

The Mirković–Vilonen basis and cluster pointedness

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Abstract

We prove that the Mirković–Vilonen basis of $\mathbb{C}[N]$ contains every cluster monomial and that each of its elements is g -pointed in every cluster, where N is a maximal unipotent subgroup of a complex semisimple group. This resolves Conjecture 6.6 in Kamnitzer’s ICM 2022 survey.

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The results were developed in an approximately 88-minute Codex session using GPT-6-Astra at high reasoning effort, followed by approximately three hours of research using Chat-6-Pro. Codex subsequently used its mathematical-paper-writing skill to revise the manuscript and conduct adversarial mathematical reviews. The manuscript passed an AI-assisted review using Rethlas-OSV [21, 22, 23].

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1 Introduction

Cluster algebras were introduced by Fomin and Zelevinsky to study canonical bases and total positivity [13]. Their distinguished generators are arranged into overlapping sets, called clusters, related by binomial exchange relations. Products of variables from one cluster are the cluster monomials. These provide a natural family of candidate basis elements, but generally do not span the algebra. For the coordinate ring of a maximal unipotent subgroup, this leads to two related problems: to identify the cluster monomials inside representation-theoretic bases, and to understand the Laurent expansions of the remaining basis elements. The cluster structures on double Bruhat cells of Berenstein–Fomin–Zelevinsky supply the reduced-word coordinates underlying this comparison [14].

1.1 Theta functions and positive bases

Fock and Goncharov proposed bases of functions on cluster varieties parametrized by integral tropical points of the dual cluster variety [15]. A seed identifies these tropical points with integer vectors, while a change of seed acts by a piecewise-linear mutation. Thus a basis element should carry leading-exponent data that transform consistently between charts. This formulation ties the basis problem to the geometry of the cluster atlas.

Theta functions arose in the mirror-symmetry construction of Gross–Hacking–Keel for log Calabi–Yau surfaces [16]. Gross–Hacking–Keel–Kontsevich developed the cluster version

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using scattering diagrams and broken lines [10]. A scattering diagram records wall-crossing transformations between torus charts; a broken line carries a monomial that changes at the walls it crosses. Summing its final monomials constructs a theta function. Their theory gives a canonical middle algebra, positive multiplication, and criteria identifying it with the upper cluster algebra. It also treats partial compactifications obtained by allowing frozen variables to vanish. In the unipotent setting, the resulting theta basis contains the cluster monomials and has a unit leading Laurent term in every seed, with the other exponents in the exchange cone [1, Theorem 6.4].

The quantum theory supplies a second source of positivity. Davison proved positivity of quantum cluster coefficients for skew-symmetric quantum cluster algebras through purity in cohomological Donaldson–Thomas theory [17]. Davison and Mandel combined that theory with scattering diagrams to construct quantum theta bases and prove positivity of their multiplication constants; their classical specializations recover the GHKK theta functions [18]. These results distinguish two positivity questions: positivity of Laurent expansions in cluster variables, and positivity of multiplication in a basis. The proof below uses the second property of the MV basis to obtain the first once cluster-monomial membership has been established.

Qin described bases parametrized by tropical points for full-rank, injective-reachable upper cluster algebras [19]. His degree and codegree comparison and his theory of triangular bases and twist automorphisms provide tools for propagating leading terms between seeds [9]. For unipotent coordinate rings, the dual canonical basis contains the cluster monomials in symmetrizable type. The categorical approach uses determinantal modules and their convolution products [6, 7]. These results suggest asking for the same cluster properties of a basis constructed by geometric Satake.

1.2 The MV basis and the cluster conjecture

The geometric Satake correspondence realizes irreducible representations through the affine Grassmannian of the Langlands dual group. Fundamental classes of Mirković–Vilonen cycles give their MV bases. Baumann–Kamnitzer–Knutson proved that these bases fit together into a basis of the unipotent coordinate ring and that its multiplication constants are intersection multiplicities in a fusion family [2]. In particular, these constants are nonnegative integers. The MV, dual canonical, and dual semicanonical bases have a common crystal parametrization, encoded by MV polytopes, although the corresponding functions can differ. Kamnitzer characterized MV polytopes by their chamber data and identified their crystal structure with that of the canonical basis [25, 26]. His ICM survey organizes these comparisons through perfect bases, polytopes, and their associated measures [1].

The cluster-monomial question already appears in the type A work of Anderson and Kogan: their Conjecture 5.1 predicts that every cluster monomial is an MV polynomial with a specified determinantal description [24]. Bai–Dranowski–Kamnitzer computed fusion products through the Mirković–Vybornov isomorphism and verified the cluster exchange relations for the upper unitriangular group of size 4 [27]. Baumann–Gaussent–Littelmann proved that the monomials in a reduced-word seed belong to the MV basis when the word satisfies their geometric condition (A) [3, Proposition 7.2]. Reduced-word seeds form only part of the mutation class, so a passage to arbitrary seeds is required.

In Conjecture 6.6 of his ICM 2022 report, Kamnitzer asked whether the MV basis contains every cluster monomial and whether each MV element is g -pointed in every cluster [1, §6.4]. Here pointedness means that a Laurent expansion has one distinguished monomial with coefficient one and all other exponents lie in its positive exchange cone. The same report asks separately whether the theta basis is biperfect (Question 6.5) and whether the MV and theta bases coincide (Conjecture 6.9). These place the cluster properties of the MV basis in a larger comparison of geometric and scattering constructions.

Subsequent work supplies additional comparison tools. Baumann's polite bases are compatible with extremal coproduct decompositions; the transition between two such bases is unitriangular with respect to polytope containment [4]. Leroux-Lapierre relates asymptotic characters to characteristic cycles and equivariant multiplicities in simply-laced type, and formulates the resulting evidence for a positive canonical-to-MV comparison [28]. Fei constructs crystal structures and proves biproductness results for generic bases of upper cluster algebras [20]. His work on universal Schubert polynomials studies finite families in type A indexed by common geometric-basis data [29]. These developments concern the comparison of bases through their representation-theoretic parameters. The present argument uses polytope containment to control Laurent support and positive multiplication to transport that control through mutation.

1.3 Main result

Let G be a complex semisimple group, N its positive maximal unipotent subgroup, and $A_0 = \mathbb{C}[N]$, with its standard unipotent cluster structure for this finite root system. Passing to the simply connected cover changes neither N nor this coordinate ring. We keep every frozen variable. The MV basis is denoted by $\mathcal{B}_{MV}$. A cluster monomial means a monomial with nonnegative integer exponents in the mutable and frozen variables of a single seed.

For a seed t , let m be the total number of variables, n the number of mutable variables, and $B_t \in \mathbb{Z}^{m \times n}$ the full exchange matrix. Here $X_1, \dots, X_m$ are the algebraically independent seed variables, e_j is the j -th standard coordinate vector, and $X^a = \prod_{i=1}^m X_i^{a_i}$ for $a \in \mathbb{Z}^m$. Write $\mathbb{N} = \mathbb{Z}_{\geq 0}$ and put $Y_j = X^{B_t e_j}$. A Laurent polynomial f is g -pointed at t if

$$f = X^g F(Y), \quad F \in \mathbb{C}[Y_1, \dots, Y_n], \quad F(0) = 1.$$

Thus the nonleading support lies in $g + B_t(\mathbb{N}^n \setminus \{0\})$. We use Qin's sign for the initial exchange matrix. Full column rank makes the Y -exponents unambiguous.

Theorem 1.1. *Every cluster monomial of the ring $\mathbb{C}[N]$ is an MV basis element. Every MV basis element is pointed in every seed. In fact, the resulting leading exponents are compatible under tropical mutation.*

The proof separates initial pointedness from its propagation. Polytope containment controls the dominance order in a reduced-word seed. We use Baumann's unitriangular comparison of polite bases [4] and the Laurent expansions of ordinary quiver Hecke simple modules [7] to obtain the initial pointed basis. A classical propagation criterion then uses positive multiplication and the unipotent twist to reach arbitrary seeds. The twist preserves the localized MV basis by cyclic rotation of invariant Satake tensors. Interval minors provide the finite mutation path needed to start the propagation.

2 Positive bases and mutation

We first work in a frozen-localized cluster algebra A contained in its upper cluster algebra. The coefficient field is $\mathbb{C}$. A positive basis means a $\mathbb{C}$ -basis whose multiplication constants are nonnegative integers. For a seed t , its Laurent ring is $\mathbb{C}[X_1(t)^{\pm 1}, \dots, X_m(t)^{\pm 1}]$; the upper cluster algebra is the intersection of these rings over all seeds in the mutation class. A seed is *adapted* if every monomial in its mutable variables, times any frozen Laurent monomial, is a basis element.

We use $[a]_+ = \max(a, 0)$, applied coordinatewise to vectors. Mutation at a mutable index k replaces X_k by

$$X'_k = X_k^{-1} \left(\prod_i X_i^{[b_{ik}]_+} + \prod_i X_i^{[-b_{ik}]_+} \right),$$

and leaves the other variables fixed. Its exchange matrix has entries

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+ + [b_{kj}]_+ - [-b_{ik}]_+ - [-b_{kj}]_+ & \text{otherwise.} \end{cases}$$

Rows run over all variables and columns over mutable variables. The Laurent support $\text{supp}_t f$ is the set of exponents with nonzero coefficient in the expansion of f at t . The dominance relation is $h \preceq_t g$ if $h - g \in B_t \mathbb{N}^n$. Full column rank makes it a partial order. A complete pointed basis has one element with each head $g \in \mathbb{Z}^m$.

The tropical mutation of a head is the piecewise-linear bijection

$$\phi_{t',t}(g) = J_k g + g_k [B_t e_k]_+ - \min(g_k, 0) B_t e_k,$$

where J_k changes the sign of the k -th coordinate. Pointed elements are *compatibly pointed* along a mutation path if their heads are related by these maps. We identify the seed Laurent rings inside their common rational function field.

2.1 Two elementary positivity facts

Lemma 2.1. *If a positive basis $\mathcal{B}$ is adapted to a seed t , then every $b \in \mathcal{B}$ has nonnegative integral Laurent coefficients at t .*

Proof. Choose a mutable monomial D large enough that Db has no negative mutable exponents. Its individual monomials, including their frozen Laurent factors, are distinct members of $\mathcal{B}$. On the other hand $D, b \in \mathcal{B}$, so their product has a nonnegative integral $\mathcal{B}$ -expansion. Uniqueness of the basis expansion identifies these coefficients with those of the Laurent polynomial. $\square$

Lemma 2.2. *Suppose the exchange column at k is nonzero. If a positive basis is adapted to t , and the new variable X'_k is a nonnegative integral basis combination, then it is adapted to $t' = \mu_k t$.*

Proof. Let

$$Z = (X'_k)^d \prod_{j \neq k} X_j^{a_j}, \quad d \in \mathbb{N},$$

where $a_j \geq 0$ for mutable $j \neq k$ and $a_j \in \mathbb{Z}$ for frozen j . Positivity of multiplication and the assumed positive expansion of X'_k give a finite expansion $Z = \sum_b n_b b$, with $n_b \in \mathbb{N}$. Write $M_{\pm} = \prod_j X_j^{[\pm b_{jk}]_+}$, so that $X_k X'_k = M_+ + M_-$, and put

$$R = \mathbb{C}[X_j^{\pm 1} : j \neq k], \quad y = M_+/M_-, \quad U = M_-^d \prod_{j \neq k} X_j^{a_j} \in R^{\times}.$$

The nonzero exchange column makes y a nonconstant Laurent monomial. By Lemma 2.1, every b with $n_b > 0$ has nonnegative integral Laurent coefficients. Since $Z = X_k^{-d} U (1 + y)^d$, its support contains the support of each such b . Thus

$$b = X_k^{-d} U P_b(y), \quad 0 \neq P_b \in \mathbb{N}[y], \quad \deg P_b \leq d.$$

Substitution of $X_k = M_-(1 + y)/X'_k$ gives

$$b = (X'_k)^d U M_-^{-d} \frac{P_b(y)}{(1 + y)^d}.$$

Laurentness in the new seed implies $P_b(y) \in (1 + y)^d R$.

The group algebra R is a free module over $\mathbb{C}[y^{\pm 1}]$: choose representatives of the cosets of the rank-one lattice generated by the exponent of y in $\mathbb{Z}^{m-1}$. The identity coset is a direct summand, so

$$(1+y)^d R \cap \mathbb{C}[y^{\pm 1}] = (1+y)^d \mathbb{C}[y^{\pm 1}].$$

Consequently $P_b(y) = (1+y)^d Q_b(y)$ for a Laurent polynomial Q_b . The least exponent in Q_b is nonnegative, since the constant term of $(1+y)^d$ is one and P_b is a polynomial. The degree bound then forces $Q_b = c_b$ to be a positive integer. Comparing constant terms in $(1+y)^d = \sum_b n_b P_b(y)$ yields $\sum_b n_b c_b = 1$. There is exactly one summand, with $n_b = c_b = 1$, and $Z = b$. $\square$

2.2 The two-cone lemma

Lemma 2.3. *Let $t' = \mu_k t$, and let $g' = \phi_{t',t}(g)$ be the tropical mutation of g . Suppose f is g -pointed at t , and*

$$\text{supp}_{t'} f \subseteq g' + B_{t'} \mathbb{N}^n.$$

Then f is g' -pointed at t' .

Proof. Write $f = X^g F(Y) = X^{g'} F'(Y')$, where the support assumption says that F' is a polynomial. Set $z = Y'_k$. The exchange relation and matrix mutation give

$$Y_k = z^{-1}, \quad Y_j = Y'_j z^{[-b_{kj}]_+} (1+z)^{b_{kj}} \quad (j \neq k), \quad X^g = X^{g'} z^{[-g_k]_+} (1+z)^{g_k}.$$

Full rank embeds the rational function field of the Y 's into that of the X 's, so these are identities in independent variables. Set the other Y'_j 's to zero in the localized ring where z and $1+z$ are invertible. If $P(y) = F(0, \dots, y, \dots, 0)$, then $P(0) = 1$, and

$$P'(z) := F'(0, \dots, z, \dots, 0) = z^{[-g_k]_+} (1+z)^{g_k} P(z^{-1}). \quad (2.1)$$

This is a polynomial.

If $g_k \geq 0$, a positive degree $d = \deg P$ would give a nonzero coefficient at z^{-d} , impossible. Thus $P = 1$ and $P'(0) = 1$. If $g_k = -r < 0$, put $d = \deg P$. Regularity at zero gives $d \leq r$, and regularity at -1 gives $(1+z)^r \mid z^d P(z^{-1})$. The polynomial on the right is monic of degree d , since $P(0) = 1$. Therefore $d = r$ and $z^d P(z^{-1}) = (1+z)^r$, so $P'(0) = 1$. This is the desired constant term of F' . $\square$

2.3 Adjacent compatibility

A seed s is *injective-reachable* if there is a seed $s[1]$ and a permutation σ of the mutable indices for which $I_j(s) := X_{\sigma(j)}(s[1])$ has mutable head $-e_j$ at s . The full heads may have frozen coordinates. We fix these shifts as in [9, Definition 2.5.1, equation (2.6), and §6.5], and write $s[r]$, $r \in \mathbb{Z}$, for their iterates; mutation of s in direction k corresponds to mutation of $s[r]$ in direction $\sigma^r(k)$. Write g_{uf} for the mutable coordinates of g . Qin's distinguished functions are

$$I_g^s = p_g X(s)^{[g_{\text{uf}}]_+} I(s)^{[-g_{\text{uf}}]_+}, \quad (2.2)$$

with the frozen factor p_g chosen so that the leading exponent at s is g . They are compatibly pointed at s and each adjacent seed [9, Lemma 6.2.7].

Proposition 2.4 (Classical adjacent propagation). *Suppose $\mathcal{B} = \{b_g : g \in \mathbb{Z}^m\}$ is a complete pointed basis at t , with distinct heads, and positive multiplication. Assume it is adapted to $t' = \mu_k t$, and every $I_j(t')$ belongs to $\mathcal{B}$, up to frozen Laurent factors which permute the basis. Then $\mathcal{B}$ is compatibly pointed at t, t' , with complete distinct labels at t' .*

Proof. Expand $I'_{\phi(g)}$ in $\mathcal{B}$. It is a finite nonnegative integral combination by (2.2). At t its head is g . The elementary finite dominance-triangular argument for pointed families with distinct heads gives coefficient 1 on b_g , and puts all other heads strictly in $g + B_t \mathbb{N}^n$. Indeed, a maximal head of a finite combination cannot be cancelled by any other incomparable or smaller head; removing maximal heads proves this assertion successively. This is also [9, Lemma 3.3.2].

Lemma 2.1 applies at t' , so no Laurent cancellation is possible in this positive basis expansion. Consequently

$$\text{supp}_{t'} b_g \subseteq \text{supp}_{t'} I'_{\phi(g)} \subseteq \phi(g) + B_{t'} \mathbb{N}^n.$$

Lemma 2.3 gives a unit head at $\phi(g)$. Tropical mutation is a bijection of $\mathbb{Z}^m$. $\square$

2.4 Propagation by the DT twist

A monomial variation is a Laurent-ring isomorphism sending mutable variables to the corresponding mutable variables times frozen Laurent monomials, and inducing an invertible integer map on the frozen exponent lattice. It identifies the exchange monomials Y_j . A Donaldson–Thomas (DT) type twist η is an automorphism of the common rational function field which has, for every seed s , a monomial variation from s to its injective shift $s[1]$, and satisfies

$$\eta(X_j(s)) = p_j I_j(s), \quad p_j \text{ a frozen Laurent monomial. } (j \text{ mutable}) \quad (2.3)$$

Variation maps preserve the Y -variables and the dominance cone, and commute with mutation with the standard permutation of mutable indices [9, §§5.1–5.2].

We record the classical calculation behind this use of variation. After relabeling mutable indices, let two exchange matrices be E, F and let the exponent map be

$$M = \begin{pmatrix} 1 & 0 \\ C & A \end{pmatrix} \in \text{GL}_m(\mathbb{Z}), \quad ME = F,$$

where the first block is the mutable identity and A acts on the frozen lattice. For $v = Ee_k$, $w = Fe_k = Mv$, put

$$d = M(-e_k + [v]_+) - (-e_k + [w]_+).$$

Its mutable part is zero. The exchanged binomials show that the map after mutation is M' , obtained by replacing the k -th column of M by $e_k + d$. The other columns and the frozen block are unchanged. Direct matrix mutation gives $M'\mu_k(E) = \mu_k(F)$; equivalently, both monomial maps identify the corresponding Y -variables and their mutation formulas. For the extended tropical formula

$$\phi_E(g) = J_k g + g_k[v]_+ - \min(g_k, 0)v,$$

where J_k negates the k -th coordinate, substitution gives $M'\phi_E(g) = \phi_F(Mg)$. Indeed $(Mg)_k = g_k$, $M'v = w$, and, writing $c = Me_k - e_k$, the remaining terms cancel by $c + d = M[v]_+ - [w]_+$. This proves the required transport of pointed functions and compatible heads, with arbitrary integral frozen shear and arbitrary unimodular frozen block.

At a seed s , an element is *copointed* at $h \in \mathbb{Z}^m$ if its coefficient at X^h is one and its support is contained in $h - B_s \mathbb{N}^n$. Being pointed at g and copointed at $g + B_s d$ is called *bipointed* with support dimension $d \in \mathbb{N}^n$.

Theorem 2.5 (Positive DT criterion). *Let A lie in a full-rank injective-reachable upper cluster algebra, with all frozen variables inverted. Suppose a $\mathbb{C}$ -basis $\mathcal{B}$ satisfies:*

1. *its multiplication constants are in $\mathbb{N}$, and frozen Laurent monomials permute it;*

2. it is a complete pointed basis and is adapted at one seed t_0 ;
 3. a DT-type twist η permutes the basis;
 4. along one finite mutation path from t_0 to $t_0[1]$, each newly created variable belongs to $\mathcal{B}$.
- Then every seed is adapted and $\mathcal{B}$ is compatibly pointed in every seed.

Proof. First apply Lemma 2.2 along the given path. Every seed on it becomes adapted. Formula (2.3) and twist/frozen stability put the injective variables at each such seed in $\mathcal{B}$. Proposition 2.4 then proves compatibility along the path, in particular at $t_0, t_0[1]$.

Transport this pair by the twist. A monomial variation maps pointed polynomials to pointed polynomials by an integral invertible linear transformation of exponents, and its mutation square commutes tropically. Since $\eta\mathcal{B} = \mathcal{B}$, this transports compatibility to every consecutive pair $t_0[r], t_0[r+1]$, $r \in \mathbb{Z}$; use η^{-1} for negative shifts. It also transports adaptedness.

Now assume these properties hold for a seed s and all its shifts. Fix a mutable k and write $s'[r] = \mu_{\sigma^r k} s[r]$. At $s[r]$, the new variable has the form

$$x' = X^g(1 + Y_{\sigma^r k}). \quad (2.4)$$

Choose the unique $b_g \in \mathcal{B}$ with this old head. Compatibility at $s[r], s[r-1]$, together with the degree/codegree comparison [9, Proposition 3.4.5], makes b_g bipointed with the same support dimension as x' , namely $e_{\sigma^r k}$. Full rank identifies its possible Y -exponents with the integral box $[0, e_{\sigma^r k}]$. This box has only two points. Both endpoint coefficients are 1, so $b_g = x'$.

Cluster monomials are already known to be compatibly pointed in every seed; see the discussion after Qin Proposition 3.4.5 and the sign-coherence results of [10, Theorem 5.11]. Thus (2.4) supplies this support dimension.

Lemma 2.2 makes each $s'[r]$ adapted; (2.3) puts its injective variables in the basis. Proposition 2.4 gives compatibility across each edge $s[r] \rightarrow s'[r]$. The rectangle consisting of two such edges and one shift commutes in the seed groupoid and under tropical mutation. Composing the three known compatible maps gives compatibility between $s'[r]$ and $s'[r-1]$. Thus the whole shifted chain has been established for the arbitrary neighbor s' .

Induction on a finite mutation word reaches every seed. If there are no mutable variables, the initial pointed assertion already says that the basis is the Laurent monomial basis. $\square$

3 The MV basis and the unipotent twist

Let I index the simple roots of G . Fix a maximal torus $T \subset G$, opposite maximal unipotent subgroups N and U^- , and put $\mathfrak{n} = \text{Lie}(N)$. Choose Chevalley generators e_i, f_i with $[e_i, f_i] = \alpha_i^\vee$. Write P and Q for the weight and root lattices, P_+ for the dominant weights, and W for the Weyl group. Its longest element is w_0 ; $\rho^\vee$ is half the sum of positive coroots. Denote the fundamental weights by ϖ_i , characterized by $\langle \alpha_j^\vee, \varpi_i \rangle = \delta_{ij}$. For $\lambda \in P_+$, let V_λ be the irreducible representation with chosen highest vector v_λ . Put $\lambda^* = -w_0\lambda$, the highest weight of the dual representation. Normalize the extremal vectors $v_{w\lambda}$ by the usual divided powers of the lowering operators along a reduced expression of w . For an element of the big cell, $[g]_-, [g]_0$, and $[g]_+$ are the factors in $g = [g]_- [g]_0 [g]_+ \in U^- T N$. For a weight γ and $h \in T$, h^γ denotes the value of its character. Pullback means $\eta^* b = b \circ \eta$, and $\text{Ad}_h(u) = h u h^{-1}$.

We use the positive pinning and the Tits lifts $\bar{s}_i = \exp(-e_i) \exp(f_i) \exp(-e_i)$. Let $\bar{w}$ lift the longest Weyl element. Transpose is the antiautomorphism fixing T and satisfying $e_i^T = f_i$. Let

$$\eta(u) = [\bar{w}u^T]_+, \quad \Delta_\lambda(u) = (v_\lambda, uv_{w_0\lambda}), \quad \Psi_\lambda(v)(u) = (v_\lambda, uv), \quad (3.1)$$

using the contravariant form normalized by $(v_\lambda, v_\lambda) = 1$. The map $\Psi_\lambda : V_\lambda \hookrightarrow \mathbb{C}[N]$ identifies the MV basis of the representation with a subset of $\mathcal{B}_{\text{MV}}$ [2, Proposition 6.1]. The MV bases arise from fundamental classes of the top-dimensional components of intersections of

semi-infinite orbits with affine Grassmannian Schubert varieties under geometric Satake. Their tensor-product version is the convolution MV basis; its invariant part is the invariant Satake basis used below [3, §§2–3]. The open set where all Δ_{ϖ_i} are nonzero is the unipotent cell. The map in (3.1) is its unipotent twist.

3.1 Extraction from invariant Satake vectors

Lemma 3.1. *For dominant root-lattice weights λ, μ, ν , let t be an invariant Satake basis vector in $(V_\lambda \otimes V_\mu \otimes V_\nu)^G$. Let $\xi_\nu \in V_\nu^*$ take the value one on v_ν and vanish on every other weight space, and let $\ell_\lambda \in V_\lambda^*$ take the value one on $v_{w_0\lambda}$ and vanish on every other weight space. Then*

$$\pi(t) = (\ell_\lambda \otimes 1 \otimes \xi_\nu)t \quad (3.2)$$

is a coefficient-one MV vector in V_μ . Conversely, each MV vector occurs in this way after allowing sufficiently large root-lattice highest weights for the other factors.

Proof. BGL’s proof of cyclic rotation explicitly identifies the lowest-first extraction with the MV basis of the appropriate isotypic highest-weight space [3, §6.1, pp. 974–975]. An isotypic summand of strictly smaller highest weight cannot have the larger highest weight in its weight support. Thus the quotient is the identity on the weight space being extracted.

The highest-last extraction is exactly the application of BGL Theorem 5.13 displayed in that proof: a tensor MV vector whose last crystal factor is highest projects to its first MV vector, with coefficient one. Highest elements in the tensor convention being used have such a last highest factor. This proves (3.2).

For the converse, choose a dominant root-lattice ν large enough that the tensor crystal pair consisting of the prescribed vector and the highest vector of V_ν is highest. Its weight $\lambda^* = \nu + \text{wt}(v)$ is dominant and in the root lattice. The isotypic identification above supplies the invariant Satake vector. Coherence of MV bases [2, Proposition 6.1] ensures that every stable MV function occurs in a sufficiently large such representation. $\square$

3.2 Rotation of invariant tensors

First work in the adjoint group, so all highest weights are in the root lattice and $\bar{w}^2 = 1$. The signed cyclic permutation in BGL Theorem 6.1 has sign $(-1)^{\langle 2\rho^\vee, \lambda \rangle} = 1$ for these weights. Let

$$\theta(g) = \bar{w}(g^T)^{-1}\bar{w}^{-1},$$

the positive pinned diagram automorphism.

On the decorated-flag space G/U^- , a vector $v \in V_\lambda$ defines the regular function $\xi_\lambda(g^{-1}v)$. Apply these three matrix coefficients to t to obtain the invariant function F_t on triples. Normalize a generic triple as

$$(\bar{w}U^-, h_1U^-, uh_2U^-).$$

Common left translation by u^{-1} , followed by (3.2), gives

$$F_t = h_1^{-\mu} h_2^{-\nu} b(u), \quad b = \Psi_\mu(\pi(t)). \quad (3.3)$$

Let $\rho(F_1, F_2, F_3) = (F_2, F_3, F_1)$ denote left rotation of decorated flags. For the cyclic-coordinate calculation, put $l = [u\bar{w}]_-$ and $h = [u\bar{w}]_0$; then $l = \eta(u)^T$. After left rotation of the triple, left multiply by $\bar{w}h_1^{-1}l^{-1}$. The normalized coordinates become

$$\begin{aligned} h'_1 &= \theta(h_1)h_2w_0(h), \\ h'_2 &= \theta(h_1), \\ u' &= \text{Ad}_{\theta(h_1)}\theta(\eta(u)). \end{aligned} \quad (3.4)$$

Here $w_0(h) = \bar{w}h\bar{w}^{-1}$. This is the cyclic-coordinate formula of [11, Lemma 5.4].

The root degree of b is $\beta = \mu + \nu - \lambda^*$. Substitution of (3.4) into (3.3) gives

$$\rho^* F_t = h_1^{-\lambda} h_2^{-\mu} \Delta_{\mu^*}(u) b(\theta(\eta(u))). \quad (3.5)$$

Pullback by left rotation of flags is right rotation of the tensor. BGL's invariant Satake basis is preserved by either cyclic direction. The same extraction calculation on that rotated tensor therefore shows that the last factor in (3.5) is one MV function. The pinned diagram automorphism preserves MV vectors: it transports MV cycles, Chevalley divisors, and their intersection pairings, with the complex orientation. Equivalently these ordered Chevalley-word pairings separate the coordinate functions. Consequently

$$\Delta_\lambda \eta^*(b) \in \mathcal{B}_{\text{MV}} \quad \text{if } \lambda \in Q \cap P_+, \quad b \in \mathcal{B}_{\text{MV}} \cap \text{im } \Psi_\lambda. \quad (3.6)$$

All coefficients in this assertion are exactly one.

3.3 Frozen closure and removal of the root-lattice restriction

Proposition 3.2. *For every dominant weight λ and $b \in \mathcal{B}_{\text{MV}} \cap \text{im } \Psi_\lambda$, $\Delta_\lambda \eta^*(b)$ is one MV basis element. For every dominant δ , multiplication by Δ_δ maps $\mathcal{B}_{\text{MV}}$ into $\mathcal{B}_{\text{MV}}$. After freezing is localized and duplicate actual functions are identified, η^* permutes the localized basis.*

Proof. The inverse cyclic rotation and the surjectivity in Lemma 3.1 show that every $c \in \mathcal{B}_{\text{MV}}$ can be written

$$c = \Delta_\kappa \eta^*(a), \quad a \in \mathcal{B}_{\text{MV}} \cap \text{im } \Psi_\kappa, \quad \kappa \in Q \cap P_+. \quad (3.7)$$

Explicitly, realize c by an invariant tensor, rotate that tensor backwards, and apply (3.5) to the resulting tensor. If its middle highest weight is μ and its extracted function is b , set $\kappa = \mu^*$ and $a = b \circ \theta$. The pinned diagram automorphism carries $\text{im } \Psi_\mu$ to $\text{im } \Psi_{\mu^*}$, with its MV basis, giving exactly (3.7). We also use the normalized identity $\Delta_{\lambda+\delta} = \Delta_\lambda \Delta_\delta$. It follows from the highest-weight inclusion $V_{\lambda+\delta} \hookrightarrow V_\lambda \otimes V_\delta$, which sends both the highest and the lowest extremal vectors to their tensor products with coefficient one. For root-lattice dominant δ , coherence puts a in $\text{im } \Psi_{\kappa+\delta}$. Applying (3.6) proves $\Delta_\delta c \in \mathcal{B}_{\text{MV}}$.

For arbitrary dominant δ , choose $r \geq 1$ with $r\delta \in Q$. The flag minor $d = \Delta_\delta$ is an MV element [2, Remark 2.10]; $d^r c$ is a single unit basis element by the preceding paragraph. If dc had two coefficient units in its nonnegative integral MV expansion, multiplication by d^{r-1} would keep at least two: the coordinate ring is a domain and every nonzero product has a nonzero positive integral expansion. This contradicts the unit expansion of $d^r c$. Thus $dc \in \mathcal{B}_{\text{MV}}$. The positivity used here is BKK's fusion multiplication [2, Theorem 7.11].

For arbitrary dominant λ , a Gauss-decomposition calculation gives the regular-function identity

$$\Delta_\lambda(u) \Psi_\lambda(v)(\eta(u)) = (v, uv_{w_0\lambda}). \quad (3.8)$$

If $\bar{w}^2 = s_G$ acts on V_λ by s_λ , write $\bar{w}u^T = lh\eta(u)$. Then $\xi_\lambda \bar{w} = s_\lambda \ell_\lambda$. Applying ξ_λ to the decomposition, and using $\xi_\lambda l = \xi_\lambda$, gives

$$h^\lambda \Psi_\lambda(v)(\eta(u)) = s_\lambda \ell_\lambda(u^T v) = s_\lambda(v, uv_{w_0\lambda}).$$

The extremal vector has norm one for the contravariant form. Taking $v = v_\lambda$ yields $h^\lambda = s_\lambda \Delta_\lambda(u)$. Substitution cancels s_λ and proves (3.8).

Choose dominant δ with $\lambda + \delta \in Q$. The function in (3.8), call it f , is regular, while $\Delta_\delta f$ is a single MV element by (3.6) and coherence. The ideal $\Delta_\delta A_0$ has basis $\Delta_\delta \mathcal{B}_{\text{MV}}$, a subset of $\mathcal{B}_{\text{MV}}$ by the frozen result already proved. Thus basis compatibility of this ideal forces $f \in \mathcal{B}_{\text{MV}}$.

The fundamental Δ 's are the frozen variables. Their multiplication is injective on the basis, so the usual direct-limit localization gives a basis with duplicate actual functions removed.

Identity (3.8), with $v = v_{w_0\lambda}$, gives $\eta^*\Delta_\lambda = \Delta_\lambda^{-1}$. Thus η^* maps every localized basis element to one. Formula (3.7) shows conversely that every original MV element, and hence every localized one, is obtained in this way. The classical unipotent twist is birational [12, Proposition 2.20], so these maps are bijective. $\square$

4 Pointedness in reduced-word seeds

4.1 An ordinary KLR polite comparison basis

For a homogeneous function $f \in \mathbb{C}[N]$ of torus weight γ , let $\text{Pol}(f)$ be the convex hull of those $\delta \in Q$ for which the $(\delta, \gamma - \delta)$ -component of its coproduct is nonzero. The coproduct is $\Delta f(u, v) = f(uv)$; the torus action is $(h \cdot f)(u) = f(h^{-1}uh)$. In particular, a function pairing with $U(\mathfrak{n})_\nu$ has weight $-\nu$. We use ordinary quiver Hecke algebras for the symmetrizable Cartan datum, in characteristic zero, and specialize their restricted integral dual at $q = 1$, paired with the universal enveloping algebra $U(\mathfrak{n})$. Here $Q_+ = \sum_i \mathbb{N}\alpha_i$. Write $R(\nu)$ for the quiver Hecke algebra of degree $\nu \in Q_+$, and use finite-dimensional graded modules, identifying grading shifts after specialization. For an $R(\nu)$ -module L , write $\deg L = \nu$ for its root degree. For a word $\mathbf{j} = (j_1, \dots, j_d)$ of weight $\nu = \alpha_{j_1} + \dots + \alpha_{j_d}$, $e(\mathbf{j})$ is its word idempotent. The product $L \circ M$ is induction from $R(\alpha) \otimes R(\beta)$ to $R(\alpha + \beta)$, and $\text{Res}_{\alpha, \beta}$ is the corresponding idempotent restriction. The notation hd means the maximal semisimple quotient; in the instances used below this quotient is simple. The symbol $\boxtimes$ denotes the external tensor product of modules. For a graded simple L , let χ_L be its direct character function:

$$\langle e_{j_1} \cdots e_{j_d}, \chi_L \rangle = \dim_{\mathbb{C}} e(j_1, \dots, j_d)L. \quad (4.1)$$

Induction gives multiplication and restriction gives coproduct in this coordinate ring. In one simple-root direction the nil-Hecke simple has character $r!i^r$, so its pairing with $e_i^{(r)}$ is one; the function is ζ_i^r , where ζ_i is the weight $-\alpha_i$ function satisfying $\langle e_i, \zeta_i \rangle = 1$. Here $e_i^{(r)} = e_i^r/r!$.

We describe the filtration and splitting required of a polite basis in the positive-degree convention. For a homogeneous function of weight $-\nu$, call $\gamma \in Q_+$ a cut if its coproduct component of weights $(-\gamma, -\nu + \gamma)$ is nonzero. Fix a real linear functional θ on the root lattice. Project a cut to $(\text{ht } \gamma, \theta(\gamma))$, where height is the sum of its simple-root coordinates, and order these pairs by increasing second coordinate, with smaller first coordinate larger when the second coordinates agree. The filtration piece at (r, d) consists of functions all of whose projected cuts have $\theta(\gamma) < d$, or $\theta(\gamma) = d$ and $\text{ht } \gamma \geq r$. Taking the coproduct components at the largest pair defines a map on the associated graded space.

This is Baumann's splitting map Δ' [4, Proposition 2.9], after changing from negative weights to positive degrees and relabeling θ by its negative. Its factors are the subspaces whose nonempty suffix cuts have strictly positive θ , and whose prefix cuts have nonpositive θ , respectively. The other splitting uses the opposite endpoint and the opposite strict/non-strict convention. A basis is polite if it is homogeneous, contains every ζ_i^r , spans every piece of these filtrations by its intersection with that piece, and each splitting maps the induced basis bijectively to the tensor product of the bases in the two factors. These are the conditions of [4, Definition 3.1].

A charge is a linear map from the root lattice to $\mathbb{C}$ taking positive roots into an open half-plane. It orders root degrees by argument. For the charge $-\theta + i \text{ht}$ used below, the slope of a nonzero $\nu \in Q_+$ means $\theta(\nu)/\text{ht}(\nu)$; increasing slope corresponds to increasing argument in $(0, \pi)$. A simple module of degree ν is semicuspidal for a charge c if every nonempty proper prefix of each word in its character, of degree γ , satisfies $\arg c(\gamma) \leq \arg c(\nu)$, with arguments chosen in the given half-plane [5, Definition 2.3]. A pair (A, B) is *unmixing* if restriction of $A \circ B$ to their ordered degrees is exactly $A \boxtimes B$.

Lemma 4.1. *The ordinary KLR simple-function basis is polite in the sense of [4, Definition 3.1], in every finite symmetrizable type.*

Proof. Homogeneity and the rank-one normalization were just checked. Restriction has nonnegative simple composition multiplicities, so its nonzero root-weight cuts are exactly the character-prefix cuts. The coproduct polytope of (4.1) is the negative of the positive character polytope.

We verify the extremal splitting condition, including adaptation to the whole filtration. Write the projected degree of a positive dimension vector ν as $(\text{ht } \nu, \theta(\nu))$, for an arbitrary linear functional θ . The relevant order maximizes the second coordinate and then minimizes height. This is the strict/non-strict cut in Baumann's splitting map Δ' ; replacing the sign of θ is just a relabeling of the family of all such cuts.

Apply the semicuspidal factorization of [5, Theorem 2.4] to the charge $-\theta + i \text{ht}$, and group the factors of positive slope into their simple head A , and those of nonpositive slope into their simple head B . Every nonempty suffix of a character word of A has positive θ , and every prefix of a word of B has nonpositive θ . These facts hold for semicuspidals by their defining slope inequality, pass to shuffles, and pass to quotients. A crossing shuffle at the cut $(\deg A, \deg B)$ would identify a nonempty suffix degree of A with a nonempty prefix degree of B . Their θ -values have opposite strict/non-strict signs, which is impossible. Thus (A, B) is unmixing by [5, Lemma 2.8]. The induction is generated by its restriction at this cut; every proper submodule has zero restriction there [5, Lemma 2.6]. Its unique simple quotient is the original L by the semicuspidal factorization theorem. Exact restriction therefore gives

$$\text{Res}_{\alpha, \beta} L = A \boxtimes B, \quad \alpha = \deg A, \quad \beta = \deg B, \quad (4.2)$$

with multiplicity one.

Any character prefix of L is a shuffle prefix and has dimension γ satisfying

$$\gamma - \alpha = -\deg(\text{remaining suffix of } A) + \deg(\text{chosen prefix of } B).$$

Its θ -value is at most $\theta(\alpha)$. Equality forces the remaining A -suffix to be empty; equality also of height forces the B -prefix to be empty. Hence the largest projected cut is a unique *root-lattice* cut, α .

The leading tensor is therefore $A \boxtimes B$. Distinct simples cannot have the same leading tensor, since the unmixing induction has a unique simple head. Leading terms of any finite linear combination cannot cancel, which proves adaptation to every filtration piece. Conversely, every pair satisfying the strict/non-strict suffix/prefix conditions is unmixing and occurs as the extremal restriction of its unique simple head. Thus the splitting map takes the induced basis bijectively to the tensor-product basis, with coefficient one.

For the second endpoint, group the semicuspidal factors of nonnegative slope before those of strictly negative slope. The same argument uses nonnegative suffix values in the first group and strictly negative prefix values in the second. It proves adaptation and bijectivity for the other splitting, with the same unit coefficient. $\square$

The coordinate convention used by KKOP is reversed relative to (4.1). KKOP [6, §§1.8.2, 10.2] define $\langle \iota(u), x \rangle = (u, \varphi(x))$, with φ an *antiautomorphism* sending e_i to f_i . Consequently

$$\Omega_L = \kappa^* \chi_L, \quad \langle e_{j_1} \cdots e_{j_d}, \Omega_L \rangle = \dim e(j_d, \dots, j_1) L. \quad (4.3)$$

Here κ fixes the simple root subgroups and reverses multiplication. Word reversal permutes gradable simples [5, §1.7]. Thus the χ -basis and the Ω -basis are the same set of actual functions, with reflected indexing. If P_L is the positive character polytope and $\nu = \deg L$, then

$$\text{Pol}(\Omega_L) = P_L - \nu, \quad \text{Pol}^\sharp(\Omega_L) := \text{wt}(\Omega_L) - \text{Pol}(\Omega_L) = -P_L. \quad (4.4)$$

The second transformation is fixed on each weight space and preserves containment.

4.2 The chamber/exchange-matrix identity

Let $\varpi_a^\vee$ be the fundamental coweight characterized by $\langle \alpha_b, \varpi_a^\vee \rangle = \delta_{ab}$. The dominant chamber in the real coweight space is

$$C = \{x \in \text{Hom}_{\mathbb{Z}}(Q, \mathbb{R}) : \langle \alpha_i, x \rangle > 0 \text{ for all } i \in I\}.$$

Let $\mathbf{i} = (i_1, \dots, i_\ell)$ be a reduced word for w_0 . Put $w_k = s_{i_1} \cdots s_{i_k}$, $\beta_k = w_{k-1} \alpha_{i_k}$, and let $p(k), s(k)$ be the previous and next occurrences of i_k . Put $p(k) = 0$ if no previous occurrence exists and $s(k) = \infty$ if no next occurrence exists. The vectors $e_1, \dots, e_\ell$ are the standard basis of the exponent lattice. Mutable indices are those with finite $s(k)$; final occurrences are frozen. Denote the set of mutable positions by J , and set $n = |J|$. Define

$$Ke_j = e_j - e_{p(j)}, \quad (Kc)_k = c_k - c_{s(k)}. \quad (4.5)$$

Use $e_0 = 0$ and $c_\infty = 0$. Set $C_{ab} = \langle \alpha_a^\vee, \alpha_b \rangle$. Qin's full matrix is

$$(B_Q)_{jk} = \begin{cases} 1 & j = p(k), \\ -1 & j = s(k), \\ C_{i_j i_k} & j < k < s(j) < s(k), \\ -C_{i_j i_k} & k < j < s(k) < s(j), \\ 0 & \text{otherwise.} \end{cases} \quad (4.6)$$

For a normalized negative MV polytope with edge data c , put

$$\mu_{w_k} = - \sum_{j>k} c_j \beta_j, \quad m_k = \min_{x \in P} \langle x, w_k \varpi_{i_k}^\vee \rangle.$$

Here the edge data $c_j \in \mathbb{N}$ record the edge increments $\mu_{w_j} - \mu_{w_{j-1}} = c_j \beta_j$; the normalization is $\mu_{w_0} = 0$. The vertices μ_w minimize the chamber functionals $w \varpi_a^\vee$. In (4.7), $m_{p(k)}$ means m_{0, i_k} when $p(k) = 0$. The initial chamber values are $m_{0,a} = -\langle \nu, \varpi_a^\vee \rangle$, where $\nu = \sum c_j \beta_j$, and the final chamber values are zero. If $q_a(k)$ is the latest occurrence of a before k , using the initial chamber value $m_{0,a}$ when there is none, the coweight reflection formula gives

$$c_k = -m_{p(k)} - m_k - \sum_{a \neq i_k} C_{i_k a} m_{q_a(k)}. \quad (4.7)$$

Indeed the new i_k -coweight is the negative of the old one minus $\sum_{a \neq i_k} C_{i_k a}$ times the other chamber coweights, and the edge increment pairs with it as $-c_k$.

Lemma 4.2. *Let $W : \mathbb{Z}^\ell \rightarrow Q$ be the degree map $Wc = \sum_j c_j \beta_j$, and let*

$$H_{kj} = \begin{cases} -\langle \beta_j, w_k \varpi_{i_k}^\vee \rangle & j > k, \\ 0 & j \leq k \end{cases} \quad (k \text{ mutable}).$$

Then

$$K\delta c = B_Q H \delta c \quad \text{whenever } W\delta c = 0. \quad (4.8)$$

The matrix B_Q has full column rank. If same-weight MV polytopes $P' \subsetneq P$ have the indicated data, then

$$Kc(P') - Kc(P) = B_Q n, \quad n \in \mathbb{N}^n \setminus \{0\}. \quad (4.9)$$

Proof. Take differences of (4.7). Fixed weight makes all initial boundary differences zero, and final boundary differences are zero as well. Subtract the equation at $s(k)$ from the one at k . For each other label, all interior occurrences cancel, leaving only the two interval endpoints. Their coefficients are precisely row k of (4.6). For a frozen row use $s(k) = \infty$ and the zero final boundary. This proves (4.8) with the full frozen rows retained.

More formally, put arbitrary variables at mutable chambers, zero at all initial and terminal chambers, and define E by (4.7). The same telescoping identity is

$$KE = B_Q. \quad (4.10)$$

It holds for any finite word and any matrix with diagonal 2. Here is a direct reconstruction which also proves the full integral-kernel assertion needed below. At step k , define x_k by pairing it with the basis $w_k \varpi_a^\vee$: its pairings are the current chamber values for the labels a . Only the i_k -value changes. The reflection formula gives $x_k - x_{k-1} = (Em)_k \beta_k$. The zero boundaries give $x_0 = x_\ell = 0$; hence $WE = 0$ and $x_k = -\sum_{j>k} (Em)_j \beta_j$, which proves $HE = 1$.

Conversely, for any integral c with $Wc = 0$, put $x_k = \sum_{j \leq k} c_j \beta_j = -\sum_{j>k} c_j \beta_j$. Its mutable chamber values are Hc ; all initial and terminal values are zero. Pairing the edge increments with the chamber coweights and using (4.7) gives $c = EHc$. Thus

$$\ker_{\mathbb{Z}} W = E\mathbb{Z}^n, \quad HE = 1.$$

Since K is triangular with diagonal 1, $B_Q = KE$ has full column rank and (4.8) holds on the whole kernel.

Containment increases every chamber minimum, so $n = m(P') - m(P) \geq 0$. Formula (4.8) gives (4.9). The inequality is strict because reduced-word edge data parametrize MV polytopes injectively. $\square$

For example, for the type A_2 word $(1, 2, 1)$, the only mutable position is 1, and

$$K = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B_Q = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}.$$

Writing $u = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$, the seed is $(X_1, X_2, X_3) = (a, ac - b, b)$. Mutation replaces a by c , and

$$c = X_1^{-1} X_3 (1 + X_2 X_3^{-1}).$$

Thus its head is $(-1, 0, 1)$, and its second exponent differs by the column B_Q .

4.3 Laurent support of KLR simples

For $0 \leq a \leq b \leq \ell$ and a fundamental weight λ , let $M(w_b \lambda, w_a \lambda)$ denote the self-dual determinantal simple module of root degree $w_a \lambda - w_b \lambda$, supplied by [7, Proposition 2.2]. Its function under (4.3) is

$$\Omega_{M(w_b \lambda, w_a \lambda)}(u) = (v_{w_a \lambda}, uv_{w_b \lambda})$$

[6, §§8–10]. In particular, the seed variables are

$$X_k = \Omega_{M(w_k \varpi_{i_k}, \varpi_{i_k})}, \quad X_k(u) = (v_{\varpi_{i_k}}, uv_{w_k \varpi_{i_k}}). \quad (4.11)$$

These are Qin's negative-unipotent minors transported by transpose and BGL's positive-unipotent flag minors.

Let $a(L) \in \mathbb{N}^\ell$ be KKOP's cuspidal vector:

$$L = \text{hd}(S_\ell^{\circ a_\ell} \circ \cdots \circ S_1^{\circ a_1}), \quad S_k = M(w_k \varpi_{i_k}, w_{k-1} \varpi_{i_k}). \quad (4.12)$$

The order is reversed. The character-polytope path traverses $\beta_\ell, \dots, \beta_1$ in that order, by the convex-order semicuspidal and polytope results of [5, §§2.2, 3.1]. Every prefix root degree of S_k belongs to $Q_+ \cap (-w_k Q_+)$, by the determinantal-module construction [7, §2.3]. A functional on the wall between $w_{k-1}C$ and $w_k C$ is negative on preceding roots, zero on β_k , and positive on following roots. It gives precisely the semicuspidal cut for the reversed convex order. The general convex-order theorem applies to this order. Negating and reversing this path in (4.4) yields

$$\mu_{w_k}(\text{Pol}^\sharp(\Omega_L)) = - \sum_{j>k} a_j(L) \beta_j. \quad (4.13)$$

Put $M_j = M(w_j \varpi_{i_j}, \varpi_{i_j})$ and, for $a \in \mathbb{N}^\ell$, write $M(a)$ for the convolution product of the $M_j^{\circ a_j}$. These modules commute up to grading shift, and $M(a)$ is simple; shifts disappear at $q = 1$. Choose $a \in \mathbb{N}^\ell$ sufficiently large that every composition factor of $L \circ M(a)$ is a determinantal monomial $M(b)$; such an a exists by [7, Corollary 3.4]. If its simple head is $M(b_0)$, then

$$b_0 - a = Ka(L)$$

by [7, Definition 3.7 and Proposition 4.5]. We determine the other exponents from the R-matrix inequalities.

A simple module is real if its convolution square is simple, and affreal if it is real and admits an affinization in the sense of [8, §2]. For a pair with an affreal factor, $\Lambda(M, N)$ is the degree of its normalized R-matrix $M \circ N \rightarrow N \circ M$, with the convention of [7, Definition 2.6]. The head $M(b_0)$ has multiplicity one [7, Proposition 2.7]. For each j , enlarge the denominator to $M(a + te_j)$, for arbitrary $t \geq 0$. Tensor the surjection $L \circ M(a) \twoheadrightarrow M(b_0)$ by $M(te_j)$. Its target $M(b_0 + te_j)$ is simple, while the enlarged source has a unique simple head by KKOP Proposition 2.7(ii). Thus that target is its head. Exact tensoring likewise keeps $M(b + te_j)$ as a constituent. All these denominators are affreal: commuting convolution affinizations are supplied by [8, Lemma 2.16], iterated over the determinantal family, whose individual affreal modules are supplied by [7, Theorem 2.4].

For a nonhead constituent, KKOP Proposition 2.7(iii)(a) gives

$$\Lambda(M(b + te_j), M(a + te_j)) < \Lambda(M(b_0 + te_j), M(a + te_j)).$$

Bilinearity on the commuting family in KKOP Lemma 2.11 cancels the terms common to the two sides, leaving

$$0 < \Lambda(M(b_0 - b), M(a)) + t\Lambda(M(b_0 - b), M_j) \quad (t \in \mathbb{N}).$$

The coefficient of t is therefore nonnegative. For the head itself the difference is zero. We obtain

$$\Lambda(M(b_0 - b), M_j) \geq 0 \quad \text{for every } j. \quad (4.14)$$

On a difference of exponent vectors Λ denotes its bilinear extension on the commuting family; the additivity is [7, Lemma 2.11]. Let $L_{st} = -\Lambda(M_s, M_t)$ be the skew matrix displayed for this seed; its equality with the actual R-matrix degrees is [7, Proposition 2.17]. For $d = b_0 - b$, (4.14) says $Ld \geq 0$. Equal constituent weights give $WK^{-1}d = 0$. By (4.8),

$$d = B_Q z, \quad z = HK^{-1}d \in \mathbb{Z}^n. \quad (4.15)$$

Qin's displayed L is the same matrix. Its compatibility identity is

$$LB_Q = -2D_J, \quad (4.16)$$

where D_J is the $\ell \times n$ matrix with $(D_J)_{rk} = d_{i_k} \delta_{rk}$ for $1 \leq r \leq \ell$, $k \in J$. Here the positive Weyl-invariant root form is normalized so that $d_i = (\alpha_i, \alpha_i)/2$ are positive integral symmetrizers. Put $v_a(k) = \varpi_a - w_k \varpi_a = \sum_{r \leq k, i_r=a} \beta_r$. The displayed matrix has

$$L_{jk} = (v_{i_j}(j), 2\varpi_{i_k} - v_{i_k}(k)) \quad (j < k).$$

Let $Q_{rs} = \text{sgn}(r-s)(\beta_r, \beta_s)$. Telescoping the repeated-label sums gives $K^T L K = Q$. Explicitly, induction on k proves

$$\sum_{\substack{r, s \leq k \\ i_r=a, i_s=b}} \text{sgn}(r-s)(\beta_r, \beta_s) = (v_a(k), 2\varpi_b - v_b(k));$$

for a new letter $a \neq b$, both increments are $(\beta_k, v_b(k-1))$, since $(\beta_k, w_{k-1} \varpi_b) = (\alpha_a, \varpi_b) = 0$. The same-label identity is zero by Weyl invariance. Splitting the sum at the earlier of two positions proves the asserted matrix identity. For $c = Em$ the chamber reconstruction above then yields

$$(Qc)_r = 2(\beta_r, x_{r-1}) + c_r(\beta_r, \beta_r) = 2d_{i_r}(m_{p(r)} - m_r).$$

Equivalently $QE = -2K^T D_J$. Together with $KE = B_Q$, this proves $LB_Q = -2D_J$, including every frozen row. Combining (4.14)–(4.16) gives $z \leq 0$, hence

$$b - b_0 = B_Q n, \quad n \in \mathbb{N}^n. \quad (4.17)$$

For a nonhead factor $n \neq 0$, by head multiplicity one and independence of determinantal monomials. This proves that the ordinary simple function is $Ka(L)$ -pointed, with the sign in (4.6).

Remark 4.3. The matrix B_K displayed in [7, §2.2, equation (2.2)] is $-B_Q$, while its skew matrix is L . Thus those displayed matrices satisfy $LB_K = 2D_J$. Equations (4.14)–(4.17) fix the support convention used here.

4.4 Transfer to MV and complete localized labels

Proposition 4.4. *Every MV basis element is pointed in the seed (4.11), for every reduced word and every finite symmetrizable root type. After frozen localization these elements have distinct complete labels $\mathbb{Z}^\ell$.*

Proof. The ordinary KLR function basis is polite by Lemma 4.1, and the MV basis is polite by [4, Theorem 4.8]. Polite triangularity [4, Proposition 3.9] indexes both bases by their coproduct polytopes. Write v_P and s_P for the MV and ordinary KLR functions indexed by the same polytope P . Then

$$v_P = s_P + \sum_{Q \subsetneq P} c_{PQ} s_Q. \quad (4.18)$$

The diagonal coefficient is one. The coefficients c_{PQ} may have either sign. On a fixed weight space the reflection in (4.4) preserves containment. Equations (4.9), (4.13), and (4.17) put the entire support of every off-diagonal summand in the strict positive B_Q -cone based at the head of s_P . It cannot alter its unit leading coefficient.

The labels before localization are $K\mathbb{N}^\ell$. For any $g \in \mathbb{Z}^\ell$, $K^{-1}g$ is integral. Adding a sufficiently large exponent at the last occurrence of each simple label adds the same positive integer to every corresponding coordinate of $K^{-1}g$. It therefore makes all these coordinates nonnegative. The added variables are exactly frozen, so every g occurs after localization. If two localized basis elements have the same head, multiply by a common frozen denominator. Proposition 3.2 turns both into original MV basis elements with the same initial head; injectivity of K and the polytope parametrization make them equal. The localized labels are consequently complete and distinct. $\square$

5 Interval minors on the DT path

A bipерfect basis is a weight basis compatible with the left and right Chevalley crystal operators, with the normalized leading coefficients specified in [2, Definition 2.2]. The MV basis is bipерfect [2, Theorem 6.2].

Proposition 5.1. *Let $u, v \in W$ satisfy $v = uw$ and $\ell(v) = \ell(u) + \ell(w)$. For every dominant λ , the function*

$$D_{u\lambda, v\lambda}(n) = (v_{u\lambda}, nv_{v\lambda}) \quad (5.1)$$

belongs to every bipерfect basis of $\mathbb{C}[N]$, with coefficient one. So does each of its nonnegative integral powers.

Proof. For $u = 1$, this is the flag-minor assertion [2, Remark 2.10]. Peel a common left prefix: $u = s_i u'$, $v = s_i v'$, with both length increases. In this proof write $U^+ = U(\mathfrak{n})$ for the universal enveloping algebra, distinct from the unipotent groups used in Section 3. Write $b' = D_{u'\lambda, v'\lambda}$ and $b = D_{u\lambda, v\lambda}$. The vector $v_{v'\lambda}$ is i -highest, so b' annihilates $U^+ e_i$. The vector $v_{u\lambda}$ is i -lowest, so b annihilates $e_i U^+$.

Let $s = \bar{s}_i$, $T_i = \text{Ad}(s)$. Positive extremal normalization and $s^T = s^{-1}$ give

$$s^{-1}v_{u\lambda} = v_{u'\lambda}, \quad s^{-1}v_{v\lambda} = v_{v'\lambda}.$$

For $a \in U^+ \cap T_i^{-1}(U^+)$, therefore,

$$\langle b, T_i(a) \rangle = (v_{u\lambda}, sas^{-1}v_{v\lambda}) = (v_{u'\lambda}, av_{v'\lambda}) = \langle b', a \rangle. \quad (5.2)$$

BKK Theorem 3.3 identifies this with the inverse Saito reflection of b' . Uniqueness is at the function level, since

$$U^+ = (U^+ \cap T_i(U^+)) \oplus e_i U^+.$$

Both the minor and the inverse-Saito basis function vanish on the second summand and agree on the first by (5.2). Prepending the common reduced prefix one reflection at a time proves (5.1).

The normalized highest-weight inclusion $V(d\lambda) \hookrightarrow V(\lambda)^{\otimes d}$ sends all extremal vectors to their tensor powers. The restricted contravariant form has the same normalization. Thus

$$D_{u\lambda, v\lambda}^d = D_{u(d\lambda), v(d\lambda)},$$

which proves the power assertion; $d = 0$ is the unit. $\square$

Let $k^{\max}$ be the last occurrence of i_k . After transport by transpose, Qin's interval minor is

$$D[j, k] = D_{w_{j-1}\varpi_{i_j}, w_k\varpi_{i_j}}, \quad 1 \leq j \leq k \leq \ell, \quad i_j = i_k.$$

This is (5.1) with $u = w_{j-1}$, $v = w_k$, so the reduced-prefix condition holds. The powers of the quantum parameter in Qin's normalization specialize to one. Qin [9, Theorem 8.2.2], using Goodearl–Yakimov's cluster construction, supplies a finite mutation path $t_0 \rightarrow t_0[1]$ whose new variables are these interval minors. Proposition 5.1 supplies their MV membership.

6 Proof of the cluster theorem

6.1 Identification of the DT twist

At the seed (4.11), take $v = v_{w_k\varpi_{i_k}}$ in (3.8). For mutable k , the result is

$$\eta^* X_k = \Delta_{\varpi_{i_k}}^{-1}(v_{w_k\varpi_{i_k}}, uv_{w_0\varpi_{i_k}}) = X_{k^{\max}}^{-1} I_k(t_0). \quad (6.1)$$

Indeed, no occurrence of i_k lies between k and its successor, so $w_{s(k)-1}\varpi_{i_k} = w_k\varpi_{i_k}$; at the last occurrence, $w_{k^{\max}}\varpi_{i_k} = w_0\varpi_{i_k}$. These are exact normalized-vector identities as well: for $j \neq i$, both e_j and f_j kill the highest vector of V_{ϖ_i} , so $\bar{s}_j v_{\varpi_i} = v_{\varpi_i}$. Each intervening or terminal Tits factor therefore fixes that vector with scalar $+1$. Qin's $I_k(t_0) = D[s(k), k^{\max}]$ is exactly this matrix coefficient [9, Lemma 9.1.3]. For frozen j , $\eta^* X_j = X_j^{-1}$.

Thus the map (3.1) agrees on a seed transcendence basis with Qin's classical specialization of the DT twist [9, Theorems 9.4.3–9.4.5]. The positive/negative unipotent identification is the same transpose used in (4.11) and (5.1). The classical negative twist of [12, Propositions 2.20 and 2.23] is $\eta^-(z) = [z^T \bar{w}]_-$; Gauss decomposition gives $\eta^-(u^T) = \eta(u)^T$. The variation theorem in [9, §5] supplies (2.3) in every seed.

6.2 The adapted starting seed

One needs only one adapted starting reduced-word seed. BGL Proposition 7.2 gives it for a word satisfying condition (A). Such a longest word exists in every finite root system: choose generic x, y in the dominant chamber C , and follow the segment from x to $-y$. Choose it to avoid codimension-two faces of the Weyl fan. It crosses every positive-root wall once, producing a reduced word for w_0 . Choose successive points x_k in its chambers. Then $x_k - x_{k+1}$ is a positive multiple of $x + y$, hence belongs to C . This is precisely condition (A). BGL's theorem puts every nonnegative monomial in that seed in $\mathcal{B}_{\text{MV}}$. Proposition 3.2 extends this to frozen Laurent factors.

Proof of Theorem 1.1. Write $\Delta_i = \Delta_{\varpi_i}$. Let $A = A_0[\Delta_i^{-1} : i \in I]$, and let $\mathcal{B}$ be the localized MV basis. Positive integral multiplication follows from BKK fusion. Proposition 3.2 proves frozen and twist stability. Proposition 4.4 supplies complete distinct pointed labels in the adapted reduced-word seed just chosen. The DT identity is (6.1), and the path variables belong to the basis by Proposition 5.1. The standard unipotent cluster realization, with the coordinate restricted-dual integral form specialized at $q = 1$, is the one in [9, §§8–9] and [7, §2.2]. In particular A lies in the upper cluster algebra. Its initial exchange matrix has full column rank by Lemma 4.2; mutation preserves that rank.

All hypotheses of Theorem 2.5 now hold. It gives all cluster monomials in $\mathcal{B}$ and compatible pointedness of every $\mathcal{B}$ -element at every seed.

It remains to remove frozen localization. For every frozen monomial D , Proposition 3.2 makes $D\mathcal{B}_{\text{MV}}$ a subset of $\mathcal{B}_{\text{MV}}$, and it is a basis of the principal ideal DA_0 . If $D^{-1}b \in \mathcal{B}$ is regular on N , then $b \in DA_0$. Since b is a single original MV basis element, basis compatibility of that ideal forces $b = Db_0$ for a single $b_0 \in \mathcal{B}_{\text{MV}}$. Consequently

$$\mathcal{B} \cap A_0 = \mathcal{B}_{\text{MV}}.$$

Ordinary cluster monomials are regular on N , so their localized membership descends to $\mathcal{B}_{\text{MV}}$. Pointedness of every original MV element is already included in the localized conclusion. This proves both assertions. $\square$

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