

Binary Reconstruction from the Fourth Deck

Torsion Phases, Local Gluing, and Sign Extinction

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Abstract

We give a candidate proof of the following statement: for every finite cyclic group C_n , the cumulative fourth deck of a binary subset determines it up to translation. Hence $r_{\text{set}}(C_n) \leq 4$ for every n , and, using the Keleti–Kolountzakis Corollary 1.1 for the lower bound, $r_{\text{set}}(C_n) = 4$ for every even $n > 10$. The proof has four components: finiteness of the binary fibre, cyclotomic torsion of the phases, gluing of primary characters, and extinction of dyadic residues by two odd-denominator projectors. The bridge between ordered analytic correlations and Pebody’s cumulative deck is proved explicitly, including stabilizers, zero and repeated offsets, and the empty pattern. The text contains the complete mathematical chain; finite checks are included only as convention and implementation checks, not as premises of the uniform proof.

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1 Statement and conventions

Pebody’s Conjecture 8.1 [3, p. 514] asserts that $r(C) = 4$ for every cyclic group of even order $|C| > 10$. His set-reconstruction number $r(C)$ is denoted here by $r_{\text{set}}(C_n)$. The candidate proof establishes the upper bound for every cyclic order; the matching even-order lower bound is imported from Keleti–Kolountzakis [4].

The strategy combines the correspondence between the cumulative deck and ordered correlations, cyclotomic torsion of the phases, gluing of primary characters, and extinction of dyadic residues by odd-denominator projectors. The four interfaces are:

- (i) ordered analytic deck versus cumulative deck;
- (ii) gluing characters across distinct complete tags;
- (iii) lifting in exact-order shells, including $p = 3$;
- (iv) frequency-by-frequency verification of the projector identities.

Let G be a finite abelian group written additively. For $k \geq 1$, set

$$D_k f(t_1, \dots, t_{k-1}) = \sum_{x \in G} f(x) \prod_{j=1}^{k-1} f(x + t_j),$$

where the offsets are ordered and may be zero or repeated. For $k = 1$, $D_1 f = \sum_x f(x)$.

Definition 1.1 (Pebody's cumulative deck). For $A \subseteq G$, the cumulative k -deck is the multiset of translation classes $[T]$ of all subsets $T \subseteq A$ with $|T| \leq k$. The multiplicity of $[T]$ is denoted by $\nu_A([T])$. Two sets are k -isomorphic if they have the same deck.

We do not quotient by reflection. This is Pebody's convention [3], and it agrees with that of Keleti–Kolountzakis [4]. Let $r_{\text{set}}(G)$ be the least $k \geq 1$ for which this deck determines every subset of G up to translation.

Lemma 1.2 (Ordered–cumulative bridge). *Let $A \subseteq G$, let $T \subseteq G$ with $|T| \leq 4$, and set*

$$\text{Stab}(T) = \{h \in G : h + T = T\}, \quad N_A(T) = \#\{x \in G : x + T \subseteq A\}.$$

Then

$$N_A(T) = |\text{Stab}(T)| \nu_A([T]). \tag{1.1}$$

Moreover, for $\mathbf{t} = (t_1, \dots, t_{j-1})$ and $T_{\mathbf{t}} = \{0, t_1, \dots, t_{j-1}\}$,

$$D_j \mathbf{1}_A(\mathbf{t}) = N_A(T_{\mathbf{t}}). \tag{1.2}$$

Thus two indicators have the same cumulative deck through order four if and only if they have the same D_j , $1 \leq j \leq 4$.

Proof. Every distinct copy B in the class $[T]$ has the form $B = x + T$. If $x_0 + T = B$, then $x + T = B$ exactly when $x - x_0 \in \text{Stab}(T)$. Hence every copy is counted exactly $|\text{Stab}(T)|$ times in $N_A(T)$, which proves (1.1). Formula (1.2) is the definition of D_j , because the indicator of the pattern $T_{\mathbf{t}}$ is one exactly when its translate with origin x is contained in A .

Zeros and repetitions in the tuple merely reduce $T_{\mathbf{t}}$ to a pattern with fewer points; they introduce no extra factors. For the empty pattern, $N_A(\emptyset) = |G|$, $\text{Stab}(\emptyset) = G$, and the empty pattern has multiplicity one in the deck, so (1.1) holds there as well. The converse follows by choosing, for each nonempty pattern T , a tuple that normalizes it with one point at zero and applying (1.1); the multiplicity of the empty pattern is independent of A . $\square$

Remark 1.3. For indicators, D_4 alone determines D_1, D_2, D_3 : set the excess offsets to zero and use $\mathbf{1}_A^j = \mathbf{1}_A$. This does not hold for arbitrary real competitors; for them we retain the cumulative data explicitly.

Theorem 1.4 (Reconstruction from the fourth deck). *For every $n \geq 1$, two subsets $A, B \subseteq C_n$ with the same fourth deck are translates. In particular*

$$r_{\text{set}}(C_n) \leq 4 \quad (n \geq 1). \quad (1.3)$$

For every even integer $n > 10$,

$$\boxed{r_{\text{set}}(C_n) = 4}. \quad (1.4)$$

Remark 1.5 (Lower-bound locator). KK06 Corollary 1.1, p. 5, states that the 3-deck reconstructs every cyclic set if and only if n is a power of an odd prime, a product of at most three odd primes, or $n \in \{2, 4, 6, 8, 10\}$. Hence for every even $n > 10$ there is a non-translate pair with the same 3-deck, and $r_{\text{set}}(C_n) \geq 4$. Theorem 2.22, p. 12, gives the explicit construction for $n = 2k$, where $k \geq 6$ is an integer, and therefore also supplies this lower bound for every even $n > 10$.

2 Fourier analysis and finiteness of the binary fibre

Let $\widehat{G}$ be the dual group of G , and set

$$\widehat{f}(\xi) = \sum_{x \in G} f(x) \overline{\xi(x)}.$$

The transform in all offsets is

$$\widehat{D_k f}(\xi_1, \dots, \xi_{k-1}) = \widehat{f}(-\xi_1 - \dots - \xi_{k-1}) \prod_{j=1}^{k-1} \widehat{f}(\xi_j). \quad (2.1)$$

From here through the final proof assume $A \neq \emptyset$. If $f = \mathbf{1}_A$ and $g = \mathbf{1}_B$ have the same $D_1, \dots, D_4$, then D_2 gives

$$\text{supp } \widehat{f} = \text{supp } \widehat{g} =: S, \quad |\widehat{f}(\xi)| = |\widehat{g}(\xi)|.$$

For $\xi \in S$, define

$$\rho(\xi) = \frac{\widehat{g}(\xi)}{\widehat{f}(\xi)} \in \mathbb{S}^1.$$

Reality and positive mass give

$$\rho(0) = 1, \quad \rho(-\xi) = \overline{\rho(\xi)}. \quad (2.2)$$

Formula (2.1) also implies

$$\prod_{j=1}^k \rho(\xi_j) = 1 \quad \text{if } \xi_1 + \dots + \xi_k = 0, \quad \xi_j \in S, \quad 1 \leq k \leq 4. \quad (2.3)$$

Lemma 2.1 (Binary moment rigidity). *If $f : G \rightarrow \{0, 1\}$ and $u : G \rightarrow \mathbb{R}$ have the same moments of orders 2, 3, 4, then u is binary.*

Proof. Equality of the diagonal moments gives

$$\sum_x u(x)^2 (u(x) - 1)^2 = \sum_x u(x)^4 - 2 \sum_x u(x)^3 + \sum_x u(x)^2 = 0.$$

Every summand is nonnegative and vanishes precisely when $u(x) = 0$ or $u(x) = 1$. $\square$

Theorem 2.2 (Finiteness of the phase fibre). *The set $\mathcal{R}_4(S)$ of maps $\rho : S \rightarrow \mathbb{S}^1$ satisfying (2.2)–(2.3) is a finite abelian group. Every value $\rho(\xi)$ is a root of unity.*

Proof. For $\rho \in \mathcal{R}_4(S)$, define $\hat{u} = \rho\hat{f}$ on S and zero outside S . The opposite-frequency relations make u real. Relations (2.3) and (2.1) give the same D_k , $k \leq 4$, for u and f . Lemma 2.1 makes u binary, and the mass places it among the subsets of cardinality $|A|$. Exactness of S makes $\rho \mapsto u$ injective, so the fibre is finite. Pointwise products and inverses preserve (2.2)–(2.3), hence the fibre is a finite group and every element has finite order. $\square$

3 Local cyclotomic characters

Let $\xi \in S$ have order d . Since f and g are rational, $\hat{f}(\xi), \hat{g}(\xi) \in \mathbb{Q}(\zeta_d)$, and the same holds for $\rho(\xi)$. Galois covariance is

$$\rho(u\xi) = \sigma_u(\rho(\xi)), \quad \gcd(u, d) = 1, \quad (3.1)$$

where $\sigma_u(\zeta_d) = \zeta_d^u$.

Proposition 3.1 (Character on the generator orbit). *For $\xi \in S$ of order d ,*

$$\rho(\xi)^d = 1, \quad \rho(u\xi) = \rho(\xi)^u \quad (\gcd(u, d) = 1). \quad (3.2)$$

Consequently, the restriction of ρ to the generator shell of $\langle \xi \rangle$ is the restriction of a character of $\langle \xi \rangle$. The statement also holds when d is divisible by three.

Proof. Theorem 2.2 makes $\rho(\xi)$ a root of unity. The roots of unity in $\mathbb{Q}(\zeta_d)$ are μ_d when d is even and μ_{2d} when d is odd [8]. In the even case, $\rho(\xi)^d = 1$ follows immediately.

Now let $d > 1$ be odd. Write $\rho(\xi) = \varepsilon z$, with $\varepsilon \in \{\pm 1\}$ and $z \in \mu_d$. Since 2 is coprime to d , $-2\xi \in S$ by Galois covariance. The three-term instance of (2.3) on $(\xi, \xi, -2\xi)$ gives $\rho(2\xi) = \rho(\xi)^2 = z^2$. On the other hand, (3.1) gives $\rho(2\xi) = \varepsilon z^2$, hence $\varepsilon = 1$. The case $d = 1$ is the relation $\rho(0) = 1$. Thus (3.1) becomes (3.2), and $\rho(\xi)$ defines the character $k\xi \mapsto \rho(\xi)^k$. $\square$

4 Galois boxes and primary gluing

Write, via CRT,

$$G = C_{2^r} \times H, \quad H = \prod_{p \mid |H|} C_{p^{a_p}}, \quad |H| \text{ odd.}$$

We allow $r = 0$ and $H = \{0\}$. In the dual group, E_i is the shell of elements of exact order 2^i . For $\tau = (\tau_p)_p$, let O_τ be the product of the shells of order p^{τ_p} . A box is

$$B_{i,\tau} = E_i \times O_\tau.$$

The boxes are the Galois orbits of the rational support S . To avoid ambiguity between different generators, put $a_2 := r$ and, for $q \in \{2\} \cup \{p : p \mid |H|\}$, write

$$C_{q^s} := \{z \in C_{q^{a_q}} : q^s z = 0\}$$

for the canonical subgroup of the ambient factor (with the evident interpretation in the dyadic factor). A shell of exact order q^s generates precisely C_{q^s} . Thus, for every occupied box, Proposition 3.1 assigns a character

$$\chi_{i,\tau} : K_{i,\tau} := C_{2^i} \times \prod_p C_{p^{\tau_p}} \longrightarrow \mathbb{S}^1,$$

and its p -primary restriction $\chi_{i,\tau,p}$ is defined on the canonical subgroup $C_{p^{\tau_p}}$, independently of the choice of a shell generator.

For each odd $p \mid |H|$, fix an ambient generator e_p of the factor $C_{p^{a_p}}$. Thus $p^{a_p-b_p}e_p$ is the canonical generator of $C_{p^{b_p}}$ whenever $0 \leq b_p \leq a_p$.

In the next lemma, for $B = B_{i,\tau}$ we abbreviate

$$\chi_{B,p} := \chi_{i,\tau,p}.$$

Lemma 4.1 (Increment comparison). *For two occupied boxes $B = B_{i,\tau}$ and $B' = B_{i',\tau'}$, put $b_p = \min(\tau_p, \tau'_p)$ and $b_2 = \min(i, i')$. Then*

$$\chi_{B,p}|_{C_{p^{b_p}}} = \chi_{B',p}|_{C_{p^{b_p}}} \quad (p \text{ odd}), \quad \chi_{B,2}|_{2C_{2^{b_2}}} = \chi_{B',2}|_{2C_{2^{b_2}}}.$$

When $b_p = 0$ or $b_2 = 0$, the corresponding assertion is vacuous.

Proof. Fix an increment $\delta \in C_{p^{b_p}}$. If the level of the first box is strictly larger than b_p , every element of the shell of order p^{τ_p} remains in that shell after adding δ , because an increment of strictly smaller order cannot change its p -adic valuation. If instead $\tau_p = b_p$, write the two elements as units modulo p : choose a class x different from 0 and $-\delta$, so that $x + \delta$ is also a unit. For $p = 3$, when the obstacles are distinct, exactly one class remains admissible, so the choice is still available. The same argument applies to the second box. If δ is already divisible by p at the common level, every unit representative works. In the dyadic part take instead $\delta \in 2C_{2^{b_2}}$: adding an even element preserves every odd unit class, with no exception.

Fix all coordinates other than the one under consideration and choose simultaneously $x, x + \delta \in B$ and $y, y + \delta \in B'$. More precisely, the four elements

$$x + \delta, \quad y, \quad -x, \quad -(y + \delta)$$

belong respectively to B, B', B, B' and satisfy $(x + \delta) + y - x - (y + \delta) = 0$. The collision

$$(x + \delta) + y = x + (y + \delta)$$

gives, by (2.3),

$$\frac{\rho(x + \delta)}{\rho(x)} = \frac{\rho(y + \delta)}{\rho(y)}.$$

The two ratios are $\chi_{B,p}(\delta)$ and $\chi_{B',p}(\delta)$. The increments δ generate $C_{p^{b_p}}$, so the p -primary restrictions agree. In the dyadic part the permitted increments generate $2C_{2^{b_2}}$, giving the second equality. The construction uses only the shells of the two boxes under consideration; it does not require the tag vectors to be comparable. $\square$

Proposition 4.2 (Gluing across distinct complete tags). *There is a translation character on the odd factor such that, after dividing ρ by it, every local character $\chi_{i,\tau}$ is trivial on all odd components. The complete tags need not be comparable.*

Proof. For every odd prime p , put

$$b_p = \max\{\tau_p : B_{i,\tau} \subseteq S\}.$$

Choose an occupied box at level b_p . By Lemma 4.1, the p -primary restriction of every other box agrees with that of the chosen box on its p -primary subgroup: the levels $0 \leq \tau_p \leq b_p$ form the canonical chain

$$C_{p^0} \subseteq C_{p^1} \subseteq \cdots \subseteq C_{p^{b_p}}.$$

If the value on the canonical generator of $C_{p^{b_p}}$ is $\exp(2\pi it/p^{b_p})$, extend it to the ambient generator by setting

$$\lambda_p(e_p) = \exp(2\pi it/p^{a_p}), \quad \lambda_p|_{C_{p^{b_p}}} = \chi_{B_p, p}.$$

For $b_p = 0$, simply set $\lambda_p = 1$. Diagrammatically, for every box B , the compatibility just proved is

$$\begin{array}{ccc} C_{p^{\tau_p}} & \hookrightarrow & C_{p^{b_p}} \\ \chi_{B, p}|_{C_{p^{\tau_p}}} & = & \chi_{B_p, p}|_{C_{p^{\tau_p}}} \end{array} \quad \text{and} \quad \lambda_p|_{C_{p^{\tau_p}}} = \chi_{B, p}.$$

The ambient extension need not be unique, but all extensions chosen this way agree on occupied frequencies. The product of the λ_p is a character of the odd factor and, by CRT, is the restriction of a global character of $\widehat{G}$, hence of a physical translation. Dividing ρ by this character means translating g . Relations (2.2)–(2.3) are unchanged: in a zero-sum relation the product of a global character on the four (or fewer) terms is one.

This construction is coordinated prime by prime. For example, the tags $(1, 0)$ and $(0, 1)$ in $C_3 \times C_5$ are incomparable as vectors, but their restrictions are compared separately in the 3-chain and the 5-chain. Exactness of the support is used only to ensure that an occupied box is entirely available when constructing the collision. $\square$

5 Lifting in exact-order shells

Lemma 5.1 (Exact-shell lifting criterion). *Let $\mathbf{B} = (B_1, \dots, B_q)$ be an indexed list of occupied boxes, $1 \leq q \leq 4$, where $B_j = B_{i_j, \tau(j)}$; repetitions are allowed and count by their index. There are frequencies $\xi_j \in B_j$ with $\sum_{j=1}^q \xi_j = 0$ if and only if, for every odd prime p , the maximum positive level*

$$b_p := \max_j \tau_p^{(j)}$$

occurs at least twice, and the maximum dyadic level

$$b_2 := \max_j i_j$$

occurs an even number of times. If a coordinate has $b_p = 0$ or $b_2 = 0$, the corresponding condition is vacuous.

Proof. Work first in a coordinate C_{p^a} . If $b > 0$ is the maximum level and occurs only once, divide the relation by p^{a-b} and reduce modulo p : one unit remains, while all lower-level terms are divisible by p , which is impossible. For $p = 2$, the same reduction says that the number k of terms at level b must be even, since every unit is 1 modulo 2. This proves necessity.

For sufficiency, fix the indexed lower-level terms and let z be the negative of their sum in C_{p^b} . They are divisible by p , hence $z \equiv 0 \pmod{p}$. Identifying C_{p^b} with the canonical subgroup of C_{p^a} , for $k = 2, 3, 4$ maximal terms choose respectively

$$(1, z-1), \quad (1, 1, z-2), \quad (1, -1, 1, z-1). \quad (5.1)$$

The tuples are ordered according to the indices of the maximal-level boxes and sum to z . If p is odd, all their entries are units: $z-1 \equiv -1$, $z-2 \equiv -2 \not\equiv 0 \pmod{p}$, and again $z-1 \equiv -1$ in the last case. For $p = 3$, in particular, $z-2 \equiv 1 \pmod{3}$, while the last tuple is $(1, -1, 1, -1) \pmod{3}$. If $p = 2$, the condition allows only $k = 2$ or $k = 4$; here z is even and all entries of

(5.1) are odd. Multiplying the normalized entries by p^{a-b} gives elements of exact order p^b in the ambient factor.

Make the choice independently in each coordinate, preserving the index j , and recombine the q coordinates by CRT. The sums vanish in each factor, hence in G , and every ξ_j belongs to its prescribed box. $\square$

6 Dyadic sign normalization

After Proposition 4.2, assume that all odd components of the local characters have been removed. For a box of dyadic level $i > 0$, put $g_i = 2^{r-i}$ and write

$$\rho(vg_i, \eta) = \exp(2\pi i v c_{i,\tau}) \quad (\eta \in O_\tau), \quad v \text{ odd}, \quad 2^i c_{i,\tau} = 0 \text{ in } \mathbb{R}/\mathbb{Z}. \quad (6.1)$$

This is simply the restriction of $\chi_{i,\tau}$ to the dyadic subgroup. In row $i = 0$, the subgroup generated by the box is entirely odd: the local character was already removed by gluing, and $\rho = 1$ on every box in this row.

Let $m > 0$ be the maximum occupied dyadic level and choose a reference box $B_* = (m, \tau_*)$. Let $c = c_{m,\tau_*}$. For a box $B = (i, \tau)$, $0 < i \leq m$, put $d = 2^{m-i}$. The four dyadic frequencies

$$g_m, \quad -(2d+1)g_m, \quad g_i, \quad g_i \quad (6.2)$$

sum to zero. Assign tag τ_* to the first two and tag τ to the last two. In every odd coordinate, regardless of how the tags compare, each positive maximum occurs either twice or four times. The coordinatewise construction in Lemma 5.1 completes the odd frequencies while leaving the dyadic frequencies in (6.2) fixed. The phase relation is

$$2(c_{i,\tau} - dc) = 0 \quad \text{in } \mathbb{R}/\mathbb{Z}. \quad (6.3)$$

Thus $c_{i,\tau} - dc$ is 0 or $1/2$. The reference character extends from the subgroup of order 2^m to the ambient dyadic factor; after removing this translation,

$$\rho|_{B_{i,\tau}} = (-1)^{e_{i,\tau}}, \quad e_{i,\tau} \in \mathbb{F}_2. \quad (6.4)$$

The reference has sign zero, and all rows $i = 0$ have sign zero.

7 Projectors and sign extinction

We fix all domain conventions here. For $h \in G$, physical translation is

$$(T_h u)(x) := u(x - h), \quad \widehat{T_h u}(\xi) = \widehat{\xi(h)} \widehat{u}(\xi). \quad (7.0)$$

For $u, v : H \rightarrow \mathbb{C}$, convolution in the odd factor is

$$(u *_H v)(h_0) := \sum_{h \in H} u(h) v(h_0 - h).$$

For a tag set Λ , put

$$Q_\Lambda := \bigcup_{\tau \in \Lambda} O_\tau, \quad S_\Lambda := \bigcup_i E_i \times Q_\Lambda.$$

The projector P_Λ is defined in the frequency domain by $\widehat{P_\Lambda u} = \mathbf{1}_{S_\Lambda} \widehat{u}$. In physical space it is convolution with the kernel K_Λ below in the odd factor alone, leaving the dyadic coordinate

unchanged. Thus every subsequent table checks Fourier coefficients of an identity already defined in physical space.

For an odd tag set Λ , let P_Λ be the Fourier projector that multiplies frequencies in

$$\bigcup_{\tau \in \Lambda} \bigcup_i E_i \times O_\tau$$

by one and all other frequencies by zero. Its kernel in the odd factor is

$$K_\Lambda(h) = \frac{1}{|H|} \sum_{\eta \in \bigcup_{\tau \in \Lambda} O_\tau} \eta(h).$$

The summed set is stable under Galois and negation. Hence $|H|K_\Lambda(h)$ is a rational algebraic integer, and therefore an integer, and K_Λ is real. By character orthogonality, $\widehat{K_\Lambda} = \mathbf{1}_{Q_\Lambda}$, so physical convolution realizes exactly the multiplier $\mathbf{1}_{S_\Lambda}$. In particular, for integer f , $P_\Lambda f \in |H|^{-1}\mathbb{Z}^G$. If $v \in |H|^{-1}\mathbb{Z}^G$ and $2^a v$ is integral, then v is integral because $\gcd(|H|, 2^a) = 1$ (Bézout's identity). If $H = \{0\}$, the only tag is the empty one and the projector is zero or the identity; the denominator is 1 and the same argument applies.

Assume $m > 0$, and let T be physical translation by 2^{m-1} in the coordinate C_{2r} . Its Fourier multiplier is $+1$ on levels $i < m$, -1 on level m , and the ambient level r need not be fixed. Indeed, for $s = v2^{r-i}$ the multiplier is $\exp(-2\pi i s 2^{m-1}/2^r) = \exp(-2\pi i v 2^{m-1-i})$. Both Fourier coefficients vanish on levels $i > m$.

Proposition 7.1 (Lower extinction). *All signs $e_{i,\tau}$ with $i < m$ vanish.*

Proof. Set $e_{0,\tau} = 0$. For two occupied lower levels $0 \leq i, j < m$ with the same tag τ , take one copy of $B_{i,\tau}$, one of $B_{j,\tau}$, and two copies of the reference box at level m . Lemma 5.1 admits the relation; the two reference signs vanish, so $e_{i,\tau} + e_{j,\tau} = 0$. Thus the lower sign is constant along each tag. In particular, if the tag is also present at level 0, its sign at every positive lower level already vanishes. If the same tag also occurs at level m , two copies of the upper box and one lower copy give an admissible triple, so the lower sign is directly zero.

Let Λ_- be the set of tags with lower-level ratio $\rho = -1$. No tag in Λ_- occurs at level 0 or level m of the support. In Fourier space, for each frequency, consider the four cases:

frequency	$t(\xi)$	$\rho(\xi)$	$(1 + t(\xi))(\widehat{f} - \widehat{g})$	$4\widehat{P_{\Lambda_-} f}$
tag in Λ_- , $i < m$	$+1$	-1	$4\widehat{f}$	$4\widehat{f}$
unselected tag, $i < m$	$+1$	$+1$	0	0
level m	-1	± 1	0	0
$\xi \notin S$	arbitrary	undefined	0	0

Therefore

$$4P_{\Lambda_-} f = (I + T)(f - g). \quad (7.1)$$

The right-hand side is integral and has modulus at most 2, so $|P_{\Lambda_-} f| \leq 1/2$. The integrality argument makes $P_{\Lambda_-} f$ integral, hence zero. If Λ_- were nonempty, exactness of the support would provide a nonzero Fourier coefficient selected by the projector, a contradiction. $\square$

Put

$$U = \{\tau : E_m \times O_\tau \subseteq S\}, \quad L = \bigcup_{i < m} \{\tau : E_i \times O_\tau \subseteq S\}.$$

All lower-level signs on L vanish.

Lemma 7.2 (Synchronization of shared tags). *The upper signs are constant on $U \cap L$.*

Proof. For $\tau, \sigma \in U \cap L$, choose lower levels $i, j < m$ and consider the four boxes

$$(m, \tau), \quad (m, \sigma), \quad (i, \tau), \quad (j, \sigma).$$

In every odd coordinate, each positive maximum occurs at least twice: if the two tags have different levels, the maximum belongs to the two occurrences of the corresponding tag; if they agree, it occurs four times. The dyadic maximum occurs twice. Lemma 5.1 realizes the relation, and the two lower signs are zero, leaving $e_{m,\tau} + e_{m,\sigma} = 0$. $\square$

Theorem 7.3 (Upper extinction). *After at most one further global translation, $f = g$.*

Proof. If $U \cap L \neq \emptyset$, Lemma 7.2 assigns one sign c . If $U \cap L = \emptyset$, set $c = 0$. Replace g by $T^c g$. This is one global translation and leaves the lower signs zero. Let $\Lambda_+ = \{\tau \in U : e_{m,\tau} = 1\}$. Then $\Lambda_+ \cap L = \emptyset$. The frequency-by-frequency verification of

$$f - g = 2P_{\Lambda_+} f \tag{7.2}$$

is:

frequency	$t(\xi)$	$\rho(\xi)$	$\widehat{f} - \widehat{g}$	$\widehat{2P_{\Lambda_+} f}$
tag in Λ_+ , $i = m$	−1	−1	$2\widehat{f}$	$2\widehat{f}$
unselected tag, $i = m$	−1	+1	0	0
lower frequency	+1	+1	0	0
$\xi \notin S$	arbitrary	undefined	0	0

By (7.2), $2P_{\Lambda_+} f$ is integral; projector integrality makes $P_{\Lambda_+} f$ integral. But (7.2) and binarity give $|P_{\Lambda_+} f| \leq 1/2$, so $P_{\Lambda_+} f = 0$ and $f = g$. $\square$

8 Proof of the theorem and scope

Proof of Theorem 1.4. By Lemma 1.2, equality of the fourth cumulative deck is equivalent to equality of $D_1, \dots, D_4$. If one of the two sets is empty, the mass makes both empty. In the nonempty case, (2.1)–(2.3) produce the unit-modulus ratio ρ . Theorem 2.2 makes it torsion, and Proposition 3.1 turns it into local characters. Proposition 4.2 removes the odd components by a translation. Section 6 leaves only dyadic signs. If there is no positive dyadic level, the ratio is already a global character. Otherwise Proposition 7.1 and Theorem 7.3 cancel all signs. Thus A and B differ by a translation, proving (1.3).

For even $n > 10$, KK06 Corollary 1.1 rules out sufficiency of the 3-deck and gives $r_{\text{set}}(C_n) \geq 4$. Together with (1.3), this proves (1.4). $\square$

Corollary 8.1 (Real competitors). *If $f = \mathbf{1}_A$ on C_n and $u : C_n \rightarrow \mathbb{R}$ has the same cumulative data $D_1, \dots, D_4$, then u is a translate of f .*

Proof. Lemma 2.1 makes u binary. Apply the theorem. $\square$

Remark 8.2 (Scope). The result does not assert reconstruction for general positive integer signals, multisets, or noncyclic abelian groups. We do not replace the deck convention by a list of only distinct patterns of cardinality exactly four, nor do we identify objects under reflection in advance. The Grünbaum–Moore phases for nonbinary rational signals and the finite-order phase-retrieval literature are adjacent prior art, not an automatic importation of the present theorem.

9 Prior art and positioning

Pebody defines the deck as the multiset of translation classes of subsets of cardinality at most k , without reflection [3]; Pebody 2004 focuses mainly on reconstruction of rational functions and multisets [2]. Grünbaum–Moore use Fourier ratios and characters, but their Theorem 5 requires a nonzero primitive coefficient and is not a uniform binary theorem [1]. KK06 provides the Fourier framework for the deck and the uniform lower bound through Corollary 1.1 [4].

Simon’s thesis documents the analytic/combinatorial bridge and maintains the distinction between translation and reflection [5]. The work of Mandereau–Ghisi–Amiot–Andreatta–Agon also studies decks up to reflection and musical phase retrieval [6]; that quotient is not used here. Amiot classifies rational spectral units of finite order [7]; the ratio considered here is defined only on the exact Fourier support, and the moment identities force the inverse-transformed signal to be binary. Thus that result is a contextual check rather than an automatic import. The bibliographic comparison is limited to the cited sources and is not a global priority or novelty claim.

10 Computational checks and limitations

The companion code provides a bounded check of the exact-shell lifting criterion. It enumerates prime-power coordinates for $p = 2, 3, 5$, exponents 1, 2, 3, and indexed lists of lengths one through four, including repetitions, and compares the criterion with direct zero-sum enumeration. These are tests of the local coordinate conditions, not an enumeration of every ambient cyclic group.

The repository also supplies manuscript-build, public-surface and hash checks. The reproduction instructions identify the shipped commands and their scope.

No finite check is a premise of the uniform proof. The checks are intended to catch convention, implementation, or typesetting errors; they do not replace independent mathematical reading, formalization, or editorial assessment. The code and control data are not used to extrapolate the theorem to noncyclic groups, general positive integer signals, or a different deck convention.

AI assistance. The author reports using GPT-6 Pro and Astra via ChatGPT, together with OpenAI Codex. AI assistance included mathematical exploration, source comparison, manuscript drafting and revision, finite sanity checks, and repository preparation under the author’s direction. Contributions are not attributed to individual models. The author is responsible for the mathematical claims and final decisions. AI reviews do not constitute human peer review or independent specialist verification.

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