed field of characteristic zero. We descend the resulting open subset to k in Section 5. We write $\mathbb{P}(V)$ for the projective space of lines in V, with homogeneous coordinate ring $\mathrm{Sym}(V^*)$. A map of finite-dimensional vector spaces has *maximal rank* when it is injective or surjective.

2. Multiplication of sections on a curve

Let C be a smooth projective integral curve. Suppose that L is a line bundle and that a base-point-free subspace $V \subset H^0(C, L)$ of dimension $n+1$ gives an exact sequence

$$0 \longrightarrow K \longrightarrow V \otimes \mathcal{O}_C \xrightarrow{e} L \longrightarrow 0. \tag{2.1}$$

Let B be another line bundle satisfying

$$H^1(C, B) = 0, \qquad h^0(C, B) = t, \qquad h^0(C, B \otimes L) = t + r. \tag{2.2}$$

Multiplication of sections gives

$$m : H^0(C, B) \otimes V \longrightarrow H^0(C, B \otimes L).$$

Its dual determines a matrix of linear forms over $S = \mathrm{Sym}(V^*)$:

$$M : H^0(C, B \otimes L)^* \otimes S(-1) \longrightarrow H^0(C, B)^* \otimes S. \tag{2.3}$$

After choosing bases, its degree-s component is the map μ_s in (1.2).

Lemma 2.1 (The cohomology of the graded maps)**.** *For every $s \geq 0$, the matrix (2.3) satisfies*

$$(\operatorname{coker} \mu_s)^* \simeq H^0(C, B \otimes \mathrm{Sym}^s K), \qquad (\ker \mu_s)^* \simeq H^1(C, B \otimes \mathrm{Sym}^s K). \tag{2.4}$$

Moreover, its transpose has constant rank t on $\mathbb{P}(V)$, and hence defines a Steiner bundle as in (1.1).

Proof. For $s \geq 1$, contraction with the quotient map e gives

$$0 \longrightarrow \mathrm{Sym}^s K \longrightarrow \mathrm{Sym}^s V \otimes \mathcal{O}_C \xrightarrow{\delta_s} \mathrm{Sym}^{s-1} V \otimes L \longrightarrow 0, \tag{2.5}$$

where

$$\delta_s(v_1 \cdots v_s) = \sum_{j=1}^{s} v_1 \cdots \widehat{v_j} \cdots v_s \otimes e(v_j).$$

To verify exactness, split (2.1) locally. The summand $\mathrm{Sym}^{s-j} K \otimes L^j$ maps to the corresponding summand of the target by multiplication by j. These scalars are invertible for $1 \leq j \leq s$ in characteristic zero. The summand with $j = 0$ is exactly the kernel.

Tensor (2.5) with B and take cohomology. Since $H^1(C, B) = 0$ and C is a curve, the result is

$$\begin{aligned} 0 \longrightarrow H^0(C, B \otimes \mathrm{Sym}^s K) &\longrightarrow H^0(C, B) \otimes \mathrm{Sym}^s V \\ &\xrightarrow{H^0(\delta_s)} H^0(C, B \otimes L) \otimes \mathrm{Sym}^{s-1} V \\ &\longrightarrow H^1(C, B \otimes \mathrm{Sym}^s K) \longrightarrow 0. \end{aligned} \tag{2.6}$$

Here $H^0(\delta_s)$ denotes the map after tensoring with B.

Choose dual bases $x_0, \ldots, x_n$ and $v_0, \ldots, v_n$. The differential pairing

$$\langle x^\alpha, v^\beta \rangle = \begin{cases} \alpha!, & \alpha = \beta, \\ 0, & \alpha \neq \beta \end{cases}$$

identifies S_s^* with $\mathrm{Sym}^s V$. Under this pairing, multiplication by x_i is adjoint to differentiation with respect to v_i. Consequently $H^0(\delta_s)$ is precisely μ_s^*, and (2.6) gives (2.4). For $s = 0$, the source of μ_0 is zero; the same identities follow directly from (2.2).

At a point $[v] \in \mathbb{P}(V)$, with $v \neq 0$, the transpose of the twisted sheaf map associated with M is

$$H^0(C, B) \longrightarrow H^0(C, B \otimes L), \qquad b \longmapsto b\, e(v).$$

The section $e(v)$ is nonzero because $V \subset H^0(C, L)$. Multiplication by a nonzero section of a line bundle on an integral curve is injective. The fiber map therefore has rank t at every point, proving the final assertion. □

Thus it suffices to choose (C, L, V, B) for which the two groups in (2.4) never occur simultaneously. The elliptic construction below achieves this by splitting the relevant bundles after a finite cover.

3. The elliptic construction

We first record the cohomological facts used in the construction.

Lemma 3.1. *Let D be an elliptic curve and let T be a line bundle on D. If $\deg T > 0$, then $H^1(D, T) = 0$ and $h^0(D, T) = \deg T$. If $\deg T < 0$, then $H^0(D, T) = 0$. If $\deg T = 0$ and $T \not\simeq \mathcal{O}_D$, both cohomology groups vanish.*

If $q : D \to C$ is a finite étale cover of degree n in characteristic zero and W is a vector bundle on C, then $H^i(C, W)$ is a direct summand of $H^i(D, q^*W)$ for every i.

Proof. The canonical bundle of an elliptic curve is trivial. A line bundle of negative degree has no nonzero section, and a degree-zero line bundle with a nonzero section is trivial. The first assertions follow from these facts, Serre duality, and Riemann–Roch.

For the last assertion, the unit and trace satisfy

$$\mathcal{O}_C \longrightarrow q_*\mathcal{O}_D \longrightarrow \mathcal{O}_C, \qquad 1 \longmapsto n.$$

Dividing the trace by n splits the unit. Tensoring with W, applying the projection formula, and taking cohomology gives the claimed direct summand. The trace construction is recalled in [8, Tag 0BVH]. □

Proposition 3.2 (A matrix for $r > n$)**.** *Suppose $n \geq 1$, $t \geq 1$, and $r > n$. There are curve data satisfying (2.1) and (2.2) for which the matrix (2.3) has maximal rank in every degree $s \geq 0$. More precisely,*

$$\begin{aligned} nt - rs > 0 &\implies \ker \mu_s = 0, \\ nt - rs < 0 &\implies \operatorname{coker} \mu_s = 0, \\ nt - rs = 0 &\implies \mu_s \text{ is an isomorphism.} \end{aligned} \tag{3.1}$$

Proof. Choose an elliptic curve D and a cyclic subgroup of order n. The quotient is a cyclic étale isogeny

$$q : D \longrightarrow C, \qquad \deg q = n.$$

For $n = 1$, take the identity. Write G for the group of deck translations, and choose a line bundle A of degree r on D. Set $Q = q_*A$. It is a vector bundle of rank n, and finite pushforward and Riemann–Roch give

$$\deg Q = \chi(C, Q) = \chi(D, A) = r, \qquad h^0(C, Q) = r.$$

Base change gives

$$q^*Q \simeq \bigoplus_{g \in G} g^*A. \tag{3.2}$$

Indeed, $D \times_C D$ is the disjoint union of the graphs of the translations in G, so each component contributes one of these line bundles.

The bundle Q is globally generated. For any $p \in C$, finite pushforward and the projection formula give

$$H^1(C, Q(-p)) \simeq H^1\big(D, A \otimes q^*\mathcal{O}_C(-p)\big) = 0,$$

because the line bundle on the right has degree $r - n > 0$. The evaluation map $H^0(C, Q) \to Q|_p$ is therefore surjective.

We next choose $n+1$ sections generating Q at every point. Let $\mathcal{T} = H^0(C, Q)^{\oplus(n+1)}$. For a fixed p, evaluation maps $\mathcal{T}$ onto the space of $n \times (n+1)$ scalar matrices. The matrices of rank at most $n-1$ have codimension two. The closed incidence of pairs $(p, \mathbf{u})$ for which the sections $\mathbf{u}$ do not span $Q|_p$ thus has dimension at most $\dim \mathcal{T} - 1$. Its image in $\mathcal{T}$ is closed, since C is projective. A tuple outside that image generates Q everywhere.

For the resulting map from $U \otimes \mathcal{O}_C$, with $\dim U = n+1$, let L^{-1} denote its kernel. We obtain

$$0 \longrightarrow L^{-1} \longrightarrow U \otimes \mathcal{O}_C \longrightarrow Q \longrightarrow 0, \qquad \deg L = r. \tag{3.3}$$

Dualizing gives (2.1) with $K = Q^*$ and $V = U^*$. By (3.2), q^*K is a direct sum of line bundles of degree $-r$, so $H^0(C, K) = 0$ by Lemma 3.1. The map $V \to H^0(C, L)$ is consequently injective. Thus V is the required base-point-free subspace.

Choose $B \in \mathrm{Pic}^t(C)$, with a restriction to be imposed below. Both B and $B \otimes L$ have positive degree, so Lemma 3.1 gives (2.2). For

$$T_s = B \otimes \mathrm{Sym}^s K$$

the splitting (3.2) yields

$$q^*T_s \simeq \bigoplus_{\substack{\alpha_g \geq 0 \\ \sum_{g\in G} \alpha_g = s}} q^*B \otimes \bigotimes_{g\in G} (g^*A)^{-\alpha_g}. \tag{3.4}$$

Every line bundle in this sum has degree

$$nt - rs. \tag{3.5}$$

If this number is positive, Lemma 3.1 gives $H^1(C, T_s) = 0$; if it is negative, it gives $H^0(C, T_s) = 0$.

There is at most one integer s for which $nt - rs = 0$. At that degree, choose B so that none of the finitely many summands in (3.4) is trivial. For each multi-index α, the forbidden condition is

$$q^*B \simeq \bigotimes_{g\in G} (g^*A)^{\alpha_g}. \tag{3.6}$$

It excludes only finitely many points of $\mathrm{Pic}^t(C)$. In fact, two solutions differ by an element of the kernel of $q^* : \mathrm{Pic}^0(C) \to \mathrm{Pic}^0(D)$. The norm identity $\mathrm{Nm}_q(q^*P) \simeq P^{\otimes n}$ puts this kernel in the finite n-torsion subgroup of $\mathrm{Pic}^0(C)$. Since $\mathrm{Pic}^t(C)$ is an elliptic curve over an infinite algebraically closed field, it has points outside the finite union of forbidden sets.

For this choice of B, every degree-zero summand is nontrivial. Both cohomology groups of T_s then vanish, again by Lemma 3.1. The one chosen line bundle B therefore provides the required vanishings for all $s \geq 0$. Lemma 2.1 now proves (3.1).

Finally, the difference between the target and source dimensions of μ_s is

$$t\binom{n+s}{n} - (t+r)\binom{n+s-1}{n} = \binom{n+s-1}{n-1}\left(t - \frac{rs}{n}\right), \tag{3.7}$$

where the second term on the left is zero at $s = 0$. Its sign is the sign of (3.5), so these vanishings give maximal rank in the required direction. □

4. The rational boundary case

Proposition 4.1 (A matrix for $r = n$)**.** *For $n, t \geq 1$ and $r = n$, there are curve data satisfying (2.1) and (2.2) whose matrix has maximal rank in every degree. It is injective for $s < t$, an isomorphism for $s = t$, and surjective for $s > t$.*

Proof. Take

$$C = \mathbb{P}^1, \qquad L = \mathcal{O}_{\mathbb{P}^1}(n), \qquad V = H^0(\mathbb{P}^1, L), \qquad B = \mathcal{O}_{\mathbb{P}^1}(t-1).$$

For the degree-n monomial basis $x^{n-j}y^j$, $0 \leq j \leq n$, the adjacent relations give a map

$$\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus n} \longrightarrow \mathcal{O}_{\mathbb{P}^1}^{\oplus(n+1)}$$

whose column indexed by $0 \leq j < n$ has $-y$ in row j, x in row $j+1$, and zero elsewhere. These columns lie in the evaluation kernel. They are independent wherever $x \neq 0$ or $y \neq 0$, and hence at every point of $\mathbb{P}^1$. Comparing ranks in the evaluation sequence proves

$$K \simeq \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus n}.$$

Also $h^0(B) = t$, $h^0(B \otimes L) = t + n$, and $H^1(B) = 0$.

It follows that

$$B \otimes \operatorname{Sym}^s K \simeq \mathcal{O}_{\mathbb{P}^1}(t-s-1)^{\oplus\binom{n+s-1}{n-1}}.$$

The line bundle on the right has only H^0 when $s < t$, no cohomology when $s = t$, and only H^1 when $s > t$. The assertions follow from Lemma 2.1. □

5. Generality and cohomology in every twist

Proof of Theorem 1.1. Let $\mathcal{M}$ be the affine space of $t \times (t+r)$ matrices with entries in S_1, and put

$$\beta = \left\lfloor \frac{nt}{r} \right\rfloor, \qquad \alpha = \left\lceil \frac{nt}{r} \right\rceil.$$

Impose on a matrix $M \in \mathcal{M}$ the conditions that μ_β be injective and μ_α be surjective. They are open rank conditions. If $\beta = 0$, the first condition is vacuous. If $\alpha = \beta$, they say that one square map is invertible. Propositions 3.2 and 4.1 give a matrix satisfying both, so their intersection is nonempty.

These two conditions imply maximal rank in every degree. Injectivity propagates downward: a nonzero homogeneous vector in $\ker \mu_s$ for $s < \beta$ could be multiplied by a nonzero monomial to give a nonzero vector in $\ker \mu_\beta$. Surjectivity propagates upward: the graded module

$$N = \operatorname{coker}(M : S(-1)^{\oplus(t+r)} \to S^{\oplus t})$$

is generated in degree zero. Hence $N_\alpha = 0$ implies $N_s = 0$ for all $s \geq \alpha$. Since α and β are equal or consecutive, every nonnegative degree is covered. The dimension comparison (3.7) shows that the injective and surjective ranges are the ones required for maximal rank.

In particular, N has finite length. Sheafifying M and twisting by $\mathcal{O}_{\mathbb{P}^n}(1)$ gives a surjection

$$\mathcal{O}_{\mathbb{P}^n}^{\oplus(t+r)} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus t}.$$

Its kernel E is locally free. Dualizing gives (1.1) with $F = E^*$.

For a twist $a = s - 1 \geq -1$, the exact sequence

$$0 \longrightarrow E(a) \longrightarrow \mathcal{O}_{\mathbb{P}^n}(a)^{\oplus(t+r)} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(a+1)^{\oplus t} \longrightarrow 0 \tag{5.1}$$

and the cohomology of line bundles on projective space yield

$$H^0(\mathbb{P}^n, E(s-1)) = \ker \mu_s, \qquad H^1(\mathbb{P}^n, E(s-1)) = \operatorname{coker} \mu_s. \tag{5.2}$$

All higher groups vanish, and maximal rank ensures that these two groups are not simultaneously nonzero.

For $a \le -2$, both line bundles in (5.1) have no global sections and no intermediate cohomology. The long exact sequence therefore leaves only $H^n(\mathbb{P}^n, E(a))$ as a possible nonzero group. When $n = 1$, the same conclusion follows directly from the vanishing of H^0 of the two line bundles. These line-bundle cohomology statements are recorded in [8, Tag 01XT]. Thus E has natural cohomology in every twist. Serre duality gives

$$H^i(\mathbb{P}^n, F(a)) \simeq H^{n-i}(\mathbb{P}^n, E(-a-n-1))^*,$$

so F has natural cohomology as well.

It remains to justify the assertion over the original field k. The two critical maps are universal matrices whose entries are integer polynomials in the coefficients of M. The construction over an algebraic closure gives maximal minors of the two maps with nonzero product. This product is a nonzero polynomial over $\mathbb{Q}$, hence also over k. Its nonvanishing defines a nonempty open subset of $\mathcal{M}_k$. An infinite field has rational points in any such open subset of affine space. The same rank and cohomology arguments apply to these matrices over k, completing the proof. □

We record the resulting dimensions explicitly.

Corollary 5.1. *For a matrix in the open subset of Theorem 1.1, let $E = F^*$ and set, for $s \ge 0$,*

$$D_s = t\binom{n+s}{n} - (t+r)\binom{n+s-1}{n},$$

with the second binomial interpreted as zero at $s = 0$. Then

$$h^0(\mathbb{P}^n, E(s-1)) = \max\{-D_s, 0\}, \qquad h^1(\mathbb{P}^n, E(s-1)) = \max\{D_s, 0\}.$$

All higher cohomology groups in these twists vanish. The graded module $N = \operatorname{coker} M$ has Hilbert series

$$\operatorname{Hilb}_N(z) = \left[\frac{t-(t+r)z}{(1-z)^{n+1}}\right]_+,$$

where positive truncation keeps the coefficients preceding the first nonpositive coefficient and sets that coefficient and all later coefficients to zero.

Proof. The formulas for E follow from (5.2), maximal rank, and the difference between the two vector-space dimensions. The coefficient of degree s in the displayed rational expression is D_s. Formula (3.7) shows that its sign changes at $s = nt/r$ and cannot become positive again. Since $\dim_k N_s = \max\{D_s, 0\}$, the Hilbert-series formula follows. □

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