

Factorial denominators of rank-two scattering coefficients

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Abstract

For the universal rank-two scattering coefficient $\tau(i, j)$, we prove that $\max(i, j)! \tau(i, j)$ has integer coefficients in the independent parameters b, c, g for every $i, j > 0$. This proves the factorial-denominator conjecture of Elgin, Reading, and Stella (Conjecture 4).

AI use and scholarly credit

The results were obtained within a three-hour Codex session using GPT-6-Astra at high reasoning effort. The arguments and exposition were subsequently reviewed and revised in Codex using the built-in mathematical writing skill. The manuscript passed an AI review using Rethlas-OSV [12, 14, 16].

By posting this manuscript on this platform, I do not seek to assert ownership of the results or claim priority. I welcome anyone interested in preparing a finished account or checking the arguments; the eventual scholarly credit should go to those who undertake that work.

1 Introduction

Cluster algebras, introduced by Fomin and Zelevinsky [7], organize families of generators by repeated mutation. Scattering diagrams encode compatible changes of coordinates by automorphisms attached to walls in a real vector space. Consistency means that the ordered composition along a closed path, avoiding wall intersections, is the identity. The constructions of Kontsevich–Soibelman [13] and Gross–Siebert [11] established this wall-crossing framework. Gross–Hacking–Keel–Kontsevich [8] used cluster scattering diagrams and broken lines to construct theta functions and canonical bases for cluster varieties under their stated hypotheses.

In rank two, walls are rays or lines in a plane. Two initial wall functions determine the outgoing wall functions by consistency. Their coefficients are accessible to computation, but even in this case there are generally infinitely many walls and coefficients. The tropical vertex construction of Gross–Pandharipande–Siebert [10] relates such factorizations to enumerative geometry. Gross–Pandharipande [9] connected the same rank-two factorizations with quiver representations. These interpretations express wall coefficients through invariants of moduli spaces.

Elgin, Reading, and Stella [6], in a preprint first posted in April 2024, proposed eighteen conjectures about rank-two cluster scattering coefficients. Their parameters b, c are positive integers specifying the two off-diagonal entries of the rank-two exchange data. For a monomial

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indexed by positive integers i, j , the primitive normalization introduces the integer $g = \gcd(ib, jc) / \gcd(i, j)$. They sought polynomial formulas $\tau(i, j)$ in the three parameters, together with degree, divisibility, symmetry, and positivity properties. Their Conjecture 4 asks that multiplication by $\max(i, j)!$ make every coefficient of this polynomial integral. Thus it bounds the denominators of the universal polynomial, before the parameters are specialized to integers.

Akagi [1] developed explicit dilogarithm computations and structural formulas for rank-two scattering diagrams. In subsequent work [2], he proved several of the polynomiality and structural conjectures of Elgin–Reading–Stella and obtained partial results on others. The proof of Conjecture 11 recorded in [2, Proposition 4.3] is credited there to Burcroff, Lee, and Mou; see [2, Remark 1.4].

Burcroff, Lee, and Mou introduced tight gradings in [3, 4]. These are nonnegative integer assignments to edges of maximal Dyck paths satisfying compatibility conditions. Their formulas express rank-two wall functions as positive sums and connect them to Euler characteristics of quiver moduli and relative Gromov–Witten invariants. They also prove positivity results for generalized cluster scattering diagrams in higher rank. Burcroff, Lee, Mou, Musiker, and Reineke [5] subsequently combined this approach with quiver moduli.

Their Theorem 1.4 proves Conjectures 1–3, 5–10, and 14–18 of Elgin–Reading–Stella. Their Remark 1.5 lists Conjectures 4, 12, and 13 as remaining open. The polynomiality theorem [5, Theorem 1.2] provides the universal coefficients to which the factorial-denominator question applies. We prove Conjecture 4 for every pair $i, j > 0$.

Our proof uses the change of lattice in [5, Proposition 2.5] and the abelian wall-crossing formula of Mozgovoy–Reineke [15, Theorem 8.3]. The latter expands a wall Hamiltonian as a sum over weighted bipartite quivers. Its geometric interpretation [15, Theorem 6.5] replaces each abelian invariant by the Euler characteristic of a moduli space for a small generic deformation. The expansion has denominators from vertex weights and permutations of equal-weight vertices. We choose a deformation retaining the permutations on the side of smaller total weight. The stabilizers of stable trees consist of permutations of leaves with a common neighbor. Semistability bounds the total weight of these leaves by the weight of that neighbor. This clears their denominators with factorials on the larger side. A set-partition count then passes from Hamiltonians to wall functions.

We first define the coefficients algebraically. Let b, c be independent indeterminates and work in the formal power series ring $\mathbb{Q}[b, c][[x, y]]$, completed by total degree in x, y . For nonnegative integers u, v, r, s , set

$$\{x^u y^v, x^r y^s\} = (us - vr)x^{u+r}y^{v+s}, \quad (1)$$

and extend this bracket linearly and continuously. A series H with zero constant term defines the derivation $D_H = \{H, -\}$. Its exponential $\exp(D_H) = \sum_{n \geq 0} D_H^n / n!$ is well defined because D_H raises total degree. Put

$$H_x = c \sum_{a \geq 1} \frac{(-1)^{a-1}}{a^2} x^a, \quad H_y = b \sum_{s \geq 1} \frac{(-1)^{s-1}}{s^2} y^s. \quad (2)$$

These flows act respectively by

$$(x, y) \mapsto (x, y(1+x)^c), \quad (x, y) \mapsto (x(1+y)^{-b}, y).$$

Composition of automorphisms is read from right to left.

For relatively prime positive integers d, e , let $t = x^d y^e$. The ordered factorization

$$\exp(D_{H_x}) \exp(D_{H_y}) = \exp(D_{H_y}) \left(\prod_{\substack{e/d \text{ decreasing} \\ d, e > 0 \\ \gcd(d, e) = 1}} \exp\left(D_{\sum_{r \geq 1} H_{rd, re} t^r}\right) \right) \exp(D_{H_x}) \quad (3)$$

defines $H_{i,j} \in \mathbb{Q}[b, c]$ for $i, j > 0$. The product is read in the indicated order from left to right. Modulo terms of degree greater than any fixed integer, it is finite. To see existence and uniqueness, filter the Lie algebra of Hamiltonians by degree. At each successive degree, the discrepancy between the two sides has a unique decomposition into its monomials on primitive rays. The map $H \mapsto D_H$ is injective on series with zero constant term: bracketing a monomial $x^u y^v$ with x and y detects it unless $u = v = 0$. The discrepancy has only mixed monomials, since brackets involving both axes have both exponents positive. Inserting the required terms corrects that degree and changes only higher degrees. Starting at degree two gives (3). Setting either $b = 0$ or $c = 0$ leaves only one axis flow, so uniqueness gives

$$H_{i,j} \in bc\mathbb{Q}[b, c]. \quad (4)$$

Define $f_{d,e}(t) \in 1 + t\mathbb{Q}[b, c][[t]]$ by requiring that the ray factor in (3) act as

$$x \mapsto x f_{d,e}(t)^{-e}, \quad y \mapsto y f_{d,e}(t)^d.$$

The factor on this ray in (3) fixes t . Indeed, for $K = \sum_{r \geq 1} H_{rd, re} t^r$, the bracket gives $D_K(x)/x = -e \sum_{r \geq 1} r H_{rd, re} t^r$ and $D_K(y)/y = d \sum_{r \geq 1} r H_{rd, re} t^r$. Both multiplying series are fixed by D_K , so exponentiation gives

$$\log f_{d,e}(t) = \sum_{r \geq 1} r H_{rd, re} t^r. \quad (5)$$

Let g be a third independent indeterminate. Define $\tau(rd, re) \in \mathbb{Q}[b, c, g]$ by

$$1 + \sum_{r \geq 1} \tau(rd, re) t^r = \exp\left(\frac{g}{bc} \log f_{d,e}(t)\right) = \exp\left(g \sum_{r \geq 1} \frac{r H_{rd, re}}{bc} t^r\right). \quad (6)$$

Here $\log(1 + z) = \sum_{n \geq 1} (-1)^{n-1} z^n / n$ and the exponential are formal series; (4) makes the right side a series over $\mathbb{Q}[b, c, g]$. Every pair of positive integers lies on exactly one primitive ray, so this defines all $\tau(i, j)$. For a series F , $[t^r]F$ denotes its coefficient of t^r ; for a polynomial in g , $[g]F$ denotes its coefficient of g^1 , with absent coefficients equal to zero.

For positive integer values of b, c , the change-of-lattice formula [5, Proposition 2.5] identifies (6) with the cluster wall function after the substitution

$$g = \gcd(db, ec) = \frac{\gcd(ib, jc)}{\gcd(i, j)} \quad \text{when } (i, j) = k(d, e), \quad k \in \mathbb{Z}_{>0}.$$

The initial tropical-vertex functions are $(1+x)^c, (1+y)^b$ in that order; this fixes the placement of b and c .

The following uniqueness statement identifies this polynomial extension with the universal coefficients of [6, 5].

Lemma 1.1. *Fix relatively prime positive integers d, e . A polynomial $P \in \mathbb{Q}[b, c, g]$ that vanishes at every positive integer pair b, c with $g = \gcd(db, ec)$ is zero.*

Proof. Fix a positive integer q . For each positive integer m congruent to 1 modulo e , and every integer $h \geq 0$, take $b = qm$ and $c = q(1 + hdm)$. Since $\gcd(dm, e) = 1$ and $\gcd(dm, 1 + hdm) = 1$, these choices have $\gcd(db, ec) = q$. For fixed q, m , the polynomial $P(qm, c, q)$ therefore vanishes at infinitely many values of c , so it is identically zero as a polynomial in c . Each of its coefficients, viewed as a polynomial in b , vanishes at the infinitely many values qm with $m \equiv 1 \pmod{e}$. Thus $P(b, c, q) = 0$. Finally, each coefficient in b, c is a polynomial in g vanishing at every positive integer q , and is consequently zero. $\square$

Theorem 1.2. *For every pair of positive integers i, j ,*

$$\max(i, j)! \tau(i, j) \in \mathbb{Z}[b, c, g].$$

Section 5 deduces the theorem from the bound for $[g]\tau(i, j) = \gcd(i, j)H_{i,j}/(bc)$ proved in Proposition 4.2.

2 Abelian quivers and the wall-crossing formula

A partition $\lambda \vdash i$ is a finite multiset of positive integers with sum i . Let m_a be the number of its parts equal to a , and $p = \sum_{a \geq 1} m_a$ its length. For $\mu \vdash j$, write n_s for the multiplicity of s and $\ell = \sum_{s \geq 1} n_s$ for the length. Thus $\sum_a a m_a = i$ and $\sum_s s n_s = j$. Products indexed by positive integers include only nonzero multiplicities; we use $0! = 1$ and the empty product 1. For example, the partition $(2, 1, 1)$ has $m_1 = 2, m_2 = 1$ and length three.

Associate to (λ, μ) a quiver with one individually labelled vertex for each part. Vertices from μ are sources and those from λ are sinks. Between a source of weight s and a sink of weight a there are as distinct arrows, all directed towards the sink. A *thin representation* assigns $\mathbb{C}$ to every vertex and a complex scalar to every arrow. Isomorphisms rescale the bases at vertices: multiplication by a nonzero scalar at a vertex multiplies each incoming arrow scalar and divides each outgoing arrow scalar by that value. A common rescaling at every vertex acts trivially.

We describe stability for any such weighted bipartite quiver. Write L for its source set, R for its sink set, and $w_v > 0$ for the integer weight of a vertex v . Set

$$A = \sum_{l \in L} w_l, \quad B = \sum_{r \in R} w_r, \quad \theta_l = B w_l, \quad \theta_r = -A w_r.$$

For a dimension vector $\alpha = (\alpha_v)_{v \in L \sqcup R}$ with nonnegative integer entries, define

$$A_\alpha = \sum_{l \in L} w_l \alpha_l, \quad B_\alpha = \sum_{r \in R} w_r \alpha_r, \quad \theta(\alpha) = B A_\alpha - A B_\alpha.$$

The slope of a nonzero such vector is

$$\mu_\theta(\alpha) = \frac{\theta(\alpha)}{A_\alpha + B_\alpha}. \quad (7)$$

The denominator is positive. For a vertex subset U , its indicator dimension vector has entry one on U and zero elsewhere; write $\theta(U)$ and $\mu_\theta(U)$ for its parameter and, when $U \neq \emptyset$, its slope. The full thin dimension has parameter and slope zero. A thin representation is θ -stable if every nonzero proper subrepresentation has negative parameter, and θ -semistable if every such parameter is nonpositive.

For the quiver of (λ, μ) , these definitions use $A = j, B = i$. Choose a sufficiently small rational deformation of the vertex parameters. For rational parameters θ'_v with $\sum_v \theta'_v = 0$, use

$$\theta'(\alpha) = \sum_v \theta'_v \alpha_v, \quad \mu_{\theta'}(\alpha) = \frac{\sum_v \theta'_v \alpha_v}{\sum_v w_v \alpha_v} \quad (\alpha \neq 0).$$

Choose θ' close to θ so that the slopes of distinct nonempty subsets are different and every originally strict slope ordering is preserved. Such deformations exist by avoidance of finitely many proper hyperplanes, as in [15, Lemma 6.1]. Let $M_{\lambda, \mu}^{\text{def}}$ be the geometric moduli space of stable thin representations for this deformation, obtained by taking the geometric quotient of the stable locus by vertex rescalings. Rational parameters may be multiplied by a common positive denominator to give an integral stability character; this leaves stability unchanged. For a complex algebraic variety X , let $H_c^n(X, \mathbb{Q})$ be its compactly supported rational cohomology in degree n , and define its compactly supported Euler characteristic by $\chi_c(X) = \sum_{n \geq 0} (-1)^n \dim_{\mathbb{Q}} H_c^n(X, \mathbb{Q})$. Write

$$G_{\lambda, \mu} = \chi_c(M_{\lambda, \mu}^{\text{def}}), \quad (8)$$

where the deformation satisfies the conditions just stated.

For dimension vectors α, β , the antisymmetric Euler pairing in the convention of [15] is

$$\langle \alpha, \beta \rangle = A_\beta B_\alpha - A_\alpha B_\beta. \quad (9)$$

Equality of slopes for nonzero dimension vectors α, β gives

$$(A + B)(A_\alpha B_\beta - A_\beta B_\alpha) = 0,$$

so (9) vanishes on vectors of equal slope. This is the hypothesis of [15, Definition 3.6]. Theorem 6.5 there identifies the abelian invariant with the virtual motive of $M_{\lambda, \mu}^{\text{def}}$. At motivic parameter one, this is (8). In particular, $G_{\lambda, \mu}$ is independent of the permitted deformation; see also [15, Remark 6.7].

Proposition 2.1. *For all $i, j > 0$, the Hamiltonian coefficient in (3) is*

$$H_{i,j} = \sum_{\substack{\lambda \vdash i \\ \mu \vdash j}} \frac{(-1)^{i+j-p-\ell} c^p b^\ell G_{\lambda, \mu}}{\prod_{a \geq 1} m_a! a^{2m_a} \prod_{s \geq 1} n_s! s^{2n_s}}. \quad (10)$$

Proof. Apply [15, Theorem 8.3, equation (31)] to the charge lattice $\mathbb{Z}^2$ with determinant pairing. Let v be an indeterminate. The quantum torus has basis elements X^α , $\alpha \in \mathbb{Z}^2$, over $\mathbb{Q}(v)[b, c]$, with multiplication

$$X^\alpha X^\beta = v^{\det(\alpha, \beta)} X^{\alpha + \beta},$$

where $\det((u_1, u_2), (r_1, r_2)) = u_1 r_2 - u_2 r_1$. We use its subalgebra on nonnegative charges, completed by total degree; the motivic parameter is v^2 . For each nonzero nonnegative charge α , let $\bar{\Omega}_\alpha^-$ denote its input coefficient in the Hamiltonian of the wall-crossing identity. The corresponding quantum exponential has exponent $\sum_\alpha \bar{\Omega}_\alpha^- X^\alpha / (v - v^{-1})$, summed along one ray. For positive integers a, s , take

$$\bar{\Omega}_{(a,0)}^- = c \frac{(-1)^{a-1}}{a^2}, \quad \bar{\Omega}_{(0,s)}^- = b \frac{(-1)^{s-1}}{s^2};$$

All input coefficients on mixed charges are zero. For a fixed total-degree truncation, the theorem is a universal polynomial identity in finitely many initial coefficients. We may therefore adjoin the independent indeterminates b, c and make these substitutions. Its associated quiver has $\det((a, 0), (0, s)) = as$ arrows from the charge $(0, s)$ to $(a, 0)$, which is exactly the quiver defined above.

We take the classical limit on conjugation actions. For elements U, V of the quantum torus write $[U, V] = UV - VU$. For an invertible element S , let $\text{Ad}(S)$ denote the conjugation $U \mapsto SUS^{-1}$. Apply the virtual Poincaré polynomial to the motivic coefficients. For nonzero nonnegative charges α, β ,

$$\frac{[X^\alpha, X^\beta]}{v - v^{-1}} = \frac{v^{\det(\alpha, \beta)} - v^{-\det(\alpha, \beta)}}{v - v^{-1}} X^{\alpha + \beta} \xrightarrow{v \rightarrow 1} \det(\alpha, \beta) X^{\alpha + \beta}. \quad (11)$$

For a positive integer n , the quotient $(v^n - v^{-n}) / (v - v^{-1})$ is $v^{n-1} + v^{n-3} + \dots + v^{1-n}$ and has value n at $v = 1$. For $n = 0$ it is zero, and negative n follow by changing the sign. This proves the limit in (11).

For a quantum series H of positive total degree with coefficients regular at $v = 1$, write $H|_{v=1}$ for its commutative specialization, where $X^{(u_1, u_2)}$ becomes $x^{u_1} y^{u_2}$. The identity

$$\text{Ad exp} \left(\frac{H}{v - v^{-1}} \right) = \sum_{n \geq 0} \frac{1}{n!} \left(\frac{[H, -]}{v - v^{-1}} \right)^n$$

has only finitely many contributing terms in each total degree. Equation (11) makes each term regular at $v = 1$; specializing gives $\exp(D_H|_{v=1})$.

Theorem 6.5 and Corollary 6.6 of [15], with (9), make the output abelian invariants Laurent polynomials in v , whose values at one are $G_{\lambda,\mu}$. In the original charge coordinates (u_1, u_2) (sink weight, source weight), take the central charge $Z(u_1, u_2) = ju_1 - iu_2 + \sqrt{-1}(u_1 + u_2)$. For a nonzero nonnegative charge, its slope is $-\operatorname{Re} Z / \operatorname{Im} Z = (iu_2 - ju_1)/(u_1 + u_2)$. The map from vertex dimensions to charges is $\alpha \mapsto (B_\alpha, A_\alpha)$, which gives (7). The two basis charges have slopes $-j < i$, so this is the chamber c_+ of [15, §8]. Its clockwise order is decreasing u_2/u_1 , with the vertical axis first. The reverse, axis-only order is therefore the left side of (3). Thus the ordered quantum identity specializes to (3), whose coefficients are unique. Equation (31) of [15] has one multiplicity factorial for each distinct input charge. Grouping its nonzero inputs into partitions gives the factors $m_a!, n_s!$, while the axis coefficients contribute a^{2m_a}, s^{2n_s} . Their signs multiply to $(-1)^{i+j-p-\ell}$, and their parameter powers to $c^p b^\ell$. This is (10). $\square$

3 Stable supports and a symmetric deformation

A *support* of a thin representation is the set of arrows whose scalars are nonzero. A subset of vertices defines a subrepresentation precisely when it is closed under all arrows in the support: an arrow with source in the subset must also have its target there. Stability therefore depends only on the support. A tree will mean a connected undirected graph with no cycles. A spanning tree contains every vertex, and a leaf is a vertex of degree one. For a tree T , write $E(T)$ for its edge set and $\deg_T(v)$ for the number of edges incident to a vertex v .

Lemma 3.1. *For any permitted generic deformation, $G_{\lambda,\mu}$ is the number of stable spanning-tree supports, with distinct parallel arrows counted separately. Equivalently,*

$$G_{\lambda,\mu} = \sum_{T \text{ stable}} \prod_v w_v^{\deg_T(v)}, \quad (12)$$

where the sum is over stable spanning trees in the underlying simple complete bipartite graph.

Proof. Every stable support is connected. Otherwise each connected component is a nonzero proper subrepresentation and hence has strictly negative parameter. Summing over components would contradict total parameter zero.

Fix a connected support with V vertices and E arrows. Its nonzero arrow scalars form $(\mathbb{C}^*)^E$. Choose one spanning tree in the support and a base vertex. Successive vertex rescalings, starting at that vertex, normalize all its tree scalars to one. The common scalar at all vertices acts trivially, and no other rescaling preserves the normalized tree scalars. Thus the geometric quotient of this support stratum is $(\mathbb{C}^*)^{E-V+1}$. The finitely many support strata are locally closed and partition the stable moduli space.

Euler characteristic is additive over such strata, and $\chi_c((\mathbb{C}^*)^n) = 0$ for $n > 0$, whereas it is one for $n = 0$. Only spanning trees contribute. For an underlying simple bipartite tree T , each edge lr admits $w_l w_r$ choices of a parallel arrow; all have the same closed vertex subsets. The number of resulting supports is

$$\prod_{lr \in E(T)} w_l w_r = \prod_v w_v^{\deg_T(v)}. \quad \square$$

For the denominator estimate we orient the smaller total-weight side as the sources, so that $A \leq B$. If the original orientation is opposite, dualize every representation and reverse all arrows, and replace θ by $-\theta$. A subrepresentation of the dual corresponds to the dual

of a quotient of the original representation. Write $\dim M$ for the vector of dimensions at the vertices and M^* for the representation obtained by dualizing its vector spaces and arrow maps. If U is a subrepresentation of M , then $(M/U)^*$ is a subrepresentation of M^* and

$$(-\theta)(\dim(M/U)^*) = -\theta(\dim M - \dim U) = \theta(\dim U).$$

Every subrepresentation of M^* arises this way, so both stability inequalities are preserved. This duality induces an isomorphism of the corresponding moduli spaces, and hence preserves $G_{\lambda,\mu}$. After this operation the parameters again have the form $\theta_l = Bw_l, \theta_r = -Aw_r$.

Lemma 3.2. *When $A \leq B$, the invariant $G_{\lambda,\mu}$ can be computed by a set of stable trees that is invariant under permutations of equal-weight source vertices. Every tree in this set is semistable for the original parameter θ .*

Proof. Choose small rational numbers ε_r , one for each sink r , and first deform the parameters by

$$\delta\theta_r = \varepsilon_r, \quad \delta\theta_l = -\frac{w_l}{A} \sum_{r \in R} \varepsilon_r. \quad (13)$$

Their total change is zero, and equal-weight sources receive equal changes. For a vertex subset U , set $A_U = \sum_{l \in U \cap L} w_l$. Then

$$\delta\theta(U) = \sum_{r \in U \cap R} \varepsilon_r - \frac{A_U}{A} \sum_{r \in R} \varepsilon_r. \quad (14)$$

For every nonempty proper U , this linear form is nonzero. If $U \cap R = \emptyset$, identically zero would imply $A_U = 0$ and hence $U = \emptyset$. If $U \cap R = R$, it would imply $A_U = A$ and hence $U = L \sqcup R$. If $U \cap R$ is nonempty and proper, its coefficients take the two distinct values $1 - A_U/A$ and $-A_U/A$, so cannot all vanish.

First choose a neighborhood of zero in the ε -space on which the ordering of every pair of nonempty subsets with distinct original slopes is preserved. Such a neighborhood exists because there are finitely many pairs and their denominators are fixed and positive. Within it, avoid the finitely many proper hyperplanes on which an originally zero-parameter proper subset retains parameter zero. Rational points outside these hyperplanes exist. The resulting first deformation has nonzero parameter on every nonempty proper subset. Hence stability and semistability coincide for the full thin dimension, and every stable representation is originally semistable. Its stable supports are invariant under the indicated source permutations.

Now make a second, smaller rational deformation within the hyperplane of total parameter zero. Require it to preserve every strict subset-slope comparison at the first stage, and make the slopes of all distinct nonempty subsets different. For a subset U , let $\mathbf{1}_U$ have entry one on U and zero elsewhere. For distinct nonempty subsets U, V , equality of their slopes is the linear equation with coefficient vector

$$\frac{\mathbf{1}_U}{\sum_{v \in U} w_v} - \frac{\mathbf{1}_V}{\sum_{v \in V} w_v},$$

For nonnested subsets this vector has opposite signs on $U \setminus V$ and $V \setminus U$. If $U \subsetneq V$, positivity of the weights makes its entries positive on U and negative on $V \setminus U$; the other inclusion reverses the signs. It is therefore not a constant vector, so its linear form is nonzero on the hyperplane of total parameter zero. The slope equalities thus define finitely many proper hyperplanes to avoid.

Every comparison of a proper subset with the full dimension was already strict after the first deformation. It is unchanged after the second, so the two full-dimensional stable loci coincide. The second deformation also preserves every originally strict subset-slope order, and therefore satisfies all the conditions of [15, Lemma 6.1]. By [15, Theorem 6.5] it computes $G_{\lambda,\mu}$. Lemma 3.1 then computes that same invariant using the tree set of the first deformation. $\square$

4 The factorial bound for Hamiltonians

We keep $A \leq B$, and choose the stable tree set of Lemma 3.2. For a sink r , let L_r be the set of source leaves adjacent to r . For a positive integer a , let $m_{r,a}$ be the number of vertices in L_r of weight a .

Lemma 4.1. *For every sink r and every tree in this set,*

$$\sum_{a \geq 1} a m_{r,a} \leq w_r. \quad (15)$$

Proof. The subset $U = L_r \cup \{r\}$ is closed under all arrows of the tree support, since each source in L_r has only r as a neighbor and r is a sink. If L_r is nonempty and U is proper, original semistability gives

$$B \sum_{a \geq 1} a m_{r,a} - A w_r \leq 0.$$

As $A \leq B$, this implies (15). If U is the full vertex set, all sources are leaves at r and $R = \{r\}$, so the same bound is $A \leq B = w_r$. If L_r is empty, the left side is zero. $\square$

Proposition 4.2. *For every $i, j > 0$, with $k = \gcd(i, j)$ and $B = \max(i, j)$,*

$$B! \frac{k H_{i,j}}{bc} \in \mathbb{Z}[b, c]. \quad (16)$$

Proof. Fix a pair of partitions in (10), orient the smaller side as the sources, and write $A = \min(i, j)$. Let m_a^L be the number of sources of weight a , and n_s^R the number of sinks of weight s . Writing S_n for the permutation group of an n -element set, the source permutation group

$$G_L = \prod_{a \geq 1} S_{m_a^L}$$

permutes equal-weight sources and fixes every sink. The stable underlying trees form a G_L -invariant set. The weight $\prod_v w_v^{\deg_T(v)}$ is constant on each orbit.

Suppose first that a tree T has at least three vertices. A permutation in G_L fixing T can move a nonleaf source only to a source with the same neighbors. Two distinct sources with the same two distinct sink neighbors would give a cycle of length four, impossible in a tree. Thus every nonleaf source is fixed. The subgroup $\text{Stab}_{G_L}(T)$ fixing T consists precisely of arbitrary permutations among equal-weight source leaves with the same sink neighbor. Its order is

$$|\text{Stab}_{G_L}(T)| = \prod_{r \in R} \prod_{a \geq 1} m_{r,a}!. \quad (17)$$

The source multiplicity factorials in (10) form $|G_L|$. By the orbit-stabilizer formula, summing one orbit and dividing by all the multiplicity and squared-weight factors gives

$$\frac{\prod_v w_v^{\deg_T(v)-2}}{\left(\prod_{s \geq 1} n_s^R! \right) \prod_{r \in R} \prod_{a \geq 1} m_{r,a}!}. \quad (18)$$

The common sign and monomial in b, c have been left out of this numerical expression.

Negative weight exponents in (18) occur only at leaves. For each nonleaf sink r , combine the factors from its source-leaf neighbors and their permutations:

$$\prod_{a \geq 1} a^{m_{r,a}} m_{r,a}! \mid w_r!.$$

Indeed, place disjoint blocks of size a , one for each such leaf, inside a set of w_r elements, which is possible by Lemma 4.1. Cyclic rotations on the blocks and permutations of the blocks of equal size give a faithful permutation group of order $\prod_a a^{m_{r,a}} m_{r,a}!$. It is a subgroup of S_{w_r} , so its order divides $w_r!$.

A sink leaf cannot be adjacent to a source leaf in a tree with at least three vertices. Its only weight denominator is therefore w_r , which also divides $w_r!$. All other weight exponents in (18) are nonnegative. It follows that the denominator of (18), in lowest terms, divides

$$\left(\prod_{s \geq 1} n_s^R! \right) \prod_{r \in R} w_r! = \prod_{s \geq 1} n_s^R! (s!)^{n_s^R}, \quad \prod_{s \geq 1} n_s^R! (s!)^{n_s^R} \mid B!. \quad (19)$$

The second divisibility follows because the quotient counts the partitions of a labelled B -element set into n_s^R unordered blocks of size s for each s . Consequently $B!$ clears every orbit contribution from trees with at least three vertices.

If the quiver has exactly two vertices, their weights are A and B . There are AB stable single-arrow supports: the only nonzero proper closed subset is the sink, whose original parameter is strictly negative and remains so under the deformation. The quotient of this tree count by the squared-weight factor is $1/(AB)$. Multiplying by $k = \gcd(A, B)$ replaces it by

$$\frac{k}{AB} = \frac{1}{\text{lcm}(A, B)}.$$

Both A and B divide $B!$, hence so does their least common multiple. This handles the two-vertex term.

Each mixed-charge partition pair has $p, \ell \geq 1$, so its monomial $c^p b^\ell$ in (10) is divisible by bc . After division by bc , the orbit estimates and the two-vertex computation show that every contribution to $B! k H_{i,j} / (bc)$ is an integer-coefficient monomial. There are finitely many partition pairs and tree orbits, which proves (16). $\square$

5 Exponentiation on a primitive ray

Proof of Theorem 1.2. Fix relatively prime positive integers d, e . For each $r \geq 1$, define the polynomial

$$u_r = \frac{r H_{rd, re}}{bc} \in \mathbb{Q}[b, c].$$

Proposition 4.2 gives $(r \max(d, e))! u_r \in \mathbb{Z}[b, c]$. For a fixed positive integer k , the coefficient of t^k in (6) is

$$\tau(kd, ke) = \sum_{\substack{m_1, \dots, m_k \geq 0 \\ \sum_{r=1}^k r m_r = k}} g^{\sum_{r=1}^k m_r} \prod_{r=1}^k \frac{u_r^{m_r}}{m_r!}. \quad (20)$$

The integer m_r here records the number of times the term $u_r t^r$ is chosen in the exponential expansion. For every tuple in this finite sum,

$$\frac{(k \max(d, e))!}{\prod_{r=1}^k ((r \max(d, e))!)^{m_r} m_r!} \in \mathbb{Z}. \quad (21)$$

This quotient counts partitions of a labelled $k \max(d, e)$ -element set into m_r unordered blocks of size $r \max(d, e)$, for $1 \leq r \leq k$. The block sizes are distinct because $\max(d, e) \geq 1$.

After multiplication by $(k \max(d, e))!$, each summand of (20) becomes

$$\frac{(k \max(d, e))!}{\prod_{r=1}^k ((r \max(d, e))!)^{m_r m_r!}} g^{\sum_{r=1}^k m_r} \prod_{r=1}^k ((r \max(d, e))! u_r)^{m_r}.$$

The first factor is an integer by (21), and each remaining factor belongs to $\mathbb{Z}[b, c, g]$. Hence $(k \max(d, e))! \tau(kd, ke) \in \mathbb{Z}[b, c, g]$. Every (i, j) equals (kd, ke) for such a primitive (d, e) , and $k \max(d, e) = \max(i, j)$. $\square$

Example 5.1. The first coefficients illustrate the normalization:

$$H_{1,1} = bc, \quad H_{1,2} = \frac{bc(b-1)}{2}, \quad H_{2,1} = \frac{bc(c-1)}{2}.$$

For $(1, 2)$, the partition $\mu = (2)$ contributes $-bc/2$. For $\mu = (1, 1)$ and $\lambda = (1)$ the only underlying tree has both sources adjacent to the sink and contributes $b^2c/2$. Therefore

$$\tau(1, 1) = g, \quad \tau(1, 2) = \frac{g(b-1)}{2}, \quad \tau(2, 1) = \frac{g(c-1)}{2}.$$

The two-vertex term at $(i, j) = (2, 2)$ contributes $bc/4$ to $H_{2,2}$. The ray multiplicity $k = 2$ in (5) changes its contribution to $[g]\tau(2, 2)$ to $1/2$, which is cleared by $2!$.

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