

Dynamics of the spherical Sherrington–Kirkpatrick model

Part II: Mixing profiles and critical dynamics

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AUTHORS' NOTE

With the exception of this note and small edits to the abstract, the entire document is machine-written. We have taken care to supervise this writing to ensure that the proofs meet a minimum standard of readability and that previous literature is appropriately cited. However, the resulting document remains lacking in exposition and overall coherence. We are nonetheless releasing this manuscript on Hexagon to disseminate the result as quickly as possible, because we believe it may be of interest to the mathematics and physics communities. We welcome suggestions concerning mathematical corrections, omitted citations, and attribution of ideas.

For publication, the manuscript is divided into two volumes: *Part I: Foundations and tools* (Sections 1–12) and *Part II: Mixing profiles and critical dynamics* (Sections 13–21). Section, theorem, equation, and citation numbers are shared across the volumes, and each volume includes the complete bibliography.

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ABSTRACT

We study the Langevin dynamics of the spherical Sherrington–Kirkpatrick model, the diffusion on the sphere of radius $\sqrt{N}$ in $\mathbb{R}^N$ that is reversible with respect to the Gibbs measure with density proportional to $\exp(\beta x^\top W x / \sqrt{2})$, where W is an $N \times N$ matrix from the Gaussian orthogonal ensemble. The dynamics is started from the uniform distribution on the sphere, independently of W . We determine the asymptotic quenched total-variation distance to equilibrium as a function of time in the fixed-temperature, mesoscopic, critical-window, and infinite-temperature crossover regimes described below, with convergence in probability or in distribution with respect to W . Let $\beta_c = 1/\sqrt{2}$ be the critical inverse temperature. For $\beta < \beta_c$ there is cutoff with a Gaussian profile at fixed $\beta \in (0, \beta_c)$, along every mesoscopic approach of β to β_c from below, and in the infinite-temperature crossover when $N\beta_N^2 \rightarrow \infty$; when $N\beta_N^2$ stays bounded, the distance relaxes over bounded times and there is no cutoff. At fixed $\beta \in (0, \beta_c)$ the mixing time is $(\log N - (3/2) \log \log N + O(1)) / (4(1 - \sqrt{2}\beta)^2)$, and the correction $-(3/2) \log \log N$ comes from the edge of the spectrum of W . For $\beta > \beta_c$, at fixed β and along every mesoscopic approach to β_c from above, there is cutoff at a time set by the gap between the two largest eigenvalues of W , of order $N^{2/3} \log N$ at fixed β . The profile is the total-variation distance between a standard Gaussian variable and its sum with an independent Cauchy variable; the Cauchy variable is the ratio of the first two eigencoordinates at an early stage of the evolution. In the critical window $\sqrt{2}\beta = 1 + \sigma N^{-1/3}$, the distance at time $N^{2/3}s$ converges in distribution to a random profile determined by the Airy point process, and there is no cutoff. As $\sigma \rightarrow \pm\infty$, this profile, suitably recentered and rescaled, converges to the profiles of the two mesoscopic regimes.

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PART III

High temperature

Fixed high temperature

In this section, we prove the second case of Theorem 1.1. Fix $c \in (0, 1)$ and let $g = (1 - c)^2$. We show in Theorem 13.1 that D_N has the cutoff profile Ψ of (3.16), the Gaussian profile that also describes the random walk on the hypercube [59, Theorem 1], with window $1/g$ and center

$$T_N = \frac{\log N - \frac{3}{2} \log \log N + \log A_c}{4g}, \quad A_c = \frac{\sqrt{c} (1 - c)^3}{4\sqrt{\pi} (1 + c)^2}.$$

We combine three results from Part II. First, Theorem 9.13 implies that the time- t law $\nu_N P_t$ is close to the Bingham comparison law μ_t of Section 7 once t is large. Second, both μ_t and $\pi_{N,c}$ are Gaussian laws conditioned on S_N , with variances $v_i(t)$ and q_i of equal sum, and Theorem 6.8 implies that the distance between them is close to $2\Phi(\sqrt{\mathcal{I}_N(t)}/2) - 1$, where $\mathcal{I}_N(t)$ is the information of the covariance evolution (Definition 3.5). The new work in this section is the computation of $\mathcal{I}_N(t)$ at the times $T_N + u/g$. We compare the covariance evolution of the GOE spectrum with the same evolution for the semicircle law, which we solve exactly. Its relative errors are of order e^{-2gt} , and their amplitude is of order one only for eigenvalues within distance $O(1/t)$ of the spectral edge. The square-root vanishing of the semicircle density at the edge then produces a factor $t^{-3/2}$ in the information, and this factor is the source of the term $-\frac{3}{2} \log \log N$ in T_N .

We use the rigidity estimate of Lemma 4.2, Theorem 6.8, the covariance evolution of Section 7, Theorem 9.13, and the results of Section 3.5 on profiles. The computations for the semicircle law (Propositions 13.4 and 13.7), the comparison of the finite and continuum evolutions (Lemma 13.11), and the uniform form of the likelihood limit (Lemma 13.2) are stated for all c in an interval $(0, c_1]$ with $c_1 < 1$, with constants depending only on c_1 . They are used again in Section 15, where $c = c_N \rightarrow 0$. The comparison bounds the difference between the finite and continuum relative errors in $L^2(0, 1)$ by a multiple of e^{-2gt} , and it gives a separate bound on the largest relative error. Because of the factor e^{-2gt} , we need the rigidity estimate only in a weak form: it suffices that the eigenvalues lie within $o((\log N)^{-7/4})$ of the semicircle quantiles, uniformly.

The continuum evolution of Section 13.3 is a deterministic approximation of the covariance evolution of Section 7. It is not a description of the diffusion, which enters the proof only through Theorem 9.13. In the thermodynamic limit, the normalization of the covariance evolution is the solution of the scalar Volterra equation of Cugliandolo and

Dean [51, (II.4)–(II.7)], which they solved by Laplace transform [51, Appendix A]. The power of t produced by the edge of the semicircle law in such solutions also appears in the work of Zippold, Kühn, and Horner [156, (14), (17)–(18)]. We follow the same route in Proposition 13.4.

Notation. Throughout this section, we fix $c \in (0, 1)$ and write

$$g = (1 - c)^2, \quad K_c = \frac{\sqrt{c}}{16\sqrt{\pi}(1 + c)^2}, \quad A_c = 4K_c g^{3/2} = \frac{\sqrt{c}(1 - c)^3}{4\sqrt{\pi}(1 + c)^2}. \quad (13.1)$$

Let $\gamma : [0, 1] \rightarrow [-2, 2]$ be the decreasing quantile function of the semicircle law, defined by $\int_{\gamma(\omega)}^2 \rho_{\text{sc}}(\lambda) d\lambda = \omega$, so that $\gamma(0) = 2$ and $\gamma(1) = -2$. Then $\gamma(i/N) = \gamma_i$ is the i th semicircle quantile of Section 4, and $\int_0^1 F(\gamma(\omega)) d\omega = \int F(\lambda) \rho_{\text{sc}}(\lambda) d\lambda$ for every bounded measurable F . We identify a vector $f \in \mathbb{R}^N$ with the step function on $(0, 1)$ equal to f_i on $((i - 1)/N, i/N]$, so that $N^{-1} \sum_i f_i g_i = \int_0^1 f g d\omega$. For real numbers $\lambda_1 \geq \dots \geq \lambda_N$, we write ℓ_N for the step function equal to λ_i on $((i - 1)/N, i/N]$.

For $I \in [0, \infty)$, let

$$\mathcal{G}(I) = 2\Phi(\sqrt{I}/2) - 1 = \|\mathcal{N}(\sqrt{I}, 1) - \mathcal{N}(0, 1)\|_{\text{TV}}, \quad (13.2)$$

where the second equality is the last part of Lemma 3.6. The function $\mathcal{G}$ is continuous and increasing, $\mathcal{G}(I) \rightarrow 1$ as $I \rightarrow \infty$, and $\Psi(u) = \mathcal{G}(e^{-4u})$. The derivative of $x \mapsto 2\Phi(x/2)$ is at most $(2\pi)^{-1/2}$, and we have

$$|\mathcal{G}(I) - \mathcal{G}(I')| \leq |\sqrt{I} - \sqrt{I'}| \quad (I, I' \geq 0). \quad (13.3)$$

For a real symmetric matrix A , we use the Lagrange multiplier z , the precisions p_i , the variances q_i , and the smallest precision d of Definition 3.3, together with the covariance evolution of Section 7: the variances $v_i(t)$, the relative errors $e_i(t) = v_i(t)/q_i - 1$, the information $\mathcal{I}_N(t) = \frac{1}{2} \sum_i e_i(t)^2$, and the Bingham comparison law μ_t . Recall that $\sum_i v_i(t) = \sum_i q_i = N$, and that $\mu_t = \mu_{v(t)}$ and $\pi_A = \mu_q$ are the Gaussian laws with variances $v_i(t)$ and q_i conditioned on S_N (Definition 5.4 and Proposition 5.6). For the model, $A = cW_N$, and we write z_N and d_N as in (3.12).

13.1 The main result

Theorem 13.1 (Fixed high temperature). *Fix $0 < c < 1$, let $g = (1 - c)^2$, and define*

$$A_c = \frac{\sqrt{c}(1 - c)^3}{4\sqrt{\pi}(1 + c)^2}, \quad T_N = \frac{\log N - \frac{3}{2} \log \log N + \log A_c}{4g}. \quad (13.4)$$

Then for every compact set $K \subset \mathbb{R}$, in probability,

$$\sup_{u \in K} |D_N(T_N + u/g) - \Psi(u)| \longrightarrow 0.$$

For each $\varepsilon \in (0, 1)$, we have $g(t_N(\varepsilon) - T_N) \rightarrow u_\varepsilon$ in probability, where $u_\varepsilon = \Psi^{-1}(\varepsilon)$ is given in (3.16). In particular, $t_N(\varepsilon)/\log N \rightarrow (4g)^{-1}$ in probability, and there is cutoff with window $1/g$. For finitely many fixed values $c \in (0, 1)$, these conclusions hold simultaneously for the same matrices W_N .

As in Definition 3.7, a supremum over $u \in K$ of the distance at a time such as $T_N + u/g$ is taken over those u for which the time is nonnegative, and an empty supremum is zero. This convention is used throughout this section and Section 15; for each compact K , the restriction is void for N large or with probability tending to one.

The constant in the center comes from the following computation, which is made precise in Proposition 13.7. At large times, the continuum approximation to the information of the covariance evolution is asymptotic to $\frac{N}{2} K_c t^{-3/2} e^{-4gt}$, with K_c as in (13.1). At $t = T_N + u/g$, we have $Ne^{-4gt} = (\log N)^{3/2} e^{-4u}/A_c$ and $t^{-3/2} \sim (4g)^{3/2} (\log N)^{-3/2}$, so that the information tends to $\frac{1}{2} K_c (4g)^{3/2} A_c^{-1} e^{-4u} = e^{-4u}$. Then $2\Phi(\sqrt{e^{-4u}}/2) - 1 = \Psi(u)$.

The proof of Theorem 13.1 is given in Section 13.5. Its steps are as follows.

1. By Theorem 9.13, $\|v_N P_t - \mu_t\|_{\text{TV}} \leq C e^{-gt/4}$ on an event of probability tending to one, on which $d_N \geq g/2$.
2. By Lemma 13.2, a uniform form of Theorem 6.8, the distance $\|\mu_t - \pi_{N,c}\|_{\text{TV}}$ is close to $\mathcal{G}(\mathcal{I}_N(t))$, because $\max_i |e_i(t)|$ and $\max_i q_i / (2 \sum_i q_i^2)^{1/2}$ tend to zero.
3. By the first bound of Lemma 13.11, which compares the relative errors with those of the semicircle law in $L^2(0, 1)$, $\mathcal{I}_N(t)$ is close to $\frac{N}{2} \int \eta_t^2 \rho_{\text{sc}}$, where η_t is the relative error of the covariance evolution of the semicircle law (Section 13.3). Proposition 13.7 evaluates this integral.
4. The pointwise profile implies the locally uniform profile and the convergence of the mixing times, by Lemma 3.11.

13.2 Total variation between Gaussian laws with equal traces

The following lemma is a uniform version of Theorem 6.8. It covers all values of the information at once, including large values.

Lemma 13.2 (A uniform form of the likelihood limit). *For every $\kappa > 0$, there exists $\delta > 0$ with the following property. Let $n \geq 1$, and let $v, \tilde{v} \in (0, \infty)^n$ satisfy $\sum_i v_i = \sum_i \tilde{v}_i$. Write $\tilde{v}_i = v_i(1 + e_i)$ and $I = \frac{1}{2} \sum_i e_i^2$, and let η_v be as in (5.1). If $\max_i |e_i| \leq \delta$ and $\eta_v \leq \delta$, then*

$$\left| \|\mu_{\tilde{v}} - \mu_v\|_{\text{TV}} - \mathcal{G}(I) \right| \leq \kappa.$$

Proof. Suppose that the claim fails for some $\kappa > 0$. Then for every $k \geq 1$, there exist $n_k \geq 1$ and sequences $v^{(k)}, \tilde{v}^{(k)} \in (0, \infty)^{n_k}$ with equal sums, whose relative differences $e^{(k)}$

and information I_k satisfy $\max_i |e_i^{(k)}| \leq 1/k$ and $\eta_{v^{(k)}} \leq 1/k$, and

$$\left| \|\mu_{\tilde{v}^{(k)}} - \mu_{v^{(k)}}\|_{\text{TV}} - \mathcal{G}(I_k) \right| > \kappa. \quad (13.5)$$

Passing to a subsequence, we may assume that $I_k \rightarrow I \in [0, \infty]$. The sequences $v^{(k)}$ and $\tilde{v}^{(k)}$ satisfy the hypotheses (6.23) of Theorem 6.8, with k in place of N . If $I < \infty$, then (6.25) implies that $\|\mu_{\tilde{v}^{(k)}} - \mu_{v^{(k)}}\|_{\text{TV}} \rightarrow \mathcal{G}(I)$, and $\mathcal{G}(I_k) \rightarrow \mathcal{G}(I)$ by continuity. If $I = \infty$, then the last part of Theorem 6.8 implies that $\|\mu_{\tilde{v}^{(k)}} - \mu_{v^{(k)}}\|_{\text{TV}} \rightarrow 1$, and $\mathcal{G}(I_k) \rightarrow 1$. In both cases, the left side of (13.5) tends to zero, a contradiction. $\square$

13.3 The covariance evolution for the semicircle law

We first write the covariance evolution of Section 7 as an evolution of functions on $(0, 1)$, so that a finite spectrum and the semicircle law can be compared directly. Let $L^\infty(0, 1)$ be the space of essentially bounded functions on $(0, 1)$, with the norm $\|\cdot\|_\infty$. For $a \in L^\infty(0, 1)$, let $\mathcal{L}_a$ be the bounded operator on $L^\infty(0, 1)$ defined by

$$\mathcal{L}_a f = 2c a f + 2 \left(\int_0^1 f d\omega \right) \mathbf{1},$$

and set

$$M_t^a = e^{t\mathcal{L}_a} \mathbf{1}, \quad \bar{M}_t^a = \int_0^1 M_t^a d\omega, \quad v_t^a = \frac{M_t^a}{\bar{M}_t^a} \quad (t \geq 0). \quad (13.6)$$

Equivalently, M^a is the unique solution of $\partial_t M_t^a = 2c a M_t^a + 2\bar{M}_t^a$ with $M_0^a = \mathbf{1}$.

Two choices of a are used. First, let $\lambda_1 \geq \dots \geq \lambda_N$ be real and $a = \ell_N$. Let $A = \text{diag}(c\lambda_i)$. The operator $\mathcal{L}_{\ell_N}$ maps step functions to step functions, and on them it acts as the matrix $2 \text{diag}(c\lambda_i) + (2/N)\mathbf{1}\mathbf{1}^\top$ of (7.19), with $\alpha_i = c\lambda_i$. We conclude that $M_t^{\ell_N}$ is the step function of the solution $m(t)$ of (7.19), and that $v_t^{\ell_N}$ is the step function of the variances $v_i(t) = Nm_i(t) / \sum_j m_j(t)$ of the covariance evolution of A (Definition 7.5).

Second, let $a = \gamma$. Functions of the form $F \circ \gamma$ form an invariant subspace of $\mathcal{L}_\gamma$, and we have $M_t^\gamma = M_t \circ \gamma$ and $v_t^\gamma = v_t \circ \gamma$, where M_t and v_t are functions on $[-2, 2]$ with

$$\partial_t M_t(\lambda) = 2c\lambda M_t(\lambda) + 2\bar{M}_t, \quad \bar{M}_t = \int M_t(\lambda) \rho_{\text{sc}}(\lambda) d\lambda, \quad M_0 = 1, \quad v_t = \frac{M_t}{\bar{M}_t}.$$

We call this the continuum covariance evolution. The Lagrange multiplier of the semicircle law is $1 + c^2$. Indeed, with the Stieltjes transform $\mathbf{m}_{\text{sc}}$ of (4.1),

$$\int \frac{\rho_{\text{sc}}(\lambda) d\lambda}{1 + c^2 - c\lambda} = \frac{1}{c} \mathbf{m}_{\text{sc}}\left(c + \frac{1}{c}\right) = \frac{1}{c} \cdot \frac{c + c^{-1} - (c^{-1} - c)}{2} = 1, \quad (13.7)$$

since $(c + c^{-1})^2 - 4 = (c^{-1} - c)^2$. Accordingly, we define the continuum precision and relative error by

$$p(\lambda) = 1 + c^2 - c\lambda, \quad \eta_t(\lambda) = p(\lambda)v_t(\lambda) - 1. \quad (13.8)$$

Note that $p(\lambda) \geq p(2) = g > 0$ on $[-2, 2]$.

The multiplication operator $f \mapsto 2caf$ does not preserve positivity when a takes negative values. The positivity of $e^{t\mathcal{L}_a}$ comes instead from the positivity of e^{2cat} and of the averaging term, as follows.

Lemma 13.3 (Positivity and comparison). *Let $a, a' \in L^\infty(0, 1)$.*

1. *For every $t \geq 0$, the operator $e^{t\mathcal{L}_a}$ maps nonnegative functions to nonnegative functions, and $M_t^a \geq e^{2cat} > 0$.*
2. *Suppose that $\|a - a'\|_\infty \leq E$. Then for all $t \geq 0$,*

$$e^{-2cEt} M_t^{a'} \leq M_t^a \leq e^{2cEt} M_t^{a'}, \quad e^{-4cEt} v_t^{a'} \leq v_t^a \leq e^{4cEt} v_t^{a'}. \quad (13.9)$$

Proof. For the first part, write $\mathcal{L}_a = \mathcal{D} + \mathcal{P}$ with $\mathcal{D}f = 2caf$ and $\mathcal{P}f = 2(\int_0^1 f d\omega)\mathbf{1}$. The operator $e^{s\mathcal{D}}$ is multiplication by $e^{2cas} > 0$, and $\mathcal{P}$ maps nonnegative functions to nonnegative constants. Define $S_0(t) = e^{t\mathcal{D}}$ and $S_{n+1}(t) = \int_0^t e^{(t-s)\mathcal{D}} \mathcal{P}S_n(s) ds$. By induction, each $S_n(t)$ maps nonnegative functions to nonnegative functions, and $\|S_n(t)\| \leq e^{t\|\mathcal{D}\|} (2t)^n / n!$. The Dyson expansion $e^{t\mathcal{L}_a} = \sum_{n \geq 0} S_n(t)$ converges in operator norm, which proves the first claim, and $M_t^a \geq S_0(t)\mathbf{1} = e^{2cat}$.

For the second part, let $U_t = e^{2cEt} M_t^{a'}$. Then

$$\partial_t U_t = \mathcal{L}_{a'} U_t + 2cE U_t = \mathcal{L}_a U_t + F_t, \quad F_t = 2c(E + a' - a) U_t \geq 0,$$

where $F_t \geq 0$ because $U_t \geq 0$ and $E + a' - a \geq 0$. Since $U_0 = M_0^a = \mathbf{1}$, we find by variation of constants that

$$U_t - M_t^a = \int_0^t e^{(t-s)\mathcal{L}_a} F_s ds \geq 0,$$

by the first part. This is the upper bound for M_t^a , and exchanging the roles of a and a' proves the lower bound. Integrating over $(0, 1)$ yields the same bounds for $\bar{M}_t^a$, and the bounds for v_t^a follow by division. $\square$

We now solve the continuum covariance evolution. Let

$$\begin{aligned} \varphi(t) &= \int e^{2c\lambda t} \rho_{\text{sc}}(\lambda) d\lambda, & j(t) &= c^2 \int \frac{e^{-2p(\lambda)t}}{p(\lambda)} \rho_{\text{sc}}(\lambda) d\lambda, \\ K(w) &= \int e^{-2(2-\lambda)w} \rho_{\text{sc}}(\lambda) d\lambda, \end{aligned} \quad (13.10)$$

for $t, w \geq 0$. Note that $j(0) = c^2$ by (13.7), and that $0 \leq j(t) \leq c^2 e^{-2gt}$ because $p \geq g$.

Proposition 13.4 (The continuum covariance evolution). *Let $c \in (0, 1)$. For all $t \geq 0$ and $\lambda \in [-2, 2]$,*

$$\bar{M} = \varphi + 2\varphi * \bar{M}, \quad v_t(\lambda) = \frac{e^{2c\lambda t} + 2 \int_0^t e^{2c\lambda(t-s)} \bar{M}_s ds}{\bar{M}_t}, \quad (13.11)$$

$$\bar{M}_t = e^{2(1+c^2)t} (1 - c^2 + j(t)), \quad (13.12)$$

$$\eta_t(\lambda) = \frac{e^{-2p(\lambda)t}}{1 - c^2 + j(t)} \left[p(\lambda) - 1 + 2c^2 \int_0^t e^{2p(\lambda)s} \int e^{-2p(a)s} \rho_{sc}(a) da ds \right], \quad (13.13)$$

where $(\varphi * \bar{M})(t) = \int_0^t \varphi(t-s) \bar{M}_s ds$. Moreover, with $\tilde{t} = ct$, for all $y \in [0, 4]$,

$$\eta_t(2-y) = \frac{c e^{-2gt} e^{-2y\tilde{t}}}{1 - c^2 + j(t)} \left[c - 2 + y + 2 \int_0^{\tilde{t}} e^{2yw} K(w) dw \right]. \quad (13.14)$$

The first identity in (13.11) is the scalar Volterra equation of [51, (II.7)], and (13.12) is its explicit solution for the semicircle law.

Proof. The proof has three steps: the renewal equation, its solution by Laplace transform, and an integration by parts.

Step 1: the renewal equation. Solving the equation for $M_t(\lambda)$ by variation of constants, with the forcing $2\bar{M}_t$, we find

$$M_t(\lambda) = e^{2c\lambda t} + 2 \int_0^t e^{2c\lambda(t-s)} \bar{M}_s ds.$$

The first identity in (13.11) follows by integrating against ρ_{sc} , and the second by dividing by $\bar{M}_t$.

Step 2: the Laplace transform. The operator $\mathcal{L}_\gamma$ has norm at most $4c + 2$, so $0 < \bar{M}_t \leq e^{(4c+2)t}$, and $\varphi(t) \leq e^{4ct}$. For $\zeta > 4c + 2$, let $\hat{f}(\zeta) = \int_0^\infty e^{-\zeta t} f(t) dt$. By Tonelli's theorem,

$$\hat{\varphi}(\zeta) = \int \frac{\rho_{sc}(\lambda) d\lambda}{\zeta - 2c\lambda} = \frac{1}{2c} m_{sc}\left(\frac{\zeta}{2c}\right) \leq \frac{1}{\zeta - 4c} < \frac{1}{2}.$$

Recall that $m = m_{sc}(E)$ solves $m^2 - Em + 1 = 0$. With $E = \zeta/(2c)$ and $m = 2c\hat{\varphi}(\zeta)$, this equation becomes

$$4c^2 \hat{\varphi}(\zeta)^2 - \zeta \hat{\varphi}(\zeta) + 1 = 0. \quad (13.15)$$

The Laplace transform of the renewal equation is $\hat{M} = \hat{\varphi} + 2\hat{\varphi}\hat{M}$, that is, $\hat{M} = \hat{\varphi}/(1 - 2\hat{\varphi})$. Next, since $(\zeta - 2c\lambda) - 2p(\lambda) = \zeta - 2(1 + c^2)$, we have the partial fraction decomposition

$$\frac{1}{p(\lambda)(\zeta - 2c\lambda)} = \frac{1}{\zeta - 2(1 + c^2)} \left[\frac{1}{p(\lambda)} - \frac{2}{\zeta - 2c\lambda} \right].$$

Integrating against ρ_{sc} and using (13.7), we obtain

$$\int \frac{\rho_{\text{sc}}(\lambda) d\lambda}{p(\lambda)(\zeta - 2c\lambda)} = \frac{1 - 2\hat{\phi}(\zeta)}{\zeta - 2(1 + c^2)}.$$

We conclude that

$$\frac{1 - c^2}{\zeta - 2(1 + c^2)} + c^2 \int \frac{\rho_{\text{sc}}(\lambda) d\lambda}{p(\lambda)(\zeta - 2c\lambda)} = \frac{1 - 2c^2\hat{\phi}}{\zeta - 2(1 + c^2)} = \frac{\hat{\phi}}{1 - 2\hat{\phi}} = \hat{M}(\zeta). \quad (13.16)$$

For the second equality, cross-multiplication shows that it is equivalent to $\zeta\hat{\phi} = 1 + 4c^2\hat{\phi}^2$, which is (13.15). Note that $e^{2(1+c^2)t}j(t) = c^2 \int p(\lambda)^{-1} e^{2c\lambda t} \rho_{\text{sc}}(\lambda) d\lambda$, because $2(1 + c^2) - 2p(\lambda) = 2c\lambda$. This implies that the left side of (13.16) is the Laplace transform of $e^{2(1+c^2)t}(1 - c^2 + j(t))$. Both this function and $\hat{M}$ are continuous and bounded by $e^{(4c+2)t}$, and their Laplace transforms agree for $\zeta > 4c + 2$. We now check that they are equal, using the compactification-and-moments proof of uniqueness of Laplace transforms in [71, Chapter XIII, Section 1, Theorem 1 and its second proof, pp. 429–430]. Let f be their difference, and fix $b > 4c + 2$. The push-forward of the finite signed measure $e^{-bt}f(t) dt$ under the map $t \mapsto x = e^{-t}$ is a finite signed measure on $[0, 1]$, whose n th moment is $\hat{f}(b + n) = 0$ for every integer $n \geq 0$. Since polynomials are dense in $C([0, 1])$, this measure is zero. Then $f = 0$ almost everywhere, and $f = 0$ everywhere by continuity. This proves (13.12).

Step 3: the relative error. Fix λ and write $p = p(\lambda)$. Inserting (13.12) into (13.11), dividing the numerator and the denominator by $e^{2(1+c^2)t}$, and using $2c\lambda - 2(1 + c^2) = -2p$, we find

$$(1 - c^2 + j(t))v_t(\lambda) = e^{-2pt} + 2 \int_0^t e^{-2p(t-s)} (1 - c^2 + j(s)) ds.$$

We have $2 \int_0^t e^{-2p(t-s)} ds = (1 - e^{-2pt})/p$, and integrating by parts, we find

$$2 \int_0^t e^{-2p(t-s)} j(s) ds = \frac{j(t) - e^{-2pt}j(0)}{p} - \frac{1}{p} \int_0^t e^{-2p(t-s)} j'(s) ds.$$

Multiplying by p , subtracting $1 - c^2 + j(t)$, and using $j(0) = c^2$, we obtain

$$(1 - c^2 + j(t))\eta_t(\lambda) = e^{-2pt} \left[p - 1 - \int_0^t e^{2ps} j'(s) ds \right].$$

This is (13.13), because $j'(s) = -2c^2 \int e^{-2p(a)s} \rho_{\text{sc}}(a) da$.

Finally, let $\lambda = 2 - y$. Then $p(2 - y) = g + cy$ and $p(2 - y) - 1 = c(c - 2 + y)$. For every a , we have $p(a) = g + c(2 - a)$, which gives $\int e^{-2p(a)s} \rho_{\text{sc}}(a) da = e^{-2gs} K(cs)$. The integral in (13.13) becomes

$$2c^2 \int_0^t e^{2(g+cy)s} e^{-2gs} K(cs) ds = 2c \int_0^{\bar{t}} e^{2yw} K(w) dw,$$

where we substituted $w = cs$. Together with $e^{-2p(2-y)t} = e^{-2gt}e^{-2y\bar{t}}$, this yields (13.14). $\square$

The kernel K in (13.14) carries the edge behavior of the semicircle law. We record its two properties that are used below.

Lemma 13.5 (The edge kernel). *For all $w \geq 0$, we have $0 \leq K(w) \leq 3(1+w)^{-3/2}$, and $\int_0^\infty K(w) dw = 1/2$.*

Proof. We have $K(w) \leq 1$, because $e^{-2(2-\lambda)w} \leq 1$ on $[-2, 2]$. For $0 \leq y \leq 4$, we have $\rho_{\text{sc}}(2-y) = (2\pi)^{-1} \sqrt{y(4-y)} \leq \pi^{-1} \sqrt{y}$. For $w > 0$, this gives

$$K(w) = \int_0^4 e^{-2yw} \rho_{\text{sc}}(2-y) dy \leq \frac{1}{\pi} \int_0^\infty \sqrt{y} e^{-2yw} dy = \frac{\sqrt{\pi}}{2\pi(2w)^{3/2}} = \frac{1}{2\sqrt{\pi}(2w)^{3/2}}.$$

We use $K(w) \leq 1 \leq 2^{3/2}(1+w)^{-3/2}$ for $w \leq 1$, and $2w \geq 1+w$ for $w \geq 1$. Finally, Tonelli's theorem and monotone convergence give $\int_0^\infty K(w) dw = \frac{1}{2} \int (2-\lambda)^{-1} \rho_{\text{sc}}(\lambda) d\lambda = \frac{1}{2} \lim_{E \downarrow 2} \mathfrak{m}_{\text{sc}}(E) = \frac{1}{2}$. $\square$

Lemma 13.6 (An exponential convolution bound). *There exists a universal constant C such that for all $b, T \geq 0$,*

$$\int_0^T e^{bs} (1+s)^{-3/2} ds \leq C \left[1 + \frac{e^{bT} (1+T)^{-1/2}}{1+bT} \right].$$

Proof. We consider three cases. If $bT \leq 2$, then $e^{bs} \leq e^2$ on $[0, T]$, and the integral is at most $e^2 \int_0^\infty (1+s)^{-3/2} ds = 2e^2$.

If $T \leq 1$ and $bT > 2$, then the integral is at most $\int_0^T e^{bs} ds \leq e^{bT}/b = Te^{bT}/(bT) \leq e^{bT}/(bT)$. Using $1+bT \leq 2bT$ and $(1+T)^{-1/2} \geq 2^{-1/2}$, we see that this is at most $2^{3/2}e^{bT}(1+T)^{-1/2}/(1+bT)$.

Let $T > 1$ and $bT > 2$. Then $1/b < T/2$. We split the integral at $s = 1/b$. On $[0, 1/b]$, we have $e^{bs} \leq e$, and this part is at most $2e$. On $[1/b, T]$, we substitute $s = x/b$ and use $(1+x/b)^{-3/2} \leq b^{3/2}x^{-3/2}$, to find

$$\int_{1/b}^T e^{bs} (1+s)^{-3/2} ds \leq b^{1/2} \int_1^{bT} e^x x^{-3/2} dx.$$

For $S \geq 2$, we have $\int_{S/2}^S e^x x^{-3/2} dx \leq 2^{3/2}e^S S^{-3/2}$ and $\int_1^{S/2} e^x x^{-3/2} dx \leq 2e^{S/2}$. Both are at most $C_0 e^S S^{-3/2}$ with $C_0 = 2^{3/2} + 2(3/e)^{3/2}$, because $\sup_{x>0} x^{3/2}e^{-x/2} = (3/e)^{3/2}$. With $S = bT$, the second part of the integral is at most $C_0 e^{bT}/(bT^{3/2})$. Since $T > 1$ and $bT > 2$, we have $(1+T)^{1/2} \leq (2T)^{1/2}$ and $1+bT \leq 2bT$, which give $e^{bT}/(bT^{3/2}) \leq 2^{3/2}e^{bT}(1+T)^{-1/2}/(1+bT)$. This completes the proof. $\square$

We now evaluate the relative errors of the continuum evolution. The following proposition holds uniformly in $c \in (0, c_1]$ for every fixed $c_1 < 1$, so that it applies both at a fixed temperature and in Section 15, where $c \rightarrow 0$. All limits are taken as $\tilde{t} = ct \rightarrow \infty$. For a fixed c , this means $t \rightarrow \infty$.

Proposition 13.7 (Edge asymptotics of the continuum relative errors). *Let $0 < c_1 < 1$. There exists C_1 depending only on c_1 such that, for all $c \in (0, c_1]$, $t \geq 0$ and $y \in [0, 4]$, with $\tilde{t} = ct$,*

$$|\eta_t(2 - y)| \leq C_1 c e^{-2gt} \left[e^{-2y\tilde{t}} + \frac{(1 + \tilde{t})^{-1/2}}{1 + y\tilde{t}} \right]. \quad (13.17)$$

In particular, $\sup_\lambda |\eta_t(\lambda)| \leq 2C_1 c e^{-2gt}$. Moreover, for all $L, \kappa > 0$, there exists T_0 depending only on c_1, L and κ such that, for all $c \in (0, c_1]$ and all t with $\tilde{t} = ct \geq T_0$,

$$\sup_{0 \leq y \leq L/\tilde{t}} \left| \frac{1 + c}{c} e^{2gt} \eta_t(2 - y) + e^{-2y\tilde{t}} \right| \leq \kappa, \quad (13.18)$$

and, for all $\kappa > 0$, there exists T'_0 depending only on c_1 and κ such that, for all $c \in (0, c_1]$ and $ct \geq T'_0$,

$$\left| \frac{\int \eta_t(\lambda)^2 \rho_{\text{sc}}(\lambda) d\lambda}{K_c t^{-3/2} e^{-4gt}} - 1 \right| \leq \kappa. \quad (13.19)$$

For a fixed c , (13.18) states that $e^{2gt} \eta_t(2 - y) = -c(1 + c)^{-1} e^{-2cyt} + o(1)$ as $t \rightarrow \infty$, uniformly for yt in compact sets. The relative errors of eigenvalues at distance of order $1/t$ from the edge are of order e^{-2gt} , while (13.17) shows that the other relative errors are smaller. The computation in (13.19) is of Laplace type, as in Watson's lemma [117, §2.3(ii)].

Proof. Throughout the proof, C denotes a constant depending only on c_1 , whose value may change from line to line. We use (13.14), whose prefactor is at most $Cce^{-2gt}e^{-2y\tilde{t}}$, because $1 - c^2 + j(t) \geq 1 - c_1^2$. Note also that $t \geq \tilde{t}$ and $g \geq (1 - c_1)^2$, which give

$$0 \leq j(t) \leq c^2 e^{-2gt} \leq e^{-2(1-c_1)^2 \tilde{t}}. \quad (13.20)$$

Step 1: the global bound. In the bracket of (13.14), we have $|c - 2 + y| \leq 3$. By Lemmas 13.5 and 13.6 with $b = 2y$,

$$2 \int_0^{\tilde{t}} e^{2yw} K(w) dw \leq 6 \int_0^{\tilde{t}} e^{2yw} (1 + w)^{-3/2} dw \leq C \left[1 + \frac{e^{2y\tilde{t}} (1 + \tilde{t})^{-1/2}}{1 + 2y\tilde{t}} \right].$$

Multiplying by the prefactor and using $1 + 2y\tilde{t} \geq 1 + y\tilde{t}$, we obtain (13.17). The bound on $\sup_\lambda |\eta_t(\lambda)|$ follows because both terms in the bracket of (13.17) are at most one.

Step 2: the amplitude at the edge. Let $\mathcal{E}(\tilde{t}, y) = 2 \int_0^{\tilde{t}} e^{2yw} K(w) dw - 1$. Since $2 \int_0^\infty K = 1$ by Lemma 13.5, we have, for $0 < R \leq \tilde{t}$,

$$\mathcal{E}(\tilde{t}, y) = 2 \int_0^R (e^{2yw} - 1) K(w) dw + 2 \int_R^{\tilde{t}} e^{2yw} K(w) dw - 2 \int_R^\infty K(w) dw.$$

For $0 \leq y \leq L/\tilde{t}$, we have $e^{2yw} - 1 \leq e^{2LR/\tilde{t}} - 1$ for $w \leq R$, and $e^{2yw} \leq e^{2L}$ for $w \leq \tilde{t}$. We obtain

$$\sup_{0 \leq y \leq L/\tilde{t}} |\mathcal{E}(\tilde{t}, y)| \leq (e^{2LR/\tilde{t}} - 1) + 2(e^{2L} + 1) \int_R^\infty K(w) dw. \quad (13.21)$$

Taking first R large and then $\tilde{t}$ large, we see that the right side of (13.21) tends to zero as $\tilde{t} \rightarrow \infty$, at a rate depending only on L . The bracket in (13.14) equals $-(1-c) + y + \mathcal{E}(\tilde{t}, y)$, and we find

$$\frac{1+c}{c} e^{2gt} \eta_t(2-y) + e^{-2y\tilde{t}} = e^{-2y\tilde{t}} \left[\frac{(1+c)(y + \mathcal{E}(\tilde{t}, y))}{1-c^2 + j(t)} + \frac{j(t)}{1-c^2 + j(t)} \right],$$

where we used $(1+c)(1-c) = 1-c^2$. For $0 \leq y \leq L/\tilde{t}$, the absolute value of the right side is at most $(1-c_1^2)^{-1} [2(L/\tilde{t} + |\mathcal{E}(\tilde{t}, y)|) + j(t)]$. Using (13.21) and (13.20), we see that this tends to zero as $\tilde{t} \rightarrow \infty$, uniformly in $c \in (0, c_1]$. This proves (13.18).

Step 3: the information. Let $\Theta = c^2(1+c)^{-2}e^{-4gt}\tilde{t}^{-3/2}$. Note that $K_c t^{-3/2} e^{-4gt} = \Theta/(16\sqrt{\pi})$, since $c^2\tilde{t}^{-3/2} = \sqrt{c} t^{-3/2}$. Also

$$\frac{1}{\pi} \int_0^\infty \sqrt{x} e^{-4x} dx = \frac{\sqrt{\pi}}{2 \cdot 4^{3/2} \pi} = \frac{1}{16\sqrt{\pi}}.$$

It therefore suffices to show that $\Theta^{-1} \int \eta_t^2 \rho_{sc} \rightarrow (16\sqrt{\pi})^{-1}$ as $\tilde{t} \rightarrow \infty$, uniformly in $c \in (0, c_1]$. Fix $L > 0$ and $\kappa \in (0, 1]$, let $\tilde{t} \geq \max\{T_0, L\}$ with T_0 as in (13.18), and split $\int \eta_t^2 \rho_{sc} = \int_0^4 \eta_t(2-y)^2 \rho_{sc}(2-y) dy$ at $y = L/\tilde{t}$.

On $[0, L/\tilde{t}]$, we write $\eta_t(2-y) = -c(1+c)^{-1}e^{-2gt}(e^{-2y\tilde{t}} + \varrho)$, where $|\varrho| \leq \kappa$ by (13.18), and $\rho_{sc}(2-y) = \pi^{-1}\sqrt{y}\sqrt{1-y/4}$. The substitution $y = x/\tilde{t}$ yields

$$\Theta^{-1} \int_0^{L/\tilde{t}} \eta_t(2-y)^2 \rho_{sc}(2-y) dy = \frac{1}{\pi} \int_0^L (e^{-2x} + \varrho)^2 \sqrt{x} \sqrt{1-x/(4\tilde{t})} dx.$$

For $x \leq L$, we have $|(e^{-2x} + \varrho)^2 - e^{-4x}| \leq 3\kappa$ and $1 - \sqrt{1-x/(4\tilde{t})} \leq L/(4\tilde{t})$. It follows that the right side differs from $\pi^{-1} \int_0^L \sqrt{x} e^{-4x} dx$ by at most $C_L(\kappa + \tilde{t}^{-1})$, with C_L depending only on L .

On $[L/\tilde{t}, 4]$, we use (13.17), $(\alpha + \beta)^2 \leq 2\alpha^2 + 2\beta^2$, and $\rho_{sc}(2-y) \leq \pi^{-1}\sqrt{y}$. Substituting $y = x/\tilde{t}$ again, we have

$$\begin{aligned} \int_{L/\tilde{t}}^4 e^{-4y\tilde{t}} \sqrt{y} dy &\leq \tilde{t}^{-3/2} \int_L^\infty \sqrt{x} e^{-4x} dx, \\ \int_0^4 \frac{\sqrt{y} dy}{(1+y\tilde{t})^2} &\leq \tilde{t}^{-3/2} \int_0^\infty \frac{\sqrt{x} dx}{(1+x)^2} = \frac{\pi}{2} \tilde{t}^{-3/2}. \end{aligned}$$

Since $(1+c)^2 \leq 4$, this gives

$$\Theta^{-1} \int_{L/\tilde{t}}^4 \eta_t(2-y)^2 \rho_{\text{sc}}(2-y) dy \leq \frac{8C_1^2}{\pi} \left[\int_L^\infty \sqrt{x} e^{-4x} dx + \frac{\pi}{2} (1+\tilde{t})^{-1} \right].$$

Combining the two parts, we have shown that

$$\limsup_{\tilde{t} \rightarrow \infty} \sup_{c \in (0, c_1]} \left| \Theta^{-1} \int \eta_t^2 \rho_{\text{sc}} - \frac{1}{16\sqrt{\pi}} \right| \leq C_L \kappa + C \int_L^\infty \sqrt{x} e^{-4x} dx.$$

Letting $\kappa \rightarrow 0$ and then $L \rightarrow \infty$, we obtain (13.19). $\square$

13.4 Comparison of the finite and continuum evolutions

We now compare the covariance evolution of a finite spectrum with the continuum evolution, at times up to a multiple of $\log N$. The comparison is made on the common parameter space $(0, 1)$. Let $E = \|\ell_N - \gamma\|_\infty$. In Lemma 13.11, we show that the difference between the finite and continuum relative errors has norm at most $C_2 c E (1+t) e^{-2gt}$ in $L^2(0, 1)$, and we bound the largest relative error by a separate argument. We first bound E for the GOE. We then prove two lemmas on operators on $L^2(0, 1)$: the gap of the operator that generates the covariance evolution, on the orthogonal complement of its stationary vector (Lemma 13.9), and a comparison of the decaying parts of two such evolutions (Lemma 13.10).

Lemma 13.8 (Semicircle quantiles and rigidity). *For all $\omega, \omega' \in [0, 1]$, we have $|\gamma(\omega) - \gamma(\omega')| \leq (4\sqrt{2}\pi|\omega - \omega'|)^{2/3}$. Let $\lambda_1 \geq \dots \geq \lambda_N$ be the eigenvalues of W_N , and fix $C \geq 1$. There exists N_C depending only on C such that, on the event F_C of (4.2) and for $N \geq N_C$,*

$$\|\ell_N - \gamma\|_\infty \leq (C+7)N^{-5/12}, \quad \max_i |\lambda_i| \leq 3, \quad \left| \frac{1}{N} \sum_i \lambda_i \right| \leq (C+7)N^{-5/12}.$$

For each fixed confidence $1 - \eta$, Lemma 4.2 supplies such a C for all sufficiently large N .

Proof. Let $J = [\alpha, \alpha + h] \subset [-2, 2]$. The set of $\lambda \in J$ with $2 - |\lambda| < h/4$ has length at most $h/2$. For the other points of J , we have $\rho_{\text{sc}}(\lambda) = (2\pi)^{-1} \sqrt{(2-\lambda)(2+\lambda)} \geq (2\pi)^{-1} \sqrt{2(2-|\lambda|)} \geq (2\pi)^{-1} \sqrt{h/2}$. Hence the semicircle mass of J is at least $(h/2)^{3/2} / (2\pi) = h^{3/2} / (4\sqrt{2}\pi)$. If $|\gamma(\omega) - \gamma(\omega')| = h$, then $|\omega - \omega'|$ is the mass of an interval of length h , which proves the first claim.

On F_C , we have $|\lambda_i - \gamma_i| \leq CN^{-5/12}$ for all i . For $\omega \in ((i-1)/N, i/N]$, we have $\gamma_i = \gamma(i/N)$, and by the first claim,

$$\begin{aligned} |\ell_N(\omega) - \gamma(\omega)| &\leq |\lambda_i - \gamma_i| + |\gamma(i/N) - \gamma(\omega)| \\ &\leq CN^{-5/12} + (4\sqrt{2}\pi)^{2/3} N^{-2/3} \leq (C+7)N^{-5/12}. \end{aligned}$$

This is at most $1/2$ for N large depending on C . The other two bounds follow from $|\gamma| \leq 2$, $\int_0^1 \gamma d\omega = \int \lambda \rho_{\text{sc}}(\lambda) d\lambda = 0$, and $N^{-1} \sum_i \lambda_i = \int_0^1 \ell_N d\omega$. $\square$

For the two lemmas on operators, we write $\langle f, f' \rangle = \int_0^1 f f' d\omega$ and $\|f\|_2 = \langle f, f \rangle^{1/2}$ for the inner product and the norm of $L^2(0, 1)$, and $\|\cdot\|$ for the norm of an operator on it. For background on orthogonal projections in Hilbert spaces, see [37, Section 5.1], and for exponentials of bounded operators, see [68, Chapter I, Section 3]. For $p \in L^\infty(0, 1)$, let H_p be the operator on $L^2(0, 1)$ defined by

$$H_p f = -2pf + 2\langle \mathbf{1}, f \rangle \mathbf{1}.$$

If p is the step function of precisions $p_1, \dots, p_N$ satisfying (7.1), then H_p maps step functions to step functions, and on them it acts as the heat matrix H of (7.3). In the proof of Lemma 13.11, we write the covariance evolutions of ℓ_N and of γ through two such operators.

Lemma 13.9 (The gap on the orthogonal complement of the stationary vector). *Let $0 < d \leq s$, and let p be a measurable function on $(0, 1)$ such that $d \leq p \leq s$ almost everywhere and $\int_0^1 p^{-1} d\omega = 1$. Let $q = p^{-1}$, and let P_q be the orthogonal projection onto the span of q in $L^2(0, 1)$. Then H_p is a bounded self-adjoint operator with $\|H_p\| \leq 2s + 2$, and its kernel is spanned by q . For every $f \in L^2(0, 1)$ with $\langle f, q \rangle = 0$,*

$$\langle f, H_p f \rangle \leq -2d \|f\|_2^2.$$

For every $t \geq 0$,

$$\|e^{tH_p}\| \leq 1, \quad \|e^{tH_p}(I - P_q)\| \leq e^{-2dt}, \quad \|H_p e^{tH_p}\| \leq (2s + 2)e^{-2dt}.$$

Finally, $1 \leq \|q\|_2 \leq d^{-1}$, and for every $t \geq 0$,

$$\langle \mathbf{1}, e^{tH_p} \mathbf{1} \rangle = \|e^{tH_p/2} \mathbf{1}\|_2^2 \geq \|q\|_2^{-2} \geq d^2.$$

Proof. Multiplication by p is self-adjoint with norm at most s , and the map $f \mapsto \langle \mathbf{1}, f \rangle \mathbf{1}$ is self-adjoint with norm $\|\mathbf{1}\|_2^2 = 1$. This proves the first claim. Since $pq = 1$ and $\langle \mathbf{1}, q \rangle = \int_0^1 p^{-1} d\omega = 1$, we have $H_p q = -2\mathbf{1} + 2\mathbf{1} = 0$. Conversely, if $H_p f = 0$, then $pf = \langle \mathbf{1}, f \rangle \mathbf{1}$, that is, $f = \langle \mathbf{1}, f \rangle q$.

Next, let $\langle f, q \rangle = 0$, let $\tilde{f} = \langle \mathbf{1}, f \rangle$, and let $\tilde{f} = f - \tilde{f}q$. All integrals below are finite, because p and q are bounded. Using $pq = 1$, $\langle \mathbf{1}, q \rangle = 1$, and $\langle f, q \rangle = 0$, we find

$$\int_0^1 p \tilde{f}^2 d\omega = \int_0^1 p f^2 d\omega - 2\tilde{f}^2 + \tilde{f}^2 = \int_0^1 p f^2 d\omega - \tilde{f}^2, \quad \|\tilde{f}\|_2^2 = \|f\|_2^2 + \tilde{f}^2 \|q\|_2^2.$$

By the definition of H_p , we have $\langle f, H_p f \rangle = -2 \int_0^1 p f^2 d\omega + 2\tilde{f}^2$, and the first identity gives

$$\langle f, H_p f \rangle = -2 \int_0^1 p \tilde{f}^2 d\omega \leq -2d \|\tilde{f}\|_2^2 \leq -2d \|f\|_2^2.$$

In the last inequality, we used the second identity.

For the bounds on the exponential, we use only that H_p is bounded and self-adjoint, that $H_p q = 0$, and the inequality just proved. Since H_p is self-adjoint and $H_p q = 0$, the orthogonal complement $q^\perp$ of q is invariant under H_p and under e^{tH_p} , which is defined by a power series that converges in norm. Let $f \in q^\perp$ and $f_t = e^{tH_p} f$. The map $t \mapsto e^{tH_p}$ is differentiable in the operator norm, with derivative $H_p e^{tH_p}$ [68, Chapter I, Proposition 3.5], and we have

$$\frac{d}{dt} \left(e^{4dt} \|f_t\|_2^2 \right) = e^{4dt} \left(4d \|f_t\|_2^2 + 2 \langle f_t, H_p f_t \rangle \right) \leq 0.$$

We conclude that $\|f_t\|_2 \leq e^{-2dt} \|f\|_2$, which is the second bound. Since $e^{tH_p} q = q$, the orthogonal decomposition $f = P_q f + (I - P_q) f$ of an arbitrary $f \in L^2(0, 1)$ gives $\|e^{tH_p} f\|_2^2 = \|P_q f\|_2^2 + \|e^{tH_p} (I - P_q) f\|_2^2 \leq \|f\|_2^2$, which is the first bound. Further, $H_p e^{tH_p} P_q = e^{tH_p} H_p P_q = 0$, so that $H_p e^{tH_p} = H_p e^{tH_p} (I - P_q)$, and the third bound follows from the second and $\|H_p\| \leq 2s + 2$.

Finally, $1 = \langle \mathbf{1}, q \rangle \leq \|q\|_2$ by the Cauchy–Schwarz inequality, and $\|q\|_2 \leq d^{-1}$ because $0 < q \leq d^{-1}$. Since $e^{tH_p/2}$ is self-adjoint and $e^{tH_p/2} q = q$, we have

$$\langle \mathbf{1}, e^{tH_p} \mathbf{1} \rangle = \|e^{tH_p/2} \mathbf{1}\|_2^2, \quad \langle e^{tH_p/2} \mathbf{1}, q \rangle = \langle \mathbf{1}, q \rangle = 1.$$

By the Cauchy–Schwarz inequality, we have $1 \leq \|e^{tH_p/2} \mathbf{1}\|_2 \|q\|_2$, which proves the last claim. $\square$

The next lemma compares the decaying parts of two such evolutions whose stationary vectors are close. It is stated on a real Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and its constants do not depend on the space.

Lemma 13.10 (Comparison of the decaying parts of two evolutions). *Let H_1 and H_2 be bounded self-adjoint operators on a real Hilbert space, and let u_1, u_2 be unit vectors with $\langle u_1, u_2 \rangle > 0$ and $H_1 u_1 = H_2 u_2 = 0$. Suppose that there exist $d > 0$ and $B \geq 0$ such that $\|H_1\|, \|H_2\| \leq B$ and, for $j = 1, 2$,*

$$\langle f, H_j f \rangle \leq -2d \|f\|^2 \quad \text{for all } f \text{ with } \langle f, u_j \rangle = 0.$$

Let $a = \|H_1 - H_2\|$ and $b = \|u_1 - u_2\|$. Then for all $t \geq 0$,

$$\|H_1 e^{tH_1} - H_2 e^{tH_2}\| \leq 20(1+B)^2(a+b)(1+t)e^{-2dt}.$$

Moreover, let x be a unit vector, let $t \geq 0$, and suppose that $m_j(t) = \langle x, e^{tH_j} x \rangle$ satisfies $m_j(t) \geq \delta$ for $j = 1, 2$, for some $\delta > 0$. Then

$$\left\| \frac{H_1 e^{tH_1} x}{m_1(t)} - \frac{H_2 e^{tH_2} x}{m_2(t)} \right\| \leq \left(\frac{20(1+B)^2(a+b)}{\delta} + \frac{Ba}{\delta^2} \right) (1+t)e^{-2dt}.$$

Proof. The proof of the bounds on the exponential in Lemma 13.9 uses only that the operator is bounded and self-adjoint, that it annihilates a given vector, and the inequality on the orthogonal complement of that vector. The same argument applies to H_1 and H_2 . For $j = 1, 2$ and $t \geq 0$, we obtain that e^{tH_j} is a contraction, that $\|e^{tH_j}f\| \leq e^{-2dt}\|f\|$ whenever $\langle f, u_j \rangle = 0$, and that $\|H_j e^{tH_j}\| \leq Be^{-2dt}$.

We begin by rotating u_1 onto u_2 . Let $\alpha = \langle u_1, u_2 \rangle \in (0, 1]$. If $u_1 = u_2$, let $U = I$. Otherwise, $\alpha < 1$, and we let $\beta = (1 - \alpha^2)^{1/2} > 0$ and $w = \beta^{-1}(u_2 - \alpha u_1)$, which is a unit vector orthogonal to u_1 . We define U by

$$Uu_1 = \alpha u_1 + \beta w = u_2, \quad Uw = -\beta u_1 + \alpha w,$$

and $Uf = f$ for f orthogonal to u_1 and w . Then U is orthogonal, that is, $U^* = U^{-1}$. On the span of u_1 and w , we have $(U - I)^*(U - I) = 2(1 - \alpha)I$, and $U - I$ vanishes on the orthogonal complement of this span. This implies that

$$\|U - I\| = \|U^{-1} - I\| = \sqrt{2(1 - \alpha)} = \|u_1 - u_2\| \leq 2b.$$

This is the case of one-dimensional subspaces of the classical construction of a unitary operator that maps one projection onto a nearby one [94, Chapter I, Section 6.8, Theorem 6.32]; we use only the explicit bound above.

Let $\tilde{H}_1 = UH_1U^{-1}$. Then $\tilde{H}_1$ is self-adjoint, $\|\tilde{H}_1\| \leq B$, and $\tilde{H}_1u_2 = UH_1u_1 = 0$. If $\langle f, u_2 \rangle = 0$, then $\langle U^{-1}f, u_1 \rangle = \langle f, Uu_1 \rangle = 0$, and the inequality for H_1 implies that $\langle f, \tilde{H}_1f \rangle = \langle U^{-1}f, H_1U^{-1}f \rangle \leq -2d\|f\|^2$. We have shown that $\tilde{H}_1$ satisfies the hypotheses imposed on H_2 , with the same vector u_2 . From the identity $\tilde{H}_1 - H_1 = (U - I)H_1U^{-1} + H_1(U^{-1} - I)$, we find

$$\|\tilde{H}_1 - H_1\| \leq 4Bb, \quad \|\tilde{H}_1 - H_2\| \leq a + 4Bb.$$

Next, we compare $\tilde{H}_1$ with H_2 , which annihilate the same vector u_2 . The operator $\tilde{H}_1 - H_2$ maps every vector into the orthogonal complement of u_2 , since $\langle (\tilde{H}_1 - H_2)f, u_2 \rangle = \langle f, (\tilde{H}_1 - H_2)u_2 \rangle = 0$. Let $\langle f, u_2 \rangle = 0$. Differentiating $s \mapsto e^{(t-s)H_2}e^{s\tilde{H}_1}f$ and integrating over $[0, t]$, we obtain the variation of constants formula [68, Chapter III, Corollary 1.7]

$$(e^{t\tilde{H}_1} - e^{tH_2})f = \int_0^t e^{(t-s)H_2}(\tilde{H}_1 - H_2)e^{s\tilde{H}_1}f ds.$$

The integrand is continuous in s . The vector $e^{s\tilde{H}_1}f$ is orthogonal to u_2 and has norm at most $e^{-2ds}\|f\|$, and $(\tilde{H}_1 - H_2)e^{s\tilde{H}_1}f$ is orthogonal to u_2 . The integrand has norm at most $e^{-2d(t-s)}\|\tilde{H}_1 - H_2\|e^{-2ds}\|f\|$, and we find $\|(e^{t\tilde{H}_1} - e^{tH_2})f\| \leq t\|\tilde{H}_1 - H_2\|e^{-2dt}\|f\|$. Now

$$\tilde{H}_1 e^{t\tilde{H}_1} - H_2 e^{tH_2} = (\tilde{H}_1 - H_2)e^{t\tilde{H}_1} + H_2(e^{t\tilde{H}_1} - e^{tH_2}),$$

and both sides vanish at u_2 . Applying the bounds above to $f - \langle f, u_2 \rangle u_2$, whose norm is at most $\|f\|$, we obtain

$$\|\tilde{H}_1 e^{t\tilde{H}_1} - H_2 e^{tH_2}\| \leq \|\tilde{H}_1 - H_2\|(1+Bt)e^{-2dt} \leq (a+4Bb)(1+Bt)e^{-2dt}.$$

We now return to H_1 . By the power series of the exponential, we have $\tilde{H}_1 e^{t\tilde{H}_1} = UH_1 e^{tH_1} U^{-1}$. As above, $\|UXU^{-1} - X\| \leq 4b\|X\|$ for $X = H_1 e^{tH_1}$, whose norm is at most Be^{-2dt} . Combining this with the last display, we find

$$\|H_1 e^{tH_1} - H_2 e^{tH_2}\| \leq (4Bb + (a+4Bb)(1+Bt))e^{-2dt}.$$

Using $4Bb \leq 4(1+B)(a+b)$, $a+4Bb \leq 4(1+B)(a+b)$, and $1+Bt \leq (1+B)(1+t)$ bounds the prefactor by $8(1+B)^2(a+b)(1+t)$, and hence by the stated $20(1+B)^2(a+b)(1+t)$.

For the second claim, the variation of constants formula for H_1 and H_2 on the whole space, together with the contraction bound, gives $\|e^{tH_1} - e^{tH_2}\| \leq ta$, which implies that $|m_1(t) - m_2(t)| \leq ta$. Let $X_j = H_j e^{tH_j} x$, so that $\|X_2\| \leq Be^{-2dt}$. We have

$$\frac{X_1}{m_1(t)} - \frac{X_2}{m_2(t)} = \frac{X_1 - X_2}{m_1(t)} + \frac{m_2(t) - m_1(t)}{m_1(t)m_2(t)} X_2,$$

and the right side has norm at most $\delta^{-1}\|X_1 - X_2\| + \delta^{-2}taBe^{-2dt}$. The first claim and $t \leq 1+t$ complete the proof. $\square$

Lemma 13.11 (Comparison with the continuum evolution). *Let $0 < c_1 < 1$. There exists C_2 depending only on c_1 such that the following holds. Let $c \in (0, c_1]$, $N \geq 1$, and $\lambda_1 \geq \dots \geq \lambda_N$, and suppose that $E = \|\ell_N - \gamma\|_\infty \leq (1 - c_1)^2/4$. Let $A = \text{diag}(c\lambda_i)$, with Lagrange multiplier z and smallest precision d , and let $e_i(t)$ and $\mathcal{I}_N(t)$ be the relative errors and the information of its covariance evolution. Then $|z - (1 + c^2)| \leq cE$ and $d \geq g/2$. Moreover, let $e_N(t, \cdot)$ be the step function of $(e_i(t))_i$. For all $t \geq 0$ with $cEt \leq 1/4$,*

$$\|e_N(t, \cdot) - \eta_t \circ \gamma\|_2 \leq C_2 cE (1+t) e^{-2gt}. \quad (13.22)$$

Further, for such t ,

$$\begin{aligned} \left| \sqrt{2\mathcal{I}_N(t)} - \left(N \int \eta_t^2 \rho_{\text{sc}} \right)^{1/2} \right| &\leq C_2 cE (1+t) \sqrt{N} e^{-2gt}, \\ \max_i |e_i(t)| &\leq C_2 cE (1+t) + 2C_1 c e^{-2gt}, \end{aligned} \quad (13.23)$$

with C_1 as in Proposition 13.7.

Proof. We first compare the Lagrange multipliers. Let $Y_N(y) = \int_0^1 (y - c\ell_N)^{-1} d\omega$, which is decreasing on $(c\lambda_1, \infty)$, and $Y(y) = \int_0^1 (y - c\gamma)^{-1} d\omega$, which is decreasing on $(2c, \infty)$.

Then $Y_N(z) = 1$, and $Y(1+c^2) = 1$ by (13.7). The inequalities $c\ell_N \leq c\gamma + cE$ and $1+c^2 > 2c$ imply that $Y_N(1+c^2+cE) \leq Y(1+c^2) = 1$, and then $z \leq 1+c^2+cE$. If $1+c^2-cE \leq c\lambda_1$, then $z > c\lambda_1 \geq 1+c^2-cE$. Otherwise, $c\ell_N \geq c\gamma - cE$ gives $Y_N(1+c^2-cE) \geq Y(1+c^2) = 1$, and again $z \geq 1+c^2-cE$. This proves $|z - (1+c^2)| \leq cE$. For small ω , we have $\lambda_1 = \ell_N(\omega) \leq \gamma(\omega) + E \leq 2 + E$. Together with $z \geq 1+c^2-cE$, this gives $d = z - c\lambda_1 \geq g - 2cE \geq g/2$, where we used $2cE \leq (1-c_1)^2/2 \leq g/2$. Let $p_N(\omega) = z - c\ell_N(\omega)$, the step function of the precisions p_i . Then

$$|p_N(\omega) - p(\gamma(\omega))| \leq |z - (1+c^2)| + c|\ell_N(\omega) - \gamma(\omega)| \leq 2cE. \quad (13.24)$$

Next, we write the relative errors through the operators of Lemma 13.9. Let $d_0 = (1-c_1)^2/2$ and $d_* = g - 2cE$. Since $2cE \leq (1-c_1)^2/2 \leq g/2$, we have $d_* \geq g/2 \geq d_0$. On $(0,1)$, we have $g \leq p \circ \gamma \leq (1+c)^2 \leq 4$, and by (13.24), we have $d_* \leq p_N \leq 4 + 2cE \leq 5$. Further, $\int_0^1 p_N^{-1} d\omega = N^{-1} \sum_i p_i^{-1} = 1$ by the definition of z , and $\int_0^1 (p \circ \gamma)^{-1} d\omega = \int p^{-1} \rho_{sc} = 1$ by (13.7). By these bounds, Lemma 13.9 applies to p_N and to $p \circ \gamma$, with d_* in place of d and with $s = 5$. Let $H_1 = H_{p_N}$ and $H_2 = H_{p \circ \gamma}$. Since $2c\ell_N = 2z - 2p_N$ and $2c\gamma = 2(1+c^2) - 2p \circ \gamma$, the operators of (13.6) satisfy $\mathcal{L}_{\ell_N} f = H_1 f + 2zf$ and $\mathcal{L}_\gamma f = H_2 f + 2(1+c^2)f$ for $f \in L^\infty(0,1)$. The partial sums of the exponential series of $t\mathcal{L}_{\ell_N}$, applied to $\mathbf{1}$, converge in $L^\infty(0,1)$ to $M_t^{\ell_N}$. Since the norm of $L^\infty(0,1)$ dominates that of $L^2(0,1)$, they also converge to $M_t^{\ell_N}$ in $L^2(0,1)$, where their limit is $e^{2zt} e^{tH_1} \mathbf{1}$. Let $h_j(t) = e^{tH_j} \mathbf{1}$ and $\bar{h}_j(t) = \langle \mathbf{1}, h_j(t) \rangle$ for $j = 1, 2$. By (13.6), we have

$$M_t^{\ell_N} = e^{2zt} h_1(t), \quad \bar{M}_t^{\ell_N} = e^{2zt} \bar{h}_1(t), \quad v_t^{\ell_N} = \frac{h_1(t)}{\bar{h}_1(t)},$$

and in the same way $v_t^\gamma = h_2(t)/\bar{h}_2(t)$. By the definition of H_1 , we have $p_N h_1(t) = -H_1 h_1(t)/2 + \bar{h}_1(t) \mathbf{1}$, and similarly for H_2 . Recall from the discussion before Lemma 13.3 that $v_t^{\ell_N}$ is the step function of $(v_i(t))_i$, and that $v_t^\gamma = v_t \circ \gamma$. Using also the definition (13.8) of η_t , we find

$$e_N(t, \cdot) = p_N v_t^{\ell_N} - \mathbf{1} = -\frac{H_1 e^{tH_1} \mathbf{1}}{2\bar{h}_1(t)}, \quad \eta_t \circ \gamma = (p \circ \gamma) v_t^\gamma - \mathbf{1} = -\frac{H_2 e^{tH_2} \mathbf{1}}{2\bar{h}_2(t)}.$$

We now check the hypotheses of Lemma 13.10. Let $q_1 = p_N^{-1}$ and $q_2 = (p \circ \gamma)^{-1}$, and let $u_j = q_j / \|q_j\|_2$. By Lemma 13.9, the operators H_j are self-adjoint, $\|H_j\| \leq 12$, $H_j u_j = 0$, and $\langle f, H_j f \rangle \leq -2d_* \|f\|_2^2$ whenever $\langle f, u_j \rangle = 0$. Also $1 \leq \|q_j\|_2 \leq d_0^{-1}$, and $\bar{h}_j(t) \geq \|q_j\|_2^{-2} \geq d_0^2$ for all $t \geq 0$. Since q_1 and q_2 are positive, we have $\langle u_1, u_2 \rangle > 0$. By (13.24),

$$\|H_1 - H_2\| \leq 2\|p_N - p \circ \gamma\|_\infty \leq 4cE, \quad \|q_1 - q_2\|_2 \leq \left\| \frac{p \circ \gamma - p_N}{p_N (p \circ \gamma)} \right\|_\infty \leq \frac{2cE}{d_0^2}.$$

Further,

$$u_1 - u_2 = \frac{q_1 - q_2}{\|q_1\|_2} + \frac{\|q_2\|_2 - \|q_1\|_2}{\|q_1\|_2} u_2,$$

and the triangle inequality and $\|q_1\|_2 \geq 1$ give $\|u_1 - u_2\|_2 \leq 2\|q_1 - q_2\|_2 \leq 4cE/d_0^2$.

We apply Lemma 13.10 with $B = 12$, with d_* in place of d , with $x = \mathbf{1}$, for which $m_j(t) = \bar{h}_j(t)$, and with $\delta = d_0^2$. Using the formulas for $e_N(t, \cdot)$ and $\eta_t \circ \gamma$ above, we find

$$\begin{aligned} \|e_N(t, \cdot) - \eta_t \circ \gamma\|_2 &\leq \frac{1}{2} \left(\frac{20 \cdot 13^2 \cdot 4cE (1 + d_0^{-2})}{d_0^2} + \frac{12 \cdot 4cE}{d_0^4} \right) (1 + t) e^{-2d_* t} \\ &\leq \frac{2^{14} cE}{d_0^4} (1 + t) e^{-2d_* t}. \end{aligned}$$

In the last inequality, we used $d_0 \leq 1/2$. Since $4cEt \leq 1$, we have $e^{-2d_* t} = e^{-2gt} e^{4cEt} \leq e \cdot e^{-2gt}$. This proves (13.22) for every $C_2 \geq 2^{14} e d_0^{-4}$. The lower bound $d_* = g - 2cE$ for the precisions, rather than $g/2$, is what gives the factor e^{-2gt} .

Next, note that $2\mathcal{I}_N(t) = \sum_i e_i(t)^2 = N\|e_N(t, \cdot)\|_2^2$ and $\int \eta_t^2 \rho_{sc} = \|\eta_t \circ \gamma\|_2^2$. The first bound in (13.23) follows from (13.22) and the triangle inequality in $L^2(0, 1)$.

It remains to prove the second bound in (13.23). We do not deduce it from (13.22), which controls only the norm in $L^2(0, 1)$, but from the positivity of the evolution. Equation (13.9), with $a = \ell_N$ and $a' = \gamma$, implies that $|v_t^{\ell_N}/v_t^\gamma - 1| \leq e^{4cEt} - 1 \leq 4e cEt$, where we used $4cEt \leq 1$. Further, (13.24) and $p \geq g$ give $|p_N/(p \circ \gamma) - 1| \leq 2cE/g \leq 1/2$. Now

$$1 + e_N(t, \cdot) = p_N v_t^{\ell_N} = (1 + \eta_t \circ \gamma) \frac{p_N}{p \circ \gamma} \frac{v_t^{\ell_N}}{v_t^\gamma},$$

and $|(1 + \alpha)(1 + \beta) - 1| \leq |\alpha| + 2|\beta|$ for $|\alpha| \leq 1$. Using $|\eta_t| \leq 2C_1$ from Proposition 13.7, we find, for every $\omega \in (0, 1)$,

$$|e_N(t, \omega) - \eta_t(\gamma(\omega))| \leq (1 + 2C_1) \left(\frac{2cE}{g} + 8e cEt \right) \leq C_\infty cE (1 + t),$$

with $C_\infty = (1 + 2C_1) \max\{2(1 - c_1)^{-2}, 8e\}$. Together with $\sup_\lambda |\eta_t(\lambda)| \leq 2C_1 c e^{-2gt}$ from Proposition 13.7, this gives the second bound in (13.23) for every $C_2 \geq C_\infty$. We take $C_2 = \max\{2^{14} e d_0^{-4}, C_\infty\}$, which depends only on c_1 . $\square$

The comparison in Lemma 13.11 is made at the actual times t , which grow with N ; no limit $N \rightarrow \infty$ at fixed time is interchanged with the limit $t \rightarrow \infty$. At the times $t = T_N + u/g$ of Theorem 13.1, the information is of order one, and $\sqrt{N} e^{-2gt}$ is of order $(\log N)^{3/4}$. The first error in (13.23) is then $O(E(\log N)^{7/4})$, and the second is $O(E \log N) + o(1)$, with constants depending on c and u . Without the factor e^{-2gt} , the first error would be of order $\sqrt{N} E \log N$. For the GOE, at every fixed confidence we have $E \leq (C + 7)N^{-5/12}$ on the event F_C for N large, by Lemma 13.8. The argument in Section 13.5 uses the eigenvalues only through this bound, and it applies on any event on which $\|\ell_N - \gamma\|_\infty \leq E_N$, for a deterministic sequence E_N with $E_N(\log N)^{7/4} \rightarrow 0$.

13.5 Proof of the main result

Fix $c \in (0, 1)$, a compact set $K \subset \mathbb{R}$, and $\eta > 0$. Choose $C \geq 1$ using Lemma 4.2 so that $\mathbb{P}(F_C) \geq 1 - \eta$ for all sufficiently large N . All constants below depend only on c , K and C , and they are chosen before $N \rightarrow \infty$. All estimates on F_C below are deterministic consequences of the bounds of Lemma 13.8. Let $E_N = (C + 7)N^{-5/12}$ and $t_N(u) = T_N + u/g$ for $u \in K$. Since $T_N \sim \log N/(4g)$, for N large we have $1 \leq t_N(u) \leq \log N/(2g)$ for all $u \in K$.

Step 1: the information. On F_C , we have $\|\ell_N - \gamma\|_\infty \leq E_N$ by Lemma 13.8, and for N large, $E_N \leq (1 - c)^2/4$ and $cE_N t_N(u) \leq 1/4$. By the definition (13.4) of T_N , we have $\sqrt{N}e^{-2gt_N(u)} = A_c^{-1/2}(\log N)^{3/4}e^{-2u}$. Let $U = \sup(\{0\} \cup \{|u| : u \in K\})$. We apply the first bound in (13.23) with $c_1 = c$, and use $1 + t_N(u) \leq (1 + (2g)^{-1}) \log N$ for $N \geq 3$. We find

$$\sup_{u \in K} \left| \sqrt{2\mathcal{I}_N(t_N(u))} - \left(N \int \eta_{t_N(u)}^2 \rho_{sc} \right)^{1/2} \right| \leq C_2 c (1 + (2g)^{-1}) A_c^{-1/2} e^{2U} E_N (\log N)^{7/4},$$

which tends to zero, because $E_N = (C + 7)N^{-5/12}$. Next, we evaluate the integral with Proposition 13.7, again with $c_1 = c$. We have

$$4gt_N(u) = \log N - \frac{3}{2} \log \log N + \log A_c + 4u, \quad Ne^{-4gt_N(u)} = \frac{(\log N)^{3/2} e^{-4u}}{A_c}.$$

In particular, $4gt_N(u)/\log N \rightarrow 1$ uniformly in $u \in K$, and $t_N(u)^{-3/2} = (4g)^{3/2}(\log N)^{-3/2}(1 + o(1))$. Using (13.19), we find

$$\begin{aligned} \frac{N}{2} \int \eta_{t_N(u)}^2 \rho_{sc} &= \frac{N}{2} K_c t_N(u)^{-3/2} e^{-4gt_N(u)} (1 + o(1)) \\ &= \frac{K_c (4g)^{3/2}}{2A_c} e^{-4u} (1 + o(1)) = e^{-4u} (1 + o(1)), \end{aligned}$$

uniformly in $u \in K$, where in the last step we used $A_c = 4K_c g^{3/2}$. Combining the two displays, we conclude that, on F_C ,

$$\sup_{u \in K} |\mathcal{I}_N(t_N(u)) - e^{-4u}| \rightarrow 0, \quad \sup_{u \in K} \max_i |e_i(t_N(u))| \rightarrow 0, \quad (13.25)$$

where the second limit follows from the second, maximum-coordinate bound in (13.23), since $E_N \log N \rightarrow 0$ and $e^{-2gt_N(u)} \rightarrow 0$ uniformly in $u \in K$.

Step 2: the Bingham comparison law and the Gibbs measure. We have $\pi_{N,c} = \mu_q$ and $\mu_t = \mu_{v(t)}$, with $\sum_i q_i = \sum_i v_i(t) = N$ and $v_i(t) = q_i(1 + e_i(t))$. Recall from Lemma 13.11 that $d_N \geq g/2$, that is, $\max_i q_i \leq 2/g$. Together with $\sum_i q_i^2 \geq N^{-1}(\sum_i q_i)^2 = N$, this gives

$$\eta_q = \frac{\max_i q_i}{(2 \sum_i q_i^2)^{1/2}} \leq \frac{2}{g\sqrt{2N}}.$$

Combining (13.25) with Lemma 13.2, we have $\sup_{u \in K} |\|\mu_{t_N(u)} - \pi_{N,c}\|_{\text{TV}} - \mathcal{G}(\mathcal{I}_N(t_N(u)))| \rightarrow 0$ on F_C . Further, (13.3) and (13.25) imply that $\mathcal{G}(\mathcal{I}_N(t_N(u))) \rightarrow \mathcal{G}(e^{-4u}) = \Psi(u)$ uniformly in $u \in K$. We conclude that

$$\sup_{u \in K} |\|\mu_{t_N(u)} - \pi_{N,c}\|_{\text{TV}} - \Psi(u)| \rightarrow 0 \quad \text{on } F_C.$$

Step 3: the time- t law. On F_C , we have $\|cW_N\| \leq 3c \leq 3$ and $d_N \geq g/2$, so that $Nd_N^2 \geq Ng^2/4 \geq C_w(3)$ for N large, where C_w is the constant of (9.35). Then Theorem 9.13, with $\Lambda = 3$, implies that for all $t \geq 0$,

$$\|v_N P_t - \mu_t\|_{\text{TV}} \leq C_3 d_N^{-2} e^{-d_N t/2} \leq 4C_3 g^{-2} e^{-gt/4},$$

where C_3 is the constant C_Λ of (9.36) with $\Lambda = 3$. This tends to zero uniformly in $u \in K$, because $t_N(u) \geq T_N - U/g \rightarrow \infty$. By the triangle inequality and Step 2, we conclude that $\sup_{u \in K} |D_N(t_N(u)) - \Psi(u)| \rightarrow 0$ on F_C .

Step 4: conclusion. Every bound in Steps 1–3 is deterministic on F_C and holds for N larger than a constant depending on c , K and C . Thus, for every $\varepsilon > 0$, the limsup of the probability that $\sup_{u \in K} |D_N(T_N + u/g) - \Psi(u)| > \varepsilon$ is at most η . Sending $\eta \downarrow 0$, we obtain $\sup_{u \in K} |D_N(T_N + u/g) - \Psi(u)| \rightarrow 0$ in probability. Because $gT_N \rightarrow \infty$, the distance D_N has cutoff profile Ψ with center T_N and window $1/g$ in the sense of Definition 3.7. By Lemma 3.9 and Corollary 3.12, we have $g(t_N(\varepsilon) - T_N) \rightarrow \Psi^{-1}(\varepsilon) = u_\varepsilon$ in probability for each $\varepsilon \in (0, 1)$, and there is cutoff. Finally, $t_N(\varepsilon)/\log N \rightarrow (4g)^{-1}$ in probability, because $T_N/\log N \rightarrow (4g)^{-1}$.

For finitely many values $c \in (0, 1)$, the event F_C does not depend on c , and on F_C the bounds above hold for all of them once N exceeds the largest of the corresponding thresholds. This proves the simultaneous statement and completes the proof.

Approach to the critical temperature from above

In this section, we prove Theorem 1.1(3). Let $c_N = 1 - \delta_N$, where $\delta_N \rightarrow 0$ and $M_N = N\delta_N^3 \rightarrow \infty$. Theorem 14.1 below states that D_N has the cutoff profile Ψ with window δ_N^{-2} and center

$$T_N = \frac{\log M_N - \frac{3}{2} \log \log M_N + \log B_{c_N}}{4\delta_N^2}, \quad B_c = \frac{\sqrt{c}}{4\sqrt{\pi}(1+c)^2}.$$

No condition is imposed on the rate at which M_N tends to infinity.

The proof compares the time- t law $\nu_N P_t$ with the Bingham comparison law μ_t of Section 7, and it has two independent parts. The first part concerns only Gaussian laws conditioned on S_N . Both μ_t and π_{N,c_N} are laws of this kind, and by Theorem 6.6 the distance between them is close to $2\Phi(\sqrt{\mathcal{I}_N(t)}/2) - 1$, where $\mathcal{I}_N(t)$ is the information of the covariance evolution (Definition 3.5). Recall from Lemma 7.3 that $\mathcal{I}_N(t)$ is determined by the mean $\bar{h}$ of the heat vector and by its second derivative. In Section 14.2, we compute $\bar{h}$ and $\bar{h}''$ at times of order $\delta_N^{-2} \log M_N$. At such times only the precisions within a distance of order $\delta_N^2 / \log M_N$ of the smallest precision matter. We describe them through a rescaled spectral measure (Theorem 14.2). This gives the asymptotics of $\mathcal{I}_N$ (Corollary 14.3) and the profile of μ_t (Theorem 14.4). The second part, in Section 14.3, is deterministic. For a matrix whose precisions near the smallest precision d are spaced like the eigenvalues of the GOE near the edge, Theorem 14.10 bounds $\|\nu_N P_t - \mu_t\|_{\text{TV}}$ by $Ce^{-(d-2/N)t}$ for $t \leq C_T d^{-1} \log M$, where $M = Nd^{3/2}$. The constant C does not depend on N or d , and the bound holds as soon as M exceeds a fixed threshold. We combine the two parts in Section 14.4.

We use the following earlier results. From Section 3, we use the properties of Ψ (Lemma 3.9), Lemma 3.8, and Corollary 3.12; from Section 4, Theorem 4.13, Lemmas 4.5 and 4.6, and Corollary 4.9; and from Sections 5 and 6, Proposition 5.6 and Theorem 6.6. From Section 7, we use the covariance evolution together with Lemmas 7.3, 7.10 to 7.12, and 7.15; from Section 9, Lemma 9.3 and Theorems 9.5 and 9.16; and from Section 10, Lemmas 10.16 and 10.21 and Theorems 10.15 and 10.18. Theorem 14.10 is used again in Section 20, where M stays fixed while $N \rightarrow \infty$, and its statement is therefore nonasymptotic.

14.1 Setting and main result

The covariance evolution. Let A be a real symmetric $N \times N$ matrix with precisions $p_1 \leq \dots \leq p_N$ and Lagrange multiplier z (Definition 3.3). Recall that $q_i = 1/p_i$, $\sum_i q_i = N$, and $d = p_1$. We work in the eigencoordinates of A . As in Section 7, we let

$$H = -2 \operatorname{diag}(p_1, \dots, p_N) + \frac{2}{N} \mathbf{1} \mathbf{1}^\top, \quad h(t) = e^{tH} \mathbf{1}, \quad \bar{h}(t) = \frac{1}{N} \sum_{i=1}^N h_i(t),$$

so that $h'_i(t) = -2p_i h_i(t) + 2\bar{h}(t)$. The variances of the covariance evolution of A are $v_i(t) = h_i(t)/\bar{h}(t)$, and the Bingham comparison law at time t is $\mu_t = \mu_{v(t)}$, the Gaussian law with variances $v_i(t)$ conditioned on S_N (Proposition 5.6). We have $v_i(0) = 1$ and $\mu_0 = \nu_N$. The numbers $v_i(t)$ are the variances of the unconditioned Gaussian vector in the definition of μ_t ; they need not equal the second moments of μ_t or those of $\nu_N P_t$. Recall the relative errors and, from Definition 3.5, the information $\mathcal{I}(t) = \mathcal{I}(q, v(t))$:

$$e_i(t) = \frac{v_i(t)}{q_i} - 1 = p_i v_i(t) - 1, \quad \mathcal{I}(t) = \frac{1}{2} \sum_{i=1}^N e_i(t)^2.$$

The matrix H is negative semidefinite, its kernel is spanned by $q = (q_1, \dots, q_N)$, and $-H \geq 2d$ on the orthogonal complement of q (the second part of Lemma 7.1). Let $\vartheta = N / \sum_i q_i^2$. We let ρ be the positive measure on $[2d, \infty)$ such that

$$\frac{1}{N} \mathbf{1}^\top g(-H) \mathbf{1} = \vartheta g(0) + \int g(x) \rho(dx) \quad (14.1)$$

for every function g on the spectrum of $-H$; this is the measure ρ of Lemma 7.1. In other words, $\vartheta \delta_0 + \rho$ is the spectral measure of $-H$ at the unit vector $N^{-1/2} \mathbf{1}$, and $\vartheta = (\mathbf{1}^\top q)^2 / (N |q|^2)$ is the mass of its atom at zero. The choice $g(x) = e^{-tx}$ gives

$$\bar{h}(t) = \vartheta + \int_{[2d, \infty)} e^{-tx} \rho(dx), \quad \rho([2d, \infty)) = 1 - \vartheta, \quad (14.2)$$

which is (7.16), and by Lemma 7.3 we have

$$\mathcal{I}(t) = \frac{N \bar{h}''(2t)}{8 \bar{h}(t)^2} \quad (t \geq 0). \quad (14.3)$$

When $A = c_N W_N$, we write d_N , ϑ_N , ρ_N , and $\mathcal{I}_N$ for these quantities.

The regime. In Sections 14.1, 14.2, and 14.4, we assume that

$$c_N = 1 - \delta_N, \quad 0 < \delta_N < 1, \quad \delta_N \rightarrow 0, \quad M_N = N \delta_N^3 \rightarrow \infty, \quad (14.4)$$

where the sequence δ_N is deterministic and need not be monotone. We write $\ell_N = \log(2 + M_N)$. The matrix is $A = c_N W_N$, its precisions are $p_i = z_N - c_N \lambda_i$, and we set

$$\gamma_i = p_i - d_N = c_N (\lambda_1 - \lambda_i) \quad (1 \leq i \leq N).$$

We have $\gamma_1 = 0$, and almost surely $\gamma_i > 0$ for $i \geq 2$. Probabilities refer to W_N , and $o_{\mathbb{P}}$ and $O_{\mathbb{P}}$ are as in Section 3.4.

The main result. The following theorem is Theorem 1.1(3). Recall the profile $\Psi(u) = 2\Phi(e^{-2u}/2) - 1$ from (3.16).

Theorem 14.1 (Profile for the approach from above). *Assume (14.4). For N so large that $M_N > e$, define*

$$B_c = \frac{\sqrt{c}}{4\sqrt{\pi}(1+c)^2}, \quad T_N = \frac{\log M_N - \frac{3}{2} \log \log M_N + \log B_{c_N}}{4\delta_N^2}, \quad T_N(u) = T_N + \frac{u}{\delta_N^2}. \quad (14.5)$$

Then for every $u_0 > 0$,

$$\sup_{|u| \leq u_0} |D_N(T_N(u)) - \Psi(u)| \longrightarrow 0 \quad (14.6)$$

in probability. For every $\varepsilon \in (0, 1)$,

$$\delta_N^2(t_N(\varepsilon) - T_N) \longrightarrow -\frac{1}{2} \log \left[2\Phi^{-1}\left(\frac{1+\varepsilon}{2}\right) \right] = \Psi^{-1}(\varepsilon) \quad (14.7)$$

in probability. Further, $4\delta_N^2 T_N / \log M_N \rightarrow 1$, and D_N has cutoff profile Ψ with center T_N and window δ_N^{-2} in the sense of Definition 3.7; in particular, there is cutoff.

The numbers T_N and δ_N^{-2} are deterministic. As $\delta_N^2 T_N \rightarrow \infty$, all the times $T_N(u)$ with $|u| \leq u_0$ are positive for large N . The comparison of T_N with the center at fixed temperature is discussed in Section 1.4.

Spectral inputs. We use the following consequences of Section 4. Recall from (4.25) in Theorem 4.13 that

$$\ell_N \left| \frac{d_N}{\delta_N^2} - 1 \right| \longrightarrow 0, \quad \ell_N \left| \frac{1 - c_N^2}{N} \sum_{i=1}^N q_i^2 - 1 \right| \longrightarrow 0 \quad (14.8)$$

in probability. As $1 - c_N^2 = \delta_N(2 - \delta_N)$ and $\vartheta_N = N / \sum_i q_i^2$, we obtain

$$\frac{d_N}{\delta_N^2} \rightarrow 1, \quad \frac{Nd_N^{3/2}}{M_N} \rightarrow 1, \quad \frac{\vartheta_N}{\sqrt{d_N}} \rightarrow 2 \quad (14.9)$$

in probability. By the third and fourth parts of Lemma 4.5, we have

$$\frac{1}{N} \sum_{i=2}^N \frac{1}{\gamma_i} = \frac{1}{c_N} + \frac{\Xi_N}{c_N N^{1/3}}, \quad \frac{1}{\delta_N} \left(\frac{1}{N} \sum_{i=2}^N \frac{1}{\gamma_i} - 1 \right) = \frac{1}{c_N} + \frac{\Xi_N}{c_N M_N^{1/3}} \longrightarrow 1, \quad (14.10)$$

where the limit holds in probability. Here Ξ_N is the regularized sum of inverse gaps of Lemma 4.5, which is tight; its convergence was proved by Landon and Sosoe [102, Theorem 6.1]. Finally, by (4.7) in Lemma 4.6, and since $1/2 \leq c_N \leq 1$ for large N , for

every $\eta \in (0, 1)$ there exist a constant $K_\eta \geq 1$ and events G^η with $\liminf_N \mathbb{P}(G^\eta) \geq 1 - \eta$ on which

$$K_\eta^{-1} \left(\frac{i-1}{N} \right)^{2/3} \leq \gamma_i \leq K_\eta \left(\frac{i-1}{N} \right)^{2/3} \quad (2 \leq i \leq N). \quad (14.11)$$

The constant K_η depends on η . As in Section 3.4, we take $N \rightarrow \infty$ at fixed η and remove the localization at the end with Lemma 3.8.

Deterministic inputs. Two results of Sections 6 and 7 about a fixed matrix are used in the following form. Let $K \geq 1$, and suppose that $0 < d \leq 1$, $Nd^{3/2} \geq 1$, and $K^{-1}((i-1)/N)^{2/3} \leq p_i - d \leq K((i-1)/N)^{2/3}$ for $2 \leq i \leq N$. By (7.41) in Lemma 7.12, there exists a constant C_K , depending only on K , such that

$$|e_i(d^{-1} + s)| \leq C_K \frac{d}{p_i} (1 + ds) e^{-2ds} \quad (s \geq 0, 1 \leq i \leq N). \quad (14.12)$$

The second input is Theorem 6.6: there exists a universal constant $\eta_{\text{lik}} > 0$, and for every $B < \infty$ a constant C_B , with the following property. Let $v, \tilde{v}$ be variance sequences on $\{1, \dots, N\}$ with $\sum_i v_i = \sum_i \tilde{v}_i = N$, write $\tilde{v}_i = v_i(1 + \tilde{e}_i)$, and suppose that $\max_i |\tilde{e}_i| \leq 1/4$, $\eta_v \leq \eta_{\text{lik}}$, and $I = \frac{1}{2} \sum_i \tilde{e}_i^2 \leq B$. Then

$$\left| \|\mu_{\tilde{v}} - \mu_v\|_{\text{TV}} - \left(2\Phi\left(\frac{\sqrt{I}}{2}\right) - 1 \right) \right| \leq C_B \left(\eta_v + \max_i |\tilde{e}_i| \right). \quad (14.13)$$

Here $\eta_v = \max_i v_i / (2 \sum_j v_j^2)^{1/2}$ is the largest standardized variance of (5.1).

14.2 The information at a smaller scale

We now compute the information at times t with $\delta_N^2 t \rightarrow \infty$. At such a time, the relative error $e_i(t)$ is small unless $p_i - d_N$ is at most of order $1/t$, which is much smaller than δ_N^2 . These coordinates correspond to the eigenvalues within a distance of order $1/t$ of λ_1 , and there are of order $Nt^{-3/2}$ of them. Throughout this section, we fix deterministic times $\bar{t}_N > 0$ that satisfy (4.15) with $t_N = \bar{t}_N$, that is,

$$\delta_N^2 \bar{t}_N \rightarrow \infty, \quad \delta_N^2 \bar{t}_N = o(M_N^{2/3}),$$

and we set $\bar{w}_N = d_N \bar{t}_N$, which is random. Recall from (4.28) that $\bar{w}_N / (\delta_N^2 \bar{t}_N) \rightarrow 1$ in probability, that $\bar{t}_N = o(N^{2/3})$ and $N \bar{t}_N^{-3/2} \rightarrow \infty$, and that $\bar{t}_N / (N \delta_N) = \delta_N^2 \bar{t}_N / M_N \rightarrow 0$. In Theorem 14.4 below, we take $\bar{t}_N = T_N$, for which $\delta_N^2 \bar{t}_N \sim \frac{1}{4} \log M_N$.

Counts at the scale $1/\bar{t}_N$. The eigenvalue counts at this scale are given by Corollary 4.9, applied with $t_N = \bar{t}_N$. Recalling that $c_N(\lambda_1 - \lambda_i) = \gamma_i$, we may state it as follows: the positive measures

$$\omega_N^{\text{gap}} = \frac{\bar{t}_N^{3/2}}{N} \sum_{i=1}^N \delta_{\bar{t}_N \gamma_i} \quad (14.14)$$

on $[0, \infty)$ satisfy, for every continuous function g on $[0, \infty)$ with $\sup_{y \geq 0} (1+y)^2 |g(y)| < \infty$,

$$\int g \, d\omega_N^{\text{gap}} \longrightarrow \frac{1}{\pi} \int_0^\infty g(y) \sqrt{y} \, dy \quad \text{in probability.} \quad (14.15)$$

The atom of ω_N^{gap} at zero comes from $i = 1$ and has mass $\bar{t}_N^{3/2}/N$, which tends to zero.

The rescaled transient measure. We rescale ρ_N at the same scale. Let $\mathfrak{m}_N$ be the finite positive measure on $[0, \infty)$ defined by

$$\int g \, d\mathfrak{m}_N = d_N \bar{t}_N^{3/2} \int g\left(\bar{t}_N \left(\frac{x}{2} - d_N\right)\right) \rho_N(dx) \quad (14.16)$$

for bounded measurable g . It is supported on $[0, \infty)$ because ρ_N is supported on $[2d_N, \infty)$. We substitute $x = 2d_N + 2y/\bar{t}_N$ in (14.2), so that $\bar{t}_N x = 2\bar{w}_N + 2y$ and $x^2 = 4d_N^2(1 + y/\bar{w}_N)^2$. We also use $d_N \bar{t}_N^{3/2} = d_N^{-1/2} \bar{w}_N^{3/2}$ and $\bar{h}''(t) = \int x^2 e^{-tx} \rho_N(dx)$. For every $a > 0$, this gives

$$\begin{aligned} \bar{h}(a\bar{t}_N) - \vartheta_N &= \sqrt{d_N} \bar{w}_N^{-3/2} e^{-2a\bar{w}_N} \int e^{-2ay} \mathfrak{m}_N(dy), \\ \bar{h}''(2a\bar{t}_N) &= 4d_N^{5/2} \bar{w}_N^{-3/2} e^{-4a\bar{w}_N} \int \left(1 + \frac{y}{\bar{w}_N}\right)^2 e^{-4ay} \mathfrak{m}_N(dy). \end{aligned} \quad (14.17)$$

These are exact identities for each N .

We say that positive measures $\mathfrak{m}_N$ on $[0, \infty)$ converge vaguely in probability to $\mathfrak{m}$ if $\int g \, d\mathfrak{m}_N \rightarrow \int g \, d\mathfrak{m}$ in probability for every continuous, compactly supported g on $[0, \infty)$.

Theorem 14.2 (The rescaled transient measure). *Assume (14.4), and let $\bar{t}_N > 0$ be deterministic times that satisfy (4.15) with $t_N = \bar{t}_N$. Then, in probability,*

$$\mathfrak{m}_N \longrightarrow \frac{\sqrt{y}}{\pi} \, dy \quad \text{vaguely,} \quad \int \frac{\mathfrak{m}_N(dy)}{1+y^2} \longrightarrow \frac{1}{\sqrt{2}}. \quad (14.18)$$

Moreover,

$$\bar{h}(\bar{t}_N) - \vartheta_N = \frac{\sqrt{d_N} \bar{w}_N^{-3/2} e^{-2\bar{w}_N}}{4\sqrt{2\pi}} (1 + o_{\mathbb{P}}(1)), \quad (14.19)$$

$$\bar{h}''(2\bar{t}_N) = \frac{d_N^{5/2} \bar{w}_N^{-3/2} e^{-4\bar{w}_N}}{4\sqrt{\pi}} (1 + o_{\mathbb{P}}(1)). \quad (14.20)$$

The limit $\pi^{-1} \sqrt{y} \, dy$ is the image of the semicircle density near the edge, $\rho_{\text{sc}}(2-r) \approx \pi^{-1} \sqrt{r}$, under the rescaling of (14.14). We derive (14.20) from the exact formula for $\bar{h}''$ in (14.17), and not by differentiating (14.19).

Proof. The proof has four steps. We first express the transforms $\int (s+y)^{-2} \mathfrak{m}_N(dy)$ through the gaps γ_i , using an exact identity for the resolvent of H at a negative real argument. We then compute their limits with (14.15) and (14.10), identify the limit of $\mathfrak{m}_N$ with Lemma 7.15, and insert the result into (14.17).

Step 1: a resolvent identity. For $\xi \in (-2d_N, 0)$, let

$$G(\xi) = \frac{1}{N} \sum_{i=1}^N \frac{1}{p_i + \xi/2},$$

as in (7.5); it has no poles on this interval. Such ξ do not lie in $\{0\} \cup [-2p_N, -2d_N]$, and the sixth part of Lemma 7.1 gives $G(\xi) \neq 1$ and

$$\frac{\vartheta_N}{\xi} + \int \frac{\rho_N(dx)}{\xi + x} = \frac{G(\xi)}{2(1 - G(\xi))} \quad (-2d_N < \xi < 0). \quad (14.21)$$

For these ξ , the left side is $N^{-1} \mathbf{1}^\top (\xi - H)^{-1} \mathbf{1}$, a resolvent of the finite matrix H at a negative argument, and not a Laplace transform of $\bar{h}$.

Both sides of (14.21) are smooth functions of $\xi \in (-2d_N, 0)$. Differentiating, and using $G'(\xi) = -\frac{1}{2N} \sum_i (p_i + \xi/2)^{-2}$, we obtain

$$\int \frac{\rho_N(dx)}{(\xi + x)^2} = \frac{1}{4(1 - G(\xi))^2} \frac{1}{N} \sum_{i=1}^N \frac{1}{(p_i + \xi/2)^2} - \frac{\vartheta_N}{\xi^2} \quad (-2d_N < \xi < 0). \quad (14.22)$$

Step 2: the squared transforms of m_N . For $s > 0$, write $\mathcal{T}_N(s) = \int (s + y)^{-2} m_N(dy)$, which is finite. Fix $s > 0$. On the event $\{\bar{w}_N > s\}$, whose probability tends to one, we set $\xi_s = -2d_N + 2s/\bar{t}_N$, which lies in $(-2d_N, 0)$. Then $p_i + \xi_s/2 = \gamma_i + s/\bar{t}_N$ and $s + \bar{t}_N(x/2 - d_N) = \frac{\bar{t}_N}{2}(x + \xi_s)$. Using (14.16) and (14.22), we find

$$\mathcal{T}_N(s) = \frac{4d_N}{\sqrt{\bar{t}_N}} \int \frac{\rho_N(dx)}{(x + \xi_s)^2} = \frac{d_N}{(1 - G(\xi_s))^2} \int \frac{\omega_N^{\text{gap}}(dy)}{(s + y)^2} - \frac{4d_N \vartheta_N}{\sqrt{\bar{t}_N} \xi_s^2}. \quad (14.23)$$

In the second equality, we used that

$$\frac{1}{\sqrt{\bar{t}_N}} \frac{1}{N} \sum_{i=1}^N \frac{1}{(\gamma_i + s/\bar{t}_N)^2} = \frac{\bar{t}_N^{3/2}}{N} \sum_{i=1}^N \frac{1}{(s + \bar{t}_N \gamma_i)^2} = \int \frac{\omega_N^{\text{gap}}(dy)}{(s + y)^2}.$$

Step 3: limits of the squared transforms. We treat the three factors in (14.23) in turn. First, the function $g(y) = (s + y)^{-2}$ satisfies $(1 + y)^2 g(y) \leq \max(1, s^{-2})$, and (14.15) implies that

$$\int \frac{\omega_N^{\text{gap}}(dy)}{(s + y)^2} \longrightarrow \frac{1}{\pi} \int_0^\infty \frac{\sqrt{y}}{(s + y)^2} dy = \frac{1}{2\sqrt{s}} \quad (14.24)$$

in probability. The value of the integral follows from the substitution $y = sr^2$ and $\int_0^\infty r^2(1 + r^2)^{-2} dr = \pi/4$.

Second, we show that $(1 - G(\xi_s))/\delta_N \rightarrow -1$ in probability. Separating the term $i = 1$, we have

$$G(\xi_s) = \frac{\bar{t}_N}{Ns} + \frac{1}{N} \sum_{i=2}^N \frac{1}{\gamma_i + s/\bar{t}_N}.$$

Fix $\eta \in (0, 1)$ and work on G^η . The function $\gamma \mapsto b/(\gamma(\gamma + b))$, with $b = s/\bar{t}_N$, is decreasing on $(0, \infty)$. Using (14.11), we bound the summand with index $i \geq 2$ in the following sum by the value of this function at $K_\eta^{-1}((i-1)/N)^{2/3}$. Comparing this right Riemann sum of a decreasing function with the corresponding integral, we find

$$\begin{aligned} 0 &\leq \frac{1}{N} \sum_{i=2}^N \left(\frac{1}{\gamma_i} - \frac{1}{\gamma_i + b} \right) \leq \int_0^1 \frac{b \, dy}{K_\eta^{-1} y^{2/3} (K_\eta^{-1} y^{2/3} + b)} \\ &\leq \frac{3}{2} K_\eta^{3/2} b \int_0^\infty \frac{r^{-1/2} \, dr}{r + b} = \frac{3\pi}{2} K_\eta^{3/2} \sqrt{b}. \end{aligned}$$

In the second inequality, we substituted $r = K_\eta^{-1} y^{2/3}$ and extended the range of integration. Recall that $\bar{t}_N/(N\delta_N) = \delta_N^2 \bar{t}_N/M_N$, and note that $\sqrt{s/\bar{t}_N}/\delta_N = (s/(\delta_N^2 \bar{t}_N))^{1/2}$. We conclude that on G^η ,

$$\left| \frac{G(\xi_s) - 1}{\delta_N} - \frac{1}{\delta_N} \left(\frac{1}{N} \sum_{i=2}^N \frac{1}{\gamma_i} - 1 \right) \right| \leq \frac{\delta_N^2 \bar{t}_N}{s M_N} + \frac{3\pi}{2} K_\eta^{3/2} \left(\frac{s}{\delta_N^2 \bar{t}_N} \right)^{1/2}.$$

The right side is deterministic and tends to zero by (4.15). Together with (14.10) and Lemma 3.8, this gives $(G(\xi_s) - 1)/\delta_N \rightarrow 1$ in probability. As $d_N/\delta_N^2 \rightarrow 1$ as well, we have

$$\frac{d_N}{(1 - G(\xi_s))^2} = \frac{d_N/\delta_N^2}{((1 - G(\xi_s))/\delta_N)^2} \rightarrow 1$$

in probability.

Third, on the event $\{\bar{w}_N \geq 2s\}$ we have $|\xi_s| = 2d_N(1 - s/\bar{w}_N) \geq d_N$, so that

$$0 \leq \frac{4d_N \vartheta_N}{\sqrt{\bar{t}_N} \xi_s^2} \leq \frac{4\vartheta_N}{d_N \sqrt{\bar{t}_N}} = \frac{4\vartheta_N}{\sqrt{d_N}} \bar{w}_N^{-1/2},$$

which tends to zero in probability by (14.9), since $\bar{w}_N \rightarrow \infty$. Inserting these three limits into (14.23), we find that for every fixed $s > 0$,

$$\mathcal{T}_N(s) \rightarrow \frac{1}{2\sqrt{s}} = \int_0^\infty \frac{\mathfrak{m}(dy)}{(s+y)^2} \quad \text{in probability,} \quad \mathfrak{m}(dy) = \frac{\sqrt{y}}{\pi} dy. \quad (14.25)$$

Step 4: identification of the limit. The measures $\mathfrak{m}_N$ and $\mathfrak{m}$ have finite transforms at $s = 1$, and (14.25) holds for every rational $s \geq 1$. Lemma 7.15, in its version for convergence in probability, then implies that $\mathfrak{m}_N \rightarrow \mathfrak{m}$ vaguely in probability, and that for every continuous f on $[0, 1]$,

$$\int \frac{f((1+y)^{-1})}{(1+y)^2} \mathfrak{m}_N(dy) \rightarrow \int \frac{f((1+y)^{-1})}{(1+y)^2} \mathfrak{m}(dy) \quad \text{in probability.} \quad (14.26)$$

The test function $(1 + y^2)^{-1}$ has the form in (14.26) with $f(r) = (r^2 + (1 - r)^2)^{-1}$, which is continuous on $[0, 1]$, since $(1 + y)^2 / (1 + y^2) = f((1 + y)^{-1})$. This gives the second claim in (14.18), with the value $\pi^{-1} \int_0^\infty \sqrt{y} (1 + y^2)^{-1} dy = 1/\sqrt{2}$. More generally, if g is continuous on $[0, \infty)$ and $(1 + y)^2 g(y) \rightarrow 0$ as $y \rightarrow \infty$, then $f(r) = r^{-2} g(r^{-1} - 1)$ for $r \in (0, 1]$ and $f(0) = 0$ define a continuous function on $[0, 1]$, and by (14.26) we have

$$\int g \, d\mathbf{m}_N \longrightarrow \frac{1}{\pi} \int_0^\infty g(y) \sqrt{y} \, dy \quad \text{in probability.} \quad (14.27)$$

This applies to e^{-2y} and e^{-4y} , and we obtain

$$\begin{aligned} \int e^{-2y} \mathbf{m}_N(dy) &\rightarrow \frac{1}{\pi} \int_0^\infty e^{-2y} \sqrt{y} \, dy = \frac{1}{4\sqrt{2\pi}}, \\ \int e^{-4y} \mathbf{m}_N(dy) &\rightarrow \frac{1}{\pi} \int_0^\infty e^{-4y} \sqrt{y} \, dy = \frac{1}{16\sqrt{\pi}}. \end{aligned} \quad (14.28)$$

The test function in the formula for $\bar{h}''$ in (14.17) depends on $\bar{w}_N$. On $\{\bar{w}_N \geq 1\}$, we have $0 \leq (1 + y/\bar{w}_N)^2 - 1 \leq 3(y + y^2)/\bar{w}_N$, which gives

$$\left| \int \left(1 + \frac{y}{\bar{w}_N}\right)^2 e^{-4y} \mathbf{m}_N(dy) - \int e^{-4y} \mathbf{m}_N(dy) \right| \leq \frac{3}{\bar{w}_N} \sup_{y \geq 0} ((1 + y)^2 (y + y^2) e^{-4y}) \mathcal{T}_N(1).$$

The right side tends to zero in probability, since $\mathcal{T}_N(1) \rightarrow 1/2$ and $\bar{w}_N \rightarrow \infty$. Inserting this and (14.28) into (14.17) with $a = 1$, we obtain (14.19) and (14.20), since $4 \cdot (16\sqrt{\pi})^{-1} = (4\sqrt{\pi})^{-1}$. This completes the proof. $\square$

Corollary 14.3 (Information at a smaller scale). *Assume (14.4), and let $\bar{t}_N > 0$ be deterministic times that satisfy (4.15) with $t_N = \bar{t}_N$. Then*

$$\mathcal{I}_N(\bar{t}_N) = \frac{Nd_N^{3/2}}{128\sqrt{\pi}} \bar{w}_N^{-3/2} e^{-4\bar{w}_N} (1 + o_{\mathbb{P}}(1)). \quad (14.29)$$

Moreover, $\max_i |e_i(\bar{t}_N)| \rightarrow 0$ in probability, and $\eta_q = \max_i q_i / (2 \sum_i q_i^2)^{1/2} = O_{\mathbb{P}}(M_N^{-1/2})$.

Proof. By (14.19) and $\bar{w}_N \rightarrow \infty$, we have $(\bar{h}(\bar{t}_N) - \vartheta_N) / \sqrt{d_N} \rightarrow 0$ in probability. As $\vartheta_N / \sqrt{d_N} \rightarrow 2$, we have $\bar{h}(\bar{t}_N) = \vartheta_N(1 + o_{\mathbb{P}}(1))$. Inserting this and (14.20) into (14.3), we find

$$\mathcal{I}_N(\bar{t}_N) = \frac{Nd_N^{5/2} \bar{w}_N^{-3/2} e^{-4\bar{w}_N}}{32\sqrt{\pi} \vartheta_N^2} (1 + o_{\mathbb{P}}(1)),$$

and since $d_N / \vartheta_N^2 \rightarrow 1/4$, this is (14.29).

For the relative errors, fix $\eta \in (0, 1)$ and work on the event $G^\eta \cap \{d_N \leq 1, Nd_N^{3/2} \geq 1, \bar{w}_N \geq 1\}$, whose probability is at least $1 - \eta - o(1)$. On this event, the hypotheses of

(14.12) hold with $K = K_\eta$. Taking $s = \bar{t}_N - d_N^{-1} \geq 0$ in (14.12), for which $1 + d_N s = \bar{w}_N$, and using $d_N \leq p_i$, we obtain

$$\max_i |e_i(\bar{t}_N)| \leq C_{K_\eta} e^2 \bar{w}_N e^{-2\bar{w}_N},$$

which tends to zero in probability. Removing the localization with Lemma 3.8, we conclude that $\max_i |e_i(\bar{t}_N)| \rightarrow 0$ in probability.

Finally, $\max_i q_i = 1/d_N$ and $\sum_i q_i^2 = N/\vartheta_N$, so that

$$\eta_q = \frac{1}{d_N} \left(\frac{\vartheta_N}{2N} \right)^{1/2} = (Nd_N^{3/2})^{-1/2} \left(\frac{\vartheta_N}{2\sqrt{d_N}} \right)^{1/2}.$$

The last factor tends to one, and $Nd_N^{3/2}/M_N \rightarrow 1$, by (14.9). $\square$

In (14.29), the factor $Nd_N^{3/2}\bar{w}_N^{-3/2}$ equals $N\bar{t}_N^{-3/2}$, which is of the order of the number of indices with $\gamma_i \leq 1/\bar{t}_N$, and the factor $e^{-4\bar{w}_N}$ comes from the decay of the relative errors $e_i(t)$ at the rate $2d_N$. The center T_N of Theorem 14.1 is the time at which the right side of (14.29) is close to one.

Theorem 14.4 (The comparison law at the times $T_N(u)$). *Assume (14.4), let $T_N(u)$ be as in (14.5), and let μ_t be the Bingham comparison law of $c_N W_N$. Then for every $u_0 > 0$,*

$$\sup_{|u| \leq u_0} \left| \|\mu_{T_N(u)} - \pi_{N,c_N}\|_{\text{TV}} - \Psi(u) \right| \rightarrow 0 \quad \text{in probability.}$$

Proof. The proof has three steps. We first prove that the information at the times $T_N(u)$ converges to e^{-4u} , uniformly in u , using one measure $\mathbf{m}_N$ for all u . We then check the remaining hypotheses of (14.13), and finally we apply it. Fix $u_0 > 0$, and let N be so large that $M_N > e$ and $T_N(-u_0) > 0$.

Step 1: the information. Recall that $B_{c_N} \rightarrow B_1 = 1/(16\sqrt{\pi})$, so that

$$\delta_N^2 T_N = \frac{1}{4} \log M_N - \frac{3}{8} \log \log M_N + \frac{1}{4} \log B_{c_N} = \frac{1}{4} \log M_N (1 + o(1)).$$

In particular, $\bar{t}_N = T_N$ satisfies (4.15) with $t_N = T_N$. Let $\mathbf{m}_N$ and $\bar{w}_N = d_N T_N$ be defined with this choice. For $|u| \leq u_0$, let $a_N(u) = T_N(u)/T_N = 1 + u/(\delta_N^2 T_N)$. These numbers are deterministic, $\sup_{|u| \leq u_0} |a_N(u) - 1| \rightarrow 0$, and $a_N(u) \in [1/2, 2]$ for large N . Note that $a_N(u)\bar{w}_N = d_N T_N(u)$. Then the identities (14.17) with $a = a_N(u)$ read

$$\begin{aligned} \bar{h}(T_N(u)) &= \vartheta_N + \sqrt{d_N} \bar{w}_N^{-3/2} e^{-2d_N T_N(u)} \Lambda_N^{(1)}(u), \\ \bar{h}''(2T_N(u)) &= 4d_N^{5/2} \bar{w}_N^{-3/2} e^{-4d_N T_N(u)} \Lambda_N^{(2)}(u), \end{aligned} \tag{14.30}$$

where

$$\Lambda_N^{(1)}(u) = \int e^{-2a_N(u)y} \mathbf{m}_N(dy), \quad \Lambda_N^{(2)}(u) = \int \left(1 + \frac{y}{\bar{w}_N}\right)^2 e^{-4a_N(u)y} \mathbf{m}_N(dy).$$

The mean value theorem gives $|e^{-2ay} - e^{-2y}| \leq 2|a - 1|ye^{-y}$ and $|e^{-4ay} - e^{-4y}| \leq 4|a - 1|ye^{-2y}$ for $a \geq 1/2$ and $y \geq 0$, and for $\bar{w}_N \geq 1$ we have $0 \leq (1 + y/\bar{w}_N)^2 - 1 \leq 3(y + y^2)/\bar{w}_N$. Multiplying these bounds by $(1 + y)^2$ and integrating against $(1 + y)^{-2}m_N(dy)$, we find that

$$\begin{aligned} & \sup_{|u| \leq u_0} \left| \Lambda_N^{(1)}(u) - \int e^{-2y} dm_N \right| + \sup_{|u| \leq u_0} \left| \Lambda_N^{(2)}(u) - \int e^{-4y} dm_N \right| \\ & \leq C \left(\sup_{|u| \leq u_0} |a_N(u) - 1| + \bar{w}_N^{-1} \right) \mathcal{T}_N(1), \end{aligned}$$

where C is a numerical constant and $\mathcal{T}_N(1) = \int (1 + y)^{-2} m_N(dy)$ as in the proof of Theorem 14.2. The right side tends to zero in probability, because $\mathcal{T}_N(1) \rightarrow 1/2$, and (14.28) implies that

$$\sup_{|u| \leq u_0} \left| \Lambda_N^{(1)}(u) - \frac{1}{4\sqrt{2\pi}} \right| \rightarrow 0, \quad \sup_{|u| \leq u_0} \left| \Lambda_N^{(2)}(u) - \frac{1}{16\sqrt{\pi}} \right| \rightarrow 0 \quad (14.31)$$

in probability.

Combining (14.30) and (14.31) with $\sqrt{d_N}/\vartheta_N \rightarrow 1/2$ and $\bar{w}_N \rightarrow \infty$, we find that $\sup_{|u| \leq u_0} |\bar{h}(T_N(u))/\vartheta_N - 1| \rightarrow 0$ in probability. Inserting (14.30) into (14.3) and using $d_N/\vartheta_N^2 \rightarrow 1/4$, we find, uniformly for $|u| \leq u_0$,

$$\mathcal{I}_N(T_N(u)) = \frac{Nd_N^{3/2}}{8} \bar{w}_N^{-3/2} e^{-4d_N T_N(u)} \Lambda_N^{(2)}(u) (1 + o_{\mathbb{P}}(1)).$$

We now replace d_N by δ_N^2 in the exponent. We have $4d_N T_N(u) - 4\delta_N^2 T_N(u) = (d_N/\delta_N^2 - 1)(4\delta_N^2 T_N + 4u)$, and $|4\delta_N^2 T_N + 4u| \leq C\ell_N$ for $|u| \leq u_0$. Using (14.8), we see that the difference is $o_{\mathbb{P}}(1)$, uniformly in $|u| \leq u_0$. From the definition of T_N , we have

$$e^{-4\delta_N^2 T_N(u)} = \frac{(\log M_N)^{3/2}}{M_N B_{c_N}} e^{-4u}.$$

Further, $\bar{w}_N = d_N T_N = \frac{1}{4} \log M_N (1 + o_{\mathbb{P}}(1))$ and $Nd_N^{3/2} = M_N(1 + o_{\mathbb{P}}(1))$. Combining these estimates with (14.31), we conclude that uniformly for $|u| \leq u_0$,

$$\begin{aligned} \mathcal{I}_N(T_N(u)) &= \frac{M_N}{8} \cdot \frac{8}{(\log M_N)^{3/2}} \cdot \frac{(\log M_N)^{3/2}}{M_N B_{c_N}} e^{-4u} \cdot \frac{1}{16\sqrt{\pi}} (1 + o_{\mathbb{P}}(1)) \\ &= \frac{e^{-4u}}{16\sqrt{\pi} B_{c_N}} (1 + o_{\mathbb{P}}(1)). \end{aligned}$$

As $16\sqrt{\pi} B_{c_N} \rightarrow 1$, this proves

$$\sup_{|u| \leq u_0} |\mathcal{I}_N(T_N(u)) - e^{-4u}| \rightarrow 0 \quad \text{in probability.} \quad (14.32)$$

Step 2: the relative errors and the largest standardized variance. Corollary 14.3 implies that $\eta_q \rightarrow 0$ in probability, and η_q does not depend on u . For the relative errors, fix $\eta \in (0, 1)$ and work on $G^\eta \cap \{d_N \leq 1, Nd_N^{3/2} \geq 1, d_N T_N(-u_0) \geq 1\}$. As in the proof of Corollary 14.3, (14.12) yields

$$\begin{aligned} \sup_{|u| \leq u_0} \max_i |e_i(T_N(u))| &\leq C_{K_\eta} e^2 \sup_{|u| \leq u_0} d_N T_N(u) e^{-2d_N T_N(u)} \\ &= C_{K_\eta} e^2 d_N T_N(-u_0) e^{-2d_N T_N(-u_0)}, \end{aligned}$$

where we used that $r \mapsto re^{-2r}$ is decreasing for $r \geq 1/2$. The right side is $o_{\mathbb{P}}(1)$, and Lemma 3.8 implies that the supremum over $|u| \leq u_0$ of $\max_i |e_i(T_N(u))|$ tends to zero in probability.

Step 3: conclusion. Let $B = e^{4u_0} + 1$. By Steps 1 and 2, the event that $\eta_q \leq \eta_{\text{lik}}$, $\max_i |e_i(T_N(u))| \leq 1/4$, and $\mathcal{I}_N(T_N(u)) \leq B$ for all $|u| \leq u_0$ has probability tending to one. On this event we apply (14.13) with $v = q$ and $\tilde{v} = v(T_N(u))$. Both sequences have sum N , the relative errors are $\tilde{e}_i = e_i(T_N(u))$, and $I = \mathcal{I}_N(T_N(u))$. Recall that $\mu_q = \pi_{N, c_N}$ by Proposition 5.6. Then (14.13) gives

$$\begin{aligned} \sup_{|u| \leq u_0} \left| \|\mu_{T_N(u)} - \pi_{N, c_N}\|_{\text{TV}} - \left(2\Phi\left(\frac{1}{2}\sqrt{\mathcal{I}_N(T_N(u))}\right) - 1\right) \right| \\ \leq C_B \left(\eta_q + \sup_{|u| \leq u_0} \max_i |e_i(T_N(u))| \right), \end{aligned}$$

and the right side tends to zero in probability. Finally, Φ is Lipschitz with constant $(2\pi)^{-1/2}$ and $|\sqrt{a} - \sqrt{b}| \leq |a - b|^{1/2}$ for $a, b \geq 0$, so that

$$\left| 2\Phi\left(\frac{1}{2}\sqrt{\mathcal{I}_N(T_N(u))}\right) - 2\Phi\left(\frac{1}{2}e^{-2u}\right) \right| \leq (2\pi)^{-1/2} |\mathcal{I}_N(T_N(u)) - e^{-4u}|^{1/2}.$$

The supremum over $|u| \leq u_0$ of the right side tends to zero in probability, by (14.32). As $2\Phi(e^{-2u}/2) - 1 = \Psi(u)$, this completes the proof. $\square$

14.3 The time- t law and the Bingham comparison law

In this section, the matrix A is deterministic, and all statements are deterministic. We compare $\nu_N P_t^A$ with the Bingham comparison law μ_t of A on the whole time interval relevant for the profile. The main result is Theorem 14.10.

Hypotheses and constants. Let A be a real symmetric $N \times N$ matrix with precisions $p_1 \leq \dots \leq p_N$ and Lagrange multiplier z , and let $d = p_1$. Fix $K \geq 1$ and $d_0 \in (0, 1)$. We assume that

$$0 < d \leq d_0, \quad K^{-1} \left(\frac{i-1}{N} \right)^{2/3} \leq p_i - d \leq K \left(\frac{i-1}{N} \right)^{2/3} \quad (2 \leq i \leq N), \quad (14.33)$$

and we write

$$M = Nd^{3/2}.$$

These are the hypotheses (7.30) of Section 7, together with $d \leq d_0$, and they are the hypotheses of the estimates for the unnormalized process in Section 10. The largest precision satisfies $p_N \leq d_0 + K$. Adding a multiple of the identity to A changes neither L_A nor the covariance evolution, and the results below depend on A only through its precisions and its eigenbasis.

We also fix $C_T > 0$ and set

$$T_* = C_T d^{-1} \log M, \quad \bar{r} = d - \frac{2}{N}.$$

The rate $\bar{r}$ is the rate in the weighted gradient estimate of Lemma 10.21; it is not the rate r_0 of Section 9. In this section, C denotes a constant that depends only on K , d_0 , and C_T , and M_0 a threshold with the same dependence; when a constant also depends on a moment order, we say so. Their values may change from line to line. Every statement holds for every $N \geq 2$, every matrix A that satisfies (14.33) with $M \geq M_0$, and every time in the stated range, with the same constants.

The coefficient differences and the forward equation. We use the notation of Section 14.1 for the covariance evolution of A , and let $\zeta(t) = \bar{h}'(t)/\bar{h}(t)$. Summing the equations $h'_i = -2p_i h_i + 2\bar{h}$ over i , we have $N^{-1} \sum_i p_i h_i = \bar{h} - \bar{h}'/2$, and hence $N^{-1} \sum_i p_i v_i(t) = 1 - \zeta(t)/2$. Writing $\alpha_i = z - p_i$ for the eigenvalues of A , the moving multiplier of Section 7 is therefore $z(t) = 1 + N^{-1} \sum_i \alpha_i v_i(t) = z + \zeta(t)/2$. The matrix $C_t = A - B_t$ of Lemma 9.3, where $B_t = \text{diag}(b_i(t))$ with $b_i(t) = z(t) - v_i(t)^{-1}$, is diagonal, and its entries are the coefficient differences $C_i(t) = \alpha_i - b_i(t)$. Since $\alpha_i = z - p_i$ and $z(t) = z + \zeta(t)/2$, they satisfy

$$C_i(t) = \frac{1}{v_i(t)} - p_i - \frac{\zeta(t)}{2} = \frac{v'_i(t)}{2v_i(t)}, \quad \text{so that} \quad \sum_{i=1}^N C_i(t) v_i(t) = \frac{1}{2} \sum_{i=1}^N v'_i(t) = 0. \quad (14.34)$$

For the second expression of $C_i(t)$, note that $v'_i/v_i = h'_i/h_i - \zeta = -2p_i + 2/v_i - \zeta$. By Lemma 9.3, the forward-equation defect (Definition 9.2) of the family $(\mu_t)_{t \geq 0}$ is

$$R_t = -\frac{Q_B(t)Q_C(t) - \mu_t(Q_B(t)Q_C(t))}{N},$$

$$Q_B(t) = N - \sum_{i=1}^N \frac{x_i^2}{v_i(t)}, \quad Q_C(t) = \sum_{i=1}^N C_i(t) x_i^2,$$

where we used that $Q_B(t) = x^\top B_t x - N(z(t) - 1)$ equals $N - \sum_i x_i^2/v_i(t)$ on S_N . In other words,

$$\partial_t \mu_t = L_A^* \mu_t + R_t \mu_t, \quad \mu_t(R_t) = 0, \quad \mu_0 = \nu_N. \quad (14.35)$$

The Poisson equation. The law μ_t is the Gibbs measure π_{B_t} of the generator L_{B_t} (Proposition 5.6). By Lemma 9.6, there exists a unique smooth solution ψ_t of

$$-L_{B_t}\psi_t = R_t, \quad \mu_t(\psi_t) = 0,$$

and we write $\mathcal{E}(t) = \mu_t\Gamma(\psi_t)$ for its energy. Recall from Corollary 9.7 that for every smooth g ,

$$\mu_t(R_t g) = \mu_t\Gamma(\psi_t, g), \quad (14.36)$$

Thus $R_t = -\operatorname{div}_{\mu_t}(\nabla_{S_N}\psi_t)$, where $\operatorname{div}_{\mu_t}j = g_t^{-1}\operatorname{div}_{S_N}(g_t j)$ and $g_t = d\mu_t/d\nu_N$; its size is measured by $\mathcal{E}(t)$.

The next lemma restates Theorem 9.16 in terms of a scale d_* , which is the square of the parameter δ there. With $\delta = d_*^{1/2}$, the variance conditions at scale δ with constants (c_0, C_0) (Definition 9.15) read

$$\begin{aligned} c_0 \leq v_i \leq C_0 d_*^{-1} \quad (1 \leq i \leq N), \quad c_0 N d_*^{-1/2} \leq \sum_i v_i^2 \leq C_0 N d_*^{-1/2}, \\ \sum_i v_i^3 \leq C_0 N d_*^{-3/2}, \quad \sum_i v_i^4 \leq C_0 N d_*^{-5/2}, \end{aligned} \quad (14.37)$$

together with the existence of a set $\mathcal{S} \subset \{1, \dots, N\}$ with $|\mathcal{S}| \geq c_0 N d_*^{3/2}$, such that $v_i \geq c_0 d_*^{-1}$ for $i \in \mathcal{S}$ and $\sum_{i \notin \mathcal{S}} v_i^2 \geq c_0 \sum_i v_i^2$.

Lemma 14.5 (The Poisson equation at a given scale). *Let $0 < c_0 < 1 \leq C_0$. There exist constants C_* and M_*^0 , depending only on c_0 and C_0 , with the following property. Let v be a variance sequence on $\{1, \dots, N\}$ with $\sum_i v_i = N$, let $d_* \in (0, 1]$ with $M_* = N d_*^{3/2} \geq M_*^0$, and suppose that v satisfies the variance conditions at scale $d_*^{1/2}$ with constants (c_0, C_0) . Let $C_1, \dots, C_N$ be real numbers with $\sum_i C_i v_i = 0$, let $Q_B = N - \sum_i x_i^2/v_i$, $Q_C = \sum_i C_i x_i^2$, and $R = -N^{-1}(Q_B Q_C - \mu_v(Q_B Q_C))$, let ψ be the solution of $-L_B \psi = R$ with $\mu_v(\psi) = 0$, where $B = -\operatorname{diag}(v_i^{-1})$, and let*

$$I_C = \sum_i C_i^2 v_i^2, \quad J_C = \sum_i C_i^2, \quad K_C = \sum_i C_i^2 v_i.$$

Then

$$\mu_v\Gamma(\psi) \leq \frac{C_*}{M_*} \left(K_C + \frac{J_C}{d_*} + d_* I_C \right).$$

Proof. This is Theorem 9.16 with $\delta = d_*^{1/2} \in (0, 1]$. Indeed, $N\delta^3 = M_*$, the conditions (9.40) and (9.41) at scale δ are (14.37) and the conditions on $\mathcal{S}$, and the bound (9.42) is $\frac{C_*}{M_*}(K_C + J_C d_*^{-1} + d_* I_C)$. The only assumption on the numbers C_i is $\sum_i C_i v_i = 0$. $\square$

The proof of Theorem 9.16 uses the lower bound of order d_* on the spectral gap of L_B from Theorem 8.8; this is where the set $\mathcal{S}$ enters. The next lemma turns bounds on the coefficient differences into bounds on the energy $\mathcal{E}$.

Lemma 14.6 (Energy bounds from bounds on the coefficient differences). Assume (14.33). Let $v(t)$, $t \geq 0$, be variance sequences with $\sum_i v_i(t) = N$, and let $C_i(t)$ be real numbers with $\sum_i C_i(t)v_i(t) = 0$. Suppose that there exist $\Lambda_0 \geq 1$, $0 < c_0 < 1 \leq C_0$, and an integer $j \geq 0$ such that the following hold.

1. For $0 \leq t \leq d^{-1}$ and all i , $|C_i(t)| \leq \Lambda_0(1+t)^{-1}(1+p_it)^{-1}$.
2. For $s \geq 0$ and all i , $|C_i(d^{-1}+s)| \leq \Lambda_0 d^2 p_i^{-1}(1+ds)^j e^{-2ds}$.
3. For $t \geq 0$ and all i , $\Lambda_0^{-1}(1+t)/(1+p_it) \leq v_i(t) \leq \Lambda_0(1+t)/(1+p_it)$.
4. For $0 \leq t \leq d^{-1}$, the sequence $v(t)$ satisfies the variance conditions at scale $(1+t)^{-1/2}$ with constants (c_0, C_0) . For $t \geq d^{-1}$, it satisfies the variance conditions at scale $d^{1/2}$ with the same constants.

For $t \geq 0$, let ψ_t be the solution of the Poisson equation of Lemma 14.5 defined from $v = v(t)$ and $C_i = C_i(t)$, and let $\mathcal{E}(t) = \mu_{v(t)}\Gamma(\psi_t)$. There exist C and M_0 , depending only on $K, d_0, \Lambda_0, c_0, C_0$, and j , such that if $M \geq M_0$, then

$$\mathcal{E}(t) \leq \frac{C}{1+t} \quad (0 \leq t \leq d^{-1}), \quad \mathcal{E}(d^{-1}+s) \leq Cd(1+ds)^{2j}e^{-4ds} \quad (s \geq 0).$$

Proof. We first record two sums over the precisions. For every integer $k \geq 2$, we claim that

$$\sum_{i=1}^N \frac{1}{(1+p_it)^k} \leq C_k N(1+t)^{-3/2} \quad (0 \leq t \leq d^{-1}), \quad \sum_{i=1}^N \frac{1}{p_i^k} \leq C_k N d^{3/2-k}, \quad (14.38)$$

provided $M \geq 2^{3/2}$, where C_k depends only on K and k . For the first bound, we note that the summand is decreasing in p_i . Using (14.33) and comparing a right Riemann sum with an integral, we find for $t > 0$ that

$$\sum_{i=1}^N \frac{1}{(1+p_it)^k} \leq 1 + N \int_0^\infty \frac{dy}{(1+K^{-1}y^{2/3}t)^k} = 1 + \frac{3}{2}K^{3/2} \left(\int_0^\infty \frac{r^{1/2} dr}{(1+r)^k} \right) N t^{-3/2},$$

where we substituted $r = K^{-1}y^{2/3}t$. The last integral is finite since $k > 3/2$. For $t \leq 1$ the sum is at most $N \leq 2^{3/2}N(1+t)^{-3/2}$. For $1 \leq t \leq d^{-1}$, we have $t^{-3/2} \leq 2^{3/2}(1+t)^{-3/2}$, and since $1+t \leq 2d^{-1}$, also $N(1+t)^{-3/2} \geq 2^{-3/2}M \geq 1$, which absorbs the term 1. The second bound follows in the same way from

$$\sum_{i=1}^N \frac{1}{p_i^k} \leq d^{-k} + N \int_0^\infty \frac{dy}{(d+K^{-1}y^{2/3})^k} = d^{-k} + \frac{3}{2}K^{3/2} \left(\int_0^\infty \frac{r^{1/2} dr}{(1+r)^k} \right) N d^{3/2-k},$$

together with $d^{-k} = N d^{3/2-k}/M$.

Let $0 \leq t \leq d^{-1}$, $d_* = (1+t)^{-1}$, and $M_t = Nd_*^{3/2} = N(1+t)^{-3/2}$. By (i), (iii), and (14.38),

$$\begin{aligned} I_C &\leq \Lambda_0^4 \sum_i \frac{1}{(1+p_it)^4} \leq CM_t, \\ J_C &\leq \frac{\Lambda_0^2}{(1+t)^2} \sum_i \frac{1}{(1+p_it)^2} \leq C(1+t)^{-2}M_t, \\ K_C &\leq \frac{\Lambda_0^3}{1+t} \sum_i \frac{1}{(1+p_it)^3} \leq C(1+t)^{-1}M_t, \end{aligned}$$

where I_C, J_C, K_C are defined from $v(t)$ and $(C_i(t))$ as in Lemma 14.5. As $1+t \leq 2d^{-1}$, we have $M_t \geq 2^{-3/2}M$, and $M_t \geq M_*^0$ when M_0 is large. Then (iv) and Lemma 14.5 imply that

$$\mathcal{E}(t) \leq \frac{C}{M_t} \left(\frac{M_t}{1+t} + (1+t) \frac{M_t}{(1+t)^2} + \frac{M_t}{1+t} \right) = \frac{3C}{1+t}.$$

Let $t = d^{-1} + s$ with $s \geq 0$, let $d_* = d$, and write $L_j(s) = (1+ds)^j e^{-2ds}$. Using (iii) and $p_it \geq 1$, we have $v_i(t) \leq \Lambda_0(1+t)/(p_it) \leq 2\Lambda_0/p_i$. By (ii) and (14.38),

$$\begin{aligned} I_C &\leq Cd^4 L_j(s)^2 \sum_i \frac{1}{p_i^4} \leq CML_j(s)^2, \quad J_C \leq Cd^4 L_j(s)^2 \sum_i \frac{1}{p_i^2} \leq Cd^2 ML_j(s)^2, \\ K_C &\leq Cd^4 L_j(s)^2 \sum_i \frac{1}{p_i^3} \leq CdML_j(s)^2. \end{aligned}$$

With (iv) and Lemma 14.5 for $M_* = M$, we obtain

$$\mathcal{E}(d^{-1} + s) \leq \frac{C}{M} (dM + d^{-1}d^2M + dM) L_j(s)^2 = 3Cd(1+ds)^{2j} e^{-4ds}.$$

This completes the proof. □

We now verify the hypotheses of Lemma 14.6 for the covariance evolution.

Lemma 14.7 (Coefficient differences along the covariance evolution). *Assume (14.33). There exist $\Lambda_0 \geq 1$, $0 < c_0 < 1 \leq C_0$, and M_0 , depending only on K and d_0 , such that if $M \geq M_0$, then the variances $v_i(t)$ of the covariance evolution of A and the entries $C_i(t)$ in (14.34) satisfy (i)–(iv) of Lemma 14.6 with $j = 1$. Moreover, after enlarging M_0 , there exists C , depending only on K and d_0 , such that*

$$\mathcal{E}(t) \leq \frac{C}{1+t} \quad (0 \leq t \leq d^{-1}), \quad \mathcal{E}(d^{-1} + s) \leq Cd(1+ds)^2 e^{-4ds} \quad (s \geq 0). \quad (14.39)$$

Proof. The proof has five steps: bounds on $\bar{h}$ and its derivatives, the comparison (iii) of the variances, the bound (i) on the coefficient differences at times $t \leq d^{-1}$, the bound (ii) at later times, and the variance conditions (iv). Throughout, $M \geq M_0$, and we write $a(t) = \sqrt{d} + (1+t)^{-1/2}$. Positive lower-bound constants denoted by c_* depend only on K and d_0 and may change from line to line.

Step 1: the function $\bar{h}$. Recall from Lemma 7.11 that, for all $t \geq 0$ and all i ,

$$\bar{h}(t) \asymp a(t), \quad h_i(t) \asymp a(t) \frac{1+t}{1+p_it}, \quad (14.40)$$

where $X \asymp Y$ means $C^{-1}Y \leq X \leq CY$. Also $\vartheta \asymp \sqrt{d}$ by Lemma 7.10, and differentiating (14.2), we have $\bar{h}^{(n)}(t) = (-1)^n \int x^n e^{-tx} \rho(dx)$ for $n \geq 1$. For $t \geq 1$, we use $x^n e^{-tx} \leq (2n/(et))^n e^{-tx/2}$ and $\bar{h}(t/2) \leq C\bar{h}(t)$, which follows from (14.40) and $a(t/2) \leq \sqrt{2}a(t)$. For $t \leq 1$, we use that ρ is supported in $[0, \|H\|]$, that $\|H\| \leq 2p_N + 2 \leq 2(d_0 + K + 1)$, and that $\bar{h}(t) \geq \bar{h}(1) \geq c_*$. This gives

$$|\bar{h}^{(n)}(t)| \leq C_n(1+t)^{-n} \bar{h}(t) \quad (t \geq 0, n = 1, 2). \quad (14.41)$$

For $t = d^{-1} + s$, we use $e^{-x(d^{-1}+s)} \leq e^{-xd^{-1}} e^{-2ds}$ on $[2d, \infty)$ and the previous bound at time d^{-1} to obtain $|\bar{h}^{(n)}(d^{-1} + s)| \leq C_n d^n \bar{h}(d^{-1}/2) e^{-2ds} \leq C_n d^{n+1/2} e^{-2ds}$. Recall that $\zeta = \bar{h}'/\bar{h}$. As $\bar{h}$ is positive and decreasing, we have $\zeta \leq 0$. By the Cauchy–Schwarz inequality for the measure $\vartheta\delta_0 + \rho$, we also have $\bar{h}''\bar{h} \geq (\bar{h}')^2$, that is, $\zeta' \geq 0$. Using (14.41) and $\bar{h} \geq \vartheta \geq c_*\sqrt{d}$, we conclude that

$$\begin{aligned} |\zeta(t)| &\leq \frac{C}{1+t}, \quad 0 \leq \zeta'(t) \leq \frac{C}{(1+t)^2} \quad (t \geq 0), \\ 0 \leq \zeta'(d^{-1} + s) &\leq C d^2 e^{-2ds} \quad (s \geq 0). \end{aligned} \quad (14.42)$$

Step 2: the variances. The comparison (iii) follows from (14.40), since $v_i = h_i/\bar{h}$. In particular, $c_* \leq v_i(t) \leq C(1+t)$ for all t . For $t \geq d^{-1}$, we have $p_it \geq 1$, so that $(1+t)/(1+p_it)$ lies between $1/(2p_i)$ and $2/p_i$; hence $v_i(t) \asymp p_i^{-1}$ and $1 + e_i(t) = p_i v_i(t) \geq c_*$.

Step 3: the coefficient differences at times $t \leq d^{-1}$. Recall from (14.34) that $2C_i(t) = h'_i(t)/h_i(t) - \zeta(t)$. Fix $t \geq 1$ and an index i with $p_it \geq 1$. For $r \geq 0$, let $g(r) = h'_i(r) - \zeta(t)h_i(r)$. Differentiating $h'_i = -2p_i h_i + 2\bar{h}$, we find $g'(r) = -2p_i g(r) + 2\bar{h}(r)(\zeta(r) - \zeta(t))$. Solving this linear equation on $[t/2, t]$ and substituting $r = t - s$, we obtain

$$h'_i(t) - \zeta(t)h_i(t) = e^{-p_it} g(t/2) + 2 \int_0^{t/2} e^{-2p_is} \bar{h}(t-s) (\zeta(t-s) - \zeta(t)) ds.$$

We divide by $h_i(t)$. From (14.40), we have $h_i(t/2) \leq C h_i(t)$ and, as $1 + p_it \leq 2p_it$, also $\bar{h}(t/2) \leq C\bar{h}(t) \leq C p_i h_i(t)$. With $|h'_i| \leq 2p_i h_i + 2\bar{h}$ and (14.42), the first term divided by $h_i(t)$ is at most $C(p_i + t^{-1})e^{-p_it} \leq C/(p_it^2)$, where we used $r^2 e^{-r} \leq C$ and $re^{-r} \leq C$ with $r = p_it \geq 1$. In the integral, $|\zeta(t-s) - \zeta(t)| \leq Cs/t^2$ for $s \leq t/2$ by (14.42), and $\bar{h}(t-s) \leq \bar{h}(t/2) \leq C p_i h_i(t)$. As $\int_0^\infty s e^{-2p_is} ds = (4p_i^2)^{-1}$, the integral divided by $h_i(t)$ is at most $C/(p_it^2)$. We conclude that

$$|C_i(t)| \leq \frac{C}{p_it^2} \leq \frac{4C}{(1+t)(1+p_it)} \quad (t \geq 1, p_it \geq 1).$$

If $t \geq 1$ and $p_i t < 1$, then $2C_i(t) = -2p_i + 2\bar{h}(t)/h_i(t) - \zeta(t)$, where $p_i < t^{-1}$, $\bar{h}(t)/h_i(t) \leq C(1 + p_i t)/(1 + t) \leq 2C/t$ by (14.40), and $|\zeta(t)| \leq C/t$. Together, these bounds give $|C_i(t)| \leq C/t \leq 4C(1 + t)^{-1}(1 + p_i t)^{-1}$. If $t < 1$, then $|C_i(t)| \leq v_i(t)^{-1} + p_i + |\zeta(t)| \leq C$, while $(1 + t)(1 + p_i t) \leq 2(1 + d_0 + K)$. This proves (i) for $0 \leq t \leq d^{-1}$.

Step 4: the coefficient differences at times $t \geq d^{-1}$. Let $t = d^{-1} + s$ with $s \geq 0$. The hypotheses of (14.12) hold by (14.33). Using also $1 + e_i(t) \geq c_*$ from Step 2, we find

$$\left| \frac{1}{v_i(t)} - p_i \right| = \frac{p_i |e_i(t)|}{1 + e_i(t)} \leq Cd(1 + ds)e^{-2ds}. \quad (14.43)$$

Differentiating (14.34) and using $v'_i = 2C_i v_i$, we find the linear equation

$$C'_i(t) = -\frac{2C_i(t)}{v_i(t)} - \frac{\zeta'(t)}{2}.$$

For $0 \leq s_1 \leq s_2$, (14.43) implies that $\int_{s_1}^{s_2} 2v_i(d^{-1} + r)^{-1} dr \geq 2p_i(s_2 - s_1) - 2C \int_0^\infty d(1 + dr)e^{-2dr} dr = 2p_i(s_2 - s_1) - \frac{3}{2}C$. Hence the integrating factor of this equation from $d^{-1} + s_1$ to $d^{-1} + s_2$ is at most $Ce^{-2p_i(s_2 - s_1)}$. From (i) at $t = d^{-1}$, we have $|C_i(d^{-1})| \leq C(1 + d^{-1})^{-1}(1 + p_i d^{-1})^{-1} \leq Cd^2/p_i$. With (14.42), we obtain

$$\begin{aligned} |C_i(d^{-1} + s)| &\leq C \frac{d^2}{p_i} e^{-2p_i s} + C \int_0^s e^{-2p_i(s-r)} d^2 e^{-2dr} dr \\ &\leq C \frac{d^2}{p_i} e^{-2ds} + Cd^2 e^{-2ds} \min \left\{ s, \frac{1}{2(p_i - d)} \right\}, \end{aligned}$$

where the minimum is s if $p_i = d$. In the last step, we wrote $e^{-2p_i(s-r)-2dr} = e^{-2ds} e^{-2(p_i-d)(s-r)}$ and integrated. If $p_i \leq 2d$, then $d^2 s \leq 2(d^2/p_i) ds$. If $p_i > 2d$, then $p_i - d > p_i/2$ and $d^2/(2(p_i - d)) \leq d^2/p_i$. In both cases,

$$|C_i(d^{-1} + s)| \leq C \frac{d^2}{p_i} (1 + ds) e^{-2ds},$$

which is (ii) with $j = 1$.

Step 5: the variance conditions. Let $0 \leq t \leq d^{-1}$, $d_* = (1 + t)^{-1}$, and $M_* = N(1 + t)^{-3/2} \geq 2^{-3/2}M$. We have $c_* \leq v_i(t) \leq Cd_*^{-1}$ by Step 2, and (iii) and (14.38) imply that $\sum_i v_i(t)^k \leq C(1 + t)^k \sum_i (1 + p_i t)^{-k} \leq CN(1 + t)^{k-3/2}$ for $k = 2, 3, 4$; these are the required upper bounds. For the lower bound on $\sum_i v_i^2$ and for the set $\mathcal{S}$, consider the indices with $i - 1 \leq \epsilon N(1 + t)^{-3/2}$, where $\epsilon \in (0, 1]$. For such i , we have $p_i \leq d + K\epsilon^{2/3}(1 + t)^{-1} \leq (2 + K)(1 + t)^{-1}$ by (14.33) and $d \leq 2(1 + t)^{-1}$, and then $v_i(t) \geq c_*(1 + t)$ by (iii). Taking $\epsilon = K^{-3/2}$, we obtain at least $K^{-3/2}M_*$ indices with $v_i(t) \geq c_*(1 + t)$, which gives

$\sum_i v_i(t)^2 \geq c_* N(1+t)^{1/2}$. Let $\mathcal{S}$ consist of the first $\lceil \epsilon M_* \rceil$ indices, with $\epsilon \leq K^{-3/2}$ to be chosen. Then $v_i(t) \geq c_*(1+t) = c_* d_*^{-1}$ for $i \in \mathcal{S}$, and if $\epsilon M_* \geq 1$,

$$\sum_{i \in \mathcal{S}} v_i(t)^2 \leq 2\epsilon M_* C^2 (1+t)^2 = 2\epsilon C^2 N(1+t)^{1/2}.$$

We choose ϵ , depending only on K and d_0 , so small that the right side is at most half of the lower bound on $\sum_i v_i(t)^2$; then the complement of $\mathcal{S}$ carries at least half of $\sum_i v_i(t)^2$. The condition $\epsilon M_* \geq 1$ holds when $M \geq M_0$.

Let $t \geq d^{-1}$, $d_* = d$, and $M_* = M$. Step 2 and (14.38) yield $c_* \leq v_i(t) \leq C/p_i \leq Cd^{-1}$ and $\sum_i v_i(t)^k \leq C \sum_i p_i^{-k} \leq CNd^{3/2-k}$. The indices with $i-1 \leq \epsilon M$ satisfy $p_i \leq (1 + K\epsilon^{2/3})d$, so that $v_i(t) \geq c_* d^{-1}$. The rest of the argument is the same as above. This proves (iv), with constants depending only on K and d_0 .

Finally, Steps 2–5 and Lemma 14.6 with $j = 1$ yield (14.39). $\square$

The initial law weighted by $1/\varphi$. As in (9.2), let

$$\varphi(x) = \frac{1}{N} \sum_{i=1}^N q_i x_i^2, \quad \text{and recall} \quad \frac{1}{\vartheta} = \frac{1}{N} \sum_{i=1}^N q_i^2,$$

which is the value of φ at $x_i^2 = q_i$. On S_N we have $1/p_N \leq \varphi \leq 1/d$, and computing its tangential gradient, we find

$$\Gamma(\log \varphi)(x) = \frac{4}{N} \left(\frac{N \sum_i q_i^2 x_i^2}{(\sum_i q_i x_i^2)^2} - 1 \right) \geq 0.$$

We write $X = (X_s)_{s \geq 0}$ for the diffusion with generator L_A , and $\mathbb{E}_x$ and $\mathbb{E}_\mu$ for expectations when $X_0 = x$ and when X_0 has law μ . The following three estimates are proved in Section 10 for the unnormalized process Y with quadratic potential coefficients $\rho_i = p_i/2$, the half-precisions. Its normalization $\sqrt{N}Y/|Y|$ is the diffusion X started from $\sqrt{N}Y_0/|Y_0|$, and the quantities below depend only on the path of X . For every $\lambda > 0$ and $r > 0$, there exist C and M_0 , depending only on K , d_0 , C_T , λ , and r , such that if $M \geq M_0$, then for every $u \geq 0$,

$$\sup_{x \in S_N} \mathbb{E}_x \exp \left(\lambda \int_0^{T_*} \Gamma(\log \varphi)(X_s) \, ds \right) \leq C, \quad (14.44)$$

$$\mathbb{E}_{\mu_u} \left[\sup_{0 \leq s \leq T_*} (\vartheta \varphi(X_s))^r \right] \leq C, \quad (14.45)$$

$$\mu_u \left[(\varphi \bar{h}(u))^r + (\varphi \bar{h}(u))^{-r} \right] \leq C. \quad (14.46)$$

The bound (14.44) is Theorem 10.15, and (14.45) is the last bound of Theorem 10.18. For (14.46), let X_0 have law μ_u and $Y_i(0) = X_i(0)/\varphi(X_0)^{1/2}$, as in Lemma 10.16. As $h(0) = \mathbf{1}$

and $|X_0|^2 = N$, we have $\sum_i h_i(0) Y_i(0)^2 = N/\varphi(X_0)$. Using Lemma 10.16 at time 0, we obtain a random variable $C_{\text{init}} \geq 1$ with $\mathbb{E} C_{\text{init}}^r \leq C_r$ and $C_{\text{init}}^{-1} \leq (\varphi(X_0) \bar{h}(u))^{-1} \leq C_{\text{init}}$, which implies (14.46).

The weighted gradient estimate (14.47) below controls $\varphi \Gamma(P_\tau^A f)$ rather than $\Gamma(P_\tau^A f)$. To bound $\mu_u \Gamma(P_\tau^A f)$, we divide by φ , which replaces the initial law μ_u by the following law.

Lemma 14.8 (The initial law weighted by $1/\varphi$). *Assume (14.33). For $u \geq 0$, let*

$$Z_u = \mu_u \left(\frac{1}{\vartheta \varphi} \right), \quad \tilde{\mu}_u(dx) = \frac{1}{Z_u} \frac{1}{\vartheta \varphi(x)} \mu_u(dx).$$

There exist C and M_0 such that if $M \geq M_0$, then for every $u \geq 0$,

$$\frac{\bar{h}(u)}{C\vartheta} \leq Z_u \leq \frac{C\bar{h}(u)}{\vartheta}, \quad \left\| \frac{d\tilde{\mu}_u}{d\mu_u} \right\|_{L^2(\mu_u)} \leq C.$$

Further, for every $r > 0$ and $\lambda > 0$, there exist $C_{r,\lambda}$ and $M_0(r, \lambda)$ such that if $M \geq M_0(r, \lambda)$, then for every $u \geq 0$,

$$\mathbb{E}_{\tilde{\mu}_u} \left[\sup_{0 \leq s \leq T_*} (\vartheta \varphi(X_s))^r \right] \leq C_{r,\lambda}, \quad \mathbb{E}_{\tilde{\mu}_u} \exp \left(\lambda \int_0^{T_*} \Gamma(\log \varphi)(X_s) ds \right) \leq C_{r,\lambda}.$$

Proof. We have $Z_u = \vartheta^{-1} \bar{h}(u) \mu_u((\varphi \bar{h}(u))^{-1})$ and

$$\frac{d\tilde{\mu}_u}{d\mu_u} = \frac{(\varphi \bar{h}(u))^{-1}}{\mu_u((\varphi \bar{h}(u))^{-1})}.$$

Using (14.46) with $r = 1$, we have $\mu_u((\varphi \bar{h}(u))^{-1}) \leq C$ and $\mu_u(\varphi \bar{h}(u)) \leq C$, and by Jensen's inequality, $\mu_u((\varphi \bar{h}(u))^{-1}) \geq \mu_u(\varphi \bar{h}(u))^{-1} \geq C^{-1}$. This proves the bounds on Z_u . With $r = 2$, (14.46) implies that the $L^2(\mu_u)$ norm of $d\tilde{\mu}_u/d\mu_u$ is at most $C^{1/2} \cdot C$.

Let $\mathcal{B} \geq 0$ be a measurable function of the path $(X_s)_{0 \leq s \leq T_*}$. Conditionally on X_0 , the law of the path is the same whether X_0 has law μ_u or $\tilde{\mu}_u$. Using the Cauchy–Schwarz inequality, we find

$$\mathbb{E}_{\tilde{\mu}_u} \mathcal{B} = \mathbb{E}_{\mu_u} \left[\frac{d\tilde{\mu}_u}{d\mu_u}(X_0) \mathcal{B} \right] \leq C (\mathbb{E}_{\mu_u} \mathcal{B}^2)^{1/2}.$$

With $\mathcal{B} = \sup_{s \leq T_*} (\vartheta \varphi(X_s))^r$, the first claim follows from (14.45) with order $2r$. The second claim holds for every initial law, since (14.44) is a supremum over initial points. $\square$

We use the weighted gradient estimate of Lemma 10.21: for every smooth g on S_N , every $x \in S_N$, and every $s \geq 0$,

$$\varphi(x) \Gamma(P_s^A g)(x) \leq e^{-2\bar{r}s} \mathbb{E}_x \left[\exp \left(2 \int_0^s \Gamma(\log \varphi)(X_v) dv \right) \varphi(X_s) \Gamma(g)(X_s) \right]. \quad (14.47)$$

It follows from the Bochner identity for the Dirichlet form weighted by φ , in which the curvature term is at least $\bar{r}$ after a square is completed, together with the Feynman–Kac formula; the term $\Gamma(\log \varphi)$ is what remains of the square. It is controlled along the paths of the diffusion by (14.44).

Lemma 14.9 (Gradient bounds under the comparison laws). *Assume (14.33). There exist C and M_0 such that if $M \geq M_0$, then $d/2 \leq \bar{r} \leq d$, and for every $u \geq 0$, every $0 < \tau \leq T_*$, and every smooth $f : S_N \rightarrow [0, 1]$,*

$$\mu_u \Gamma(P_\tau^A f) \leq \frac{C \bar{r} Z_u}{e^{2\bar{r}\tau} - 1} \leq \frac{C \bar{r} \bar{h}(u)}{\vartheta (e^{2\bar{r}\tau} - 1)}. \quad (14.48)$$

The factor $\bar{h}(u)$, which is of order $(1+u)^{-1/2}$ for $u \leq d^{-1}$, comes from the normalization Z_u of the weighted initial law. In the proof of Theorem 14.10, it makes the contribution of times $u \leq d^{-1}$ bounded independently of d .

Proof. Note that $Nd = Md^{-1/2} \geq M$. Then $2/N \leq d/2$ when $M \geq 4$, and $d/2 \leq \bar{r} \leq d$. The proof has three steps.

Step 1: integration of (14.47). Fix $\tau \in (0, T_*]$ and a smooth $f : S_N \rightarrow [0, 1]$. For $0 \leq s \leq \tau$, we apply (14.47) to $g = P_{\tau-s}^A f$, multiply by $e^{2\bar{r}s}$, and integrate over $s \in [0, \tau]$. Since $P_s^A P_{\tau-s}^A = P_\tau^A$, the left side becomes $\varphi(x) \Gamma(P_\tau^A f)(x) (e^{2\bar{r}\tau} - 1) / (2\bar{r})$. On the right side, we use $\Gamma(\log \varphi) \geq 0$ to bound the exponential by $e^{2\mathcal{O}_\tau}$ and $\varphi(X_s)$ by $\vartheta^{-1} \varphi_\tau^*$, where

$$\mathcal{O}_\tau = \int_0^\tau \Gamma(\log \varphi)(X_v) dv, \quad \varphi_\tau^* = \sup_{0 \leq s \leq \tau} \vartheta \varphi(X_s), \quad \mathcal{V}_\tau = \int_0^\tau \Gamma(P_{\tau-s}^A f)(X_s) ds.$$

By Tonelli's theorem, we obtain for every $x \in S_N$

$$\varphi(x) \Gamma(P_\tau^A f)(x) \frac{e^{2\bar{r}\tau} - 1}{2\bar{r}} \leq \frac{1}{\vartheta} \mathbb{E}_x [e^{2\mathcal{O}_\tau} \varphi_\tau^* \mathcal{V}_\tau]. \quad (14.49)$$

Step 2: the quadratic variation. Let $m_s = P_{\tau-s}^A f(X_s)$ for $0 \leq s \leq \tau$. Using Itô's formula and the backward equation $\partial_s P_{\tau-s}^A f = -L_A P_{\tau-s}^A f$, we see that m is a continuous martingale. Its quadratic variation is $[m]_s = 2 \int_0^s \Gamma(P_{\tau-v}^A f)(X_v) dv$, because the noise coefficient of the diffusion is $\sqrt{2}$; see [123, Chapter IV, Section 3]. In particular $[m]_\tau = 2\mathcal{V}_\tau$. As $0 \leq f \leq 1$, we have $0 \leq m_s \leq 1$ and $\sup_s |m_s - m_0| \leq 1$, and the Burkholder–Davis–Gundy inequality with exponent 6 [123, Chapter IV, Section 4] implies that

$$\mathbb{E}_x \mathcal{V}_\tau^3 \leq C \quad (x \in S_N), \quad (14.50)$$

where C is a numerical constant. The same bound holds for every initial law.

Step 3: averaging over the initial law. We divide (14.49) by $\varphi(x)$ and integrate over x with respect to μ_u . By the definition of $\tilde{\mu}_u$, we have $\int \mu_u(dx) (\vartheta\varphi(x))^{-1} \mathbb{E}_x[\cdot] = Z_u \mathbb{E}_{\tilde{\mu}_u}[\cdot]$, and hence

$$\mu_u \Gamma(P_\tau^A f) \frac{e^{2\bar{r}\tau} - 1}{2\bar{r}} \leq Z_u \mathbb{E}_{\tilde{\mu}_u} [e^{2\mathcal{O}_\tau} \varphi_\tau^* \mathcal{V}_\tau] \leq Z_u (\mathbb{E}_{\tilde{\mu}_u} e^{6\mathcal{O}_\tau})^{1/3} (\mathbb{E}_{\tilde{\mu}_u} (\varphi_\tau^*)^3)^{1/3} (\mathbb{E}_{\tilde{\mu}_u} \mathcal{V}_\tau^3)^{1/3},$$

where we used Hölder's inequality with three exponents equal to 3. The quantities $\mathcal{O}_\tau$ and φ_τ^* are at most their values at T_* , because $\tau \leq T_*$. Using Lemma 14.8 with $\lambda = 6$ and $r = 3$, together with (14.50), we find that the three expectations are bounded by a constant. This proves the first inequality in (14.48), and the second follows from the bound $Z_u \leq C\bar{h}(u)/\vartheta$ of Lemma 14.8. $\square$

We now prove the main estimate of this section. The constants are chosen in the following order.

Quantity	Role	Dependence
K, d_0, C_T	hypotheses (14.33) and time T_*	given
$\lambda = 6, r = 3$	orders in Lemma 14.8	fixed
M_0	threshold for Lemmas 14.7 to 14.9	K, d_0, C_T
C	constant in (14.51)	K, d_0, C_T
N, A, t	any, with $M \geq M_0$ and $0 \leq t \leq T_*$	last

Theorem 14.10 (Distance to the Bingham comparison law on $[0, C_T d^{-1} \log M]$). *For every $K \geq 1$, $d_0 \in (0, 1)$, and $C_T > 0$, there exist constants $C \geq 1$ and $M_0 \geq 4$, depending only on K , d_0 , and C_T , with the following property. Let $N \geq 2$, and let A be a real symmetric $N \times N$ matrix whose precisions $p_1 \leq \dots \leq p_N$ satisfy (14.33) and $M = Nd^{3/2} \geq M_0$, where $d = p_1$. Let μ_t be the Bingham comparison law of A . Then for $0 \leq t \leq C_T d^{-1} \log M$,*

$$\|v_N P_t^A - \mu_t\|_{TV} \leq \min \{1, C e^{-(d-2/N)t}\}. \quad (14.51)$$

The bound contains no factor depending on d or on Nd^2 . At times with $dt \geq \frac{1}{8} \log M$ and $t \leq C_T d^{-1} \log M$, we have $2t/N \leq 2C_T \log M / (Nd) \leq 2C_T \log M / M$, and the right side of (14.51) is at most $C e^{2C_T} M^{-1/8}$; at $dt = \frac{1}{4} \log M + O(\log \log M)$ it is $M^{-1/4} (\log M)^{O(1)}$. In both cases it tends to zero as $M \rightarrow \infty$, however slowly.

Proof. We first bound the distance by a Duhamel integral, using part (c) of Theorem 9.5, and then estimate the integral. Let $M \geq M_0$, and let $T_* = C_T d^{-1} \log M$ and $\bar{r} = d - 2/N \in [d/2, d]$ be as above.

Step 1: the Duhamel bound. Fix $t \in (0, T_*]$. We apply part (c) of Theorem 9.5 to the family $(\mu_u)_{u \geq 0}$, whose forward-equation defect is R_u by (14.35), and to the semigroup P_t^A . By Corollary 9.7, the flux hypothesis (9.20) holds with $j_u = \nabla_{S_N} \psi_u$ and with $\mathcal{J}(u)$ equal to the right side of (14.39), that is, $\mathcal{J}(u) = C(1+u)^{-1}$ for $u \leq d^{-1}$ and $\mathcal{J}(d^{-1} + s) = Cd(1+ds)^2 e^{-4ds}$. For the gradient hypothesis, let

$$\mathcal{G}(u, \tau) = \frac{C \bar{r} \bar{h}(u)}{\vartheta(e^{2\bar{r}\tau} - 1)} \quad (u, \tau \in (0, t)),$$

where C is the constant of Lemma 14.9. For $\tau < t \leq T_*$, that lemma states that $\mu_u \Gamma(P_\tau^A f) \leq \mathcal{G}(u, \tau)$ for every smooth $f : S_N \rightarrow [0, 1]$. Both $\mathcal{J}$ and $\mathcal{G}$ are Borel measurable. As $\mu_0 = \nu_N$, (9.21) gives

$$\|v_N P_t^A - \mu_t\|_{\text{TV}} \leq \int_0^t \sqrt{\mathcal{J}(u) \mathcal{G}(u, t-u)} \, du.$$

We have $\mathcal{G}(u, t-u) = C \bar{r} \vartheta^{-1} \bar{h}(u) e^{-2\bar{r}(t-u)} (1 - e^{-2\bar{r}(t-u)})^{-1}$, which yields

$$\|v_N P_t^A - \mu_t\|_{\text{TV}} \leq C e^{-\bar{r}t} \int_0^t \frac{e^{\bar{r}u} \sqrt{\bar{r} \vartheta^{-1} \bar{h}(u) \mathcal{J}(u)}}{\sqrt{1 - e^{-2\bar{r}(t-u)}}} \, du \quad (0 < t \leq T_*). \quad (14.52)$$

Step 2: the numerator. We have $\bar{r} \leq d$, $\vartheta^{-1} \leq Cd^{-1/2}$ by Lemma 7.10, and $\bar{h}(u) \leq C(\sqrt{d} + (1+u)^{-1/2})$ by Lemma 7.11. For $0 \leq u \leq d^{-1}$, we have $\sqrt{d} \leq \sqrt{2}(1+u)^{-1/2}$, $e^{\bar{r}u} \leq e$, and $\mathcal{J}(u) = C(1+u)^{-1}$. For $u = d^{-1} + s$ with $s \geq 0$, we have $\bar{h}(u) \leq C\sqrt{d}$, $e^{\bar{r}u} \leq e^{1+ds}$, and $\mathcal{J}(u) = Cd(1+ds)^2 e^{-4ds}$. We conclude that

$$e^{\bar{r}u} \sqrt{\bar{r} \vartheta^{-1} \bar{h}(u) \mathcal{J}(u)} \leq \begin{cases} Cd^{1/4} (1+u)^{-3/4}, & 0 \leq u \leq d^{-1}, \\ Cd (1+ds) e^{-ds}, & u = d^{-1} + s, s \geq 0. \end{cases} \quad (14.53)$$

The first line integrates over $[0, d^{-1}]$ to at most $4Cd^{1/4}(1+d^{-1})^{1/4} \leq 4 \cdot 2^{1/4}C$, and the second over $s \in [0, \infty)$ to $C \int_0^\infty (1+r) e^{-r} \, dr = 2C$. Thus

$$\int_0^\infty e^{\bar{r}u} \sqrt{\bar{r} \vartheta^{-1} \bar{h}(u) \mathcal{J}(u)} \, du \leq C. \quad (14.54)$$

Step 3: the denominator. Let $2/\bar{r} \leq t \leq T_*$, and split the integral in (14.52) at $u = t - 1/\bar{r}$. For $u \leq t - 1/\bar{r}$, the denominator is at least $(1 - e^{-2})^{1/2}$, and by (14.54) this part of the integral is at most $C(1 - e^{-2})^{-1/2}$. For $t - 1/\bar{r} \leq u \leq t$, we have $u \geq 1/\bar{r} \geq d^{-1}$, so the second line of (14.53) applies, and since $(1+r)e^{-r} \leq 1$, the numerator is at most Cd . By concavity, $1 - e^{-2y} \geq (1 - e^{-2})y$ for $0 \leq y \leq 1$, and this part of the integral is at most

$$Cd \int_0^{1/\bar{r}} \frac{dv}{\sqrt{(1 - e^{-2})\bar{r}v}} = \frac{2Cd}{\sqrt{1 - e^{-2}} \bar{r}} \leq \frac{4C}{\sqrt{1 - e^{-2}}}$$

where we used $\bar{r} \geq d/2$. By (14.52), we conclude that $\|v_N P_t^A - \mu_t\|_{\text{TV}} \leq C' e^{-\bar{r}t}$ for $2/\bar{r} \leq t \leq T_*$, with C' depending only on K, d_0 , and C_T . If $0 \leq t < 2/\bar{r}$, then $e^{-\bar{r}t} > e^{-2}$, and the total variation distance is at most $1 \leq e^2 e^{-\bar{r}t}$. Taking $C = \max(C', e^2)$ proves (14.51). $\square$

Remark 14.11 (The parameters M and M_N). In Theorem 14.10, $M = Nd^{3/2}$ is defined from the smallest precision of a fixed matrix, and the estimate holds for every N and every such matrix with $M \geq M_0$. For $A = c_N W_N$ in the regime (14.4), $M = Nd_N^{3/2}$ is random, and it is related to the deterministic $M_N = N\delta_N^3$ by $Nd_N^{3/2}/M_N \rightarrow 1$ in probability, by (14.9); more precisely, $\ell_N |Nd_N^{3/2}/M_N - 1| \rightarrow 0$. In Section 20, Theorem 14.10 is applied to deterministic matrices with M fixed while $N \rightarrow \infty$. The estimate requires no relation between N and d beyond $M \geq M_0$, and it therefore applies however slowly M_N tends to infinity.

14.4 Proof of the profile

Proof of Theorem 14.1. We first prove (14.6). Fix $u_0 > 0$ and $\varepsilon_0 > 0$. The parameters are chosen in the following order.

Parameter	Role	When it is fixed
u_0, ε_0	range of offsets and tolerance	first
$\eta \in (0, 1)$	confidence of the events G^η in (14.11)	second
$K = K_\eta$	constant in (14.11)	depends on η
$d_0 = 1/2, C_T = 1$	constants in (14.33) and in T_*	absolute
C, M_0	constants of Theorem 14.10	depend on K_η
$N \rightarrow \infty$	limit in probability	third
$\eta \rightarrow 0$	removal of the localization (Lemma 3.8)	last

Let $X_N = \sup_{|u| \leq u_0} |D_N(T_N(u)) - \Psi(u)|$. Fix $\eta \in (0, 1)$, and let C and M_0 be the constants of Theorem 14.10 for $K = K_\eta, d_0 = 1/2$, and $C_T = 1$. Let $\mathcal{E}_N$ be the intersection of G^η with the event that

$$d_N \leq \frac{1}{2}, \quad Nd_N^{3/2} \geq M_0, \quad \frac{1}{8} \log M_N \leq d_N T_N(u) \leq \log(Nd_N^{3/2}) \quad \text{for all } |u| \leq u_0.$$

We claim that $\mathbb{P}(G^\eta \setminus \mathcal{E}_N) \rightarrow 0$. Indeed, by (14.9), $d_N \rightarrow 0$ and $Nd_N^{3/2}/M_N \rightarrow 1$ in probability. Further, $\delta_N^2 T_N(u) = \frac{1}{4} \log M_N (1 + o(1))$ uniformly for $|u| \leq u_0$, so $d_N T_N(u) / \log M_N = (d_N / \delta_N^2) \delta_N^2 T_N(u) / \log M_N \rightarrow 1/4$ in probability, uniformly for $|u| \leq u_0$, while $\log(Nd_N^{3/2}) / \log M_N \rightarrow 1$.

On $\mathcal{E}_N$, the matrix $A = c_N W_N$ satisfies the hypotheses of Theorem 14.10. Indeed, its precisions in increasing order are $p_i = d_N + \gamma_i$, and (14.11) and $d_N \leq 1/2$ give (14.33). Further $M = Nd_N^{3/2} \geq M_0$, and $0 < T_N(u) \leq d_N^{-1} \log M$ for $|u| \leq u_0$. By (14.51), on $\mathcal{E}_N$,

$$\sup_{|u| \leq u_0} \|\nu_N P_{T_N(u)} - \mu_{T_N(u)}\|_{\text{TV}} \leq C \sup_{|u| \leq u_0} e^{-d_N T_N(u) + 2T_N(u)/N} \leq C e^{2T_N(u_0)/N} M_N^{-1/8}.$$

The number $T_N(u_0)/N$ is deterministic and is at most $C\delta_N^{-2} \log M_N/N = C\delta_N \log M_N/M_N$, which tends to zero. In particular, the supremum is at most $2CM_N^{-1/8}$ on $\mathcal{E}_N$ for large N . Using the triangle inequality, we have

$$X_N \leq \sup_{|u| \leq u_0} \|\nu_N P_{T_N(u)} - \mu_{T_N(u)}\|_{\text{TV}} + \sup_{|u| \leq u_0} \left| \|\mu_{T_N(u)} - \pi_{N, c_N}\|_{\text{TV}} - \Psi(u) \right|.$$

For N so large that $2CM_N^{-1/8} < \varepsilon_0/2$, we therefore have

$$\mathbb{P}(G^\eta \cap \{X_N > \varepsilon_0\}) \leq \mathbb{P}(G^\eta \setminus \mathcal{E}_N) + \mathbb{P}\left(\sup_{|u| \leq u_0} \left| \|\mu_{T_N(u)} - \pi_{N, c_N}\|_{\text{TV}} - \Psi(u) \right| > \frac{\varepsilon_0}{2}\right),$$

and both terms tend to zero, the second by Theorem 14.4. Now Lemma 3.8 implies that $X_N \rightarrow 0$ in probability, which is (14.6).

The center and the window are deterministic, and

$$\begin{aligned} \frac{4\delta_N^2 T_N}{\log M_N} &= 1 - \frac{3 \log \log M_N}{2 \log M_N} + \frac{\log B_{c_N}}{\log M_N} \rightarrow 1, \\ \frac{\delta_N^{-2}}{T_N} &= \frac{4}{\log M_N - \frac{3}{2} \log \log M_N + \log B_{c_N}} \rightarrow 0. \end{aligned}$$

Together with (14.6), this shows that D_N has cutoff profile Ψ with center T_N and window δ_N^{-2} in the sense of Definition 3.7. Recall from Lemma 3.9 that Ψ is continuous and strictly decreasing from 1 to 0, with $\Psi^{-1}(\varepsilon) = -\frac{1}{2} \log[2\Phi^{-1}((1+\varepsilon)/2)]$. Then Corollary 3.12 implies that $(t_N(\varepsilon) - T_N)/\delta_N^{-2} \rightarrow \Psi^{-1}(\varepsilon)$ in probability for every $\varepsilon \in (0, 1)$, which is (14.7), and there is cutoff. $\square$

The infinite-temperature crossover

In this section, we let the rescaled inverse temperature $c = c_N$ tend to zero and prove Theorem 1.2 and the first case of Theorem 1.1. The section has a deterministic part and a part about the GOE.

In the deterministic part, Theorem 15.1, G is a fixed symmetric matrix of bounded norm, and $A = cG$ with c small. We show that the time- t law $\nu_N P_t^A$ is within $O(c^2)$ of the Bingham comparison law μ_t uniformly in time, that the information $\mathcal{I}_N(t)$ of the covariance evolution (Definition 3.5) decays at rate $4 + O(c)$, and that $D_N(t)$ is close to $2\Phi(\sqrt{\mathcal{I}_N(t)}/2) - 1$ uniformly in $t \geq 0$. The distance is determined by the single function $\mathcal{I}_N$. If $\mathcal{I}_N(0) \rightarrow \infty$, there is cutoff at the time τ_N at which $\mathcal{I}_N(\tau_N) = 1$, with a window of order one.

For the GOE, we show in Theorem 15.2 that $\mathcal{I}_N(0)$ is asymptotic to $J_N/2$, where $J_N = Nc_N^2$. If J_N stays bounded, the distance relaxes over a bounded time and there is no cutoff. If $J_N \rightarrow \infty$, there is cutoff at τ_N . In Theorem 15.3, we determine τ_N up to $o(1)$. In the regime $J_N \rightarrow \infty$, let $h_N = (c_N/4) \log J_N$ for all sufficiently large N . When h_N is bounded, the information at times of order $\log J_N$ is governed by the orthogonal compression of multiplication by the eigenvalues to the subspace of vectors with mean zero, and its semicircle limit is computed with Chebyshev polynomials. The resulting center involves the modified Bessel function I_1 . When $h_N \rightarrow \infty$, the relative errors are concentrated near the spectral edge. The computation of Section 13 holds uniformly for small c , and it yields the center of Theorem 13.1 evaluated at $c = c_N$. Table 15.1 summarizes these results.

We use the covariance evolution of Section 7 (Lemma 7.6), Lemma 9.1 and Proposition 9.14, and the results of Section 3.5. From Section 13, we use the uniform form of the likelihood limit (Lemma 13.2), which rests on Theorem 6.8. About the GOE, the proof of Theorem 15.2 uses only the norm of W_N and the traces of W_N and W_N^2 (Lemma 15.8). The proof of Theorem 15.3 uses in addition the rigidity estimate of Lemma 4.2, through Lemma 13.8, the computations for the semicircle law (Proposition 13.7 and Lemma 13.11), and the evolution on vectors with mean zero of Section 15.5.

Notation. We use the notation of Section 13: the function $\mathcal{G}(I) = 2\Phi(\sqrt{I}/2) - 1$ of (13.2), the semicircle quantile function γ , and the identification of vectors with step functions on $(0,1)$. For $f, f' \in \mathbb{R}^N$, we write $\langle f \rangle = N^{-1} \sum_i f_i$, $\langle f, f' \rangle_N = N^{-1} \sum_i f_i f'_i$,

and $\|f\|_{2,N} = \langle f, f \rangle_N^{1/2}$. Let G be a real symmetric $N \times N$ matrix with eigenvalues $\lambda_1 \geq \dots \geq \lambda_N$, let $c \geq 0$, and let $A = cG$. We write $D_N(t) = \|\nu_N P_t^A - \pi_A\|_{\text{TV}}$ and define $t_N(\varepsilon)$ by (3.15) with this D_N . For $A = c_N W_N$, these are the quantities of Section 3. We use the Lagrange multiplier z , the precisions p_i , the variances q_i , and the numbers d and s of Definition 3.3, and the covariance evolution of Section 7: the variances $v_i(t)$, the moving multiplier $z(t)$, the Bingham comparison law $\mu_t = \mu_{v(t)}$, the relative errors $e_i(t) = v_i(t)/q_i - 1$, and the information $\mathcal{I}_N(t) = \frac{1}{2} \sum_i e_i(t)^2$. We write $e(t) = (e_i(t))_i$ and $\bar{e}(t) = \langle e(t) \rangle$.

Adding a multiple of the identity. Replacing A by $A + \tau I$ replaces z by $z + \tau$ and leaves the precisions p_i and the variances q_i unchanged. In (7.19), it multiplies $m(t)$ by $e^{2\tau t}$, so that the variances $v_i(t)$ do not change, and neither do $e_i(t)$, $\mathcal{I}_N(t)$ and $\mu_t = \mu_{v(t)}$. The quantities D_N and t_N do not change either, by (3.7). In the proofs, we may therefore replace G by $G - N^{-1}(\text{Tr } G)I$, which has trace zero and norm at most $2\|G\|$.

15.1 Statements

Theorem 15.1 (Small inverse temperature: comparison and information). *Let $M > 0$. There exist $c_0 \in (0, 1)$, N_0 and C , depending only on M , such that the following holds for every $N \geq N_0$, every $c \in [0, c_0]$, and every real symmetric $N \times N$ matrix G with $\|G\| \leq M$, where $A = cG$.*

1. For all $t \geq 0$,

$$\|\nu_N P_t^A - \mu_t\|_{\text{TV}} \leq Cc^2 \min(t, 1). \quad (15.1)$$

2. For all $t \geq 0$, we have $\max_i |e_i(t)| \leq Cce^{-t}$. If $\mathcal{I}_N(0) = 0$, then $\mathcal{I}_N(t) = 0$ for all $t \geq 0$. If $\mathcal{I}_N(0) > 0$, then $\mathcal{I}_N(t) > 0$ for all $t \geq 0$, and

$$-4 - Cc \leq \frac{d}{dt} \log \mathcal{I}_N(t) \leq -4 + Cc. \quad (15.2)$$

3. For every $\kappa > 0$, there exist $c_\kappa \in (0, c_0]$ and $N_\kappa \geq N_0$, depending only on κ and M , such that if $c \leq c_\kappa$ and $N \geq N_\kappa$, then

$$\sup_{t \geq 0} |D_N(t) - \mathcal{G}(\mathcal{I}_N(t))| \leq \kappa. \quad (15.3)$$

4. If $\mathcal{I}_N(0) > 1$, then there exists a unique $\tau_N > 0$ with $\mathcal{I}_N(\tau_N) = 1$. It satisfies

$$\frac{\log \mathcal{I}_N(0)}{4 + Cc} \leq \tau_N \leq \frac{\log \mathcal{I}_N(0)}{4 - Cc}, \quad |\log \mathcal{I}_N(\tau_N + u) + 4u| \leq Cc|u| \quad (u \geq -\tau_N). \quad (15.4)$$

5. Let $c_N \rightarrow 0$, and for each N let G_N be a real symmetric $N \times N$ matrix with $\|G_N\| \leq M$, such that the information of $A = c_N G_N$ satisfies $\mathcal{I}_N(0) \rightarrow \infty$. Then $\tau_N / \log \mathcal{I}_N(0) \rightarrow 1/4$, and for every compact set $K \subset \mathbb{R}$ and every compact interval $I \subset (0, 1)$,

$$\sup_{u \in K} |D_N(\tau_N + u) - \Psi(u)| \rightarrow 0, \quad \sup_{\varepsilon \in I} |t_N(\varepsilon) - \tau_N - u_\varepsilon| \rightarrow 0.$$

In the last part, the window is of order one while $\tau_N \rightarrow \infty$, so there is cutoff for every sequence along which the initial information diverges. The relative bound $\tau_N = \frac{1}{4} \log \mathcal{I}_N(0)(1 + O(c_N))$ given by (15.4) does not determine τ_N up to $o(1)$; this is done for the GOE in Theorem 15.3.

Theorem 15.2 (Relaxation and cutoff near infinite temperature for the GOE). *Let $c_N \geq 0$ be deterministic with $c_N \rightarrow 0$, let $J_N = Nc_N^2$, and consider the model with $A = c_N W_N$. All limits below are in probability.*

1. Suppose that $\sqrt{N} c_N \rightarrow a \in [0, \infty)$. Then

$$\sup_{t \geq 0} \left| D_N(t) - \left(2\Phi\left(\frac{ae^{-2t}}{2\sqrt{2}}\right) - 1 \right) \right| \rightarrow 0, \quad (15.5)$$

and for every $\varepsilon \in (0, 1)$,

$$t_N(\varepsilon) \rightarrow \max \left\{ \frac{1}{2} \log \frac{a}{2\sqrt{2} \Phi^{-1}((1+\varepsilon)/2)}, 0 \right\}, \quad (15.6)$$

where the limit is zero if $a = 0$. If $\varepsilon > 2\Phi(a/(2\sqrt{2})) - 1$, then $\mathbb{P}(t_N(\varepsilon) = 0) \rightarrow 1$. If $c_N = 0$, then $\pi_{N,0} = \nu_N$ and $D_N(t) = 0$ for all $t \geq 0$.

2. Suppose that $J_N \rightarrow \infty$. Define

$$\tau_N = \begin{cases} \inf\{t \geq 0 : \mathcal{I}_N(t) \leq 1\} & \text{if } \mathcal{I}_N(0) > 1, \\ 1 & \text{if } \mathcal{I}_N(0) \leq 1. \end{cases} \quad (15.7)$$

Then

$$\mathbb{P}(\mathcal{I}_N(0) > 1 \text{ and } \tau_N \text{ is the only solution of } \mathcal{I}_N(t) = 1 \text{ in } [0, \infty)) \rightarrow 1. \quad (15.8)$$

Moreover, $\tau_N / \log J_N \rightarrow 1/4$, and for every compact set $K \subset \mathbb{R}$ and every compact interval $I \subset (0, 1)$,

$$\sup_{\substack{u \in K \\ \tau_N + u \geq 0}} |D_N(\tau_N + u) - \Psi(u)| \rightarrow 0, \quad \sup_{\varepsilon \in I} |t_N(\varepsilon) - \tau_N - u_\varepsilon| \rightarrow 0. \quad (15.9)$$

In particular, D_N has cutoff profile Ψ with center τ_N and window 1, and there is cutoff.

The time τ_N in (15.7) is defined for every realization of W_N , and $\mathcal{I}_N(\tau_N) = 1$ whenever $\mathcal{I}_N(0) > 1$. Indeed, for every real symmetric matrix A , the variances $v_i(t)$ are continuous in t , and they converge to q_i as $t \rightarrow \infty$ by (7.22), because $d > 0$ (Definition 3.3). Then $\mathcal{I}_N$ is continuous on $[0, \infty)$, and $\mathcal{I}_N(t) \rightarrow 0$ as $t \rightarrow \infty$. If $\mathcal{I}_N(0) > 1$, the set $\{t \geq 0 : \mathcal{I}_N(t) \leq 1\}$ is nonempty and closed, and its infimum τ_N belongs to it. Then $\tau_N > 0$ and $\mathcal{I}_N > 1$ on $[0, \tau_N)$, and continuity gives $\mathcal{I}_N(\tau_N) \geq 1$. We conclude that $\mathcal{I}_N(\tau_N) = 1$, and that τ_N is the smallest solution of $\mathcal{I}_N(t) = 1$ on the event $\{\mathcal{I}_N(0) > 1\}$. The theorem asserts that it is the only solution with probability tending to one; we do not claim uniqueness on all of $\{\mathcal{I}_N(0) > 1\}$. Under the assumptions of the fourth part of Theorem 15.1, the solution is unique, and τ_N is the time given there.

The time τ_N is a Borel function of W_N ; this is needed for the limits in probability in the second part. To see this, fix N and c_N , and write $\mathcal{I}_N(W, t)$ for the information at time t when W is a real symmetric $N \times N$ matrix and $A = c_N W$. This function is jointly continuous in (W, t) . Indeed, the ordered eigenvalues of W are continuous functions of W [4, Corollary A.6]. The Lagrange multiplier is the unique solution $z > c_N \lambda_1$ of (3.10), whose left side is continuous in (z, W) and strictly decreasing in z there; bracketing the solution between $z - \epsilon$ and $z + \epsilon$ shows that it depends continuously on W . Finally, the solution of (7.19) depends continuously on the eigenvalues of A and on t , and its normalization $\sum_j m_j(t)$ is positive by Lemma 7.6. We conclude that q_i , $v_i(t)$, and $\mathcal{I}_N(t)$ are continuous in (W, t) . For $r \geq 0$, the function $W \mapsto \min_{0 \leq s \leq r} \mathcal{I}_N(W, s)$ is Borel, because by continuity it is the infimum over a countable dense subset of $[0, r]$. On $\{\mathcal{I}_N(0) > 1\}$, we have $\tau_N \leq r$ if and only if $\min_{0 \leq s \leq r} \mathcal{I}_N(s) \leq 1$, and on $\{\mathcal{I}_N(0) \leq 1\}$, we have $\tau_N \leq r$ if and only if $r \geq 1$. This proves that τ_N is Borel. The event in (15.8) is the intersection of $\{\mathcal{I}_N(0) > 1\}$ with the event that $\mathcal{I}_N(t) = 1$ has exactly one solution, and it is Borel as well. For $m, k \geq 1$, the sets of matrices W for which $\mathcal{I}_N(W, t) = 1$ for some $t \in [0, m]$, or for two times $s, t \in [0, m]$ with $|s - t| \geq 1/k$, are closed, by joint continuity and the compactness of $[0, m]$. The set of W for which $\mathcal{I}_N(W, t) = 1$ has exactly one solution in $[0, \infty)$ is the union over m of the sets of the first kind, minus the union over m and k of the sets of the second kind.

In (15.9), the supremum is over $u \in K$ with $\tau_N + u \geq 0$, and an empty supremum is zero, as in Definition 3.7. Since $\tau_N \rightarrow \infty$ in probability, the restriction is void with probability tending to one. The same convention applies to the other suprema of this kind in this section.

At equality $\varepsilon = 2\Phi(a/(2\sqrt{2})) - 1$ with $a > 0$, the first part gives $t_N(\varepsilon) \rightarrow 0$ in probability, but not that $t_N(\varepsilon) = 0$ with probability tending to one. Along a sequence with $J_N \rightarrow J \in [0, \infty]$, the first part applies if $J < \infty$, with $a = \sqrt{J}$, and the second if $J = \infty$. A cutoff profile with a diverging center therefore appears exactly when $J_N \rightarrow \infty$.

For $h \in \mathbb{R}$, let

$$\mathcal{B}(h) = \frac{1}{2} \int_{-2}^2 e^{4h\lambda} \rho_{\text{sc}}(\lambda) d\lambda = \frac{I_1(8h)}{8h}, \quad \mathcal{B}(0) = \frac{1}{2}, \quad (15.10)$$

where I_1 is the modified Bessel function of the first kind (Remark 15.4). The proofs use only the integral.

Theorem 15.3 (Explicit centers). *In the setting of the second part of Theorem 15.2, let $h_N = (c_N/4) \log J_N$ for all N large enough that $J_N > 0$.*

1. *Suppose that (h_N) is bounded, and let*

$$T_N^{\text{bulk}} = \frac{1}{4} [\log J_N + \log \mathcal{B}(h_N)]. \quad (15.11)$$

Then $\tau_N - T_N^{\text{bulk}} \rightarrow 0$ in probability, and (15.9) holds with T_N^{bulk} in place of τ_N . If moreover $h_N \rightarrow 0$, then $T_N^{\text{bulk}} - \frac{1}{4} \log(J_N/2) \rightarrow 0$.

2. *Suppose that $h_N \rightarrow \infty$. Let $g_N = (1 - c_N)^2$ and let $T_N(c_N)$ be the center (13.4) of Theorem 13.1 evaluated at $c = c_N$, that is,*

$$T_N(c_N) = \frac{\log N - \frac{3}{2} \log \log N + \log A_{c_N}}{4g_N}, \quad A_c = \frac{\sqrt{c}(1-c)^3}{4\sqrt{\pi}(1+c)^2}. \quad (15.12)$$

Then $\tau_N - T_N(c_N) \rightarrow 0$ in probability, and for every compact set $K \subset \mathbb{R}$ and every compact interval $I \subset (0, 1)$, in probability,

$$\sup_{u \in K} |D_N(T_N(c_N) + u/g_N) - \Psi(u)| \rightarrow 0, \quad \sup_{\varepsilon \in I} |g_N(t_N(\varepsilon) - T_N(c_N)) - u_\varepsilon| \rightarrow 0. \quad (15.13)$$

In the second part, no condition such as $c_N^2 \log N \rightarrow 0$ is imposed, and g_N cannot be replaced by 1 in the center. Indeed, $g_N^{-1} - 1 \geq 2c_N$ and $c_N \log N \geq 4h_N \rightarrow \infty$, so replacing $4g_N$ by 4 in the denominator of (15.12) changes the center by an amount that tends to infinity. Sequences along which h_N has several limit points in $[0, \infty]$ are still covered by Theorem 15.2, with the center τ_N .

Remark 15.4 (The Bessel function). By [117, (10.32.2)] with $\nu = 1$, we have $I_1(x) = \pi^{-1} x \int_{-1}^1 \sqrt{1-y^2} e^{xy} dy$. The substitution $\lambda = 2y$ yields $\int e^{s\lambda} \rho_{\text{sc}}(\lambda) d\lambda = I_1(2s)/s$ for $s \neq 0$, and with $s = 4h$ we obtain the second expression in (15.10). We also note that $\mathcal{B}(h) \geq \frac{1}{2} e^{4h \int \lambda \rho_{\text{sc}}} = \frac{1}{2}$ by Jensen's inequality, and that $\mathcal{B}(h) \leq \frac{1}{2} e^{8|h|}$ and $|\mathcal{B}'(h)| \leq 4e^{8|h|}$, since $|\lambda| \leq 2$ on the support of ρ_{sc} .

Table 15.1. The regimes of Theorems 15.2 and 15.3, with $J_N = Nc_N^2$. The parameter $h_N = (c_N/4) \log J_N$ is used only in the regime $J_N \rightarrow \infty$, for N large.

Regime	Center	Window	Statement
$\sqrt{N} c_N \rightarrow a < \infty$	no cutoff	—	Theorem 15.2, part 1
$J_N \rightarrow \infty$	τ_N as defined in (15.7)	1	Theorem 15.2, part 2
$J_N \rightarrow \infty, h_N$ bounded	$\frac{1}{4}[\log J_N + \log \mathcal{B}(h_N)]$	1	Theorem 15.3, part 1
$J_N \rightarrow \infty, h_N \rightarrow \infty$	$T_N(c_N)$ in (15.12)	$(1 - c_N)^{-2}$	Theorem 15.3, part 2

15.2 Small inverse temperature

The following lemmas concern a matrix $A = cG$ with $\text{Tr } G = 0$, $\|G\| \leq M$, and $cM \leq 1/8$. Then $|\lambda_i| \leq M$ for all i , and $\sum_i \lambda_i = 0$. We write $\lambda = (\lambda_i)_i$ for the vector of eigenvalues.

Lemma 15.5 (The Lagrange multiplier and the covariance evolution). *Let $\text{Tr } G = 0$, $\|G\| \leq M$, and $cM \leq 1/8$. Then*

$$|z - 1| \leq 8c^2M^2, \quad |q_i - 1 - c\lambda_i| \leq 8c^2M^2, \quad |p_i - 1| \leq 2cM, \quad |q_i - 1| \leq 2cM, \quad (15.14)$$

and $3/4 \leq d \leq 1 \leq s \leq 5/4$. For all $t \geq 0$ and all i ,

$$|v_i(t) - q_i| \leq 4cMe^{-2dt}, \quad |e_i(t)| \leq 5cMe^{-3t/2}. \quad (15.15)$$

Proof. Since $N^{-1} \text{Tr } A = 0$ and $\|A\| \leq cM \leq 1/8$, Lemma 9.1 with $\epsilon = \|A\|$ implies that $|z - 1| \leq 8c^2M^2$, $|q_i - 1 - c\lambda_i| \leq 8c^2M^2$, and $|d - 1|, |s - 1| \leq 2cM \leq 1/4$. Then $p_i = z - c\lambda_i$ satisfies $|p_i - 1| \leq 8c^2M^2 + cM \leq 2cM$, and similarly $|q_i - 1| \leq cM + 8c^2M^2 \leq 2cM$. We have $|v_i(t) - q_i| \leq (s - d)d^{-2}e^{-2dt}$ by (7.22), and the first bound in (15.15) follows from $s - d = c(\lambda_1 - \lambda_N) \leq 2cM$ and $d^{-2} \leq 16/9$. Finally, $|e_i(t)| = p_i|v_i(t) - q_i| \leq \frac{5}{4} \cdot 4cMe^{-2dt}$, and $2d \geq 3/2$. $\square$

Lemma 15.6 (Relative errors and information). *Let A be a real symmetric matrix. For all $t \geq 0$,*

$$e'_i(t) = -2p_i e_i(t) + 2\bar{e}(t)(1 + e_i(t)), \quad \sum_i q_i e_i(t) = 0, \quad z(t) = z - \bar{e}(t), \quad (15.16)$$

and

$$\mathcal{I}'_N(t) = -2 \sum_i p_i e_i(t)^2 + 2N\bar{e}(t)^2 + 2\bar{e}(t) \sum_i e_i(t)^2. \quad (15.17)$$

If moreover $A = cG$ with $\text{Tr } G = 0$, $\|G\| \leq M$, and $cM \leq 1/8$, then $|\bar{e}(t)| \leq 2cM\|e(t)\|_{2,N}$ and $|\mathcal{I}'_N(t) + 4\mathcal{I}_N(t)| \leq 16cM\mathcal{I}_N(t)$ for all $t \geq 0$.

Proof. We have $\sum_i q_i e_i = \sum_i v_i - \sum_i q_i = N - N = 0$. Recall that $z(t) = 1 + N^{-1} \sum_i \alpha_i v_i(t)$ with $\alpha_i = z - p_i$, and that $p_i v_i = 1 + e_i$. Using $\sum_i v_i = N$, we find $z(t) = 1 + z - N^{-1} \sum_i (1 + e_i) = z - \bar{e}$. Recall also from (7.21) that $v'_i = 2 + 2(\alpha_i - z(t))v_i$, where now $\alpha_i - z(t) = -p_i + \bar{e}$. Multiplying by p_i , we have

$$e'_i = p_i v'_i = 2p_i - 2p_i(1 + e_i) + 2\bar{e}(1 + e_i) = -2p_i e_i + 2\bar{e}(1 + e_i),$$

which proves (15.16). Equation (15.17) follows from $\mathcal{I}'_N = \sum_i e_i e'_i$ and $\sum_i e_i = N\bar{e}$.

Now let $A = cG$ as in the second statement. The second identity in (15.16) implies that $\bar{e} = N^{-1} \sum_i (1 - q_i) e_i$. We obtain $|\bar{e}| \leq 2cM \|e\|_{2,N}$ from (15.14) and the Cauchy-Schwarz inequality. We also have $|\bar{e}| \leq 2cM \max_i |e_i| \leq 10c^2 M^2$ by (15.15). Recall that $\|e\|_{2,N}^2 = 2\mathcal{I}_N/N$. We bound the terms of (15.17) as follows:

$$\begin{aligned} \left| 2 \sum_i p_i e_i^2 - 4\mathcal{I}_N \right| &\leq 2 \max_i |p_i - 1| \sum_i e_i^2 \leq 8cM \mathcal{I}_N, \\ 2N\bar{e}^2 &\leq 16c^2 M^2 \mathcal{I}_N, \quad 2|\bar{e}| \sum_i e_i^2 \leq 40c^2 M^2 \mathcal{I}_N. \end{aligned}$$

As $56c^2 M^2 \leq 7cM$, we conclude that $|\mathcal{I}'_N + 4\mathcal{I}_N| \leq 15cM \mathcal{I}_N$. $\square$

At bounded times, the relative errors are close to those of a linear evolution with rate two.

Lemma 15.7 (Relative errors at bounded times). *Let $\text{Tr } G = 0$, $\|G\| \leq M$, and $cM \leq 1/8$. For all $t \geq 0$,*

$$\|e(t) + ce^{-2t}\lambda\|_{2,N} \leq 24c^2 M^2 (1+t)e^{-t}.$$

Proof. We first note that Lemma 15.6 implies that $\mathcal{I}_N(s) \leq \mathcal{I}_N(0)e^{-(4-16cM)s}$, that is, $\|e(s)\|_{2,N} \leq \|e(0)\|_{2,N} e^{-(2-8cM)s}$. We have $e(0) = (p_i - 1)_i$, because $v_i(0) = 1$, and $\|e(0)\|_{2,N} \leq 2cM$ by (15.14). We write (15.16) as $e' = -2e + F$ with $F_i = -2(p_i - 1)e_i + 2\bar{e}(1 + e_i)$. Using Lemmas 15.5 and 15.6 and $1 + 5cM \leq 2$, we have $\|F\|_{2,N} \leq (4cM + 8cM)\|e\|_{2,N} = 12cM\|e\|_{2,N}$. By variation of constants,

$$e(t) = e^{-2t}e(0) + \int_0^t e^{-2(t-s)}F(s)ds.$$

We have $e(0) = (z - 1)\mathbf{1} - c\lambda$, and therefore $\|e^{-2t}(e(0) + c\lambda)\|_{2,N} = e^{-2t}|z - 1| \leq 8c^2 M^2 e^{-2t}$. The integral has norm at most

$$\int_0^t e^{-2(t-s)} 12cM \cdot 2cM e^{-(2-8cM)s} ds = 24c^2 M^2 e^{-2t} \int_0^t e^{8cMs} ds \leq 24c^2 M^2 t e^{-(2-8cM)t}.$$

Adding the two bounds and using $8cM \leq 1$, we obtain the claim. $\square$

15.3 Proof of the deterministic theorem

Proof of Theorem 15.1. As explained at the beginning of the section, we may assume that $\text{Tr } G = 0$ and $\|G\| \leq M'$, with $M' = 2M$. Let ϵ_0 , N'_0 and C_1 be the universal constants of Proposition 9.14. We set $C = \max\{C_1 M'^2, 16M', 5M'\}$, $c_0 = \min\{1/2, 1/(8M'), \epsilon_0/M', 2/C\}$, and $N_0 = N'_0$. Let $c \leq c_0$ and $N \geq N_0$.

For the first part, we have $\|A - N^{-1}(\text{Tr } A)I\| = \|A\| \leq cM' \leq \epsilon_0$, and (15.1) is the first bound of Proposition 9.14.

For the second part, the bound on $e_i(t)$ is (15.15). We have $|\mathcal{I}'_N| \leq 6\mathcal{I}_N$ by Lemma 15.6. If $\mathcal{I}_N(0) = 0$, we obtain $\mathcal{I}_N \equiv 0$ from Gronwall's inequality. If $\mathcal{I}_N(0) > 0$, the same inequality gives $\mathcal{I}_N(t) \geq \mathcal{I}_N(0)e^{-6t} > 0$, and dividing $|\mathcal{I}'_N + 4\mathcal{I}_N| \leq 16cM'\mathcal{I}_N$ by $\mathcal{I}_N$ yields (15.2).

For the third part, fix $\kappa > 0$, and let $\delta > 0$ be given by Lemma 13.2 with $\kappa/2$ in place of κ . Let $c_\kappa \leq c_0$ satisfy $C_1 c_\kappa^2 M'^2 \leq \kappa/2$ and $5c_\kappa M' \leq \delta$, and $N_\kappa \geq N_0$ with $(4/3)(2N_\kappa)^{-1/2} \leq \delta$. Let $c \leq c_\kappa$, $N \geq N_\kappa$ and $t \geq 0$. Using (15.1) and the triangle inequality, we have

$$|D_N(t) - \|\mu_t - \pi_A\|_{\text{TV}}| \leq \|\nu_N P_t^A - \mu_t\|_{\text{TV}} \leq \kappa/2.$$

We have $\mu_t = \mu_{v(t)}$ and $\pi_A = \mu_q$, with $v_i(t) = q_i(1 + e_i(t))$ and $\max_i |e_i(t)| \leq 5cM' \leq \delta$. Note that $\eta_q \leq (4/3)(2N)^{-1/2} \leq \delta$, because $q_i \leq 1/d \leq 4/3$ and $\sum_i q_i^2 \geq N$. Then Lemma 13.2 implies that $|\|\mu_t - \pi_A\|_{\text{TV}} - \mathcal{G}(\mathcal{I}_N(t))| \leq \kappa/2$, and (15.3) follows.

For the fourth part, (15.2) and $Cc \leq 2$, which holds since $c \leq c_0$, show that $\mathcal{I}_N$ is continuous and strictly decreasing, with $\mathcal{I}_N(t) \leq \mathcal{I}_N(0)e^{-2t} \rightarrow 0$. If $\mathcal{I}_N(0) > 1$, it follows that there exists a unique $\tau_N > 0$ with $\mathcal{I}_N(\tau_N) = 1$. We integrate (15.2) from 0 to τ_N to obtain the first bound in (15.4), and from τ_N to $\tau_N + u$ to obtain the second, using $\log \mathcal{I}_N(\tau_N) = 0$.

For the fifth part, (15.4) implies that $\tau_N / \log \mathcal{I}_N(0) \rightarrow 1/4$, and in particular that $\tau_N \rightarrow \infty$. Fix $\kappa > 0$ and a compact set K , and let $U = \sup(\{0\} \cup \{|u| : u \in K\})$. For N large, we have $c_N \leq c_\kappa$, $N \geq N_\kappa$, and $\tau_N \geq U$. For $u \in K$, we have $\mathcal{I}_N(\tau_N + u) = e^{-4u+r}$ with $|r| \leq Cc_N U$ by (15.4), and then

$$|\mathcal{I}_N(\tau_N + u)^{1/2} - e^{-2u}| \leq e^{2U}(e^{Cc_N U/2} - 1).$$

Combining this with (15.3) and (13.3), we find $|D_N(\tau_N + u) - \Psi(u)| \leq \kappa + e^{2U}(e^{Cc_N U/2} - 1)$ for all $u \in K$. As the parameter κ is arbitrary, the first limit follows. The function D_N is nonincreasing and tends to zero as $t \rightarrow \infty$, by the arguments in the proofs of Lemma 3.6 and Corollary 3.12, which apply to any symmetric matrix A . The second limit then follows from the deterministic version of Lemma 3.11, with $b_n = \tau_N$, $w_n = 1$ and $V = \Psi$, and from $\Psi^{-1}(\epsilon) = u_\epsilon$ (Lemma 3.9). $\square$

15.4 Proof of the classification for the GOE

Throughout the rest of this section, $\lambda_1 \geq \dots \geq \lambda_N$ are the eigenvalues of W_N . Let $\bar{\lambda} = N^{-1} \sum_i \lambda_i = N^{-1} \text{Tr } W_N$, let $\tilde{\lambda}_i = \lambda_i - \bar{\lambda}$, and let $\tilde{G} = W_N - \bar{\lambda}I$, which has eigenvalues $\tilde{\lambda}_i$. Let $\sigma_N \geq 0$ be defined by

$$\sigma_N^2 = \|\tilde{\lambda}\|_{2,N}^2 = \int_0^1 (\ell_N - \bar{\lambda})^2 d\omega.$$

The proof of Theorem 15.2 uses the following elementary facts about the GOE, which concern only the norm of W_N and the traces of W_N and W_N^2 . The rigidity estimate of Lemma 4.2 is used only for the explicit centers, from Section 15.5 on.

Lemma 15.8 (Norm and traces of the GOE). *Let $m_{2,N} = N^{-1} \text{Tr}(W_N^2)$. For every $N \geq 2$,*

$$\begin{aligned} \mathbb{E}\bar{\lambda} &= 0, & \mathbb{E}\bar{\lambda}^2 &= \frac{2}{N^2}, \\ \mathbb{E}(m_{2,N} - 1)^2 &\leq \frac{18}{N^2}. \end{aligned}$$

Moreover, $\mathbb{P}(\|W_N\| \leq 3) \rightarrow 1$. In particular, with $b_N = N^{-1/4}$, the event

$$Q_N = \{\|W_N\| \leq 3, |\bar{\lambda}| \leq b_N, |m_{2,N} - 1| \leq b_N\} \quad (15.18)$$

satisfies $\mathbb{P}(Q_N) \rightarrow 1$. On Q_N , we have

$$\text{Tr } \tilde{G} = 0, \quad \|\tilde{G}\| \leq 4, \quad \sigma_N^2 = m_{2,N} - \bar{\lambda}^2, \quad |\sigma_N - 1| \leq 2b_N. \quad (15.19)$$

Proof. For a centered Gaussian variable Y with variance v , we have $\mathbb{E}Y^2 = v$, $\mathbb{E}Y^4 = 3v^2$, and $\text{Var}(Y^2) = 2v^2$. The trace $\text{Tr } W_N = \sum_i W_{ii}$ is a sum of N independent Gaussian variables with mean zero and variance $2/N$. This gives $\mathbb{E}\bar{\lambda} = 0$ and $\mathbb{E}\bar{\lambda}^2 = N^{-2} \cdot N \cdot 2/N = 2/N^2$. Next,

$$m_{2,N} = \frac{1}{N} \sum_{i=1}^N W_{ii}^2 + \frac{2}{N} \sum_{i < j} W_{ij}^2.$$

Taking expectations, we find $\mathbb{E}m_{2,N} = 2/N + (N-1)/N = 1 + 1/N$. The variables W_{ij}^2 , $i \leq j$, are independent, with $\text{Var}(W_{ii}^2) = 8/N^2$ and $\text{Var}(W_{ij}^2) = 2/N^2$ for $i < j$. Summing these variances, we find

$$\text{Var}(m_{2,N}) = \frac{1}{N^2} \cdot N \cdot \frac{8}{N^2} + \frac{4}{N^2} \cdot \frac{N(N-1)}{2} \cdot \frac{2}{N^2} = \frac{8 + 4(N-1)}{N^3} = \frac{4(N+1)}{N^3},$$

and $\mathbb{E}(m_{2,N} - 1)^2 = \text{Var}(m_{2,N}) + N^{-2} = 5/N^2 + 4/N^3 \leq 18/N^2$.

For the norm, we use the convergence of the largest eigenvalue of a Wigner matrix to 2. Let X_N be the real symmetric matrix whose entries X_{ij} , $i \leq j$, are independent

Gaussian variables with mean zero and variance $1/N$, including the diagonal entries. This is a Wigner matrix in the sense of [4, Section 2.1], with standard Gaussian entries before scaling, and it satisfies the moment condition of [4, Theorem 2.1.22]: for a standard Gaussian variable Y and every integer $k \geq 1$, we have $\mathbb{E}|Y|^k \leq (\mathbb{E}Y^{2k})^{1/2} = ((2k-1)!!)^{1/2} \leq k^k$. This theorem implies that the largest eigenvalue of X_N converges to 2 in probability. Applying this also to $-X_N$, which has the law of X_N , we find that $\mathbb{P}(\|X_N\| > 5/2) \rightarrow 0$. Let $Z_1, \dots, Z_N$ be standard Gaussian variables, independent of each other and of X_N , and let $Y_N = \text{diag}(Z_i/\sqrt{N})$. Then $X_N + Y_N$ has the law of W_N , since its diagonal entries have variance $2/N$. By a union bound and Markov's inequality,

$$\mathbb{P}(\|Y_N\| > 1/2) \leq \sum_{i=1}^N \mathbb{P}(Z_i^4 > N^2/16) \leq N \cdot \frac{16 \cdot 3}{N^2} = \frac{48}{N}.$$

We conclude that $\mathbb{P}(\|W_N\| > 3) \leq \mathbb{P}(\|X_N\| > 5/2) + 48/N \rightarrow 0$.

By Chebyshev's inequality and the moment bounds above, we have $\mathbb{P}(|\bar{\lambda}| > b_N) \leq 2N^{-2}b_N^{-2} = 2N^{-3/2}$ and $\mathbb{P}(|m_{2,N} - 1| > b_N) \leq 18N^{-2}b_N^{-2} = 18N^{-3/2}$. Together with the bound on the norm, this gives $\mathbb{P}(Q_N^c) \leq \mathbb{P}(\|W_N\| > 3) + 20N^{-3/2} \rightarrow 0$.

Finally, on Q_N , we have $\text{Tr } \tilde{G} = 0$ and $\|\tilde{G}\| \leq \|W_N\| + |\bar{\lambda}| \leq 3 + b_N \leq 4$. Since $N^{-1} \sum_i \lambda_i^2 = m_{2,N}$, expanding the square gives $\sigma_N^2 = m_{2,N} - \bar{\lambda}^2$. Then $|\sigma_N^2 - 1| \leq |m_{2,N} - 1| + \bar{\lambda}^2 \leq b_N + b_N^2 \leq 2b_N$. As $\sigma_N \geq 0$, we have $|\sigma_N - 1| = |\sigma_N^2 - 1|/(\sigma_N + 1) \leq |\sigma_N^2 - 1|$, which completes the proof. $\square$

We now work on the event Q_N , on which Theorem 15.1 applies with $G = W_N$ and $M = 3$. We write C, c_0, c_K , and N_K for its constants with $M = 3$. As explained at the beginning of the section, we may compute $e_i(t)$, $\mathcal{I}_N(t)$, and τ_N with the matrix $c_N \tilde{G}$ in place of $c_N W_N$. By (15.19), on Q_N , Lemmas 15.5 to 15.7 apply to $c_N \tilde{G}$ with $M = 4$ once $32c_N \leq 1$, which holds for all N larger than a threshold depending only on the sequence (c_N) . Let $\tilde{z}$ be the Lagrange multiplier of $c_N \tilde{G}$.

The initial information. We have $e_i(0) = \tilde{z} - 1 - c_N \tilde{\lambda}_i$, because $v_i(0) = 1$. Expanding the square and using $\sum_i \tilde{\lambda}_i = 0$, we find

$$\mathcal{I}_N(0) = \frac{c_N^2}{2} \sum_i \tilde{\lambda}_i^2 + \frac{N}{2} (\tilde{z} - 1)^2 = \frac{J_N}{2} \sigma_N^2 + \frac{N}{2} (\tilde{z} - 1)^2. \quad (15.20)$$

By (15.14) with $M = 4$, we have $|\tilde{z} - 1| \leq 128c_N^2$, and hence

$$\frac{J_N}{2} \sigma_N^2 \leq \mathcal{I}_N(0) \leq \frac{J_N}{2} (\sigma_N^2 + 2^{14} c_N^2). \quad (15.21)$$

These formulas also hold when $c_N = 0$.

Proof of Theorem 15.2. If $c_N = 0$, then $L = \Delta_{S_N}$ and $\pi_{N,0} = \nu_N$ is invariant, so $D_N \equiv 0$. We now prove the two parts.

The first part. Let $\sqrt{N}c_N \rightarrow a \in [0, \infty)$. On Q_N and for N large, we use Lemma 15.7 for $c_N\tilde{G}$ with $M = 4$, together with $(1+t)e^{-t} \leq 1$, and find that for all $t \geq 0$

$$\left| \sqrt{2\mathcal{I}_N(t)} - \sqrt{N}c_N\sigma_N e^{-2t} \right| \leq \sqrt{N} \|e(t) + c_N e^{-2t} \tilde{\lambda}\|_{2,N} \leq 384\sqrt{N}c_N^2.$$

Here we used $\sqrt{2\mathcal{I}_N(t)} = \sqrt{N}\|e(t)\|_{2,N}$ and $\|c_N e^{-2t} \tilde{\lambda}\|_{2,N} = c_N\sigma_N e^{-2t}$. Using also $|\sigma_N - 1| \leq 2b_N$ from (15.19), we find that, on Q_N and for N large,

$$\sup_{t \geq 0} \left| \sqrt{2\mathcal{I}_N(t)} - a e^{-2t} \right| \leq R_N, \quad R_N = 384\sqrt{N}c_N^2 + |\sqrt{N}c_N - a| + 2b_N\sqrt{N}c_N.$$

The number R_N does not depend on W_N , and it tends to zero, because $\sqrt{N}c_N$ is bounded and $\sqrt{N}c_N^2 = (\sqrt{N}c_N)c_N \rightarrow 0$. Let $D_\infty(t) = \mathcal{G}(a^2 e^{-4t}/2) = 2\Phi(ae^{-2t}/(2\sqrt{2})) - 1$. By (13.3),

$$\sup_{t \geq 0} |\mathcal{G}(\mathcal{I}_N(t)) - D_\infty(t)| \leq 2^{-1/2} \sup_{t \geq 0} \left| \sqrt{2\mathcal{I}_N(t)} - a e^{-2t} \right| \leq 2^{-1/2} R_N.$$

Fix $\kappa > 0$. For $N \geq N_\kappa$ with $c_N \leq c_\kappa$, we have $\sup_t |D_N(t) - \mathcal{G}(\mathcal{I}_N(t))| \leq \kappa$ on Q_N , by (15.3). Let $\delta_N = \sup_{t \geq 0} |D_N(t) - D_\infty(t)|$. Since D_N is continuous in t (see the proof of Corollary 3.12), the supremum may be taken over rational t , and δ_N is a random variable. Then $\delta_N \leq \kappa + 2^{-1/2}R_N$ on Q_N , for all N larger than a threshold depending only on κ and the sequence (c_N) . Since $R_N \rightarrow 0$, we have $\mathbb{P}(\delta_N > 2\kappa) \leq \mathbb{P}(Q_N^c)$ for N large, and $\mathbb{P}(Q_N^c) \rightarrow 0$ by Lemma 15.8. As κ is arbitrary, $\delta_N \rightarrow 0$ in probability, which is (15.5).

We now turn to the mixing times. Fix $\varepsilon \in (0, 1)$, and let s_* be the right side of (15.6). If $a > 0$, then D_∞ is continuous and strictly decreasing, with $D_\infty(0) = 2\Phi(a/(2\sqrt{2})) - 1$ and $D_\infty(t) \rightarrow 0$, and $D_\infty(s) = \varepsilon$ exactly when $s = \frac{1}{2} \log(a/(2\sqrt{2}\Phi^{-1}((1+\varepsilon)/2)))$. Recall that D_N is nonincreasing (Lemma 3.6). We consider three cases.

- If $\varepsilon < D_\infty(0)$, then $a > 0$ and $s_* > 0$. For $h \in (0, s_*)$, we have $D_\infty(s_* - h) > \varepsilon > D_\infty(s_* + h)$. When δ_N is smaller than both differences, we have $D_N(s_* + h) < \varepsilon$, which gives $t_N(\varepsilon) \leq s_* + h$, and $D_N(t) \geq D_N(s_* - h) > \varepsilon$ for all $t \leq s_* - h$, which gives $t_N(\varepsilon) \geq s_* - h$.
- If $\varepsilon > D_\infty(0)$, then $s_* = 0$. When $\delta_N < \varepsilon - D_\infty(0)$, we have $D_N(0) < \varepsilon$ and $t_N(\varepsilon) = 0$. This also covers the case $a = 0$, in which $D_\infty \equiv 0$.
- If $\varepsilon = D_\infty(0)$ and $a > 0$, then $s_* = 0$. For $h > 0$, we have $D_\infty(h) < \varepsilon$, and when $\delta_N < \varepsilon - D_\infty(h)$, we have $t_N(\varepsilon) \leq h$.

In each case, the convergence $\delta_N \rightarrow 0$ in probability implies that $t_N(\varepsilon) \rightarrow s_*$ in probability, and in the second case $\mathbb{P}(t_N(\varepsilon) = 0) \rightarrow 1$.

The second part. Let $J_N \rightarrow \infty$. Then $c_N > 0$ for N large. On Q_N , (15.21) and (15.19) imply that

$$\mathcal{I}_N(0) \geq \frac{J_N}{2}(1 - 2b_N)^2, \quad \left| \frac{2\mathcal{I}_N(0)}{J_N} - 1 \right| \leq b_N + b_N^2 + 2^{14}c_N^2.$$

Since the right sides do not depend on W_N , we have $\mathcal{I}_N(0) \rightarrow \infty$ and $\log \mathcal{I}_N(0) - \log(J_N/2) \rightarrow 0$ uniformly on Q_N . Let N be so large that $\mathcal{I}_N(0) > 1$ on Q_N , $c_N \leq c_0$, and $N \geq N_0$. For every matrix in Q_N , the fourth part of Theorem 15.1 then implies that the equation $\mathcal{I}_N(t) = 1$ has exactly one solution in $[0, \infty)$, which is positive and satisfies (15.4). By the discussion after Theorem 15.2, the time τ_N of (15.7) is a solution of this equation, and it is the only one. We have shown that Q_N is contained in the event in (15.8) for N large. As $\mathbb{P}(Q_N) \rightarrow 1$, this proves (15.8).

By (15.4), the ratio $\tau_N / \log \mathcal{I}_N(0)$ lies between $(4 + Cc_N)^{-1}$ and $(4 - Cc_N)^{-1}$ on Q_N , for N large. Since $Cc_N \leq 2$ for N large, we have

$$\left| \frac{\tau_N}{\log \mathcal{I}_N(0)} - \frac{1}{4} \right| \leq Cc_N \quad \text{on } Q_N.$$

Together with $\log \mathcal{I}_N(0) - \log(J_N/2) \rightarrow 0$ uniformly on Q_N and $\log J_N \rightarrow \infty$, this gives $\sup_{Q_N} |\tau_N / \log J_N - 1/4| \rightarrow 0$. As $\mathbb{P}(Q_N) \rightarrow 1$, we have $\tau_N / \log J_N \rightarrow 1/4$ in probability, and in particular $\tau_N \rightarrow \infty$ in probability.

Next, fix a compact set $K \subset \mathbb{R}$ and $\kappa > 0$, and let $U = \sup(\{0\} \cup \{|u| : u \in K\})$. For N large, on Q_N , we have $\tau_N \geq U$, $c_N \leq c_\kappa$, and $N \geq N_\kappa$. For $u \in K$, we have $\mathcal{I}_N(\tau_N + u) = e^{-4u+r}$ with $|r| \leq Cc_N U$, by (15.4), and then $|\mathcal{I}_N(\tau_N + u)^{1/2} - e^{-2u}| \leq e^{2U}(e^{Cc_N U/2} - 1)$. Using (15.3) and (13.3) as in the proof of the fifth part of Theorem 15.1, we find that, on Q_N and for N large,

$$\sup_{u \in K} |D_N(\tau_N + u) - \Psi(u)| \leq \kappa + e^{2U}(e^{Cc_N U/2} - 1).$$

Here the restriction $\tau_N + u \geq 0$ is void, because $\tau_N \geq U$. The left side is a random variable: τ_N is Borel, D_N is jointly measurable and continuous in t (see the proof of Corollary 3.12), and the supremum may be taken over a countable dense subset of K . As κ is arbitrary and $\mathbb{P}(Q_N) \rightarrow 1$, the first limit in (15.9) holds in probability. The center τ_N is a positive random variable, by the discussion after Theorem 15.2, and $1/\tau_N \rightarrow 0$ in probability. Then D_N has cutoff profile Ψ with center τ_N and window 1 in the sense of Definition 3.7, and Corollary 3.12 gives the second limit in (15.9) and cutoff. $\square$

15.5 The center when h_N is bounded

For the explicit centers, we use the approximation of the eigenvalues by the semicircle quantiles, in addition to Lemma 15.8. In this section and the next, fix $C_{\text{gr}} \geq 1$ and work on the event $F_{C_{\text{gr}}}$ of Lemma 4.2. Put $E_N = (C_{\text{gr}} + 7)N^{-5/12}$. For every $\eta \in (0, 1)$, the constant C_{gr} can be chosen so that $\mathbb{P}(F_{C_{\text{gr}}}) \geq 1 - \eta$ for all sufficiently large N . We keep C_{gr} fixed until the deterministic estimates are complete, and then let $\eta \downarrow 0$. Recall the notation $\bar{\lambda}$ and $\tilde{\lambda}_i$ of Section 15.4. By Lemma 13.8, we have, on $F_{C_{\text{gr}}}$ and for N large,

$$\|\ell_N - \gamma\|_\infty \leq E_N, \quad \max_i |\lambda_i| \leq 3, \quad |\bar{\lambda}| \leq E_N, \quad \max_i |\tilde{\lambda}_i| \leq 4. \quad (15.22)$$

The event $F_{C_{\text{gr}}}$ is contained in the event Q_N of Lemma 15.8 for N large. Indeed, $\int_0^1 \gamma^2 d\omega = \int \lambda^2 \rho_{\text{sc}}(\lambda) d\lambda = 1$ and $m_{2,N} = \int_0^1 \ell_N^2 d\omega$. Then (15.22) implies

$$|m_{2,N} - 1| \leq \int_0^1 |\ell_N - \gamma| |\ell_N + \gamma| d\omega \leq (4 + E_N)E_N \leq 5E_N.$$

For N large, we have $5E_N \leq b_N$, and also $|\bar{\lambda}| \leq E_N \leq b_N$ and $\|W_N\| = \max_i |\lambda_i| \leq 3$. Every conclusion of the proof of Theorem 15.2 that holds on Q_N for N large then also holds on $F_{C_{\text{gr}}}$ for N large.

When ct is bounded, the relative errors $e(t)$ are of order ce^{-2t} , and their profile is obtained from a linear evolution on vectors with mean zero. Let $\Lambda = \text{diag}(\lambda_i)$ act on $\mathbb{R}^N$, let $\Pi f = f - \langle f \rangle \mathbf{1}$ be the orthogonal projection onto the subspace $\mathcal{H}_0 = \{f : \langle f \rangle = 0\}$, and let $\mathcal{T} = \Pi \Lambda$, restricted to $\mathcal{H}_0$. For $f, f' \in \mathcal{H}_0$, we have $\langle f, \mathcal{T} f' \rangle_N = \langle f, \Lambda f' \rangle_N = \langle \Lambda f, f' \rangle_N$, so that $\mathcal{T}$ is self-adjoint on $\mathcal{H}_0$, with $\|\mathcal{T}\| \leq \max_i |\lambda_i|$. When $\text{Tr } G = 0$, the vector λ lies in $\mathcal{H}_0$. The operator $\mathcal{T}$ is applied only to vectors in $\mathcal{H}_0$; equations in $\mathbb{R}^N$ are first projected by Π .

Lemma 15.9 (Evolution on vectors with mean zero). *Let $\text{Tr } G = 0$, $\|G\| \leq M$, and $0 < cM \leq 1/8$, and let $\hat{e}(t) = c^{-1}e^{2t}e(t)$. For every $H > 0$ and every $t \geq 0$ with $ct \leq H$,*

$$\|\hat{e}(t) + e^{2ct\mathcal{T}} \lambda\|_{2,N} \leq C_{H,M} c, \quad C_{H,M} = (54HM^3 + 3M^2)e^{9MH}. \quad (15.23)$$

Proof. We write $\hat{e} = \hat{e}^0 + \langle \hat{e} \rangle \mathbf{1}$ with $\hat{e}^0 = \Pi \hat{e}$, derive an equation for $\hat{e}^0$, and compare it with $\partial_t f = 2c\mathcal{T}f$.

Step 1: the equation for $\hat{e}$. Since $p_i = 1 + (z - 1) - c\lambda_i$, (15.16) reads $e'_i = -2e_i - 2(z - 1)e_i + 2c\lambda_i e_i + 2\bar{e} + 2\bar{e}e_i$. Multiplying by $c^{-1}e^{2t}$ and using $c^{-1}e^{2t}\bar{e} = \langle \hat{e} \rangle$, we find

$$\hat{e}' = 2c\Lambda \hat{e} + 2\langle \hat{e} \rangle \mathbf{1} + R_1, \quad R_1 = (2\bar{e} - 2(z - 1))\hat{e}.$$

Recall that $|z - 1| \leq 8c^2M^2$ and $|\bar{e}| \leq 10c^2M^2$ (Lemmas 15.5 and 15.6), which give $\|R_1\|_{2,N} \leq 36c^2M^2\|\hat{e}\|_{2,N}$. Next, $\sum_i q_i \hat{e}_i = 0$ by (15.16), so that $\langle \hat{e} \rangle = \langle (\mathbf{1} - q)\hat{e} \rangle$, where $q = (q_i)_i$. Using (15.14), we find $|\langle \hat{e} \rangle| \leq 2cM\|\hat{e}\|_{2,N} \leq \frac{1}{4}\|\hat{e}\|_{2,N}$. Together with $\|\hat{e}\|_{2,N} \leq \|\hat{e}^0\|_{2,N} + |\langle \hat{e} \rangle|$, this gives

$$|\langle \hat{e} \rangle| \leq 2cM\|\hat{e}\|_{2,N}, \quad \|\hat{e}\|_{2,N} \leq \frac{4}{3}\|\hat{e}^0\|_{2,N}. \quad (15.24)$$

Step 2: the projected equation. Applying Π , and using $\Pi \mathbf{1} = 0$ and $\Pi \Lambda \mathbf{1} = \Pi \lambda = \lambda$, we obtain

$$(\hat{e}^0)' = 2c\mathcal{T}\hat{e}^0 + R_2, \quad R_2 = 2c\langle \hat{e} \rangle \lambda + \Pi R_1, \quad \hat{e}^0(0) = c^{-1}\Pi((z - 1)\mathbf{1} - c\lambda) = -\lambda.$$

Using Step 1 and $\|\lambda\|_{2,N} \leq M$, we find $\|R_2\|_{2,N} \leq 40c^2M^2\|\hat{e}\|_{2,N} \leq 54c^2M^2\|\hat{e}^0\|_{2,N}$. Then $\frac{d}{dt}\|\hat{e}^0\|_{2,N}^2 \leq (4cM + 108c^2M^2)\|\hat{e}^0\|_{2,N}^2 \leq 18cM\|\hat{e}^0\|_{2,N}^2$, where we used $cM \leq 1/8$, and therefore

$$\|\hat{e}^0(t)\|_{2,N} \leq Me^{9cMt}.$$

Step 3: comparison. By variation of constants on $\mathcal{H}_0$,

$$\hat{e}^0(t) + e^{2ct\mathcal{T}}\lambda = \int_0^t e^{2c(t-s)\mathcal{T}} R_2(s) ds.$$

Since $\|e^{2c(t-s)\mathcal{T}}\| \leq e^{2cM(t-s)}$, the right side has norm at most

$$\int_0^t e^{2cM(t-s)} 54c^2 M^3 e^{9cMs} ds \leq 54c^2 M^3 t e^{9cMt} \leq 54cM^3 H e^{9MH} \quad (ct \leq H).$$

Finally, $\|\hat{e} - \hat{e}^0\|_{2,N} = |\langle \hat{e} \rangle| \leq \frac{8}{3}cM\|\hat{e}^0\|_{2,N} \leq 3cM^2 e^{9MH}$ by (15.24). Adding the two bounds proves (15.23). $\square$

The limit of $\mathcal{T}$ for the semicircle law is identified with Chebyshev polynomials.

Lemma 15.10 (The compressed multiplication for the semicircle law). *Let $L_0^2(\rho_{\text{sc}}) = \{f \in L^2(\rho_{\text{sc}}) : \int f \rho_{\text{sc}} = 0\}$, and let $\mathcal{T}_{\text{sc}} f = \lambda f - (\int \lambda f \rho_{\text{sc}}) \mathbf{1}$ for $f \in L_0^2(\rho_{\text{sc}})$. Then $\mathcal{T}_{\text{sc}}$ is a bounded self-adjoint operator on $L_0^2(\rho_{\text{sc}})$, and for every $h \in \mathbb{R}$,*

$$\|e^{2h\mathcal{T}_{\text{sc}}}\lambda\|_{L^2(\rho_{\text{sc}})}^2 = \int e^{4h\lambda} \rho_{\text{sc}}(\lambda) d\lambda = 2\mathcal{B}(h).$$

Proof. Self-adjointness follows as for $\mathcal{T}$. Let U_n be the Chebyshev polynomials of the second kind, and let $u_n(\lambda) = U_n(\lambda/2)$. The U_n are orthogonal on $(-1, 1)$ with weight $(1-x^2)^{1/2}$ and $\int_{-1}^1 U_n^2(1-x^2)^{1/2} dx = \pi/2$, and they satisfy $U_0 = 1$, $U_1(x) = 2x$ and $U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x)$ [117, Table 18.3.1 and (18.9.1), Table 18.9.1]. Substituting $\lambda = 2x$, we find that $(u_n)_{n \geq 0}$ is orthonormal in $L^2(\rho_{\text{sc}})$, with

$$u_0 = 1, \quad u_1 = \lambda, \quad \lambda u_0 = u_1, \quad \lambda u_n = u_{n+1} + u_{n-1} \quad (n \geq 1).$$

Polynomials are dense in $L^2(\rho_{\text{sc}})$, so $(u_n)_{n \geq 0}$ is an orthonormal basis, and $(u_n)_{n \geq 1}$ is an orthonormal basis of $L_0^2(\rho_{\text{sc}})$. In these bases, $\mathcal{T}_{\text{sc}} u_1 = u_2$ and $\mathcal{T}_{\text{sc}} u_n = u_{n+1} + u_{n-1}$ for $n \geq 2$, because the projection removes only the multiple of u_0 . Let $S : L^2(\rho_{\text{sc}}) \rightarrow L_0^2(\rho_{\text{sc}})$ be the unitary map with $Su_n = u_{n+1}$. Comparing the two recursions, we find $\mathcal{T}_{\text{sc}} S = S\Lambda_{\text{sc}}$, where Λ_{sc} is multiplication by λ on $L^2(\rho_{\text{sc}})$. Hence $e^{4h\mathcal{T}_{\text{sc}}} = S e^{4h\Lambda_{\text{sc}}} S^{-1}$, and, since $\lambda = u_1 = Su_0$,

$$\|e^{2h\mathcal{T}_{\text{sc}}}\lambda\|^2 = \langle u_1, e^{4h\mathcal{T}_{\text{sc}}} u_1 \rangle = \langle u_0, e^{4h\Lambda_{\text{sc}}} u_0 \rangle = \int e^{4h\lambda} \rho_{\text{sc}}(\lambda) d\lambda.$$

In the first step, we used that $\mathcal{T}_{\text{sc}}$ is self-adjoint. $\square$

For the GOE, we compare $\mathcal{T}$ with $\mathcal{T}_{\text{sc}}$ on the common space $L^2(0, 1)$. We write $\mathcal{T}_N$ for the operator $\mathcal{T}$ built from the centered eigenvalues $\tilde{\lambda}_i$.

Lemma 15.11 (The information when ct is bounded). *Let $H > 0$. On $F_{C_{\text{gr}}}$, for N large, all $c \in (0, 1/32]$ and all $h \in [0, H]$,*

$$\left| \|e^{2h\mathcal{T}_N} \tilde{\lambda}\|_{2,N} - \sqrt{2\mathcal{B}(h)} \right| \leq 2(2 + 4H)e^{8H}E_N. \quad (15.25)$$

Moreover, there exists C'_H depending only on H such that, on $F_{C_{\text{gr}}}$, for N large, all $c \in (0, 1/32]$ and all $t \geq 0$ with $ct \leq H$, the information of $A = cW_N$ satisfies

$$\left| \frac{2e^{4t}}{Nc^2} \mathcal{I}_N(t) - 2\mathcal{B}(ct) \right| \leq C'_H(c + E_N). \quad (15.26)$$

Proof. Let $\hat{\Pi}$ be the orthogonal projection onto functions with mean zero in $L^2(0, 1)$, and let $\mathcal{T}'_N = \hat{\Pi}\Lambda_N\hat{\Pi}$ and $\mathcal{T}' = \hat{\Pi}\Lambda_\gamma\hat{\Pi}$, where Λ_N and Λ_γ are multiplication by $\ell_N - \bar{\lambda}$ and by γ . These are self-adjoint operators on $L^2(0, 1)$ of norm at most 4 and 2, respectively, by (15.22). The identification of $(\mathbb{R}^N, \langle \cdot, \cdot \rangle_N)$ with step functions is an isometry that commutes with Π and Λ , so that $\|e^{2h\mathcal{T}_N} \tilde{\lambda}\|_{2,N} = \|e^{2h\mathcal{T}'_N}(\ell_N - \bar{\lambda})\|_{L^2(0,1)}$. Similarly, the map $F \mapsto F \circ \gamma$ is an isometry from $L^2(\rho_{\text{sc}})$ into $L^2(0, 1)$, which maps $L^2_0(\rho_{\text{sc}})$ into functions with mean zero and intertwines $\mathcal{T}_{\text{sc}}$ with $\mathcal{T}'$. Using Lemma 15.10, we find $\|e^{2h\mathcal{T}'}\gamma\|_{L^2(0,1)} = \sqrt{2\mathcal{B}(h)}$.

Next, (15.22) implies that $\|\mathcal{T}'_N - \mathcal{T}'\| \leq \|\ell_N - \bar{\lambda} - \gamma\|_\infty \leq 2E_N$. Using the identity $e^X - e^Y = \int_0^1 e^{sX}(X - Y)e^{(1-s)Y}ds$ for bounded operators, we find $\|e^{2h\mathcal{T}'_N} - e^{2h\mathcal{T}'}\| \leq 4hE_Ne^{8h}$. Using also $\|\ell_N - \bar{\lambda} - \gamma\|_{L^2(0,1)} \leq 2E_N$ and $\|\gamma\|_{L^2(0,1)} = 1$, we obtain

$$\left| \|e^{2h\mathcal{T}'_N}(\ell_N - \bar{\lambda})\| - \|e^{2h\mathcal{T}'}\gamma\| \right| \leq (2 + 4h)e^{8h}E_N \leq 2(2 + 4H)e^{8H}E_N,$$

which proves (15.25).

For (15.26), we work with $c\tilde{G}$, $\tilde{G} = W_N - \bar{\lambda}I$, which has the same information, and we apply Lemma 15.9 with $M = 4$. Since $2\mathcal{I}_N(t) = N\|e(t)\|_{2,N}^2 = Nc^2e^{-4t}\|\hat{e}(t)\|_{2,N}^2$, the left side of (15.26) equals $|\|\hat{e}(t)\|_{2,N}^2 - 2\mathcal{B}(ct)|$. Combining (15.23) with (15.25) for $h = ct$, we have $|\|\hat{e}(t)\|_{2,N} - \sqrt{2\mathcal{B}(ct)}| \leq C_{H,4}c + 2(2 + 4H)e^{8H}E_N$. This proves (15.26), because $\sqrt{2\mathcal{B}(ct)} \leq e^{4H}$. $\square$

15.6 Proof of the explicit centers

Proof of Theorem 15.3. Fix $\eta \in (0, 1)$ and choose C_{gr} as in Section 15.5, so that $\mathbb{P}(F_{C_{\text{gr}}}) \geq 1 - \eta$ for all sufficiently large N . We work on $F_{C_{\text{gr}}}$, with the notation of Sections 15.4 and 15.5. Since $F_{C_{\text{gr}}} \subset Q_N$ for N large, the proof of Theorem 15.2 shows that, on $F_{C_{\text{gr}}}$ and for N large, $\mathcal{I}_N(0) > 1$ and τ_N is the unique solution of $\mathcal{I}_N(\tau_N) = 1$. The function $\log \mathcal{I}_N$ is decreasing with derivative at most $-(4 - Cc_N)$, by (15.2), and it vanishes at τ_N . This implies that, for every $s \geq 0$,

$$|s - \tau_N| \leq \frac{|\log \mathcal{I}_N(s)|}{4 - Cc_N}. \quad (15.27)$$

In both regimes, we show that the information at the proposed center plus u (or plus u/g_N) tends to e^{-4u} , uniformly for u in compact sets.

Bounded h_N . Let $H = \sup_N h_N + 1$, fix a compact set $K \subset \mathbb{R}$, and let $U = \sup(\{0\} \cup \{|u| : u \in K\})$. For $u \in K$, let $t = T_N^{\text{bulk}} + u$. Using $\frac{1}{2} \leq \mathcal{B}(h) \leq \frac{1}{2}e^{8h}$ (Remark 15.4), we have

$$|c_N t - h_N| \leq \frac{c_N}{4} |\log \mathcal{B}(h_N)| + c_N U \leq c_N(2H + 1 + U) \rightarrow 0.$$

For N large, we have $0 \leq c_N t \leq H$. Note that $J_N e^{-4t} = e^{-4u} / \mathcal{B}(h_N)$ by (15.11). Then (15.26) gives

$$\mathcal{I}_N(t) = \frac{J_N e^{-4t}}{2} \left(2\mathcal{B}(c_N t) + O_H(c_N + E_N) \right) = e^{-4u} \frac{\mathcal{B}(c_N t)}{\mathcal{B}(h_N)} + O_{H,U}(c_N + E_N).$$

We also have $|\mathcal{B}(c_N t) - \mathcal{B}(h_N)| \leq 4e^{8H}|c_N t - h_N| \rightarrow 0$ and $\mathcal{B}(h_N) \geq 1/2$, by Remark 15.4. We conclude that $\sup_{u \in K} |\mathcal{I}_N(T_N^{\text{bulk}} + u) - e^{-4u}| \rightarrow 0$ on $F_{C_{\text{gr}}}$. With $u = 0$, (15.27) gives $|T_N^{\text{bulk}} - \tau_N| \rightarrow 0$ on $F_{C_{\text{gr}}}$, which proves the convergence in probability after letting $\eta \downarrow 0$. Using (15.3) and (13.3), we also find that $\sup_{u \in K} |D_N(T_N^{\text{bulk}} + u) - \Psi(u)| \rightarrow 0$ in probability. Because $T_N^{\text{bulk}} \rightarrow \infty$, Corollary 3.12 then gives the second limit in (15.9) with T_N^{bulk} in place of τ_N . Finally, if $h_N \rightarrow 0$, then $\mathcal{B}(h_N) \rightarrow \mathcal{B}(0) = 1/2$, and $T_N^{\text{bulk}} - \frac{1}{4} \log(J_N/2) = \frac{1}{4} \log(2\mathcal{B}(h_N)) \rightarrow 0$.

Diverging h_N . Let $h_N \rightarrow \infty$. For N large, $c_N \leq 1/2$, and since $J_N \leq N$, we have $c_N \log N \geq c_N \log J_N = 4h_N \rightarrow \infty$. For N large, it follows that $c_N \geq 1/\log N$ and $|\log c_N| \leq \log \log N$. Recalling that $\log A_c = \frac{1}{2} \log c + 3 \log(1 - c) - 2 \log(1 + c) - \log(4\sqrt{\pi})$, we find $|\log A_{c_N}| \leq \frac{1}{2} \log \log N + C_0$ with a universal C_0 . Fix a compact set K , and for $u \in K$ let $t = t_N(u) = T_N(c_N) + u/g_N$. By (15.12),

$$4g_N t = \log N - \frac{3}{2} \log \log N + \log A_{c_N} + 4u, \quad N e^{-4g_N t} = \frac{(\log N)^{3/2} e^{-4u}}{A_{c_N}}.$$

The first identity gives $4g_N t / \log N \rightarrow 1$ uniformly in $u \in K$. As $g_N \in [1/4, 1]$, for N large we have $t \leq 2 \log N$ and $c_N t \geq c_N \log N / 5 \rightarrow \infty$.

We now use Lemma 13.11 and Proposition 13.7 with $c_1 = 1/2$. On $F_{C_{\text{gr}}}$, for N large, we have $\|\ell_N - \gamma\|_\infty \leq E_N \leq 1/16$ and $c_N E_N t \leq E_N \log N \leq 1/4$ for all $u \in K$. Let $U = \sup(\{0\} \cup \{|u| : u \in K\})$. By the second identity in the last display, $\sqrt{N} e^{-2g_N t} = A_{c_N}^{-1/2} (\log N)^{3/4} e^{-2u}$. Then the first bound in (13.23), with C_2 the constant of Lemma 13.11 for $c_1 = 1/2$, together with $1 + t \leq 3 \log N$, implies that

$$\sup_{u \in K} \left| \sqrt{2\mathcal{I}_N(t)} - \left(N \int \eta_t^2 \rho_{\text{sc}} \right)^{1/2} \right| \leq 3C_2 e^{2U} \frac{c_N}{\sqrt{A_{c_N}}} E_N (\log N)^{7/4} \leq 36C_2 e^{2U} E_N (\log N)^{7/4},$$

where η_t is the continuum relative error of Section 13.3 with $c = c_N$. In the last inequality, we used that $c A_c^{-1/2} = 2\pi^{1/4} c^{3/4} (1+c)(1-c)^{-3/2} \leq 6\sqrt{2} \pi^{1/4} c^{3/4} \leq 12$ for $0 < c \leq 1/2$.

The right side tends to zero, because $E_N = (C_{\text{gr}} + 7)N^{-5/12}$. No assumption that A_{c_N} is bounded away from zero is needed, since the factor c_N in (13.23) compensates $A_{c_N}^{-1/2}$. Because $c_N t \rightarrow \infty$ uniformly in $u \in K$, we may use (13.19), and we find

$$\begin{aligned} \frac{N}{2} \int \eta_i^2 \rho_{\text{sc}} &= \frac{N}{2} K_{c_N} t^{-3/2} e^{-4g_N t} (1 + o(1)) \\ &= \frac{K_{c_N} (4g_N)^{3/2}}{2A_{c_N}} e^{-4u} (1 + o(1)) = e^{-4u} (1 + o(1)), \end{aligned}$$

uniformly in $u \in K$. Here we used $t^{-3/2} = (4g_N)^{3/2} (\log N)^{-3/2} (1 + o(1))$ and the identity $A_c = 4K_c g^{3/2}$ of (13.1), valid for every $c \in (0, 1)$. Combining the last two displays, we obtain $\sup_{u \in K} |\mathcal{I}_N(t_N(u)) - e^{-4u}| \rightarrow 0$ on $F_{C_{\text{gr}}}$.

Taking $u = 0$ in (15.27), we find that $|T_N(c_N) - \tau_N| \rightarrow 0$ on $F_{C_{\text{gr}}}$, and letting $\eta \downarrow 0$ gives the convergence in probability. By (15.3) and (13.3), we also have $\sup_{u \in K} |D_N(T_N(c_N) + u/g_N) - \Psi(u)| \rightarrow 0$ in probability, which is the first limit in (15.13). Because $g_N T_N(c_N) \rightarrow \infty$, the distance D_N has cutoff profile Ψ with center $T_N(c_N)$ and window $1/g_N$, and Corollary 3.12 gives the second limit in (15.13). This completes the proof. $\square$

PART IV

Low temperature

Fixed low temperature

In this section, we determine the total-variation profile of the dynamics at a fixed rescaled inverse temperature $c > 1$, which is case (5) of Theorem 1.1. The result is Theorem 16.1, with its consequence for cutoff Corollary 16.2. Both are deduced in Section 16.1 from a deterministic statement about sequences of spectra, Theorem 16.5, whose proof occupies the rest of the section. Along the way, we prove Proposition 16.9, which compares the Gibbs measure, written in graph coordinates over the top eigencoordinate, with a product of Gaussian laws. That estimate is uniform in the top fraction θ_N , and Section 17 uses it when $\theta_N \rightarrow 0$.

We use the conventions of Section 3: the reflection symmetry Lemma 3.1, the facts on total variation in Lemma 3.6, the profiles and their properties in Lemma 3.9, the passage from pointwise convergence to profiles and mixing times in Lemma 3.11, and the removal of localizations in Lemma 3.8. The random-matrix inputs to the proof of Theorem 16.1 are the eigenvalue gaps at fixed confidence (Lemma 4.6) and rigidity and convergence at the edge (Lemma 4.2), from which we deduce a law of large numbers for the inverse gaps (Lemma 16.6). We also use the representation of the dynamics through the unnormalized process (Lemma 10.2) and the following results of Section 11: the Bismut-type bound for scores over a final time interval (Theorem 11.5), the generator and the scores in graph coordinates and the score of a marginal (Lemmas 11.9, 11.10, and 11.14), the Gaussian logarithmic Sobolev inequality (11.50), the time derivative of the conditional relative entropy (Proposition 11.18), the bounds on weighted logarithms (Lemma 11.17), the passage from weak convergence to total variation by Gaussian smoothing (Theorem 11.8 and Lemma 11.7), and the final comparison of product measures (Lemma 11.21).

16.1 Results and outline

The main result

Fix $c > 1$ and let W_N be a GOE matrix with eigenvalues $\lambda_1 \geq \dots \geq \lambda_N$. As in (3.8) and (3.14), we set

$$a_i = \frac{c}{2}(\lambda_1 - \lambda_i), \quad F(x) = \sum_{i=2}^N a_i x_i^2, \quad \theta_N = 1 - \frac{1}{N} \sum_{i=2}^N \frac{1}{2a_i}, \quad R_N = Na_2,$$

so that $L = \Delta_{S_N} - \nabla_{S_N} F \cdot \nabla_{S_N}$ and $\pi_{N,c} \propto e^{-F} \nu_N$. Almost surely the eigenvalues are distinct, so $0 = a_1 < a_2 < \dots < a_N$. Let $\theta = 1 - 1/c$, and define the centers

$$T_N'' = \begin{cases} \frac{\log R_N}{4a_2}, & R_N > 1, \\ 1, & R_N \leq 1, \end{cases} \quad T_N = \begin{cases} \frac{\log(2\theta_N R_N)}{4a_2}, & \theta_N > 0 \text{ and } 2\theta_N R_N > 1, \\ 1, & \text{otherwise.} \end{cases} \quad (16.1)$$

Both centers are positive. The events on which the second formulas are used have probabilities tending to zero (Lemma 16.7 below), so that $T_N'' = \log R_N / (4a_2)$ and $T_N = \log(2\theta_N R_N) / (4a_2)$ with probability tending to one. For $t < 0$ we set $D_N(t) = D_N(0)$, so that expressions such as $D_N(T_N'' + u/a_2)$ are defined for all $u \in \mathbb{R}$; this convention does not change the mixing times. Recall the profiles $\mathcal{V}$ and $\mathcal{V}_\theta$ from (3.17), and recall from Lemma 3.9 that $\mathcal{V}_\theta(u) = \mathcal{V}(u - \log(2\theta)/4)$.

Theorem 16.1 (Profile at fixed low temperature). *Fix $c > 1$ and let $\theta = 1 - 1/c$. Then D_N has cutoff profile $\mathcal{V}$ with center T_N and window a_2^{-1} , in the sense of Definition 3.7. Equivalently, for every compact set $K \subset \mathbb{R}$,*

$$\sup_{u \in K} |D_N(T_N'' + u/a_2) - \mathcal{V}_\theta(u)| \rightarrow 0 \quad (16.2)$$

in probability. For every compact interval $I \subset (0, 1)$,

$$\sup_{\varepsilon \in I} |a_2(t_N(\varepsilon) - T_N'') - \mathcal{V}_\theta^{-1}(\varepsilon)| \rightarrow 0, \quad \sup_{\varepsilon \in I} |a_2(t_N(\varepsilon) - T_N) - \mathcal{V}^{-1}(\varepsilon)| \rightarrow 0,$$

in probability, and $a_2 t_N(\varepsilon) / \log N \rightarrow 1/12$ in probability for every $\varepsilon \in (0, 1)$. For finitely many fixed values $c_1, \dots, c_m > 1$, these conclusions hold simultaneously for the same matrices W_N .

The centers are random through the first spectral gap $\lambda_1 - \lambda_2$, which is of order $N^{-2/3}$, and through θ_N . By Lemma 4.2, $N^{2/3}a_2$ converges in distribution to $c\Delta_2/2$, where Δ_2 is almost surely positive and finite (see Lemma 16.7). We make no statement with a deterministic center.

Corollary 16.2 (Cutoff at fixed low temperature). *Fix $c > 1$. For every fixed $\eta \in (0, 1)$,*

$$D_N((1 - \eta)T_N'') \rightarrow 1, \quad D_N((1 + \eta)T_N'') \rightarrow 0$$

in probability. For every $\varepsilon \in (0, 1)$, we have $t_N(\varepsilon) = T_N'' + O_{\mathbb{P}}(a_2^{-1})$, and $\log(a_2^{-1}) / \log N \rightarrow 2/3$ in probability. Further, $D_N(T_N'' + \omega_N/a_2) \rightarrow 0$ in probability for every deterministic sequence $\omega_N \rightarrow \infty$.

The proofs of Theorem 16.1 and Corollary 16.2 are given in Section 16.1.

Deterministic hypotheses

The dynamical estimates of this section are statements about a fixed sequence of spectra. Throughout the section, $N \geq 3$, and $0 = a_1 < a_2 \leq a_3 \leq \dots \leq a_N$ are real numbers, which may depend on N . We define F , L , π , θ_N , and R_N by the formulas above, write $P_t = e^{tL}$, and set

$$D_N(t) = \|\nu_N P_t - \pi\|_{\text{TV}}, \quad t_N(\varepsilon) = \inf\{t \geq 0 : D_N(t) \leq \varepsilon\},$$

with $D_N(t) = D_N(0)$ for $t < 0$. For the model, these are the quantities of Section 3, by (3.8) and the reduction to eigencoordinates.

Definition 16.3 (Hypotheses at low temperature). Let $\theta_* \in (0, 1)$ and $B, c_*, C_* > 0$. We say that (a_i) satisfies $H(\theta_*, B, c_*, C_*)$ if

$$\theta_N \geq \theta_*, \quad a_N \leq B, \quad a_j \geq c_*(j-1)^{2/3}N^{-2/3} \quad (2 \leq j \leq N), \quad a_2 \leq C_*N^{-2/3}. \quad (16.3)$$

For $k \geq 3$, $\omega > 0$, and $D_k \geq 1$, we say that (a_i) satisfies $H_k(\omega, D_k)$ if

$$\frac{a_3}{a_2} \geq 1 + \omega, \quad \frac{a_k}{a_2} \leq D_k. \quad (16.4)$$

A sequence of spectra satisfies these hypotheses if it does so for all sufficiently large N , with constants independent of N .

Recall that the first condition in (16.3) says that the Gaussian vector $(Z_i)_{i \geq 2}$ with independent coordinates of variances $(2a_i)^{-1}$ has mean squared norm at most $(1 - \theta_*)N$. The other conditions give a lower bound of order $((j-1)/N)^{2/3}$ on each a_j and the two-sided bound $c_*N^{-2/3} \leq a_2 \leq C_*N^{-2/3}$; an upper bound on a_k/a_2 is imposed separately in $H_k(\omega, D_k)$. The condition $H_3(\omega, D_3)$ is used in two places. Its lower bound separates the third gap from the second, and in Theorem 16.20 it ensures that the standardized coordinates x_j , $j \geq 3$, have standard Gaussian limits. Its upper bound controls the singular factor of the Gibbs measure in graph coordinates in Proposition 16.9. We record some elementary consequences.

Lemma 16.4 (Consequences of the hypotheses). *Assume $H(\theta_*, B, c_*, C_*)$. Then the following hold.*

1. We have $c_*N^{1/3} \leq R_N \leq C_*N^{1/3}$, and $F \leq BN$ on S_N .
2. We have $a_2^2 \sum_{j=2}^N a_j^{-2} \leq C_0$, where $C_0 = (C_*/c_*)^2 \sum_{n \geq 1} n^{-4/3}$.
3. For every $t > 0$, we have $\sum_{j=2}^N e^{-4a_j t} \leq C_1 N t^{-3/2}$, where C_1 depends only on c_* .
4. We have $\theta_N \leq 1 - (N-1)/(2BN)$.

Proof. The first claim follows from $a_2 \geq c_* N^{-2/3}$, $a_2 \leq C_* N^{-2/3}$, and $F(x) \leq a_N |x|^2$. For the second, we have $a_2^2 a_j^{-2} \leq C_*^2 c_*^{-2} (j-1)^{-4/3}$. For the third, we note that the function $u \mapsto e^{-4c_* t u^{2/3} N^{-2/3}}$ is decreasing on $[0, \infty)$, which implies that

$$\sum_{j=2}^N e^{-4a_j t} \leq \sum_{n \geq 1} e^{-4c_* t n^{2/3} N^{-2/3}} \leq \int_0^\infty e^{-4c_* t u^{2/3} N^{-2/3}} du = \frac{3\sqrt{\pi}}{4} (4c_* t)^{-3/2} N,$$

where we substituted $w = 4c_* t u^{2/3} N^{-2/3}$ and used $\int_0^\infty w^{1/2} e^{-w} dw = \sqrt{\pi}/2$. The last claim follows from $a_j \leq B$. $\square$

The deterministic theorem

For $u \in \mathbb{R}$, we write

$$t_u = T_N'' + \frac{u}{a_2}$$

for the time at offset u . Under $H(\theta_*, B, c_*, C_*)$, we have $R_N \geq c_* N^{1/3} > 1$ for large N , and then $t_u = \log R_N / (4a_2) + u/a_2$ and $a_2 t_u = \log R_N / 4 + u \rightarrow \infty$. In particular, $t_u > 0$ for large N .

Theorem 16.5 (Deterministic profile). *Let (a_i) be a sequence of spectra satisfying $H(\theta_*, B, c_*, C_*)$ and $H_3(\omega, D_3)$, and suppose that $\theta_N \rightarrow \theta$ for some $\theta \in (0, 1)$. Then the following hold.*

1. *For every compact set $K \subset \mathbb{R}$, we have $\sup_{u \in K} |D_N(t_u) - \mathcal{V}_\theta(u)| \rightarrow 0$.*
2. *For every compact interval $I \subset (0, 1)$, we have $\sup_{\varepsilon \in I} |a_2(t_N(\varepsilon) - T_N'') - \mathcal{V}_\theta^{-1}(\varepsilon)| \rightarrow 0$.*
3. *Let T_N be as in (16.1), so that $T_N = \log(2\theta_N R_N) / (4a_2)$ for large N . For every compact set $K \subset \mathbb{R}$ and every compact interval $I \subset (0, 1)$, we have $\sup_{u \in K} |D_N(T_N + u/a_2) - \mathcal{V}(u)| \rightarrow 0$ and $\sup_{\varepsilon \in I} |a_2(t_N(\varepsilon) - T_N) - \mathcal{V}^{-1}(\varepsilon)| \rightarrow 0$.*

The second part gives exact limits for the differences of mixing times on the scale a_2^{-1} : for $0 < \varepsilon < \varepsilon' < 1$, we have $a_2(t_N(\varepsilon) - t_N(\varepsilon')) \rightarrow \mathcal{V}_\theta^{-1}(\varepsilon) - \mathcal{V}_\theta^{-1}(\varepsilon') > 0$. The proof of Theorem 16.5 is completed in Section 16.11.

Outline of the proof

We first describe the mechanism, which is visible in the unnormalized process of Section 10. Let Y solve

$$dY_j = -2a_j Y_j dt + \sqrt{2/N} |Y| dB_j, \quad Y(0) = X_0 \sim \nu_N, \quad (16.5)$$

so that $X = \sqrt{N} Y / |Y|$ is the diffusion generated by L and started from ν_N . The first coordinate Y_1 is a martingale, because $a_1 = 0$, while the coordinates Y_j , $j \geq 2$, are pulled toward zero at rate $2a_j$.

- *Gaussian approximation at time t_p .* At the time $t_p = (\log N)/100$, the coordinates $(Y_1(t_p), Y_2(t_p))$ are close in total variation to a pair $\theta^{-1/2}(g_1, g_2)$ of independent centered Gaussian variables with variance θ^{-1} . They are built from the initial point and the noise up to time t_p .
- *Growth of the top coordinate.* The martingale Y_1 barely moves after t_p , while $|Y|^2$ relaxes to $Y_1(t_p)^2/\theta_N$. After normalization, the top eigencoordinate carries the fraction θ_N of the squared norm, and X_1 grows from order one to $\sqrt{\theta N}$.
- *The ratio of the first two coordinates.* The second coordinate decays like $e^{-2a_2 t}$. At time t_u , its standardized value $\sqrt{2a_2} X_2(t_u)$ is approximately

$$\sqrt{2a_2 N \theta_N} e^{-2a_2 t_u} \frac{Y_2(t_p)}{|Y_1(t_p)|} + G = \sqrt{2\theta_N} e^{-2u} \frac{Y_2(t_p)}{|Y_1(t_p)|} + G,$$

where G is a standard Gaussian variable produced by the noise in coordinate 2 during the last time interval of length of order a_2^{-1} , independent of (g_1, g_2) . The ratio $g_2/|g_1|$ has the standard Cauchy law, which gives the profile $\mathcal{V}_\theta$.

- *The remaining coordinates.* We retain all coordinates x_j with $a_j \leq Aa_2$, where $A = 16/r$ and $r > 0$ is a fixed constant chosen in Section 16.8; by Lemma 16.16, their number is bounded. On the support of a cutoff function that vanishes on $\{x_1^2 < rN/2\}$, the Gaussian logarithmic Sobolev inequality bounds the conditional relative entropy of the other coordinates given the retained ones by $(32a_2)^{-1}$ times its dissipation. The resulting contraction rate $16a_2$ is four times the rate $4a_2$ that sets the profile, and it absorbs an initial value of order N . We show that the conditional law of the coordinates x_j with $a_j > Aa_2$ given the retained ones is close to its equilibrium at time t_u .

The ratio of two independent centered Gaussian variables is a classical construction of the Cauchy law; see [110]. The exponential relaxation of the dynamics on the time scale $N^{2/3}$, governed by the random first spectral gap, was identified at zero temperature by Fyodorov, Perret, and Schehr [73]; see also the study of finite-size effects by Barbier, de Freitas Pimenta, Cugliandolo, and Stariolo [13]. The hitting-time bounds of orders $N^{2/3}$ and $N^{2/3} \log N$ in [155, Section 3, Theorem 3.3] concern gradient descent on the sphere started from the uniform distribution, and the time it needs to reach a fixed overlap with an extreme eigenvector. In a spherical quadratic model with a large spike, Gheissari, Grant, and Yu proved fast mixing from symmetric initial laws at low temperature [78, Theorem 1.2]. The renewal structure of the mean squared norm in Section 16.3 is the finite-dimensional counterpart of the self-consistent equation of Cugliandolo and Dean [51, (II.4)–(II.7)].

The proof proceeds in the following steps.

1. In Section 16.2, we compare the Gibbs measure in graph coordinates with the Gaussian product law $G = \otimes_{j \geq 2} \mathcal{N}(0, (2a_j)^{-1})$, including the singular factor $(1 - |y|^2/N)^{-1/2}$ of the graph parametrization (Proposition 16.9).
2. In Section 16.3, we compute the mean of $|Y|^2$ through a renewal equation (Lemma 16.12). In Section 16.4, we deduce that at the time $T_e = \chi_N/a_2$, with $\chi_N \rightarrow \infty$ slowly, the diffusion lies in a fixed spherical cap $\{x_1^2 \geq \sigma N\}$ outside an event of small fixed probability (Proposition 16.13). We also show that the uniformly initialized path, reweighted by a bounded measurable function of its past at entrance, stays in a slightly larger cap for times of order $\log R_N/a_2$ (Proposition 16.14).
3. In Section 16.5, we show that for every fixed $A > 1$ the number of coordinates x_j with $a_j \leq Aa_2$ is bounded (Lemma 16.16). For densities that vanish outside a cap, we prove a logarithmic Sobolev inequality for the conditional relative entropy of the other coordinates given these, with constant of order $(Aa_2)^{-1}$ (Lemma 16.17).
4. In Section 16.6, we prove the Gaussian limit of the first coordinates of $Y(t_p)$ (Proposition 16.18). In Section 16.7, we prove that $X_1(t_u)^2/N \rightarrow \theta$ (Proposition 16.19) and identify the weak limit of finitely many standardized coordinates (Theorem 16.20).
5. In Section 16.8, we weight the path law by a smooth function of the position at the time T_e , with values in $[0, 1]$, which vanishes outside a cap and has mean at least $1 - \varepsilon$. We multiply the endpoint law by a cutoff function that depends only on x_1^2 and vanishes outside a larger cap. We describe the density of this law and prove bounds on the cutoff, on the density, and on the scores (Lemmas 16.21 to 16.26).
6. In Section 16.9, we show that the relative entropy of the conditional law of the coordinates x_j with $a_j > Aa_2$ given the retained ones, under the weighted law multiplied by the cutoff and normalized, with respect to its Gaussian equilibrium, tends to zero at the times t_u (Lemma 16.27 and Proposition 16.28). The initial value is of order N , and the contraction rate is $16a_2$.
7. In Section 16.10, we upgrade the weak limit of finitely many standardized coordinates to convergence in total variation (Proposition 16.29), and Theorem 16.30 is its case of the second coordinate. In Section 16.11, we combine these estimates and prove Theorem 16.5. The number of retained coordinates may depend on N , and we pass to subsequences on which it is constant.

Table 16.1 lists the parameters of the proof in the order in which they are chosen. All constants in the estimates below depend only on the parameters chosen before them in the table.

Proof of the main result

We now deduce Theorem 16.1 and Corollary 16.2 from Theorem 16.5. The spectral input consists of the eigenvalue gaps at fixed confidence of Lemma 4.6, rigidity and

Parameter	Role	Depends on
$\theta_*, B, c_*, C_*; \varpi, D_3$	spectral constants, (16.3)–(16.4)	fixed
$\varepsilon \in (0, 1/8)$	loss allowed in the weight Ψ of (16.53)	fixed
$h, M; \sigma_e, \sigma_0$	bounds of Proposition 16.13 with loss $\varepsilon/2$; caps $\{x_1^2 \geq \sigma_e N\}$ and $\{x_1^2 > \sigma_0 N\}$ for the weight	ε , spectral constants
r	cutoff cap, (16.57); residence in $\{x_1^2 \geq 2rN\}$	σ_0, θ_*
$A = 16/r, M_A; J_N$	threshold for the retained coordinates and bound on their number; last retained index, (16.34)	$r, c_*, C_*; J_N$ also on N and the spectrum
ψ, χ, η	cutoff functions in (16.59)	r
$1/8$	Young's inequality for the two gradient terms in the entropy estimate	numerical
u , or a compact set K	offset of the time t_u	fixed
$N \rightarrow \infty$	determines $\chi_N = \sqrt{\log \log N}$, $T_e = \chi_N/a_2$, t_0 , t_1 , $t_p = (\log N)/100$, and p_N in (16.58)	all of the above
$M_* \rightarrow \infty, r_0 \downarrow 0$	last noise interval and restriction to $\{ Y_1(t_p) \geq r_0\}$ in Theorem 16.20	after $N \rightarrow \infty$
$\delta \downarrow 0$	loss in the weight used in Proposition 16.29; δ is fixed before $N \rightarrow \infty$, and the smoothing scale of Theorem 11.8 tends to zero before $\delta \downarrow 0$	after $N \rightarrow \infty$
$\varepsilon \downarrow 0$	removal of the weight	last

Table 16.1. Parameters in the proof of Theorem 16.5, in the order in which they are chosen.

convergence at the edge (Lemma 4.2), and the following law of large numbers for the inverse gaps, which we deduce from them.

Lemma 16.6 (A law of large numbers for the inverse gaps). *Let $\lambda_1 \geq \dots \geq \lambda_N$ be the eigenvalues of W_N . Then*

$$\frac{1}{N} \sum_{i=2}^N \frac{1}{\lambda_1 - \lambda_i} \longrightarrow 1 \quad \text{in probability.} \quad (16.6)$$

The left side of (16.6) is defined on the event $\{\lambda_1 > \lambda_2\}$, and we set it equal to zero on its complement. Since the ordered eigenvalues are continuous functions of the entries of W_N , this defines a random variable. The complement has probability zero, because the eigenvalues of W_N are almost surely distinct: the discriminant of the characteristic polynomial is a nonzero polynomial in the independent entries, whose zero set has Lebesgue measure zero, and the entries have a joint density. The limit 1 equals $\lim_{E \downarrow 2} m_{sc}(E)$, with m_{sc} as in (4.1).

Proof. We write I_N for the left side of (16.6). Fix $\eta \in (0, 1)$, and let $\mathcal{G}_N^{\eta/2}$ and $c_{\eta/2}$ be as in Lemma 4.6. Then $\liminf_N \mathbb{P}(\mathcal{G}_N^{\eta/2}) \geq 1 - \eta/2$, and by (4.7), on $\mathcal{G}_N^{\eta/2}$ and for large N ,

$$\lambda_1 - \lambda_i \geq c_{\eta/2}(i-1)^{2/3}N^{-2/3} \quad (2 \leq i \leq N). \quad (16.7)$$

In particular $\lambda_1 > \lambda_2$ there. Fix $\alpha \in (0, 1/2)$ and let $m = \lfloor \alpha N \rfloor$, so that $m \geq 2$ for large N . We split the sum defining I_N at the index m .

The first indices. By (16.7), on $\mathcal{G}_N^{\eta/2}$,

$$0 \leq \frac{1}{N} \sum_{i=2}^m \frac{1}{\lambda_1 - \lambda_i} \leq \frac{N^{-1/3}}{c_{\eta/2}} \sum_{k=1}^{m-1} k^{-2/3} \leq \frac{3}{c_{\eta/2}} \left(\frac{m}{N}\right)^{1/3} \leq \frac{3}{c_{\eta/2}} \alpha^{1/3}. \quad (16.8)$$

In the second inequality, we compared the sum with $\int_0^{m-1} s^{-2/3} ds = 3(m-1)^{1/3}$.

Comparison with the quantiles. By Lemma 4.2, choose $C = C_\eta$ and N_0 such that the event $F = F_C$ in (4.2) has probability at least $1 - \eta/2$ for $N \geq N_0$. On F , we have $|\lambda_i - \gamma_i| \leq CN^{-5/12}$ for all i , and Lemma 4.1 with $i = 1$ gives $|\lambda_1 - 2| \leq CN^{-5/12} + 7N^{-2/3}$. Then

$$\max_{2 \leq i \leq N} |(\lambda_1 - \lambda_i) - (2 - \gamma_i)| \leq e_N, \quad e_N = 2CN^{-5/12} + 7N^{-2/3}.$$

For $i > m$, we have $i/N > \alpha$, and Lemma 4.1 gives $2 - \gamma_i \geq d_\alpha$ with $d_\alpha = (3\pi\alpha/2)^{2/3}$. For large N , we have $e_N \leq d_\alpha/2$, so that $\lambda_1 - \lambda_i \geq d_\alpha/2$ for $i > m$. Writing the difference of the inverses over a common denominator, we obtain, on F and for large N ,

$$\frac{1}{N} \sum_{i=m+1}^N \left| \frac{1}{\lambda_1 - \lambda_i} - \frac{1}{2 - \gamma_i} \right| \leq \frac{2e_N}{d_\alpha^2}. \quad (16.9)$$

The sum over the quantiles. The map $x \mapsto \int_x^2 \rho_{\text{sc}}(y) dy$ is a continuous, strictly decreasing bijection from $[-2, 2]$ onto $[0, 1]$. Let $Q_{\text{sc}}: [0, 1] \rightarrow [-2, 2]$ be its inverse, which is continuous, and let $\phi(s) = (2 - Q_{\text{sc}}(s))^{-1}$ for $0 < s \leq 1$. Then $\gamma_i = Q_{\text{sc}}(i/N)$. Since $\rho_{\text{sc}}(x) \leq \pi^{-1}\sqrt{2-x}$, we have $s \leq 2(2 - Q_{\text{sc}}(s))^{3/2}/(3\pi)$, that is,

$$\phi(s) \leq \left(\frac{2}{3\pi s}\right)^{2/3} \quad (0 < s \leq 1), \quad (16.10)$$

which is the lower bound of Lemma 4.1 for a continuous index. The function ϕ is continuous on $[\alpha/2, 1]$, and we let ω_α be its modulus of continuity there. For large N , we have $m/N \geq \alpha/2$, and comparing each term $N^{-1}\phi(i/N)$ with $\int_{(i-1)/N}^{i/N} \phi(s) ds$, we find

$$\left| \frac{1}{N} \sum_{i=m+1}^N \phi\left(\frac{i}{N}\right) - \int_{m/N}^1 \phi(s) ds \right| \leq \omega_\alpha(1/N).$$

Since $|m/N - \alpha| \leq 1/N$ and $\phi \leq C_\alpha = (4/(3\pi\alpha))^{2/3}$ on $[\alpha/2, 1]$ by (16.10), we obtain

$$\left| \frac{1}{N} \sum_{i=m+1}^N \frac{1}{2 - \gamma_i} - \int_\alpha^1 \phi(s) ds \right| \leq \omega_\alpha(1/N) + \frac{C_\alpha}{N}. \quad (16.11)$$

Substituting $x = Q_{\text{sc}}(s)$, we find

$$\int_0^1 \phi(s) ds = \int_{-2}^2 \frac{\rho_{\text{sc}}(x)}{2 - x} dx = 1. \quad (16.12)$$

For the last equality, we substitute $x = 2 \cos t$ with $0 \leq t \leq \pi$, under which $\rho_{\text{sc}}(x) dx / (2 - x)$ becomes $\pi^{-1}(1 + \cos t) dt$, up to orientation. By (16.10), we have

$$0 \leq 1 - \int_\alpha^1 \phi(s) ds = \int_0^\alpha \phi(s) ds \leq C_{\text{sc}} \alpha^{1/3}, \quad C_{\text{sc}} = 3 \left(\frac{2}{3\pi} \right)^{2/3}.$$

Conclusion. Combining (16.8), (16.9), (16.11), and the last display, we find that on $\mathcal{G}_N^{\eta/2} \cap F$, for large N ,

$$|I_N - 1| \leq \left(\frac{3}{c_{\eta/2}} + C_{\text{sc}} \right) \alpha^{1/3} + \frac{2e_N}{d_\alpha^2} + \omega_\alpha(1/N) + \frac{C_\alpha}{N}.$$

Let $\gamma > 0$. We choose α such that the first term is less than $\gamma/2$. The other terms are deterministic and tend to zero as $N \rightarrow \infty$, and we obtain $\limsup_N \mathbb{P}(|I_N - 1| > \gamma) \leq \limsup_N \mathbb{P}((\mathcal{G}_N^{\eta/2} \cap F)^c) \leq \eta$. As $\eta \in (0, 1)$ is arbitrary, this proves (16.6). $\square$

In the proof, the confidence η is fixed before the index $m = \lfloor \alpha N \rfloor$, because the first indices are controlled by the lower bounds (16.7) at fixed confidence; the convergence of the empirical eigenvalue distribution alone would not control them. The constants c_η need not be bounded below as $\eta \rightarrow 0$, and the proof gives no rate of convergence.

Lemma 16.7 (Hypotheses for the GOE). *Fix $c > 1$, let $a_i = c(\lambda_1 - \lambda_i)/2$, and let $\theta = 1 - 1/c$. Then $\theta_N \rightarrow \theta$ in probability, and $N^{2/3}a_2$ and $(N^{2/3}a_2)^{-1}$ are tight. Further, for every $\eta \in (0, 1)$ there exist constants $\theta_* \in (0, \theta)$, $B, c_*, C_*, \omega > 0$, and $D_3 \geq 1$, depending only on c and η , and events $\mathcal{G}_N^\eta$ with $\liminf_N \mathbb{P}(\mathcal{G}_N^\eta) \geq 1 - \eta$, such that for all sufficiently large N , on $\mathcal{G}_N^\eta$, the spectrum has distinct eigenvalues and satisfies $H(\theta_*, B, c_*, C_*)$ and $H_3(\omega, D_3)$.*

Proof. Since $2a_i = c(\lambda_1 - \lambda_i)$, we have $\theta_N = 1 - c^{-1}N^{-1} \sum_{i \geq 2} (\lambda_1 - \lambda_i)^{-1}$, and Lemma 16.6 implies that $\theta_N \rightarrow 1 - 1/c = \theta$ in probability. By the second part of Lemma 4.2, $N^{2/3}a_2 = c\Delta_{2,N}/2 \Rightarrow c\Delta_2/2$, where Δ_2 is almost surely positive and finite. This implies that $N^{2/3}a_2$ and its reciprocal are tight; for the reciprocal, we use the portmanteau theorem [30, Section 2] for the closed sets $[0, 1/M]$, $M > 0$. In particular, $R_N = N^{1/3} \cdot N^{2/3}a_2 \rightarrow \infty$ and $\theta_N R_N \rightarrow \infty$ in probability, so that the events on which (16.1) uses the value 1 have probabilities tending to zero.

Now fix $\eta \in (0, 1)$. Let $\mathcal{G}'_N$ be the events of Lemma 4.6 with $\eta/3$ in place of η , and let $c_{\eta/3}, C_{\eta/3}$ be the corresponding constants in (4.7). We set

$$c_* = \frac{c}{2} c_{\eta/3}, \quad B = C_* = \frac{c}{2} C_{\eta/3}, \quad \theta_* = \frac{\theta}{2}, \quad D_3 = 2^{2/3} \frac{C_*}{c_*}.$$

On $\mathcal{G}'_N$, for large N , we have $c_*(i-1)^{2/3}N^{-2/3} \leq a_i \leq C_*(i-1)^{2/3}N^{-2/3}$ for $2 \leq i \leq N$. In particular, $a_N \leq C_* = B$, $a_2 \leq C_*N^{-2/3}$, and $a_3/a_2 \leq D_3$, where $D_3 \geq 1$ because $C_* \geq c_*$. Further, $a_3/a_2 = \Delta_{3,N}/\Delta_{2,N}$ converges in distribution to Δ_3/Δ_2 , by the second part of Lemma 4.2 and the continuous mapping theorem [30, Section 2], and $\Delta_3/\Delta_2 > 1$ almost surely, because the reflected GOE Airy point process is simple. We choose $\omega > 0$ such that $\mathbb{P}(\Delta_3/\Delta_2 \leq 1 + \omega) < \eta/3$, which is possible because these probabilities decrease to $\mathbb{P}(\Delta_3/\Delta_2 \leq 1) = 0$ as $\omega \downarrow 0$. The portmanteau theorem for the closed set $(-\infty, 1 + \omega]$ gives $\limsup_N \mathbb{P}(a_3/a_2 \leq 1 + \omega) < \eta/3$. Let $\mathcal{G}^\eta_N$ be the intersection of $\mathcal{G}'_N$, the events $\{a_3/a_2 > 1 + \omega\}$ and $\{\theta_N \geq \theta/2\}$, and the event that the eigenvalues are distinct. Since $\mathbb{P}(\theta_N < \theta/2) \rightarrow 0$ and the eigenvalues are almost surely distinct, we have $\liminf_N \mathbb{P}(\mathcal{G}^\eta_N) \geq 1 - 2\eta/3 \geq 1 - \eta$. These events are measurable, since they are defined by inequalities between the ordered eigenvalues. For large N , on $\mathcal{G}^\eta_N$, the spectrum has distinct eigenvalues and satisfies $H(\theta_*, B, c_*, C_*)$ and both inequalities of $H_3(\omega, D_3)$, with constants depending only on c and η . $\square$

Proof of Theorem 16.1. Fix $u \in \mathbb{R}$ and $\gamma > 0$. We first show that $D_N(t_u) \rightarrow \mathcal{V}_\theta(u)$ in probability. Because $\theta_N \rightarrow \theta$ in probability, there exists a deterministic sequence $\zeta_N \downarrow 0$ such that $\mathbb{P}(|\theta_N - \theta| > \zeta_N) \rightarrow 0$. Fix $\eta \in (0, 1)$ and let $E_N = \mathcal{G}^\eta_N \cap \{|\theta_N - \theta| \leq \zeta_N\}$. We claim that $\mathbb{P}(E_N \cap \{|D_N(t_u) - \mathcal{V}_\theta(u)| > \gamma\}) \rightarrow 0$. Suppose that this fails. Then there exist $\alpha > 0$ and a subsequence (N_k) along which this probability is at least α ; in particular, the event is nonempty. Recall that $D_N(t)$ depends on W_N only through its eigenvalues (Section 3.1). We may therefore choose, for each k , a spectrum in this event. The resulting deterministic sequence of spectra, indexed by N_k , satisfies $H(\theta_*, B, c_*, C_*)$ and $H_3(\omega, D_3)$ with $\theta_{N_k} \rightarrow \theta$, and $|D_{N_k}(t_u) - \mathcal{V}_\theta(u)| > \gamma$ for all k . This contradicts the first part of Theorem 16.5, whose proof applies verbatim along any subsequence of dimensions. We conclude from Lemma 3.8 that $D_N(t_u) \rightarrow \mathcal{V}_\theta(u)$ in probability.

Next, D_N is nonincreasing (Lemma 3.6), $D_N(t) \rightarrow 0$ as $t \rightarrow \infty$ (see the proof of Corollary 3.12), and $a_2 T''_N = \log R_N/4$ on $\{R_N > 1\}$, which tends to infinity in probability by Lemma 16.7. Recall from Lemma 3.9 that $\mathcal{V}_\theta$ is continuous and strictly decreasing from 1 to 0. Using Lemma 3.11 with $b_N = T''_N$, $w_N = a_2^{-1}$, and $V = \mathcal{V}_\theta$, we obtain (16.2) and the convergence of $a_2(t_N(\varepsilon) - T''_N)$, uniformly for $\varepsilon \in I$.

For the center T_N , we work on the event $\{R_N > 1, \theta_N > 0, 2\theta_N R_N > 1\}$, whose probability tends to one by Lemma 16.7. On this event, $T_N + u/a_2 = t_{u+\ell_N}$ with $\ell_N = \log(2\theta_N)/4$, and $\ell_N \rightarrow \log(2\theta)/4$ in probability. By (16.2) applied on a compact neighborhood of $u + \log(2\theta)/4$ and the continuity of $\mathcal{V}_\theta$, we have $D_N(T_N + u/a_2) \rightarrow$

$\mathcal{V}_\theta(u + \log(2\theta)/4) = \mathcal{V}(u)$ in probability, where we used the third part of Lemma 3.9. On the same event, $a_2 T_N = \log(2\theta_N R_N)/4$, which tends to infinity in probability, so that $a_2^{-1}/T_N \rightarrow 0$. Then Lemma 3.11 with $b_N = T_N$ and $V = \mathcal{V}$ implies that D_N has cutoff profile $\mathcal{V}$ with center T_N and window a_2^{-1} , together with the convergence of $a_2(t_N(\varepsilon) - T_N)$.

For the last claim, $a_2 t_N(\varepsilon) = \log R_N/4 + \mathcal{V}_\theta^{-1}(\varepsilon) + o_{\mathbb{P}}(1)$ and $\log R_N = (\log N)/3 + \log(N^{2/3}a_2) = (\log N)/3 + O_{\mathbb{P}}(1)$ by Lemma 16.7, so $a_2 t_N(\varepsilon)/\log N \rightarrow 1/12$ in probability. Finally, each of the conclusions above is a convergence in probability of a function of W_N . For finitely many temperatures, each of them holds separately for the same matrices W_N , and they therefore hold simultaneously. $\square$

Proof of Corollary 16.2. Fix $\eta \in (0,1)$ and $A > 0$. As in the proof of Theorem 16.1, $a_2 T_N'' \rightarrow \infty$ in probability. Then $(1 - \eta)T_N'' \leq t_{-A}$ with probability tending to one, and on this event $D_N((1 - \eta)T_N'') \geq D_N(t_{-A})$, because D_N is nonincreasing. Then (16.2) implies that $D_N(t_{-A}) \rightarrow \mathcal{V}_\theta(-A)$ in probability, and $\mathcal{V}_\theta(-A) \rightarrow 1$ as $A \rightarrow \infty$. This proves the first limit, and the second is proved in the same way with t_A (this is the argument of the third part of Lemma 3.11). The same bracketing with t_A proves the claim for $\omega_N \rightarrow \infty$. The statement $t_N(\varepsilon) = T_N'' + O_{\mathbb{P}}(a_2^{-1})$ is part of Theorem 16.1, and $\log(a_2^{-1})/\log N \rightarrow 2/3$ because $N^{2/3}a_2$ and its reciprocal are tight (Lemma 16.7). $\square$

16.2 The Gibbs measure in graph coordinates

In this section, we compare the Gibbs measure with a Gaussian product law. The comparison is uniform in the top fraction θ_N , which may tend to zero, so that it also applies in Section 17. We use only the spectral quantities, not the dynamics.

Graph coordinates. Let $Q_{\{1\}}(x) = (|x_1|, x_2, \dots, x_N)$ be the map that folds the first eigencoordinate (see Section 3.1). We identify the closed upper hemisphere $\{x \in S_N : x_1 \geq 0\}$ with the closed ball $\{y \in \mathbb{R}^{N-1} : |y|^2 \leq N\}$ through

$$y = (x_2, \dots, x_N), \quad x_1 = \sqrt{N - |y|^2}.$$

We call y the graph coordinates and write $y = (y_2, \dots, y_N)$, so that $y_j = x_j$. For a probability measure μ on S_N , we write μ^+ for the image of μ under $x \mapsto (x_2, \dots, x_N)$; when μ is invariant under the reflection R_1 of the first eigencoordinate, this is the image of μ under $Q_{\{1\}}$ in graph coordinates, and by the second part of Lemma 3.1 we have

$$\|\mu - \mu'\|_{\text{TV}} = \|\mu^+ - \mu'^+\|_{\text{TV}} \quad (16.13)$$

for any two R_1 -invariant probability measures μ, μ' on S_N ; see also (11.30). On the open upper hemisphere, the surface measure of S_N is $\sqrt{N} x_1^{-1} dy$. In particular, ν_N^+ has a Lebesgue density proportional to $(1 - |y|^2/N)^{-1/2}$ on $\{|y|^2 < N\}$.

Let G be the centered Gaussian measure on $\mathbb{R}^{N-1}$ with independent coordinates of variances $(2a_j)^{-1}$, $2 \leq j \leq N$, that is, $G(dy) = \prod_{j=2}^N \sqrt{a_j/\pi} e^{-a_j y_j^2} dy_j$. Write

$$J(y) = \begin{cases} (1 - |y|^2/N)^{-1/2}, & |y|^2 < N, \\ 0, & |y|^2 \geq N, \end{cases} \quad Z_G = \mathbb{E}_G[J].$$

Recall that $\pi \propto e^{-F} \nu_N$ and $F(x) = \sum_{j \geq 2} a_j y_j^2$. It follows that the folded Gibbs measure π^+ satisfies

$$\frac{d\pi^+}{dG}(y) = \frac{J(y)}{Z_G}. \quad (16.14)$$

This identity is exact; the factor J is the Jacobian of the graph parametrization. Under G , the squared radius $S = |y|^2$ has mean $\mathbb{E}_G S = \sum_{j \geq 2} (2a_j)^{-1} = (1 - \theta_N)N$. When $\theta_N > 0$, a typical sample of G lies inside the ball, and the remaining squared radius, about $\theta_N N$, corresponds to the top coordinate. We set

$$m_N = \theta_N N a_2 = \theta_N R_N.$$

Lemma 16.8 (Sums of Gaussian squares). *Assume $a_2^2 \sum_{j \geq 2} a_j^{-2} \leq C_0$, and let $S = |y|^2$ under G . Then the following hold.*

1. We have $\mathbb{E}_G S = (1 - \theta_N)N$ and $\text{Var}_G(S) = \sum_{j \geq 2} a_j^{-2}/2 \leq C_0/(2a_2^2)$.
2. For $|\lambda| \leq a_2/4$, we have

$$\log \mathbb{E}_G e^{\lambda(S - \mathbb{E}_G S)} \leq \frac{\lambda^2}{2} \sum_{j \geq 2} a_j^{-2} \leq \frac{C_0}{32}. \quad (16.15)$$

In particular, $G(|S - \mathbb{E}_G S| > x) \leq 2e^{C_0/32} e^{-a_2 x/4}$ for every $x \geq 0$.

3. Let $0 \leq \lambda \leq a_2/4$, and let $G_\lambda(dy) = e^{\lambda|y|^2} G(dy) / \mathbb{E}_G e^{\lambda S}$. Under G_λ , the coordinates are independent centered Gaussian variables with variances $(2(a_j - \lambda))^{-1}$, and S has a density bounded by $\sqrt{(a_2 - \lambda)(a_3 - \lambda)} \leq \sqrt{a_2 a_3}$.

Proof. We write $S = \sum_{j \geq 2} (2a_j)^{-1} g_j^2$ with independent standard Gaussian variables g_j . The first part follows from $\mathbb{E} g^2 = 1$ and $\text{Var}(g^2) = 2$. For the second, the identity $\mathbb{E} e^{s g^2} = (1 - 2s)^{-1/2}$ for $s < 1/2$ gives

$$\log \mathbb{E}_G e^{\lambda(S - \mathbb{E}_G S)} = \frac{1}{2} \sum_{j \geq 2} [-\log(1 - x_j) - x_j], \quad x_j = \frac{\lambda}{a_j} \in [-\frac{1}{4}, \frac{1}{4}].$$

For $|x| \leq 1/4$, we have $-\log(1 - x) - x = \sum_{n \geq 2} x^n/n \leq (x^2/2) \sum_{n \geq 0} |x|^n \leq x^2$. We obtain (16.15), where in the last step we used $\lambda^2 \leq a_2^2/16$. The tail bound follows from Markov's inequality applied to $e^{\pm a_2(S - \mathbb{E}_G S)/4}$.

For the third part, the density of G_λ is proportional to $\prod_j e^{-(a_j - \lambda)y_j^2}$, and $a_j - \lambda \geq 3a_j/4 > 0$. If y_j has density $\sqrt{p/\pi} e^{-py^2}$, then y_j^2 has density $\sqrt{p/\pi} s^{-1/2} e^{-ps}$ on $(0, \infty)$. With $p_j = a_j - \lambda$, we find that the density of $y_2^2 + y_3^2$ at $v > 0$ is

$$\frac{\sqrt{p_2 p_3}}{\pi} \int_0^v \frac{e^{-p_2 w - p_3(v-w)}}{\sqrt{w(v-w)}} dw \leq \frac{\sqrt{p_2 p_3}}{\pi} \int_0^v \frac{dw}{\sqrt{w(v-w)}} = \sqrt{p_2 p_3}.$$

The density of S is the convolution of this density with the law of the independent variable $\sum_{j \geq 4} y_j^2$, and convolution with a probability measure does not increase the supremum of a density. $\square$

Proposition 16.9 (The Gibbs measure in graph coordinates). *Let $B, D, C_0 > 0$, and assume that $\theta_N > 0$ and*

$$a_N \leq B, \quad a_3 \leq D a_2, \quad a_2^2 \sum_{j=2}^N a_j^{-2} \leq C_0. \quad (16.16)$$

There exist constants $c_0, C' > 0$, depending only on B, D , and C_0 , such that

$$|\sqrt{\theta_N} Z_G - 1| + \|\pi^+ - G\|_{\text{TV}} \leq \frac{C'}{m_N} + C' e^{-c_0 m_N}. \quad (16.17)$$

Further, consider a sequence of spectra satisfying (16.16) with fixed constants and $m_N \rightarrow \infty$. Then $x_1^2/(\theta_N N) \rightarrow 1$ in probability under π , and for every fixed $k \geq 2$ the law of $(\sqrt{2a_j} x_j)_{j=2}^k$ under π converges in total variation to the standard Gaussian law on $\mathbb{R}^{k-1}$.

The estimate (16.17) is uniform in θ_N : the constants do not depend on a lower bound for the top fraction. The concentration of S alone does not control the singular factor J near the sphere $|y|^2 = N$. For this part, we use the bound on the density of S under an exponential tilt from Lemma 16.8.

Proof. Let $W = \sqrt{\theta_N} J$, so that $\sqrt{\theta_N} Z_G = \mathbb{E}_G W$. On $\{S < N\}$, we have $1 - S/N = \theta_N - (S - \mathbb{E}_G S)/N$, so that

$$W = \left(1 - \frac{S - \mathbb{E}_G S}{\theta_N N}\right)^{-1/2}.$$

We bound $\mathbb{E}_G |W - 1|$ separately on three events. Here and below, C and c_0 denote positive constants depending only on B, D , and C_0 .

The central event. On $\{|S - \mathbb{E}_G S| \leq \theta_N N/2\}$, the derivative of $s \mapsto (1 - s/(\theta_N N))^{-1/2}$ at $s = S - \mathbb{E}_G S$ is at most $(\theta_N N)^{-1/2} 2^{3/2} \leq 2/(\theta_N N)$. As $W = 1$ at $S = \mathbb{E}_G S$, the first part of Lemma 16.8 implies that

$$\mathbb{E}_G [|W - 1|; |S - \mathbb{E}_G S| \leq \theta_N N/2] \leq \frac{2}{\theta_N N} \mathbb{E}_G |S - \mathbb{E}_G S| \leq \frac{2}{\theta_N N} \left(\frac{C_0}{2a_2^2}\right)^{1/2} \leq \frac{C}{m_N}.$$

The lower complement. On $\{S < \mathbb{E}_G S - \theta_N N/2\}$, we have $1 - S/N > 3\theta_N/2$, which gives $W \leq 1$ and $|W - 1| \leq 1$. The probability of this event is at most $Ce^{-m_N/8}$, by the second part of Lemma 16.8 with $x = \theta_N N/2$.

The upper complement. Let $\ell = \theta_N N/2$, so that $\mathbb{E}_G S + \ell = N - \ell$. On $\{S \geq N\}$ we have $W = 0$, and on $\{N - \ell < S < N\}$ we have $|W - 1| \leq W + 1$. The probability of $\{S > N - \ell\}$ is at most $Ce^{-m_N/8}$, as before. It remains to bound $\mathbb{E}_G[W; N - \ell < S < N]$. Let $\lambda = a_2/4$ and let G_λ be the tilted measure of Lemma 16.8. For a nonnegative function ϕ ,

$$\mathbb{E}_G[\phi(S)] = \mathbb{E}_G[e^{\lambda(S - \mathbb{E}_G S)}] \mathbb{E}_{G_\lambda}[\phi(S)e^{-\lambda(S - \mathbb{E}_G S)}] \leq e^{C_0/32} \mathbb{E}_{G_\lambda}[\phi(S)e^{-\lambda(S - \mathbb{E}_G S)}].$$

We use this identity with $\phi(S) = W \mathbf{1}_{\{N - \ell < S < N\}}$, and we write $S = N - r$ with $0 < r < \ell$. Then $S - \mathbb{E}_G S = 2\ell - r$ and $W = \sqrt{\theta_N N} r^{-1/2}$. The density of S under G_λ is at most $\sqrt{a_2 a_3} \leq \sqrt{D} a_2$, and we find

$$\mathbb{E}_G[W; N - \ell < S < N] \leq e^{C_0/32} \sqrt{D} a_2 \sqrt{\theta_N N} e^{-\lambda\ell} \int_0^\ell e^{-\lambda(\ell-r)} r^{-1/2} dr.$$

On $[0, \ell/2]$ the integrand is at most $e^{-\lambda\ell/2} r^{-1/2}$, and on $[\ell/2, \ell]$ it is at most $(\ell/2)^{-1/2} e^{-\lambda(\ell-r)}$. The integral is then at most $2\sqrt{\ell} e^{-\lambda\ell/2} + \sqrt{2} \ell^{-1/2} \lambda^{-1}$. Using $\lambda\ell = m_N/8$, $a_2 \sqrt{\theta_N N} \sqrt{\ell} = m_N/\sqrt{2}$, and $a_2 \sqrt{\theta_N N} \ell^{-1/2} \lambda^{-1} = 4\sqrt{2}$, we obtain

$$\mathbb{E}_G[W; N - \ell < S < N] \leq C(m_N e^{-3m_N/16} + e^{-m_N/8}) \leq C e^{-c_0 m_N}.$$

Conclusion. Combining the three bounds, we find

$$\mathbb{E}_G|W - 1| \leq \frac{C}{m_N} + C e^{-c_0 m_N}. \quad (16.18)$$

This bounds $|\sqrt{\theta_N N} Z_G - 1| = |\mathbb{E}_G(W - 1)|$. For the total-variation distance, by (16.14) and the first part of Lemma 3.6, we have

$$\|\pi^+ - G\|_{\text{TV}} = \frac{1}{2} \mathbb{E}_G \left| \frac{W}{\mathbb{E}_G W} - 1 \right| \leq \frac{\mathbb{E}_G|W - 1| + |\mathbb{E}_G W - 1|}{2 \mathbb{E}_G W} \leq \frac{\mathbb{E}_G|W - 1|}{\mathbb{E}_G W}.$$

If $\mathbb{E}_G|W - 1| \leq 1/2$, then $\mathbb{E}_G W \geq 1/2$, and the right side is at most $2\mathbb{E}_G|W - 1|$. Otherwise, the total-variation distance is at most $1 \leq 2\mathbb{E}_G|W - 1|$. In both cases, (16.18) gives (16.17) after enlarging C' .

The limits as $m_N \rightarrow \infty$. Under π , we have $x_1^2 = N - |y|^2$. Under G , the first part of Lemma 16.8 gives

$$\mathbb{E}_G \left| \frac{N - S}{\theta_N N} - 1 \right|^2 = \frac{\text{Var}_G(S)}{\theta_N^2 N^2} \leq \frac{C_0}{2m_N^2} \rightarrow 0.$$

Using $\|\pi^+ - G\|_{\text{TV}} \rightarrow 0$, we find that the event $\{|x_1^2/(\theta_N N) - 1| > \gamma\}$ has π -probability tending to zero for every $\gamma > 0$. The image of G under $y \mapsto (\sqrt{2a_j} y_j)_{j=2}^k$ is the standard Gaussian law, and the image of π is the image of π^+ , since the map does not involve x_1 . The second part of Lemma 3.6 implies that the total-variation distance between the two images is at most $\|\pi^+ - G\|_{\text{TV}} \rightarrow 0$. $\square$

The following consequence makes precise the statement that the Gibbs measure places the fraction θ_N of $|x|^2$ on the top eigencoordinate.

Corollary 16.10 (The top fraction under the Gibbs measure). *Under the hypotheses of Proposition 16.9, for every $\gamma > 0$,*

$$\pi\left(\left|\frac{x_1^2}{\theta_N N} - 1\right| > \gamma\right) \leq \frac{C'}{m_N} + C'e^{-c_0 m_N} + \frac{C_0}{2\gamma^2 m_N^2}. \quad (16.19)$$

For the GOE model, in either case (F) or case (M) of Lemma 4.7, the left side of (16.19) with $\pi = \pi_{N, c_N}$ tends to zero in probability for every $\gamma > 0$. In particular, at a fixed temperature $c > 1$, we have $\pi_{N, c}(|x_1^2/N - (1 - 1/c)| > \gamma) \rightarrow 0$ in probability for every $\gamma > 0$.

Proof. Recall that $x_1^2 = N - |y|^2$ and $\mathbb{E}_G |y|^2 = (1 - \theta_N)N$. By Chebyshev's inequality and Lemma 16.8, we have $G(|N - |y|^2 - \theta_N N| > \gamma \theta_N N) \leq \text{Var}_G(|y|^2)/(\gamma \theta_N N)^2 \leq C_0/(2\gamma^2 m_N^2)$. The event in (16.19) is invariant under R_1 , and its π -probability equals its π^+ -probability. Then (16.17) implies (16.19). For the GOE model, $m_N \rightarrow \infty$ in probability by the first part of Lemma 4.7, and on the events of its second part (intersected with $\{\theta_N > 0\}$ in case (M)) the hypotheses (16.16) hold with $D = 2^{2/3}C_*/c_*$ and $C_0 = 4(C_*/c_*)^2$, by (4.11). The right side of (16.19) then tends to zero in probability, by Lemma 3.8. In case (F), we also use $\theta_N \rightarrow 1 - 1/c$ in probability. $\square$

The statement and proof of Proposition 16.9 use only (16.16). Under $H(\theta_*, B, c_*, C_*)$ and $H_3(\omega, D_3)$, these hold with $D = D_3$ and C_0 from Lemma 16.4, and $m_N \geq \theta_* c_* N^{1/3}$. The following static bounds on a cap give an upper bound on the density of the Gibbs measure in graph coordinates away from the sphere $\{|y|^2 = N\}$, which we use in Section 16.8.

Lemma 16.11 (The Gibbs measure on a cap). *Assume $H(\theta_*, B, c_*, C_*)$, and let $0 < r \leq \theta_*/2$. Let $C_r = \{x \in S_N : x_1 \geq \sqrt{rN}\}$, let $D_r = \{y : |y|^2 \leq (1 - r)N\}$ be its image in graph coordinates, and let π_{C_r} be the probability measure π conditioned on C_r . Then, for N larger than a constant depending only on the spectral constants and r ,*

$$G(D_r^c) \leq e^{C_0/32 - m_N/8}, \quad \frac{d\pi_{C_r}}{dG} = \frac{J\mathbf{1}_{D_r}}{\mathbb{E}_G[J\mathbf{1}_{D_r}]} \leq 2r^{-1/2}. \quad (16.20)$$

Further, $Z_G \geq \theta_N \geq \theta_*$, and $d\pi^+/dG \leq r^{-1/2}\theta_*^{-1}$ on D_r .

Proof. Since $r \leq \theta_N/2$, we have $D_r^c \subset \{S - \mathbb{E}_G S > \theta_N N/2\}$, and the first bound follows from (16.15) and Markov's inequality applied to $e^{a_2(S - \mathbb{E}_G S)/4}$. The formula for the density follows from (16.14). On D_r , we have $1 \leq J \leq r^{-1/2}$. This gives $\mathbb{E}_G[J \mathbf{1}_{D_r}] \geq G(D_r) \geq 1/2$ for large N , and the bound on the density follows. We also have $G(S \geq N) \leq \mathbb{E}_G S/N = 1 - \theta_N$ by Markov's inequality, which implies $Z_G \geq G(S < N) \geq \theta_N$. The last claim follows from (16.14). $\square$

16.3 The unnormalized process and its mean

From now on, we assume $H(\theta_*, B, c_*, C_*)$ and $N \geq 3$. Unless said otherwise, constants denoted C and c_0 depend only on θ_* , B , c_* , and C_* , and they may change from line to line.

The unnormalized process. We use Lemma 10.2 with the quadratic potential coefficients $\rho_i = a_i$. Let $\mathcal{B} = (\mathcal{B}_{ij})_{i,j=1}^N$ be a matrix of independent standard Brownian motions, independent of $X_0 \sim \nu_N$, and let Φ solve the linear matrix equation

$$d\Phi = -2D_a\Phi dt + \sqrt{2/N} d\mathcal{B}\Phi, \quad \Phi(0) = I, \quad D_a = \text{diag}(a_1, \dots, a_N). \quad (16.21)$$

By the fifth part of Lemma 10.2, $\Phi(t)$ is invertible for all $t \geq 0$, and $Y = \Phi X_0$ is the unnormalized process with initial value X_0 , with respect to the filtration $(\mathcal{F}_t)$ generated by X_0 and $\mathcal{B}$. That is, there exists a standard Brownian motion $B = (B_1, \dots, B_N)$ for this filtration such that Y solves (16.5). The first two parts of that lemma imply that Y never vanishes, and $X = \sqrt{N}Y/|Y|$ is the diffusion generated by L , started from ν_N ; it solves

$$dX_j = \sqrt{2} (dB_j - X_j N^{-1} X \cdot dB) + \left[-2a_j + \frac{2F(X)}{N} - \frac{N-1}{N} \right] X_j dt. \quad (16.22)$$

In particular, the law of X_t is $\nu_N P_t$. We also know from the first part of the same lemma that Y is a strong solution, and Y and X are adapted to the filtration generated by X_0 and B . Note that Y_1 is a martingale, since $a_1 = 0$. We write

$$v(t) = \mathbb{E}|Y(t)|^2, \quad h_i(t) = \mathbb{E}Y_i(t)^2.$$

The process Y has moments of all orders at every fixed time, since $|Y(t)| \leq \|\Phi(t)\|\sqrt{N}$ and Φ solves a linear equation.

The renewal kernel. Let

$$k(t) = \frac{2}{N} \sum_{j=2}^N e^{-4a_j t}, \quad Y = \sum_{n \geq 1} k^{*n}, \quad H(t) = 1 + \int_0^t Y(s) ds,$$

where k^{*n} is the n -fold convolution of k on $[0, \infty)$. Note that $\int_0^\infty k = (2/N) \sum_{j \geq 2} (4a_j)^{-1} = 1 - \theta_N \leq 1 - \theta_*$. In the terminology of renewal theory, k is a defective renewal kernel.

Lemma 16.12 (Renewal estimates). Assume $H(\theta_*, B, c_*, C_*)$. There exists $C > 0$, depending only on θ_* and c_* , such that the following hold.

1. For $t \geq 0$, we have $k(t) \leq C(1+t)^{-3/2}$, $Y(t) \leq C(1+t)^{-3/2}$, $\int_t^\infty Y \leq C(1+t)^{-1/2}$, $\int_0^\infty Y = (1 - \theta_N)/\theta_N$, and $H(t) \leq 1/\theta_N \leq 1/\theta_*$.
2. We have the identity

$$v = H + \frac{N}{2} Y + \frac{2}{N} H * v, \quad (16.23)$$

and, for $0 \leq t \leq N$,

$$v(t) \leq C e^{2t/(\theta_* N)} [1 + N(1+t)^{-3/2}], \quad \int_0^t v(s) ds \leq C(t + N). \quad (16.24)$$

3. For $1 \leq t \leq N$, we have $|h_1(t) - \theta_N^{-1}| \leq C[(1+t)^{-1/2} + t/N]$. For every i and $t \geq 0$, we have $0 \leq h_1(t) - h_i(t) \leq 4a_i t h_1(t)$.
4. The first column Φe_1 satisfies $\mathbb{E}|\Phi(t)e_1|^2 \leq \theta_*^{-1} e^{2t/(\theta_* N)}$, and $\mathbb{E}|\Phi_{11}(t) - 1|^2 \leq Ct/N$ for $0 \leq t \leq N$.

Proof. First part. By the third part of Lemma 16.4, $k(t) \leq 2C_1 t^{-3/2}$, and $k(t) \leq 2$ for all t . Together, these bounds imply $k(t) \leq C(1+t)^{-3/2}$. For $n \geq 1$, write

$$k^{*n}(t) = \int_{\{s \in [0, \infty)^n : s_1 + \dots + s_n = t\}} \prod_{i=1}^n k(s_i),$$

where the integral is with respect to the Lebesgue measure in the variables $(s_1, \dots, s_{n-1})$, with $s_n = t - s_1 - \dots - s_{n-1}$. On this simplex at least one coordinate is at least t/n . It follows that $k^{*n}(t)$ is at most the sum over i of the integrals over $\{s_i \geq t/n\}$. The integrand is symmetric under permutations of the coordinates, and each of these integrals equals the one with $i = n$. In that integral we bound $k(s_n) \leq C(1+t/n)^{-3/2}$ and integrate $s_1, \dots, s_{n-1}$ over $[0, \infty)$. We obtain

$$k^{*n}(t) \leq nC \left(1 + \frac{t}{n}\right)^{-3/2} (1 - \theta_N)^{n-1} \leq Cn^{5/2} (1+t)^{-3/2} (1 - \theta_*)^{n-1},$$

where we used $1 + t/n \geq (1+t)/n$. Summing over n , we obtain the bound on Y , and integrating it, we find the bound on $\int_t^\infty Y$. As $\int k^{*n} = (1 - \theta_N)^n$, we have $\int Y = (1 - \theta_N)/\theta_N$, which implies $H(t) \leq 1 + (1 - \theta_N)/\theta_N = 1/\theta_N$.

Second part. Using Itô's formula and (16.5), we have $d(Y_j^2) = -4a_j Y_j^2 dt + (2/N)|Y|^2 dt + dM_j$ with a martingale M_j . Taking expectations, and using $\mathbb{E}X_0 X_0^\top = I$, we find

$$h_j(t) = e^{-4a_j t} + \frac{2}{N} \int_0^t e^{-4a_j(t-s)} v(s) ds. \quad (16.25)$$

Summing over j and separating $j = 1$, for which $a_1 = 0$, we obtain the renewal equation

$$v = 1 + \frac{N}{2} k + k * v + \frac{2}{N} 1 * v,$$

where $1 * v(t) = \int_0^t v$. The measure $\delta_0 + Y$ is the convolution inverse of $\delta_0 - k$, since $Y = k + k * Y$. Convolving the renewal equation with $\delta_0 + Y$, and using $(\delta_0 + Y) * 1 = H$ and $(\delta_0 + Y) * k = Y$, we obtain (16.23). The convolutions are well defined, because all terms are nonnegative.

The first part shows that $g = H + (N/2)Y$ satisfies $g(t) \leq C[1 + N(1+t)^{-3/2}]$ and $\int_0^t g \leq C(t+N)$. Using $H \leq 1/\theta_*$ in (16.23), we find $v(t) \leq g(t) + \alpha \int_0^t v$ with $\alpha = 2/(\theta_* N)$. By Gronwall's inequality,

$$v(t) \leq g(t) + \alpha \int_0^t e^{\alpha(t-s)} g(s) ds \leq g(t) + \alpha e^{\alpha t} C(t+N),$$

and for $t \leq N$ we have $\alpha(t+N) \leq 4/\theta_*$. This proves the first bound in (16.24). Integrating the same inequality over $[0, t]$ and using $\alpha t e^{\alpha t} \leq C e^{2/\theta_*}$ for $t \leq N$ gives the second bound.

Third part. By (16.25) with $j = 1$ and by (16.23),

$$h_1(t) = 1 + \frac{2}{N} \int_0^t v = H(t) + \frac{2}{N} \int_0^t H(s) ds + \frac{4}{N^2} \int_0^t (H * v)(s) ds.$$

We have $\theta_N^{-1} - H(t) = \int_t^\infty Y \leq C(1+t)^{-1/2}$, the second term is at most $2t/(\theta_* N)$, and the third is at most $(4t/(\theta_* N^2)) \int_0^t v \leq Ct(t+N)/N^2 \leq Ct/N$ for $t \leq N$. This proves the bound on h_1 . Further, (16.25) shows that $h_i(t) \leq 1 + (2/N) \int_0^t v = h_1(t)$ and $h_i(t) \geq e^{-4a_i t} h_1(t)$, so that $0 \leq h_1 - h_i \leq (1 - e^{-4a_i t}) h_1 \leq 4a_i t h_1$.

Fourth part. The column $V = \Phi e_1$ solves $dV = -2D_a V dt + \sqrt{2/N} dB V$ with $V(0) = e_1$. As in the second part, $f_j = \mathbb{E} V_j^2$ satisfies $f'_j = -4a_j f_j + (2/N)f$ with $f = \sum_j f_j$ and $f_j(0) = \mathbf{1}_{\{j=1\}}$. Then $f = 1 + k * f + (2/N)1 * f$, and convolving with $\delta_0 + Y$ as before, we find $f = H + (2/N)H * f \leq \theta_*^{-1} + (2/(\theta_* N)) \int_0^t f$. Gronwall's inequality now gives the first claim. The entry $\Phi_{11} = V_1$ is a martingale with $\Phi_{11}(0) = 1$ and quadratic variation $(2/N) \int_0^t |V(s)|^2 ds$, which implies $\mathbb{E} |\Phi_{11}(t) - 1|^2 = (2/N) \int_0^t f \leq Ct/N$ for $t \leq N$. $\square$

16.4 Entrance into a cap and residence

For $\sigma \in (0, 1)$, let $\mathcal{C}_\sigma = \{x \in S_N : x_1 \geq \sqrt{\sigma N}\}$ be the spherical cap around the top eigenvector, and let $-\mathcal{C}_\sigma = R_1 \mathcal{C}_\sigma$. In this section, we show that the diffusion enters $\mathcal{C}_\sigma \cup (-\mathcal{C}_\sigma)$ at a time much smaller than $\log R_N/a_2$, outside an event of small fixed probability (Proposition 16.13), and that the original uniform-start path, after a bounded reweighting at a deterministic entrance time supported in these caps, stays in a slightly larger cap for times of order $\log R_N/a_2$ (Proposition 16.14).

Proposition 16.13 (Entrance into a cap). *Assume $H(\theta_*, B, c_*, C_*)$, let $\chi_N \rightarrow \infty$ with $\chi_N = o(\log R_N)$, and let $T_e = \chi_N/a_2$. For every $\varepsilon > 0$ there exist $h > 0$ and $M < \infty$ with $h^2 < M$, depending only on $\varepsilon, \theta_*, c_*$, and C_* , such that*

$$\mathbb{P}(|Y_1(T_e)| \geq h, |Y(T_e)|^2 \leq M) \geq 1 - \varepsilon - o(1). \quad (16.26)$$

In particular, $\mathbb{P}(X_1(T_e)^2 \geq \sigma N) \geq 1 - \varepsilon - o(1)$ with $\sigma = h^2/M$.

Proof. We have $T_e \geq \chi_N N^{2/3}/C_*$ by Lemma 16.4, and $T_e/N = \chi_N/R_N \rightarrow 0$. It follows that $N(1 + T_e)^{-3/2} \leq C_*^{3/2} \chi_N^{-3/2} \leq C$, and by (16.24), we have $\mathbb{E}|Y(T_e)|^2 \leq C_2$ for large N , with C_2 depending only on θ_* , c_* , and C_* . Then Markov's inequality gives $\mathbb{P}(|Y(T_e)|^2 > M) \leq C_2/M$.

For the first coordinate, we condition on the matrix Φ , which is independent of X_0 . We have $Y_1(T_e) = \langle \varphi, X_0 \rangle$, where φ is the first row of $\Phi(T_e)$. For a fixed vector $\varphi \neq 0$, rotation invariance of ν_N shows that $\langle \varphi, X_0 \rangle$ has the law of $|\varphi| x_1$ under ν_N . The density of x_1 under ν_N , for $|x| < \sqrt{N}$, is

$$\frac{\Gamma(N/2)}{\Gamma((N-1)/2)\sqrt{\pi N}} \left(1 - \frac{x^2}{N}\right)^{(N-3)/2} \leq \frac{\Gamma(N/2)}{\Gamma((N-1)/2)\sqrt{\pi N}} \leq \frac{1}{\sqrt{2\pi}},$$

and we set it equal to zero for $|x| \geq \sqrt{N}$. For the bounds, we used $N \geq 3$ and $\Gamma(z + 1/2) \leq \sqrt{z} \Gamma(z)$ for $z = (N-1)/2$, which follows from the log-convexity of Γ . On the event $\{\Phi_{11}(T_e) \geq 1/2\}$, we have $|\varphi| \geq 1/2$, and then $\mathbb{P}(|Y_1(T_e)| \leq h \mid \Phi) \leq 4h/\sqrt{2\pi} \leq 2h$. The fourth part of Lemma 16.12 and Chebyshev's inequality give $\mathbb{P}(\Phi_{11}(T_e) < 1/2) \leq 4\mathbb{E}|\Phi_{11}(T_e) - 1|^2 \leq CT_e/N = o(1)$. Combining these bounds, we obtain

$$\mathbb{P}(|Y_1(T_e)| < h) \leq 2h + o(1).$$

Let $\varepsilon_0 = \min\{\varepsilon, 1/2\}$ and choose $h = \varepsilon_0/4$ and $M = \max\{2C_2/\varepsilon_0, 1\}$. Then $2h \leq \varepsilon_0/2$, $C_2/M \leq \varepsilon_0/2$, and $0 < h^2/M \leq 1/64 < 1$. Thus the probability in (16.26) is at least $1 - \varepsilon_0 - o(1)$, which is positive for large N and implies the stated bound for every $\varepsilon > 0$. On the event in (16.26), $X_1(T_e)^2 = NY_1(T_e)^2/|Y(T_e)|^2 \geq h^2N/M$. $\square$

The estimate (16.26) concerns the whole initial law ν_N . It does not assert that a single fixed cap carries all but a vanishing fraction of the mass; the fraction left out is a fixed number ε , which is sent to zero only after $N \rightarrow \infty$.

For $K \geq 1$, let

$$U_K = \frac{K \log R_N}{a_2}.$$

Note that $U_K/N = K \log R_N/R_N \rightarrow 0$, by Lemma 16.4.

Proposition 16.14 (Residence in a cap). *Assume $H(\theta_*, B, c_*, C_*)$, and fix $\sigma \in (0, 1)$. Let $\gamma = \theta_*/1024$, $\zeta = 32/\theta_*$, and choose σ' with*

$$0 < \sigma' < \sigma, \quad \sigma' \leq \min \left\{ \frac{\theta_*}{8}, \frac{\theta_* \sigma}{16(1 + \zeta)} \right\}. \quad (16.27)$$

Let P be the law of the uniformly initialized path (16.5), let $T \geq 0$ be deterministic, and let w be an $\mathcal{F}_T$ -measurable random variable with $0 \leq w \leq 1$ and $\mathbb{E}_P w > 0$. Suppose that $w \neq 0$ implies

$X_1(T)^2 \geq \sigma N$, P -almost surely, and set $Q = wP/\mathbb{E}_P w$. For every $K \geq 1$ there exists $c_0 > 0$, depending only on K, σ', θ_* , and B , such that for N larger than a constant depending on the same quantities and c_*, C_* , uniformly in T and w ,

$$\begin{aligned} Q(\exists s \in [0, U_K] : X_1(T+s)^2 < \sigma' N) &\leq \varepsilon_N, \\ \varepsilon_N &= 2e^{-c_0 R_N / \log R_N} + (N^3 + 2)e^{-3\theta_*^2 R_N / 1024} + 2(N^3 + 2)e^{-c_0 N^3}. \end{aligned} \quad (16.28)$$

The reweighting changes the law up to time T ; the Brownian increments after T remain Brownian under Q .

Note that ε_N tends to zero faster than any power of N , by Lemma 16.4. The constant σ' depends only on σ and θ_* . The proof follows the ratios X_j/X_1 , $j \geq 2$, whose evolution does not contain the radial term $2F(X)/N$ of (16.22).

Proof. Under Q , restart at T and reflect the first coordinate by $s_T = \text{sgn } X_1(T)$. Since w and s_T are $\mathcal{F}_T$ -measurable, the reflected increments $(s_T(B_1(T+t) - B_1(T)), B_2(T+t) - B_2(T), \dots)$ form a Brownian motion for the restarted filtration. In this proof only, write X_t for this reflected restarted path; then $X_0 \in \mathcal{C}_\sigma$, Q -almost surely. Let $\tau = \inf\{t \geq 0 : X_1(t) \leq \sqrt{\sigma' N}\}$ and $U = U_K$. The proof has four steps: an exact formula for the ratios, a bound on an integrating factor, a bound on the noise at a fixed time, and a bound between the points of a fine grid.

Step 1: the ratios. For $t < \tau$, we have $X_1(t)^2 \geq \sigma' N > 0$, and we set $r_j = X_j/X_1$ for $j \geq 2$ and $\mathcal{S} = \sum_{j \geq 2} r_j^2 = (N - X_1^2)/X_1^2$. The covariation of (16.22) is $d\langle X_i, X_k \rangle = 2(\delta_{ik} - X_i X_k / N) dt$. Itô's formula for the quotient gives

$$dr_j = \left(-2a_j + \frac{2}{X_1^2}\right)r_j dt + \frac{\sqrt{2}}{X_1} dB_j - \frac{\sqrt{2}r_j}{X_1} dB_1. \quad (16.29)$$

Indeed, the common factor $2F/N - (N-1)/N$ in the drifts of X_j and X_1 cancels in the quotient, and so do the terms containing $X \cdot dB$ in the noise. The second-order terms are $-d\langle X_j, X_1 \rangle / X_1^2 = 2X_j/(NX_1) dt$ and $X_j d\langle X_1 \rangle / X_1^3 = (2X_j/X_1^3 - 2X_j/(NX_1)) dt$, whose sum is $2r_j X_1^{-2} dt$. Let $\mathcal{E}$ solve

$$d\mathcal{E}_t = \frac{2}{X_1^2} \mathcal{E}_t dt - \frac{\sqrt{2}}{X_1} \mathcal{E}_t dB_1, \quad \mathcal{E}_0 = 1,$$

so that $\log \mathcal{E}_t = \int_0^t X_1^{-2} ds - \int_0^t \sqrt{2} X_1^{-1} dB_1$. Itô's formula shows that $d(1/\mathcal{E}) = \sqrt{2} X_1^{-1} \mathcal{E}^{-1} dB_1$, and then that $d(r_j/\mathcal{E}) = -2a_j(r_j/\mathcal{E}) dt + \sqrt{2}(X_1 \mathcal{E})^{-1} dB_j$; the drift $2r_j/X_1^2$ cancels against the covariation of r_j and $1/\mathcal{E}$. Integrating, we obtain, for $t < \tau$,

$$r_j(t) = \mathcal{E}_t [e^{-2a_j t} r_j(0) + M_j(t)], \quad M_j(t) = \int_0^t e^{-2a_j(t-s)} \frac{\sqrt{2}}{X_1(s) \mathcal{E}_s} dB_j(s). \quad (16.30)$$

Step 2: the integrating factor. Let $\tau_\varepsilon = \inf\{t : |\log \mathcal{E}_t| \geq \gamma\}$ and $\hat{\tau} = \tau \wedge \tau_\varepsilon$. Before τ , the drift of $\log \mathcal{E}$ is at most $(\sigma'N)^{-1}$ per unit time and the quadratic variation of its martingale part grows at rate at most $2(\sigma'N)^{-1}$. For large N , we have $U/(\sigma'N) \leq \gamma/2$. Then, on $\{\tau_\varepsilon \leq U \wedge \tau\}$, the martingale part reaches $\gamma/2$ in absolute value before time $U \wedge \tau$. We use the exponential inequality for a continuous martingale m with $m_0 = 0$ and $\langle m \rangle_t \leq y$, for $x, y > 0$:

$$\mathbb{P}\left(\sup_{s \leq t} |m_s| \geq x\right) \leq 2e^{-x^2/(2y)}. \quad (16.31)$$

It follows from Doob's maximal inequality [123, Chapter II.1] applied to the supermartingales $\exp(\pm \lambda m_s - \lambda^2 \langle m \rangle_s / 2)$, with $\lambda = x/y$. By (16.31),

$$Q(\tau_\varepsilon \leq U \wedge \tau) \leq 2 \exp\left(-\frac{(\gamma/2)^2}{2 \cdot 2U/(\sigma'N)}\right) = 2 \exp\left(-\frac{\gamma^2 \sigma' R_N}{16K \log R_N}\right).$$

In (16.30) and below, we replace the integrand of M_j by zero after $\hat{\tau}$. This defines $M_j(t)$ for all t , and (16.30) holds for $t < \hat{\tau}$. The integrand is predictable and bounded by $\sqrt{2}e^\gamma(\sigma'N)^{-1/2}$.

Step 3: the noise at a fixed time. Fix $t \leq U$ and a deterministic vector $v = (v_j)_{j \geq 2}$. The process $s \mapsto \sum_j v_j \int_0^s e^{-2a_j(t-s')} \sqrt{2}(X_1 \mathcal{E})^{-1} \mathbf{1}_{\{s' < \hat{\tau}\}} dB_j(s')$, $s \leq t$, is a continuous martingale, and since the B_j are independent, its quadratic variation at time t is at most $\sum_j d_j v_j^2$, where

$$d_j = \frac{e^{2\gamma}}{2\sigma' N a_j}, \quad \sum_{j \geq 2} d_j = \frac{e^{2\gamma}(1 - \theta_N)}{\sigma'}, \quad \max_j d_j = \frac{e^{2\gamma}}{2\sigma' R_N}.$$

Since the exponential of a continuous martingale minus half its quadratic variation is a supermartingale, we have $\mathbb{E}_Q \exp(\langle v, M(t) \rangle) \leq \exp(\sum_j d_j v_j^2 / 2)$. We use the Gaussian linearization technique of [86, Theorem 1 and Remark 2]. Let $\mathbf{g}$ be a standard Gaussian vector independent of everything else. For $u_0 > 0$ and $z \in \mathbb{R}^{N-1}$, we have $e^{u_0 |z|^2} = \mathbb{E}_{\mathbf{g}} e^{\sqrt{2u_0} \langle \mathbf{g}, z \rangle}$. By Fubini's theorem,

$$\mathbb{E}_Q e^{u_0 |M(t)|^2} \leq \mathbb{E}_{\mathbf{g}} \exp\left(u_0 \sum_j d_j \mathbf{g}_j^2\right) = \prod_j (1 - 2u_0 d_j)^{-1/2}$$

if $2u_0 \max_j d_j < 1$. The auxiliary vector $\mathbf{g}$ is used only for this identity; we make no claim about the conditional law of $M(t)$. If $2u_0 \max_j d_j \leq 1/2$, the inequality $-\log(1-x) \leq x + x^2$ for $0 \leq x \leq 1/2$ gives $\log \mathbb{E}_Q e^{u_0 |M(t)|^2} \leq u_0 \sum_j d_j + 2u_0^2 \max_j d_j \sum_j d_j$. We choose

$$u_0 = \frac{\theta_* \sigma' R_N e^{-2\gamma}}{32}, \quad T_0 = \frac{e^{2\gamma}(1 - \theta_N + \theta_*/8)}{\sigma'}.$$

Then $2u_0 \max_j d_j = \theta_*/32$, $u_0(T_0 - \sum_j d_j) = \theta_*^2 R_N / 256$, and $2u_0^2 \max_j d_j \sum_j d_j \leq \theta_*^2 R_N / 1024$. By Markov's inequality,

$$Q(|M(t)|^2 > T_0) \leq \exp\left(-\frac{\theta_*^2 R_N}{256} + \frac{\theta_*^2 R_N}{1024}\right) = e^{-3\theta_*^2 R_N / 1024}. \quad (16.32)$$

Now suppose that $t < \hat{\tau}$ and $|M(t)|^2 \leq T_0$. By (16.30) and $(a+b)^2 \leq (1+\zeta)a^2 + (1+\zeta^{-1})b^2$,

$$\mathcal{S}(t) \leq (1+\zeta)e^{2\gamma}\mathcal{S}(0) + (1+\zeta^{-1})e^{2\gamma}T_0.$$

Since $X_0 \in \mathcal{C}_\sigma$ almost surely under Q , we have $\mathcal{S}(0) \leq \sigma^{-1}$, and by (16.27) and $e^{2\gamma} \leq 2$, the first term is at most $\theta_*/(8\sigma')$. For the second term, we use $e^{4\gamma} \leq 1 + \theta_*/128$, which gives $(1+\zeta^{-1})e^{4\gamma} \leq (1+\theta_*/32)(1+\theta_*/128) \leq 1 + \theta_*/16$, and $1 - \theta_N + \theta_*/8 \leq 1 - 7\theta_*/8$. The product of the last two bounds is at most $1 - \theta_*/2$. We conclude that

$$\mathcal{S}(t) \leq \frac{1 - 3\theta_*/8}{\sigma'} \leq \left(\frac{1}{\sigma'} - 1\right) - \frac{\theta_*}{4\sigma'}, \quad (16.33)$$

where the last inequality uses $\sigma' \leq \theta_*/8$. Recall that the boundary of $\mathcal{C}_{\sigma'}$ is $\{\mathcal{S} = 1/\sigma' - 1\}$. The bound (16.33) is a margin of $\theta_*/(4\sigma')$ below the boundary.

Step 4: between the points of a grid. By Itô's formula and (16.29),

$$d\mathcal{S} = \left[-4 \sum_{j \geq 2} a_j r_j^2 + \frac{6\mathcal{S}}{X_1^2} + \frac{2(N-1)}{X_1^2} \right] dt + dM_{\mathcal{S}}, \quad d\langle M_{\mathcal{S}} \rangle = \frac{8(\mathcal{S} + \mathcal{S}^2)}{X_1^2} dt.$$

Before τ , we have $\mathcal{S} \leq 1/\sigma'$ and $X_1^2 \geq \sigma'N$. It follows that the drift is at most a constant $C_{\sigma',B}$ in absolute value and the quadratic variation grows at rate at most $C_{\sigma'}/N$. We consider the stopped process $\mathcal{S}(t \wedge \hat{\tau})$ and the grid $t_i = iN^{-2}$, $0 \leq i \leq \lfloor UN^2 \rfloor + 1$, which has at most $N^3 + 2$ points for large N , since $U \leq N$. On each interval $[t_i, t_{i+1}]$, the drift changes $\mathcal{S}(\cdot \wedge \hat{\tau})$ by at most $C_{\sigma',B}N^{-2} \leq \theta_*/(16\sigma')$ for large N , and (16.31) shows that the martingale part deviates from its value at t_i by more than $\theta_*/(16\sigma')$ on $[t_i, t_{i+1}]$ with probability at most $2\exp(-c_0N^3)$. Let $\mathcal{A}$ be the event on which $\tau_{\mathcal{E}} > U \wedge \tau$, the bound $|M(t_i)|^2 \leq T_0$ holds at every grid point $t_i \leq U$, and

$$\sup_{s \in [t_i, t_{i+1}]} |\mathcal{S}(s \wedge \hat{\tau}) - \mathcal{S}(t_i \wedge \hat{\tau})| < \frac{\theta_*}{8\sigma'}$$

for every grid interval. Combining Step 2, (16.32), and a union bound, we find $Q(\mathcal{A}^c) \leq \varepsilon_N$.

We claim that $\tau > U$ on $\mathcal{A}$. Suppose that $\tau \leq U$. We have $\tau > 0$, because $X_0 \in \mathcal{C}_\sigma$ almost surely and $\sigma' < \sigma$. Let t_i be the last grid point with $t_i < \tau$. On $\mathcal{A}$, we have $\hat{\tau} = \tau$, and in particular $t_i < \hat{\tau}$, so that (16.33) holds at t_i . As $\tau \leq t_{i+1}$, we have $\mathcal{S}(\tau) < \mathcal{S}(t_i) + \theta_*/(8\sigma')$, and by (16.33), this implies $\mathcal{S}(\tau) < 1/\sigma' - 1$, which contradicts $X_1(\tau)^2 = \sigma'N$. We conclude that $Q(\tau \leq U) \leq Q(\mathcal{A}^c) \leq \varepsilon_N$. If the original path has $X_1(T+s)^2 < \sigma'N$ for some $s \leq U$, continuity forces $\tau \leq U$, which proves (16.28). $\square$

Corollary 16.15 (Entrance and residence). *Assume $H(\theta_*, B, C_*, C_*)$, let T_e and ε be as in Proposition 16.13, and choose h, M as in its proof. Let $\sigma = h^2/M$, choose σ' as in (16.27), and fix $K \geq 1$. Then*

$$\mathbb{P}(X_1(T_e)^2 \geq \sigma N, X_1(t)^2 \geq \sigma' N \text{ for all } t \in [T_e, T_e + U_K]) \geq 1 - \varepsilon - o(1).$$

On this event, the sign of X_1 is constant on $[T_e, T_e + U_K]$.

Proof. Let E_N be the event in (16.26). It belongs to $\mathcal{F}_{T_e}$, has probability at least $1 - \min\{\varepsilon, 1/2\} - o(1) > 0$ for large N , and implies $X_1(T_e)^2 \geq h^2 N/M = \sigma N$. Apply Proposition 16.14 to the original uniform-start path with $T = T_e$ and $w = \mathbf{1}_{E_N}$. Under the resulting conditional law, the probability of leaving $\{x_1^2 \geq \sigma' N\}$ before $T_e + U_K$ is at most $\varepsilon_N = o(1)$. Multiplying by $\mathbb{P}(E_N)$ proves the claim. Continuity and the positive lower bound on X_1^2 give sign constancy. $\square$

16.5 Retained coordinates and a logarithmic Sobolev inequality on a cap

In this section, we prove two deterministic facts that Sections 16.8 and 16.9 use. For every fixed $A > 1$, the number of coordinates x_j with $a_j \leq Aa_2$ is bounded under $H(\theta_*, B, c_*, C_*)$ (Lemma 16.16). We retain these coordinates. For a density that vanishes outside a cap, the conditional relative entropy of the other coordinates given the retained ones is at most a constant of order $(Aa_2)^{-1}$ times its dissipation (Lemma 16.17).

Lemma 16.16 (The number of retained coordinates). *Assume $H(\theta_*, B, c_*, C_*)$ and fix $A > 1$. There exists an integer $M_A \geq 2$, depending only on A, c_*, C_* , such that $c_* M_A^{2/3} > AC_*$. Fix one such M_A . For $N > M_A$, let $J_N = \max\{j \in \{2, \dots, N\} : a_j \leq Aa_2\}$. Then*

$$2 \leq J_N \leq M_A < N, \quad a_j \leq Aa_2 \quad (2 \leq j \leq J_N), \quad a_j > Aa_2 \quad (J_N < j \leq N). \quad (16.34)$$

Proof. The set in the definition of J_N contains 2, because $A > 1$. Choose M_A large enough that $c_* M_A^{2/3} > AC_*$. Such an integer exists because $m^{2/3} \rightarrow \infty$. With this choice, the hypotheses $H(\theta_*, B, c_*, C_*)$ give

$$a_{M_A+1} \geq c_* M_A^{2/3} N^{-2/3} > AC_* N^{-2/3} \geq Aa_2.$$

As the a_j are nondecreasing in j , every $j > M_A$ satisfies $a_j > Aa_2$, which implies $J_N \leq M_A$. The other claims follow from the monotonicity of the a_j and the definition of J_N . $\square$

Every coordinate with $a_j = Aa_2$ is retained, and equal values among the a_j cause no difficulty. The index J_N may depend on N and on the spectrum, and we use only the bound $J_N \leq M_A$. The hypotheses $H(\theta_*, B, c_*, C_*)$ give no upper bound on a_k/a_2 for a prescribed index $k \geq 3$, and we do not assume one.

For the second fact, we use the notation of Sections 11.4 and 11.5. Let $2 \leq J < N$. As in Section 11, J is the index of the last retained coordinate, and it is unrelated to the function $J(y)$ of Section 16.2, which does not appear in the rest of this section. We split $y = (z, v)$ with $z = (y_2, \dots, y_J)$ and $v = (y_{J+1}, \dots, y_N)$, so that $G = G_z \otimes G_v$. We use the matrix $\mathfrak{g}(y) = I - yy^\top/N$, the carré du champ $\Gamma(\varphi) = \nabla \varphi^\top \mathfrak{g} \nabla \varphi$, and the quantities $\mathcal{H}(w)$ and $\mathcal{D}(w)$ of (11.46). In graph coordinates, we write $q(y) = 1 - |y|^2/N$, so that $q = x_1^2/N$.

Lemma 16.17 (A logarithmic Sobolev inequality on a cap). *Let $2 \leq J < N$, $A > 0$, and $r_0 \in (0, 1)$, and suppose that $a_j \geq Aa_2$ for $J < j \leq N$. Let $w \in C^2(\mathbb{R}^{N-1})$ be a nonnegative probability density with respect to G , with compact support in U_N , such that $w = 0$ on $\{q < r_0\}$. Then $\mathcal{H}(w)$ and $\mathcal{D}(w)$, the latter defined by the formula in (11.46), are finite, and*

$$0 \leq \mathcal{H}(w) \leq \frac{1}{4Aa_2r_0} \mathcal{D}(w). \quad (16.35)$$

Proof. Let $y \in U_N$ and $b \in \mathbb{R}^{N-1}$. The Cauchy–Schwarz inequality $(y \cdot b)^2 \leq |y|^2 |b|^2$ implies that

$$b^\top \mathfrak{g}(y)b = |b|^2 - \frac{(y \cdot b)^2}{N} \geq \left(1 - \frac{|y|^2}{N}\right) |b|^2 = q(y) |b|^2. \quad (16.36)$$

We first show that both quantities are finite. Since $\int w dG = 1$ and w is bounded, we have $\text{KL}(wG \| G) = \int w \log w dG \leq \log \sup w < \infty$, and the chain rule (11.47) gives $0 \leq \mathcal{H}(w) \leq \text{KL}(wG \| G)$. Let $M_2 > 0$ be an upper bound for the operator norm of the Hessian of w . For y with $\nabla w(y) \neq 0$ and $s \geq 0$, Taylor’s formula along $-\nabla w(y)/|\nabla w(y)|$ gives $0 \leq w(y) - s|\nabla w(y)| + M_2 s^2/2$, and the choice $s = |\nabla w(y)|/M_2$ yields $|\nabla w|^2 \leq 2M_2 w$. In particular, $\nabla w = 0$ where $w = 0$, and $\int |\nabla w|^2/w dG \leq 2M_2$, where the integrand is zero on $\{w = 0\}$. The function $\bar{w}(z) = \int w(z, v) G_v(dv)$ is of class C^2 , since w has compact support and we may differentiate under the integral, and the Cauchy–Schwarz inequality on each fiber gives $|\nabla \bar{w}|^2 \leq \bar{w} \int |\nabla_z w|^2/w dG_v$. As $\mathfrak{g} \leq I$ and $\nabla \mathfrak{g} = \nabla w/w - \nabla \bar{w}/\bar{w}$ on $\{w > 0\}$, we obtain

$$\mathcal{D}(w) \leq 2 \int \frac{|\nabla w|^2}{w} dG + 2 \int \frac{|\nabla \bar{w}|^2}{\bar{w}} dG_z \leq 4 \int \frac{|\nabla w|^2}{w} dG < \infty.$$

Now fix z with $\bar{w}(z) > 0$, and let $\phi = w(z, \cdot)/\bar{w}(z)$, a nonnegative function of class C^2 with compact support and $\int \phi dG_v = 1$. Then $\int \phi \log \phi dG_v = \mathbb{E}_w[g \mid z]$, and $(\partial_{v_i} \phi)^2/\phi = \phi (\partial_{v_i} g)^2$ on $\{\phi > 0\}$. By the Gaussian logarithmic Sobolev inequality [79], in the form (11.51), which follows from (11.50) [76, Theorem 2.7, (9), p. 5] after a diagonal change of variables, and by $a_i \geq Aa_2$ for $i > J$, we have

$$\mathbb{E}_w[g \mid z] \leq \sum_{i>J} \frac{1}{4a_i} \mathbb{E}_w[(\partial_{v_i} g)^2 \mid z] \leq \frac{1}{4Aa_2} \mathbb{E}_w[|\nabla_v g|^2 \mid z].$$

Here (11.51), which is stated for smooth functions, applies to ϕ , with the integrand on the right set to zero on $\{\phi = 0\}$. Indeed, the mollifications ϕ_n of ϕ are smooth and nonnegative, they are supported in a common compact set, and they converge to ϕ in C^2 , because the derivatives of ϕ up to order two are uniformly continuous. Fix $e > 0$, and apply (11.51) to $(\phi_n + e)/(c_n + e)$, where $c_n = \int \phi_n dG_v \rightarrow 1$. The integrands on both sides are bounded uniformly in n , since $\phi_n + e \geq e$, and they are constant outside the common support. Dominated convergence then gives the inequality for $(\phi + e)/(1 + e)$.

As $e \downarrow 0$, the left side converges by bounded convergence, because ϕ is bounded and $s \log s$ is continuous on $[0, \infty)$. The right side is $\sum_{i>J} (4a_i)^{-1} \int (\partial_{v_i} \phi)^2 / ((\phi + e)(1 + e)) dG_v$, and it converges by monotone convergence, since $\nabla \phi = 0$ on $\{\phi = 0\}$. Since $w = 0$ on $\{q < r_0\}$, we have $|\nabla_v g|^2 \leq |\nabla g|^2 \leq \Gamma(g)/r_0$ on $\{w > 0\}$, by (16.36). Multiplying by $\bar{w}(z)$ and integrating with respect to G_z , we obtain (16.35). Fibers with $\bar{w}(z) = 0$ contribute zero to both sides. $\square$

For $r \in (0, 1)$, we use Lemma 16.17 with $r_0 = r/2$ and $A = 16/r$. Then $4Aa_2r_0 = 32a_2$, and under the hypotheses of the lemma,

$$\mathcal{H}(w) \leq \frac{1}{32a_2} \mathcal{D}(w). \quad (16.37)$$

The constant in (16.35) is cruder than that of Proposition 11.16, which keeps the matrix A_v of Lemma 11.15. Here we bound g from below by q times the identity, which loses the fixed factor r_0 , and we compensate for this loss by retaining all coordinates with $a_j \leq Aa_2$ for a large A . The dissipation $\mathcal{D}(w)$ contains the derivatives in all variables, and the proof uses only its part in v .

16.6 Gaussian approximation of the coordinates at time t_p

In this section, we show that at the time

$$t_p = \frac{\log N}{100},$$

finitely many coordinates of the unnormalized process are close in total variation to independent centered Gaussian variables with variance θ^{-1} , and that the first coordinate barely moves afterwards. These Gaussian variables are not the initial coordinates of X_0 ; they are built from X_0 and the noise up to time t_p . The proof conditions on the matrix Φ , which is independent of X_0 , and uses the rotation invariance of ν_N ; it does not use a central limit theorem for martingales.

Proposition 16.18 (Gaussian approximation at time t_p). *Assume $H(\theta_*, B, c_*, C_*)$.*

1. Suppose that $\theta_N \rightarrow \theta \in (0, 1)$, and let $k \geq 1$ be fixed with $a_k \log N \rightarrow 0$. Then

$$\| \text{Law}(Y_1(t_p), \dots, Y_k(t_p)) - \mathcal{N}(0, \theta^{-1} I_k) \|_{\text{TV}} \rightarrow 0. \quad (16.38)$$

2. For every $K \geq 1$ and every U with $t_p \leq U \leq U_K$,

$$\mathbb{E} \sup_{t_p \leq t \leq U} |Y_1(t) - Y_1(t_p)|^2 \leq C \left[\frac{U}{N} + (1 + t_p)^{-1/2} \right], \quad (16.39)$$

where C depends only on θ_* , c_* , and C_* . The right side tends to zero.

The hypothesis $a_k \log N \rightarrow 0$ holds for $k \leq 2$ under $H(\theta_*, B, c_*, C_*)$, and for every fixed k under $H_k(\omega, D_k)$ as well, since then $a_k \leq D_k C_* N^{-2/3}$. The second part, combined with the first, gives the weak limit $\mathcal{N}(0, \theta^{-1})$ for $Y_1(t)$ at every later time $t \leq U_K$, jointly with $(Y_1(t_p), \dots, Y_k(t_p))$. It does not give convergence in total variation at later times.

Proof of the first part. Let $\Sigma = \Phi \Phi^\top$, with entries Σ_{ij} . As X_0 is independent of Φ and $\mathbb{E} X_0 X_0^\top = I$, we have $\mathbb{E}[Y(t)Y(t)^\top \mid \Phi] = \Sigma(t)$, and in particular $\mathbb{E}\Sigma_{ii}(t) = h_i(t)$. The proof has three steps: we bound the second moments of Σ , show that the first $k \times k$ block of $\Sigma(t_p)$ concentrates around $\theta^{-1}I_k$, and then use the conditional law of Y given Φ .

Step 1: second moments of Σ . By Itô's formula and (16.21),

$$d\Sigma = \left[-2(D_a \Sigma + \Sigma D_a) + \frac{2}{N} \text{Tr}(\Sigma)I \right] dt + \sqrt{\frac{2}{N}} (d\mathcal{B} \Sigma + \Sigma d\mathcal{B}^\top). \quad (16.40)$$

The drift term $(2/N) \text{Tr}(\Sigma)I$ comes from $(d\mathcal{B} \Sigma d\mathcal{B}^\top)_{ik} = \delta_{ik} \text{Tr}(\Sigma) dt$. Expanding $d \text{Tr}(\Sigma^2) = 2 \text{Tr}(\Sigma d\Sigma) + \text{Tr}(d\Sigma d\Sigma)$, the noise contributes four contractions: $\text{Tr}(d\mathcal{B} \Sigma d\mathcal{B} \Sigma)$ and $\text{Tr}(\Sigma d\mathcal{B}^\top \Sigma d\mathcal{B}^\top)$ each equal $\text{Tr}(\Sigma^2) dt$, and $\text{Tr}(d\mathcal{B} \Sigma^2 d\mathcal{B}^\top)$ and $\text{Tr}(\Sigma d\mathcal{B}^\top d\mathcal{B} \Sigma)$ each equal $N \text{Tr}(\Sigma^2) dt$. We obtain

$$\frac{d}{dt} \mathbb{E} \text{Tr}(\Sigma^2) = \mathbb{E} \left[-8 \text{Tr}(D_a \Sigma^2) + \frac{4}{N} (\text{Tr} \Sigma)^2 + \left(4 + \frac{4}{N}\right) \text{Tr}(\Sigma^2) \right].$$

Note that $\text{Tr}(D_a \Sigma^2) \geq 0$, because $a_i \geq 0$, and that $(\text{Tr} \Sigma)^2 \leq N \text{Tr}(\Sigma^2)$. With $\text{Tr}(\Sigma(0)^2) = N$ and $N \geq 2$, we obtain $\mathbb{E} \text{Tr}(\Sigma(t)^2) \leq N e^{12t}$. Next, let $\mathcal{U}_i = (\Sigma^2)_{ii} = \sum_j \Sigma_{ij}^2$. We see from (16.40) that the quadratic variation of Σ_{ij} grows at rate $(2/N)(\mathcal{U}_i + \mathcal{U}_j)$ for $i \neq j$, and that of Σ_{ii} at rate $(8/N)\mathcal{U}_i$. Summing $d(\Sigma_{ij}^2)$ over j , we find that the drift of $\mathcal{U}_i$ is

$$-4a_i \mathcal{U}_i - 4 \sum_j a_j \Sigma_{ij}^2 + \frac{4}{N} \text{Tr}(\Sigma) \Sigma_{ii} + \frac{2}{N} [(N+2)\mathcal{U}_i + \text{Tr}(\Sigma^2)].$$

Using $\Sigma_{ii} \leq \mathcal{U}_i^{1/2}$, we find $(4/N) \text{Tr}(\Sigma) \Sigma_{ii} \leq 4(\text{Tr}(\Sigma^2)/N)^{1/2} \mathcal{U}_i^{1/2} \leq 2\mathcal{U}_i + 2 \text{Tr}(\Sigma^2)/N$. Then $u_i = \mathbb{E} \mathcal{U}_i$ satisfies $u'_i \leq 6u_i + 4e^{12t}$ and $u_i(0) = 1$, and Gronwall's inequality gives

$$\mathbb{E} \mathcal{U}_i(t) \leq 2e^{12t}. \quad (16.41)$$

These computations are justified first up to bounded stopping times and then by monotone convergence, since all moments of $\Phi(t)$ are finite.

Step 2: concentration of the covariance. The diagonal of (16.40) reads

$$d(\Sigma_{11}, \dots, \Sigma_{NN})^\top = A(\Sigma_{11}, \dots, \Sigma_{NN})^\top dt + d\mathcal{M}, \quad A = -4D_a + \frac{2}{N} \mathbf{1}\mathbf{1}^\top,$$

where $d\mathcal{M}_i = 2\sqrt{2/N} \sum_k \Sigma_{ki} d\mathcal{B}_{ik}$. Since the martingale $\mathcal{M}_i$ uses only the i -th row of $\mathcal{B}$, we have $\langle \mathcal{M}_i, \mathcal{M}_j \rangle = 0$ for $i \neq j$, and $d\langle \mathcal{M}_i \rangle = (8/N)\mathcal{U}_i dt$. The vector $h = (h_i)$

solves $h' = Ah$ with $h(0) = \mathbf{1}$. The off-diagonal entries of A are nonnegative, so e^{tA} has nonnegative entries; together with $A\mathbf{1} \leq 2 \cdot \mathbf{1}$ entrywise, this shows that its row sums are at most e^{2t} . By variation of constants, the orthogonality of the $\mathcal{M}_i$, and (16.41),

$$\mathbb{E}|\Sigma_{ii}(t) - h_i(t)|^2 = \frac{8}{N} \int_0^t \sum_j (e^{(t-s)A})_{ij}^2 \mathbb{E} \mathcal{U}_j(s) ds \leq \frac{16}{N} \int_0^t e^{4(t-s)+12s} ds \leq \frac{2e^{12t}}{N}.$$

For $i \neq j$, the entry Σ_{ij} starts at zero and has drift $-2(a_i + a_j)\Sigma_{ij}$, so that

$$\mathbb{E} \Sigma_{ij}(t)^2 \leq \frac{2}{N} \int_0^t \mathbb{E} (\mathcal{U}_i(s) + \mathcal{U}_j(s)) ds \leq \frac{2e^{12t}}{3N}.$$

At $t = t_p$, we have $e^{12t_p}/N = N^{-22/25}$. Recall from the third part of Lemma 16.12 that $|h_1(t_p) - \theta_N^{-1}| \leq C[(1+t_p)^{-1/2} + t_p/N]$ and $|h_1(t_p) - h_i(t_p)| \leq 4a_k t_p h_1(t_p)$ for $i \leq k$. Using $\theta_N \rightarrow \theta$ and $a_k t_p \rightarrow 0$, we conclude that the block $\Sigma_{[k]} = (\Sigma_{ij}(t_p))_{i,j \leq k}$ satisfies

$$\mathbb{E} \|\Sigma_{[k]} - \theta^{-1} I_k\|_{\text{HS}}^2 \rightarrow 0. \quad (16.42)$$

Step 3: the conditional law given Φ . Let $\Phi_{[k]}$ be the $k \times N$ matrix formed by the first k rows of $\Phi(t_p)$. As $\Phi(t_p)$ is invertible, the matrix $\Sigma_{[k]} = \Phi_{[k]} \Phi_{[k]}^\top$ is positive definite, and $O = \Sigma_{[k]}^{-1/2} \Phi_{[k]}$ has orthonormal rows. We complete O to an orthogonal matrix. By the rotation invariance of ν_N and the independence of X_0 and Φ , conditionally on Φ the vector OX_0 has the law of $Z_{N,k} = (x_1, \dots, x_k)$ under ν_N . It follows that $\text{Law}(Y_{[k]}(t_p) \mid \Phi) = \text{Law}(\Sigma_{[k]}^{1/2} Z_{N,k})$. Since an invertible linear map preserves total variation, we obtain

$$\|\text{Law}(Y_{[k]}(t_p) \mid \Phi) - \mathcal{N}(0, \Sigma_{[k]})\|_{\text{TV}} = \varepsilon_{N,k}, \quad \varepsilon_{N,k} = \|\text{Law}(Z_{N,k}) - \mathcal{N}(0, I_k)\|_{\text{TV}}.$$

For $N > k + 2$, the law of $Z_{N,k}$ has the density

$$\frac{\Gamma(N/2)}{\Gamma((N-k)/2)(\pi N)^{k/2}} \left(1 - \frac{|z|^2}{N}\right)_+^{(N-k-2)/2},$$

which converges pointwise to the standard Gaussian density on $\mathbb{R}^k$, since $\Gamma(N/2)/\Gamma((N-k)/2) \sim (N/2)^{k/2}$. Scheffé's lemma then implies $\varepsilon_{N,k} \rightarrow 0$. This is the classical Gaussian approximation of the coordinates of a uniform point on a high-dimensional sphere; quantitative bounds are in [57, (1)]. Further, for positive definite Σ_1, Σ_0 , Pinsker's inequality and the formula for the relative entropy of Gaussian laws give

$$\|\mathcal{N}(0, \Sigma_1) - \mathcal{N}(0, \Sigma_0)\|_{\text{TV}}^2 \leq \frac{1}{4} \left[\text{Tr}(\Sigma_0^{-1} \Sigma_1) - k - \log \det(\Sigma_0^{-1} \Sigma_1) \right],$$

and the right side tends to zero as $\Sigma_1 \rightarrow \Sigma_0$. Using (16.42) and bounded convergence, we find $\mathbb{E} \|\mathcal{N}(0, \Sigma_{[k]}) - \mathcal{N}(0, \theta^{-1} I_k)\|_{\text{TV}} \rightarrow 0$. By the convexity of total variation under mixtures, we obtain

$$\|\text{Law}(Y_{[k]}(t_p)) - \mathcal{N}(0, \theta^{-1} I_k)\|_{\text{TV}} \leq \varepsilon_{N,k} + \mathbb{E} \|\mathcal{N}(0, \Sigma_{[k]}) - \mathcal{N}(0, \theta^{-1} I_k)\|_{\text{TV}} \rightarrow 0,$$

which proves (16.38). $\square$

Proof of the second part. Recall that Y_1 is a martingale with $d\langle Y_1 \rangle = (2/N)|Y|^2 dt$. By Doob's inequality in L^2 and (16.24), we have, for $t_p \leq U \leq U_K \leq N$,

$$\begin{aligned} \mathbb{E} \sup_{t_p \leq t \leq U} |Y_1(t) - Y_1(t_p)|^2 &\leq \frac{8}{N} \int_{t_p}^U v(s) ds \\ &\leq \frac{C}{N} \int_{t_p}^U [1 + N(1+s)^{-3/2}] ds \leq C \left[\frac{U}{N} + (1+t_p)^{-1/2} \right]. \end{aligned}$$

The right side tends to zero, since $U_K/N = K \log R_N / R_N \rightarrow 0$ by Lemma 16.4. $\square$

16.7 The late-time squared norm and finitely many coordinates

In this section, we fix

$$\chi_N = \sqrt{\log \log N}, \quad T_e = \frac{\chi_N}{a_2}, \quad Z_N = Y_1(t_p).$$

Under $H(\theta_*, B, c_*, C_*)$, we have $\chi_N = o(\log R_N)$, and Proposition 16.13 applies at the time T_e . Further, $t_p < T_e < t_u$ for large N , and

$$a_2(t_u - T_e) = \frac{\log R_N}{4} + u - \chi_N \rightarrow \infty, \quad t_u - T_e \leq U_1 \quad (16.43)$$

for every fixed u and large N . In this section, “in probability” refers to the randomness of the initial point and the noise; the spectra are deterministic.

Proposition 16.19 (The squared norm at time t_u). *Assume $H(\theta_*, B, c_*, C_*)$, and fix $u \in \mathbb{R}$. Then*

$$|Y(t_u)|^2 - \frac{Z_N^2}{\theta_N} \rightarrow 0, \quad Y_1(t_u) - Z_N \rightarrow 0 \quad (16.44)$$

in probability. If, in addition, $\theta_N \rightarrow \theta \in (0, 1)$, then $X_1(t_u)^2 / N \rightarrow \theta$ in probability.

The first convergence says that the squared norm of the unnormalized process at time t_u is determined by its first coordinate: the remaining coordinates have relaxed, and their total squared norm is the fraction $(1 - \theta_N)/\theta_N$ of Y_1^2 . The proof conditions the path law on the event of Proposition 16.13, stops the process when it leaves a cap, and solves the renewal equation for $|Y|^2$ along the stopped process.

Proof. Fix $\varepsilon \in (0, 1/4)$, let h, M be as in Proposition 16.13, let $E_N = \{|Y_1(T_e)| \geq h, |Y(T_e)|^2 \leq M\}$, and choose $\sigma' > 0$ so that $2\sigma'$ satisfies (16.27) with $\sigma = h^2/M$. Let $U = U_1$. Throughout the proof, $\mathbb{P}_E$ and $\mathbb{E}_E$ denote probability and expectation conditionally on E_N , and C denotes constants depending only on ε and the spectral constants.

Step 1: the stopped process. Let $s_N = \text{sgn } Y_1(T_e)$. Since $E_N \in \mathcal{F}_{T_e}$ and the increments of B after T_e are independent of $\mathcal{F}_{T_e}$, the process $r \mapsto (s_N Y_1, Y_2, \dots, Y_N)(T_e + r)$ solves (16.5) under $\mathbb{P}_E$ with the Brownian motion $r \mapsto (s_N B_1, B_2, \dots, B_N)(T_e + r) - (s_N B_1, B_2, \dots, B_N)(T_e)$, which we denote by $\bar{B}$. Let $\tau = \inf\{r \geq 0 : s_N X_1(T_e + r) \leq \sqrt{\sigma' N}\}$. On E_N , the first squared coordinate at T_e is at least σN . Apply Proposition 16.14 directly to the original uniform-start path with $T = T_e$, $w = \mathbf{1}_{E_N}$, and residence threshold $2\sigma'$. Since hitting the stopping threshold σ' entails falling below $2\sigma'$, we obtain

$$p_N = \mathbb{P}_E(\tau \leq U) \leq \varepsilon_N \longrightarrow 0.$$

We define $\bar{Y}(r) = (s_N Y_1, Y_2, \dots, Y_N)(T_e + r)$ for $r \leq \tau$; for $r > \tau$, let $\bar{Y}_1(r) = \bar{Y}_1(\tau)$ and $\bar{Y}_j(r) = e^{-2a_j(r-\tau)} \bar{Y}_j(\tau)$ for $j \geq 2$. In other words, $\bar{Y}$ follows the unnormalized process until τ , and afterwards its noise is removed and its first coordinate is frozen. Let $\bar{A} = |\bar{Y}|^2$ and $y = \bar{Y}_1(0) = |Y_1(T_e)|$, so that $h \leq y \leq \sqrt{M}$ on E_N . Then

$$d\bar{Y}_1 = \sqrt{2/N} \bar{A}^{1/2} \mathbf{1}_{\{r < \tau\}} d\bar{B}_1, \quad d\bar{Y}_j = -2a_j \bar{Y}_j dr + \sqrt{2/N} \bar{A}^{1/2} \mathbf{1}_{\{r < \tau\}} d\bar{B}_j \quad (j \geq 2). \quad (16.45)$$

Before τ , we have $\bar{Y}_1^2 \geq \sigma' \bar{A}$. After τ , the first coordinate is frozen and the others decrease, which gives $\bar{A}(r) \leq \bar{A}(\tau) \leq \bar{Y}_1(\tau)^2 / \sigma'$. We conclude that

$$\bar{A}(r) \leq \frac{\bar{Y}_1(r)^2}{\sigma'} \quad (r \geq 0). \quad (16.46)$$

Step 2: moments of the first coordinate. For an integer $\ell \geq 1$, Itô's formula, (16.45), and (16.46) show that the drift of $\bar{Y}_1^{2\ell}$ is at most $2\ell(2\ell - 1)(\sigma' N)^{-1} \bar{Y}_1^{2\ell}$. We then have $\mathbb{E}_E \bar{Y}_1^{2\ell}(r) \leq M^\ell e^{2\ell(2\ell-1)r/(\sigma' N)}$ by Gronwall's inequality. As $U/N \rightarrow 0$, the moments of order up to eight of $\bar{Y}_1(r)$ and, by (16.46), of $\bar{A}(r)^{1/2}$ are bounded uniformly for $r \leq U$. The process $\bar{Y}_1^2 - y^2$ has drift $(2/N) \bar{A} \mathbf{1}_{\{r < \tau\}}$ and a martingale part with quadratic variation $(8/N) \bar{A} \bar{Y}_1^2 \mathbf{1}_{\{r < \tau\}} dr$. By Doob's inequality in L^2 and Minkowski's inequality, we obtain

$$\left\| \sup_{r \leq U} |\bar{Y}_1(r)^2 - y^2| \right\|_{L^2(\mathbb{P}_E)} \leq C \left(\sqrt{\frac{U}{N}} + \frac{U}{N} \right). \quad (16.47)$$

Step 3: the renewal equation for the stopped process. For $j \geq 2$, Itô's formula and variation of constants give

$$\begin{aligned}\bar{Y}_j(r)^2 &= e^{-4a_j r} \bar{Y}_j(0)^2 + \frac{2}{N} \int_0^r e^{-4a_j(r-w)} \bar{A}(w) \mathbf{1}_{\{w < \tau\}} dw \\ &\quad + \sqrt{\frac{8}{N}} \int_0^r e^{-4a_j(r-w)} \bar{A}(w)^{1/2} \bar{Y}_j(w) \mathbf{1}_{\{w < \tau\}} d\bar{B}_j(w).\end{aligned}$$

Summing over $j \geq 2$, we obtain

$$\bar{A} = \bar{Y}_1^2 + \ell_0 + k * \bar{A} + n - e, \quad (16.48)$$

where k is the kernel of Lemma 16.12 and

$$\ell_0(r) = \sum_{j \geq 2} e^{-4a_j r} \bar{Y}_j(0)^2 \leq M e^{-4a_2 r}, \quad e(r) = \int_0^r k(r-w) \bar{A}(w) \mathbf{1}_{\{\tau \leq w\}} dw,$$

and $n(r)$ is the sum of the stochastic integrals. The term e is present because the noise was removed after τ . Using the independence of the $\bar{B}_j$ and Step 2, we find

$$\mathbb{E}_E n(r)^2 = \frac{8}{N} \sum_{j \geq 2} \int_0^r e^{-8a_j(r-w)} \mathbb{E}_E [\bar{A} \bar{Y}_j^2 \mathbf{1}_{\{w < \tau\}}] dw \leq \frac{8}{N} \int_0^r \mathbb{E}_E \bar{A}(w)^2 dw \leq \frac{CU}{N}.$$

Using Hölder's inequality, we find $\|\bar{A}(w) \mathbf{1}_{\{\tau \leq w\}}\|_{L^2(\mathbb{P}_E)} \leq (\mathbb{E}_E \bar{A}(w)^4)^{1/4} p_N^{1/4} \leq C p_N^{1/4}$, and Minkowski's inequality and $\int k \leq 1$ then give $\|e(r)\|_{L^2(\mathbb{P}_E)} \leq C p_N^{1/4}$ for $r \leq U$.

Step 4: solving the renewal equation. Convolving (16.48) with $\delta_0 + Y$, which has total mass $1/\theta_N$, we obtain $\bar{A} = (\delta_0 + Y) * (\bar{Y}_1^2 + \ell_0 + n - e)$. Recall that $(\delta_0 + Y) * 1 = H$ and that $\theta_N^{-1} - H(t) = \int_t^\infty Y \leq C(1+t)^{-1/2}$ by Lemma 16.12. We obtain

$$\left\| (\delta_0 + Y) * \bar{Y}_1^2(t) - \frac{y^2}{\theta_N} \right\|_{L^2(\mathbb{P}_E)} \leq \frac{1}{\theta_*} \left\| \sup_{r \leq U} |\bar{Y}_1(r)^2 - y^2| \right\|_{L^2(\mathbb{P}_E)} + CM(1+t)^{-1/2}.$$

Splitting the convolution at $t/2$, we find

$$(\delta_0 + Y) * \ell_0(t) \leq M e^{-4a_2 t} + M \int_{t/2}^\infty Y + \frac{M}{\theta_*} e^{-2a_2 t} \leq C [e^{-2a_2 t} + (1+t)^{-1/2}].$$

The convolutions of n and e with $\delta_0 + Y$ are bounded in $L^2(\mathbb{P}_E)$, uniformly for $t \leq U$, by θ_*^{-1} times the bounds of Step 3. Combining these bounds with (16.47), we obtain, for every $0 \leq v_N \leq U$,

$$\sup_{v_N \leq t \leq U} \left\| \bar{A}(t) - \frac{y^2}{\theta_N} \right\|_{L^2(\mathbb{P}_E)} \leq C \left[\sqrt{\frac{U}{N}} + \frac{U}{N} + p_N^{1/4} + e^{-2a_2 v_N} + (1+v_N)^{-1/2} \right]. \quad (16.49)$$

The supremum is over the values of the norms at fixed times; this is not a bound on the norm of a supremum. If $a_2 v_N \rightarrow \infty$, then also $v_N \rightarrow \infty$, and the right side tends to zero.

Step 5: conclusion. By (16.43), we may take $t = t_u - T_e$ and $v_N = t$ in (16.49). On $E_N \cap \{\tau > U\}$, we have $\bar{A}(t_u - T_e) = |Y(t_u)|^2$ and $y^2 = Y_1(T_e)^2$. Then, for every $\gamma > 0$, Markov's inequality and Cauchy–Schwarz imply

$$\mathbb{P}(E_N \cap \{||Y(t_u)|^2 - Y_1(T_e)^2/\theta_N| > \gamma\}) \leq p_N + \gamma^{-1} \left\| \bar{A}(t_u - T_e) - \frac{y^2}{\theta_N} \right\|_{L^2(\mathbb{P}_E)} \longrightarrow 0.$$

Further, (16.39) gives $\mathbb{E}|Y_1(T_e) - Z_N|^2 \rightarrow 0$ and $\mathbb{E}|Y_1(t_u) - Z_N|^2 \rightarrow 0$, and $\mathbb{E}Y_1(t)^2 = h_1(t) \leq C$ for $t \leq N$ by Lemma 16.12. It follows that $Y_1(T_e)^2 - Z_N^2 \rightarrow 0$ in probability, and

$$\limsup_N \mathbb{P}\left(\left||Y(t_u)|^2 - \frac{Z_N^2}{\theta_N}\right| > 2\gamma\right) \leq \limsup_N \mathbb{P}(E_N^c) \leq \varepsilon.$$

As ε is arbitrary, (16.44) follows. Finally, assume $\theta_N \rightarrow \theta$. We know from Proposition 16.18 with $k = 1$ that Z_N converges in distribution to $\mathcal{N}(0, \theta^{-1})$. For $r_0 > 0$, on $\{|Z_N| \geq r_0\}$ the ratio $X_1(t_u)^2/N = Y_1(t_u)^2/|Y(t_u)|^2$ differs from θ_N by a quantity tending to zero in probability, by (16.44), and $\limsup_N \mathbb{P}(|Z_N| < r_0) \leq \mathbb{P}(|\mathcal{N}(0, \theta^{-1})| \leq r_0)$, which tends to zero as $r_0 \rightarrow 0$. $\square$

Theorem 16.20 (Finitely many coordinates). *Assume $H(\theta_*, B, c_*, C_*)$ and $\theta_N \rightarrow \theta \in (0, 1)$. Fix $k \geq 2$, and if $k \geq 3$ assume also $H_k(\omega, D_k)$. Let $G_2, \dots, G_k$ be independent standard Gaussian variables and C an independent standard Cauchy variable. Then for every fixed $u \in \mathbb{R}$,*

$$(\sqrt{2a_j} X_j(t_u))_{j=2}^k \Longrightarrow (G_2 + \sqrt{2\theta} e^{-2u} C, G_3, \dots, G_k). \quad (16.50)$$

If, in addition, a_3/a_2 is bounded uniformly in N , then Proposition 16.9 shows that the same vector under π converges in total variation to $(G_2, \dots, G_k)$. This upper bound on a_3/a_2 is included in $H_3(\omega, D_3)$ in the main deterministic theorem. Under these hypotheses, since the total-variation distance is at least half the difference of expectations of a function with values in $[-1, 1]$, (16.50) for $k = 2$ gives $\liminf_N D_N(t_u) \geq \mathcal{V}_\theta(u)$. The matching upper bound requires control of all N coordinates, which is the content of Sections 16.8 to 16.11.

Proof. We write $\varkappa_j = a_j/a_2$ for $2 \leq j \leq k$, so that $1 \leq \varkappa_j \leq D_k$ and $\varkappa_j \geq 1 + \omega$ for $j \geq 3$; for $k = 2$, set $D_2 = 1$. Note that $a_k t_p \leq D_k C_* N^{-2/3} t_p \rightarrow 0$. By Proposition 16.18, we have

$$\left\| \text{Law}(Y_1(t_p), \dots, Y_k(t_p)) - \text{Law}(\theta^{-1/2}(H_1, \dots, H_k)) \right\|_{\text{TV}} \rightarrow 0, \quad (16.51)$$

where $H_1, \dots, H_k$ are independent standard Gaussian variables. Let $\Lambda_N = \sqrt{\theta_N} |Y(t_u)|/|Z_N|$. Then Proposition 16.19 implies that, for every $r_0 > 0$, $\Lambda_N \rightarrow 1$ in probability on the event $\{|Z_N| \geq r_0\}$.

Step 1: decomposition. For $2 \leq j \leq k$, variation of constants in (16.5) from time t_p gives

$$\sqrt{2a_j} X_j(t_u) = \frac{\sqrt{2a_j N} Y_j(t_u)}{|Y(t_u)|} = \Lambda_N^{-1} \left[S_{j,N} + \frac{\sqrt{\theta_N}}{|Z_N|} (I_j^{(1)} + I_j^{(2)} + I_j^{(3)}) \right],$$

where

$$S_{j,N} = \frac{\sqrt{2a_j N \theta_N} e^{-2a_j(t_u - t_p)} Y_j(t_p)}{|Z_N|} = \sqrt{2\theta_N \varkappa_j} R_N^{(1-\varkappa_j)/2} e^{-2\varkappa_j u} e^{2a_j t_p} \frac{Y_j(t_p)}{|Z_N|},$$

and $I_j^{(1)}, I_j^{(2)}, I_j^{(3)}$ are the integrals $\sqrt{4a_j} \int e^{-2a_j(t_u - r)} |Y(r)| dB_j(r)$ over the intervals $[t_p, T_e]$, $[T_e, l_N]$, and $[l_N, t_u]$, with $l_N = t_u - M_*/a_2$ for a fixed $M_* > 0$. In the formula for $S_{j,N}$, we used $e^{-2a_j t_u} = R_N^{-\varkappa_j/2} e^{-2\varkappa_j u}$ and $\sqrt{2a_j N \theta_N} = \sqrt{2\theta_N \varkappa_j} R_N^{1/2}$.

Step 2: the initial terms. For $j = 2$, we have $S_{2,N} = \sqrt{2\theta_N} e^{-2u} e^{2a_2 t_p} Y_2(t_p) / |Z_N|$. For $j \geq 3$, we have $R_N^{(1-\varkappa_j)/2} \leq R_N^{-\omega/2} \rightarrow 0$, while $e^{-2\varkappa_j u} e^{2a_j t_p} \leq e^{2D_k|u|+1}$ for large N . Since $Y_j(t_p)$ is tight by (16.51), we conclude that $S_{j,N} \rightarrow 0$ in probability on $\{|Z_N| \geq r_0\}$ for $j \geq 3$.

Step 3: the early noise. By Itô's isometry and (16.24),

$$\mathbb{E} |I_j^{(1)}|^2 = 4a_j \int_{t_p}^{T_e} e^{-4a_j(t_u - r)} v(r) dr \leq 4a_j e^{-4a_j(t_u - T_e)} C [T_e + N(1 + t_p)^{-1/2}].$$

Recall from (16.43) that $e^{-4a_j(t_u - T_e)} = R_N^{-\varkappa_j} e^{-4\varkappa_j(u - \chi_N)}$. Using $a_j T_e = \varkappa_j \chi_N$ and $R_N^{1-\varkappa_j} \leq 1$, we obtain

$$\mathbb{E} |I_j^{(1)}|^2 \leq C D_k e^{4D_k(|u| + \chi_N)} \left[\frac{\chi_N}{R_N} + (1 + t_p)^{-1/2} \right] \rightarrow 0,$$

because $e^{4D_k \chi_N} = (\log N)^{4D_k/\chi_N}$ grows more slowly than every positive power of $\log N$. For $T_e \leq r \leq T_e + U_1$, we have $v(r) \leq C$ by (16.24), because $N(1 + T_e)^{-3/2} \leq C$. It follows that

$$\mathbb{E} |I_j^{(2)}|^2 \leq 4a_j C \int_{-\infty}^{l_N} e^{-4a_j(t_u - r)} dr = C e^{-4\varkappa_j M_*} \leq C e^{-4M_*}.$$

Step 4: the late noise. Let $\mathcal{G}_{j,N} = \sqrt{4a_j} \int_{l_N}^{t_u} e^{-2a_j(t_u - r)} dB_j(r)$. These variables are independent centered Gaussian variables with variances $1 - e^{-4\varkappa_j M_*}$, and they are independent of $\mathcal{F}_{l_N}$, in particular of $\mathcal{F}_{T_e}$ and of $(Y_1, \dots, Y_k)(t_p)$. We compare $I_j^{(3)}$ with $|Z_N| \theta_N^{-1/2} \mathcal{G}_{j,N}$. Let $E_N, \mathbb{P}_E, \tau, \bar{A}, y$, and $U = U_1$ be as in the proof of Proposition 16.19, for a fixed ε . On $E_N \cap \{\tau > U\}$, we have $I_j^{(3)} = \tilde{I}_j$, where

$$\tilde{I}_j = \sqrt{4a_j} \int_{l_N}^{t_u} e^{-2a_j(t_u - r)} \bar{A}(r - T_e)^{1/2} \mathbf{1}_{\{r - T_e < \tau\}} dB_j(r)$$

is the integral with the stopped amplitude. Let $c_e = y/\sqrt{\theta_N}$, which is $\mathcal{F}_{T_e}$ -measurable and at most $(M/\theta_*)^{1/2}$ on E_N . By Itô's isometry and $4a_j \int_{l_N}^{t_u} e^{-4a_j(t_u-r)} dr \leq 1$,

$$\begin{aligned} \mathbb{E}_E |\tilde{I}_j - c_e \mathcal{G}_{j,N}|^2 &\leq \sup_{l_N \leq r \leq t_u} \mathbb{E}_E |\bar{A}(r - T_e)^{1/2} \mathbf{1}_{\{r - T_e < \tau\}} - c_e|^2 \\ &\leq 2 \sup_{l_N \leq r \leq t_u} \mathbb{E}_E \left| \bar{A}(r - T_e) - \frac{y^2}{\theta_N} \right| + \frac{2M}{\theta_*} p_N, \end{aligned}$$

where we used $|\sqrt{a} - \sqrt{b}|^2 \leq |a - b|$ for $a, b \geq 0$. As $a_2(l_N - T_e) \rightarrow \infty$, the right side tends to zero by (16.49). Further, $\mathcal{G}_{j,N}$ is independent of $\mathcal{F}_{T_e}$ and has variance at most one, and we have

$$\mathbb{E}_E \left| \left(c_e - \frac{|Z_N|}{\sqrt{\theta_N}} \right) \mathcal{G}_{j,N} \right|^2 \leq \frac{\mathbb{E} |Y_1(T_e) - Z_N|^2}{\theta_* \mathbb{P}(E_N)} \rightarrow 0,$$

by (16.39). We conclude that $\sqrt{\theta_N} |Z_N|^{-1} I_j^{(3)} - \mathcal{G}_{j,N} \rightarrow 0$ in probability on $E_N \cap \{|Z_N| \geq r_0\}$.

Step 5: conclusion. Combining Steps 1–4, for fixed ε , r_0 , M_* , and $\gamma > 0$,

$$\begin{aligned} \limsup_N \mathbb{P} \left(\max_{2 \leq j \leq k} \left| \sqrt{2a_j} X_j(t_u) - (S_{j,N} \mathbf{1}_{\{j=2\}} + \mathcal{G}_{j,N}) \right| > \gamma \right) \\ \leq \varepsilon + \mathbb{P}(|H_1| \leq \sqrt{\theta} r_0) + \frac{Ck}{\gamma^2 r_0^2} e^{-4M_*}, \end{aligned}$$

where we used that $S_{j,N}$ and $\mathcal{G}_{j,N}$ are tight, Chebyshev's inequality for $I_j^{(2)}$, and the factor $\sqrt{\theta_N}/|Z_N| \leq r_0^{-1}$. The vector $(\mathcal{G}_{j,N})_{2 \leq j \leq k}$ is independent of $(Y_1, Y_2)(t_p)$, and its law is within total-variation distance Cke^{-4M_*} of the standard Gaussian law on $\mathbb{R}^{k-1}$. Using (16.51), $e^{2a_2 t_p} \rightarrow 1$, and $\theta_N \rightarrow \theta$, we see that the variable $S_{2,N}$ converges in distribution to $\sqrt{2\theta} e^{-2u} H_2/|H_1|$. Testing against bounded Lipschitz functions, letting first $N \rightarrow \infty$, then $M_* \rightarrow \infty$, then $\gamma \downarrow 0$ and $r_0 \downarrow 0$, and finally $\varepsilon \downarrow 0$, we obtain that $(\sqrt{2a_j} X_j(t_u))_{j=2}^k$ converges in distribution to $(G_2 + \sqrt{2\theta} e^{-2u} H_2/|H_1|, G_3, \dots, G_k)$, where $G_2, \dots, G_k$ are independent standard Gaussian variables, independent of (H_1, H_2) . Finally, the density of $H_2/|H_1|$ at $w \in \mathbb{R}$ is

$$2 \int_0^\infty s \phi(s) \phi(ws) ds = \frac{1}{\pi} \int_0^\infty s e^{-(1+w^2)s^2/2} ds = \frac{1}{\pi(1+w^2)},$$

and $H_2/|H_1|$ has the standard Cauchy law. This proves (16.50). $\square$

The Cauchy variable in (16.50) is the ratio $H_2/|H_1|$ of the Gaussian approximations of the first two coordinates at time t_p , and the Gaussian variable G_2 is produced by the

noise in the second coordinate during the last time interval $[t_u - M_*/a_2, t_u]$. They are independent because the Brownian increments after time l_N are independent of $\mathcal{F}_{l_N}$. We never assign a Gaussian law to the stochastic integrals with the random amplitude $|Y(r)|$; they are compared with integrals of deterministic functions after the amplitude has been replaced by an $\mathcal{F}_{T_e}$ -measurable variable.

16.8 The weighted law, the cutoff function, and scores

In this section, we weight the path law by a smooth function of the position at the time T_e , introduce a cutoff function, and prove the estimates that Sections 16.9 and 16.10 use: a formula for the density of the weighted law and its forward equation (Lemma 16.21), the probability that the cutoff differs from one and the fourth moments of the coordinates (Lemma 16.22), bounds on the derivatives of the cutoff (Lemma 16.23), bounds on the scores (Lemmas 16.24 and 16.26), and an upper bound on the density (Lemma 16.25). Throughout the section, we assume $H(\theta_*, B, c_*, C_*)$; no bound on a_3/a_2 is used.

Times and the weight. Fix $\varepsilon \in (0, 1/8)$. With $\chi_N = \sqrt{\log \log N}$ and $T_e = \chi_N/a_2$ as in Section 16.7, let

$$t_0 = T_e + \frac{1}{a_2}, \quad t_1 = T_e + \frac{\log R_N}{a_2}, \quad (16.52)$$

and recall that $U_2 = 2 \log R_N/a_2$. For every fixed $u \in \mathbb{R}$, we have $t_0 \leq t_u \leq t_1$ for large N , by (16.43), and $t_1 < T_e + U_2$. Fix once and for all the smooth transition $\psi(s) = b(s)/(b(s) + b(1-s))$, where $b(s) = e^{-1/s}$ for $s > 0$ and $b(s) = 0$ for $s \leq 0$. It satisfies $\psi = 0$ on $(-\infty, 0]$ and $\psi = 1$ on $[1, \infty)$. Only bounds on ψ' and ψ'' enter the estimates below.

Let h, M be given by Proposition 16.13 with $\varepsilon/2$ in place of ε , so that they depend only on $\varepsilon, \theta_*, c_*$, and C_* . Let $\sigma_e = \min\{h^2/M, 1/2\}$ and $\sigma_0 = \sigma_e/2$, and write $q(x) = x_1^2/N$. By Proposition 16.13, we have $\mathbb{P}(q(X_{T_e}) \geq \sigma_e) \geq 1 - \varepsilon/2 - o(1)$. We define

$$\Psi_0(x) = \psi\left(\frac{q(x)}{\sigma_0} - 1\right), \quad \Psi = \Psi_0(X_{T_e}), \quad Z_\Psi = \mathbb{E}\Psi. \quad (16.53)$$

The function Ψ_0 is smooth on S_N , invariant under R_1 , equal to one on $\{q \geq \sigma_e\}$, and equal to zero on $\{q \leq \sigma_0\}$. It follows that $Z_\Psi \geq \mathbb{P}(q(X_{T_e}) \geq \sigma_e) \geq 1 - \varepsilon$ for large N . We write P for the law of the whole process (X_0, B) , as in Section 11.1, and we call

$$Q = \frac{\Psi}{Z_\Psi} P$$

the weighted law. For $t \geq 0$, we let Q_t be the law of X_t under Q .

The variable Ψ is $\mathcal{F}_{T_e}$ -measurable. By the Markov property of (16.22), conditionally on $\mathcal{F}_{T_e}$, the process $(X_{T_e+s})_{s \geq 0}$ is the diffusion generated by L started from X_{T_e} , and the

increments of B after T_e are independent of $\mathcal{F}_{T_e}$. Both statements remain true under Q , because $dQ/dP = \Psi/Z_\Psi$ is $\mathcal{F}_{T_e}$ -measurable. In particular, $Q_t = Q_{T_e}P_{t-T_e}$ for $t \geq T_e$, and Q_{T_e} is concentrated on $\{q > \sigma_0\}$. We make no claim about the law of B before T_e under Q . The law of (X_0, B) is invariant under the reflection of the first coordinates of X_0 and B , which maps X to R_1X and leaves Ψ unchanged. It follows that every Q_t is invariant under R_1 . By Lemma 11.3, applied to the law of X_t and the measure $A \mapsto \mathbb{E}[\Psi \mathbf{1}_{\{X_t \in A\}}]$, and since $0 \leq \Psi \leq 1$, we have

$$Q_t \leq \frac{\nu_N P_t}{Z_\Psi}, \quad \|Q_t - \nu_N P_t\|_{TV} \leq 1 - Z_\Psi \leq \varepsilon. \quad (16.54)$$

Lemma 16.21 (The density of the weighted law). *For $t > T_e$, the law Q_t has a density $\rho_t = dQ_t/d\nu_N$ which is twice continuously differentiable in space and once continuously differentiable in time. The density, its first and second spatial derivatives, and its time derivative are jointly continuous on $(T_e, \infty) \times S_N$. Let $Z_\pi = \int e^{-F} d\nu_N$. For $s \geq 0$,*

$$\frac{dQ_{T_e}}{d\pi} = \frac{\Psi_0}{Z_\Psi} P_{T_e}(Z_\pi e^F), \quad \frac{dQ_{T_e+s}}{d\pi} = P_s\left(\frac{dQ_{T_e}}{d\pi}\right). \quad (16.55)$$

In graph coordinates, a nonnegative representative $f_t = dQ_t^+/dG$ has the same C^2 spatial and C^1 time regularity, with $f_t, \nabla f_t, \nabla^2 f_t$, and $\partial_t f_t$ jointly continuous, hence bounded on compact cylinders in $(T_e, \infty) \times U_N$, and satisfies

$$\partial_t f_t = L_G f_t - \operatorname{div}_G \left(\frac{y}{N} f_t \right), \quad L_G = \operatorname{div}_G(\mathfrak{g} \nabla), \quad (16.56)$$

with $\mathfrak{g}$ and div_G as in Section 11.5.

Proof. The density of ν_N with respect to π is $Z_\pi e^F$, which is bounded. For bounded measurable φ , the reversibility of π gives $\int \varphi d(\nu_N P_{T_e}) = \int P_{T_e} \varphi Z_\pi e^F d\pi = \int \varphi P_{T_e}(Z_\pi e^F) d\pi$, so that $d(\nu_N P_{T_e})/d\pi = P_{T_e}(Z_\pi e^F)$. As $Q_{T_e}(A) = Z_\Psi^{-1} \mathbb{E}[\Psi_0(X_{T_e}) \mathbf{1}_{\{X_{T_e} \in A\}}]$, we obtain the first formula in (16.55). For $s \geq 0$, the identity $Q_{T_e+s} = Q_{T_e} P_s$ and reversibility give

$$\int \varphi dQ_{T_e+s} = \int \frac{dQ_{T_e}}{d\pi} P_s \varphi d\pi = \int \varphi P_s \left(\frac{dQ_{T_e}}{d\pi} \right) d\pi,$$

which is the second formula.

For the regularity, let $\sigma_t(A) = \mathbb{E}[\Psi \mathbf{1}_{\{X_t \in A\}}] = Z_\Psi Q_t(A)$ be the endpoint measure of Lemma 11.4 with $T = T_e$, the weight Ψ , and $D = S_N$. Apply that lemma to the original uniformly initialized diffusion and choose its density representative u , retaining its first time derivative and first two spatial derivatives. Set $\rho_t = u(t, \cdot)/Z_\Psi$. The graph formula in Lemma 11.9 gives $f_t(y) = 2\omega_N \rho_t(x^+(y))/(x_1(y)\gamma(y))$ on U_N , where γ is the density (11.32) of G . The chart map and its prefactor are smooth on U_N , so the chain rule gives

the asserted spatial derivatives and their joint continuity; the same holds for the time derivative. Let $\varphi \in C_c^\infty(U_N)$. By (11.55), divided by Z_Ψ ,

$$\frac{d}{dt} \int \varphi f_t dG = \int f_t \left(L_G \varphi + \frac{y}{N} \cdot \nabla \varphi \right) dG = \int \varphi \left(L_G f_t - \operatorname{div}_G \left(\frac{y}{N} f_t \right) \right) dG.$$

In the second step, we used (11.44) and the divergence theorem for the weighted divergence div_G . Differentiating under the integral on the left, which is allowed by the joint continuity of $\partial_t f_t$ and the compact support of φ , and using that φ is arbitrary, we obtain (16.56). $\square$

We work with this representative of the density throughout. The estimates below use only this finite regularity near the support of the cutoff function, and no lower bound on it.

The cap of the cutoff and the residence error. Let

$$r = \frac{1}{4} \min \left\{ \sigma_0, \frac{\theta_*}{8}, \frac{\theta_* \sigma_0}{16(1 + 32/\theta_*)} \right\}, \quad A = \frac{16}{r}, \quad (16.57)$$

and let M_A and J_N be as in Lemma 16.16, for $N > M_A$. Then $\sigma' = 2r$ satisfies (16.27) with $\sigma = \sigma_0$, with the strict inequality $2r < \sigma_0$. Let $c_0 > 0$ be small enough that the bounds in Proposition 16.14 and its ratio construction hold with $\sigma = \sigma_0$, $\sigma' = 2r$, for both $K = 1$ and $K = 2$; it depends only on r , θ_* , and B . We set

$$p_N = 2 \exp \left(- \frac{c_0 c_* N^{1/3}}{\log(C_* N^{1/3})} \right) + (N^3 + 2) e^{-3\theta_*^2 c_* N^{1/3}/1024} + 2(N^3 + 2) e^{-c_0 N^3}. \quad (16.58)$$

By Lemma 16.4, we have $c_* N^{1/3} \leq R_N \leq C_* N^{1/3}$, and therefore, for large N , the bound ε_N of (16.28) with these parameters satisfies $\varepsilon_N \leq p_N$. The sequence p_N does not depend on the spectrum, and $N^b p_N \rightarrow 0$ for every $b > 0$: the first exponent is of order $N^{1/3}/\log N$, and the other two exponentials decay faster than every power of N . Here and below, “for large N ” means for N larger than a constant depending only on ε , the spectral constants, and ψ .

The cutoff function. Let

$$\chi(s) = \psi \left(\frac{2s}{r} - 1 \right), \quad \zeta = \chi \circ q, \quad \eta = \zeta^8. \quad (16.59)$$

Then $\chi = 0$ on $(-\infty, r/2]$ and $\chi = 1$ on $[r, \infty)$, and we have $\|\chi'\|_\infty \leq 2\|\psi'\|_\infty/r$ and $\|\chi''\|_\infty \leq 4\|\psi''\|_\infty/r^2$. The functions ζ and η are smooth on S_N , invariant under every reflection R_j , equal to one on $\{q \geq r\}$, and equal to zero on $\{q \leq r/2\}$. They depend only on x_1^2 , and they do not restrict the other coordinates. Let

$$\mathcal{K} = \{q < r\}, \quad M_t = \mathbb{E}_Q \eta(X_t), \quad Q_t^\eta = \frac{\eta Q_t}{M_t}.$$

Outside $\mathcal{K}$, we have $\eta = 1$, and all derivatives of η vanish. The probability measure Q_t^η is the normalized cutoff-weighted law of Section 11.1, with normalizing mass M_t .

Lemma 16.22 (The region outside which the cutoff equals one). *There exists $C > 0$, depending only on ε and the spectral constants, such that for large N and all $t \in [T_e, T_e + U_2]$,*

$$Q(q(X_t) < 2r) \leq p_N, \quad Q(X_t \in \mathcal{K}) \leq p_N, \quad 1 - M_t \leq p_N, \quad Q(\eta(X_t) \neq 1) \leq p_N, \quad (16.60)$$

and, for $t \in [T_e, t_1]$ with $s = t - T_e$, for every $2 \leq j \leq N$,

$$\mathbb{E}_Q X_j(t)^4 \leq CN^2 [e^{-8a_j s} + (Na_j)^{-2} + p_N]. \quad (16.61)$$

The constant in (16.61) does not depend on j , N , or t . The bounds (16.60) hold at each fixed time through $T_e + U_2$; the fourth-moment bound is asserted only through $t_1 = T_e + \log R_N/a_2$. This shorter interval contains every fixed-offset profile time t_u for all sufficiently large N .

Proof. Apply Proposition 16.14 to the original uniform-start path with $T = T_e$, $w = \Psi$, $\sigma = \sigma_0$, $\sigma' = 2r$, and $K = 2$. Its support hypothesis holds because $\Psi \neq 0$ implies $q(X_{T_e}) > \sigma_0$. The resulting weighted law is exactly Q , so the first estimate in (16.60) follows, with error at most $\varepsilon_N \leq p_N$. For the ratio and moment estimates, use the same weighted-path construction over the shorter interval $0 \leq s \leq \log R_N/a_2$. Reflecting the first coordinate by its $\mathcal{F}_{T_e}$ -measurable sign leaves the later Brownian increments Brownian and changes neither X_j for $j \geq 2$ nor any cap event. The construction in the proof of Proposition 16.14, now with $K = 1$, supplies an event $\mathcal{A}$ with $Q(\mathcal{A}^c) \leq p_N$, on which $X_1(T_e + s)^2 \geq 2rN$, $|\log \mathcal{E}_s| \leq \gamma$ with $\gamma = \theta_*/1024$, and

$$\frac{X_j(T_e + s)}{|X_1(T_e + s)|} = \mathcal{E}_s \left[e^{-2a_j s} \frac{X_j(T_e)}{|X_1(T_e)|} + M_j(s) \right], \quad j \geq 2, \quad (16.62)$$

where $\mathcal{E}$ and the stopped integrals M_j are those of (16.30) for the reflected process.

The other three bounds in (16.60) follow from $\mathcal{K} \subset \{q < 2r\}$, $\{\eta \neq 1\} \subset \mathcal{K}$, and $1 - M_t \leq Q(\eta(X_t) \neq 1)$, which holds because $0 \leq \eta \leq 1$.

For the fourth moments, fix $0 \leq s \leq \log R_N/a_2$ and $j \geq 2$. As in Step 3 of the proof of Proposition 16.14, with $\sigma' = 2r$, the variable $M_j(s)$ is the terminal value of a continuous martingale whose quadratic variation is at most $d_j = e^{2\gamma}/(4rNa_j)$. Under Q , the Brownian increments after T_e remain Brownian, so the third bound in (11.14) with $p = 2$ directly gives $\mathbb{E}_Q M_j(s)^4 \leq 16d_j^2$. We use only this moment bound, and we make no claim about the conditional law of $M_j(s)$. On $\mathcal{A}$, we have $X_1^2 \leq N$, $X_j(T_e)^2/X_1(T_e)^2 \leq \sigma_0^{-1}$, and $e^{4\gamma} \leq 2$. Using (16.62) and $(b_1 + b_2)^4 \leq 8(b_1^4 + b_2^4)$, we find that on $\mathcal{A}$,

$$X_j(T_e + s)^4 \leq N^2 \frac{X_j(T_e + s)^4}{X_1(T_e + s)^4} \leq 16N^2 [\sigma_0^{-2} e^{-8a_j s} + M_j(s)^4].$$

On $\mathcal{A}^c$, we use $X_j^4 \leq N^2$. Combining these bounds, we obtain

$$\mathbb{E}_Q X_j(T_e + s)^4 \leq 16N^2 [\sigma_0^{-2} e^{-8a_j s} + 16d_j^2] + N^2 p_N,$$

and $16d_j^2 = e^{4\gamma} r^{-2} (Na_j)^{-2} \leq 2r^{-2} (Na_j)^{-2}$. This proves (16.61). $\square$

Lemma 16.23 (Derivatives of the cutoff). *There exists $C > 0$, depending only on r , B , and ψ , such that on S_N , for $2 \leq j \leq N$,*

$$\frac{\Gamma(\eta)}{\eta} \leq \frac{C}{N} \eta^{3/4} \mathbf{1}_{\mathcal{K}}, \quad |L\eta| \leq C\eta^{3/4} \mathbf{1}_{\mathcal{K}}, \quad |K_{1j}\eta| \leq C\eta^{7/8} \leq C\eta^{1/2}, \quad (16.63)$$

where the ratio is zero where $\eta = 0$, and $K_{1j} = x_1 \partial_j - x_j \partial_1$.

Proof. Recall from (3.2) that, for a function f smooth near S_N , we have $\Gamma(f) = |\Pi(x) \nabla f|^2$ and $Lf = \text{Tr}(\Pi(x) \nabla^2 f) - (1 - 1/N) \langle x, \nabla f \rangle - \langle \Pi(x) \nabla F, \nabla f \rangle$. For q , a direct computation gives

$$\Gamma(q) = \frac{4q(1-q)}{N} \leq \frac{1}{N}, \quad Lq = \frac{2}{N} - 2q + \frac{4qF}{N}, \quad |Lq| \leq 2 + 4B,$$

where we used $\Pi(x) \nabla F \cdot \nabla q = -4qF/N$, which holds because $(\nabla F)_1 = 0$ and $x \cdot \nabla F = 2F$, and $0 \leq F \leq BN$. Further, $K_{1j}q = -2x_1 x_j / N$, so that $|K_{1j}q| \leq (x_1^2 + x_j^2)/N \leq 1$. By the chain rule, with χ and its derivatives evaluated at q ,

$$\nabla \eta = 8\chi^7 \chi' \nabla q, \quad L\eta = 8\chi^7 \chi' Lq + (56\chi^6 (\chi')^2 + 8\chi^7 \chi'') \Gamma(q), \quad K_{1j}\eta = 8\chi^7 \chi' K_{1j}q.$$

Since $\eta^{3/4} = \chi^6$ and $\eta^{7/8} = \chi^7$, we obtain $\Gamma(\eta)/\eta = 64\chi^6 (\chi')^2 \Gamma(q) \leq 64\|\chi'\|_\infty^2 N^{-1} \eta^{3/4}$,

$$|L\eta| \leq (8\|\chi'\|_\infty(2 + 4B) + 56\|\chi'\|_\infty^2 + 8\|\chi''\|_\infty) \eta^{3/4},$$

and $|K_{1j}\eta| \leq 8\|\chi'\|_\infty \eta^{7/8}$. In the bound on $L\eta$, we used $\chi^7 \leq \chi^6$ and $\Gamma(q) \leq 1$. All these derivatives vanish unless $\chi'(q) \neq 0$ or $\chi''(q) \neq 0$, which happens only in $\{r/2 < q < r\} \subset \mathcal{K}$. The bounds on χ' and χ'' stated after (16.59) complete the proof. $\square$

Lemma 16.24 (Scores of the weighted law). *There exists a numerical constant $C > 0$ such that for large N , $t \in [t_0, t_1]$, and $1 \leq i < j \leq N$, the law Q_t has a weak K_{ij} -score $K_{ij} \log \rho_t$, and*

$$\mathbb{E}_Q |K_{ij} \log \rho_t(X_t)|^4 \leq CN^2 a_j^2. \quad (16.64)$$

In particular, $\mathbb{E}_Q |K_{12} \log \rho_t(X_t)|^2 \leq CR_N$.

Proof. We use Theorem 11.5 with $p = 2$, the domain $D = S_N$ (so that there is no killing), $H = N$, the Killing field K_{ij} , the final interval $[t - \ell, t]$ with $\ell = 1/a_j$, the linear ramp, and the weight $W = \Psi$. We check its hypotheses. The domain S_N is invariant under all rotations, and $x_i^2 + x_j^2 \leq N$ on S_N . The process X solves (16.22), which is (11.2), and

it is adapted to the filtration $(\mathcal{F}_s)$ generated by X_0 and $\mathcal{B}$, for which B is a Brownian motion (Section 16.3). Since $\ell \leq 1/a_2$ and $t \geq t_0 = T_e + 1/a_2$, we have $t - \ell \geq T_e$. In the notation of Theorem 11.5, the weight is $W = \Phi(\xi, X|_{[t-\ell, t]})$ with the $\mathcal{F}_{t-\ell}$ -measurable variable $\xi = \Psi$ and $\Phi(e, \omega) = e$; it takes values in $[0, 1]$, and it is unchanged by the rotations, which act only on $[t - \ell, t]$. The normalizing constant is $\mathbb{E}W = Z_\Psi \geq 1/2$, and the weighted endpoint law is Q_t . With $\Delta = a_j - a_i \leq a_j$, (11.13) implies

$$\mathbb{E}_Q |K_{ij} \log \rho_t(X_t)|^4 \leq \frac{C_2}{Z_\Psi} [N(\ell^{-1} + \Delta^2 \ell)]^2 \leq 8C_2 N^2 a_j^2,$$

where C_2 is the constant of Theorem 11.5 with $p = 2$. For $(i, j) = (1, 2)$, this is $8C_2 R_N^2$, and the Cauchy–Schwarz inequality gives the last claim. $\square$

The bound (16.64) requires no bound on the scores of the law at earlier times: the rotation acts only on the last interval of length $1/a_j$, and it starts from the identity, also when $t - \ell = T_e$.

Graph coordinates for the weighted law. We use the graph coordinates $y = (x_2, \dots, x_N)$ of Section 16.2 and the splitting $y = (z, v)$ of Section 11.4 with $J = J_N$. Thus $z = (x_2, \dots, x_{J_N})$ are the retained coordinates and $v = (x_{J_N+1}, \dots, x_N)$ the remaining ones, and $G = G_z \otimes G_v$. Recall that J_N is the index of Lemma 16.16 with $A = 16/r$; it may depend on N and on the spectrum, and $2 \leq J_N \leq M_A$. Recall also from Lemma 16.21 that $f_t = dQ_t^+ / dG$ for $t > T_e$. Viewed as a function of y , η is smooth with compact support in U_N , since $\eta(y) \neq 0$ only if $|y|^2 < (1 - r/2)N$. Let

$$w_t = \frac{\eta f_t}{M_t}, \quad \bar{w}_t(z) = \int w_t(z, v) G_v(dv), \quad (16.65)$$

extended by zero outside U_N . Then $w_t G = (Q_t^\eta)^+$, and $\bar{w}_t$ is the density of the law of z under Q_t^η with respect to G_z . The function w_t vanishes on $\{q \leq r/2\}$.

Lemma 16.25 (An upper bound on the density). *For $t \geq T_e$, we have $dQ_t / d\pi \leq e^{BN} / Z_\Psi$ on S_N . For $t > T_e$ and large N ,*

$$f_t \leq \frac{e^{BN}}{Z_\Psi \theta_* (r/2)^{1/2}} \leq e^{CN} \quad \text{on } D_{r/2} = \{y : |y|^2 \leq (1 - r/2)N\}, \quad (16.66)$$

with C depending only on B, θ_* , and r .

Proof. The density of ν_N with respect to π is $Z_\pi e^F$, with $Z_\pi \leq 1$, and it is at most e^{BN} by Lemma 16.4. By the reversibility of π , the density of $\nu_N P_t$ with respect to π is $P_t(Z_\pi e^F) \leq e^{BN}$, by the positivity of P_t and $P_t 1 = 1$. Then (16.54) implies $dQ_t / d\pi \leq e^{BN} / Z_\Psi$. Both Q_t and π are invariant under R_1 , and the same bound holds for $dQ_t^+ / d\pi^+$. Finally, $r/2 \leq \theta_*/2$, and Lemma 16.11 gives $d\pi^+ / dG \leq (r/2)^{-1/2} \theta_*^{-1}$ on $D_{r/2}$. This proves the first bound in (16.66), and the second follows from $Z_\Psi \geq 1/2$. $\square$

The bound (16.66) is crude, but it holds for Q_t itself, at every time. We use it only through logarithms, where it costs a factor of order N .

The score of the retained coordinates. For $t \in [t_0, t_1]$, let

$$S_t(z) = z \cdot \nabla_z \log \bar{w}_t(z), \quad B(t) = \int w_t S_t^2 dG, \quad (16.67)$$

where S_t is defined on $\{\bar{w}_t > 0\}$. The function $\bar{w}_t$ is C^2 , since w_t is C^2 with compact support. The quantity $B(t)$ is the second moment, under Q_t^η , of the derivative of $\log \bar{w}_t$ along the radial vector field $z \cdot \nabla_z$ of the retained coordinates.

Lemma 16.26 (The score of the retained coordinates). *There exists $C > 0$, depending only on $\varepsilon, \theta_*, B, c_*, C_*$, and ψ , such that for large N and all $t \in [t_0, t_1]$,*

$$B(t) \leq C \left\{ 1 + R_N e^{-4a_2(t-T_e)} + R_N^2 e^{-8a_2(t-T_e)} \right\}. \quad (16.68)$$

The constant depends on the spectral constants also through A and M_A , and it is uniform over the index J_N and over the ratios a_j/a_2 with $j \leq J_N$. We use no convergence of these ratios. The last term in (16.68) comes from the Gaussian reference G_Z . It is large at the beginning of the interval, and it is integrated against a contracting factor in Section 16.9.

Proof. For $2 \leq j \leq N$, let $s_j = K_{1j} \log \rho_t$ be the weak score of Lemma 16.24. The law Q_t^η is invariant under R_1 and vanishes on $\{x_1^2 < rN/2\}$. Since $s_j \in L^2(Q_t)$ and $|K_{1j}\eta| \leq C\eta^{1/2}$ (Lemma 16.23), Lemma 11.2 with $p = 1$ implies that Q_t^η has the weak K_{1j} -score $s_j + \mathbf{1}_{\{\eta > 0\}} K_{1j} \log \eta$. We apply (11.34) of Lemma 11.10 with $J = J_N$ and $r_0 = r/2$. For $2 \leq j \leq J_N$, almost everywhere on $\{\bar{w}_t > 0\}$, we obtain

$$z_j \partial_{z_j} \log \bar{w}_t = \mathbb{E}_w \left[\frac{z_j}{x_1} s_j \circ x^+ + \frac{z_j}{x_1} (K_{1j} \log \eta) \circ x^+ + \frac{z_j^2}{x_1^2} \middle| z \right] + 2a_j z_j^2, \quad (16.69)$$

where $\mathbb{E}_w[\cdot | z]$ is the conditional expectation under $w_t G$ and $x_1 = x_1(y)$. The term z_j^2/x_1^2 comes from the Jacobian x_1^{-1} of the graph coordinates (Lemma 11.9), and it has a positive sign. The term $2a_j z_j^2$ comes from the Gaussian reference G_Z .

Using the conditional Jensen inequality and $(b_1 + b_2 + b_3 + b_4)^2 \leq 4(b_1^2 + b_2^2 + b_3^2 + b_4^2)$, we see that $\int w_t (z_j \partial_{z_j} \log \bar{w}_t)^2 dG$ is at most four times the sum of the second moments under $w_t G$ of the four terms. The functions $|s_j|$, $|K_{1j}\eta|$, x_j^2 , and x_1^2 are invariant under R_1 (see the discussion after Lemma 11.9), and these second moments are equal to the corresponding second moments under Q_t^η . On the support of η , we have $x_1^2 \geq rN/2$, and $Q_t^\eta \leq Q_t/M_t \leq 2Q_t$ for large N , by (16.60). We write $\mathbb{E}_Q$ for expectations of functions of X_t under Q , and recall that $z_j = x_j$.

The first term. By the Cauchy–Schwarz inequality, (16.61), and Lemma 16.24,

$$\begin{aligned} \mathbb{E}_{Q_t^\eta} \frac{x_j^2 s_j^2}{x_1^2} &\leq \frac{4}{rN} (\mathbb{E}_Q X_j^4)^{1/2} (\mathbb{E}_Q |s_j|^4)^{1/2} \\ &\leq C [Na_j e^{-4a_j(t-T_e)} + 1 + Na_j p_N^{1/2}]. \end{aligned}$$

The second term. Using $K_{1j}q = -2x_1 x_j / N$, we find

$$\frac{z_j K_{1j} \log \eta}{x_1} = -\frac{16z_j^2}{N} \frac{\chi'(q)}{\chi(q)}.$$

In the second moment under Q_t^η , the factor $\eta = \chi^8$ cancels the denominator, and this term contributes at most $C\mathbb{E}_Q X_j^4 / N^2$. *The third term* contributes at most $8\mathbb{E}_Q X_j^4 / (r^2 N^2)$, since $x_1^4 \geq r^2 N^2 / 4$ on the support of η . *The fourth term* contributes at most $8a_j^2 \mathbb{E}_Q X_j^4$.

For $2 \leq j \leq J_N$, we have $a_2 \leq a_j \leq Aa_2$ by Lemma 16.16, and (16.61) gives

$$\frac{\mathbb{E}_Q X_j^4}{N^2} + a_j^2 \mathbb{E}_Q X_j^4 \leq C [1 + R_N^2 e^{-8a_2(t-T_e)} + R_N^2 p_N]$$

and

$$Na_j e^{-4a_j(t-T_e)} + Na_j p_N^{1/2} \leq A [R_N e^{-4a_2(t-T_e)} + R_N p_N^{1/2}].$$

As p_N tends to zero faster than any power of N , the terms $R_N p_N^{1/2}$ and $R_N^2 p_N$ are at most one for large N . We conclude that, for $2 \leq j \leq J_N$,

$$\int w_t (z_j \partial_{z_j} \log \bar{w}_t)^2 dG \leq C \{1 + R_N e^{-4a_2(t-T_e)} + R_N^2 e^{-8a_2(t-T_e)}\}.$$

Finally, $S_t = \sum_{j=2}^{J_N} z_j \partial_{z_j} \log \bar{w}_t$, and the inequality $(\sum_{j=2}^{J_N} b_j)^2 \leq (J_N - 1) \sum_{j=2}^{J_N} b_j^2$ with $J_N \leq M_A$ proves (16.68). $\square$

16.9 Conditional relative entropy of the remaining coordinates

In this section, we show that, for the weighted law multiplied by the cutoff and normalized, at the times t_u , the conditional law of the remaining coordinates $v = (x_{J_N+1}, \dots, x_N)$ given the retained coordinates $z = (x_2, \dots, x_{J_N})$ is close to the Gaussian law G_v in relative entropy. We use the notation of Section 16.8, and for $t \in (T_e, T_e + U_2)$ we let

$$g_t = \log w_t - \log \bar{w}_t, \quad \mathcal{H}(t) = \int w_t g_t dG, \quad \mathcal{D}(t) = \int w_t \Gamma(g_t) dG, \quad (16.70)$$

where, in graph coordinates, $\Gamma(\varphi) = \nabla \varphi^\top (I - yy^\top / N) \nabla \varphi$ is the carré du champ of L . Logarithms and their derivatives are evaluated only on $\{w_t > 0\}$, and integrands

that contain the factor w_t are set to zero elsewhere. These are the quantities (11.46) of Section 11.5, with $J = J_N$. In particular, $\mathcal{H}(t) = \text{KL}(w_t G \| (\bar{w}_t G_z) \otimes G_v) \in [0, \infty]$ is the conditional relative entropy of the remaining coordinates given z , averaged over the law of z . Throughout the section, we assume $H(\theta_*, B, c_*, C_*)$, and C denotes constants depending only on ε , the spectral constants, and ψ .

The argument has three parts. The initial value $\mathcal{H}(t_0)$ is of order N , by the upper bound on the density (Lemma 16.27). The conditional relative entropy identity of Section 11, together with Lemma 16.17, gives a differential inequality with rate $16a_2$. During the time $t_u - t_0$, which is close to $\log R_N / (4a_2)$, this rate multiplies the entropy by about R_N^{-4} , and this beats the initial value $N \leq c_*^{-3} R_N^3$.

Lemma 16.27 (Initial entropy). *For large N and every $t \in (T_e, T_e + U_2)$,*

$$0 \leq \mathcal{H}(t) \leq \text{KL}(w_t G \| G) \leq BN + \log \frac{4}{\theta_*(r/2)^{1/2}} \leq CN. \quad (16.71)$$

In particular, $\mathcal{H}(t_0) \leq CN$.

Proof. We have $M_t \geq 1/2$ by (16.60), and η vanishes outside $D_{r/2}$. Since $Z_\Psi \geq 1/2$, Lemma 16.25 implies that $w_t \leq 4e^{BN}/(\theta_*(r/2)^{1/2})$. As $\int w_t dG = 1$, we have $\text{KL}(w_t G \| G) = \int w_t \log w_t dG \leq \log \sup w_t$, which is the third inequality. Here the positive part of $w_t \log w_t$ is integrable because w_t is bounded, and its negative part is bounded by $1/e$. The chain rule (11.47) [46, Theorem 2.5.3 and Section 8.5] gives $\text{KL}(w_t G \| G) = \text{KL}(\bar{w}_t G_z \| G_z) + \mathcal{H}(t)$, and both terms on the right are nonnegative. This proves the first two inequalities, and the last follows from $N \geq 1$. $\square$

The bound (16.71) concerns the weighted law itself, and it uses no comparison with another process. It is of order N , which is of order R_N^3 , and the rate $16a_2$ below compensates for this.

Logarithms. Fix $t \in (T_e, T_e + U_2)$ and let $\bar{f} = f_t \mathbf{1}_{D_{r/2}}$. Then $\int \bar{f} dG \leq 1$, and Lemma 16.25 gives $\int \bar{f}^2 dG \leq e^{CN}$. We use Lemma 11.17 with $p = 2$, its parameter A equal to N , $z_0 = 1/2$, $\alpha_0 = 1/2$, and the product $G = G_z \otimes G_v$. Since η vanishes outside $D_{r/2}$, the functions w_t , $\bar{w}_t$, and g_t are the same for $\bar{f}$ and for f_t , and $\int \eta \bar{f} dG = M_t \geq 1/2$. We obtain, for $\alpha \in [1/2, 1]$,

$$\int_{\{w_t > 0\}} f_t \eta^\alpha (|g_t|^2 + |\log w_t|^2 + |\log \bar{w}_t|^2) dG \leq CN^2. \quad (16.72)$$

Together with (16.71) and $\int f_t dG = 1$, this gives

$$\int f_t \eta^{1/2} (g_t - \mathcal{H}(t))^2 dG \leq CN^2. \quad (16.73)$$

Proposition 16.28 (Conditional relative entropy at the times t_u). Assume $H(\theta_*, B, c_*, C_*)$, fix $\varepsilon \in (0, 1/8)$, and let $\mathcal{H}$ be as in (16.70). For every compact set $K \subset \mathbb{R}$, there exist $C_K, N_K < \infty$, depending only on $K, \varepsilon, \theta_*, B, c_*, C_*$, and ψ , such that for $N \geq N_K$ and $u \in K$,

$$0 \leq \mathcal{H}(t_u) \leq C_K \left\{ R_N^{-1} e^{16\chi_N} + R_N^5 p_N^{1/2} \right\}. \quad (16.74)$$

The right side tends to zero as $N \rightarrow \infty$.

The bound is uniform over all spectra that satisfy $H(\theta_*, B, c_*, C_*)$ with $N \geq N_K$, because p_N does not depend on the spectrum. It concerns the splitting after the coordinate x_{J_N} of Lemma 16.16, and its constants may deteriorate as $\theta_* \rightarrow 0$.

Proof. We write $\mathcal{H} = \mathcal{H}(t)$, $\mathcal{D} = \mathcal{D}(t)$, and $S = S_t$, and we write $\mathbb{E}_w$ for integration against $w_t G$.

Step 1: the entropy identity. We apply Proposition 11.18 with $J = J_N$, the domain $D = S_N$ (no killing), the time $T = T_e$, the weight Ψ of (16.53), the function $\zeta = \chi \circ q$ of (16.59), $k = 8$, and the interval $\mathcal{T} = (T_e, T_e + U_2)$. We check its hypotheses. The weight is $\mathcal{F}_{T_e}$ -measurable with values in $[0, 1]$. The endpoint measure at time T_e , $Z_\Psi Q_{T_e}$, and the domain S_N are invariant under R_1 . In graph coordinates, ζ is smooth, takes values in $[0, 1]$, and has compact support in $\{|y|^2 < N\} = D^+$, since $\zeta(y) \neq 0$ only if $|y|^2 < (1 - r/2)N$. Finally, $M_t \geq 1 - p_N > 0$ on $\mathcal{T}$, by (16.60). The endpoint subprobability measures of Lemma 11.4 are $Z_\Psi Q_t$, whose densities in graph coordinates are $Z_\Psi f_t$; the quantities $w_t, \bar{w}_t, g_t, \mathcal{H}, \mathcal{D}$, and the identity below are unchanged when f_t and M_t are multiplied by the same constant. By Proposition 11.18, $\mathcal{H}(t)$ and $\mathcal{D}(t)$ are finite for $t \in \mathcal{T}$, the function $\mathcal{H}$ is absolutely continuous on $[t_0, t_1]$, and for almost every $t \in \mathcal{T}$,

$$\mathcal{H}' = -\mathcal{D} + \frac{1}{N} \mathbb{E}_w [(1 + S) v \cdot \nabla_v g] + 2 \mathbb{E}_w \Gamma(g, \log \eta) + \frac{1}{M_t} \int f_t (g - \mathcal{H}) L \eta dG. \quad (16.75)$$

The second term on the right comes from the drift $N^{-1}y \cdot \nabla$, by which the generator in graph coordinates differs from the symmetric operator L_G (see (11.45)); the coefficient one in the factor $1 + S$ is the coefficient of this drift. The terms that contain only derivatives in z cancel, because $\int w_t \nabla_z g_t dG_v = 0$ for every z with $\bar{w}_t(z) > 0$.

The proof of Proposition 11.18 uses only the stated C^2 spatial and C^1 time regularity of f_t near the support of ζ , and no lower bound on f_t . For completeness, we recall its steps in the present setting. Fix a compact interval $\mathcal{I} \subset \mathcal{T}$ and a compact neighborhood $\mathcal{O} \subset U_N$ of $D_{r/2}$. By Lemma 16.21, the functions $f_t, \nabla f_t, \nabla^2 f_t$, and $\partial_t f_t$ are bounded on $\mathcal{I} \times \mathcal{O}$, and M_t is bounded below by a positive constant on $\mathcal{I}$. Let d_0 be the distance from $D_{r/2}$ to the complement of $\mathcal{O}$, and let $M_2 > 0$ bound $\nabla^2 f_t$ on $\mathcal{I} \times \mathcal{O}$. On $D_{r/2}$, Taylor's formula along $-\nabla f_t / |\nabla f_t|$ gives $0 \leq f_t - \varsigma |\nabla f_t| + M_2 \varsigma^2 / 2$ for $0 \leq \varsigma \leq d_0$. The choice $\varsigma = |\nabla f_t| / M_2$ if this is at most d_0 , and $\varsigma = d_0$ otherwise, yields $|\nabla f_t|^2 \leq C' f_t$ on $D_{r/2}$.

In this paragraph, the constant C' may depend on N and $\mathcal{I}$; such constants justify the identity, and they do not enter the estimates below. Expanding $w_t = \zeta^8 f_t / M_t$, we deduce that, uniformly for $t \in \mathcal{I}$,

$$\int \frac{|\nabla w_t|^2}{w_t} dG < \infty, \quad \int \frac{|\nabla \bar{w}_t|^2}{\bar{w}_t} dG_z \leq \int w_t |\nabla_z \log w_t|^2 dG < \infty. \quad (16.76)$$

Here the first integrand is bounded by constants times $f_t \zeta^6 |\nabla \zeta|^2$ and $\zeta^8 |\nabla f_t|^2 / f_t$, and the second inequality is the Cauchy–Schwarz inequality on each fiber. In particular, $\mathcal{D}(t)$ is finite. For $0 < e \leq 1$, let $\Phi_e(\zeta) = (\zeta + e) \log(\zeta + e) - \zeta$, $g_e = \log(w_t + e) - \log(\bar{w}_t + e)$, and $\mathcal{H}_e(t) = \int \Phi_e(w_t) dG - \int \Phi_e(\bar{w}_t) dG_z$. Then $\mathcal{H}_e' = \int \partial_t w_t g_e dG$. Using (16.56), integration by parts with the compactly supported function ηg_e , and $M_t' = \int f_t L\eta dG$, we find

$$\begin{aligned} \mathcal{H}_e' = & - \int \Gamma(w_t, g_e) dG + \int w_t \frac{y}{N} \cdot \nabla g_e dG + \frac{1}{M_t} \int f_t g_e L\eta dG \\ & + \frac{2}{M_t} \int f_t \Gamma(\eta, g_e) dG - \frac{M_t'}{M_t} \int w_t g_e dG. \end{aligned} \quad (16.77)$$

As $e \downarrow 0$, the integrands converge pointwise, and they are dominated by means of (16.76), (16.72) with $\alpha = 3/4$, the bound $|L\eta| \leq C\zeta^6$, and $|\log(\zeta + e)| \leq |\log \zeta| + \log 2$ for $\zeta > 0$. Here we used that the first derivatives of f_t , w_t , and $\bar{w}_t$ vanish where these functions vanish, that $f_t = 0$ on $\{\eta > 0, w_t = 0\}$, and that $L\eta = 0$ where $\zeta = 0$. Integrating (16.77) over subintervals of $\mathcal{I}$ and letting $e \downarrow 0$ by dominated convergence, we find that $\mathcal{H}$ is absolutely continuous on $\mathcal{I}$, and that its derivative is, almost everywhere, the limit of the right side of (16.77). The identity $\int w_t \nabla_z g_t dG_v = 0$ and the expansion of Γ in the variables (z, v) turn this limit into (16.75). This is the classical entropy dissipation calculation [12, Section 5.2], with the additional conditional and cutoff terms, which are derived in the proof of Proposition 11.18.

Step 2: the drift term. On the support of η , we have $q \geq r/2$, and (16.36) gives $|\nabla g|^2 \leq 2\Gamma(g)/r$. Since $|v|^2 \leq N$, we have $|v \cdot \nabla_v g|^2 \leq 2N\Gamma(g)/r$ there. The Cauchy–Schwarz inequality and $xy \leq x^2/8 + 2y^2$ give

$$\begin{aligned} \frac{1}{N} |\mathbb{E}_w[(1+S) v \cdot \nabla_v g]| & \leq \mathcal{D}^{1/2} \left(\frac{2\mathbb{E}_w(1+S)^2}{rN} \right)^{1/2} \\ & \leq \frac{\mathcal{D}}{8} + \frac{4\mathbb{E}_w(1+S)^2}{rN} \leq \frac{\mathcal{D}}{8} + \frac{C}{N} (1+B(t)). \end{aligned} \quad (16.78)$$

In the last step, we used $(1+S)^2 \leq 2+2S^2$ and the definition (16.67).

Step 3: the terms containing the cutoff. By the Cauchy–Schwarz inequality, $2xy \leq x^2/8 + 8y^2$, Lemma 16.23, $M_t \geq 1/2$, and (16.60),

$$2|\mathbb{E}_w \Gamma(g, \log \eta)| \leq \frac{\mathcal{D}}{8} + 8\mathbb{E}_w \Gamma(\log \eta) = \frac{\mathcal{D}}{8} + \frac{8}{M_t} \int f_t \frac{\Gamma(\eta)}{\eta} dG \leq \frac{\mathcal{D}}{8} + \frac{C}{N} p_N. \quad (16.79)$$

Here we used that $\mathcal{K}$ is invariant under R_1 , so that $\int f_t \mathbf{1}_{\mathcal{K}} dG = Q(X_t \in \mathcal{K}) \leq p_N$. For the last term in (16.75), we write $\eta^{3/4} = \eta^{1/4} \eta^{1/2}$, and we use Lemma 16.23, the Cauchy–Schwarz inequality, (16.73), and (16.60):

$$\begin{aligned} \left| \frac{1}{M_t} \int f_t(g - \mathcal{H}) L\eta dG \right| &\leq C \left(\int f_t \eta^{1/2} (g - \mathcal{H})^2 dG \right)^{1/2} \left(\int f_t \eta \mathbf{1}_{\mathcal{K}} dG \right)^{1/2} \\ &\leq CN p_N^{1/2}. \end{aligned} \quad (16.80)$$

All these estimates concern the same weighted law, and no relative entropy is transferred through a total variation bound.

Step 4: the differential inequality. Combining (16.75), (16.78), (16.79), (16.80), and Lemma 16.26, we obtain, for almost every $t \in [t_0, t_1]$,

$$\mathcal{H}' \leq -\frac{3}{4}\mathcal{D} + \frac{C}{N} \left\{ 1 + R_N e^{-4a_2(t-T_e)} + R_N^2 e^{-8a_2(t-T_e)} \right\} + CN p_N^{1/2}. \quad (16.81)$$

For each $t \in [t_0, t_1]$, the function w_t is C^2 , has compact support in U_N , and vanishes on $\{q < r/2\}$, and $a_j > Aa_2$ for $j > J_N$ by Lemma 16.16. Then (16.37) implies that $\mathcal{D} \geq 32a_2\mathcal{H}$, and since $\mathcal{H} \geq 0$, we have $-3\mathcal{D}/4 \leq -24a_2\mathcal{H} \leq -16a_2\mathcal{H}$. We conclude that, for almost every $t \in [t_0, t_1]$,

$$\mathcal{H}' \leq -16a_2\mathcal{H} + \frac{C}{N} \left\{ 1 + R_N e^{-4a_2(t-T_e)} + R_N^2 e^{-8a_2(t-T_e)} \right\} + CN p_N^{1/2}. \quad (16.82)$$

The threshold $A = 16/r$ compensates for the factor $r/2$ lost in (16.36). No comparison of metrics finer than (16.36) is used.

Step 5: integration. Let $K \subset \mathbb{R}$ be compact, let $u \in K$, and let $S_u = a_2(t_u - T_e) = \log R_N/4 + u - \chi_N$. For N larger than a constant depending on K , we have $1 \leq S_u \leq \log R_N$, that is, $t_0 \leq t_u \leq t_1$. Since $\mathcal{H}$ is absolutely continuous on $[t_0, t_1]$, so is $e^{16a_2t}\mathcal{H}(t)$, and by (16.82) its derivative is at most e^{16a_2t} times the last two terms of (16.82), almost everywhere. Integrating from t_0 to t_u , changing variables to $\varsigma = a_2(t - T_e)$, and using $1/(Na_2) = 1/R_N$ and Lemma 16.27, we obtain

$$\begin{aligned} \mathcal{H}(t_u) &\leq CN e^{-16(S_u-1)} + \frac{C}{R_N} \int_1^{S_u} e^{-16(S_u-\varsigma)} (1 + R_N e^{-4\varsigma} + R_N^2 e^{-8\varsigma}) d\varsigma \\ &\quad + \frac{CN}{16a_2} p_N^{1/2}. \end{aligned} \quad (16.83)$$

Only the inequality (16.82) almost everywhere is used here. Factoring out e^{-16S_u} , we see that the three parts of the integral are at most $C/(16R_N)$, $Ce^{-4S_u}/12$, and $CR_N e^{-8S_u}/8$. By Lemma 16.4, we have $c_* N^{1/3} \leq R_N \leq C_* N^{1/3}$, and therefore

$$N \leq c_*^{-3} R_N^3, \quad \frac{N}{a_2} = \frac{N^2}{R_N} \leq c_*^{-6} R_N^5. \quad (16.84)$$

Using $e^{-4S_u} = R_N^{-1}e^{4(\chi_N - u)}$, we find that the first term of (16.83) is at most $Cc_*^{-3}e^{16}R_N^{-1}e^{16(\chi_N - u)}$, that the second and third parts of the integral are at most $CR_N^{-1}e^{4(\chi_N - u)}/12$ and $CR_N^{-1}e^{8(\chi_N - u)}/8$, and that the last term is at most $Cc_*^{-6}R_N^5p_N^{1/2}/16$. As $\chi_N \geq 0$ and $|u|$ is bounded on K , we obtain (16.74) from these bounds.

Finally, $\log R_N \geq (\log N)/3 + \log c_*$ and $16\chi_N = 16\sqrt{\log \log N} = o(\log N)$, which implies that $R_N^{-1}e^{16\chi_N} \rightarrow 0$. Further, $R_N^5p_N^{1/2} \rightarrow 0$, because p_N tends to zero faster than any power of N . $\square$

The rate $16a_2$ in (16.82) is four times the rate $4a_2$ that sets the profile. The initial entropy is of order $N \asymp R_N^3$, and it is multiplied by $e^{-16a_2(t_u - t_0)} = e^{16}R_N^{-4}e^{16(\chi_N - u)}$, which leaves a quantity of order $R_N^{-1}e^{16\chi_N}$. This is why we retain all coordinates with $a_j \leq 16a_2/r$: the inequality (16.37) loses the factor $r/2$ from the metric on the cap, and the threshold restores the rate. The lower bound on a_3/a_2 in $H_3(\omega, D_3)$ is not used in this section.

16.10 Finitely many coordinates in total variation

In this section, we upgrade the weak convergence of finitely many standardized coordinates in Theorem 16.20 to convergence in total variation. The tool is Theorem 11.8: a weak limit with finite Fisher information, together with laws of bounded Fisher information that are close in total variation, gives convergence in total variation. We obtain such laws from the weighted law of Section 16.8 and the cutoff, with a loss that is sent to zero after $N \rightarrow \infty$. Their Fisher information is bounded through the scores of Lemma 16.24, which use only the last time interval of length a_j^{-1} before t_u . The weak limit is a Gaussian convolution, although it has no finite variance.

Proposition 16.29 (Finitely many coordinates in total variation). *Assume $H(\theta_*, B, c_*, C_*)$ and $\theta_N \rightarrow \theta \in (0, 1)$. Fix $k \geq 2$, and if $k \geq 3$ assume also $H_k(\omega, D_k)$. Let $G_2, \dots, G_k$ be independent standard Gaussian variables and C an independent standard Cauchy variable. Then for every fixed $u \in \mathbb{R}$,*

$$\left\| \text{Law}((\sqrt{2a_j} X_j(t_u))_{j=2}^k) - \text{Law}(G_2 + \sqrt{2\theta}e^{-2u}C, G_3, \dots, G_k) \right\|_{\text{TV}} \rightarrow 0. \quad (16.85)$$

Like the other results of Section 16, Proposition 16.29 holds along every increasing sequence of dimensions, since its proof uses only limits as $N \rightarrow \infty$. No convergence of the ratios a_j/a_2 is assumed.

Proof. Let $\mathbf{z}_N = (\sqrt{2a_j} X_j(t_u))_{j=2}^k$, let μ_N be its law, and let μ be the law of $(G_2 + \sqrt{2\theta}e^{-2u}C, G_3, \dots, G_k)$. Theorem 16.20 implies that $\mu_N \Rightarrow \mu$. The law μ is the convolution of the standard Gaussian law on $\mathbb{R}^{k-1}$ with the law of $(\sqrt{2\theta}e^{-2u}C, 0, \dots, 0)$, and Lemma 11.7 implies that $\text{FI}(\mu) \leq k - 1$; no moment of C is needed (see Remark 11.13).

This is the classical Fisher information bound for a Gaussian convolution, related to the Stam–Blachman convolution inequality and its score proof [33, 138]. By the second statement of Theorem 11.8, it suffices to construct, for every $\delta \in (0, 1/8)$, probability measures $\hat{\mu}_N^\delta$ on $\mathbb{R}^{k-1}$ such that $\limsup_N \|\mu_N - \hat{\mu}_N^\delta\|_{\text{TV}} \leq \delta$ and $\text{FI}(\hat{\mu}_N^\delta) \leq C_\delta$ for large N , where C_δ does not depend on N .

Fix $\delta \in (0, 1/8)$. We repeat the construction of the weight and of the cutoff in Section 16.8 with δ in place of ε , and we keep k fixed. This gives a weight Ψ^δ with mean $Z^\delta \geq 1 - \delta$, the law Q^δ with the time marginals Q_t^δ and their densities $\rho_t^\delta = dQ_t^\delta/d\nu_N$, a cap parameter r^δ , a cutoff η^δ that vanishes on $\{q \leq r^\delta/2\}$, its mass $M_t^\delta = \mathbb{E}_{Q^\delta} \eta^\delta(X_t)$, and the sequence p_N^δ of (16.58). Below, $t = t_u$, which lies in $[t_0, t_1]$ for large N , and C_δ denotes constants depending only on δ , k , and the spectral constants. Let $\hat{\rho} = \eta^\delta Q_t^\delta / M_t^\delta$, and let $\hat{\mu}_N^\delta$ be the law of $\mathbf{z}_N$ under $\hat{\rho}$.

Total variation. The law $\hat{\mu}_N^\delta$ is the law of $\mathbf{z}_N$ under $(U/\mathbb{E}U)P$, where $U = \Psi^\delta \eta^\delta(X_t)$ takes values in $[0, 1]$ and $\mathbb{E}U = Z^\delta M_t^\delta$. By Lemma 11.3 and Lemma 16.22,

$$\|\mu_N - \hat{\mu}_N^\delta\|_{\text{TV}} \leq 1 - Z^\delta M_t^\delta \leq (1 - Z^\delta) + (1 - M_t^\delta) \leq \delta + p_N^\delta,$$

and $p_N^\delta \rightarrow 0$. We do not identify a weak limit of $\hat{\mu}_N^\delta$.

Scores. For $2 \leq j \leq k$, let $s_j = K_{1j} \log \rho_t^\delta$. By Lemma 16.24 and the Cauchy–Schwarz inequality, $\mathbb{E}_{Q^\delta} s_j^2 \leq C_\delta N a_j$. By Lemma 11.2 with $p = 1$ and Lemma 16.23, the law $\hat{\rho}$ has the weak K_{1j} -score $\vartheta_j = s_j + \mathbf{1}_{\{\eta^\delta > 0\}} K_{1j} \log \eta^\delta$, and

$$\mathbb{E}_{\hat{\rho}} \vartheta_j^2 \leq \frac{2}{M_t^\delta} (\mathbb{E}_{Q^\delta} s_j^2 + C_\delta) \leq C_\delta (N a_j + 1).$$

Fisher information. The law $\hat{\rho}$ is invariant under R_1 and vanishes on $\{x_1^2 < r^\delta N/2\}$. Let f^+ be the Lebesgue density of $\hat{\rho}^+$, extended by zero outside U_N . By the third part of Lemma 11.9, for $2 \leq j \leq k$, the weak derivative of f^+ in y_j is $f^+(\vartheta_j \circ x^+ / x_1 + y_j / x_1^2)$, and it vanishes where $f^+ = 0$. We change variables to $(\mathbf{z}, v)$, with $\mathbf{z}_j = \sqrt{2a_j} y_j$ for $2 \leq j \leq k$ and $v = (y_{k+1}, \dots, y_N)$, and we let $\tilde{f}(\mathbf{z}, v) = f^+(y) \prod_{j=2}^k (2a_j)^{-1/2}$ be the joint density. Then $\partial_{\mathbf{z}_j} = (2a_j)^{-1/2} \partial_{y_j}$, and on $\{\tilde{f} > 0\}$,

$$|\partial_{\mathbf{z}_j} \log \tilde{f}|^2 \leq \frac{1}{a_j} \left(\frac{(\vartheta_j \circ x^+)^2}{x_1^2} + \frac{y_j^2}{x_1^4} \right).$$

On the support of $\hat{\rho}$, we have $x_1^2 \geq r^\delta N/2$ and $y_j^2 \leq N$. Integrating, and using that $|\vartheta_j|$ is invariant under R_1 (see the discussion after Lemma 11.9), we find

$$\int \tilde{f} |\partial_{\mathbf{z}_j} \log \tilde{f}|^2 d\mathbf{z} dv \leq \frac{2\mathbb{E}_{\hat{\rho}} \vartheta_j^2}{r^\delta N a_j} + \frac{4}{(r^\delta)^2 N a_j} \leq C_\delta \left(1 + \frac{1}{N a_j} \right) \leq C_\delta,$$

since $Na_j \geq R_N \geq 1$ for large N . Let $p(\mathbf{z}) = \int \tilde{f}(\mathbf{z}, v) dv$ be the density of $\hat{\mu}_N^\delta$. As in the proof of Lemma 11.11, we have $\partial_j p(\mathbf{z}) = \int \partial_{z_j} \tilde{f}(\mathbf{z}, v) dv$, and the Cauchy–Schwarz inequality on each fiber gives

$$\frac{|\partial_j p(\mathbf{z})|^2}{p(\mathbf{z})} \leq \int \tilde{f}(\mathbf{z}, v) |\partial_{z_j} \log \tilde{f}(\mathbf{z}, v)|^2 dv$$

where $p(\mathbf{z}) > 0$, and $\partial_j p = 0$ where $p = 0$. Integrating in $\mathbf{z}$ and summing over j , we obtain $\text{FI}(\hat{\mu}_N^\delta) \leq (k-1)C_\delta$ for large N .

Applying the second statement of Theorem 11.8, we obtain (16.85). In its proof, the Gaussian smoothing scale is sent to zero after $N \rightarrow \infty$ and before $\delta \downarrow 0$, and the translation estimate (11.29) rests on the Sobolev approximation of [37, Theorem 9.2, p. 265]. The constants C_δ need not be bounded as $\delta \downarrow 0$. $\square$

The weighted laws are used only to obtain laws of bounded Fisher information that are close to μ_N in total variation. The limit in (16.85) concerns the law of the diffusion started from ν_N . The second coordinate is the case $k = 2$.

Theorem 16.30 (The second coordinate in total variation). *Assume $H(\theta_*, B, c_*, C_*)$ and $\theta_N \rightarrow \theta \in (0, 1)$. Let G be a standard Gaussian variable and C an independent standard Cauchy variable. Then for every fixed $u \in \mathbb{R}$,*

$$\| \text{Law}(\sqrt{2a_2} X_2(t_u)) - \text{Law}(G + \sqrt{2\theta} e^{-2u} C) \|_{\text{TV}} \rightarrow 0. \quad (16.86)$$

Proof. This is the case $k = 2$ of Proposition 16.29, in which $H_k(\omega, D_k)$ is not assumed. $\square$

Theorem 16.30 concerns the law of $X_2(t_u)$ under the whole initial law ν_N . The weight and the cutoff are used only to obtain a score bound for a law that is close to it in total variation; we do not identify a weak limit of the weighted law.

16.11 Proof of the deterministic theorem

Proof of Theorem 16.5. Pointwise convergence. Fix $u \in \mathbb{R}$ and $\varepsilon \in (0, 1/8)$, and use the notation of Sections 16.8 and 16.9 with this ε . In particular, $J_N \in \{2, \dots, M_A\}$ is the index of the last retained coordinate. We first show that

$$\limsup_{N \rightarrow \infty, N \in \mathcal{N}_J} |D_N(t_u) - \mathcal{V}_\theta(u)| \leq 2\varepsilon \quad (16.87)$$

for every J such that the set $\mathcal{N}_J = \{N : J_N = J\}$ is infinite. Fix such a J . If $J \geq 3$, then $a_3/a_2 \geq 1 + \omega$ by $H_3(\omega, D_3)$ and $a_J/a_2 \leq A$ by Lemma 16.16, so that the spectra with $N \in \mathcal{N}_J$ satisfy $H_J(\omega, A)$. For $N \in \mathcal{N}_J$, we use Lemma 11.21 on the product space $\mathbb{R}^{J-1} \times \mathbb{R}^{N-J}$ of the coordinates (z, v) , where measures on the ball $\{|y|^2 \leq N\}$ are extended by zero, with the product probability $G = G_z \otimes G_v$ and the following choices:

- $\rho = (\nu_N P_{t_u})^+$, the law of $(X_2, \dots, X_N)(t_u)$;
- $\sigma(E) = \mathbb{E}_P[\Psi \eta(X_{t_u}) \mathbf{1}_{\{(X_2, \dots, X_N)(t_u) \in E\}}]$ for Borel sets E , which satisfies $\sigma \leq \rho$ since $0 \leq \Psi \eta \leq 1$, and has mass $Z_\Psi M_{t_u}$, so that $\sigma / (Z_\Psi M_{t_u}) = (Q_{t_u}^\eta)^+ = w_{t_u} G$;
- the target law μ of $((2a_j)^{-1/2} \Gamma_j)_{j=2}^J$ for the first factor, where $(\Gamma_2, \dots, \Gamma_J) = (G_2 + \sqrt{2\theta} e^{-2u} C, G_3, \dots, G_J)$ with G_j and C as in Proposition 16.29, and the folded Gibbs measure π^+ in the role of π .

With these choices, the relative entropy H_v in Lemma 11.21 is $\text{KL}(w_{t_u} G \| (\bar{w}_{t_u} G_Z) \otimes G_v) = \mathcal{H}(t_u)$, and the first marginal ρ_Z of ρ is the law of $(X_2, \dots, X_J)(t_u)$ under the original law P , without weight. By (11.85),

$$\left| \|\rho - \pi^+\|_{\text{TV}} - \|\mu - G_Z\|_{\text{TV}} \right| \leq 2(1 - Z_\Psi M_{t_u}) + \sqrt{\mathcal{H}(t_u)/2} + \|\rho_Z - \mu\|_{\text{TV}} + \|\pi^+ - G\|_{\text{TV}}. \quad (16.88)$$

We identify each term. Both $\nu_N P_{t_u}$ and π are invariant under R_1 , and (16.13) gives $\|\rho - \pi^+\|_{\text{TV}} = D_N(t_u)$. The map $(y_j)_{j=2}^J \mapsto (\sqrt{2a_j} y_j)_{j=2}^J$ is a bijection of $\mathbb{R}^{J-1}$ that maps G_Z to the standard Gaussian law and μ to the law of $(\Gamma_2, \dots, \Gamma_J)$. For probability measures α, α' on $\mathbb{R}$ and a probability measure β on $\mathbb{R}^{J-2}$, we have $\|\alpha \otimes \beta - \alpha' \otimes \beta\|_{\text{TV}} = \|\alpha - \alpha'\|_{\text{TV}}$, as in the proof of Lemma 11.21. As $\Gamma_3, \dots, \Gamma_J$ are standard Gaussian variables independent of Γ_2 , we obtain

$$\|\mu - G_Z\|_{\text{TV}} = \|\text{Law}(G_2 + \sqrt{2\theta} e^{-2u} C) - \text{Law}(G_2)\|_{\text{TV}} = \mathcal{V}_\theta(u). \quad (16.89)$$

Using (16.54) and Lemma 16.22, we find $1 - Z_\Psi M_{t_u} \leq (1 - Z_\Psi) + (1 - M_{t_u}) \leq \varepsilon + p_N$. Proposition 16.28 implies that $\mathcal{H}(t_u) \rightarrow 0$. By Proposition 16.29 with $k = J$, applied along $\mathcal{N}_J$, and the invariance of total variation under the above bijection, $\|\rho_Z - \mu\|_{\text{TV}} \rightarrow 0$ along $\mathcal{N}_J$; for $J \geq 3$, this uses $H_J(\omega, A)$, and with it the lower bound on a_3/a_2 . Finally, $\|\pi^+ - G\|_{\text{TV}} \rightarrow 0$ by Proposition 16.9 with $D = D_3$, the constant C_0 of Lemma 16.4, and $m_N = \theta_N R_N \geq \theta_* c_* N^{1/3}$; this uses the upper bound $a_3 \leq D_3 a_2$. We obtain (16.87).

The index J_N takes at most $M_A - 1$ values. If $\limsup_N |D_N(t_u) - \mathcal{V}_\theta(u)|$ were larger than 2ε , there would exist $\kappa > 0$ and an infinite set of dimensions N with $|D_N(t_u) - \mathcal{V}_\theta(u)| \geq 2\varepsilon + \kappa$. One of the values of J_N would be taken for infinitely many of these N , which contradicts (16.87). We conclude that

$$\limsup_N |D_N(t_u) - \mathcal{V}_\theta(u)| \leq 2\varepsilon.$$

All the estimates used here hold with constants fixed after ε and before $N \rightarrow \infty$, and we do not assume that J_N converges. Letting $\varepsilon \downarrow 0$, we obtain $D_N(t_u) \rightarrow \mathcal{V}_\theta(u)$ for every fixed u .

Uniform convergence and mixing times. The function D_N is nonincreasing by the second part of Lemma 3.6, and $D_N(t) \rightarrow 0$ as $t \rightarrow \infty$, by the argument in the proof of

Corollary 3.12, which applies to the generator L of this section. For large N , we have $a_2 T_N'' = \log R_N / 4 \rightarrow \infty$ by Lemma 16.4, and $\mathcal{V}_\theta$ is continuous and strictly decreasing from one to zero by Lemma 3.9. The deterministic version of Lemma 3.11, with $b_N = T_N''$, $w_N = a_2^{-1}$, and $V = \mathcal{V}_\theta$, gives the first two parts of Theorem 16.5.

The recentered profile. For large N , we have $\theta_N > 0$ and $2\theta_N R_N > 1$, so that $T_N = \log(2\theta_N R_N) / (4a_2)$ and $T_N + u/a_2 = t_{u+\ell_N}$ with $\ell_N = \log(2\theta_N)/4 \rightarrow \log(2\theta)/4$. By the first part and the continuity of $\mathcal{V}_\theta$, $D_N(T_N + u/a_2) \rightarrow \mathcal{V}_\theta(u + \log(2\theta)/4) = \mathcal{V}(u)$ for every u , where the last identity is the third part of Lemma 3.9. As $a_2 T_N = \log(2\theta_N R_N)/4 \rightarrow \infty$, Lemma 3.11 with $b_N = T_N$ and $V = \mathcal{V}$ gives the third part. $\square$

The proof compares four measures on the same space: the law of the diffusion at time t_u in graph coordinates; the measure obtained from it by weighting with $\Psi\eta(X_{t_u})$, normalized; the product of the marginal of this normalized measure on the retained coordinates $(x_2, \dots, x_{J_N})$ with G_v ; and the Gibbs measure in graph coordinates. Relative entropy is used only for the normalized weighted law Q_{t_u}'' , for which it was estimated directly. The weight is removed in total variation, through (11.85), and the marginal on $(x_2, \dots, x_{J_N})$ that enters there is that of the diffusion itself.

Approach to the critical temperature from below

In this section, we prove case (4) of Theorem 1.1: when $c_N = 1 + \epsilon_N$ with $\epsilon_N \rightarrow 0$ and $N\epsilon_N^3 \rightarrow \infty$, the distance D_N has the Gaussian–Cauchy profile $\mathcal{V}$, with the random center $\log(2\theta_N Na_2)/(4a_2)$ and the random window a_2^{-1} . As elsewhere in the paper, the dynamical statements are deterministic. In Theorem 17.3, we prove the profile for every sequence of scaled eigenvalue gaps that satisfy the bounds of Definition 17.1, provided that the top fraction θ tends to zero and that $m = \theta Na_2$ tends to infinity. We deduce the statement for the GOE in Theorem 17.5, and the limits of the mixing times in Corollary 17.6. We also show in Corollary 17.4 that the convergence is uniform over all vectors in $\mathcal{A}_N$ with $\theta \leq 1/M$ and $m \geq M$, in the iterated limit in which $N \rightarrow \infty$ first and $M \rightarrow \infty$ afterwards; this is used in Section 21.

The proof follows the strategy of Section 16 for a fixed temperature $c > 1$. Now every estimate must be uniform in the top fraction, which tends to zero, and the errors must be expressed through the parameter m alone, which may diverge arbitrarily slowly. We use the profile lemmas of Section 3 (Lemmas 3.9 and 3.11), the spectral geometry at low temperature (Lemma 4.7), and the unnormalized process (Lemma 10.2). We also use the score bounds, Gaussian smoothing, and conditional relative entropy estimates of Section 11, the density bounds of Section 12 (Theorem 12.8), two functional inequalities of Section 8 (Corollaries 8.9 and 8.13), and the comparison of the Gibbs measure with a Gaussian product law in Proposition 16.9.

Section 17.1 states the hypotheses and the main results, and describes the proof. Sections 17.2 and 17.3 estimate the means and all fixed moments of the unnormalized process. Section 17.4 shows that the radial part of the unnormalized process is determined, along the time interval that we study, by its first coordinate at an earlier time. Section 17.5 gives a Gaussian approximation of the coordinates at an early time τ and controls the domain visited by the path. Section 17.6 identifies the limit of finitely many coordinates and proves the lower bound on the distance. Section 17.7 bounds the density of the time- t law at time θ^{-2} , Section 17.8 bounds the conditional relative entropy of the remaining coordinates, and Section 17.9 proves the main results.

17.1 Setting and main results

The dynamics. Throughout this section, $N \geq 3$ and $a = (a_1, \dots, a_N)$ is a deterministic vector with

$$0 = a_1 < a_2 \leq a_3 \leq \dots \leq a_N.$$

We consider the diffusion on S_N with generator and invariant measure

$$L = \Delta_{S_N} - \nabla_{S_N} F \cdot \nabla_{S_N}, \quad F(x) = \sum_{i=2}^N a_i x_i^2, \quad \pi(dx) = \frac{e^{-F(x)} \nu_N(dx)}{\int_{S_N} e^{-F} d\nu_N}. \quad (17.1)$$

This is the operator L_A of (3.3) with $A = -2 \operatorname{diag}(a_1, \dots, a_N)$, and by (3.8) the model at rescaled inverse temperature c is the case $a_i = c(\lambda_1 - \lambda_i)/2$, written in the eigencoordinates. We write $P_t = e^{tL}$, $\rho_t = \nu_N P_t$ for the time- t law, and $D(t) = \|\rho_t - \pi\|_{\text{TV}}$ for $t \geq 0$. As in Section 16, we set $D(t) = D(0)$ for $t < 0$, so that expressions such as $D(t_m(u))$ below are defined for all $u \in \mathbb{R}$. This convention does not change the mixing times, and D remains nonincreasing on $\mathbb{R}$. When we consider a sequence of vectors a , we add the corresponding index to these symbols.

Spectral hypotheses. The following bounds play the role of the spectral hypotheses of the other sections.

Definition 17.1 (Scaled-gap bounds and separation of a_3 from a_2). Let $0 < c_* \leq C_*$. We say that a satisfies the indexed bounds with constants (c_*, C_*) if

$$c_*(i-1)^{2/3} N^{-2/3} \leq a_i \leq C_*(i-1)^{2/3} N^{-2/3} \quad (2 \leq i \leq N). \quad (17.2)$$

Let moreover $\varepsilon_* > 0$. We say that a is admissible with constants $(c_*, C_*, \varepsilon_*)$ if

$$c_*(i-1)^{2/3} N^{-2/3} \leq a_i \leq C_*(i-1)^{2/3} N^{-2/3} \quad (2 \leq i \leq N), \quad \frac{a_3}{a_2} \geq 1 + \varepsilon_*. \quad (17.3)$$

We write $\mathcal{A}_N = \mathcal{A}_N(c_*, C_*, \varepsilon_*)$ for the set of admissible vectors of length N .

By Lemma 4.7, the eigenvalue gaps of the model satisfy (17.3) at every fixed confidence. Note that the upper bound in (17.2) with $i = N$ implies that $a_N \leq C_*$.

Parameters. We write

$$a = a_2, \quad \theta = 1 - \frac{1}{N} \sum_{i=2}^N \frac{1}{2a_i}, \quad m = \theta N a. \quad (17.4)$$

For the model, these are a_2 , the top fraction θ_N , and $m_N = \theta_N R_N$ of (3.14). A centered Gaussian vector $(x_2, \dots, x_N)$ with independent coordinates of variances $(2a_i)^{-1}$ has mean squared norm $(1 - \theta)N$; when $\theta > 0$ is small, the Gibbs measure places the small fraction

θ of $|x|^2 = N$ on the first coordinate (Proposition 16.9). We will see that m is comparable to $(N\theta^3)^{1/3}$, and that it measures how far the system is from the critical window: for the model, m_N is of order $(N\epsilon_N^3)^{1/3}$. Finally, for $m > 0$ and $u \in \mathbb{R}$ we define

$$t_m(u) = \frac{\log(2m)}{4a} + \frac{u}{a}, \quad (17.5)$$

and we recall from (3.17) that $\mathcal{V}(u) = \|\text{Law}(G + e^{-2u}C) - \text{Law}(G)\|_{\text{TV}}$, where G is a standard Gaussian random variable and C is an independent standard Cauchy random variable.

We use the following elementary consequences of the indexed bounds throughout the section.

Lemma 17.2 (Consequences of the indexed bounds). *Suppose that a satisfies (17.2). There exists $C \geq 1$, depending only on c_* and C_* , such that the following hold.*

1. We have $c_*N^{-2/3} \leq a \leq C_*N^{-2/3}$, and $a_k \leq (C_*/c_*)(k-1)^{2/3}a$ for $2 \leq k \leq N$.
2. If $\theta > 0$, then

$$\frac{m^3}{C_*^3} \leq N\theta^3 \leq \frac{m^3}{c_*^3}, \quad \frac{m^2}{C_*^3} \leq \frac{\theta^2}{a} \leq \frac{m^2}{c_*^3}, \quad m \leq C_*N^{1/3}.$$

3. For every $x > 0$,

$$\min \left\{ N-1, \left\lfloor N \left(\frac{x}{C_*} \right)^{3/2} \right\rfloor \right\} \leq \#\{i \geq 2 : a_i \leq x\} \leq N \left(\frac{x}{c_*} \right)^{3/2}.$$

4. For all $t \geq 0, h > 0, s > 0$, and integers $1 \leq k \leq N-1$,

$$\begin{aligned} \sum_{i=2}^N e^{-4a_i t} + \sum_{i=2}^N \frac{1}{(1+a_i t)^2} &\leq CN(1+t)^{-3/2}, & \sum_{i=2}^N \min\{t, a_i^{-1}\}^2 &\leq CN\sqrt{t}, \\ \sum_{i=2}^N \frac{\min\{h, a_i^{-1}\}}{a_i} &\leq CN\sqrt{h}, & \sum_{i=2}^N \frac{e^{-a_i s}}{a_i} &\leq \frac{CN}{\sqrt{s}}, \\ \sum_{i \geq k+1} \left(\frac{a}{a_i} \right)^2 &\leq Ck^{-1/3}, & \sum_{i=2}^{k+1} \frac{1}{a_i} &\leq CN^{2/3}k^{1/3}. \end{aligned}$$

Proof. The first part is (17.2) with $i = 2$ and $i = k$. For the second part, we have $\theta = m/(Na)$, so that $N\theta^3 = m^3/(N^2a^3)$ and $\theta^2/a = m^2/(N^2a^3)$, and the first part implies that $c_*^3 \leq N^2a^3 \leq C_*^3$. Also $\theta \leq 1$, and $m \leq Na \leq C_*N^{1/3}$. For the third part, if $a_i \leq x$ then $c_*(i-1)^{2/3}N^{-2/3} \leq x$, that is, $i-1 \leq N(x/c_*)^{3/2}$; conversely, every $i \geq 2$ with $i-1 \leq N(x/c_*)^{3/2}$ satisfies $a_i \leq x$.

For the fourth part, the first two bounds at $t = 0$ follow from $2(N-1) \leq 2N$ and $\sum_{i=2}^N \min\{0, a_i^{-1}\}^2 = 0$. In proving those bounds below, assume $t > 0$. We write $j = i-1$

and compare the sums with integrals of decreasing functions of j . For the first sum, using the lower bound in (17.2), we find

$$\sum_{i \geq 2} e^{-4a_i t} \leq \int_0^\infty e^{-4c_* t N^{-2/3} y^{2/3}} dy = \frac{N}{(4c_* t)^{3/2}} \int_0^\infty e^{-x^{2/3}} dx,$$

and the sum is also at most N ; the sum of $(1 + a_i t)^{-2}$ is bounded in the same way by $N(c_* t)^{-3/2} \int_0^\infty (1 + x^{2/3})^{-2} dx$, which is finite. For the second sum, the indices with $a_i \leq 1/t$ number at most $N(c_* t)^{-3/2}$ and contribute at most t^2 each, while

$$\sum_{i: a_i > 1/t} a_i^{-2} \leq \frac{N^{4/3}}{c_*^2} \sum_{j > N(C_* t)^{-3/2}} j^{-4/3} \leq CN\sqrt{t}$$

when $N(C_* t)^{-3/2} \geq 1$; when $N(C_* t)^{-3/2} < 1$, the full sum $\sum_{i \geq 2} a_i^{-2} \leq 4c_*^{-2} N^{4/3}$ is at most $CN\sqrt{t}$. The third sum is the same computation with the summand $\min\{h, a_i^{-1}\} a_i^{-1}$, which equals h/a_i when $a_i \leq 1/h$ and a_i^{-2} otherwise; for the first group we use $\sum_{j \leq x} j^{-2/3} \leq 3x^{1/3}$. For the fourth sum, we have

$$\sum_{i \geq 2} \frac{e^{-a_i s}}{a_i} \leq \frac{N^{2/3}}{c_*} \int_0^\infty y^{-2/3} e^{-c_* s N^{-2/3} y^{2/3}} dy = \frac{3\sqrt{\pi}}{2c_*} \frac{N^{2/3}}{(c_* s N^{-2/3})^{1/2}} = \frac{CN}{\sqrt{s}},$$

where we substituted $x = y^{2/3}$. For the last two sums, we use $a/a_i \leq (C_*/c_*)j^{-2/3}$, $a_i^{-1} \leq c_*^{-1} N^{2/3} j^{-2/3}$, $\sum_{j \geq k} j^{-4/3} \leq 4k^{-1/3}$, and $\sum_{j \leq k} j^{-2/3} \leq 3k^{1/3}$. $\square$

Main results

The first theorem is the deterministic form of the main result. We recall that $\mathcal{V}$ is continuous and strictly decreasing from 1 to 0 (Lemma 3.9).

Theorem 17.3 (The profile below the critical temperature). *Fix $0 < c_* \leq C_*$ and $\varepsilon_* > 0$. For each $n \geq 1$, let $a^{(n)} \in \mathcal{A}_{N_n}(c_*, C_*, \varepsilon_*)$, and let θ_n , m_n , D_n , and $t_{m_n}(u)$ be the quantities defined above for $a^{(n)}$. Suppose that $\theta_n > 0$ for all n , that $\theta_n \rightarrow 0$, and that $m_n \rightarrow \infty$. Then, for every compact interval $I \subset \mathbb{R}$,*

$$\sup_{u \in I} |D_n(t_{m_n}(u)) - \mathcal{V}(u)| \longrightarrow 0 \quad (n \rightarrow \infty). \quad (17.6)$$

The hypotheses force $N_n \rightarrow \infty$, because $m_n \leq C_* N_n^{1/3}$ by Lemma 17.2, and the times $t_{m_n}(u)$, $u \in I$, are positive for all large n ; for the finitely many other n , the convention $D(t) = D(0)$ for $t < 0$ applies. The dimensions N_n need not be distinct or increasing; this matters for the next corollary, which is proved by applying Theorem 17.3 along a diagonal sequence.

Corollary 17.4 (Uniformity in the gaps). *Fix $0 < c_* \leq C_*$, $\varepsilon_* > 0$, and a nonempty compact interval $I \subset \mathbb{R}$. For $N \geq 3$ and $M \geq 1$ with $\log(2M) + 4 \min I > 0$, let*

$$\mathcal{E}_N(M) = \sup \left\{ \sup_{u \in I} |D(t_m(u)) - \mathcal{V}(u)| : a \in \mathcal{A}_N(c_*, C_*, \varepsilon_*), 0 < \theta \leq \frac{1}{M}, m \geq M \right\},$$

where D , θ , and m are computed from a , and the supremum of the empty set is zero. The condition on M ensures that $t_m(u) > 0$ for $u \in I$ and $m \geq M$. Then

$$\lim_{M \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathcal{E}_N(M) = 0. \quad (17.7)$$

Moreover, the following holds. Let $S_0 > 0$, and for $S \geq S_0$ and $N \geq 3$ let $a^{(N,S)} \in \mathcal{A}_N(c_*, C_*, \varepsilon_*)$, with parameters $\theta_{N,S} > 0$ and $m_{N,S}$. Suppose that for each fixed S we have $\theta_{N,S} \rightarrow 0$ and $m_{N,S} \rightarrow m_S \in (0, \infty)$ as $N \rightarrow \infty$, and that $m_S \rightarrow \infty$ as $S \rightarrow \infty$. Then

$$\lim_{S \rightarrow \infty} \limsup_{N \rightarrow \infty} \sup_{u \in I} |D_{N,S}(t_{m_{N,S}}(u)) - \mathcal{V}(u)| = 0, \quad (17.8)$$

where $D_{N,S}$ and $t_{m_{N,S}}(u)$ are computed from $a^{(N,S)}$, with $D_{N,S}(t) = D_{N,S}(0)$ for $t < 0$. For all sufficiently large fixed S , the times $t_{m_{N,S}}(u)$, $u \in I$, are positive for all large N .

The point of Corollary 17.4 is the order of the limits. At a fixed value of m the error need not be small; the corollary states that the error is small in the limit $N \rightarrow \infty$, uniformly over $\mathcal{A}_N$, once m is large and θ is small. This is the form in which the result enters the iterated limit of Section 21, where at each fixed outer parameter S the vectors $a^{(N,S)}$ have a finite limit m_S of $m_{N,S}$.

We now turn to the model. Recall from (3.14) that $a_i = c_N(\lambda_1 - \lambda_i)/2$, $\theta_N = 1 - N^{-1} \sum_{i \geq 2} (2a_i)^{-1}$, and $m_N = \theta_N N a_2$.

Theorem 17.5 (The profile for the GOE). *Let $c_N = 1 + \epsilon_N$, where $\epsilon_N > 0$, $\epsilon_N \rightarrow 0$, and $N\epsilon_N^3 \rightarrow \infty$. Almost surely the eigenvalues are distinct, so that $a_2 > 0$. Let*

$$T_N = \begin{cases} \frac{\log(2m_N)}{4a_2}, & \theta_N > 0 \text{ and } 2m_N > 1, \\ 1, & \text{otherwise,} \end{cases}$$

and let $D_N(t) = D_N(0)$ for $t < 0$. Then $T_N > 0$, $\mathbb{P}(\theta_N > 0, 2m_N > 1) \rightarrow 1$, $a_2 T_N \rightarrow \infty$ in probability, and for every compact interval $I \subset \mathbb{R}$,

$$\sup_{u \in I} |D_N(T_N + u/a_2) - \mathcal{V}(u)| \rightarrow 0 \quad \text{in probability.}$$

In particular, D_N has cutoff profile $\mathcal{V}$ with center T_N and window a_2^{-1} in the sense of Definition 3.7.

Corollary 17.6 (Mixing times). *The following hold.*

1. *In the setting of Theorem 17.3, let $T_n = \log(2m_n)/(4a_2^{(n)})$, which is positive for all large n , and let $t_n(\varepsilon) = \inf\{t \geq 0 : D_n(t) \leq \varepsilon\}$. Then, for every compact interval $I' \subset (0, 1)$,*

$$\sup_{\varepsilon \in I'} \left| a_2^{(n)}(t_n(\varepsilon) - T_n) - \mathcal{V}^{-1}(\varepsilon) \right| \longrightarrow 0,$$

and for every fixed $\eta \in (0, 1)$, we have $D_n((1 - \eta)T_n) \rightarrow 1$ and $D_n((1 + \eta)T_n) \rightarrow 0$. Moreover, $(a_2^{(n)}T_n)^{-1} = 4/\log(2m_n) \rightarrow 0$, and for $0 < \varepsilon_1 < \varepsilon_2 < 1$,

$$a_2^{(n)}(t_n(\varepsilon_1) - t_n(\varepsilon_2)) \longrightarrow \mathcal{V}^{-1}(\varepsilon_1) - \mathcal{V}^{-1}(\varepsilon_2) > 0.$$

2. *In the setting of Theorem 17.5, let $t_N(\varepsilon)$ be the mixing time (3.15) and T_N as in Theorem 17.5. Then the convergence assertions of the first part hold in probability: $a_2(t_N(\varepsilon) - T_N) \rightarrow \mathcal{V}^{-1}(\varepsilon)$ uniformly for ε in compact subsets of $(0, 1)$, $D_N((1 - \eta)T_N) \rightarrow 1$ and $D_N((1 + \eta)T_N) \rightarrow 0$ for every fixed $\eta \in (0, 1)$, $(a_2T_N)^{-1} \rightarrow 0$, and $a_2(t_N(\varepsilon_1) - t_N(\varepsilon_2)) \rightarrow \mathcal{V}^{-1}(\varepsilon_1) - \mathcal{V}^{-1}(\varepsilon_2)$ for $0 < \varepsilon_1 < \varepsilon_2 < 1$, all in probability. On the event $\{\theta_N > 0, 2m_N > 1\}$, whose probability tends to one, we have $(a_2T_N)^{-1} = 4/\log(2m_N)$. In particular, there is cutoff.*

Outline of the proof

We explain the mechanism and then list the steps of the proof. Let X be the diffusion started from ν_N , and write $X = \sqrt{N}Y/|Y|$, where Y is the unnormalized process of Section 10:

$$dY_i = -2a_i Y_i dt + \sqrt{2/N} |Y| dB_i, \quad Y(0) \in S_N \text{ uniform.}$$

The second moment $h_i(t) = \mathbb{E}Y_i(t)^2$ solves a linear equation whose kernel is $k(t) = (2/N) \sum_{i \geq 2} e^{-4a_i t}$, a defective probability density with total mass $1 - \theta$. Since $a_1 = 0$, the first coordinate Y_1 is a martingale, and its mean square increases to order θ^{-1} . The mean squared norm $\mathbb{E}|Y(t)|^2$ decreases, like $N(1 + t)^{-1/2}$ up to the time θ^{-2} and like $N\theta^{-2}t^{-3/2}$ afterwards, until a time of order a^{-1} . After that time the mean squared norm is of order θ^{-2} , while the mean square of the first coordinate is of order θ^{-1} , so that $X_1^2 = NY_1^2/|Y|^2$ is of order θN . In the first part of the proof (Sections 17.2 and 17.3), we estimate these means, and all fixed moments, uniformly in θ .

Let $\tau = m^a \theta^{-2}$ for a fixed $a \in (0, 1)$, and let $H_i = \sqrt{\theta} Y_i(\tau)$. We show in Section 17.5 that, for each fixed J , the vector $(H_1, \dots, H_J)$ is close to a standard Gaussian vector. After a time T of order $a^{-1} \log m$, the process $\theta^2 |Y(t)|^2$ stays close to H_1^2 along the whole interval $[T, K_0 a^{-1} \log m]$, with an error bounded by a negative power of m in every fixed L^p norm (Section 17.4). At the time $t = t_m(u)$, the standardized second coordinate is then

$$\sqrt{2a} X_2(t) = \frac{\sqrt{2aN} \theta Y_2(t)}{\theta |Y(t)|} \approx \sqrt{2m} e^{-2at} \frac{H_2}{|H_1|} + \sqrt{4a} \int_T^t e^{-2a(t-s)} dB_2(s) \approx e^{-2u} \frac{H_2}{|H_1|} + G_2,$$

where G_2 is a standard Gaussian variable independent of (H_1, H_2) . The ratio $H_2/|H_1|$ is approximately a ratio of independent standard Gaussian variables, that is, approximately standard Cauchy. The coordinates $3 \leq i \leq J$ are approximately independent standard Gaussian variables, because the separation $a_3 \geq (1 + \varepsilon_*)a$ makes their inherited part $(2m)^{(1-a_i/a)/2}H_i/|H_1|$ negligible. Under the Gibbs measure, the same coordinates are approximately independent standard Gaussian variables (Proposition 16.9). This identifies the limit of finitely many coordinates and gives the lower bound $\liminf D(t_m(u)) \geq \mathcal{V}(u)$ (Section 17.6).

For the upper bound we must control all coordinates at once. We keep the first J coordinates, where J is fixed so large that $b = a_{J+1} \geq 6a$, and we compare the remaining coordinates with independent Gaussian coordinates of variances $(2a_i)^{-1}$. Section 17.7 shows that the density of the time- t law with respect to the Gibbs measure has all moments of size $e^{O(N\theta^3)} = e^{O(m^3)}$ at time θ^{-2} ; this gives the initial conditional relative entropy, of order m^3 . The conditional relative entropy of the remaining coordinates then decays at a rate of at least $4 \cdot (7/8) \cdot b$ up to errors of order m^{-1} (Section 17.8). Over the interval from T to $t_m(u)$, of length about $(1/4 - 1/100)a^{-1} \log m$, the entropy is therefore multiplied by at most $\exp(-4 \cdot (7/8) \cdot (b/a) \cdot (24/100) \log m) \leq m^{-5}$, since $b/a \geq 6$, and this beats the initial value m^3 . The entropy estimate is proved for the endpoint measure weighted by the indicator of an event at time τ and killed on leaving a domain, multiplied by an endpoint cutoff function and normalized; the mass removed by the event restriction, the killing, and the cutoff is controlled in Section 17.5. Finally, the finitely many retained coordinates converge in total variation, by the Gaussian smoothing argument of Section 11, and Section 17.9 combines the two estimates with the stationary comparison.

The parameters of the proof of Theorem 17.3 are chosen in the order of Table 17.1. All constants depend only on the parameters above them in the table, and every estimate holds for $m \geq m_0$ and $\theta \leq 1/2$, with a threshold m_0 that depends on the same parameters.

The moment order p in the table is the order used for the ratio ξ in Corollary 17.18. The proof of Corollary 17.17 uses a larger moment order p_{res} of the path estimate of Theorem 17.14, chosen with $2p_{\text{res}}\theta \geq B_0$. For the exponents above, $\theta = 1/800$, and $p_{\text{res}} = 8000$ suffices. All these orders are fixed before the threshold for m .

The main difficulty compared with Section 16 is that the time scales θ^{-2} and a^{-1} and the mean square θ^{-1} of the first coordinate all depend on θ , which tends to zero at an arbitrary rate. The renewal estimates of Section 17.2 are therefore proved with constants that do not depend on θ , the moment bounds of Section 17.3 are proved by a comparison that keeps the small deficit θ , and every probability of failure is bounded by an explicit negative power of m .

Table 17.1. Parameters in the proof of Theorem 17.3, in the order in which they are fixed.

Parameter	Role	Value or dependence
c_*, C_*, ε_*	constants of Definition 17.1	given
J	number of retained coordinates, with $a_{J+1} \geq 6a$	c_*, C_*
$\mathfrak{a}, \mathfrak{b}, \zeta, K_0$	exponents: $\tau = m^{\mathfrak{a}}\theta^{-2}$, $T = \mathfrak{b}a^{-1}\log(em)$, cone width $\delta = m^{-\zeta}$; horizon $K_0a^{-1}\log(em)$	$\mathfrak{a} = 1/10$, $\mathfrak{b} = \zeta = 1/100$, $K_0 = 1$
γ	tolerance for the mass removed by the event restriction	fixed; $\gamma \rightarrow 0$ last
ℓ_-, ℓ_+	the event $\{\ell_- \leq H_1 \leq \ell_+, H_i \leq \ell_+\}$ at time τ	γ, J
s, L_0, L_1, L	cutoff and killing levels	$\ell_{\pm}, J$
p, B_0	moment order in the bound on the mass of the set $\mathcal{C}$ outside which the cutoff equals one; failure exponent	$p = 200$, $B_0 = 20$
m_0	threshold for m	all of the above

17.2 The unnormalized process and its mean

In this section and in Sections 17.3 to 17.6, the vector a satisfies the indexed bounds (17.2) with constants (c_*, C_*) , and $0 < \theta \leq 1/2$. We fix $K_0 \geq 1$ and write

$$T_{\max} = \frac{K_0 \log(em)}{a}$$

for the end of the time interval that we study; the times $t_m(u)$ with u in a compact set lie in $[0, T_{\max}]$ once m is large. We also write

$$f_{\theta}(t) = (1+t)^{-1/2} \min \{1, [\theta^2(1+t)]^{-1}\} \quad (t \geq 0). \quad (17.9)$$

Unless said otherwise, the constants $C, c_0 > 0$ depend only on c_*, C_* , and K_0 , and the estimates hold for $m \geq m_0$, with a threshold m_0 that depends on the same parameters. Positive constants in lower bounds are written c_1, c_2 , with the same dependence; the letter c alone is the rescaled inverse temperature. In particular, no constant depends on θ or on N , and the thresholds involve no negative power of θ and no power of N .

The unnormalized process

Let $B = (B_1, \dots, B_N)$ be a standard Brownian motion in $\mathbb{R}^N$, let g be a standard Gaussian vector in $\mathbb{R}^N$ independent of B , and let $X_0 = \sqrt{N}g/|g|$, which has law ν_N . Let Y and Y^G

be the solutions of

$$dY_i = -2a_i Y_i dt + \sqrt{2/N} |Y| dB_i \quad (1 \leq i \leq N) \quad (17.10)$$

with $Y(0) = X_0$ and $Y^G(0) = g$, respectively. By Lemma 10.2, applied with the quadratic potential coefficients $\rho_i = a_i$, both solutions exist, are unique, and never vanish, and $X = \sqrt{N} Y / |Y|$ is the diffusion with generator L of (17.1) started from ν_N . Then ρ_t is the law of $X(t)$. We call Y the unnormalized process with the uniform start, and Y^G the unnormalized process with the Gaussian start. Moments are easier to compute for the Gaussian start. The next lemma transfers upper bounds to the uniform start.

Lemma 17.7 (Gaussian and uniform starts). *Almost surely, $Y^G(t) = (|g|/\sqrt{N}) Y(t)$ for all $t \geq 0$, and $|g|$ is independent of (X_0, B) , hence of the path Y . In particular, if $\Phi \geq 0$ is a measurable function of the path such that $\Phi(\lambda y) = \lambda^{2p} \Phi(y)$ for all $\lambda > 0$, for some real $p \geq 1$, then $\mathbb{E}\Phi(Y) \leq \mathbb{E}\Phi(Y^G)$, with equality if $p = 1$.*

Proof. The drift in (17.10) is linear, and the diffusion coefficient $|y|$ is positively homogeneous of degree one. By the third part of Lemma 10.2, applied with the random variable $\lambda = |g|/\sqrt{N}$, which is measurable at time zero, the process λY is the unnormalized process with initial value $\lambda X_0 = g$, driven by the same Brownian motion B . Pathwise uniqueness implies that $Y^G = (|g|/\sqrt{N}) Y$. The variables $|g|$ and $g/|g|$ are independent by the rotation invariance of g , and both are independent of B . We obtain $\mathbb{E}\Phi(Y^G) = \mathbb{E}[(|g|^2/N)^p] \mathbb{E}\Phi(Y)$, and $\mathbb{E}(|g|^2/N)^p \geq (\mathbb{E}|g|^2/N)^p = 1$ by Jensen's inequality, with equality if $p = 1$. $\square$

The second moments. Let $h_i(t) = \mathbb{E}Y_i(t)^2$, and let $\bar{h}(t) = N^{-1} \sum_i h_i(t) = N^{-1} \mathbb{E}|Y(t)|^2$ be their mean. The variable $\sup_{s \leq t} |Y(s)|$ has finite moments of every order for each t , since the coefficients of (17.10) are globally Lipschitz, and the stochastic integrals below are martingales. By Itô's formula,

$$d(Y_i^2) = \left(-4a_i Y_i^2 + \frac{2}{N} |Y|^2 \right) dt + 2\sqrt{\frac{2}{N}} |Y| Y_i dB_i. \quad (17.11)$$

Recall that $\mathbb{E}X_{0,i}^2 = 1$. The second moments solve $h' = Hh$ with $h(0) = \mathbf{1}$, where

$$H = -4 \operatorname{diag}(a_1, \dots, a_N) + \frac{2}{N} \mathbf{1}\mathbf{1}^\top. \quad (17.12)$$

Using Lemma 17.7 with $p = 1$, we see that the Gaussian start has the same second moments. In coordinates, since $a_1 = 0$,

$$h_1(t) = 1 + 2 \int_0^t \bar{h}(s) ds, \quad h_i(t) = e^{-4a_i t} + 2 \int_0^t e^{-4a_i(t-s)} \bar{h}(s) ds \quad (i \geq 2). \quad (17.13)$$

In particular, h_1 is nondecreasing. The difference $h_1 - h_i$ vanishes at zero and has derivative $4a_i h_i \geq 0$. Therefore $h_i \leq h_1$, and we have

$$0 \leq h_1(t) - h_i(t) = \int_0^t 4a_i h_i(s) ds \leq 4a_i t h_1(t). \quad (17.14)$$

Renewal kernels. For a vector $\kappa = (\kappa_i)_{i \geq 2}$ of positive numbers, we define

$$k_\kappa(t) = \frac{2}{N} \sum_{i=2}^N \kappa_i e^{-4a_i t}, \quad U_\kappa = \sum_{n \geq 1} k_\kappa^{*n}, \quad \theta_\kappa = 1 - \int_0^\infty k_\kappa(t) dt = 1 - \frac{1}{N} \sum_{i=2}^N \frac{\kappa_i}{2a_i}, \quad (17.15)$$

where k^{*n} is the n -fold convolution on $[0, \infty)$. We write k, U for the case $\kappa_i = 1$ for all i , for which $\theta_\kappa = \theta$. If $\theta_\kappa > 0$, the series defining U_κ converges in L^1 , $\int_0^\infty U_\kappa = (1 - \theta_\kappa)/\theta_\kappa$, and U_κ is the unique locally bounded solution of the renewal equation $U_\kappa = k_\kappa + k_\kappa * U_\kappa$. Summing the second identity in (17.13) over $i \geq 2$, we find $N\bar{h} - h_1 = (N/2)k + k * (N\bar{h})$, and convolving with $\delta_0 + U$ yields

$$\bar{h} = \frac{1}{2} U + \frac{1}{N} (\delta_0 + U) * h_1, \quad (17.16)$$

where we used $k + U * k = U$. The next lemma collects the properties of these kernels. Its constants do not depend on θ , and this is what makes all later estimates uniform in the top fraction.

Lemma 17.8 (Renewal estimates uniform in the top fraction). *Let $K_1 \geq 1$, and suppose that $1/2 \leq \kappa_i \leq K_1$ for all $i \geq 2$ and that $\theta/2 \leq \theta_\kappa \leq \min\{2\theta, 2/3\}$. There exist $C \geq 1$ and $c_0 \in (0, 1]$, depending only on c_* , C_* , and K_1 , such that if $m \geq C$, the following hold.*

1. *Let $\bar{u}_i = \sqrt{\kappa_i}$ and $H_\kappa = -4 \operatorname{diag}(a_2, \dots, a_N) + (2/N) \bar{u} \bar{u}^T$. Then $U_\kappa(t) = (2/N) \bar{u}^T e^{tH_\kappa} \bar{u}$, and all eigenvalues of the symmetric matrix H_κ are smaller than $-2a$. In particular, $U_\kappa(t) = \int e^{-zt} \mu_\kappa(dz)$ for a finite positive measure μ_κ supported in $(2a, \infty)$, and U_κ is positive and nonincreasing.*
2. *For $t \geq 1$, $\int_t^\infty U_\kappa \leq C\theta^{-2}t^{-1/2}$.*
3. *We have $U_\kappa(t) \leq Cf_\theta(t)$ for $0 \leq t \leq c_0/a$, and $U_\kappa(t) \leq C(N\theta^2)^{-1}e^{-2a(t-c_0/a)}$ for $t \geq c_0/a$. In particular, $U_\kappa(t) \leq C\theta^{-2}(1+t)^{-3/2}$ for all $t \geq 0$.*
4. *If $\kappa_i = 1$ for all i , then $U(t) \geq C^{-1}f_\theta(t)$ for $0 \leq t \leq 1/a$.*

Proof. We prove the four parts in order. Throughout the proof, C denotes a constant depending only on c_* , C_* , and K_1 .

Step 1: spectral representation. By the Duhamel formula for the rank-one perturbation,

$$e^{tH_\kappa} = e^{-4tD} + \int_0^t e^{-4(t-s)D} \frac{2}{N} \bar{u} \bar{u}^T e^{sH_\kappa} ds, \quad D = \operatorname{diag}(a_2, \dots, a_N).$$

Multiplying by $(2/N) \bar{u}^T$ on the left and by $\bar{u}$ on the right, we find that $\Phi(t) = (2/N) \bar{u}^T e^{tH_\kappa} \bar{u}$ solves $\Phi = k_\kappa + k_\kappa * \Phi$, so $\Phi = U_\kappa$. For $v \in \mathbb{R}^{N-1}$, we have $(\bar{u} \cdot v)^2 \leq \sum_i \kappa_i (4a_i)^{-1} \cdot \sum_i 4a_i v_i^2$ by the Cauchy–Schwarz inequality, and therefore

$$v^T H_\kappa v \leq -4 \sum_i a_i v_i^2 \left(1 - \frac{2}{N} \sum_i \frac{\kappa_i}{4a_i}\right) = -4\theta_\kappa \sum_i a_i v_i^2 < 0 \quad (v \neq 0).$$

This shows that H_κ is negative definite, and the spectral theorem yields $U_\kappa(t) = \int e^{-zt} \mu_\kappa(dz)$, where μ_κ puts mass $(2/N) \langle \bar{u}, \varphi \rangle^2$ at z for each unit eigenvector φ of $-H_\kappa$ with eigenvalue $z > 0$. It remains to locate the eigenvalues. Let $G_\kappa(y) = (2/N) \sum_{i \geq 2} \kappa_i (4a_i + y)^{-1}$, which is decreasing in $y > -4a$, with $G_\kappa(0) = 1 - \theta_\kappa$. If $-z$ is an eigenvalue of H_κ with eigenvector φ and $z \notin \{4a_i\}$, then $(4D - z)\varphi = (2/N) \bar{u}(\bar{u} \cdot \varphi)$, which implies that $\bar{u} \cdot \varphi \neq 0$ and $G_\kappa(-z) = 1$. Using $4a_i - 2a \geq 2a_i$, we find

$$G_\kappa(-2a) - G_\kappa(0) = \frac{2}{N} \sum_{i \geq 2} \frac{2a\kappa_i}{4a_i(4a_i - 2a)} \leq \frac{K_1 a}{2N} \sum_{i \geq 2} a_i^{-2} \leq \frac{C}{Na} = \frac{C\theta}{m},$$

by Lemma 17.2. Together with $\theta_\kappa \geq \theta/2$, this implies that $G_\kappa(-2a) \leq 1 - \theta/2 + C\theta/m < 1$ for $m > 2C$. If $0 < z \leq 2a$, then $G_\kappa(-z) \leq G_\kappa(-2a) < 1$, and $-z$ is not an eigenvalue; the remaining candidates $z = 4a_i$ satisfy $z \geq 4a$. Every eigenvalue of H_κ is therefore smaller than $-2a$.

Step 2: the tail. Note that $1 - \theta_\kappa \geq 1/3$. Then the function $p_\kappa = k_\kappa/(1 - \theta_\kappa)$ is a probability density with $p_\kappa(t) \leq 3K_1 k(t) \leq C(1+t)^{-3/2}$, by Lemma 17.2. For $t \geq 1$, its tail beyond t is at most $Ct^{-1/2}$ and its first moment truncated at t is at most $C\sqrt{t}$. Let S_n be a sum of n independent random variables $X_1, \dots, X_n$ with density p_κ . Bounding separately the event that some X_j exceeds t , and the event that the truncated sum exceeds t (by Markov's inequality), we obtain

$$\mathbb{P}(S_n > t) \leq n\mathbb{P}(X_1 > t) + \frac{n}{t} \mathbb{E}[X_1 \mathbf{1}_{\{X_1 \leq t\}}] \leq \frac{Cn}{\sqrt{t}}.$$

Using $U_\kappa = \sum_n (1 - \theta_\kappa)^n p_\kappa^{*n}$, we find $\int_t^\infty U_\kappa = \sum_n (1 - \theta_\kappa)^n \mathbb{P}(S_n > t) \leq Ct^{-1/2} \theta_\kappa^{-2} \leq 4Ct^{-1/2} \theta^{-2}$.

Step 3: the upper bound before time c_0/a . For $y > 0$, the Laplace transform of U_κ is $\hat{U}_\kappa(y) = G_\kappa(y)/(1 - G_\kappa(y))$, by the renewal equation. Let $C' = 2^{2/3} C_*/c_*$, so that $y \geq C'a$ implies $N(y/C_*)^{3/2} \geq 2$, and let $C'a \leq y \leq 1$. By the third part of Lemma 17.2, at least $\min\{N-1, N(y/C_*)^{3/2}/2\}$ indices $i \geq 2$ satisfy $a_i \leq y$. For these indices $4a_i + y \leq 5y$ and $a_i^{-1} \geq y^{-1}$, and since $\kappa_i \geq 1/2$, we find

$$1 - G_\kappa(y) = \theta_\kappa + \frac{2}{N} \sum_{i \geq 2} \frac{\kappa_i y}{4a_i(4a_i + y)} \geq \theta_\kappa + \frac{1}{N} \sum_{i \geq 2: a_i \leq y} \frac{1}{20a_i} \geq \theta_\kappa + c_2 \sqrt{y},$$

where in the last step we used $y \leq 1$ when the minimum is $N-1$. Splitting the sum at $a_i = y$ and using the third and fourth parts of Lemma 17.2, we also find

$$-G'_\kappa(y) = \frac{2}{N} \sum_i \frac{\kappa_i}{(4a_i + y)^2} \leq \frac{2K_1}{N} \left(\frac{\#\{i : a_i \leq y\}}{y^2} + \sum_{i: a_i > y} \frac{1}{16a_i^2} \right) \leq \frac{C}{\sqrt{y}}.$$

Since U_κ is nonincreasing, for $t > 0$,

$$-\widehat{U}'_\kappa(1/t) = \int_0^\infty se^{-s/t} U_\kappa(s) ds \geq U_\kappa(t) \int_0^t se^{-1} ds = \frac{t^2 U_\kappa(t)}{2e}.$$

On the other hand, $-\widehat{U}'_\kappa = -G'_\kappa/(1 - G_\kappa)^2$. Let $c_0 = 1/C'$. For $1 \leq t \leq c_0/a$, the two bounds at $y = 1/t$ yield

$$U_\kappa(t) \leq \frac{2eC t^{-3/2}}{(\theta_\kappa + c_2 t^{-1/2})^2} \leq C \min\{t^{-1/2}, \theta^{-2} t^{-3/2}\} \leq C f_\theta(t).$$

For $t \leq 1$, the renewal equation and $k_\kappa \leq C$ imply that $U_\kappa(t) \leq C + C \int_0^t U_\kappa$, and by Gronwall's inequality $U_\kappa \leq C$ on $[0, 1]$, while $f_\theta(t) \geq 2^{-1/2}$ there because $\theta \leq 1/2$.

Step 4: the upper bound after time c_0/a . Let $t_0 = c_0/a$. From Step 1, we have $U_\kappa(t) \leq e^{-2a(t-t_0)} U_\kappa(t_0)$ for $t \geq t_0$. Step 3 gives $U_\kappa(t_0) \leq C f_\theta(t_0) \leq C \theta^{-2} (a/c_0)^{3/2} \leq C (N\theta^2)^{-1}$, where we used $a^{3/2} \leq C_*^{3/2} N^{-1}$. This is the second bound of the third part. For the last bound, $f_\theta(t) \leq \theta^{-2} (1+t)^{-3/2}$ covers $t \leq t_0$, and for $t \geq t_0$ we have $(1+t)^{3/2} e^{-2at} \leq C a^{-3/2} \leq CN$.

Step 5: the lower bound. Let $\kappa_i = 1$ for all i . Let $1 \leq t \leq 1/a$. We claim that at least $c_2 N t^{-3/2}$ indices $i \geq 2$ satisfy $a_i \leq 1/t$. If $N(tC_*)^{-3/2} \geq 1$, this follows from the third part of Lemma 17.2. Otherwise, the index $i = 2$ qualifies, because $a \leq 1/t$, and $N t^{-3/2} < C_*^{3/2}$. It follows that $k(t) \geq (2/N) e^{-4\#\{i \geq 2 : a_i \leq 1/t\}} \geq c_2 t^{-3/2}$, and also $k(t) \geq 2(N-1)e^{-4C_*}/N \geq c_2$ for $t \leq 1$. In particular, the probability density $p = k/(1-\theta)$ satisfies $p(t) \geq k(t) \geq c_2 (1+t)^{-3/2}$ on $[0, 1/a]$. Fix $t \in [1, 1/a]$ and $n \geq 2$. The density of S_n at t is an integral over the set where the n summands add up to t . Up to a null set, the parts of this set where the j th summand is at least $t/2$, for $j = 1, \dots, n$, are disjoint, and by symmetry they contribute equally. On the part with $j = n$, the last summand lies in $[t/2, t]$ and the others add up to at most $t/2$. We obtain

$$p^{*n}(t) \geq n \min_{[t/2, t]} p \cdot \mathbb{P}(S_{n-1} \leq t/2) \geq c_2 n t^{-3/2} \mathbb{P}(S_{n-1} \leq t/2).$$

Using Step 2, we choose $c_1 > 0$, depending only on c_* and C_* , such that $\mathbb{P}(S_{n-1} > t/2) \leq 1/2$ for $n \leq 1 + c_1 \sqrt{t}$. Let $n_* = \min\{\lfloor 1 + c_1 \sqrt{t} \rfloor, \lceil 1/\theta \rceil\}$. Using $1 - \theta \geq e^{-2\theta}$ and $\theta n_* \leq 2$, we obtain

$$U(t) \geq \sum_{n=1}^{n_*} (1-\theta)^n p^{*n}(t) \geq c_2 t^{-3/2} e^{-4} \sum_{n=1}^{n_*} n \geq c_2 t^{-3/2} \min\{t, \theta^{-2}\} \geq c_2 f_\theta(t).$$

For $t \leq 1$, we have $U \geq k \geq c_2 \geq c_2 f_\theta$. This completes the proof. $\square$

A weighted mean

The proof of the moment bounds in Section 17.3 compares the moments with products of the solutions of a linear system in which the coupling coefficients of the first coordinates are increased. The next lemma bounds these solutions. Its constants do not depend on the size C_1 of the perturbation of the remaining coordinates, which is chosen afterwards in terms of these constants.

Lemma 17.9 (Heat estimates uniform under bounded perturbations). *Fix $C_0 \geq 1$ and $C_1 \geq 0$, let $n_m = \lfloor m \rfloor$, and let $\kappa_i = C_0$ for $1 \leq i \leq n_m$ and $\kappa_i = 1 + C_1\theta/n_m$ for $i > n_m$. Let $\omega(t) = (\omega_1(t), \dots, \omega_N(t))$ solve*

$$\omega'_i = -4a_i\omega_i + \frac{2}{N}\kappa_i\omega_\Sigma, \quad \omega_\Sigma = \sum_{j=1}^N \omega_j, \quad \omega_i(0) = 1.$$

There exist $C \geq 1$, depending only on c_ , C_* , C_0 , and K_0 , and m_0 , depending in addition on C_1 , such that the following hold for $m \geq m_0$ and $0 \leq t \leq T_{\max}$.*

1. *We have $\omega_1(t) \leq C/\theta$, $\omega_\Sigma(t) \leq C[Nf_\theta(t) + \theta^{-2}]$, and $\int_0^{T_{\max}} \omega_\Sigma \leq CN/\theta$.*
2. *For $i \geq 2$,*

$$\omega_i(t) \leq \begin{cases} C \left[\frac{\sqrt{1+t}}{1+a_it} + \frac{\min\{t, a_i^{-1}\}}{N\theta^2} \right], & t \leq \theta^{-2}, \\ C \left[\frac{e^{-2a_it}}{\theta} + \frac{\min\{t, a_i^{-1}\}}{\theta^2 t^{3/2}} + \frac{\min\{t, a_i^{-1}\}}{N\theta^2} \right], & t \geq \theta^{-2}. \end{cases}$$

3. *The mean satisfies $N\bar{h}(t) \geq C^{-1}[Nf_\theta(t) + \theta^{-2}]$.*
4. *We have*

$$N\bar{h}(t) \leq \omega_\Sigma(t) \leq CN\bar{h}(t), \quad \omega_i(t) \leq Ch_i(t) \quad (1 \leq i \leq N), \quad \max_{i > n_m} \frac{\omega_i(t)}{\omega_\Sigma(t)} \leq \frac{C\theta}{n_m}.$$

The case $C_0 = 1$, $C_1 = 0$ is $\omega = h$, and then the first two parts hold for the second moments h_i and $N\bar{h} = \omega_\Sigma$.

Proof. The proof has four steps: we compare the perturbed kernel with the unperturbed one, bound the first coordinate and the total, bound the other coordinates, and finally bound $\bar{h}$ from below and compare. Throughout, C denotes a constant depending only on c_* , C_* , C_0 , and K_0 .

Step 1: the perturbed kernel. Let $\kappa_b = (\kappa_i)_{i \geq 2}$. Using Lemma 17.2, we have $\sum_{2 \leq i \leq n_m} a_i^{-1} \leq CN^{2/3}n_m^{1/3}$ and $N^{-1/3} = \theta N^{2/3}a/m \leq C_*\theta/m$. It follows that

$$0 \leq \theta - \theta_{\kappa_b} = \frac{1}{N} \sum_{i \geq 2} \frac{\kappa_i - 1}{2a_i} \leq \frac{C_0}{N} \sum_{2 \leq i \leq n_m} \frac{1}{a_i} + \frac{C_1\theta}{n_m} \leq \frac{C\theta n_m^{1/3}}{m} + \frac{C_1\theta}{n_m},$$

which is at most $\theta/2$ for $m \geq m_0(C_1)$. Also $1 \leq \kappa_i \leq C_0 + 1$ once $C_1\theta/n_m \leq 1$. We may therefore use Lemma 17.8 for κ_b with $K_1 = C_0 + 1$, and its constants depend only on c_* , C_* , and C_0 . This is the only place where C_1 enters, through the threshold m_0 .

Step 2: the first coordinate and the total. As in (17.16), summing the equations for $i \geq 2$, we find

$$\omega_\Sigma = W + (\delta_0 + U_{\kappa_b}) * \omega_1, \quad W = (\delta_0 + U_{\kappa_b}) * K_b, \quad K_b(t) = \sum_{i \geq 2} e^{-4a_i t},$$

and $\omega_1 = 1 + (2C_0/N) \int_0^t \omega_\Sigma$. Because $\kappa_i \geq 1$, we have $K_b \leq (N/2)k_{\kappa_b}$ and $W \leq (N/2)(k_{\kappa_b} + U_{\kappa_b} * k_{\kappa_b}) = (N/2)U_{\kappa_b}$. The function ω_1 is nondecreasing, and $((\delta_0 + U_{\kappa_b}) * \omega_1)(t) \leq \omega_1(t)(1 + \int U_{\kappa_b}) = \omega_1(t)/\theta_{\kappa_b} \leq 2\omega_1(t)/\theta$. We conclude that

$$\omega_\Sigma(t) \leq \frac{N}{2}U_{\kappa_b}(t) + \frac{2}{\theta}\omega_1(t), \quad (17.17)$$

and, integrating,

$$\omega_1(t) \leq 1 + \frac{C_0}{\theta_{\kappa_b}} + \frac{4C_0}{N\theta} \int_0^t \omega_1(s) ds \leq \frac{3C_0}{\theta} + \frac{4C_0}{N\theta} \int_0^t \omega_1(s) ds.$$

Gronwall's inequality implies that $\omega_1(t) \leq 3C_0\theta^{-1}e^{4C_0t/(N\theta)}$. For $t \leq T_{\max}$, we have $t/(N\theta) \leq K_0 \log(em)/(aN\theta) = K_0 \log(em)/m \leq K_0$, and $\omega_1(t) \leq C/\theta$. Inserting this and the third part of Lemma 17.8 into (17.17), we find $\omega_\Sigma(t) \leq C[Nf_\theta(t) + \theta^{-2}]$ for $t \leq c_0/a$, and $\omega_\Sigma(t) \leq C\theta^{-2}$ for $t \geq c_0/a$. This proves the first two bounds of the first part. For the third, we integrate (17.17): $\int_0^{T_{\max}} \omega_\Sigma \leq N/\theta + C\theta^{-2}T_{\max}$, and $\theta^{-2}T_{\max} = (N/\theta)K_0 \log(em)/m \leq CN/\theta$.

Step 3: the other coordinates. Let $i \geq 2$. By variation of constants,

$$\omega_i(t) = e^{-4a_i t} + \frac{2\kappa_i}{N} \int_0^t e^{-4a_i(t-s)} \omega_\Sigma(s) ds. \quad (17.18)$$

Let $t \leq \theta^{-2}$. Then $f_\theta(s) \leq (1+s)^{-1/2}$, and splitting the integral at $s = t/2$, we have

$$\int_0^t e^{-4a_i(t-s)}(1+s)^{-1/2} ds \leq 2e^{-2a_i t} \sqrt{1+t} + \frac{\sqrt{2} \min\{t, a_i^{-1}\}}{\sqrt{1+t}} \leq \frac{C\sqrt{1+t}}{1+a_i t},$$

where in the last step we used $a_i \leq C_*$; the term with θ^{-2} contributes $\theta^{-2} \min\{t, (4a_i)^{-1}\}$. This proves the first case of the second part. Now let $t \geq \theta^{-2}$, and split (17.18) at $s = t/2$. For $s \leq t/2$, the exponential is at most $e^{-2a_i t}$, and the first part gives $(2\kappa_i/N) \int_0^{T_{\max}} \omega_\Sigma \leq C/\theta$. For $s \in [t/2, t]$, we have $f_\theta(s) \leq \theta^{-2}(1+s)^{-3/2} \leq 3\theta^{-2}t^{-3/2}$, which implies that $\omega_\Sigma(s) \leq C[N\theta^{-2}t^{-3/2} + \theta^{-2}]$, and $\int_{t/2}^t e^{-4a_i(t-s)} ds \leq \min\{t, a_i^{-1}\}$. Together with $e^{-4a_i t} \leq e^{-2a_i t} \leq \theta^{-1}e^{-2a_i t}$, this proves the second case.

Step 4: lower bound and comparisons. By (17.16), we have $\bar{h} \geq U/2$, and the fourth part of Lemma 17.8 implies that $N\bar{h}(t) \geq c_1 N f_\theta(t)$ for $t \leq 1/a$. Recall from (17.13) that $h_1(s) = 1 + 2 \int_0^s \bar{h} \geq 1 + \int_0^s U$. Let $t \geq 2$. Since h_1 is nondecreasing, (17.16) implies that

$$N\bar{h}(t) \geq h_1(t) + \int_0^{t/2} U(s)h_1(t-s) ds \geq h_1(t/2) \left(1 + \int_0^{t/2} U\right) \geq \left(1 + \int_0^{t/2} U\right)^2.$$

The second part of Lemma 17.8 implies that $1 + \int_0^{t/2} U = \theta^{-1} - \int_{t/2}^\infty U \geq \theta^{-1} - C\theta^{-2}t^{-1/2}$. We conclude that $N\bar{h}(t) \geq (4\theta^2)^{-1}$ for $t \geq C_2\theta^{-2}$, with C_2 depending only on c_*, C_* . We now combine these bounds. For $t \leq C_2\theta^{-2}$, we have $f_\theta(t) \geq c_1\theta$, and the second part of Lemma 17.2 implies that $Nf_\theta(t) \geq c_1N\theta \geq c_1\theta^{-2}N\theta^3 \geq c_1\theta^{-2}$. Then $N\bar{h}(t) \geq c_1Nf_\theta(t) \geq c_1[Nf_\theta(t) + \theta^{-2}]$. For $C_2\theta^{-2} \leq t \leq 1/a$, both lower bounds apply; note that $C_2\theta^{-2} \leq 1/a$ for large m , because $\theta^2/a \geq m^2/C_*^3$. For $1/a \leq t \leq T_{\max}$, we have $Nf_\theta(t) \leq N\theta^{-2}a^{3/2} \leq C_*^{3/2}\theta^{-2}$, and $N\bar{h}(t) \geq (4\theta^2)^{-1}$. This proves the third part.

For the fourth part, note that the matrix of the system for ω is $H + (2/N)(\kappa - 1)\mathbf{1}\mathbf{1}^T$, where H is the matrix of (17.12). Since $\kappa_i \geq 1$, this matrix dominates H entrywise. The difference $e = \omega - h$ solves $e' = He + \phi$ with $e(0) = 0$ and $\phi_i = (2/N)(\kappa_i - 1)\omega_\Sigma \geq 0$, and $e(t) = \int_0^t e^{(t-s)H}\phi(s) ds \geq 0$, because e^{tH} has nonnegative entries. We conclude that $\omega \geq h$ componentwise, and in particular $\omega_\Sigma \geq N\bar{h}$. Combining the first and third parts, we have $\omega_\Sigma \leq C[Nf_\theta + \theta^{-2}] \leq CN\bar{h}$. Inserting $\omega_\Sigma \leq CN\bar{h}$ into (17.18), and comparing with (17.13), we obtain $\omega_i \leq (C_0 + 1)Ch_i$ for $i \geq 2$; similarly $\omega_1 = 1 + (2C_0/N) \int \omega_\Sigma \leq C_0Ch_1$. Finally, let $i > n_m$, so that $i - 1 \geq n_m$. Let $t \leq \theta^{-2}$. Then $f_\theta(t) \geq (1+t)^{-1/2}/2$, and using the second and third parts, we find

$$\frac{\omega_i(t)}{\omega_\Sigma(t)} \leq \frac{\omega_i(t)}{N\bar{h}(t)} \leq C \left[\frac{1+t}{N(1+a_it)} + \frac{\sqrt{1+t}}{N^2\theta^2a_i} \right] \leq \frac{C}{Na_i} \left(1 + \frac{1}{N\theta^3} \right) \leq \frac{C}{Na_i},$$

where we used $a_i \leq C_*$ and $\sqrt{1+t} \leq \sqrt{2}\theta^{-1}$. Let $t \geq \theta^{-2}$. Then $N\bar{h}(t) \geq c_1[N\theta^{-2}t^{-3/2} + \theta^{-2}]$, and dividing the three terms of the second part by this bound, we obtain

$$\frac{\omega_i(t)}{N\bar{h}(t)} \leq C \left[\frac{\theta t^{3/2} e^{-2a_it}}{N} + \frac{1}{Na_i} + \frac{1}{Na_i} \right] \leq C \left[\frac{\theta}{Na_i^{3/2}} + \frac{1}{Na_i} \right],$$

where we used $\sup_t t^{3/2} e^{-2a_it} \leq Ca_i^{-3/2}$. Note that $Na_i^{3/2} \geq c_*^{3/2}n_m$ and $Na_i \geq c_*n_m^{2/3}N^{1/3}$, by (17.2). Using $N^{1/3}\theta \geq m/C_*$ and $n_m \leq m$, we find $(Na_i)^{-1} \leq C\theta m^{-1}n_m^{-2/3} \leq C\theta/n_m$. This completes the proof. $\square$

We record the consequences for the second moments h_i and their mean $\bar{h}$.

Corollary 17.10 (Bounds for the mean). *For $m \geq m_0$ and $0 \leq t \leq T_{\max}$, the following hold.*

1. $h_i(t) \leq h_1(t) \leq \theta^{-1}e^{2t/(N\theta)} \leq C/\theta$ for all i , and $\theta h_1(t) \geq 1 - C(\theta^2 t)^{-1/2}$ for $t \geq 1$.
2. $\bar{h}(t) \leq U(t)/2 + C/(N\theta^2)$, and $\int_t^\infty U \leq C\theta^{-2}t^{-1/2}$ for $t \geq 1$.

3. $C^{-1}[f_\theta(t) + (N\theta^2)^{-1}] \leq \bar{h}(t) \leq C[f_\theta(t) + (N\theta^2)^{-1}]$.
4. For $t \geq c_0/a$, we have $\bar{h}(t) \leq C(N\theta^2)^{-1}$ and $h_i(t) \leq C[\theta^{-1}e^{-2a_it} + (N\theta^2a_i)^{-1}]$ for $i \geq 2$.
5. For $0 \leq t \leq \theta^{-2}$, we have $C^{-1}(1+t)^{-1/2} \leq U(t) \leq C(1+t)^{-1/2}$ and $C^{-1}(1+t)^{-1/2} \leq \bar{h}(t) \leq C[(1+t)^{-1/2} + (N\theta^2)^{-1}]$.

Proof. Using (17.16) and the monotonicity of h_1 , we have $\bar{h}(t) \leq U(t)/2 + h_1(t)/(N\theta)$. Integrating and using $1 + \int U = \theta^{-1}$, we find $h_1(t) \leq \theta^{-1} + (2/(N\theta)) \int_0^t h_1$, and Gronwall's inequality yields the upper bound in the first part; the bound by C/θ follows as in Step 2 of the previous proof. For the lower bound, $\theta h_1(t) \geq \theta(1 + \int_0^t U) = 1 - \theta \int_t^\infty U$, and the second part of Lemma 17.8 applies. The second part follows from $\bar{h} \leq U/2 + h_1/(N\theta)$ and the first part. The third part is Lemma 17.9 with $C_0 = 1$ and $C_1 = 0$. For the fourth part, $c_0/a \geq \theta^{-2}$ when m is large, and the second case of Lemma 17.9 applies, with $\theta^{-2}t^{-3/2} \min\{t, a_i^{-1}\} \leq \theta^{-2}(a/c_0)^{3/2}a_i^{-1} \leq C(N\theta^2a_i)^{-1}$; the bound on $\bar{h}$ is part of Step 2 there. For the fifth part, $\theta^{-2} \leq c_0/a$, and Lemma 17.8 implies that $C^{-1}f_\theta \leq U \leq Cf_\theta$ on $[0, \theta^{-2}]$, and $(1+t)^{-1/2}/2 \leq f_\theta(t) \leq (1+t)^{-1/2}$ there; the bounds on $\bar{h}$ follow from the third part and from $\bar{h} \geq U/2$. $\square$

17.3 Moments of all orders

The time- t law depends on the unnormalized process through the ratio $Y/|Y|$, and the path estimates of Section 17.4 require moments of $|Y(t)|^2$ and $Y_i(t)^2$ of every fixed order. The noise amplitude $|Y|$ is random, and the coordinates are not Gaussian. We bound the moments by comparing the exact linear equations they satisfy with an explicit supersolution. The comparison keeps the small deficit θ : the coupling coefficient of the remaining coordinates exceeds one only by $C_1\theta/n_m$, which the bound of Lemma 17.9 on ω_i/ω_Σ allows.

Proposition 17.11 (All fixed-order moment bounds for the uniform start). *Let Y have the uniform start $Y(0) \sim \nu_N$. For every real $r \geq 1$ there exist C_r and m_0 , depending only on $r, c_*, C_*,$ and K_0 , such that for $m \geq m_0, 0 \leq t \leq T_{\max}$, and $1 \leq i \leq N$,*

$$\| |Y(t)|^2 \|_r \leq C_r N \bar{h}(t), \quad \| Y_i(t)^2 \|_r \leq C_r h_i(t), \quad \| |Y(t)|^2 Y_i(t)^2 \|_r \leq C_r N \bar{h}(t) h_i(t),$$

where $\| \cdot \|_r$ is the norm of $L^r(\mathbb{P})$.

Proof. We first prove the bounds directly for Y and integer orders $p \geq 2$.

Step 1: the moment equations. For a multi-index $n \in \mathbb{N}^N$ with $|n| = n_1 + \cdots + n_N = p$, let $S_n(t) = \mathbb{E} \prod_i Y_i(t)^{2n_i}$. For $f(y) = \prod_i y_i^{2n_i}$, we have $y_i \partial_i f = 2n_i f$ and $y_i^2 \partial_i^2 f = 2n_i(2n_i - 1)f$. Applying Itô's formula to $f(Y)$, using (17.10), and taking expectations, we obtain

$$S'_n = -4(n \cdot a) S_n + \frac{2}{N} \sum_{i: n_i \geq 1} n_i(2n_i - 1) \sum_{j=1}^N S_{n-e_i+e_j}, \quad (17.19)$$

where e_i is the i th unit vector and we used $|Y|^2 = \sum_j Y_j^2$. The stochastic integrals have mean zero because Y has finite moments of all orders on bounded time intervals. The system (17.19) is linear in $(S_n)_{|n|=p}$, and its off-diagonal coefficients are nonnegative. By independence of Gaussian radius and direction, as in Lemma 17.7, we have $S_n(0) \leq \mathbb{E} \prod_i g_i^{2n_i} = \prod_i (2n_i - 1)!!$, with the convention $(-1)!! = 1$.

Step 2: a supersolution. Let C_b be the constant of the last bound in the fourth part of Lemma 17.9 for $C_0 = 2p - 1$; it does not depend on C_1 . Lemma 17.9 applies with $C_0 = 2p - 1$ and $C_1 = 2(p - 1)C_b$. Let $\eta_m = \max_{i > n_m} \omega_i / \omega_\Sigma$, so that $\eta_m \leq C_b \theta / n_m$ on $[0, T_{\max}]$. Increasing m_0 , we may assume that $(2p - 1)\eta_m \leq 1$. Define

$$\bar{S}_n(t) = (2p - 1)!! \prod_{i=1}^N \omega_i(t)^{n_i} \prod_{i > n_m} (2n_i - 1)!!.$$

We call the indices $i \leq n_m$ low indices and the indices $i > n_m$ high indices. From $\omega'_i = -4a_i \omega_i + (2/N)\kappa_i \omega_\Sigma$, we find

$$\bar{S}'_n = -4(n \cdot a) \bar{S}_n + \frac{2}{N} \sum_{i: n_i \geq 1} n_i \kappa_i \frac{\omega_\Sigma}{\omega_i} \bar{S}_n.$$

The function $\bar{S}$ is a supersolution of (17.19) if, for every n and every i with $n_i \geq 1$,

$$\kappa_i \frac{\omega_\Sigma}{\omega_i} \geq (2n_i - 1) \sum_{j=1}^N \frac{\bar{S}_{n-e_i+e_j}}{\bar{S}_n}. \quad (17.20)$$

The ratio $\bar{S}_{n-e_i+e_j} / \bar{S}_n$ equals one if $j = i$. If $j \neq i$, it equals ω_j / ω_i times a factor from the double factorials: a factor $(2n_i - 1)^{-1}$ if i is a high index, and a factor $2n_j + 1$ if j is a high index.

Let i be a high index. Multiplying by ω_i / ω_Σ , the right side of (17.20) becomes

$$\begin{aligned} \frac{1}{\omega_\Sigma} \left[(2n_i - 1)\omega_i + \sum_{j \leq n_m} \omega_j + \sum_{j > n_m, j \neq i} (2n_j + 1)\omega_j \right] &= 1 + \frac{2}{\omega_\Sigma} \left(\sum_{j > n_m} n_j \omega_j - \omega_i \right) \\ &\leq 1 + 2(p - 1)\eta_m, \end{aligned}$$

because $\sum_{j > n_m} n_j \omega_j - \omega_i = \sum_{j > n_m} (n_j - \delta_{ij})\omega_j$ and $\sum_j (n_j - \delta_{ij}) \leq p - 1$. The cancellation of the factor $2n_i - 1$ is what keeps the coefficient of the high indices close to one. The condition (17.20) holds, because $\kappa_i = 1 + C_1 \theta / n_m \geq 1 + 2(p - 1)\eta_m$.

Let i be a low index. The same computation gives the right side

$$(2n_i - 1) \left(1 + \frac{2}{\omega_\Sigma} \sum_{j > n_m} n_j \omega_j \right) \leq (2n_i - 1) (1 + 2(p - n_i)\eta_m) \leq 2n_i - 1 + 2(p - n_i) = 2p - 1,$$

where we used that the high indices carry at most $p - n_i$ units of n , and $(2n_i - 1)\eta_m \leq 1$. Here $\kappa_i = 2p - 1$, and the condition (17.20) holds.

At time zero, $\bar{S}_n(0) \geq S_n(0)$, because $\prod_{i \leq n_m} (2n_i - 1)!! \leq (2p - 1)!!$: the left side counts the pairings of $2 \sum_{i \leq n_m} n_i$ points that respect a partition into groups of sizes $2n_i$, and these are among the pairings of $2p$ points.

Step 3: comparison. Let A be the matrix of the system (17.19). Then $\bar{S}' \geq A\bar{S}$ component-wise, $\bar{S}(0) \geq S(0)$, and $S' = AS$, and the difference satisfies $(\bar{S} - S)' = A(\bar{S} - S) + \phi$ with $\phi \geq 0$. Since A has nonnegative off-diagonal entries, choosing λ with $A + \lambda I \geq 0$ we have $e^{tA} = e^{-\lambda t} \sum_k t^k (A + \lambda I)^k / k! \geq 0$ entrywise, and therefore

$$\bar{S}(t) - S(t) = e^{tA}(\bar{S}(0) - S(0)) + \int_0^t e^{(t-s)A} \phi(s) ds \geq 0.$$

This proves the comparison principle used here.

Step 4: conclusion. The double factorials in $\bar{S}_n$ are at most $((2p - 1)!!)^2$. Using the multinomial formula and $S \leq \bar{S}$, we obtain

$$\mathbb{E}|Y(t)|^{2p} = \sum_{|n|=p} \frac{p!}{\prod_i n_i!} S_n \leq ((2p - 1)!!)^2 \omega_\Sigma^p,$$

and $\mathbb{E}Y_i^{2p} = S_{pe_i} \leq ((2p - 1)!!)^2 \omega_i^p$. Recall that $\omega_\Sigma \leq CN\bar{h}$ and $\omega_i \leq Ch_i$, by the fourth part of Lemma 17.9. This proves the radial and coordinate bounds for integer orders $p \geq 2$; the case $p = 1$ follows from the identities $\mathbb{E}|Y|^2 = N\bar{h}$ and $\mathbb{E}Y_i^2 = h_i$. For real $r \geq 1$, we increase r to the integer $\lceil r \rceil$ and use $\||Y|^2 Y_i^2\|_r \leq \||Y|^2\|_{2r} \|Y_i^2\|_{2r}$. $\square$

For $p = 2$, these are the fourth-moment bounds

$$\mathbb{E}|Y(t)|^4 \leq C(N\bar{h}(t))^2, \quad \mathbb{E}[|Y(t)|^2 Y_i(t)^2] \leq CN\bar{h}(t)h_i(t) \quad (0 \leq t \leq T_{\max}). \quad (17.21)$$

17.4 Concentration of the radial part along the path

At late times, the radial part $|Y(t)|^2$ has size θ^{-2} and the square of the first coordinate has size θ^{-1} . The first coordinate determines the noise amplitude at late times through $\theta|Y|^2 \approx Y_1^2$. In this section, we show that after a time of order a^{-1} the quantity $\theta^2|Y(t)|^2$ is close to H_1^2 , where $H_1 = \sqrt{\theta}Y_1(\tau)$ is the scaled first coordinate at an earlier time τ , uniformly in t up to time $T_{\max}$. The difference $\theta^2|Y|^2 - H_1^2$ involves the projection of the vector of squares $(Y_i^2)_i$ onto the stable directions of the mean matrix H ; Lemma 17.12 collects the properties of the projected kernel, and Theorem 17.14 proves the concentration.

The projected kernel

The matrix H of (17.12) has a positive eigenvalue, because $a_1 = 0$, and this eigenvalue drives the growth of the mean. We remove the corresponding component.

Lemma 17.12 (Projected kernel estimates). *There exist C and m_0 , depending only on c_* , C_* , and K_0 , such that the following hold for $m \geq m_0$.*

1. *The matrix H has exactly one positive eigenvalue λ_+ . It is the unique positive solution of $(2/N) \sum_i (4a_i + \lambda)^{-1} = 1$, with eigenvector $u = ((4a_i + \lambda_+)^{-1})_i$, and all other eigenvalues of H are smaller than $-2a$. Let $\bar{\theta} = 2/(N\lambda_+)$. Then*

$$\theta \leq \bar{\theta} \leq \theta(1 + Cm^{-2}), \quad \lambda_+ \leq \frac{2a}{m}.$$

2. *Let $d(t) = e^{tH}(\bar{\theta}\mathbf{1} - e_1)$, and let $\tilde{U} = U_{\tilde{\kappa}}$ be the resolvent (17.15) with $\tilde{\kappa}_i = 4a_i/(4a_i + \lambda_+)$. Then $d(t)$ is orthogonal to u , and*

$$\begin{aligned} d_i &= \bar{\theta} e^{-4a_i t} * (\delta_0 + \tilde{U}) \quad (i \geq 2), \\ d_1(t) &= -\bar{\theta} \int_t^\infty \tilde{U}(s) ds, \quad \sum_{i=1}^N d_i(t) = \frac{N\bar{\theta}}{2} \tilde{U}(t). \end{aligned} \quad (17.22)$$

In particular $d_i \geq 0$ for $i \geq 2$, and $-d_1$ is nonincreasing, with values in $[0, 1]$.

3. *For $t \geq 1/a$, we have $\theta \sum_i d_i(t) \leq Ce^{-2at}$ and*

$$\|d(t)\|_2 \leq Ce^{-2at}, \quad |d_1(t)| \leq \frac{C}{m} \left(\sum_{i \geq 2} d_i(t)^2 \right)^{1/2}. \quad (17.23)$$

4. *We have*

$$\int_0^{T_{\max}} d_1^2 \leq C\theta^{-2} \log(em), \quad \int_0^\infty d_i^2 \leq \frac{1}{8a_i} \quad (i \geq 2), \quad \int_0^\infty \sum_{i \geq 2} \frac{d_i^2}{a_i} \leq \frac{C}{a^2}. \quad (17.24)$$

5. *Extend each d_i by zero on $(-\infty, 0)$, and write $\|\cdot\|_2$ for the norm of $L^2(\mathbb{R})$. For $h > 0$,*

$$\|d_i(\cdot + h) - d_i\|_2^2 \leq 2 \min\{h, (4a_i)^{-1}\} \quad (i \geq 2), \quad (17.25)$$

$$\sum_{i \geq 2} \frac{\|d_i(\cdot + h) - d_i\|_2^2}{a_i} \leq CN\sqrt{h} \leq \frac{C}{a^2} \sqrt{ah} \quad (0 < h \leq 1/a), \quad (17.26)$$

$$\|d_1(\cdot + h) - d_1\|_2^2 \leq 2h. \quad (17.27)$$

6. *For every $s > 0$,*

$$\|d'(s)\|_2 \leq \frac{2}{es} \|d(s/2)\|_2. \quad (17.28)$$

Proof. For the first part, let φ be an eigenvector of H with eigenvalue μ . Then $(\mu + 4a_i)\varphi_i = (2/N)\mathbf{1} \cdot \varphi$ for all i . If $\mu \notin \{-4a_i\}$, then $\mathbf{1} \cdot \varphi \neq 0$ and $\Phi(\mu) = 1$, where $\Phi(\mu) = (2/N)\sum_i(\mu + 4a_i)^{-1}$. On $(0, \infty)$, Φ decreases strictly from $+\infty$ to 0, which implies that there exists exactly one positive eigenvalue λ_+ , with eigenvector u ; it exists because $e_1^\top H e_1 = 2/N > 0$. On $(-4a, 0)$, Φ decreases strictly from $+\infty$ to $-\infty$, and H has at most one eigenvalue μ_2 in this interval. The other eigenvalues are at most $-4a$. With the notation of the proof of Lemma 17.8, we have $\Phi(-2a) = -(Na)^{-1} + G_1(-2a) \leq -\theta/m + 1 - \theta + C\theta/m < 1$ for m large, and since Φ is decreasing on $(-4a, 0)$, any eigenvalue μ_2 in this interval satisfies $\mu_2 < -2a$. Next, $\Phi(\lambda_+) = 1$ implies that $\bar{\theta} = 2/(N\lambda_+) = 1 - (2/N)\sum_{i \geq 2}(4a_i + \lambda_+)^{-1} \geq \theta$, and hence $\lambda_+ \leq 2/(N\theta) = 2a/m$. Also

$$\bar{\theta} - \theta = \frac{2}{N} \sum_{i \geq 2} \frac{\lambda_+}{4a_i(4a_i + \lambda_+)} \leq \frac{\lambda_+}{8N} \sum_{i \geq 2} a_i^{-2} \leq \frac{C}{N^2 \theta a^2} = \frac{C\theta}{m^2},$$

by Lemma 17.2.

For the second part, the secular equation yields $\sum_i u_i = N/2$ and $u_1 = \lambda_+^{-1} = N\bar{\theta}/2$, and $u \cdot (\bar{\theta}\mathbf{1} - e_1) = 0$. The matrix H is symmetric. Then the orthogonal complement of u is invariant, and $d(t) \perp u$ for all t . In coordinates, $d_1/\lambda_+ + \sum_{i \geq 2} d_i/(4a_i + \lambda_+) = 0$, that is, $d_1 = -\sum_{i \geq 2} (1 - \tilde{\kappa}_i)d_i$, and $\sum_i d_i = \sum_{i \geq 2} \tilde{\kappa}_i d_i$. The vector $d_b = (d_i)_{i \geq 2}$ therefore solves $d'_b = [-4 \operatorname{diag}(a_2, \dots, a_N) + (2/N)\mathbf{1}\tilde{\kappa}^\top]d_b$ with $d_b(0) = \bar{\theta}\mathbf{1}$. As in (17.16), the sum $\Omega = \sum_{i \geq 2} \tilde{\kappa}_i d_i$ solves $\Omega = (N\bar{\theta}/2)k_{\tilde{\kappa}} + k_{\tilde{\kappa}} * \Omega$, which implies that $\Omega = (N\bar{\theta}/2)\tilde{U}$, and variation of constants gives the first identity in (17.22). The third identity is $\sum_i d_i = \Omega$. Since $d'_1 = (2/N)\sum_j d_j = \bar{\theta}\tilde{U}$, $d_1(0) = \bar{\theta} - 1$, and $\int_0^\infty \tilde{U} = (1 - \theta_{\tilde{\kappa}})/\theta_{\tilde{\kappa}}$ with $\theta_{\tilde{\kappa}} = 1 - (2/N)\sum_{i \geq 2}(4a_i + \lambda_+)^{-1} = \bar{\theta}$, we obtain the second identity. The coefficients satisfy $1 \geq \tilde{\kappa}_i \geq 1 - \lambda_+/(4a_i) \geq 1/2$, and $\theta_{\tilde{\kappa}} = \bar{\theta} \in [\theta, 2\theta]$ with $\bar{\theta} \leq 2/3$ for m large, so Lemma 17.8 applies to $\tilde{U}$ with $K_1 = 1$.

For the third part, the third identity in (17.22) and Lemma 17.8 give $\theta \sum_i d_i(t) \leq CN\theta^2(N\theta^2)^{-1}e^{-2a(t-c_0/a)} \leq Ce^{-2at}$ for $t \geq c_0/a$. Let $t_0 = c_0/a$. Splitting the convolution in (17.22) at $t_0/2$ and using the third part of Lemma 17.8, we find $d_i(t_0) \leq \bar{\theta}e^{-4a_it_0} + e^{-2a_it_0} + C\theta^{-1}t_0^{-3/2}\min\{t_0, a_i^{-1}\}$. Using Lemma 17.2 and $a^{3/2} \leq C_*^{3/2}/N$, we have $\sum_{i \geq 2} e^{-4a_it_0} \leq CNt_0^{-3/2} \leq C$ and $\sum_{i \geq 2} \min\{t_0, a_i^{-1}\}^2 \leq CN\sqrt{t_0}$. We obtain $\sum_{i \geq 2} d_i(t_0)^2 \leq C + C\theta^{-2}Nt_0^{-5/2} \leq C + C\theta^{-2}a \leq C$, since $\theta^2/a \geq m^2/C_*^3$. Let $P = \operatorname{diag}(\sqrt{\tilde{\kappa}_i})_{i \geq 2}$. Then $P[-4 \operatorname{diag}(a) + (2/N)\mathbf{1}\tilde{\kappa}^\top]P^{-1} = H_{\tilde{\kappa}}$, the symmetric matrix of Lemma 17.8, all of whose eigenvalues are smaller than $-2a$. Because $\|P\| \|P^{-1}\| \leq \sqrt{2}$, we have $\|d_b(t)\|_2 \leq \sqrt{2}e^{-2a(t-t_0)}\|d_b(t_0)\|_2 \leq Ce^{-2at}$ for $t \geq t_0$. Finally, by the Cauchy-Schwarz inequality, $|d_1| \leq \lambda_+(\sum_{i \geq 2}(4a_i)^{-2})^{1/2}\|d_b\|_2 \leq (2a/m)Ca^{-1}\|d_b\|_2$.

For the fourth part, by (17.22), $\bar{\theta}(\delta_0 + \tilde{U})$ is a probability measure on $[0, \infty)$, since its mass is $\bar{\theta}(1 + (1 - \bar{\theta})/\bar{\theta}) = 1$. Using Young's inequality, we have $\|d_i\|_{L^2} \leq \|e^{-4a_i \cdot}\|_{L^2} = (8a_i)^{-1/2}$, and summing $a_i^{-1}\|d_i\|_{L^2}^2$ with Lemma 17.2 yields the third bound. For the first bound, we have $|d_1(t)| \leq 1$ for every $t \geq 0$, and the second part of Lemma 17.8 also

gives $|d_1(t)| \leq C\theta^{-1}t^{-1/2}$ for $t \geq 1$. Consequently $|d_1(t)| \leq \min\{1, C\theta^{-1}(1+t)^{-1/2}\}$ for all $t \geq 0$, after increasing C . Integrating its square gives $\int_0^{T_{\max}} d_1(t)^2 dt \leq C\theta^{-2} \log(e + \theta^2 T_{\max}) \leq C\theta^{-2} \log(em)$, since $\theta^2 T_{\max} \leq Cm^2 \log(em)$.

For the fifth part, because translation commutes with convolution, we have $d_i(\cdot + h) - d_i = \bar{\theta}(\delta_0 + \tilde{U}) * (g_i(\cdot + h) - g_i)$ with $g_i(t) = e^{-4a_i t} \mathbf{1}_{\{t \geq 0\}}$, and Young's inequality yields

$$\|d_i(\cdot + h) - d_i\|_2^2 \leq \|g_i(\cdot + h) - g_i\|_2^2 = \int_0^h e^{-8a_i s} ds + \frac{(1 - e^{-4a_i h})^2}{8a_i} \leq 2 \min\left\{h, \frac{1}{4a_i}\right\},$$

where in the last step we used $(1 - e^{-x})^2 \leq \min\{x^2, 1\}$. Summing with the weights a_i^{-1} and using Lemma 17.2, we obtain (17.26). The second inequality there uses $N\sqrt{h} = a^{-2}\sqrt{ah}Na^{3/2}$. For (17.27), on $[-h, 0)$ the difference is $d_1(\cdot + h)$, which is bounded by one. On $[0, \infty)$, the difference $d_1(t+h) - d_1(t)$ lies in $[0, 1]$, and its square is at most itself, and $\int_0^\infty (d_1(t+h) - d_1(t)) dt = \int_0^h |d_1| \leq h$.

For the sixth part, $d(s) = e^{(s/2)H}d(s/2)$ and $d(s/2) \perp u$. On the orthogonal complement of u , the matrix H is symmetric with eigenvalues $-z$, $z > 2a$, and $\|He^{(s/2)H}\varphi\| \leq \sup_{z \geq 0} ze^{-zs/2}\|\varphi\| = (2/(es))\|\varphi\|$ for $\varphi \perp u$. $\square$

A maximal inequality

The path estimates use a supremum over a time interval of length of order $a^{-1} \log m$. The following deterministic chaining inequality reduces them to bounds at fixed times and for increments.

Lemma 17.13 (Dyadic maximal inequality). *For every $p > 2$ there exists a finite constant $C_p > 0$ with the following property. Let $v_0 \leq v_1$ and $A, B \geq 0$, and let Z be a real process with measurable one-time evaluations and almost surely continuous paths on $[v_0, v_1]$. Assume separately that*

$$\|Z(t)\|_{2p} \leq A \quad \text{for every } t \in [v_0, v_1],$$

and that

$$\|Z(t+h) - Z(t)\|_{2p} \leq Bh^{1/4} \quad \text{whenever } t, t+h \in [v_0, v_1], \quad 0 < h \leq 1.$$

Let $n = \max\{1, \lceil v_1 - v_0 \rceil\}$. Then

$$\left\| \sup_{v_0 \leq t \leq v_1} |Z(t)| \right\|_{2p} \leq n^{1/(2p)} (A + C_p B).$$

Proof. If $v_0 = v_1$, the conclusion follows directly from the one-time moment bound and $B \geq 0$. Otherwise, divide $[v_0, v_1]$ into n intervals of equal length $0 < \ell \leq 1$ and fix one of them, $[s, s + \ell]$. For $j \geq 1$, let Δ_j be the maximum of $|Z(s + (k+1)\ell 2^{-j}) - Z(s + k\ell 2^{-j})|$ over $0 \leq k < 2^j$. Bounding the $2p$ -th power of the maximum by the sum of the $2p$ -th

powers, we have $\mathbb{E}\Delta_j^{2p} \leq 2^j B^{2p} (\ell 2^{-j})^{p/2}$, that is, $\|\Delta_j\|_{2p} \leq B 2^{j/(2p)-j/4}$. Every dyadic point t of $[s, s + \ell]$ is reached from s by at most one increment of each level j , and therefore $|Z(t)| \leq |Z(s)| + \sum_{j \geq 1} \Delta_j$. This holds for all $t \in [s, s + \ell]$ by continuity, and the triangle inequality in L^{2p} bounds the supremum norm by A plus the sum of the increment norms. The increment bounds form a convergent geometric series because $p > 2$; choose C_p to bound its sum. Finally, the supremum over $[v_0, v_1]$ is the maximum of the n suprema, and bounding the $2p$ -th power of the maximum by the sum gives the factor $n^{1/(2p)}$. $\square$

Concentration of the radial part

Fix $\alpha \in (0, 1)$. We define the earlier time and the scaled coordinates at that time,

$$\tau = \frac{m^\alpha}{\theta^2}, \quad H_i = \sqrt{\theta} Y_i(\tau) \quad (1 \leq i \leq N), \quad (17.29)$$

and, for $8 \leq \chi \leq (K_0/2) \log(em)$, the time $T = \chi/\alpha$. Since $\alpha\tau \leq C_*^3 m^{\alpha-2}$ by Lemma 17.2, we have $\tau \leq T/2$ for $m \geq m_0$. Let

$$\varepsilon_m = m^{-\alpha/2} + \frac{\log(em)}{m}, \quad F_p(m, \chi) = (\log(em))^{1/(2p)} (e^{-\chi/4} + m^{-1/2}) + \varepsilon_m^{1/2} + m^{-2}.$$

Theorem 17.14 (Concentration of the radial part along the path). *For every integer $p > 2$ there exist C_p and m_0 , depending only on p , α , c_* , C_* , and K_0 , such that for $m \geq m_0$ and every $\chi \in [8, (K_0/2) \log(em)]$,*

$$\begin{aligned} \left\| \sup_{T \leq t \leq T_{\max}} |\theta^2 |Y(t)|^2 - H_1^2| \right\|_{2p} &\leq C_p F_p(m, \chi), \\ \left\| \sup_{\tau \leq t \leq T_{\max}} |\sqrt{\theta} Y_1(t) - H_1| \right\|_{2p} &\leq C_p \varepsilon_m^{1/2}. \end{aligned} \quad (17.30)$$

In particular, if $\chi = \mathfrak{b} \log(em)$ for a fixed $\mathfrak{b} \in (0, K_0/2]$, then both right sides are at most $C_p m^{-\vartheta}$ for $m \geq m_0$, where $\vartheta = \min\{\alpha, \mathfrak{b}, 1\}/8$ does not depend on p , and C_p, m_0 depend also on $\mathfrak{b}$.

The proof writes $\bar{\theta}|Y|^2 - Y_1^2$ as a stochastic convolution against the projected kernel d of Lemma 17.12. We first bound its terms at fixed times and their increments, for the Gaussian start, and apply Lemma 17.13. Then we transfer the bound to the uniform start, treat the first coordinate, which is a martingale, and replace $\bar{\theta}$ by θ .

Proof. Throughout, C_p denotes a constant depending only on p , α , c_* , C_* , and K_0 ; $t, t+h \in [T, T_{\max}]$ and $0 < ah \leq 1$.

Step 1: decomposition. Let $\mathcal{D}(t) = \bar{\theta}|Y^G(t)|^2 - Y_1^G(t)^2 = (\bar{\theta}\mathbf{1} - e_1) \cdot Q(t)$, where $Q_i = (Y_i^G)^2$. By (17.11), $dQ = HQ dt + dM$, where $dM_i = 2\sqrt{2/N}|Y^G|Y_i^G dB_i$ are orthogonal martingales. By variation of constants and the symmetry of H ,

$$\begin{aligned}\theta\mathcal{D}(t) &= \theta \sum_i d_i(t) + \mathcal{I}(t) + \mathcal{Z}(t), \quad \mathcal{I}(t) = \theta \sum_i d_i(t)(g_i^2 - 1), \\ \mathcal{Z}(t) &= \theta \sqrt{\frac{8}{N}} \sum_i \int_0^t d_i(t-s) |Y^G(s)| Y_i^G(s) dB_i(s).\end{aligned}$$

The process $\mathcal{Z}$ has continuous paths, since $\mathcal{D}$ and d are continuous.

Step 2: the mean and the initial term. The third part of Lemma 17.12 shows that $\theta \sum_i d_i(t) \leq Ce^{-2at} \leq Ce^{-2\chi}$. For a nonzero vector v , the centered Gaussian quadratic form $\sum_i v_i(g_i^2 - 1)$ satisfies $\log \mathbb{E}e^{\lambda \sum_i v_i(g_i^2 - 1)} \leq 2\lambda^2|v|^2$ for $|\lambda| \leq (4|v|)^{-1}$, because $-\log(1-x) - x \leq x^2$ for $|x| \leq 1/2$. It follows that $\mathbb{E}e^{|\sum_i v_i(g_i^2 - 1)|/(4|v|)} \leq 2e^{1/8}$, and hence $\|\sum_i v_i(g_i^2 - 1)\|_{2p} \leq C_p|v|$. This last bound also holds for $v = 0$, when the quadratic form vanishes. With (17.23) and (17.28), we find $\|\mathcal{I}(t)\|_{2p} \leq C_p\theta\|d(t)\|_2 \leq C_p e^{-2\chi}$ and

$$\|\mathcal{I}(t+h) - \mathcal{I}(t)\|_{2p} \leq C_p h \sup_{s \geq t} \|d'(s)\|_2 \leq C_p \frac{h}{t} \sup_{s \geq t} \|d(s/2)\|_2 \leq C_p(ah)e^{-\chi},$$

where we used $t \geq T = \chi/a$ and $s/2 \geq T/2 \geq 1/a$.

Step 3: the noise at a fixed time. Put $R = |g|^2/N$. For every integer $p \geq 1$, the Gaussian radial moments satisfy $\mathbb{E}R^{2p} = \prod_{j=0}^{2p-1} (1 + 2j/N) \leq (2p)!$, since $N \geq 2$. By Lemma 17.7, $Y^G = \sqrt{R}Y$ and R is independent of Y . Thus Proposition 17.11 gives the particular Gaussian-start bound needed here:

$$\| |Y^G(s)|^2 Y_i^G(s)^2 \|_p = (\mathbb{E}R^{2p})^{1/p} \| |Y(s)|^2 Y_i(s)^2 \|_p \leq C_p N \bar{h}(s) h_i(s). \quad (17.31)$$

For real $p \geq 1$, the same bound follows by increasing p to $\lceil p \rceil$. For deterministic kernels $v_i(s)$, the Burkholder–Davis–Gundy inequality [123, Chapter IV, Theorem 4.1, p. 160] for continuous local martingales vanishing at zero, Minkowski's inequality, and this bound give

$$\begin{aligned}\left\| \theta \sqrt{\frac{8}{N}} \sum_i \int v_i(s) |Y^G| Y_i^G dB_i \right\|_{2p}^2 &\leq \frac{C_p \theta^2}{N} \int \sum_i v_i(s)^2 \| |Y^G(s)|^2 Y_i^G(s)^2 \|_p ds \\ &\leq C_p \theta^2 \int \sum_i v_i(s)^2 \bar{h}(s) h_i(s) ds.\end{aligned} \quad (17.32)$$

This uses only the quadratic variation. No Gaussian property of the stochastic integrals is used. We apply (17.32) with $v_i(s) = d_i(t-s)$, $s \leq t$, and split the integral at $s = t/2$.

- For $s \leq t/2$, we use $h_i \leq h_1$, and (17.23) implies that $\sum_i d_i(t-s)^2 \leq Ce^{-2at}$. The identity $h'_1 = 2\bar{h}$ and Corollary 17.10 imply that $\int_0^{T_{\max}} \bar{h} h_1 = (h_1(T_{\max})^2 - 1)/4 \leq C\theta^{-2}$. This part contributes $C_p e^{-2\chi}$.
- For $s \geq t/2$, we have $s \geq c_0/a$, so that $\bar{h}(s) \leq C(N\theta^2)^{-1}$, $h_1(s) \leq C\theta^{-1}$, and $h_i(s) \leq C[\theta^{-1}e^{-2a_i s} + (N\theta^2 a_i)^{-1}]$, by Corollary 17.10. By (17.24), the first coordinate contributes at most $C_p \theta^2 (N\theta^3)^{-1} \theta^{-2} \log(em) = C_p \log(em)/(N\theta^3)$; the part $(N\theta^2 a_i)^{-1}$ of h_i contributes at most $C_p (N\theta)^{-2} \sum_{i \geq 2} a_i^{-2} \leq C_p (N\theta a)^{-2} = C_p m^{-2}$; and the part $\theta^{-1}e^{-2a_i s} \leq \theta^{-1}e^{-a_i t}$ contributes at most $C_p (N\theta)^{-1} \sum_{i \geq 2} e^{-a_i t}/a_i$, which is at most $C_p (N\theta)^{-1} e^{-at/2} \sum_{i \geq 2} e^{-a_i t/2}/a_i \leq C_p m^{-1} e^{-\chi/2}$, by the fourth sum in Lemma 17.2 with $s = t/2 \geq 1/a$.

Recall that $N\theta^3 \geq m^3/C_*^3$. We conclude that

$$\|\mathcal{Z}(t)\|_{2p}^2 \leq C_p [e^{-\chi/2} + m^{-2} + \log(em)/(N\theta^3)] \leq C_p [e^{-\chi/2} + m^{-1}]. \quad (17.33)$$

Step 4: increments of the noise. We extend d by zero on $(-\infty, 0)$. Then $\mathcal{Z}(t+h) - \mathcal{Z}(t)$ is the stochastic integral (17.32) with $v_i(s) = d_i(t+h-s) - d_i(t-s)$, $0 \leq s \leq t+h$. It includes the new noise on $[t, t+h]$. We split the integral at $s = T/2$.

- For $s \leq T/2$, we have $t-s \geq T/2$, and (17.28) and (17.23) give $\|d(t+h-s) - d(t-s)\|_2 \leq h \sup_{r \geq T/2} \|d'(r)\|_2 \leq C(ah)e^{-\chi/2}$. As in Step 3, this part contributes $C_p(ah)^2 e^{-\chi}$.
- For $s \geq T/2$, we use the bounds on $\bar{h}$ and h_i of Step 3 and the translation bounds of Lemma 17.12. The first coordinate contributes $C_p \theta^2 (N\theta^3)^{-1} \cdot 2h = C_p ah/m$, by (17.27). The part $(N\theta^2 a_i)^{-1}$ contributes $C_p (N\theta)^{-2} \sum_{i \geq 2} a_i^{-1} \|d_i(\cdot + h) - d_i\|_2^2 \leq C_p (N\theta a)^{-2} \sqrt{ah} = C_p \sqrt{ah}/m^2$, by (17.26). The part $\theta^{-1}e^{-2a_i s} \leq \theta^{-1}e^{-a_i T}$ contributes $C_p (N\theta)^{-1} h \sum_{i \geq 2} e^{-a_i T} \leq C_p (ah/m) e^{-\chi/2}$, by (17.25) and $\sum_{i \geq 2} e^{-a_i T} \leq e^{-\chi/2} \sum_{i \geq 2} e^{-a_i T/2} \leq Ce^{-\chi/2}$.

We conclude that

$$\|\mathcal{Z}(t+h) - \mathcal{Z}(t)\|_{2p}^2 \leq C_p [(ah)^2 e^{-\chi} + ah/m + \sqrt{ah}/m^2]. \quad (17.34)$$

Step 5: the supremum for the Gaussian start. We apply Lemma 17.13 to $s \mapsto \mathcal{Z}(s/a)$ and $s \mapsto \mathcal{I}(s/a)$ on $[\chi, aT_{\max}]$, an interval of length at most $K_0 \log(em)$. For $0 < h' \leq 1$, (17.34) yields the increment bound $C_p h'^{1/4} (e^{-\chi/2} + m^{-1/2})$, and (17.33) gives the pointwise bound $C_p (e^{-\chi/4} + m^{-1/2})$. Step 2 provides the corresponding bounds for $\mathcal{I}$, with $e^{-\chi}$. Together with the bound on the mean, we obtain

$$\left\| \sup_{T \leq t \leq T_{\max}} \theta |\mathcal{D}(t)| \right\|_{2p} \leq C_p (\log(em))^{1/(2p)} (e^{-\chi/4} + m^{-1/2}).$$

Step 6: transfer to the uniform start. By Lemma 17.7, $\mathcal{D}(t) = (|g|^2/N) \mathcal{D}^u(t)$ for all t , where $\mathcal{D}^u = \bar{\theta}|Y|^2 - Y_1^2$ is computed for the uniform start, and $|g|$ is independent of the

path Y . The functional $\sup_{[T, T_{\max}]} |\mathcal{D}^u|^{2p}$ is homogeneous of degree $4p$, and Lemma 17.7 shows that the bound of Step 5 holds for $\mathcal{D}^u$. This transfer involves no inverse moment of $|g|$.

Step 7: the first coordinate. The coordinate Y_1 is a martingale, because $a_1 = 0$, and $d\langle Y_1 \rangle = (2/N)|Y|^2 dt$. By the Burkholder–Davis–Gundy inequality, Proposition 17.11, and $h'_1 = 2\bar{h}$,

$$\left\| \sup_{\tau \leq t \leq T_{\max}} \sqrt{\theta} |Y_1(t) - Y_1(\tau)| \right\|_{2p}^2 \leq C_p \theta \int_{\tau}^{T_{\max}} \frac{2}{N} \| |Y(s)|^2 \|_p ds \leq C_p \theta (h_1(T_{\max}) - h_1(\tau)).$$

From Corollary 17.10, we have $\theta h_1(T_{\max}) \leq e^{2T_{\max}/(N\theta)} = e^{2K_0 \log(em)/m} \leq 1 + C \log(em)/m$ and $\theta h_1(\tau) \geq 1 - C(\theta^2 \tau)^{-1/2} = 1 - Cm^{-\alpha/2}$. This proves the second bound in (17.30). Using Doob's inequality, we find $\| \sup_{t \leq T_{\max}} \sqrt{\theta} |Y_1(t)| \|_{4p} \leq C_p (\theta h_1(T_{\max}))^{1/2} \leq C_p$, and the Cauchy–Schwarz inequality yields

$$\left\| \sup_t |\theta Y_1(t)^2 - H_1^2| \right\|_{2p} \leq \left\| \sup_t |\sqrt{\theta} Y_1(t) - H_1| \right\|_{4p} \left\| \sup_t \sqrt{\theta} |Y_1(t)| + |H_1| \right\|_{4p} \leq C_p \varepsilon_m^{1/2},$$

where the suprema are over $\tau \leq t \leq T_{\max}$.

Step 8: conclusion. From the definition of $\mathcal{D}^u$, we have the exact identity

$$\theta^2 |Y|^2 - H_1^2 = \frac{\theta}{\bar{\theta}} \theta \mathcal{D}^u + \frac{\theta}{\bar{\theta}} (\theta Y_1^2 - H_1^2) + \left(\frac{\theta}{\bar{\theta}} - 1 \right) H_1^2.$$

Since $\theta \leq \bar{\theta}$, $|\theta/\bar{\theta} - 1| \leq Cm^{-2}$, and $\|H_1^2\|_{2p} \leq C_p$, Steps 6 and 7 prove the first bound in (17.30). No moment of $\sup_t |Y(t)|^2$ is used in this step. For the last statement, $e^{-\chi/4} = (em)^{-b/4}$, and all powers of $\log(em)$ are absorbed by the margin between ϑ and $\min\{\alpha, b, 1\}/4$. $\square$

We record the consequences at fixed times that are used in Section 17.6.

Corollary 17.15 (Radial part and first coordinate at fixed times). *Fix $\alpha \in (0, 1)$ and $b \in (0, K_0/2]$, and let $\chi = b \log(em)$. There exist C and m_0 , depending only on α, b, c_*, C_* , and K_0 , such that for $m \geq m_0$,*

$$\theta \mathbb{E} \sup_{\tau \leq t \leq T_{\max}} |Y_1(t) - Y_1(\tau)|^2 \leq C \left[m^{-\alpha/2} + \frac{\log(em)}{m} \right], \quad (17.35)$$

and

$$\sup_{T \leq t \leq T_{\max}} \left(\mathbb{E} |\theta^2 |Y(t)|^2 - H_1^2| + \mathbb{E} |\theta |Y(t)| - |H_1||^2 \right) \leq Cm^{-\vartheta}.$$

Proof. The first bound is Step 7 of the previous proof with the exponent 2 in place of $2p$. The second follows from Theorem 17.14 with $p = 3$ and the inequality $|\sqrt{x} - \sqrt{y}|^2 \leq |x - y|$ for $x, y \geq 0$. $\square$

17.5 Gaussian approximation at time τ and the domain of the path

In this section, we show that the scaled coordinates $H_1, \dots, H_J$ at the time τ are close to independent standard Gaussian variables, and that, outside an event whose probability is smaller than any fixed power of m^{-1} , the path stays after time T in a domain where the first coordinate carries a fraction of order θ of the squared norm. We then define the endpoint measure, weighted by the indicator of an event and killed on leaving a domain, to which the entropy estimate of Section 11 is applied, and bound its loss of mass, its moments, and the mass of the set outside which the cutoff function equals one. We keep the notation (17.29).

Coordinates at time τ

Lemma 17.16 (Gaussian approximation of the coordinates at time τ). *Fix $\alpha \in (0, 1)$ and an integer $J \geq 2$, and let*

$$g(m) = m^{-\alpha/2} + m^{-3(1-\alpha)/2} + m^{\alpha-2}.$$

There exist C_J and m_0 , depending only on J , α , c_ , and C_* , such that for $m \geq m_0$ the following holds. There exist independent standard Gaussian variables $G_1^*, \dots, G_J^*$, which are functions of g and B , such that*

$$\left(\mathbb{E} \sum_{i=1}^J |H_i - G_i^*|^2 \right)^{1/2} \leq C_J g(m).$$

Moreover, for $0 < \ell_- < \ell_+$, the event

$$E = E_{\ell_-, \ell_+, J} = \{ \ell_- \leq |H_1| \leq \ell_+, |H_i| \leq \ell_+ \ (2 \leq i \leq J) \} \quad (17.36)$$

satisfies, for every $0 < \varepsilon_0 < (\ell_+ - \ell_-)/2$,

$$\mathbb{P}(E) \geq \mathbb{P}(\ell_- + \varepsilon_0 \leq |G_1^*| \leq \ell_+ - \varepsilon_0, |G_i^*| \leq \ell_+ - \varepsilon_0 \ (2 \leq i \leq J)) - \frac{C_J g(m)^2}{\varepsilon_0^2}.$$

Note that $g(m) \rightarrow 0$. A prescribed loss of probability can therefore be achieved with fixed $\ell_{\pm}$ and a finite threshold for m . The approximation by a Gaussian vector is not a statement about the coordinates at the later times $t_m(u)$, where the amplitude $|Y|$ of the noise is random; at time τ , this amplitude is still close to its mean.

Proof. We work with the Gaussian start, for which $H_i^G = \sqrt{\theta} Y_i^G(\tau)$, and then pass to the uniform start. The proof has four steps.

Step 1: deterministic bounds on $[0, \tau]$. Note that $\tau \leq c_0/a$ for $m \geq m_0$, since $a\tau \leq C_*^3 m^{a-2}$. We claim that, for $0 \leq t \leq \tau$,

$$\sum_i h_i(t)^2 \leq C[Nf_\theta(t) + \theta^{-2}], \quad N\bar{h}(t)h_i(t) \leq CN, \quad N\bar{h}(t) \geq c_1 Nf_\theta(t). \quad (17.37)$$

The third bound is the third part of Corollary 17.10. For the first, $h_1^2 \leq C\theta^{-2}$, and we use the coordinate bounds of Lemma 17.9 with $C_0 = 1$, $C_1 = 0$. If $t \leq \theta^{-2}$, the sum of $(1+t)(1+a_it)^{-2}$ is at most $CN(1+t)^{-1/2} = CNf_\theta(t)$ by Lemma 17.2, and the sum of $\min\{t, a_i^{-1}\}^2(N\theta^2)^{-2}$ is at most $C\sqrt{t}/(N\theta^4) \leq C\theta^{-2}/(N\theta^3)$. If $\theta^{-2} \leq t \leq \tau$, the three terms give at most $C\theta^{-2}Nt^{-3/2}$, $C\theta^{-4}t^{-3}N\sqrt{t} \leq C\theta^{-2}Nt^{-3/2}$, and $C\sqrt{\tau}/(N\theta^4) = C\theta^{-2}m^{a/2}/(N\theta^3) \leq C\theta^{-2}$; and $\theta^{-2}Nt^{-3/2} \leq CNf_\theta(t)$ in this range. For the second bound in (17.37), if $t \leq \theta^{-2}$ we have $N\bar{h}(t) \leq CN(1+t)^{-1/2}$ and $h_i(t) \leq C\sqrt{1+t} + Ct/(N\theta^2)$, whose product is at most $C[N + \sqrt{t}\theta^{-2}] \leq C[N + \theta^{-3}] \leq CN$; if $t \geq \theta^{-2}$, then $N\bar{h}(t) \leq C[N\theta^{-2}t^{-3/2} + \theta^{-2}] \leq C[N\theta + \theta^{-2}]$ and $h_i \leq h_1 \leq C\theta^{-1}$, whose product is at most CN , because $N\theta^3 \geq 1$.

Step 2: the relative variance of the radial part. Let $A^G(t) = |Y^G(t)|^2 = \mathbf{1} \cdot Q(t)$, where $Q_i = (Y_i^G)^2$. As in Step 1 of the proof of Theorem 17.14, and since $h(t) = e^{tH}\mathbf{1}$, we have $A^G(t) = \sum_i h_i(t)g_i^2 + \sum_i \int_0^t h_i(t-s) dM_i(s)$. Since the two terms are uncorrelated and the martingales M_i are orthogonal, we have

$$\text{Var } A^G(t) = 2 \sum_i h_i(t)^2 + \frac{8}{N} \sum_i \int_0^t h_i(t-s)^2 \mathbb{E}[A^G(s)Y_i^G(s)^2] ds. \quad (17.38)$$

By (17.31) with $p = 1$ and (17.37), we have $\mathbb{E}[A^G(s)Y_i^G(s)^2] \leq CN\bar{h}(s)h_i(s) \leq CN$, and therefore, using $\int_0^\infty f_\theta \leq 5\theta^{-1}$, the second term is at most $C \int_0^t [Nf_\theta + \theta^{-2}] \leq C[N\theta^{-1} + t\theta^{-2}]$. We divide by $(N\bar{h}(t))^2 \geq c_1(Nf_\theta(t))^2$. For $t \leq \theta^{-2}$, where $f_\theta(t) \geq c_1\theta$, the four resulting terms are at most $C(N\theta^3)^{-1} + C(N\theta^3)^{-2}$. For $t = w\theta^{-2}$ with $1 \leq w \leq m^a$, where $f_\theta(t) \geq c_1\theta w^{-3/2}$, they are at most $Cw^{3/2}(N\theta^3)^{-1}$, $Cw^3(N\theta^3)^{-1}$, $Cw^3(N\theta^3)^{-1}$, and $Cw^4(N\theta^3)^{-2}$. Using $N\theta^3 \geq m^3/C_*^3$, we conclude that

$$\varepsilon'_m = \sup_{0 \leq t \leq \tau} \frac{\text{Var } A^G(t)}{(N\bar{h}(t))^2} \leq C[m^{3a-3} + m^{4a-6}] \leq Cm^{-3(1-a)}. \quad (17.39)$$

Step 3: coupling with a Gaussian process. On the same probability space, let

$$G_i(t) = e^{-2a_it}g_i + \sqrt{2/N} \int_0^t e^{-2a_i(t-s)} \sqrt{N\bar{h}(s)} dB_i(s).$$

These are independent centered Gaussian variables with variances $h_i(t)$, by (17.13). Using the mild form $Y_i^G(t) = e^{-2a_it}g_i + \sqrt{2/N} \int_0^t e^{-2a_i(t-s)} |Y^G(s)| dB_i(s)$, the Itô isometry, and $(\sqrt{x} - \sqrt{y})^2 \leq (x - y)^2/y$, we obtain

$$\mathbb{E}|Y_i^G(t) - G_i(t)|^2 = \frac{2}{N} \int_0^t e^{-4a_i(t-s)} \mathbb{E}(|Y^G(s)| - \sqrt{N\bar{h}(s)})^2 ds \leq \varepsilon'_m h_i(t).$$

Let $G_i^* = G_i(\tau) / \sqrt{h_i(\tau)}$, which are independent standard Gaussian variables. Then

$$\mathbb{E}|H_i^G - G_i^*|^2 \leq 2\theta \varepsilon'_m h_i(\tau) + 2(\sqrt{\theta h_i(\tau)} - 1)^2 \leq C\varepsilon'_m + 2(\theta h_i(\tau) - 1)^2.$$

From Corollary 17.10, we have $|\theta h_1(\tau) - 1| \leq C m^{-a/2} + e^{2\tau/(N\theta)} - 1 \leq C[m^{-a/2} + m^{a-3}]$, and by (17.14) and Lemma 17.2, $\theta|h_1(\tau) - h_i(\tau)| \leq 4a_i \tau \theta h_1(\tau) \leq C_J m^{a-2}$ for $i \leq J$. Combining this with (17.39), we obtain $(\mathbb{E} \sum_{i \leq J} |H_i^G - G_i^*|^2)^{1/2} \leq C_J [m^{-3(1-a)/2} + m^{-a/2} + m^{a-2}]$.

Step 4: the uniform start. Recall from Lemma 17.7 that $H^G = (|g|/\sqrt{N})H$, with $|g|$ independent of H . We obtain $\mathbb{E}|H^G - H|^2 = \mathbb{E}(|g|/\sqrt{N} - 1)^2 \mathbb{E}|H|^2 \leq (2/N) \theta \sum_{i \leq J} h_i(\tau) \leq C_J/N$, where we used $(\sqrt{x} - 1)^2 \leq (x - 1)^2$ and $\text{Var } |g|^2 = 2N$. Since $N^{-1/2} \leq C_*^{3/2} m^{-3/2} \leq C m^{-3(1-a)/2}$ by Lemma 17.2, this proves the first claim. For the second, if $|H_i - G_i^*| \leq \varepsilon_0$ for all $i \leq J$ and G^* lies in the smaller rectangle, then E holds; and $\mathbb{P}(\max_i |H_i - G_i^*| > \varepsilon_0) \leq C_J g(m)^2 / \varepsilon_0^2$ by Markov's inequality. $\square$

Residence of the path

Fix $\mathfrak{b} \in (0, 1/4)$, let $\chi = \mathfrak{b} \log(em)$ and $T = \chi/a$, and fix $J \geq 2$ and $0 < \ell_- < \ell_+$. We write $|x_{\leq J}|^2 = \sum_{k \leq J} x_k^2$ and let

$$C_{\text{res}} = \frac{4}{3\ell_-^2} \left[\left(\frac{5\ell_+}{4} \right)^2 + (J-1)(\ell_+ + 1)^2 \right]. \quad (17.40)$$

Corollary 17.17 (Residence of the path). *For every $B_0 > 0$ there exist C and m_0 , depending only on $B_0, \ell_{\pm}, J, a, \mathfrak{b}, c_*, C_*$, and K_0 , such that for $m \geq m_0$,*

$$\mathbb{P} \left(E \cap \left\{ \exists t \in [T, T_{\max}] : |X_1(t)|^2 \notin \left[\frac{\theta N}{4}, 4\theta N \right] \text{ or } |X_{\leq J}(t)|^2 > C_{\text{res}} \theta N \right\} \right) \leq C m^{-B_0}.$$

The probability of E intersected with the event that X_1 changes sign on $[\tau, T_{\max}]$ is also at most $C m^{-B_0}$. If $\mathbb{P}(E) \geq c_1 > 0$, the same bounds hold for the law conditioned on E , with the right sides divided by c_1 .

This is a statement about the path started from ν_N on the event E , which is measurable at time τ . It does not concern other initial laws.

Proof. Throughout, C denotes a constant depending only on the parameters listed in the statement and on the moment orders chosen below. By Theorem 17.14 with $\chi = \mathfrak{b} \log(em)$, the right sides of (17.30) are at most $C_p m^{-\vartheta}$, where $\vartheta > 0$ does not depend on p . Let G_1 be the event that

$$\sup_{T \leq t \leq T_{\max}} |\theta^2 |Y(t)|^2 - H_1^2| \leq \frac{\ell_-^2}{4} \quad \text{and} \quad \sup_{\tau \leq t \leq T_{\max}} |\sqrt{\theta} Y_1(t) - H_1| \leq \frac{\ell_-}{4}.$$

We have $\mathbb{P}(\mathcal{G}_1^c) \leq C_p m^{-2p\vartheta}$ by Markov's inequality. We fix the integer $p > 2$ with $2p\vartheta \geq B_0$. On $E \cap \mathcal{G}_1$, we have $|H_1| \geq \ell_-$, so that $\sqrt{\theta}Y_1(t)$ has the sign of H_1 and $|\sqrt{\theta}Y_1(t)| \in [3|H_1|/4, 5|H_1|/4]$ for $t \in [\tau, T_{\max}]$, and $\theta^2|Y(t)|^2 \in [H_1^2 - \ell_-^2/4, H_1^2 + \ell_-^2/4]$ for $t \in [T, T_{\max}]$. For these t , we find

$$\frac{X_1(t)^2}{\theta N} = \frac{\theta Y_1(t)^2}{\theta^2|Y(t)|^2} \in \left[\frac{9}{16} \cdot \frac{H_1^2}{H_1^2 + \ell_-^2/4}, \frac{25}{16} \cdot \frac{H_1^2}{H_1^2 - \ell_-^2/4} \right] \subset \left[\frac{1}{4}, 4 \right].$$

It remains to bound the coordinates $2 \leq i \leq J$.

The coordinates at time T . For $2 \leq i \leq J$, the mild form of (17.10) gives $\sqrt{\theta}Y_i(T) - e^{-2a_i(T-\tau)}H_i = \sqrt{2\theta/N} \int_{\tau}^T e^{-2a_i(T-s)}|Y(s)|dB_i(s)$. By the Burkholder–Davis–Gundy inequality and Proposition 17.11, its L^{2p} norm squared is at most $C_p\theta \int_{\tau}^T e^{-4a_i(T-s)}\bar{h}(s)ds$. We split at $s = c_0/a$. For $s \leq c_0/a$, the exponential is at most Ce^{-4a_iT} since $a_i \leq C_J a$, and $2\theta \int_{\tau}^{c_0/a} \bar{h} = \theta(h_1(c_0/a) - h_1(\tau)) \leq C[m^{-a/2} + m^{-1}]$ by Corollary 17.10. For $s \geq c_0/a$, we have $\bar{h}(s) \leq C(N\theta^2)^{-1}$, and the integral is at most $C(N\theta a_i)^{-1} \leq C/m$. We conclude that

$$\|\sqrt{\theta}Y_i(T) - e^{-2a_i(T-\tau)}H_i\|_{2p}^2 \leq C_p[m^{-a/2}e^{-4a_iT} + m^{-1}],$$

and the event $\mathcal{G}_2$ that $|\sqrt{\theta}Y_i(T) - e^{-2a_i(T-\tau)}H_i| \leq 1/2$ for all $2 \leq i \leq J$ satisfies $\mathbb{P}(\mathcal{G}_2^c) \leq Cm^{-B_0}$, after increasing p . On $E \cap \mathcal{G}_2$, we have $|\sqrt{\theta}Y_i(T)| \leq \ell_+ + 1/2$.

The coordinates after time T . Let $T_{\text{amp}} = \inf\{s \geq T : \theta^2|Y(s)|^2 > 2\ell_+^2\}$, with $T_{\text{amp}} = T$ if $\theta^2|Y(T)|^2 > 2\ell_+^2$. For $i \geq 2$ and $t \geq T$, define the process with stopped amplitude

$$\tilde{Y}_i(t) = e^{-2a_i(t-T)}Y_i(T) + M_i(t), \quad M_i(t) = \sqrt{\frac{2}{N}} \int_T^t e^{-2a_i(t-s)}|Y(s)|\mathbf{1}_{\{s < T_{\text{amp}}\}}dB_i(s). \quad (17.41)$$

This process is used only for estimates. On $E \cap \mathcal{G}_1$, we have $\theta^2|Y|^2 \leq \ell_+^2 + \ell_-^2/4 < 2\ell_+^2$ on $[T, T_{\max}]$. Then $T_{\text{amp}} > T_{\max}$ and $\tilde{Y}_i = Y_i$ on $[T, T_{\max}]$. The integrand of $\sqrt{\theta}M_i$ is bounded by $2\ell_+(N\theta)^{-1/2}$. In the time variable $v = at$, the process $\sqrt{\theta}M_i(v/a)$ has the form $\int e^{-2(a_i/a)(v-r)}\phi(r)dB'(r)$ for a Brownian motion B' and an adapted ϕ with $|\phi| \leq 2\ell_+m^{-1/2}$. Using the Burkholder–Davis–Gundy inequality, we bound its L^{2p} norm at fixed times by $C_p m^{-1/2}$. For increments of length $h' \leq 1$, we write the increment as $(e^{-2(a_i/a)h'} - 1)$ times the old value plus the integral over the new interval, which yields the bound $C_p m^{-1/2}(h' + h'^{1/2}) \leq C_p m^{-1/2}h'^{1/4}$. Applying Lemma 17.13 on an interval of length at most $K_0 \log(em)$, we obtain $\|\sup_{[T, T_{\max}]} \sqrt{\theta}|M_i|\|_{2p} \leq C_p(\log(em))^{1/(2p)}m^{-1/2}$. Let $\mathcal{G}_3$ be the event that $\sup_{[T, T_{\max}]} \sqrt{\theta}|M_i| \leq 1/2$ for $2 \leq i \leq J$; then $\mathbb{P}(\mathcal{G}_3^c) \leq Cm^{-B_0}$ after increasing p .

Conclusion. Let $\mathcal{G} = \mathcal{G}_1 \cap \mathcal{G}_2 \cap \mathcal{G}_3$, so that $\mathbb{P}(\mathcal{G}^c) \leq Cm^{-B_0}$. On $E \cap \mathcal{G}$, for $t \in [T, T_{\max}]$ and $2 \leq i \leq J$, we have $|\sqrt{\theta}Y_i(t)| \leq |\sqrt{\theta}Y_i(T)| + \sqrt{\theta}|M_i(t)| \leq \ell_+ + 1$, and therefore

$$\frac{|X_{\leq J}(t)|^2}{\theta N} = \sum_{i \leq J} \frac{\theta Y_i(t)^2}{\theta^2 |Y(t)|^2} \leq \frac{(5\ell_+/4)^2 + (J-1)(\ell_+ + 1)^2}{3\ell_-^2/4} = C_{\text{res}}.$$

This proves the first claim. We have already proved that X_1 , which has the sign of Y_1 , does not change sign on $[\tau, T_{\max}]$ on $E \cap \mathcal{G}_1$. $\square$

The endpoint measure weighted by $\mathbf{1}_E$ and killed on leaving D

We now define the endpoint subprobability measure to which Theorem 11.19 is applied. Fix L_0, L_1, L with $C_{\text{res}} < L_0 < L_1 < L$, and let

$$D = \{x \in S_N : |x_{\leq J}|^2 < L\theta N\}, \quad T_{\text{exit}} = \inf\{t \geq T : X(t) \notin D\},$$

with $T_{\text{exit}} = T$ if $X(T) \notin D$. For $t \geq T$, let

$$\sigma_t(A) = \mathbb{P}(E, T_{\text{exit}} > t, X(t) \in A). \quad (17.42)$$

This is the endpoint subprobability measure (11.7) of Section 11.1, weighted by $\Psi = \mathbf{1}_E$ and killed on leaving D . It is dominated by the time- t law: $\sigma_t \leq \rho_t$ as measures. The weight and the killing change the mass of the measure, not the sample paths. We also use the weighted ratio of the remaining coordinates to the first ones,

$$\xi(x) = \frac{a_{J+1} \sum_{i>J} x_i^2 / a_i}{|x_{\leq J}|^2},$$

with $\xi(x) = 0$ when $|x_{\leq J}| = 0$, as in (11.62). We write $\alpha_J = a_{J+1}/a$.

Corollary 17.18 (Moments, survival, and the mass where the cutoff may differ from one). *For every $p \geq 1$ and $B_0 > 0$ there exist C and m_0 , depending only on $p, B_0, \ell_{\pm}, J, L, \mathbf{a}, \mathbf{b}, c_*, C_*$, and K_0 , such that the following hold for $m \geq m_0$ and $t \in [T, T_{\max}]$.*

1. The killing loss satisfies

$$\mathbb{P}(E, T_{\text{exit}} \leq T_{\max}) \leq Cm^{-B_0}. \quad (17.43)$$

2. For $2 \leq j \leq J$,

$$\left(\int |x_j|^{2p} d\sigma_t \right)^{1/p} \leq C[\theta N e^{-4a_j(t-T)} + a_j^{-1}] + C\theta N m^{-B_0/p}. \quad (17.44)$$

If $B_0 \geq p$, the last term is at most Ca_j^{-1} .

3. For $0 < \delta \leq 1$,

$$\sigma_t\{\xi > \delta/2\} \leq Cm^{-B_0} + C\delta^{-p}[e^{-2\alpha_J \chi} + m^{-1}]^p. \quad (17.45)$$

4. Let $0 < s < 1/4$, let $0 < \delta \leq 1$, and let

$$\mathcal{C} = \{x_1^2 < s\theta N\} \cup \{|x_{\leq J}|^2 > L_0\theta N\} \cup \{\xi > \delta/2\}.$$

Then $\sigma_t(\mathcal{C}) \leq Cm^{-B_0} + \sigma_t\{\xi > \delta/2\}$, and for every measurable $\varphi : S_N \rightarrow [0, 1]$ that equals one outside $\mathcal{C}$,

$$\int \varphi d\sigma_t \geq \mathbb{P}(E) - Cm^{-B_0} - \sigma_t\{\xi > \delta/2\}.$$

In the third part, the bound concerns each fixed time t . It does not state that the path stays in $\{\xi \leq \delta/2\}$. If $\alpha_J \geq \alpha_* > 0$ and $\delta = m^{-\zeta}$ with $0 < \zeta < \min\{1, 2\alpha_*\mathfrak{b}\}$, the second term in (17.45) is at most $Cm^{-p \min\{2\alpha_*\mathfrak{b}-\zeta, 1-\zeta\}}$, which is smaller than any prescribed power of m^{-1} once p is large. For example, if $\alpha_J \geq 6$ and $\mathfrak{b} = \zeta = 1/100$, then the exponent is $p \min\{11/100, 99/100\}$, and with $p = 200$ we obtain m^{-22} .

Proof. Let $\mathcal{G}$ be the event of the proof of Corollary 17.17, for the exponent B_0 . Recall that $\mathbb{P}(\mathcal{G}^c) \leq Cm^{-B_0}$. On $E \cap \mathcal{G}$, the path satisfies $|X_{\leq J}|^2 \leq C_{\text{res}}\theta N < L\theta N$ on $[T, T_{\text{max}}]$. Then $T_{\text{exit}} > T_{\text{max}}$, and this proves the first part. We keep the definition of σ_t , and use the event $\mathcal{G}$ only to split integrals.

For the second part, let $2 \leq j \leq J$. On $E \cap \mathcal{G}$, we have $\theta^2|Y(t)|^2 \geq 3\ell_-^2/4$ and $Y_j = \tilde{Y}_j$. Then $X_j(t)^2 = NY_j(t)^2/|Y(t)|^2 \leq (4N\theta^2/(3\ell_-^2))\tilde{Y}_j(t)^2$. On $E \setminus \mathcal{G}$, the measure σ_t only sees the event $\{T_{\text{exit}} > t\}$, on which $X_j(t)^2 \leq |X_{\leq J}(t)|^2 < L\theta N$. We obtain

$$\int |x_j|^{2p} d\sigma_t \leq \left(\frac{4N\theta^2}{3\ell_-^2}\right)^p \mathbb{E}|\tilde{Y}_j(t)|^{2p} + (L\theta N)^p \mathbb{P}(\mathcal{G}^c).$$

By (17.41), the Burkholder–Davis–Gundy inequality with the bounded integrand, and Proposition 17.11,

$$\|M_i(t)^2\|_p \leq \frac{C_p \ell_+^2}{N\theta^2 a_i}, \quad \|\tilde{Y}_i(t)^2\|_p \leq C_p \left[e^{-4a_i(t-T)} h_i(T) + \frac{1}{N\theta^2 a_i} \right] \quad (i \geq 2), \quad (17.46)$$

and $h_j(T) \leq h_1(T) \leq C\theta^{-1}$. Taking the power $1/p$ proves (17.44). If $B_0 \geq p$, then $\theta N m^{-B_0/p} \leq \theta N/m = a^{-1} \leq C_J a_j^{-1}$.

For the third part, the normalization cancels in ξ :

$$\xi(X(t)) = \frac{a_{J+1} \sum_{i>J} Y_i(t)^2 / a_i}{\sum_{i \leq J} Y_i(t)^2}.$$

On $E \cap \mathcal{G}$, the denominator is at least $Y_1(t)^2 \geq 9\ell_-^2/(16\theta)$, and $Y_i = \tilde{Y}_i$ for all $i \geq 2$. By Minkowski's inequality and (17.46),

$$\left\| a_{J+1} \theta \sum_{i>J} \frac{\tilde{Y}_i(t)^2}{a_i} \right\|_p \leq C_p a_{J+1} \theta \sum_{i>J} \frac{1}{a_i} \left[e^{-4a_i(t-T)} h_i(T) + \frac{1}{N\theta^2 a_i} \right].$$

Using Corollary 17.10 and $T \geq c_0/a$, we have $h_i(T) \leq C[\theta^{-1}e^{-2a_iT} + (N\theta^2a_i)^{-1}]$. The terms with $(N\theta^2a_i)^{-1}$ contribute at most $C(a_{J+1}/(N\theta)) \sum_{i \geq 2} a_i^{-2} \leq C\alpha_J/m$, by Lemma 17.2. For the terms with e^{-2a_iT} , the indices with $a_{J+1} \leq a_i \leq 2a_{J+1}$ number at most $(2C_*/c_*)^{3/2}J$, by (17.2), and each contributes at most $e^{-2a_{J+1}T}$. For the indices with $a_i > 2a_{J+1}$, we have $e^{-2a_iT} \leq e^{-2a_{J+1}T}e^{-a_iT}$, and $a_{J+1} \sum_{i \geq 2} e^{-a_iT}/a_i \leq CNa_{J+1}T^{-1/2} \leq C_J$, by the fourth sum in Lemma 17.2 and $T \geq 1/a$. Using $a_{J+1} \leq C_Ja$, we have

$$\left\| a_{J+1}\theta \sum_{i > J} \frac{\tilde{Y}_i(t)^2}{a_i} \right\|_p \leq C[e^{-2\alpha_J\chi} + m^{-1}],$$

where we used $a_{J+1}T = \alpha_J\chi$. Markov's inequality on $E \cap \mathcal{G}$ and the bound on $\mathbb{P}(\mathcal{G}^c)$ give (17.45).

For the fourth part, on $E \cap \mathcal{G}$ we have $x_1^2 \geq \theta N/4 > s\theta N$ and $|x_{\leq J}|^2 \leq C_{\text{res}}\theta N < L_0\theta N$ at time t , by Corollary 17.17. It follows that $\sigma_t(\mathcal{C}) \leq \mathbb{P}(E \setminus \mathcal{G}) + \sigma_t\{\xi > \delta/2\}$. Finally, $\int \varphi d\sigma_t \geq \sigma_t(S_N) - \sigma_t(\mathcal{C})$ and $\sigma_t(S_N) \geq \mathbb{P}(E) - \mathbb{P}(E, T_{\text{exit}} \leq T_{\text{max}})$. $\square$

17.6 The limit of finitely many coordinates

In this section, we identify the limit in distribution of finitely many standardized coordinates at the times $t_m(u)$, and we deduce the lower bound on the distance. For the rest of this section, we fix the following choices of the parameters of Sections 17.4 and 17.5:

$$\begin{aligned} K_0 &= 1, & \mathfrak{a} &= \frac{1}{10}, & \mathfrak{b} &= \frac{1}{100}, \\ \tau &= \frac{m^{1/10}}{\theta^2}, & \chi &= \frac{\log(em)}{100}, & T &= \frac{\chi}{a}, & H_i &= \sqrt{\theta} Y_i(\tau). \end{aligned} \quad (17.47)$$

For a fixed $u \in \mathbb{R}$, the time $t_m(u)$ of (17.5) lies in $[T + 1/a, T_{\text{max}}]$ for $m \geq m_0(u)$, because $at_m(u) = \log(2m)/4 + u$. For $J \geq 2$, we write $\mathbf{z}(x) = (\sqrt{2a_2}x_2, \dots, \sqrt{2a_J}x_J) \in \mathbb{R}^{J-1}$ for the standardized coordinates, as in (11.36), and we let

$$\mu_u = \text{Law}(G_2 + e^{-2u}C, G_3, \dots, G_J), \quad (17.48)$$

where $G_2, \dots, G_J$ are independent standard Gaussian variables and C is an independent standard Cauchy variable.

Theorem 17.19 (The limit of finitely many coordinates and the lower bound). *Let $a^{(n)}$ be as in Theorem 17.3. For every fixed $u \in \mathbb{R}$ and $J \geq 2$, the law of $\mathbf{z}(X(t_{m_n}(u)))$, where X is the diffusion for $a^{(n)}$ started from ν_{N_n} , converges weakly to μ_u . Moreover, for every compact interval $I \subset \mathbb{R}$,*

$$\sup_{u \in I} (\mathcal{V}(u) - D_n(t_{m_n}(u)))_+ \longrightarrow 0.$$

We describe the proof. For $2 \leq i \leq J$, the mild form of (17.10) on $[\tau, t]$ splits $Y_i(t)$ into the inherited part $e^{-2a_i(t-\tau)}Y_i(\tau)$, the noise on $[\tau, T]$, and the noise on $[T, t]$. After normalization by $|Y(t)|$, the inherited part of the second coordinate is $e^{-2u}H_2/|H_1|$, up to a vanishing error, and the inherited parts of the other coordinates vanish because $a_i \geq (1 + \varepsilon_*)a$. The noise on $[\tau, T]$ is negligible. On $[T, t]$, we replace the amplitude $\theta|Y(s)|$ of the noise by the variable $|H_1|$, which is measurable at time τ ; after division by $\theta|Y(t)| \approx |H_1|$, this noise becomes a centered Gaussian variable independent of $(H_1, \dots, H_J)$. No Gaussian property of a stochastic integral with random amplitude is used. We do not claim that $H_2/|H_1|$ has a Cauchy law for finite N . It converges in distribution to the ratio of two independent standard Gaussian variables, which is standard Cauchy [110, Section 1, p. 193].

Proof. We fix u and J , drop the index n , and write $t = t_m(u)$. Throughout, C denotes a constant depending only on $u, J, c_*, C_*,$ and ε_* , and $\alpha_i = a_i/a$, so that $1 \leq \alpha_i \leq C$ for $i \leq J$ and $\alpha_i \geq 1 + \varepsilon_*$ for $3 \leq i \leq J$.

Step 1: decomposition. For $2 \leq i \leq J$, we have $\sqrt{2a_i}X_i(t) = \sqrt{2a_iN}\theta Y_i(t)/(\theta|Y(t)|)$, and the mild form of (17.10) gives $\sqrt{2a_iN}\theta Y_i(t) = I_i + Q_i + R_i$, where

$$I_i = \sqrt{2a_iN\theta} e^{-2a_i(t-\tau)} H_i,$$

$$Q_i = \sqrt{4a_i} \int_{\tau}^T e^{-2a_i(t-s)} \theta|Y(s)| dB_i(s), \quad R_i = \sqrt{4a_i} \int_T^t e^{-2a_i(t-s)} \theta|Y(s)| dB_i(s).$$

Step 2: the inherited part. Since $2a_iN\theta = 2\alpha_i m$ and $e^{-2a_it} = (2m)^{-\alpha_i/2} e^{-2\alpha_i u}$, we have

$$I_i = \sqrt{\alpha_i} (2m)^{(1-\alpha_i)/2} e^{-2\alpha_i u} e^{2a_i\tau} H_i.$$

Here $a_i\tau \leq C_*^3 \alpha_i m^{a-2} \rightarrow 0$. For $i = 2$, we have $\alpha_2 = 1$ and $I_2 = e^{-2u} e^{2a\tau} H_2$. For $3 \leq i \leq J$, the coefficient is at most $C(2m)^{-\varepsilon_*/2} \rightarrow 0$. By Lemma 17.16, $(H_1, \dots, H_J)$ converges in distribution to a standard Gaussian vector. This sequence is tight, and $I_2 - e^{-2u} H_2 \rightarrow 0$ and $I_i \rightarrow 0$ ($i \geq 3$) in probability.

Step 3: the noise on $[\tau, T]$. Using the Itô isometry and $\mathbb{E}|Y(s)|^2 = N\bar{h}(s)$, we have

$$\mathbb{E}Q_i^2 = 4a_i\theta^2 e^{-4a_it} \int_{\tau}^T e^{4a_is} N\bar{h}(s) ds \leq Cm^{1-\alpha_i} \theta \int_{\tau}^T e^{4a_is} \bar{h}(s) ds,$$

where we used $4a_i\theta^2 Ne^{-4a_it} = 4\alpha_i\theta m(2m)^{-\alpha_i} e^{-4\alpha_i u}$. We split the integral at $s = c_0/a$. For $s \leq c_0/a$, we have $e^{4a_is} \leq C$ and, by Corollary 17.10, $\bar{h} \leq U/2 + C(N\theta^2)^{-1}$ and $\theta \int_{\tau}^{\infty} U \leq C\theta^{-1}\tau^{-1/2} = Cm^{-a/2}$; also $\theta \cdot (N\theta^2)^{-1} c_0/a \leq C/m$. For $c_0/a \leq s \leq T$, we have $\bar{h}(s) \leq C(N\theta^2)^{-1}$ and $\theta \int_T^T e^{4a_is} (N\theta^2)^{-1} ds \leq Ce^{4a_iT}/(N\theta a_i) \leq Cm^{-1} e^{4\alpha_i\chi}$. Since $\alpha_i \geq 1$ and $e^{4\alpha_i\chi} = (em)^{\alpha_i/25}$, we obtain

$$\mathbb{E}Q_i^2 \leq C[m^{-a/2} + m^{-1} + m^{-\alpha_i}(em)^{\alpha_i/25}] \rightarrow 0.$$

Step 4: the noise on $[T, t]$. The variable $|H_1|$ is measurable at time $\tau < T$; in particular, the process $s \mapsto |H_1| \mathbf{1}_{\{s \geq T\}}$ is predictable. Let

$$N_i = \sqrt{4a_i} \int_T^t e^{-2a_i(t-s)} dB_i(s) \quad (2 \leq i \leq J).$$

The variables N_i are independent centered Gaussian variables with variances $1 - e^{-4a_i(t-T)}$, and they are independent of the σ -algebra of time T , which contains $(H_1, \dots, H_J)$. Their variances tend to one, because $a_i(t-T) \geq \log(2m)/4 - \log(em)/100 + u \rightarrow \infty$. By the Itô isometry and Corollary 17.15,

$$\begin{aligned} \mathbb{E}(R_i - |H_1|N_i)^2 &= 4a_i \int_T^t e^{-4a_i(t-s)} \mathbb{E}(\theta|Y(s)| - |H_1|)^2 ds \\ &\leq \sup_{T \leq s \leq T_{\max}} \mathbb{E}(\theta|Y(s)| - |H_1|)^2 \rightarrow 0. \end{aligned}$$

Step 5: the limit. Let $\Phi_u(h, n) = (e^{-2u}h_2/|h_1| + n_2, n_3, \dots, n_J)$ for $h \in \mathbb{R}^J$ with $h_1 \neq 0$ and $n \in \mathbb{R}^{J-1}$, and let $N = (N_2, \dots, N_J)$. Fix $\ell \in (0, 1)$, and consider the event $\mathcal{E}_\ell = \{|H_1| \geq \ell, |\theta|Y(t)| - |H_1| \leq \ell/2\}$, on which $\theta|Y(t)| \geq \ell/2$. On $\mathcal{E}_\ell$, Steps 1–4 give, for the coordinate with index i ,

$$\begin{aligned} (\mathbf{z}(X(t)) - \Phi_u(H, N))_i &= \frac{(I_i - \mathbf{1}_{\{i=2\}}e^{-2u}H_2) + Q_i + (R_i - |H_1|N_i)}{\theta|Y(t)|} \\ &\quad + \left(\frac{|H_1|}{\theta|Y(t)|} - 1 \right) \frac{\mathbf{1}_{\{i=2\}}e^{-2u}H_2 + |H_1|N_i}{|H_1|}, \end{aligned}$$

and every term tends to zero in probability, because $\theta|Y(t)| \geq \ell/2$ and $||H_1|/(\theta|Y(t)|) - 1| \leq 2\ell^{-1}|\theta|Y(t)| - |H_1||$. Then $\mathbb{P}(|\theta|Y(t)| - |H_1| > \ell/2) \rightarrow 0$ by Corollary 17.15. We conclude that, for every $\varepsilon > 0$,

$$\limsup \mathbb{P}(|\mathbf{z}(X(t)) - \Phi_u(H, N)| > \varepsilon) \leq \limsup \mathbb{P}(|H_1| < \ell) = \mathbb{P}(|G_1| < \ell),$$

where G_1 is standard Gaussian, and letting $\ell \rightarrow 0$ shows that $\mathbf{z}(X(t)) - \Phi_u(H, N) \rightarrow 0$ in probability. The vectors H and N are independent, H converges in distribution to a standard Gaussian vector $(G'_1, \dots, G'_J)$ by Lemma 17.16, and N converges in distribution to a standard Gaussian vector $(G_2, \dots, G_J)$. The map Φ_u is continuous on $\{h_1 \neq 0\}$, which has full measure for the limit, and the continuous mapping theorem implies that $\Phi_u(H, N) \Rightarrow \Phi_u(G', G)$. The ratio $G'_2/|G'_1|$ has the law of G'_2/G'_1 , the standard Cauchy law, because G'_2 is symmetric and independent of $|G'_1|$; it is also independent of $(G_2, \dots, G_J)$. We conclude that $\Phi_u(G', G)$ has law μ_u , and Slutsky's lemma proves the first claim.

Step 6: the lower bound. Let $\varphi : \mathbb{R} \rightarrow [0, 1]$ be continuous. By Proposition 16.9, the law of $\sqrt{2a} x_2$ under π converges in total variation to the standard Gaussian law. Its hypotheses hold with constants depending only on c_* and C_* , by Lemma 17.2: $a_N \leq C_*$, $a_3 \leq 2^{2/3}(C_*/c_*)a$, and $a^2 \sum_{i=2}^N a_i^{-2} \leq C$. By the first claim with $J = 2$, and the second part of Lemma 3.6,

$$\liminf D(t) \geq \lim \left| \mathbb{E}\varphi(\sqrt{2a} X_2(t)) - \mathbb{E}_\pi \varphi(\sqrt{2a} x_2) \right| = \left| \mathbb{E}\varphi(G + e^{-2u}C) - \mathbb{E}\varphi(G) \right|.$$

The two limiting laws have continuous densities, and the supremum of the right side over continuous φ with values in $[0, 1]$ is their total variation distance $\mathcal{V}(u)$, by the first part of Lemma 3.6 and the density of continuous functions in $L^1(\mathbb{R})$. This proves $\liminf D(t_m(u)) \geq \mathcal{V}(u)$ for each u .

For the uniformity, fix $\delta > 0$ and a mesh $u_0 < \dots < u_k$ with $I \subset [u_0, u_k]$ and $\mathcal{V}(u_j) - \mathcal{V}(u_{j+1}) \leq \delta$, which exists because $\mathcal{V}$ is uniformly continuous on $[u_0, u_k]$ (Lemma 3.9). Since D is nonincreasing, for $u \in [u_j, u_{j+1}]$ we have $D(t_m(u)) \geq D(t_m(u_{j+1}))$, and therefore

$$\mathcal{V}(u) - D(t_m(u)) \leq \delta + \max_{0 \leq j \leq k} (\mathcal{V}(u_j) - D(t_m(u_j))).$$

The maximum over the finite mesh has nonpositive limit superior, and δ is arbitrary. $\square$

17.7 Density moment bounds for $t \geq \theta^{-2}$

In this section, we bound all moments of the density of the time- t law with respect to the Gibbs measure, for $t \geq \theta^{-2}$. The bound $e^{O(N\theta^3)} = e^{O(m^3)}$ is far from one. It is used in Section 17.8 only as the initial value of a conditional relative entropy that then decays. The proof is an application of Theorem 12.8, after the quadratic potential is shifted so that it has the form considered in Section 12.

Theorem 17.20 (Density moment bounds for $t \geq \theta^{-2}$). *Suppose that a satisfies the indexed bounds (17.2) and that $0 < \theta \leq 1/2$. For every $p > 1$ there exist C_p and m_0 , depending only on p , c_* , and C_* , such that if $m \geq m_0$, then for every $t \geq \theta^{-2}$,*

$$\log \int_{S_N} \left(\frac{d\rho_t}{d\pi} \right)^p d\pi \leq C_p N \theta^3, \quad \text{KL}(\rho_t \| \pi) \leq C_2 N \theta^3. \quad (17.49)$$

Proof. Let λ_+ be the eigenvalue of Lemma 17.12, and let $r_i = a_i + \lambda_+/4$. The secular equation $(2/N) \sum_i (4a_i + \lambda_+)^{-1} = 1$ says that $\sum_i (2r_i)^{-1} = N$, that is, $r = (r_1, \dots, r_N)$ satisfies (12.1). On S_N , we have $\sum_i r_i x_i^2 = F(x) + \lambda_+ N/4$, so that the generator, the semigroup, and the Gibbs measure of (12.2) are L , P_t , and π . Let $S = \theta^{-2}$. We verify the hypotheses (H1)–(H5) of Definition 12.2 on $[0, S]$, with a constant C_H depending only on c_* and C_* . Then Theorem 12.8 yields $\log \int (d\rho_S/d\pi)^p d\pi \leq CN(1+S)^{-3/2} \leq CN\theta^3$, the monotonicity of the p -th moment in t , and the bound on the relative entropy with $p = 2$. Throughout, C denotes a constant depending only on c_* and C_* .

(H1) and (H4). We have $\lambda_+ \leq 2a/m$ by Lemma 17.12. Then $\max_i r_i \leq C_* + 1$ and $\lambda_+ S \leq 2/(N\theta^3) \leq 2C_*^3/m^3$. Also $N(1+S)^{-3/2} \geq 2^{-3/2}N\theta^3 \geq 2^{-3/2}m^3/C_*^3 \geq 1$ for large m . For (H4), the indices $i = 1, 2, 3$ satisfy $r_i S \leq (a_3 + \lambda_+/4)\theta^{-2} \leq C(a/\theta^2) + C/m^3 \leq C$, by Lemma 17.2.

(H3). For $i \geq 2$, we have $r_i \geq a_i \geq c_*((i-1)/N)^{2/3}$, and Lemma 12.3 applies with $k_0 = 1$ and gives $\sum_i (1+r_i t)^{-2} \leq 1 + CN(1+t)^{-3/2}$; the term 1 is at most $N(1+t)^{-3/2}$ for $t \leq S$, by (H1).

(H2). The heat vector of Section 12 for the half-precisions r is $e^{t(H-\lambda_+I)}\mathbf{1} = e^{-\lambda_+ t}h(t)$, since $-4\text{diag}(r) + (2/N)\mathbf{1}\mathbf{1}^T = H - \lambda_+ I$. For $t \leq S$, we have $e^{-\lambda_+ t} \in [e^{-1}, 1]$, and it suffices to prove the bounds of (H2) for $\bar{h}$ and h_i . From the fifth part of Corollary 17.10, we have $C^{-1}(1+t)^{-1/2} \leq \bar{h}(t) \leq C[(1+t)^{-1/2} + (N\theta^2)^{-1}]$, and $(N\theta^2)^{-1} \leq (1+t)^{-1/2}$ for $t \leq S$ because $N\theta^3 \geq 2$. This proves the bounds on $\bar{h}$, and the following bounds hold for $t \leq S$:

$$C^{-1}(1+t)^{-1/2} \leq U(t) \leq C(1+t)^{-1/2}, \quad C^{-1}(1+t)^{-1/2} \leq \bar{h}(t) \leq C(1+t)^{-1/2}. \quad (17.50)$$

Since $\lambda_+ t \leq 1$, we have $1 + a_i t \leq 1 + r_i t \leq (5/4)(1 + a_i t)$, and we may replace r_i by a_i in the bounds on h_i . For the upper bound with $i \geq 2$, the first case of Lemma 17.9 (with $C_0 = 1$, $C_1 = 0$) gives $h_i(t) \leq C[\sqrt{1+t}/(1+a_i t) + \min\{t, a_i^{-1}\}/(N\theta^2)]$, and $\min\{t, a_i^{-1}\} \leq 2(1+t)/(1+a_i t)$ and $\sqrt{1+t} \leq 2/\theta$ bound the second term by the first. For $i = 1$, we have $h_1(t) = 1 + 2 \int_0^t \bar{h} \leq C\sqrt{1+t}$, by (17.50). For the lower bound, let $\ell_i = \min\{t, (4a_i)^{-1}\}$, with $\ell_1 = t$; by (17.13) and (17.50),

$$h_i(t) \geq e^{-4a_i t} + 2 \int_{t-\ell_i}^t e^{-4a_i(t-s)} \bar{h}(s) ds \geq e^{-4a_i t} + \frac{2\ell_i}{Ce} (1+t)^{-1/2}.$$

If $a_i t \leq 1/4$, the right side is at least $c_1 \sqrt{1+t}$; otherwise it is at least $c_1/(a_i \sqrt{1+t}) \geq c'_1 \sqrt{1+t}/(1+a_i t)$, since $a_i(1+t) \leq (4C_* + 1)a_i t$ when $a_i t \geq 1/4$. Together, we obtain

$$C^{-1} \frac{\sqrt{1+t}}{1+r_i t} \leq e^{-\lambda_+ t} h_i(t) \leq C \frac{\sqrt{1+t}}{1+r_i t} \quad (0 \leq t \leq S, 1 \leq i \leq N). \quad (17.51)$$

(H5). Let $v_i(t)$ be the normalized variances of (12.3) for the half-precisions r , and $\mu_t = \mu_{v(t)}$ the Bingham comparison law, whose half-precisions are $k_i(t) = (2v_i(t))^{-1}$. By (H2) and the first part of Lemma 12.4, there exists $C_3 \geq 1$, depending only on c_* and C_* , with $C_3^{-1}(1+t)/(1+r_i t) \leq v_i(t) \leq C_3(1+t)/(1+r_i t)$. In particular, $\max_i v_i(t) \leq C_3(1+t)$ and $k_{\min}(t) \geq (2C_3(1+t))^{-1}$, and every index with $r_i S \leq 1$ satisfies $v_i(t) \geq (1+t)/(2C_3)$, that is, $k_i(t) \leq L_H k_{\min}(t)$ with $L_H = 2C_3^2$. Let $k_* = \lceil 64L_H^2 \rceil + 2$. Using Lemma 17.2, we have $r_{k_*} S \leq C(k_*)^{2/3} a \theta^{-2} + C/m^3 \leq 1$ and $Nk_{\min}(t) \geq N\theta^2/(4C_3) \geq 4$ for large m . Corollary 8.9, in its form for half-precisions applied to the indices $1, \dots, k_*$, gives

$C_P(\mu_t) \leq 1/k_{\min}(t) \leq 2C_3(1+t)$. The indices 1, 2, 3 satisfy $v_i(t) \geq (2C_3^2)^{-1} \max_j v_j(t)$, and Corollary 8.13 with $\delta = (2C_3^2)^{-1}$ gives

$$C_{LS}(\mu_t) \leq C_\delta \max_i v_i(t) + C_\delta C_P(\mu_t) \leq C(1+t) \quad (0 \leq t \leq S). \quad (17.52)$$

This is (H5), and the logarithmic Sobolev inequality refers to the spherical Dirichlet form of μ_t . The constants L_H and k_* depend only on c_* and C_* , and they are fixed before the threshold for m is chosen. This completes the proof. $\square$

The proof of Theorem 12.8 compares the time- t law with the comparison law $\mu_{\gamma t}$ at the earlier time γt . The coefficients of the comparison law are controlled by Lemma 12.4, and its exponential moments by Lemma 12.5. In the present setting, the counting bound (H3) contains the extra index $i = 1$, with $r_1 = \lambda_+/4$ of order $(N\theta)^{-1}$. Its contribution is absorbed because $N(1+S)^{-3/2}$ is of order $N\theta^3 \geq 1$.

The next corollary bounds the relative entropy of a law obtained from the time- t law by an endpoint cutoff, relative to the Gaussian product law in the chart. It is not used in the proof of the main results, where the corresponding bound is obtained inside Theorem 11.19. Let G be the centered Gaussian measure on $\mathbb{R}^{N-1}$ with independent coordinates of variances $(2a_i)^{-1}$, $2 \leq i \leq N$, as in (11.32), and for a measure ρ on S_N let ρ^+ be its image under $x \mapsto (x_2, \dots, x_N)$.

Corollary 17.21 (Relative entropy after an endpoint cutoff). *In the setting of Theorem 17.20, let $t \geq \theta^{-2}$, $s > 0$, $z_0 > 0$, and let $\varphi : S_N \rightarrow [0, 1]$ be measurable and equal to zero on $\{x_1^2 < s\theta N\}$, with $Z = \int \varphi d\rho_t \geq z_0$. Let $\rho^\varphi = \varphi \rho_t / Z$. There exist C and m_0 , depending only on s , z_0 , c_* , and C_* , such that for $m \geq m_0$,*

$$D((\rho^\varphi)^+ \| G) \leq CN\theta^3. \quad (17.53)$$

In particular, for every splitting $G = G_z \otimes G_v$ of the coordinates, the relative entropy of the conditional law of the remaining coordinates given the first ones, averaged over the first ones, is at most $CN\theta^3$.

Proof. We have $\varphi/Z \leq 1/z_0$, and Theorem 17.20 with $p = 2$ implies that $\int (d\rho^\varphi/d\pi)^2 d\pi \leq z_0^{-2} e^{C_2 N\theta^3}$. Then $\text{KL}(\rho^\varphi \| \pi) \leq \log \int (d\rho^\varphi/d\pi)^2 d\pi \leq C_2 N\theta^3 + 2 \log(1/z_0)$ by Jensen's inequality. By the second part of Lemma 3.6, applied to the map $x \mapsto (x_2, \dots, x_N)$, we have $\text{KL}((\rho^\varphi)^+ \| \pi^+) \leq \text{KL}(\rho^\varphi \| \pi)$. No symmetry of φ is needed for this. Recall from (16.14) that $d\pi^+/dG(y) = (1 - |y|^2/N)^{-1/2}/Z_G$, which is at most $(s\theta)^{-1/2}/Z_G \leq 2s^{-1/2}$ on the support of $(\rho^\varphi)^+$, where $x_1^2 = N - |y|^2 \geq s\theta N$; here we used $\sqrt{\theta}Z_G \geq 1/2$, which holds for large m by Proposition 16.9, whose hypotheses hold by Lemma 17.2. We obtain

$$D((\rho^\varphi)^+ \| G) = D((\rho^\varphi)^+ \| \pi^+) + \int \log \frac{d\pi^+}{dG} d(\rho^\varphi)^+ \leq C_2 N\theta^3 + 2 \log \frac{1}{z_0} + \log \frac{2}{\sqrt{s}},$$

which is at most $CN\theta^3$, since $N\theta^3 \geq 1$. The last claim follows from the chain rule (11.47), which applies because the relative entropy in (17.53) is finite. $\square$

17.8 Conditional relative entropy of the remaining coordinates

In this section, we verify the hypotheses of Theorem 11.19 for the endpoint subprobability measures σ_t of (17.42), weighted by $\mathbf{1}_E$ and killed on leaving D , and we integrate its differential inequality up to the times $t_m(u)$. We use the notation of Section 11.5: the retained coordinates are $x_1, \dots, x_J$, we write $y = (z, v)$ with $z = (x_2, \dots, x_J)$ and $v = (x_{J+1}, \dots, x_N)$, $b = a_{J+1}$, $u = x_1^2/(\theta N)$, $\bar{u} = |x_{\leq J}|^2/(\theta N)$, ξ as in (11.62), the cutoff function $\eta = \xi^k$ of (11.63), and the set $\mathcal{C}$ of (11.64), outside which the cutoff equals one. For $t > T$, f_t is the density of σ_t^+ with respect to $G = G_z \otimes G_v$, $Z_t = \int \eta f_t dG$, $w_t G$ with $w_t = \eta f_t / Z_t$ is the image of the normalized cutoff-weighted law $\eta \sigma_t / Z_t$, with normalizing mass Z_t , $\bar{w}_t$ is the density of its first marginal, and $\mathcal{H}(t)$ and $\mathcal{D}(t)$ are the conditional relative entropy of the remaining coordinates and its dissipation, as in (11.46) and (11.56).

The filtration. The weights below depend on the unnormalized process. Let $\mathcal{F}_s$ be the σ -algebra generated by g and $(B_r)_{r \leq s}$. The process B is an $(\mathcal{F}_s)$ -Brownian motion, because g is independent of B , and Y , Y^G , and $X = \sqrt{N}Y/|Y|$ are $(\mathcal{F}_s)$ -adapted. Itô's formula for $\sqrt{N}Y_i/|Y|$ shows that X solves the equation (11.2) driven by the same Brownian motion B , with noise $\sqrt{2}\Pi(X)dB$; this is the diffusion of the second part of Lemma 10.2. In particular, the setting of Section 11.1 holds with this filtration, and the event E of (17.36), which is determined by $Y(\tau)$, belongs to $\mathcal{F}_\tau$.

Lemma 17.22 (Mass, collar, and fourth-moment bounds for the entropy estimate). *Let a satisfy the indexed bounds (17.2) with $0 < \theta \leq 1/2$, fix $2 \leq J < N$, suppose that $b = a_{J+1} \geq 6a$, and use the parameters (17.47): $\mathfrak{a} = 1/10$, $\mathfrak{b} = \zeta = 1/100$, and $K_0 = 1$. Fix $0 < \ell_- < \ell_+$, let E and C_{res} be as in (17.36) and (17.40), and set*

$$s = \frac{1}{8}, \quad L_0 = C_{\text{res}} + 1, \quad L_1 = C_{\text{res}} + 2, \quad L = C_{\text{res}} + 3, \quad \delta = m^{-1/100}, \quad k = 16.$$

Fix the smooth transition

$$b_0(v) = \begin{cases} e^{-1/v}, & v > 0, \\ 0, & v \leq 0, \end{cases} \quad \psi(v) = \frac{b_0(v)}{b_0(v) + b_0(1-v)},$$

and use the cutoff (11.63) with

$$\chi_1(v) = \psi(16v - 1), \quad \chi_2(v) = 1 - \psi(v - C_{\text{res}} - 1), \quad \chi_3(v) = 1 - \psi(2v - 1).$$

Let D and σ_t be as in (17.42), let $\mathcal{C}$ be as in (11.64), and write $Z_t = \int \eta d\sigma_t$. There exist C and m_0 , depending only on J , $\ell_\pm$, c_ , and C_* , such that for $m \geq m_0$ and every $t \in [T + 1/a, T_{\text{max}}]$,*

$$Z_t \geq \mathbb{P}(E) - Cm^{-20}, \quad \sigma_t(\mathcal{C}) \leq Cm^{-20},$$

and, for $2 \leq j \leq J$,

$$\left(\int x_j^4 d\sigma_t \right)^{1/2} \leq C \{ \theta N e^{-4a_j(t-T)} + a_j^{-1} \}.$$

Proof. Apply Corollary 17.18 with $B_0 = 20$ and $p = 200$. Since $\alpha_f = b/a \geq 6$ and $\mathfrak{b} = \zeta = 1/100$, the example after that corollary bounds the cone term by Cm^{-22} . Thus $\sigma_t\{\xi > \delta/2\} \leq Cm^{-20}$. The cutoff η takes values in $[0, 1]$ and equals one outside $\mathcal{C}$, by the displayed definitions of χ_1, χ_2, χ_3 . The fourth part of Corollary 17.18 therefore gives both the mass and collar bounds. The fourth-moment bound is the second part of that corollary with $p = 2$ and $B_0 = 20$; its last term is absorbed into Ca_j^{-1} because $B_0 \geq p$. $\square$

We now integrate the conclusion of Theorem 11.19 over the time interval from $t_0 = T + 1/a$ to the times $t_m(u)$. The integration uses only (11.69), (11.70), and the absolute continuity of $\mathcal{H}$, and it is deterministic.

Corollary 17.23 (Scalar entropy bound at the times $t_m(u)$). *Fix $\mathfrak{b} \in (0, 1/8)$, $\zeta > 0$, $K > 0$, $C_0 > 0$, $\alpha_0 > \max\{2, 3/(1 - 4\mathfrak{b})\}$, and a compact interval $I = [u_-, u_+]$. There exist C and m_0 , depending only on these parameters, with the following property. Let $a > 0$ and $m \geq m_0$, let $T = \mathfrak{b} \log(em)/a$, $t_0 = T + 1/a$, $t_m(u) = \log(2m)/(4a) + u/a$, and $t_1 = t_m(u_+)$. Suppose that $\mathcal{H}, D : [t_0, t_1] \rightarrow \mathbb{R}$, that $\mathcal{H}$ is absolutely continuous, that $\mathcal{H}(t_0) \leq C_0 m^3$, and that, for every $t \in [t_0, t_1]$,*

$$0 \leq \mathcal{H}(t), \quad 0 \leq D(t), \quad 4a\alpha_0\mathcal{H}(t) \leq D(t),$$

and, for almost every $t \in [t_0, t_1]$,

$$\mathcal{H}'(t) \leq -D(t) + \frac{C_0 a}{m} + C_0 a m e^{-8a(t-T)} + C_0 a m^{5+2\zeta-K/2}.$$

Then, for every $u \in I$,

$$\mathcal{H}(t_m(u)) \leq E_m, \quad E_m = C[m^{3-\alpha_0(1-4\mathfrak{b})} + m^{-1+8\mathfrak{b}} + m^{5+2\zeta-K/2}]. \quad (17.54)$$

For example, with $\mathfrak{b} = \zeta = 1/100$, $K = 20$, and $\alpha_0 = 4$, the three exponents in (17.54) are $3 - 4 \cdot 24/25 = -21/25$, $-1 + 8/100 = -23/25$, and $5 + 2/100 - 10 = -249/50$, so that $E_m \leq Cm^{-4/5}$. To apply the corollary to Theorem 11.19, take $D = (1 - 2\varepsilon)\mathcal{D}$ and $\delta = m^{-\zeta}$. If $p_* \leq C_5 m^{-K}$, its conclusions give the scalar hypotheses, with a fixed C_0 , provided that

$$\alpha_m = \frac{(1 - 2\varepsilon)b}{a(1 + 2L_1\delta/s)} \geq \alpha_0. \quad (17.55)$$

The values in the example are used in Section 17.9; there $b \geq 6a$ and $\varepsilon = 1/16$, so that $\alpha_m \geq (7/8) \cdot 6/(1 + 2L_1 m^{-1/100}/s) \geq 4$ for large m .

Proof. Throughout, C denotes a constant depending only on the parameters listed in the statement. For m large, $t_m(u_-) > t_0$, since $a(t_m(u_-) - t_0) = \log(2m)/4 + u_- - \mathfrak{b} \log(em) - 1 \rightarrow \infty$. Using $D \geq 4a\alpha_0\mathcal{H}$ in the assumed differential inequality, we obtain, for almost every $t \in [t_0, t_1]$,

$$\mathcal{H}'(t) \leq -4a\alpha_0\mathcal{H}(t) + \frac{Ca}{m} + C a m e^{-8a(t-T)} + C a m^{5+2\zeta-K/2}.$$

Since $\mathcal{H}$ is absolutely continuous, we may multiply by $e^{4a\alpha_0 t}$ and integrate from t_0 to $t = t_m(u)$:

$$\mathcal{H}(t) \leq \mathcal{H}(t_0)e^{-4a\alpha_0(t-t_0)} + \int_{t_0}^t e^{-4a\alpha_0(t-r)} \left[\frac{Ca}{m} + Cam e^{-8a(r-T)} + Cam^{5+2\zeta-K/2} \right] dr.$$

We bound the four terms. By the initial bound, $\mathcal{H}(t_0) \leq Cm^3$, and $a(t-t_0) \geq \log(2m)/4 + u_- - \mathfrak{b} \log(em) - 1$, which bounds the first term by $Cm^{3-\alpha_0(1-4\mathfrak{b})}$. The constant forcing Ca/m contributes at most $C/(4\alpha_0 m)$, and the last forcing contributes at most $Cm^{5+2\zeta-K/2}$. For the third forcing, since $\alpha_0 > 2$,

$$Cam \int_{t_0}^t e^{-4a\alpha_0(t-r)} e^{-8a(r-T)} dr \leq \frac{Cm}{4\alpha_0 - 8} e^{-8a(t-T)} \leq Cm^{-1+8\mathfrak{b}},$$

because $e^{-8a(t-T)} = e^{-2\log(2m)-8u+8\mathfrak{b}\log(em)} \leq Cm^{-2+8\mathfrak{b}}$. The term C/m is at most $Cm^{-1+8\mathfrak{b}}$, and (17.54) follows. $\square$

17.9 Proof of the main results

The profile under the scaled-gap bounds

Proof of Theorem 17.3. For each n , the function $t \mapsto D_n(t)$ is nonincreasing and tends to zero as $t \rightarrow \infty$. This follows as in the proof of Corollary 3.12, since π is the reversible invariant measure of P_t and ν_N has a bounded density with respect to π . Discarding finitely many n , we may assume that $2m_n > 1$. Then the centers $b_n = \log(2m_n)/(4a_2^{(n)})$ and windows $w_n = 1/a_2^{(n)}$ are positive and satisfy $b_n/w_n = \log(2m_n)/4 \rightarrow \infty$, and $\mathcal{V}$ is continuous and strictly decreasing from one to zero (Lemma 3.9). It therefore suffices, by the deterministic version of Lemma 3.11, to prove that $D_n(t_{m_n}(u)) \rightarrow \mathcal{V}(u)$ for each fixed $u \in \mathbb{R}$. The lower bound $\liminf_n D_n(t_{m_n}(u)) \geq \mathcal{V}(u)$ is part of Theorem 17.19, and it remains to prove the upper bound.

We fix u , drop the index n , and write $t = t_m(u)$. We choose the parameters in the order of Table 17.1, and all thresholds m_0 below depend only on them and on u . All estimates below hold for n large, because $m \rightarrow \infty$.

Step 1: parameters. Let $J \geq 2$ be the smallest integer with $(c_*/C_*)J^{2/3} \geq 6$. Since $m \leq (C_*/2)N^{1/3}$, we have $N \rightarrow \infty$ and thus $J < N$ for large n . By (17.2), $b = a_{J+1} \geq c_*J^{2/3}N^{-2/3} \geq 6C_*N^{-2/3} \geq 6a$, so that $\alpha_J = b/a \geq 6$. We use the parameters (17.47), the exponent $\zeta = 1/100$ with $\delta = m^{-1/100}$, and the cutoff power $k = 16$. Fix a tolerance $\gamma \in (0, 1/4)$. Let $0 < \ell_- < \ell_+$ and $\varepsilon_0 \in (0, (\ell_+ - \ell_-)/2)$ be such that a standard Gaussian vector $(G_1, \dots, G_J)$ satisfies $\ell_- + \varepsilon_0 \leq |G_1| \leq \ell_+ - \varepsilon_0$ and $|G_i| \leq \ell_+ - \varepsilon_0$ for $2 \leq i \leq J$ with probability at least $1 - \gamma$. Let E and C_{res} be as in (17.36) and (17.40), let $s = 1/8$,

$L_0 = C_{\text{res}} + 1$, $L_1 = C_{\text{res}} + 2$, $L = C_{\text{res}} + 3$, and use the three fixed cutoff functions of Lemma 17.22. Let σ_t be the endpoint subprobability measure (17.42), weighted by $\mathbf{1}_E$ and killed on leaving D , and let η be the cutoff function (11.63).

Step 2: the mass removed by the event restriction and the cutoff. For $\alpha = 1/10$, the function $g(m)$ of Lemma 17.16 is at most $3m^{-1/20}$. That lemma gives

$$1 - \mathbb{P}(E) \leq \gamma + C_\gamma m^{-1/10}. \quad (17.56)$$

By Lemma 17.22, for every $t' \in [T + 1/a, T_{\max}]$,

$$\sigma_{t'}(C) \leq C_\gamma m^{-20}, \quad d_{t'} = 1 - Z_{t'} \leq \gamma + C_\gamma m^{-1/10}, \quad Z_{t'} \geq \frac{1}{2}. \quad (17.57)$$

Here C_γ depends on γ through $\ell_\pm$, and on the other parameters of Table 17.1. The fourth-moment bound of the lemma gives (11.68). For the density hypothesis (11.65), note that $T \geq \theta^{-2}$ for large m , since $\theta^2 T = \chi \theta^2 / a \geq \chi m^2 / C_*^3 \rightarrow \infty$, and apply Theorem 17.20 with $p = 2$. For (11.66), the indexed bounds imply the hypotheses of Proposition 16.9 with $B = C_*$, $D = 2^{2/3} C_* / c_*$, and $C_0 = 4(C_* / c_*)^2$. Thus $\sqrt{\theta} Z_G \geq 1/2$ for large m , and (16.14) gives, on $\{x_1^2 \geq s\theta N/4\}$,

$$\frac{d\pi^+}{dG} = \frac{(N/x_1^2)^{1/2}}{Z_G} \leq \frac{(s\theta/4)^{-1/2}}{Z_G} \leq 4s^{-1/2}.$$

All these are statements about the single family of measures $\sigma_{t'}$, which is dominated by the time- t' law.

Step 3: the remaining coordinates. We verify the setting of Section 11.5 directly. By Lemma 17.2, $N\theta^3 \leq c_*^{-3} m^3$ and $a_J \leq (C_*/c_*)(J-1)^{2/3}a$. The initial law is ν_N , the time T is deterministic, and $\Psi = \mathbf{1}_E$ is $\mathcal{F}_T$ -measurable because $\tau < T$. The set $D = \{|x_{\leq J}|^2 < L\theta N\}$ is invariant under R_1 , and (17.42) is precisely the law weighted by Ψ and killed after T in this set. The law of (g, B) is invariant under $(g, B) \mapsto (R_1 g, R_1 B)$; by pathwise uniqueness this transformation sends Y to $R_1 Y$, while preserving E , which depends only on $|Y_i(\tau)|$. Hence $A \mapsto \mathbb{E}[\mathbf{1}_E \mathbf{1}_{\{X_T \in A\}}]$ is invariant under R_1 as required. The preceding bounds now verify the hypotheses of Theorem 11.19 on $[t_0, t_1] = [T + 1/a, t_m(u + 1)]$, with $p_* = C_\gamma m^{-20}$ and $z_0 = 1/2$, for m large. Note that $t_m(u + 1) \leq T_{\max}$ for large m . Take $\varepsilon = 1/16$ and $D_{\text{sc}} = (1 - 2\varepsilon)\mathcal{D}$, and apply Corollary 17.23 with scalar dissipation $D = D_{\text{sc}}$, $I = [u - 1, u + 1]$, $\mathfrak{b} = \zeta = 1/100$, $K = 20$, $\alpha_0 = 4$, and a sufficiently large fixed C_0 depending on γ . The consumer gives the scalar differential and initial bounds, and (17.55) holds for large m by the remark after that corollary, since $\alpha_J \geq 6$. The corollary gives the entropy bound below. By (11.46), $\mathcal{H}(t)$ is the relative entropy of $w_t G$ with respect to $(\bar{w}_t G_Z) \otimes G_v$, so Pinsker's inequality (Lemma 3.6) gives the second bound:

$$\mathcal{H}(t) \leq C_\gamma m^{-4/5}, \quad \|w_t G - (\bar{w}_t G_Z) \otimes G_v\|_{\text{TV}} \leq C_\gamma m^{-2/5}. \quad (17.58)$$

The initial entropy Cm^3 , which comes from the density bound of Theorem 17.20, is removed by the contraction over the interval $[t_0, t]$, whose length is about $(1/4 - 1/100)a^{-1} \log m$, because $b \geq 6a$.

Step 4: the first J coordinates. Let μ_N be the law of $\mathbf{z}(X(t))$ under the time- t law itself, and let μ_u be as in (17.48). We verify the hypotheses of Corollary 11.12, along the sequence of gap vectors $a^{(n)}$. The bounds (11.35) hold with $C_J = (C_*/c_*)(J-1)^{2/3}$ and $m \rightarrow \infty$, and $t \geq 1/a$. By Theorem 17.19, $\mu_N \Rightarrow \mu_u$, and μ_u is the convolution of the law of $(e^{-2u}C, 0, \dots, 0)$ with the standard Gaussian law on $\mathbb{R}^{J-1}$. Lemma 11.7 then implies that $\text{FI}(\mu_u) \leq J-1$. Let $\delta' \in (0, 1)$ be the tolerance of Corollary 11.12, and choose $\gamma = \min\{\delta', 1/8\}$ in Step 1. Let $W = \mathbf{1}_E \mathbf{1}_{\{T_{\text{exit}} > t\}}$ and $\eta^0 = \eta$.

1. The weight W equals $\Phi(\xi', X|_{[t-1/a, t]})$, where $\xi' = \mathbf{1}_E \mathbf{1}_{\{X(r) \in D \text{ for } r \in [T, t-1/a]\}}$ is $\mathcal{F}_{t-1/a}$ -measurable (since $t - 1/a \geq T$) and $\Phi(e, \omega) = e \mathbf{1}_{\{\omega_r \in D \text{ for all } r\}}$. The set D is open, invariant under all rotations of $(x_1, \dots, x_J)$, and contained in $\{|x_{\leq J}|^2 < L\theta N\}$. A rotation of the path in the $(1, j)$ -plane, $2 \leq j \leq J$, preserves D , and therefore Φ is invariant under these rotations. The measure $A \mapsto \mathbb{E}[W \mathbf{1}_{\{X(t) \in A\}}]$ is σ_t , which is invariant under R_1 by the third part of Lemma 11.4, because D and σ_T are.
2. The function η is smooth with values in $[0, 1]$, depends on x_1 only through x_1^2 , and vanishes on $\{x_1^2 < (s/2)\theta N\}$. Then $|K_{1j}\eta| \leq C\eta^{1-1/k} \leq C\eta^{1/2}$ by (11.75).
3. $\mathbb{E}[W\eta(X(t))] = Z_t \geq 1 - \gamma - C_\gamma m^{-1/10}$, and its limit inferior is at least $1 - \delta'$.

Using Corollary 11.12 with $s/2$ in place of s , we therefore obtain

$$\|\mu_N - \mu_u\|_{\text{TV}} \longrightarrow 0. \quad (17.59)$$

This statement concerns the law of the first coordinates under the time- t law itself, not under the weighted measure σ_t .

Step 5: combination. We use Lemma 11.21 on $\mathbb{R}^{J-1} \times \mathbb{R}^{N-J}$, with $\rho = \rho_t^+$, the image of the time- t law under $x \mapsto y = (x_2, \dots, x_N)$, extended by zero outside the ball $\{|y|^2 < N\}$; with $\sigma = (\eta\sigma_t)^+$, which satisfies $\sigma \leq \rho$ and has mass Z_t ; with $G = G_z \otimes G_v$; with $\mu = \tilde{\mu}_u$, the law of $(x_2, \dots, x_J) = (\mathbf{z}_j / \sqrt{2a_j})_{2 \leq j \leq J}$ when $\mathbf{z} \sim \mu_u$; and with π^+ in the role of π . Note that the relative entropy H_v of the lemma is $\mathcal{H}(t)$, by (11.46). As the standardization $z \mapsto \mathbf{z}$ is a bijection, we have $\|\rho_{t,z}^+ - \tilde{\mu}_u\|_{\text{TV}} = \|\mu_N - \mu_u\|_{\text{TV}}$ and $\|\tilde{\mu}_u - G_z\|_{\text{TV}} = \|\mu_u - \mathcal{N}(0, I_{J-1})\|_{\text{TV}} = \mathcal{V}(u)$, where the last identity holds because μ_u and $\mathcal{N}(0, I_{J-1})$ differ only in the first coordinate, by a common independent factor. By (11.30), $\|\rho_t^+ - \pi^+\|_{\text{TV}} = \|\rho_t - \pi\|_{\text{TV}} = D(t)$, because ρ_t and π are invariant under R_1 (Lemma 3.1). Recall from Proposition 16.9 that $\|\pi^+ - G\|_{\text{TV}} \leq C/m + Ce^{-c_1 m}$, with C and $c_1 > 0$ depending only on c_* and C_* . Inserting these bounds and (17.58) into (11.85), we obtain

$$|D(t_m(u)) - \mathcal{V}(u)| \leq 2d_t + C_\gamma m^{-2/5} + \|\mu_N - \mu_u\|_{\text{TV}} + \frac{C}{m} + Ce^{-c_1 m}. \quad (17.60)$$

The limit superior of the right side is at most 2γ by (17.57) and (17.59). Since $\delta' \in (0, 1)$ is arbitrary and $\gamma \leq \delta'$, this proves $D_n(t_{m_n}(u)) \rightarrow \mathcal{V}(u)$. $\square$

The proof uses only finite bounds in which the tolerance γ and the parameters of Table 17.1 are fixed before $m \rightarrow \infty$, together with two convergences along the sequence: the weak convergence of Theorem 17.19, and its upgrade (17.59). No step exchanges these limits.

Uniformity and the GOE

The next lemma converts convergence along every sequence of vectors in $\mathcal{A}_N$ into the uniform statement (17.7) and into convergence in probability for the GOE.

Lemma 17.24 (From sequences to uniform bounds). *Fix $0 < c_* \leq C_*$ and $\varepsilon_* > 0$, and for each $N \geq 3$ let $Q_N : \mathcal{A}_N(c_*, C_*, \varepsilon_*) \rightarrow [0, \infty]$. Suppose that $Q_{N_n}(a^{(n)}) \rightarrow 0$ for every sequence $a^{(n)} \in \mathcal{A}_{N_n}(c_*, C_*, \varepsilon_*)$ with $\theta_n > 0$, $\theta_n \rightarrow 0$, and $m_n \rightarrow \infty$. Then*

$$\lim_{M \rightarrow \infty} \limsup_{N \rightarrow \infty} \sup \{Q_N(a) : a \in \mathcal{A}_N(c_*, C_*, \varepsilon_*), 0 < \theta \leq 1/M, m \geq M\} = 0,$$

where the supremum of the empty set is zero.

Proof. Suppose the conclusion fails. Since the quantity inside the limit in M is nonincreasing in M , there exists $\varepsilon > 0$ such that, for every M , the limit superior in N exceeds ε . We can therefore choose $M_l \rightarrow \infty$, dimensions $N_l \rightarrow \infty$, and vectors $a^{(l)} \in \mathcal{A}_{N_l}$ with $0 < \theta_l \leq 1/M_l$, $m_l \geq M_l$, and $Q_{N_l}(a^{(l)}) > \varepsilon$. This sequence satisfies the hypothesis, and $Q_{N_l}(a^{(l)}) \not\rightarrow 0$, a contradiction. $\square$

Proof of Corollary 17.4. We use Lemma 17.24 with $Q_N(a) = \sup\{|D_a(t_m(u)) - \mathcal{V}(u)| : u \in I, t_m(u) \geq 0\}$ when $\theta > 0$, and $Q_N(a) = 0$ otherwise; the supremum over an empty set is also set to 0. Here D_a is the distance for the vector a . The hypothesis of the lemma is Theorem 17.3, because along a sequence with $m_n \rightarrow \infty$ the times $t_{m_n}(u)$, $u \in I$, are positive for large n . If $\log(2M) + 4 \min I > 0$, then all times in the definition of $\mathcal{E}_N(M)$ are positive, and $\mathcal{E}_N(M)$ is at most the supremum in the lemma. This proves (17.7). For (17.8), fix $M \geq 1$ with $\log(2M) + 4 \min I > 0$. For every S with $m_S > M$, we have $\theta_{N,S} \leq 1/M$ and $m_{N,S} \geq M$ for all large N , so that the error at (N, S) is at most $\mathcal{E}_N(M)$, and hence

$$\limsup_{N \rightarrow \infty} \sup_{u \in I} |D_{N,S}(t_{m_{N,S}}(u)) - \mathcal{V}(u)| \leq \limsup_{N \rightarrow \infty} \mathcal{E}_N(M).$$

This holds for all large S , because $m_S \rightarrow \infty$. Letting $S \rightarrow \infty$ and then $M \rightarrow \infty$ in (17.7), we obtain (17.8). $\square$

The same argument applies to random families. Let the vectors $a^{(N,S)}$ be random, and let $e_{N,S} = Q_N(a^{(N,S)})$, with the preceding nonnegative-time and empty-supremum

conventions. Suppose that for every $\eta > 0$ there exist constants c_*, C_*, ε_* , not depending on N and S , and events $\mathcal{G}_{N,S}^\eta$ on which $a^{(N,S)} \in \mathcal{A}_N(c_*, C_*, \varepsilon_*)$, such that

$$\limsup_{S \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{P}((\mathcal{G}_{N,S}^\eta)^c) \leq \eta,$$

and suppose that for every fixed $M \geq 1$,

$$\lim_{S \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{P}(\theta_{N,S} \leq 0 \text{ or } \theta_{N,S} > 1/M \text{ or } m_{N,S} < M) = 0.$$

Then $\lim_{S \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{P}(e_{N,S} > r) = 0$ for every $r > 0$. Indeed, fix $r > 0$ and $\eta > 0$, and use (17.7), with the constants for η , to choose M with $\log(2M) + 4 \min I > 0$ and $\limsup_N \mathcal{E}_N(M) < r$. On $\mathcal{G}_{N,S}^\eta \cap \{0 < \theta_{N,S} \leq 1/M, m_{N,S} \geq M\}$, we have $e_{N,S} \leq \mathcal{E}_N(M) < r$ for all large N , and therefore $\limsup_S \limsup_N \mathbb{P}(e_{N,S} > r) \leq \eta$. Then let $\eta \rightarrow 0$. This is a statement in probability, and the limit in N is taken before the limit in S .

Proof of Theorem 17.5. By Lemma 4.7, case (M), we have $\mathbb{P}(\theta_N > 0) \rightarrow 1$, $\theta_N \rightarrow 0$, and $m_N \rightarrow \infty$ in probability. In particular, the event $\mathcal{L}_N = \{\theta_N > 0, 2m_N > 1\}$ has probability tending to one, and $a_2 T_N = \log(2m_N)/4$ on $\mathcal{L}_N$, which implies that $a_2 T_N \rightarrow \infty$ in probability. In the eigencoordinates, the model is the diffusion (17.1) with $a_i = c_N(\lambda_1 - \lambda_i)/2$, by (3.8), and $D_N(t)$ is the distance $D(t)$ of this vector a . Fix $\eta \in (0, 1)$, and let $\mathcal{G}_N^\eta$ be the events of the second part of Lemma 4.7, with constants c_*, C_*, ε_* depending only on η ; on $\mathcal{G}_N^\eta$, the vector a belongs to $\mathcal{A}_N(c_*, C_*, \varepsilon_*)$ for all large N . Fix a compact interval I . The assertion is immediate when $I = \emptyset$; assume $I \neq \emptyset$. Fix $\varepsilon > 0$, and let $e_N = \sup_{u \in I} |D_N(T_N + u/a_2) - \mathcal{V}(u)|$, with the convention $D_N(t) = D_N(0)$ for $t < 0$. By Corollary 17.4, there exists $M \geq \exp(4 \max_I |u|)$ with $\limsup_N \mathcal{E}_N(M) < \varepsilon$. On the event $\{0 < \theta_N \leq 1/M, m_N \geq M\}$, which is contained in $\mathcal{L}_N$, we have $T_N + u/a_2 = t_{m_N}(u) > 0$ for $u \in I$. Then $e_N \leq \mathcal{E}_N(M) < \varepsilon$ on $\mathcal{G}_N^\eta \cap \{0 < \theta_N \leq 1/M, m_N \geq M\}$ for all large N , and we obtain

$$\limsup_N \mathbb{P}(\mathcal{G}_N^\eta \cap \{e_N \geq \varepsilon\}) \leq \limsup_N \mathbb{P}(\theta_N \leq 0 \text{ or } \theta_N > 1/M \text{ or } m_N < M) = 0,$$

and Lemma 3.8 shows that $e_N \rightarrow 0$ in probability. The center T_N and the window a_2^{-1} are positive, and $a_2^{-1}/T_N = (a_2 T_N)^{-1} \rightarrow 0$ in probability. Since the supremum in Definition 3.7 is over a subset of I , this proves that D_N has cutoff profile $\mathcal{V}$ with center T_N and window a_2^{-1} . $\square$

Mixing times

Proof of Corollary 17.6. For the first part, we discard the finitely many n with $2m_n \leq 1$ and apply the deterministic version of Lemma 3.11 with $b_n = T_n$, $w_n = 1/a_2^{(n)}$, and $V = \mathcal{V}$; its hypotheses were checked at the beginning of the proof of Theorem 17.3. Its second part

gives the uniform convergence of $a_2^{(n)}(t_n(\varepsilon) - T_n)$ to $\mathcal{V}^{-1}(\varepsilon)$ on compact subsets of $(0, 1)$. Since $(1 - \eta)T_n = T_n - w_n \cdot \eta \log(2m_n)/4$ and $\eta \log(2m_n)/4 \rightarrow \infty$, its third part implies that $D_n((1 - \eta)T_n) \rightarrow 1$. For every fixed u_0 , we have $(1 + \eta)T_n \geq T_n + u_0 w_n$ for large n , so that $\limsup_n D_n((1 + \eta)T_n) \leq \lim_n D_n(T_n + u_0 w_n) = \mathcal{V}(u_0)$, and $\mathcal{V}(u_0) \rightarrow 0$ as $u_0 \rightarrow \infty$. The identity $(a_2^{(n)}T_n)^{-1} = 4/\log(2m_n)$ is the definition of T_n . For $0 < \varepsilon_1 < \varepsilon_2 < 1$, the uniform convergence implies that $a_2^{(n)}(t_n(\varepsilon_1) - t_n(\varepsilon_2)) \rightarrow \mathcal{V}^{-1}(\varepsilon_1) - \mathcal{V}^{-1}(\varepsilon_2)$, which is positive because $\mathcal{V}$ is strictly decreasing.

For the second part, Theorem 17.5 and Corollary 3.12 give the convergence of $a_2(t_N(\varepsilon) - T_N)$ in probability, uniformly for ε in compact subsets of $(0, 1)$, and cut-off. They also give $(a_2 T_N)^{-1} \rightarrow 0$ in probability, and $(a_2 T_N)^{-1} = 4/\log(2m_N)$ on $\mathcal{L}_N$ by the definition of T_N . Let $\omega_N = \eta a_2 T_N$, which tends to infinity in probability. The identities $(1 \pm \eta)T_N = T_N \pm \omega_N/a_2$ hold everywhere, and $(1 - \eta)T_N \geq 0$. The third part of Lemma 3.11, applied with ω_N , implies that $D_N((1 - \eta)T_N) \rightarrow 1$ in probability. For the last limit, fix u_0 and $\varepsilon > 0$. Since D_N is nonincreasing on $\mathbb{R}$, we have $D_N((1 + \eta)T_N) \leq D_N(T_N + u_0/a_2)$ on $\{\omega_N \geq u_0\}$, and

$$\begin{aligned} \mathbb{P}(D_N((1 + \eta)T_N) > \mathcal{V}(u_0) + \varepsilon) &\leq \mathbb{P}(\omega_N < u_0) \\ &\quad + \mathbb{P}(|D_N(T_N + u_0/a_2) - \mathcal{V}(u_0)| > \varepsilon) \rightarrow 0, \end{aligned}$$

by Theorem 17.5. Finally, $\mathcal{V}(u_0) \rightarrow 0$ as $u_0 \rightarrow \infty$. The limit of $a_2(t_N(\varepsilon_1) - t_N(\varepsilon_2))$ follows as in the first part. $\square$

PART V

The critical window

The critical limit process and the limiting Gibbs law

In this section and the next, we study the dynamics in the critical window $c_N = 1 + \sigma N^{-1/3}$ on the time scale $N^{2/3}$. The goal of this section is to construct the critical limit process and the limiting Gibbs law, in terms of which Section 19 expresses the critical profile F_σ , and to prove that the rescaled dynamics started from ν_N converge to its normalization. Lemma 18.5 identifies the limit ψ_R of the mean of the rescaled squared norm of the unnormalized process; we call ψ_R the mean function. Theorem 18.9 constructs the critical limit process, a countable system of stochastic differential equations whose coordinates share a common random noise amplitude, and shows that finitely many rescaled eigencoordinates of the Langevin diffusion started from ν_N converge to the corresponding normalized coordinates $\xi_i = U_i/\sqrt{Z}$. Corollary 18.12 identifies the limit ρ_R of the rescaled Gibbs measures, a Gaussian sequence conditioned on a renormalized radial constraint. The uniqueness statement behind Theorem 18.9 is proved in Lemma 18.8. In Section 19, the proof of the results on the critical profile uses the critical limit process of Theorem 18.9 and the limiting Gibbs law, and the continuity, positivity, and strict decrease of the profile are proved there with estimates for finite-dimensional semigroups.

We use the unnormalized process of Lemma 10.2, the heat matrix (7.13) and its estimates in Lemma 7.14, the moment bounds of Lemma 10.10, and the countable conditioned Gaussian laws of Section 5. All results of this section are deterministic statements about a fixed sequence of half-precisions satisfying the hypotheses of Definition 18.1 below; probabilities refer only to the initial point and the driving Brownian motions. For the GOE and every fixed $\eta > 0$, Lemma 4.17 gives constants c_0, C_0 and events $\mathcal{A}_N$ with $\liminf_N \mathbb{P}(\mathcal{A}_N) \geq 1 - \eta$ on which the trace condition and the Weyl bounds hold. The inverse scaled half-precisions converge in distribution in ℓ^2 , and their limits have the Weyl asymptotic almost surely. Section 19 applies the deterministic results of this section to the GOE through a localized form of the continuous mapping theorem (Lemma 19.23). The results of this section that Section 19 uses are Lemma 18.5, Theorem 18.9, and Corollary 18.12, together with the estimates in their proofs; Lemma 18.5 and Corollary 18.12 are also used in Sections 20 and 21.

Before entering into details, we describe the critical limit process informally. Its coordinates $U_i, i \geq 1$, solve

$$dU_i = -2R_i U_i ds + \sqrt{2Z} dB_i, \quad U_i(0) = 0,$$

where the B_i are independent Brownian motions and the noise amplitude $\sqrt{2Z}$ is common to all coordinates. The ratios $\xi_i = U_i/\sqrt{Z}$ are the limits of the rescaled eigencoordinates $N^{-1/3}X_i(N^{2/3}s)$. In finite dimension, Z corresponds to the rescaled squared norm $Z_N = N^{-1/3}\sum_i U_{i,N}^2$ of the unnormalized process. We call Z the rescaled squared-norm process. Passing to the limit in this identity coordinate by coordinate is not justified. In the limit, Z is determined by the stochastic Volterra equation (18.51) below. The start is singular: every coordinate starts at zero, while $Z(s)$ is of order $s^{-1/2}$ as $s \downarrow 0$ and the weighted norm $\sum_i U_i(s)^2/R_i$ tends to 2. We explain this in Remark 18.10.

18.1 Scaled half-precisions and the rescaled unnormalized process

Half-precisions. Throughout this section, N ranges over an infinite set $\mathcal{N}$ of integers $N \geq 2$, and limits as $N \rightarrow \infty$ are taken along $\mathcal{N}$. For each $N \in \mathcal{N}$, let $0 < r_{1,N} \leq \dots \leq r_{N,N}$. The main example is given by the half-precisions $r_{i,N} = p_i/2 = (z - \alpha_i)/2$ of a symmetric matrix A with eigenvalues $\alpha_1 \geq \dots \geq \alpha_N$ (Section 3). Then $\sum_i (2r_{i,N})^{-1} = \sum_i q_i = N$, and on S_N we have $\sum_i r_{i,N} x_i^2 = zN/2 - x^\top A x/2$. The first term is constant on the sphere, so that the generator L_A and the Gibbs measure π_A are those of the potential $\sum_i r_{i,N} x_i^2$:

$$L_A = \Delta_{S_N} - \nabla_{S_N} \left(\sum_i r_{i,N} x_i^2 \right) \cdot \nabla_{S_N}, \quad \pi_A(dx) \propto e^{-\sum_i r_{i,N} x_i^2} \nu_N(dx). \quad (18.1)$$

Conversely, every positive vector $(r_{i,N})$ with $\sum_i (2r_{i,N})^{-1} = N$ is of this form, with $A = -2 \operatorname{diag}(r_{i,N})$ and Lagrange multiplier $z = 0$. We use (18.1) to define L_A and π_A for every such vector. We write $R_{i,N} = N^{2/3} r_{i,N}$ for the scaled half-precisions, and we set $R_{i,N}^{-1} = 0$ for $i > N$.

Definition 18.1 (Critical sequences). Let $0 < c_0 < C_0 < \infty$, and let $R = (R_i)_{i \geq 1}$ be a sequence of positive numbers. We say that $(r_{i,N})_{i \leq N, N \in \mathcal{N}}$ is a critical sequence with constants (c_0, C_0) and limit R if the following three conditions hold.

1. For every $N \in \mathcal{N}$, we have the trace condition

$$\sum_{i=1}^N \frac{1}{2r_{i,N}} = N. \quad (18.2)$$

2. For every $N \in \mathcal{N}$, we have the Weyl bounds

$$c_0 \left(\frac{i}{N} \right)^{2/3} \leq r_{i,N} \leq C_0 \left(\frac{i}{N} \right)^{2/3}, \quad 1 \leq i \leq N; \quad (18.3)$$

equivalently, $c_0 i^{2/3} \leq R_{i,N} \leq C_0 i^{2/3}$ for $1 \leq i \leq N$.

3. The inverse scaled half-precisions converge in ℓ^2 , and the limit has a Weyl asymptotic:

$$\lim_{N \rightarrow \infty} \sum_{i \geq 1} |R_{i,N}^{-1} - R_i^{-1}|^2 = 0, \quad \lim_{i \rightarrow \infty} \frac{R_i}{i^{2/3}} = \gamma \quad \text{for some } \gamma \in (0, \infty). \quad (18.4)$$

Let $(r_{i,N})$ be a critical sequence with limit R . Since the inverses converge coordinate-wise, the Weyl bounds pass to the limit, and R is nondecreasing with

$$c_0 i^{2/3} \leq R_i \leq C_0 i^{2/3}, \quad i \geq 1. \quad (18.5)$$

In the scaled variables, the trace condition (18.2) reads $\sum_i R_{i,N}^{-1} = 2N^{1/3}$. We have $\sum_i R_{i,N}^{-2} \leq c_0^{-2} \sum_{i \geq 1} i^{-4/3} \leq 4c_0^{-2}$ by the Weyl bounds, and the same bound holds for R . In particular, $\sum_i R_i^{-2} < \infty$, while $\sum_i R_i^{-1} = \infty$ by (18.5).

The critical window. For the model with $c_N = 1 + \sigma N^{-1/3}$, we take $A = c_N W_N$ and $r_{i,N} = p_i/2$, with the Lagrange multiplier $z = z_N(c_N)$. Lemma 4.17 shows that for every $\eta > 0$ there exist constants c_0, C_0 and events of probability at least $1 - \eta$ for all large N on which (18.2) and (18.3) hold, and that the inverse scaled half-precisions converge in distribution in ℓ^2 to $((R_i^\sigma)^{-1})_{i \geq 1}$, where

$$R_1^\sigma = \frac{\varrho_\sigma}{2}, \quad R_i^\sigma = \frac{\Delta_i + \varrho_\sigma}{2} \quad (i \geq 2).$$

Here $\Delta_i = \chi_i - \chi_1$, where (χ_i) is the reflected GOE Airy point process, the limit of $N^{2/3}(2 - \lambda_i)$, and ϱ_σ is the root of $\mathcal{S}(\varrho_\sigma) = \sigma$ for the function $\mathcal{S}$ of (4.29). Almost surely, $\chi_i / (3\pi i/2)^{2/3} \rightarrow 1$ by [62, Proposition 1.4(a)], so that the second assertion of (18.4) holds with $\gamma = \gamma_{\text{GOE}} = \frac{1}{2}(3\pi/2)^{2/3}$. Section 19 transfers the deterministic results of this section to these random sequences with a localized form of the continuous mapping theorem (Lemma 19.23), using the critical spectral conditions with fixed constants at each confidence level. We write $\rho_\sigma = \rho_{R^\sigma}$ for the limiting Gibbs law of Corollary 18.12 in this case.

The rescaled unnormalized process. Let $B_1, \dots, B_N$ be independent standard Brownian motions with respect to a filtration $(\mathcal{F}_t)_{t \geq 0}$, and let Y be the unnormalized process of Lemma 10.2 with the quadratic potential coefficients $\rho_i = r_{i,N}$:

$$dY_i = -2r_{i,N}Y_i dt + \sqrt{2/N}|Y| dB_i, \quad 1 \leq i \leq N, \quad (18.6)$$

with an $\mathcal{F}_0$ -measurable initial vector $Y(0) \neq 0$. Parts (a) and (b) of Lemma 10.2, which allow any $\mathcal{F}_0$ -measurable initial vector that is nonzero almost surely, and (18.1) imply that (18.6) has a unique strong solution, that Y never vanishes, and that $X = \sqrt{N}Y/|Y|$ is the Langevin diffusion with generator L_A started from the law of $\sqrt{N}Y(0)/|Y(0)|$. If $Y(0) = g$ is a standard Gaussian vector, then X starts from ν_N . We rescale time by $N^{2/3}$ and define, for $s \geq 0$,

$$U_{i,N}(s) = N^{-1/6}Y_i(N^{2/3}s), \quad Z_N(s) = N^{-2/3}|Y(N^{2/3}s)|^2, \quad B_i^N(s) = N^{-1/3}B_i(N^{2/3}s). \quad (18.7)$$

The B_i^N are independent standard Brownian motions with respect to the filtration $\mathcal{F}_s^N = \mathcal{F}_{N^{2/3}s}$. For $i > N$, we set $U_{i,N} = 0$ and let B_i^N be further independent Brownian motions, independent of everything else. The rescaled eigencoordinates of the Langevin diffusion are

$$N^{-1/3}X_i(N^{2/3}s) = \frac{U_{i,N}(s)}{\sqrt{Z_N(s)}}. \quad (18.8)$$

Let $H = -4 \operatorname{diag}(r_{i,N}) + \frac{2}{N} \mathbf{1}\mathbf{1}^\top$ be the heat matrix (7.13); this is the matrix $-2 \operatorname{diag}(p) + \frac{2}{N} \mathbf{1}\mathbf{1}^\top$ of Section 7 with $p_i = 2r_{i,N}$. Let $h(t) = e^{tH} \mathbf{1}$. We define the scaled kernels and the scaled mean by

$$k_{i,N}(s) = N^{-1/3}h_i(N^{2/3}s), \quad \psi_N(s) = N^{-2/3} \sum_{j=1}^N h_j(N^{2/3}s) = N^{-1/3} \sum_{j=1}^N k_{j,N}(s). \quad (18.9)$$

In terms of the mean $\bar{h}(t) = N^{-1} \sum_j h_j(t)$ of Section 7, we have $\psi_N(s) = N^{1/3} \bar{h}(N^{2/3}s)$. For $i > N$, we set $k_{i,N} = 0$; all sums and suprema involving $R_{i,N}k_{i,N}$ are over $1 \leq i \leq N$. We recall from Section 7 that H is symmetric, has nonnegative off-diagonal entries, and is negative semidefinite with kernel spanned by $(r_{i,N}^{-1})_{i \leq N}$; that e^{tH} has nonnegative entries; and that $t \mapsto \mathbf{1}^\top e^{tH} \mathbf{1}$ is completely monotone, being the Laplace transform of the spectral measure of $-H$ at $\mathbf{1}$.

Table 18.1 lists the objects of this section, in finite dimension and in the limit.

Object	Finite dimension	Limit
scaled half-precisions	$R_{i,N} = N^{2/3}r_{i,N}$	R_i , (18.4)
coordinates	$U_{i,N}(s) = N^{-1/6}Y_i(N^{2/3}s)$	$U_i(s)$, Theorem 18.9
rescaled squared norm	$Z_N(s) = N^{-2/3} Y(N^{2/3}s) ^2$	$Z(s) = \psi_R(s) + V(s)$
mean function	$\psi_N(s) = \mathbb{E}Z_N(s)$ for $Y(0) = g$	$\psi_R(s)$, Lemma 18.5
kernels	$k_{i,N}(s) = N^{-1/3}h_i(N^{2/3}s)$	$k_i(s)$, (18.28)
normalized coordinates	$N^{-1/3}X_i(N^{2/3}s)$	$\zeta_i(s) = U_i(s)/\sqrt{Z(s)}$
Gibbs law	law of $(N^{-1/3}x_i)_{i \leq N}$ under π_A	ρ_R , Corollary 18.12

Table 18.1. The rescaled objects of Section 18. Times s are in units of $N^{2/3}$.

The following lemma collects the exact identities satisfied by the rescaled process. It requires only the trace condition.

Lemma 18.2 (Exact identities). *Assume (18.2), and let $Y(0)$ be an $\mathcal{F}_0$ -measurable random vector with $Y(0) \neq 0$ and $\mathbb{E}|Y(0)|^{2q} < \infty$ for every $q \geq 1$. Then the following hold for all $s \geq 0$.*

1. We have $\sum_i U_{i,N}(s)^2 = N^{1/3}Z_N(s)$, and

$$dU_{i,N} = -2R_{i,N}U_{i,N} ds + \sqrt{2Z_N} dB_i^N. \quad (18.10)$$

2. For every $s_0 \geq 0$,

$$\begin{aligned} Z_N(s_0 + s) &= \sum_{i=1}^N k_{i,N}(s) U_{i,N}(s_0)^2 \\ &\quad + \sqrt{8} \sum_{i=1}^N \int_0^s k_{i,N}(s-v) \sqrt{Z_N(s_0+v)} U_{i,N}(s_0+v) dB_i^N(s_0+v). \end{aligned} \quad (18.11)$$

3. The weighted norm $\|U_N(s)\|_{R_N}^2 = \sum_i U_{i,N}(s)^2 / R_{i,N}$ is a martingale, and

$$\|U_N(s)\|_{R_N}^2 = \|U_N(0)\|_{R_N}^2 + \sqrt{8} \sum_{i=1}^N \int_0^s R_{i,N}^{-1} \sqrt{Z_N(v)} U_{i,N}(v) dB_i^N(v). \quad (18.12)$$

4. The kernels satisfy

$$k_{i,N}(s) = N^{-1/3} e^{-4R_{i,N}s} + 2 \int_0^s e^{-4R_{i,N}(s-v)} \psi_N(v) dv, \quad \sum_{i=1}^N \frac{k_{i,N}(s)}{R_{i,N}} = 2. \quad (18.13)$$

5. If $Y(0) = g$ is a standard Gaussian vector, then $\psi_N(s) = \mathbb{E} Z_N(s)$, and

$$Z_N(s) = \psi_N(s) + I_N(s) + \Theta_N(s), \quad (18.14)$$

where

$$\begin{aligned} I_N(s) &= N^{-1/3} \sum_{i=1}^N k_{i,N}(s) (g_i^2 - 1), \\ \Theta_N(s) &= \sqrt{8} \sum_{i=1}^N \int_0^s k_{i,N}(s-v) \sqrt{Z_N(v)} U_{i,N}(v) dB_i^N(v). \end{aligned} \quad (18.15)$$

Proof. We work in the unscaled time t and write $Q_i = Y_i^2$ and $\mathcal{A} = |Y|^2 = \sum_i Q_i$; we use the letter $\mathcal{A}$ to distinguish the squared norm from the matrix A . Since the coefficients of (18.6) are globally Lipschitz, we have $\mathbb{E} \sup_{t \leq T} |Y(t)|^{2q} < \infty$ for all q and T . Every stochastic integral below with a fixed terminal time is square integrable and has mean zero, and the stochastic integrals in (18.12) are true martingales. By Itô's formula,

$$dQ_i = \left(-4r_{i,N} Q_i + \frac{2}{N} \mathcal{A} \right) dt + 2\sqrt{2/N} |Y| Y_i dB_i, \quad (18.16)$$

that is, $dQ = HQ dt + dM$ with $dM_i = 2\sqrt{2/N} |Y| Y_i dB_i$. Itô's product rule applied to $e^{-tH} Q(t)$ gives, for $t_0 \geq 0$ and $t \geq 0$,

$$Q(t_0 + t) = e^{tH} Q(t_0) + \int_{t_0}^{t_0+t} e^{(t_0+t-u)H} dM(u).$$

We multiply by $\mathbf{1}^\top$ and use the symmetry of H , which gives $\mathbf{1}^\top e^{tH} = h(t)^\top$. This yields

$$\mathcal{A}(t_0 + t) = \sum_i h_i(t) Q_i(t_0) + 2\sqrt{2/N} \sum_i \int_{t_0}^{t_0+t} h_i(t_0 + t - u) |Y(u)| Y_i(u) dB_i(u).$$

We now substitute $t_0 = N^{2/3}s_0$, $t = N^{2/3}s$, $u = N^{2/3}(s_0 + v)$, and use $|Y| = N^{1/3}\sqrt{Z_N}$, $Y_i = N^{1/6}U_{i,N}$, $h_i = N^{1/3}k_{i,N}$, and $dB_i(N^{2/3}\cdot) = N^{1/3}dB_i^N$. The powers of N in the stochastic integral are $N^{-2/3} \cdot N^{-1/2} \cdot N^{1/3} \cdot N^{1/3} \cdot N^{1/6} \cdot N^{1/3} = 1$, and those in the first term are $N^{-2/3} \cdot N^{1/3} \cdot N^{1/3} = 1$. This proves (18.11). The identity $\sum_i U_{i,N}^2 = N^{1/3}Z_N$ and (18.10) follow from the definitions in the same way.

Since $(r_{i,N}^{-1})$ spans the kernel of the symmetric matrix H , we have $\sum_i r_{i,N}^{-1}(HQ)_i = 0$. Then $\sum_i Q_i/r_{i,N}$ is a martingale with differential $\sum_i r_{i,N}^{-1}dM_i$, and rescaling gives (18.12). For the same reason, $\frac{d}{dt} \sum_i h_i(t)/r_{i,N} = 0$. Hence $\sum_i h_i(t)/r_{i,N} = \sum_i r_{i,N}^{-1} = 2N$, where we used (18.2). After rescaling, we obtain the second identity in (18.13). The i th coordinate of $h' = Hh$ reads $h'_i = -4r_{i,N}h_i + \frac{2}{N} \sum_j h_j$, with $h_i(0) = 1$; solving this linear equation and rescaling gives the first identity in (18.13).

Finally, let $Y(0) = g$. Taking expectations in the identity for $\mathcal{A}$ with $t_0 = 0$, and using $\mathbb{E}g_i^2 = 1$, we find $\mathbb{E}\mathcal{A}(t) = \sum_i h_i(t)$, that is, $\mathbb{E}Z_N(s) = \psi_N(s)$. With $s_0 = 0$, the first term of (18.11) is $N^{-1/3} \sum_i k_{i,N}(s)g_i^2 = \psi_N(s) + I_N(s)$, which proves (18.14). $\square$

We use (18.11) with $s_0 = 0$ for the process started from ν_N .

We use the following elementary sums repeatedly. Here and below, C denotes a large constant and c_* a small positive constant; they depend only on the parameters indicated, and their values may change from line to line.

Lemma 18.3 (Sums over the scaled half-precisions). *Let $(R_i)_{i \in I}$ be positive numbers, indexed by $I = \{1, \dots, N\}$ or $I = \{1, 2, \dots\}$, with $R_i \geq c_0 i^{2/3}$ for all $i \in I$. There exists $C = C(c_0)$ such that for all $t, h > 0$,*

$$\sum_{i \in I} \frac{e^{-4R_i t}}{R_i} \leq Ct^{-1/2}, \quad \sum_{i \in I} e^{-4R_i t} \leq Ct^{-3/2}, \quad (18.17)$$

$$\sum_{i \in I} \frac{1}{(1 + R_i t)^2} \leq Ct^{-3/2}, \quad \sum_{i \in I} \frac{1}{R_i(1 + R_i t)^2} \leq Ct^{-1/2}, \quad (18.18)$$

$$\sum_{i \in I} \frac{\min\{1, R_i h\}^2}{R_i^2} \leq C\sqrt{h}, \quad \sum_{i \in I} \frac{\min\{h, R_i^{-1}\}}{R_i} \leq C\sqrt{h}. \quad (18.19)$$

Moreover, for every $p > 2$ there exists $C = C(c_0, p)$ such that for all $J \geq 1$,

$$\sum_{i > J} \frac{\log(2 + R_i)}{R_i^3} \leq C \frac{\log(2 + J)}{J}, \quad \sum_{i > J} R_i^{-2+1/p} \leq CJ^{-1/3+2/(3p)}. \quad (18.20)$$

Proof. Each summand has the form $f(R_i)$ for a nonnegative nonincreasing function f on $(0, \infty)$. Using $R_i \geq c_0 i^{2/3}$, we have

$$\sum_{i \in I} f(R_i) \leq \sum_{i \geq 1} f(c_0 i^{2/3}) \leq \int_0^\infty f(c_0 x^{2/3}) dx = \frac{3}{2} c_0^{-3/2} \int_0^\infty f(y) y^{1/2} dy,$$

where we substituted $y = c_0 x^{2/3}$. For the six functions in (18.17)–(18.19), the last integral equals, respectively, $\sqrt{\pi} (4t)^{-1/2}$, $(\sqrt{\pi}/2)(4t)^{-3/2}$, $\frac{\pi}{2} t^{-3/2}$, $\frac{\pi}{2} t^{-1/2}$, $\frac{8}{3} \sqrt{h}$, and $4\sqrt{h}$. For the last two, we split the integral at $y = 1/h$. The bounds (18.20) follow in the same way from $\sum_{i > J} f(c_0 i^{2/3}) \leq \int_J^\infty f(c_0 x^{2/3}) dx$, using $\log(2 + c_0 x^{2/3}) \leq C \log(2 + x)$ and that $\frac{4}{3} - \frac{2}{3p} > 1$ for $p > 2$. $\square$

18.2 Moment and kernel bounds at the critical scale

The next lemma restates the heat estimates of Lemma 7.14 and the moment bounds of Lemma 10.10 in the rescaled variables, and adds bounds on the derivatives of the kernels. Throughout, $N^{-2/3}$ plays the role of a short-time cutoff: in the rescaled time, $s + N^{-2/3}$ corresponds to $(1 + t)N^{-2/3}$ in the original time $t = N^{2/3}s$.

Lemma 18.4 (Moment and kernel bounds). *Let $(r_{i,N})$ satisfy (18.2) and (18.3), and fix $K > 0$. There exist $C, c_* > 0$, depending only on c_0, C_0, K , such that for all $N \in \mathcal{N}$, $0 \leq s \leq K$, and $1 \leq i \leq N$,*

$$\begin{aligned} \frac{c_*}{\sqrt{s + N^{-2/3}}} &\leq \psi_N(s) \leq \frac{C}{\sqrt{s + N^{-2/3}}}, & |\psi'_N(s)| &\leq \frac{C}{(s + N^{-2/3})^{3/2}}, & (18.21) \\ \frac{c_* \sqrt{s + N^{-2/3}}}{1 + R_{i,N}s} &\leq k_{i,N}(s) \leq \frac{C \sqrt{s + N^{-2/3}}}{1 + R_{i,N}s}, & |k'_{i,N}(s)| &\leq \frac{C}{\sqrt{s + N^{-2/3}} (1 + R_{i,N}s)}, & (18.22) \end{aligned}$$

$$\sum_{j=1}^N k_{j,N}(s)^2 \leq \frac{C}{(s + N^{-2/3})^{1/2}}, \quad \sum_{j=1}^N k'_{j,N}(s)^2 \leq \frac{C}{(s + N^{-2/3})^{5/2}}. \quad (18.23)$$

Moreover, for every $p > 0$ there exist C_p and N_0 , depending only on c_0, C_0, K, p , such that the following holds. If $Y(0) = g$ is a standard Gaussian vector, then for all $N \geq N_0$, $0 \leq s \leq K$, and $1 \leq i \leq N$,

$$\begin{aligned} \|Z_N(s)\|_p &\leq \frac{C_p}{\sqrt{s + N^{-2/3}}}, & \mathbb{E} Z_N(s)^{-p} &\leq C_p (s + N^{-2/3})^{p/2}, \\ \|U_{i,N}(s)^2\|_p &\leq \frac{C_p \sqrt{s + N^{-2/3}}}{1 + R_{i,N}s}, & \|Z_N(s) U_{i,N}(s)^2\|_p &\leq \frac{C_p}{1 + R_{i,N}s}. \end{aligned} \quad (18.24)$$

Here $\|F\|_p = (\mathbb{E}|F|^p)^{1/p}$.

Proof. We write $t = N^{2/3}s$, so that $1 + t = N^{2/3}(s + N^{-2/3})$ and $r_{i,N}t = R_{i,N}s$.

Step 1: two-sided bounds. By Lemma 7.14, uniformly in $0 \leq t \leq KN^{2/3}$, we have $\sum_j h_j(t) \asymp N(1+t)^{-1/2}$ and $h_i(t) \asymp (1+t)^{1/2}(1+r_{i,N}t)^{-1}$, with constants depending only on c_0, C_0, K . Multiplying by $N^{-2/3}$ and $N^{-1/3}$, respectively, gives the two-sided bounds in (18.21) and (18.22).

Step 2: the derivative of ψ_N . The function ψ_N is completely monotone, so $\psi_N(s) = \int e^{-\lambda s} m(d\lambda)$ for a positive measure m on $[0, \infty)$. Using $\lambda e^{-\lambda s/2} \leq 2/(es)$, we find $|\psi'_N(s)| \leq 2s^{-1}\psi_N(s/2)$. If $s \geq N^{-2/3}$, then $s + N^{-2/3} \leq 2s$, and by Step 1 we have $|\psi'_N(s)| \leq Cs^{-3/2} \leq C(s + N^{-2/3})^{-3/2}$. If $s \leq N^{-2/3}$, then $\psi'_N(s) = \mathbf{1}^\top H e^{tH} \mathbf{1}$ with $t \leq 1$. Since e^{tH} is a contraction and $\|H\| \leq 4 \max_i r_{i,N} + 2 \leq 4C_0 + 2$, we find $|\psi'_N(s)| \leq (4C_0 + 2)N = (4C_0 + 2)(N^{-2/3})^{-3/2}$, which is at most $C(s + N^{-2/3})^{-3/2}$.

Step 3: the derivative of $k_{i,N}$. Fix i and abbreviate $R = R_{i,N}$ and $k = k_{i,N}$. Differentiating (18.13), we have $k' = -4Rk + 2\psi_N$. Suppose first that $Rs \leq 1$. Using also $RN^{-2/3} = r_{i,N} \leq C_0$, we find $R(s + N^{-2/3}) \leq 1 + C_0$, and Step 1 gives

$$|k'(s)| \leq 4Rk(s) + 2\psi_N(s) \leq \frac{4CR(s + N^{-2/3}) + 2C}{\sqrt{s + N^{-2/3}}} \leq \frac{C}{\sqrt{s + N^{-2/3}}},$$

which is the claim, since $1 + Rs \leq 2$. Suppose next that $Rs > 1$. Then $s \geq R^{-1} \geq N^{-2/3}/C_0$, which gives $s + N^{-2/3} \leq (1 + C_0)s$. Since $8R \int_0^s e^{-4R(s-v)} dv = 2(1 - e^{-4Rs})$, we may write $2\psi_N(s) = 8R \int_0^s e^{-4R(s-v)} \psi_N(s) dv + 2\psi_N(s)e^{-4Rs}$, and (18.13) gives

$$k'(s) = (2\psi_N(s) - 4RN^{-1/3})e^{-4Rs} - 8R \int_0^s e^{-4R(s-v)} (\psi_N(v) - \psi_N(s)) dv. \quad (18.25)$$

This subtraction is the point of the argument: estimating the two terms of $k' = -4Rk + 2\psi_N$ separately would lose a factor Rs . On $[s/2, s]$, we have $|\psi_N(v) - \psi_N(s)| \leq C(s-v)s^{-3/2}$ by Step 2, and $8R \int_0^\infty \ell e^{-4R\ell} d\ell = (2R)^{-1}$; this part of the integral is at most $C(Rs^{3/2})^{-1} \leq Cs^{-1/2}(1 + Rs)^{-1}$. On $[0, s/2]$, we have $0 \leq \psi_N(v) - \psi_N(s) \leq \psi_N(v)$, and $\int_0^{s/2} \psi_N \leq C\sqrt{s}$ by Step 1; this part is at most $CR\sqrt{s}e^{-2Rs} \leq Cs^{-1/2}(1 + Rs)^{-1}$, because $(Rs)(1 + Rs)e^{-2Rs}$ is bounded. For the first term in (18.25), we use $\psi_N(s) \leq Cs^{-1/2}$ and $N^{-1/3} \leq \sqrt{C_0/R}$, which give

$$|2\psi_N(s) - 4RN^{-1/3}|e^{-4Rs} \leq Cs^{-1/2}(1 + (Rs)^{1/2})e^{-4Rs} \leq \frac{C}{\sqrt{s}(1 + Rs)}.$$

Combining these bounds with $s \asymp s + N^{-2/3}$ proves the derivative bound in (18.22).

Step 4: the sums. Using (18.22), we have $\sum_j k_{j,N}(s)^2 \leq C(s + N^{-2/3}) \sum_j (1 + R_{j,N}s)^{-2}$. If $s \geq N^{-2/3}$, then (18.18) bounds the last sum by $Cs^{-3/2}$, and $s + N^{-2/3} \leq 2s$. If $s < N^{-2/3}$, then the sum has at most $N = (N^{-2/3})^{-3/2}$ terms. In both cases, the sum is at most $C(s + N^{-2/3})^{-3/2}$, which gives the first bound in (18.23). The second bound follows in the same way from the derivative bound in (18.22).

Step 5: moments. Let $Y(0) = g$ and $\mathcal{A} = |Y|^2$. By Lemma 10.10, for every $p > 0$ there exist C_p and N_0 such that for $N \geq N_0$ and $0 \leq t \leq KN^{2/3}$,

$$\mathbb{E} \left[\frac{\mathcal{A}(t)}{\sum_j h_j(t)} \right]^p + \mathbb{E} \left[\frac{\sum_j h_j(t)}{\mathcal{A}(t)} \right]^p \leq C_p,$$

and, for $p \geq 1$, $\|Y_i(t)^2\|_p \leq C_p h_i(t)$. Since $Z_N(s)/\psi_N(s) = \mathcal{A}(t)/\sum_j h_j(t)$, Step 1 implies the first two bounds in (18.24). Using $U_{i,N}(s)^2 = N^{-1/3} Y_i(t)^2$, we have $\|U_{i,N}(s)^2\|_p \leq C_p k_{i,N}(s)$ for $p \geq 1$, and the third bound follows from Step 1; for $p < 1$ it follows from the case $p = 1$. The fourth bound follows from the first and third by the Cauchy–Schwarz inequality, $\|Z_N U_{i,N}^2\|_p \leq \|Z_N\|_{2p} \|U_{i,N}^2\|_{2p}$. $\square$

18.3 The mean function

For a positive sequence $R = (R_i)_{i \geq 1}$ with $\sum_i R_i^{-2} < \infty$, we define

$$\Lambda_R(p) = \frac{p}{8} \sum_{i \geq 1} \frac{1}{R_i(R_i + p/4)} = \frac{1}{2} \sum_{i \geq 1} \left(\frac{1}{R_i} - \frac{1}{R_i + p/4} \right), \quad p \geq 0. \quad (18.26)$$

The function Λ_R is finite, continuous, and increasing, with $\Lambda_R(0) = 0$. If $\sum_i R_i^{-1} = \infty$, then $\Lambda_R(p) \rightarrow \infty$ as $p \rightarrow \infty$, by monotone convergence in the second expression. For finite vectors $(R_{i,N})_{i \leq N}$, we write Λ_N for the same expression with the sum over $i \leq N$. We write $\mathcal{L}f(p) = \int_0^\infty e^{-ps} f(s) ds$ for the Laplace transform.

Lemma 18.5 (The mean function). *Let $(r_{i,N})$ be a critical sequence with limit R . There exists a unique continuous function $\psi_R: (0, \infty) \rightarrow (0, \infty)$ such that*

$$\int_0^\infty e^{-ps} \psi_R(s) ds = \frac{1}{2\Lambda_R(p)}, \quad p > 0. \quad (18.27)$$

The function ψ_R is completely monotone. For every $j \geq 0$, the derivatives $\psi_N^{(j)}$ converge to $\psi_R^{(j)}$ uniformly on compact subsets of $(0, \infty)$. For every $i \geq 1$, $k_{i,N}$ converges uniformly on compact subsets of $[0, \infty)$ to

$$k_i(s) = 2 \int_0^s e^{-4R_i(s-v)} \psi_R(v) dv. \quad (18.28)$$

Finally, with γ as in (18.4),

$$\lim_{s \downarrow 0} \sqrt{s} \psi_R(s) = \psi_*, \quad \psi_* = \frac{4\gamma^{3/2}}{3\pi^{3/2}}, \quad \text{and} \quad \lim_{s \rightarrow \infty} \psi_R(s) = \frac{4}{\sum_i R_i^{-2}}. \quad (18.29)$$

The function ψ_R depends only on R , and not on the critical sequence converging to R . For the GOE value $\gamma_{\text{GOE}} = \frac{1}{2}(3\pi/2)^{2/3}$, we have $\gamma_{\text{GOE}}^{3/2} = 3\pi/2^{5/2}$ and hence $\psi_* = 1/\sqrt{2\pi}$.

Proof. We first derive the Laplace transform of ψ_N , then prove compactness of (ψ_N) , identify the limit through its transform, and finally compute the two asymptotics in (18.29).

Step 1: the transform of ψ_N . Let $\bar{h}(t) = N^{-1} \sum_j h_j(t)$, so that $\psi_N(s) = N^{1/3} \bar{h}(N^{2/3}s)$, and let

$$G(\lambda) = \frac{2}{N} \sum_{i=1}^N \frac{1}{\lambda + 4r_{i,N}}, \quad \lambda > 0.$$

We recall from (7.14) that the Laplace transform of $\bar{h}$ is $G(\lambda)/(2(1 - G(\lambda)))$, and

$$1 - G(\lambda) = \frac{\lambda}{2N} \sum_{i=1}^N \frac{1}{r_{i,N}(\lambda + 4r_{i,N})};$$

the last identity follows from (18.2) by writing $1 = \frac{1}{N} \sum_i (2r_{i,N})^{-1}$. Substituting $t = N^{2/3}s$, we find that $\mathcal{L}\psi_N(p)$ equals $N^{-1/3}$ times the Laplace transform of $\bar{h}$ at $\lambda = pN^{-2/3}$. At this λ , we have $1 - G(\lambda) = N^{-1/3} \Lambda_N(p)$, and therefore

$$\mathcal{L}\psi_N(p) = \frac{G(pN^{-2/3})}{2\Lambda_N(p)} = \frac{1}{2\Lambda_N(p)} - \frac{1}{2N^{1/3}}, \quad p > 0. \quad (18.30)$$

Step 2: convergence of Λ_N . Let $u_{i,N} = R_{i,N}^{-1}$ and $u_i = R_i^{-1}$. Then $\Lambda_N(p) = \frac{p}{8} \sum_i g_p(u_{i,N})$ with $g_p(u) = u^2/(1 + pu/4)$. A direct computation gives $g'_p(u) = (2u + pu^2/4)(1 + pu/4)^{-2} \in [0, 2u]$ for $u, p \geq 0$. It follows that $|g_p(u) - g_p(w)| \leq 2 \max\{u, w\}|u - w|$, and by the Cauchy–Schwarz inequality we obtain

$$\left| \sum_i g_p(u_{i,N}) - \sum_i g_p(u_i) \right| \leq 2(\|u_N\|_{\ell^2} + \|u\|_{\ell^2})\|u_N - u\|_{\ell^2}.$$

Then (18.4) implies that $\Lambda_N \rightarrow \Lambda_R$ uniformly on bounded subsets of $[0, \infty)$. Then (18.30) implies that $\mathcal{L}\psi_N(p) \rightarrow (2\Lambda_R(p))^{-1}$ for every $p > 0$, uniformly on compact subsets of $(0, \infty)$.

Step 3: compactness. Let f be a completely monotone function, $f(s) = \int e^{-\lambda s} \mathbf{m}(d\lambda)$. Since f is nonincreasing, $f(s)se^{-ps} \leq \int_0^s e^{-pv} f(v) dv \leq \mathcal{L}f(p)$, so that

$$f(s) \leq \frac{e^{ps}}{s} \mathcal{L}f(p), \quad |f^{(j)}(s)| = \int \lambda^j e^{-\lambda s} \mathbf{m}(d\lambda) \leq j! \left(\frac{2}{s}\right)^j f(s/2), \quad (18.31)$$

where in the second bound we used $\lambda^j e^{-\lambda s/2} \leq (2j/(es))^j \leq j!(2/s)^j$. Using Step 2 and the first bound in (18.31) with $p = 2/s_0$, we find that $\sup_N \psi_N(s_0/2) < \infty$ for every $s_0 > 0$; the second bound then shows that every derivative of ψ_N is bounded on $[s_0, \infty)$, uniformly in large N . By the Arzelà–Ascoli theorem and a diagonal argument, every subsequence of (ψ_N) has a further subsequence along which all derivatives converge uniformly on compact subsets of $(0, \infty)$. Each limit ψ is nonnegative and completely monotone, since $(-1)^j \psi_N^{(j)} \geq 0$ for all j .

Step 4: identification of the limit. Let ψ be such a subsequential limit, and fix $p > 0$. For $0 < \delta < K$, we split $\mathcal{L}\psi_N(p)$ into integrals over $[0, \delta]$, $[\delta, K]$, and $[K, \infty)$. Using (18.21), we have $\int_0^\delta \psi_N \leq 2C\sqrt{\delta + N^{-2/3}}$. The tail satisfies $\int_K^\infty e^{-ps} \psi_N(s) ds \leq e^{-pK/2} \mathcal{L}\psi_N(p/2)$, since ψ_N is nonnegative, and this is at most $Ce^{-pK/2}$ for large N by Step 2. On $[\delta, K]$, $\psi_N \rightarrow \psi$ uniformly along the subsequence. Letting $N \rightarrow \infty$, then $\delta \downarrow 0$ and $K \rightarrow \infty$, and using Fatou's lemma for the integrability of ψ near zero, we find that $\mathcal{L}\psi(p) = (2\Lambda_R(p))^{-1}$ for every $p > 0$. A continuous function on $(0, \infty)$ whose Laplace transform is finite on $(0, \infty)$ is determined by it [71, Section XIII.1, corollary on p. 433]. All subsequential limits therefore coincide, the whole sequence (ψ_N) converges together with all derivatives, and the limit ψ_R is the unique continuous function satisfying (18.27). It is completely monotone by Step 3, and positive by the second limit in (18.29), which we prove below.

For the kernels, we use (18.13). For fixed i , we have $R_{i,N} \rightarrow R_i$, the term $N^{-1/3}e^{-4R_{i,N}s}$ tends to zero uniformly, and the integral $2 \int_0^s e^{-4R_{i,N}(s-v)} \psi_N(v) dv$ converges to $k_i(s)$ uniformly for s in compact subsets of $[0, \infty)$. Indeed, the contribution of $v \in [0, \delta]$ is at most $2 \int_0^\delta \psi_N \leq C\sqrt{\delta + N^{-2/3}}$ for both ψ_N and ψ_R , and on $[\delta, K]$ the integrands converge uniformly.

Step 5: the asymptotic as $s \downarrow 0$. We first show that

$$\lim_{p \rightarrow \infty} \sqrt{p} \sum_{i \geq 1} \frac{1}{R_i(R_i + p/4)} = 3\pi\gamma^{-3/2}. \quad (18.32)$$

Fix $\epsilon \in (0, 1)$ and choose i_0 such that $(1 - \epsilon)\gamma i^{2/3} \leq R_i \leq (1 + \epsilon)\gamma i^{2/3}$ for $i \geq i_0$. The terms with $i < i_0$ contribute at most $4p^{-1/2} \sum_{i < i_0} R_i^{-1} \rightarrow 0$. Since the function $x \mapsto (x(x + p/4))^{-1}$ is decreasing, we may compare the remaining terms, for $a = (1 \pm \epsilon)\gamma$, with

$$\sqrt{p} \sum_{i \geq i_0} \frac{1}{ai^{2/3}(ai^{2/3} + p/4)}.$$

The summand is decreasing in i , so this sum differs from the integral of the same function over $[i_0, \infty)$ by at most its value at $i = i_0$, which tends to zero. The integral over $[0, i_0]$ is at most $\sqrt{p} \int_0^{i_0} 4(ax^{2/3})^{-1} dx \rightarrow 0$. Finally, the substitution $r = ax^{2/3}$ gives

$$\sqrt{p} \int_0^\infty \frac{dx}{ax^{2/3}(ax^{2/3} + p/4)} = \frac{3}{2}a^{-3/2} \sqrt{p} \int_0^\infty \frac{r^{-1/2} dr}{r + p/4} = \frac{3}{2}a^{-3/2} \cdot 2\pi = 3\pi a^{-3/2},$$

where we used $\int_0^\infty r^{-1/2}(r + b)^{-1} dr = \pi b^{-1/2}$ with $b = p/4$. Letting $\epsilon \downarrow 0$ proves (18.32). By (18.27), we obtain

$$\mathcal{L}\psi_R(p) = \frac{1}{2\Lambda_R(p)} \sim C_* p^{-1/2} \quad (p \rightarrow \infty), \quad C_* = \frac{4\gamma^{3/2}}{3\pi}. \quad (18.33)$$

The function ψ_R is nonnegative and nonincreasing. By Karamata's Tauberian theorem for Laplace transforms at infinity [71, Section XIII.5, Theorems 2 and 3, p. 445], (18.33) implies

that $F(h) = \int_0^h \psi_R(s) ds$ satisfies $F(h) \sim C_* h^{1/2} / (\sqrt{\pi}/2) = 2C_* \pi^{-1/2} h^{1/2}$ as $h \downarrow 0$. For completeness, we recall the argument. For $h > 0$, let ν_h be the measure on $[0, \infty)$ with $\nu_h([0, x]) = h^{-1/2} F(hx)$. Its Laplace transform at $q > 0$ is $h^{-1/2} \mathcal{L}\psi_R(q/h)$, which tends to $C_* q^{-1/2}$ as $h \downarrow 0$, the Laplace transform of the measure $C_* \pi^{-1/2} x^{-1/2} dx$. By the extended continuity theorem for Laplace transforms [71, Section XIII.1, Theorem 2a, p. 433], ν_h converges vaguely to this measure, which has no atoms; hence $h^{-1/2} F(h) \rightarrow 2C_* \pi^{-1/2}$. We now pass from F to ψ_R by monotonicity. For $\lambda > 1$,

$$\frac{F(\lambda h) - F(h)}{(\lambda - 1)h} \leq \psi_R(h) \leq \frac{F(h) - F(h/\lambda)}{(1 - \lambda^{-1})h}.$$

Multiplying by $\sqrt{h}$ and letting $h \downarrow 0$, the left and right sides tend to $2C_* \pi^{-1/2}(\lambda^{1/2} - 1)/(\lambda - 1)$ and $2C_* \pi^{-1/2}(1 - \lambda^{-1/2})/(1 - \lambda^{-1})$. Letting $\lambda \downarrow 1$, both tend to $C_* \pi^{-1/2}$. This proves the first limit in (18.29), with $\psi_* = C_* \pi^{-1/2} = 4\gamma^{3/2}/(3\pi^{3/2})$.

Step 6: the limit as $s \rightarrow \infty$. The limit $\ell = \lim_{s \rightarrow \infty} \psi_R(s)$ exists because ψ_R is nonincreasing and nonnegative. For $K > 0$, we have $p \int_0^K e^{-ps} \psi_R(s) ds \leq p \int_0^K \psi_R(s) ds \rightarrow 0$ as $p \downarrow 0$, while $p \int_K^\infty e^{-ps} \psi_R(s) ds$ lies between ℓe^{-pK} and $\psi_R(K) e^{-pK}$. Letting $p \downarrow 0$ and then $K \rightarrow \infty$, we find $\ell = \lim_{p \downarrow 0} p \mathcal{L}\psi_R(p)$. Using monotone convergence, we find that $\Lambda_R(p)/p \rightarrow \frac{1}{8} \sum_i R_i^{-2}$ as $p \downarrow 0$. We conclude that $\ell = 4/\sum_i R_i^{-2}$, which is positive. This completes the proof. $\square$

The finite-dimensional analog of the second limit in (18.29) is the weight of the zero mode of H : $\lim_{s \rightarrow \infty} \psi_N(s) = 4/\sum_i R_{i,N}^{-2}$.

Lemma 18.6 (The limiting kernels). *Let $(r_{i,N})$ be a critical sequence with limit R , and fix $K > 0$. There exist $C, c_* > 0$, depending only on c_0, C_0, K , such that for all $0 < t \leq K$ and all $i \geq 1$,*

$$\frac{c_*}{\sqrt{t}} \leq \psi_R(t) \leq \frac{C}{\sqrt{t}}, \quad |\psi'_R(t)| \leq \frac{C}{t^{3/2}}, \quad (18.34)$$

$$\frac{c_* \sqrt{t}}{1 + R_i t} \leq k_i(t) \leq \frac{C \sqrt{t}}{1 + R_i t}, \quad |k'_i(t)| \leq \frac{C}{\sqrt{t}(1 + R_i t)}, \quad (18.35)$$

$$\sum_i k_i(t)^2 \leq \frac{C}{\sqrt{t}}, \quad \sum_i k'_i(t)^2 \leq \frac{C}{t^{5/2}}. \quad (18.36)$$

Moreover, for $0 < t \leq K$ and $u > 0$,

$$\sup_{i \geq 1} R_i k_i(t)^2 \leq C, \quad \sum_{i \geq 1} k_i(t)^2 e^{-4R_i u} \leq \frac{Ct}{(t+u)^{3/2}}, \quad (18.37)$$

and for all $t > 0$,

$$\sum_{i \geq 1} \frac{k_i(t)}{R_i} = 2. \quad (18.38)$$

Proof. The bounds (18.34) and (18.35) follow from (18.21) and (18.22) by letting $N \rightarrow \infty$, using Lemma 18.5, $R_{i,N} \rightarrow R_i$, and, for the derivatives, $k'_{i,N} = -4R_{i,N}k_{i,N} + 2\psi_N \rightarrow -4R_i k_i + 2\psi_R = k'_i$. The sums (18.36) follow from (18.35), (18.5), and (18.18). The first bound in (18.37) follows from $R_i k_i(t)^2 \leq C^2 R_i t(1 + R_i t)^{-2} \leq C^2$. For the second, if $u \leq t$, it follows from the first bound in (18.36) and $(t + u)^{3/2} \leq (2t)^{3/2}$; if $u > t$, it follows from $k_i(t)^2 \leq C^2 t$, (18.17), and $u^{-3/2} \leq 2^{3/2}(t + u)^{-3/2}$. Finally, by (18.13), $\sum_i k_{i,N}(t)/R_{i,N} = 2$ for every N . For fixed $t > 0$, each term converges to $k_i(t)/R_i$, and by (18.22) we have the summable bound $k_{i,N}(t)/R_{i,N} \leq C\sqrt{K+1}(tR_{i,N}^2)^{-1} \leq C\sqrt{K+1}t^{-1}c_0^{-2}i^{-4/3}$, uniformly in N . We obtain (18.38) by dominated convergence. $\square$

By Tonelli's theorem, $\sum_i k_i(t)/R_i = 2 \int_0^t \psi_R(v) \Sigma_R(t-v) dv$, where

$$\Sigma_R(t) = \sum_{i \geq 1} \frac{e^{-4R_i t}}{R_i}, \quad t > 0. \quad (18.39)$$

In other words, (18.38) states that the mean function solves the Volterra equation $\int_0^t \psi_R(v) \Sigma_R(t-v) dv = 1$ for $t > 0$. In finite dimension, the corresponding object is the renewal equation for $\sum_j h_j$ behind (7.14). This renewal equation is the finite-dimensional analog of the scalar Volterra equation for the spherical constraint derived by Cugliandolo and Dean [51, (II.4), (II.6)–(II.7)] for the dynamics with a deterministic Lagrange multiplier in the thermodynamic limit, and the equation above is its rescaled limit at the critical temperature. In our setting, the mean function describes only the mean of the random amplitude Z of the critical limit process.

18.4 A clipped system and uniqueness

The critical limit process is obtained in Section 18.5 as a limit point of the rescaled processes, and its law is identified through a uniqueness statement for the limiting equations. These equations are singular in two ways: the noise amplitude $\sqrt{2Z}$ is not Lipschitz near zero and is unbounded at the entrance, and the equation for Z involves infinitely many stochastic integrals. We first solve a clipped version of the equations, in which the noise amplitude and a weighted norm of the coordinates are truncated, by a contraction argument. We then show that every solution of the untruncated equations agrees with the clipped solutions up to the time at which the truncation becomes active.

The weighted space. For a positive sequence R , let E_R be the Hilbert space of real sequences $u = (u_i)_{i \geq 1}$ with

$$\|u\|_R^2 = \sum_{i \geq 1} \frac{u_i^2}{R_i} < \infty.$$

If R satisfies (18.5), then $\|\cdot\|_R$ is equivalent to the norm $(\sum_i u_i^2 i^{-2/3})^{1/2}$, with constants depending only on c_0, C_0 ; in particular, the sets E_R coincide for all such R . The Borel

σ -field of E_R is the trace of the product σ -field of $\mathbb{R}^{\mathbb{N}}$, since the coordinates are continuous on E_R and $\|u - w\|_R$ is a supremum of countably many finite sums.

Definition 18.7 (Entrance solutions). Let R be a positive nondecreasing sequence with $R_i/i^{2/3} \rightarrow \gamma \in (0, \infty)$, let $\psi = \psi_R$ and (k_i) be the mean and kernels of Lemma 18.5, and let $K > 0$. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, K]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, carrying a sequence $B = (B_i)_{i \geq 1}$ of independent $(\mathcal{F}_t)$ -Brownian motions. An entrance solution on $[0, K]$ is a pair (U, V) of adapted processes, $U = (U_i)_{i \geq 1}$, with the following properties. Each U_i is continuous with $U_i(0) = 0$. The process V is continuous with $V(0) = 0$, and $Z = \psi + V$ is positive on $(0, K]$. For $t \in (0, K]$, $U(t) \in E_R$; the map $t \mapsto U(t)$ is continuous from $(0, K]$ into E_R ; and $\|U(t)\|_R^2 \rightarrow 2$ as $t \downarrow 0$. Finally, for every $t \in [0, K]$, almost surely,

$$\begin{aligned} U_i(t) &= \sqrt{2} \int_0^t e^{-2R_i(t-v)} \sqrt{Z(v)} dB_i(v), \quad i \geq 1, \\ V(t) &= \sqrt{8} \sum_{i \geq 1} \int_0^t k_i(t-v) \sqrt{Z(v)} U_i(v) dB_i(v), \end{aligned} \quad (18.40)$$

where the series converges in probability. A global entrance solution is a pair defined on $[0, \infty)$ that is an entrance solution on $[0, K]$ for every K .

The stochastic integrals are well defined. We have $\sqrt{t} Z(t) \rightarrow \psi_*$ as $t \downarrow 0$, because V is continuous with $V(0) = 0$; hence Z is integrable on $[0, K]$, and the integrands $k_i(t-v)^2 Z(v) U_i(v)^2$ are integrable in v as well.

We introduce the clipping maps. Let $\omega(t) = t^{-1/2}$ for $t > 0$, and let $m > 1$. We set

$$a_m(t, z) = \left(\min \{ m \omega(t), \max \{ m^{-1} \omega(t), z \} \} \right)^{1/2}, \quad \Pi_m u = \min \left\{ 1, \frac{\sqrt{m}}{\|u\|_R} \right\} u, \quad (18.41)$$

with $\Pi_m 0 = 0$. The map Π_m is the metric projection of E_R onto the closed ball of radius $\sqrt{m}$.

Lemma 18.8 (A clipped system and uniqueness). *Let R be positive and nondecreasing, with $R_i/i^{2/3} \rightarrow \gamma \in (0, \infty)$. Set $\psi = \psi_R$ and let k_i be the kernels (18.28). These objects are defined by Lemma 18.5: the construction of Lemma 19.1, with its first-gap assertion omitted when $R_1 = R_2$, supplies a critical sequence with limit R . In particular, there exist $c_0, C_0 > 0$ with $c_0 i^{2/3} \leq R_i \leq C_0 i^{2/3}$; by Lemmas 18.5 and 18.6, $\sqrt{t} \psi(t) \rightarrow \psi_* \in (0, \infty)$, $\sum_i k_i(t)/R_i = 2$, and for every $K > 0$ there exists C_K such that*

$$k_i(t) \leq \frac{C_K \sqrt{t}}{1 + R_i t}, \quad |k'_i(t)| \leq \frac{C_K}{\sqrt{t} (1 + R_i t)}, \quad 0 < t \leq K, \quad i \geq 1. \quad (18.42)$$

Fix $K > 0$ and a filtered probability space as in Definition 18.7. There exists $m_0 > \max\{2, \psi_*, \psi_*^{-1}\}$, allowed to depend on R , K , and the filtered Brownian driver fixed above, such that part (a) holds for every $m \geq m_0$.

1. The clipped system

$$\begin{aligned}
U_i^m(t) &= \sqrt{2} \int_0^t e^{-2R_i(t-v)} a_m(v, \psi(v) + V^m(v)) dB_i(v), \\
V^m(t) &= \sqrt{8} \sum_{i \geq 1} \int_0^t k_i(t-v) a_m(v, \psi(v) + V^m(v)) (\Pi_m U^m(v))_i dB_i(v)
\end{aligned} \tag{18.43}$$

has a solution on $[0, K]$ such that V^m is continuous with $V^m(0) = 0$, each U_i^m is continuous with $U_i^m(0) = 0$, and the series converges in probability in $C([0, K])$. The map $t \mapsto U^m(t)$ is continuous from $(0, K]$ into E_R , and $\|U^m(t)\|_R^2 \rightarrow 2$ as $t \downarrow 0$. Any two such solutions are indistinguishable.

2. There exist $m_{\text{can}} > \max\{2, \psi_*, \psi_*^{-1}\}$ and Borel maps $\mathfrak{T}_n$, $n \geq 0$, from countable Wiener space $C([0, K])^{\mathbb{N}}$ to $C([0, K])^{\mathbb{N}} \times C([0, K])$, such that $\mathfrak{T}_n$ represents a clipped solution on that space at level $m_{\text{can}} + n$. For every filtered Brownian driver there is a deterministic n_0 , allowed to depend on that driver, such that for every $n \geq n_0$ and every clipped solution on its space at level $m_{\text{can}} + n$,

$$(U^{m_{\text{can}}+n}, V^{m_{\text{can}}+n}) = \mathfrak{T}_n(B) \quad \text{almost surely on } [0, K].$$

3. There is a Borel map $\mathfrak{S}$ on the same input and output spaces as in (b) such that every entrance solution (U, V) on $[0, K]$ satisfies $(U, V) = \mathfrak{S}(B)$ almost surely. In particular, writing $\mathbb{W}$ for countable Wiener measure, the joint law of (U, V, B) is the image of $\mathbb{W}$ under $b \mapsto (\mathfrak{S}(b), b)$.

Part (c) is a uniqueness statement only. The existence of an entrance solution is proved in Theorem 18.9. No first-gap assumption on R is needed.

Proof. On the fixed filtered space, choose $m_0 > \max\{\psi_*, \psi_*^{-1}, 2\}$ large enough for the construction below and fix $m \geq m_0$. We first record deterministic bounds (Step 1). We then construct the clipped solution as the fixed point of a contraction on a weighted L^2 space of progressively measurable processes (Step 2), show that the fixed point has continuous paths and that the series converge in path space (Steps 3 and 4), and compute the weighted norm at the entrance (Step 5). Step 6 constructs the Borel maps in (b), and Step 7 proves the uniqueness statement (c) by localization. Throughout, C denotes a constant depending only on c_0, C_0, K, m, ψ , and the constants C_K in (18.42), unless further dependence is indicated; it never depends on the input process.

Step 1: deterministic bounds. By Lemma 18.3 and (18.42), for $0 < t < t+h \leq K$,

$$\begin{aligned}
\Sigma_R(t) &\leq Ct^{-1/2}, \quad \sup_i R_i k_i(t)^2 \leq C, \quad \sup_i R_i k'_i(t)^2 \leq Ct^{-2}, \\
\sup_i R_i |k_i(t+h) - k_i(t)|^2 &\leq C \min \left\{ 1, \frac{h^2}{t^2} \right\},
\end{aligned} \tag{18.44}$$

with Σ_R as in (18.39). For the second and third bounds, we used $R_i t(1 + R_i t)^{-2} \leq 1$. For the fourth, we used the second bound and $|k_i(t+h) - k_i(t)| \leq h \sup_{[t, t+h]} |k'_i|$ together with the third. The clipping maps satisfy

$$\begin{aligned} a_m(t, z)^2 &\leq m \omega(t), & |a_m(t, z) - a_m(t, z')|^2 &\leq \frac{m}{4\omega(t)} |z - z'|^2, \\ \|\Pi_m u\|_R^2 &\leq m, & \|\Pi_m u - \Pi_m w\|_R &\leq \|u - w\|_R. \end{aligned} \quad (18.45)$$

The second bound holds because clipping is 1-Lipschitz and the square root has Lipschitz constant $(4m^{-1}\omega(t))^{-1/2}$ on $[m^{-1}\omega(t), \infty)$, and the last holds because metric projections onto closed convex subsets of a Hilbert space are 1-Lipschitz. The function $\omega(t) = t^{-1/2}$ is nonincreasing, $\omega(t)^{-1} \leq \sqrt{K}$ on $(0, K]$, and $\int_0^t \omega(v) dv = 2\sqrt{t}$.

Step 2: a contraction. For $\lambda > 0$, let $\mathcal{H}_\lambda$ be the space of real progressively measurable processes V on $[0, K]$, modulo $dt \otimes d\mathbb{P}$ -null sets, with

$$\|V\|_\lambda^2 = \int_0^K e^{-\lambda t} \mathbb{E}|V(t)|^2 dt < \infty.$$

This is an L^2 space, hence complete, and the norms $\|\cdot\|_\lambda$ are equivalent for different λ . Given $V \in \mathcal{H}_\lambda$, let $U[V]$ be defined by the first line of (18.43) with V in place of V^m . Since each coordinate $U_i[V]$ is continuous and adapted, the process $\|U[V](t)\|_R^2$ is progressively measurable, and so is the integrand

$$\zeta_i(v) = a_m(v, \psi(v) + V(v)) (\Pi_m U[V](v))_i.$$

Note that $\sum_i \zeta_i(v)^2 / R_i \leq m^2 \omega(v)$ pathwise, by (18.45). Using (18.44), we obtain

$$\sum_i \mathbb{E} \int_0^t k_i(t-v)^2 \zeta_i(v)^2 dv \leq \int_0^t \sup_i R_i k_i(t-v)^2 \mathbb{E} \sum_i \frac{\zeta_i(v)^2}{R_i} dv \leq C m^2 \int_0^t \omega(v) dv \leq C.$$

The stochastic integrals in the second line of (18.43) are orthogonal in L^2 , and the series converges in L^2 ; we define $\mathcal{T}_m V(t)$ as its sum. We show in Step 3 that $\mathcal{T}_m V$ has a continuous adapted version; in particular, $\mathcal{T}_m V \in \mathcal{H}_\lambda$. Changing V on a $dt \otimes d\mathbb{P}$ -null set changes neither $U[V]$ nor $\mathcal{T}_m V$. The coordinates satisfy, by Itô's isometry and (18.45),

$$\begin{aligned} \mathbb{E} \|U[V](t)\|_R^2 &= 2 \int_0^t \Sigma_R(t-v) \mathbb{E} a_m(v, \psi(v) + V(v))^2 dv \\ &\leq C m \int_0^t \frac{\omega(v)}{\sqrt{t-v}} dv \leq C, \end{aligned} \quad (18.46)$$

The last integral equals π for $t > 0$. In particular, $U[V](t) \in E_R$ almost surely.

Let $V, \tilde{V} \in \mathcal{H}_\lambda$ and $Y(v) = \mathbb{E}|V(v) - \tilde{V}(v)|^2$. Itô's isometry and (18.45) give

$$\mathbb{E} \|U[V](v) - U[\tilde{V}](v)\|_R^2 \leq 2 \int_0^v \Sigma_R(v-w) \frac{m}{4\omega(w)} Y(w) dw \leq C \int_0^v \frac{Y(w)}{\omega(w) \sqrt{v-w}} dw. \quad (18.47)$$

In the radial integrand, we write $a \Pi_m U - \tilde{a} \Pi_m \tilde{U} = (a - \tilde{a}) \Pi_m U + \tilde{a} (\Pi_m U - \Pi_m \tilde{U})$, with the obvious abbreviations. By Itô's isometry, (18.44), and (18.45),

$$\begin{aligned} \mathbb{E} |\mathcal{T}_m V(s) - \mathcal{T}_m \tilde{V}(s)|^2 &\leq 16 \int_0^s \sup_i R_i k_i(s-v)^2 \frac{m^2 Y(v)}{4\omega(v)} dv \\ &\quad + 16 \int_0^s \sup_i R_i k_i(s-v)^2 m \omega(v) \mathbb{E} \|U[V](v) - U[\tilde{V}](v)\|_R^2 dv \\ &\leq C \int_0^s \frac{Y(v)}{\omega(v)} dv \\ &\quad + C \int_0^s \omega(v) \int_0^v \frac{Y(w)}{\omega(w) \sqrt{v-w}} dw dv. \end{aligned}$$

Note that $\omega(v)/\omega(w) \leq 1$ for $w \leq v$, because ω is nonincreasing, and that ω^{-1} is bounded on $(0, K]$. Exchanging the order of integration in the second term, we find

$$\mathbb{E} |\mathcal{T}_m V(s) - \mathcal{T}_m \tilde{V}(s)|^2 \leq C \int_0^s (1 + (s-w)^{1/2}) Y(w) dw.$$

We multiply by $e^{-\lambda s}$, integrate over $s \in [0, K]$, and exchange the order of integration once more. Since $\int_0^\infty e^{-\lambda \ell} (1 + \ell^{1/2}) d\ell = \lambda^{-1} + (\sqrt{\pi}/2) \lambda^{-3/2}$, we obtain

$$\|\mathcal{T}_m V - \mathcal{T}_m \tilde{V}\|_\lambda^2 \leq C(\lambda^{-1} + (\sqrt{\pi}/2) \lambda^{-3/2}) \|V - \tilde{V}\|_\lambda^2. \quad (18.48)$$

We fix λ so large that the coefficient in (18.48) is at most $1/4$. By the Banach fixed point theorem, $\mathcal{T}_m$ has a unique fixed point $V^m \in \mathcal{H}_\lambda$, and $(U[V^m], V^m)$ solves (18.43). Conversely, the V -component of any solution of (18.43) with the properties stated in (a) is a fixed point of $\mathcal{T}_m$, since its integrands are bounded as above. This proves uniqueness up to modification; by continuity, up to indistinguishability. The fixed point on $[0, K]$ restricts to the fixed point on $[0, K']$ for $K' < K$.

Step 3: continuous paths and convergence of the series. Let $V \in \mathcal{H}_\lambda$ be any input and ζ_i as in Step 2. For $0 \leq s < s+h \leq K$,

$$\begin{aligned} \mathcal{T}_m V(s+h) - \mathcal{T}_m V(s) &= \sqrt{8} \sum_i \int_0^s (k_i(s+h-v) - k_i(s-v)) \zeta_i(v) dB_i(v) \\ &\quad + \sqrt{8} \sum_i \int_s^{s+h} k_i(s+h-v) \zeta_i(v) dB_i(v). \end{aligned}$$

By (18.44) and $\sum_i \zeta_i^2 / R_i \leq m^2 \omega$, the quadratic variations of the two terms are bounded pathwise by

$$C \int_0^s \omega(v) \min \left\{ 1, \frac{h^2}{(s-v)^2} \right\} dv \quad \text{and} \quad C \int_s^{s+h} \omega(v) dv.$$

The second is at most $C\sqrt{h}$. For the first, if $s \leq 2h$ we bound the minimum by 1 and obtain at most $2\sqrt{s} \leq C\sqrt{h}$. If $s > 2h$, the part $v \leq s/2$ is at most $h^2(2/s)^2 \cdot 2\sqrt{s} \leq C\sqrt{h}$, and the part $v \geq s/2$ is at most $\sqrt{2/s} \int_0^\infty \min\{1, h^2\ell^{-2}\} d\ell = 2h\sqrt{2/s} \leq C\sqrt{h}$. By the Burkholder–Davis–Gundy inequality [123, Theorem IV.4.1 and Corollary IV.4.2], for every $p \geq 2$,

$$\mathbb{E}|\mathcal{T}_m V(s+h) - \mathcal{T}_m V(s)|^p \leq C_p h^{p/4}, \quad (18.49)$$

uniformly over inputs V . Fix $p > 4$. By Kolmogorov's continuity criterion [123, Theorem I.2.1], $\mathcal{T}_m V$ has a continuous version, with $\mathbb{E}[\|\mathcal{T}_m V\|_\alpha^p] \leq C_{p,\alpha}$ for the Hölder seminorm of order $\alpha < 1/4 - 1/p$, and $\mathcal{T}_m V(0) = 0$. The same bounds hold, uniformly in J , for the tails $\mathcal{T}_m^{>J} V$ obtained by restricting the series to $i > J$. At each fixed s , Itô's isometry gives

$$\mathbb{E}|\mathcal{T}_m^{>J} V(s)|^2 \leq 8m^2 \int_0^s \sup_{i>J} R_i k_i(s-v)^2 \omega(v) dv \rightarrow 0 \quad (J \rightarrow \infty),$$

by dominated convergence, since $R_i k_i(\ell)^2 \leq C(R_i \ell)^{-1}$ tends to zero as $i \rightarrow \infty$ for each $\ell > 0$. Fix $\eta > 0$ and a mesh $\delta > 0$. On the event that the Hölder seminorm is at most M , the supremum of $|\mathcal{T}_m^{>J} V|$ over $[0, K]$ exceeds η only if its value at some point of the grid $\delta\mathbb{Z} \cap [0, K]$ exceeds $\eta - M\delta^\alpha$. Choosing M , then δ , then J , we find that $\sup_{[0, K]} |\mathcal{T}_m^{>J} V| \rightarrow 0$ in probability. This proves that the series converges in probability in $C([0, K])$. Since the fixed point V^m equals $\mathcal{T}_m V^m$, it has a continuous version vanishing at zero; replacing it by this version does not change any stochastic integral. Each coordinate U_i^m is continuous, since $U_i^m(t) = \sqrt{2}e^{-2R_i t} \int_0^t e^{2R_i v} a_m dB_i$, and the last integral is a continuous martingale because $\int_0^K a_m^2 \leq m \int_0^K \omega < \infty$. We have $U_i^m(0) = 0$.

Step 4: continuity in E_R . Fix $\delta \in (0, K)$ and put $t_0 = \delta/2$. For $t \geq t_0$, write

$$U_i^m(t) = e^{-2R_i(t-t_0)} U_i^m(t_0) + \mathcal{S}_i(t), \quad \mathcal{S}_i(t) = \sqrt{2} \int_{t_0}^t e^{-2R_i(t-v)} a_m dB_i.$$

By (18.46), $U^m(t_0) \in E_R$ almost surely. The deterministic semigroup orbit $t \mapsto e^{-2R(t-t_0)} U^m(t_0)$ is continuous into E_R by dominated convergence. On $[t_0, K]$, the clipped amplitude satisfies $a_m^2 \leq m \omega(t_0)$. For $t_0 \leq t < t+h \leq K$,

$$\mathcal{S}_i(t+h) - \mathcal{S}_i(t) = (e^{-2R_i h} - 1) \mathcal{S}_i(t) + \sqrt{2} \int_t^{t+h} e^{-2R_i(t+h-v)} a_m dB_i.$$

By the Burkholder–Davis–Gundy inequality, $\|\mathcal{S}_i(t)\|_p^2 \leq C_{p,\delta}/R_i$ and $\|\int_t^{t+h} \dots\|_p^2 \leq C_{p,\delta} \min\{h, R_i^{-1}\}$. Using Minkowski's inequality in $L^{p/2}$ and (18.19), we obtain

$$\left\| \|\mathcal{S}(t+h) - \mathcal{S}(t)\|_R^2 \right\|_{p/2} \leq C_{p,\delta} \sum_i \frac{\min\{1, R_i h\}^2}{R_i^2} + C_{p,\delta} \sum_i \frac{\min\{h, R_i^{-1}\}}{R_i} \leq C_{p,\delta} \sqrt{h}.$$

For $p > 4$, Kolmogorov's criterion gives a continuous E_R -valued version of $\mathcal{S}$, whose coordinates are the continuous processes $\mathcal{S}_i$. Thus U^m is continuous into E_R on $[\delta, K]$. Taking a countable sequence $\delta \downarrow 0$ gives continuity from $(0, K]$ into E_R .

Step 5: the weighted norm at the entrance. Abbreviate $a(v) = a_m(v, \psi(v) + V^m(v))$. By Itô's formula and variation of constants, for each i and $t > 0$,

$$U_i^m(t)^2 = 2 \int_0^t e^{-4R_i(t-v)} a(v)^2 dv + 2\sqrt{2} \int_0^t e^{-4R_i(t-v)} a(v) U_i^m(v) dB_i(v).$$

We divide by R_i and sum over i . For the first terms we use Tonelli's theorem. For the second terms, by the Burkholder–Davis–Gundy inequality and $a^2 \leq m\omega$, we have $\|U_i^m(v)^2\|_p \leq C_p \int_0^v e^{-4R_i(v-w)} \omega^{-1/2} dw \leq C_p \sqrt{v} (1 + R_i v)^{-1}$, so that $\|a(v)^2 U_i^m(v)^2\|_p \leq C_p (1 + R_i v)^{-1}$; since $\sum_i R_i^{-2} < \infty$, this shows that the stochastic series converges in every L^p . We obtain, for each $t > 0$, almost surely,

$$\begin{aligned} \|U^m(t)\|_R^2 &= 2 \int_0^t a(v)^2 \Sigma_R(t-v) dv + \mathcal{J}(t), \\ \mathcal{J}(t) &= 2\sqrt{2} \sum_{i \geq 1} \int_0^t \frac{e^{-4R_i(t-v)}}{R_i} a(v) U_i^m(v) dB_i(v). \end{aligned} \tag{18.50}$$

From the same bounds, we obtain $\|\mathcal{J}(t)\|_p^2 \leq C_p \sum_i R_i^{-2} \min\{t, R_i^{-1} \log(1 + R_i t)\} \leq C_p t$, and increments with $\|\mathcal{J}(t+h) - \mathcal{J}(t)\|_p^2 \leq C_p \sqrt{h}$: the fresh part is bounded using $\sum_i R_i^{-2} \leq C$, and the old part using (18.19). Then $\mathcal{J}$ has a continuous version with $\mathcal{J}(0) = 0$ by Kolmogorov's criterion. Since $V^m(0) = 0$ and $\sqrt{v} \psi(v) \rightarrow \psi_* \in (m^{-1}, m)$, there exists a random $\tau_0 > 0$ such that $a(v)^2 = \psi(v) + V^m(v)$ for $v \leq \tau_0$. For $t \leq \tau_0$, Tonelli's theorem and the hypothesis $\sum_i k_i(t)/R_i = 2$ give

$$2 \int_0^t a(v)^2 \Sigma_R(t-v) dv = \sum_i \frac{k_i(t)}{R_i} + 2 \int_0^t V^m(v) \Sigma_R(t-v) dv = 2 + O\left(\sqrt{t} \sup_{v \leq t} |V^m(v)|\right),$$

where we used $\int_0^t \Sigma_R \leq C\sqrt{t}$. The right side of (18.50) is therefore continuous on $(0, \tau_0]$, as is the left side by Step 4. Both sides agree almost surely at all rational times, hence at all times in $(0, \tau_0]$, and we conclude that $\|U^m(t)\|_R^2 \rightarrow 2$ as $t \downarrow 0$.

Step 6: Borel representations. Perform (a) on canonical countable Wiener space $C([0, K])^{\mathbb{N}}$, with its usual augmented filtration, and choose one threshold $m_{\text{can}} > \max\{2, \psi_*, \psi_*^{-1}\}$ there. At the countable levels $m_{\text{can}} + n$, choose Borel versions $\mathfrak{T}_n$ of these canonical solutions.

To transfer the construction, approximate each stochastic integral by elementary adapted integrands. If F is a progressively measurable process on canonical Wiener space with $\mathbb{E} \int_0^K |F|^2 < \infty$, then F is an L^2 limit of elementary processes whose coefficients are

measurable functions of the Wiener paths up to their left endpoints. Substituting B on another filtered space preserves both these elementary integrals and their L^2 limits, since B has law W . Applying this at every Picard iterate and then at their limit pulls each canonical solution back under the Brownian input map; completing the filtration does not change the stochastic integrals. On the other space, (a) supplies its own threshold m_0 . For all n large enough that $m_{\text{can}} + n \geq m_0$, uniqueness in (a) identifies every clipped solution with $\mathfrak{T}_n(B)$. This proves (b).

The same approximation argument also shows that the joint law of stochastic integrals of measurable adapted functionals of (U, V, B) is determined by the joint law of (U, V, B) , whenever those integrals are well defined. For locally square-integrable integrands, first stop at increasing bounds for their quadratic integrals and then remove the stopping.

Step 7: uniqueness by localization. Let (U, V) be an entrance solution, as in Definition 18.7, on the same filtered space as (U^m, V^m) . Let τ_m be the first time $t \in (0, K]$ at which one of the strict inequalities

$$m^{-1} < \frac{Z(t)}{\omega(t)} < m, \quad \|U(t)\|_R^2 < m$$

fails, with $\tau_m = K$ if none fails. Then $\tau_m > 0$ almost surely, because $Z(t)/\omega(t) = \sqrt{t} Z(t) \rightarrow \psi_* \in (m^{-1}, m)$ and $\|U(t)\|_R^2 \rightarrow 2 < m$. The function Z/ω extends to a positive continuous function on $[0, K]$, and $\|U(t)\|_R^2$ is bounded on $(0, K]$. Then $\tau_m = K$ for all large m , almost surely.

For $v < \tau_m$, the clipping is inactive: $\sqrt{Z(v)} = a_m(v, \psi(v) + V(v))$ and $U(v) = \Pi_m U(v)$. We define the stopped processes (U^τ, V^τ) by inserting the predictable indicator $\mathbf{1}_{\{v \leq \tau_m\}}$ into all stochastic integrands of (18.40), keeping the full kernels $e^{-2R_i(t-v)}$ and $k_i(t-v)$. Then (U^τ, V^τ) coincides with (U, V) on $[0, \tau_m]$, and it solves the system obtained from (18.43) by inserting the same indicator into the stochastic integrands. The clipped solution (U^m, V^m) , stopped in the same way, solves the same modified system. All the estimates of Step 2 remain valid with the indicator, and the modified map is again a contraction on $\mathcal{H}_\lambda$, with both stopped processes as fixed points. By uniqueness of the fixed point, V^τ and the stopped V^m agree, and by continuity $(U, V) = (U^m, V^m)$ on $[0, \tau_m]$ almost surely. We stress that stopping the process V itself, as $V(t \wedge \tau_m)$, would not preserve the Volterra kernels; this is why we stop the integrands.

Use the countable maps $\mathfrak{T}_n$ of (b), and define $\mathfrak{S}(b)$ to be their eventual constant value whenever it exists, and a fixed path otherwise. The eventual-constancy set and the resulting map are Borel, since the output path space is Polish and the sequence is countable. On any filtered space carrying an existing entrance solution, Step 6 identifies the clipped solutions for all sufficiently large indices, and the preceding localization argument makes them equal to (U, V) once $\tau_m = K$. Countably many almost-sure identities can be imposed on one event. Thus the maps are eventually constant at (U, V)

almost surely, proving (c). No existence assertion for the untruncated equations is used here. $\square$

18.5 Construction of the critical limit process

We now construct the critical limit process as the limit of the rescaled unnormalized processes started from a standard Gaussian vector, whose radial normalizations are the Langevin diffusions started from ν_N .

Theorem 18.9 (The critical limit process). *Let $(r_{i,N})$ be a critical sequence with limit R , let ψ_R and k_i be as in Lemma 18.5, and let $Y(0) = g$ be a standard Gaussian vector. Then the following hold.*

1. *There exists, on some filtered probability space carrying independent Brownian motions $B = (B_i)_{i \geq 1}$, a global entrance solution (U, V) in the sense of Definition 18.7, with $\psi = \psi_R$. That is, with $Z = \psi_R + V$,*

$$\begin{aligned} U_i(s) &= \sqrt{2} \int_0^s e^{-2R_i(s-v)} \sqrt{Z(v)} dB_i(v), \quad i \geq 1, \\ V(s) &= \sqrt{8} \sum_{i \geq 1} \int_0^s k_i(s-v) \sqrt{Z(v)} U_i(v) dB_i(v), \\ \|U(s)\|_R^2 &= 2 + \sqrt{8} \sum_{i \geq 1} \int_0^s \frac{1}{R_i} \sqrt{Z(v)} U_i(v) dB_i(v), \quad s > 0. \end{aligned} \tag{18.51}$$

The joint law of (U, V, B) is unique. We call (U, Z) the critical limit process, and $\xi = (\xi_i)_{i \geq 1}$, $\xi_i = U_i / \sqrt{Z}$, the normalized critical limit process.

2. *For every K , the series in (18.51) converge in probability in $C([0, K])$. The process $s \mapsto \|U(s)\|_R^2$, extended by the value 2 at $s = 0$, is continuous on $[0, \infty)$. Almost surely, $Z(s) > 0$ for all $s > 0$, and $\sqrt{s} Z(s) \rightarrow \psi_*$ as $s \downarrow 0$. For every $p > 0$, $K > 0$, $0 < s \leq K$, and $i \geq 1$,*

$$\begin{aligned} \|Z(s)\|_p &\leq \frac{C_p}{\sqrt{s}}, \quad \mathbb{E} Z(s)^{-p} \leq C_p s^{p/2}, \\ \|U_i(s)^2\|_p &\leq \frac{C_p \sqrt{s}}{1 + R_i s}, \quad \|Z(s) U_i(s)^2\|_p \leq \frac{C_p}{1 + R_i s}, \end{aligned} \tag{18.52}$$

with C_p as in (18.24).

3. *As $N \rightarrow \infty$,*

$$((U_{i,N})_{i \geq 1}, Z_N - \psi_N, \|U_N\|_{R,N}^2, (B_i^N)_{i \geq 1}) \Longrightarrow ((U_i)_{i \geq 1}, V, \|U\|_R^2, (B_i)_{i \geq 1})$$

in law in $C([0, K])^{\mathbb{N}} \times C([0, K])^2 \times C([0, K])^{\mathbb{N}}$, with the product topology, for every K . In particular, for every $J \geq 1$ and $0 < s_0 < K$,

$$(N^{-1/3} X_i(N^{2/3} s))_{i \leq J} \Longrightarrow (\xi_i(s))_{i \leq J} \quad \text{in law in } C([s_0, K], \mathbb{R}^J). \tag{18.53}$$

The denominator Z in $\xi_i = U_i/\sqrt{Z}$ is random. The coordinates U_i are driven by independent Brownian motions but are not independent Ornstein–Uhlenbeck processes: they share the noise amplitude $\sqrt{2Z}$, and Z depends on all coordinates through the second line of (18.51).

Remark 18.10 (Behavior as $s \downarrow 0$). The start of the critical limit process is singular. Each coordinate starts at zero, because $U_{i,N}(0) = N^{-1/6}g_i$. On the other hand, $Z(s) \sim \psi_* s^{-1/2}$ as $s \downarrow 0$, and the weighted norm $\|U(s)\|_R^2$ tends to 2, not to 0. In finite dimension, $\|U_N(s)\|_{R_N}^2$ is a martingale by (18.12), with initial value $N^{-1/3} \sum_i g_i^2 / R_{i,N}$, which has mean 2 and variance $2N^{-2/3} \sum_i R_{i,N}^{-2} \rightarrow 0$. At time zero, the contribution of any fixed finite set of coordinates to this weighted norm tends to zero in every L^p , since $U_{i,N}(0)^2 / R_{i,N} = N^{-1/3} g_i^2 / R_{i,N}$. At a small positive time s , the kernel bounds give $\mathbb{E} U_i(s)^2 / R_i \asymp \sqrt{s} R_i^{-1} (1 + R_i s)^{-1}$, and the weighted norm is carried by the coordinates whose index is of order $s^{-3/2}$. As $s \downarrow 0$ these indices tend to infinity, and $U(s)$ has no limit in E_R . The normalized coordinates $\xi_i(s)$ tend to zero as $s \downarrow 0$, in agreement with the fact that the coordinates of a uniform point of S_N , multiplied by $N^{-1/3}$, are of order $N^{-1/3}$.

In Step 4 of the proof below, we identify the stochastic integrals of a limit point through the joint laws of the rescaled processes. A stochastic integral is not a continuous function of its integrand and of the driving path, and we use the following elementary lemma, in which only the function f and the approximating functions S are required to be continuous.

Lemma 18.11 (Identification from joint laws). *Let E be a Polish space, and let $\mathfrak{Y}_n$, $n \geq 1$, and $\mathfrak{Y}$ be E -valued random variables, defined on possibly different probability spaces, such that the laws of the $\mathfrak{Y}_n$ converge weakly to the law of $\mathfrak{Y}$. Let $f: E \rightarrow \mathbb{R}$ be continuous, and let $\Xi = G(\mathfrak{Y})$ for a Borel function $G: E \rightarrow \mathbb{R}$. Assume that for every $\varepsilon > 0$ there exists a continuous function $S: E \rightarrow \mathbb{R}$ such that*

$$\mathbb{E}|f(\mathfrak{Y}_n) - S(\mathfrak{Y}_n)|^2 \leq \varepsilon \quad \text{for all sufficiently large } n, \quad \text{and} \quad \mathbb{E}|\Xi - S(\mathfrak{Y})|^2 \leq \varepsilon. \quad (18.54)$$

Then $\Xi = f(\mathfrak{Y})$ almost surely. More precisely, every such pair (ε, S) satisfies $\mathbb{E}|f(\mathfrak{Y}) - \Xi|^2 \leq 4\varepsilon$.

The first condition in (18.54) concerns only the laws of the $\mathfrak{Y}_n$. The functions f and S need not be bounded, and G need not be continuous. In Step 4 of the proof of Theorem 18.9, Ξ is a stochastic integral of the limit point, and S is a Riemann sum.

Proof. Fix ε and S as in the statement. For $L > 0$, the function $\Phi_L = \min\{L, |f - S|^2\}$ is bounded and continuous on E , and $\Phi_L \leq |f - S|^2$. By weak convergence [30, Section 2], we have

$$\mathbb{E}\Phi_L(\mathfrak{Y}) = \lim_{n \rightarrow \infty} \mathbb{E}\Phi_L(\mathfrak{Y}_n) \leq \varepsilon.$$

Letting $L \rightarrow \infty$, we find $\mathbb{E}|f(\mathfrak{Y}) - S(\mathfrak{Y})|^2 \leq \varepsilon$ by the monotone convergence theorem. Since $|f(\mathfrak{Y}) - \Xi|^2 \leq 2|f(\mathfrak{Y}) - S(\mathfrak{Y})|^2 + 2|S(\mathfrak{Y}) - \Xi|^2$, we obtain $\mathbb{E}|f(\mathfrak{Y}) - \Xi|^2 \leq 4\varepsilon$. As $\varepsilon > 0$ is arbitrary, $\Xi = f(\mathfrak{Y})$ almost surely. $\square$

Proof of Theorem 18.9. We first prove increment bounds for the rescaled processes, which give tightness (Step 2), and estimates for the coordinates with large index (Step 3). We then identify the equations satisfied by any limit point (Step 4), show that its rescaled squared norm Z is positive (Step 5), and conclude with the uniqueness statement of Lemma 18.8 (Step 6). Table 18.2 lists the parameters and the order in which they are chosen.

Parameter	Role	Order of choice
K	time horizon	fixed first
p	moment order ($p > 4$; $p = 16$ in Step 5)	fixed with K
η	accuracy of a probability bound	after K and p
M	bound on a Hölder seminorm	after η
δ, h	cutoff near the entrance; mesh of a time grid	after M
J	number of coordinates treated explicitly	after δ and h
N	dimension	first limit: $N \rightarrow \infty$, then $J \rightarrow \infty$

Table 18.2. Parameters in the proof of Theorem 18.9, in the order used in Step 3. In Step 4, a coordinate or J is fixed first, then δ and h , and $N \rightarrow \infty$ is the first limit. In Step 5, δ is fixed first, and the limits $h \downarrow 0$ and then $M \rightarrow \infty$ are taken.

Constants in the estimates for whole sums depend only on c_0, C_0, K, p , and on δ where indicated. Constants in the estimates for a fixed coordinate, and in the arguments of Step 4 for a fixed J , may also depend on that coordinate or on J . No constant depends on N, h , or M . The moment bounds (18.24) and the kernel bounds of Lemma 18.4 are used with $K + 1$ in place of K and for $N \geq N_0$. We use two standard facts. First, for progressively measurable f_i and $p \geq 2$, the Burkholder–Davis–Gundy inequality [123, Theorem IV.4.1 and Corollary IV.4.2] and Minkowski’s inequality in $L^{p/2}$ give

$$\left\| \sum_i \int_a^b f_i(v) dB_i^N(v) \right\|_p^2 \leq C_p \int_a^b \sum_i \|f_i(v)\|_p^2 dv. \quad (18.55)$$

Second, by Lemma 5.14, for independent standard Gaussian variables g_i and real numbers w_i , of which finitely many are nonzero,

$$\left\| \sum_i w_i (g_i^2 - 1) \right\|_p \leq C_p \left(\sum_i w_i^2 \right)^{1/2}. \quad (18.56)$$

Step 1: the exact equations. We recall from Lemma 18.2 that $U_{i,N}$ satisfies (18.10) with $U_{i,N}(0) = N^{-1/6}g_i$, the rescaled squared norm satisfies $Z_N = \psi_N + I_N + \Theta_N$, and the weighted norm satisfies $\|U_N(s)\|_{R_N}^2 = \|U_N(0)\|_{R_N}^2 + \mathcal{M}_N(s)$, where

$$\mathcal{M}_N(s) = \sqrt{8} \sum_{i=1}^N \int_0^s R_{i,N}^{-1} \sqrt{Z_N(v)} U_{i,N}(v) dB_i^N(v).$$

Recall that $\sum_i R_{i,N}^{-1} = 2N^{1/3}$. The initial value is therefore $\|U_N(0)\|_{R_N}^2 = 2 + N^{-1/3} \sum_i R_{i,N}^{-1} (g_i^2 - 1)$. It converges to 2 in every L^p by (18.56) and $\sum_i R_{i,N}^{-2} \leq 4c_0^{-2}$. For $J \geq 1$, we write $\Theta_N^{\leq J}$ and $\mathcal{M}_N^{\leq J}$ for the sums over $i \leq J$ in Θ_N and $\mathcal{M}_N$, and $\Theta_N^{>J}$, $\mathcal{M}_N^{>J}$ for the remaining sums.

Step 2: increment bounds and tightness. We first show that for $0 < h \leq 1$,

$$\int_0^h \sum_i k_{i,N}(u)^2 du + \int_0^K \sum_i (k_{i,N}(u+h) - k_{i,N}(u))^2 du \leq C\sqrt{h}. \quad (18.57)$$

By (18.23), the first integral is at most $C \int_0^h u^{-1/2} du \leq C\sqrt{h}$, and so is the part $u < h$ of the second. For $u \geq h$, the Cauchy–Schwarz inequality and (18.23) give

$$\sum_i (k_{i,N}(u+h) - k_{i,N}(u))^2 \leq h \int_u^{u+h} \sum_i k'_{i,N}(w)^2 dw \leq Ch^2 u^{-5/2},$$

whose integral over $[h, \infty)$ is $\frac{2}{3}Ch^{1/2}$. This proves (18.57).

We next bound the increments of Θ_N . For $0 \leq s < s+h \leq K$, the increment $\Theta_N(s+h) - \Theta_N(s)$ is the sum of a stochastic integral over $[0, s]$ with integrands $\sqrt{8}(k_{i,N}(s+h-v) - k_{i,N}(s-v))\sqrt{Z_N}U_{i,N}$ and a stochastic integral over $[s, s+h]$ with integrands $\sqrt{8}k_{i,N}(s+h-v)\sqrt{Z_N}U_{i,N}$. Note that $\|\sqrt{Z_N(v)}U_{i,N}(v)\|_p^2 = \|Z_N(v)U_{i,N}(v)^2\|_{p/2} \leq C$, by (18.24). By (18.55) and (18.57), for $p \geq 2$,

$$\|\Theta_N(s+h) - \Theta_N(s)\|_p \leq C_p h^{1/4}. \quad (18.58)$$

The same bound holds for $\Theta_N^{\leq J}$ and $\Theta_N^{>J}$, uniformly in J . For the initial fluctuation, we find from (18.56) and (18.23) that $\|I_N(s)\|_p \leq C_p N^{-1/3} (\sum_i k_{i,N}(s)^2)^{1/2} \leq C_p N^{-1/6}$ and

$$\|I_N(s+h) - I_N(s)\|_p \leq C_p N^{-1/3} \left(h^2 \sup_{0 \leq w \leq K} \sum_i k'_{i,N}(w)^2 \right)^{1/2} \leq C_p N^{1/2} h.$$

Note that $\min\{N^{-1/6}, N^{1/2}h\} \leq h^{1/4}$ (consider $h \geq N^{-2/3}$ and $h < N^{-2/3}$ separately). We obtain $\|I_N(s+h) - I_N(s)\|_p \leq C_p h^{1/4}$. For fixed i , (18.10), (18.55), and (18.24) give

$$\begin{aligned} \|U_{i,N}(s+h) - U_{i,N}(s)\|_p &\leq 2R_{i,N} \int_s^{s+h} \|U_{i,N}(v)\|_p dv + C_p \left(\int_s^{s+h} \|Z_N(v)\|_{p/2} dv \right)^{1/2} \\ &\leq C_{p,i} h^{1/4}, \end{aligned}$$

where we used $R_{i,N} \leq C_0 i^{2/3}$ and $\int_s^{s+h} (v + N^{-2/3})^{-1/2} dv \leq 2\sqrt{h}$. Finally, the integrands of $\mathcal{M}_N$ satisfy $\sum_i R_{i,N}^{-2} \|Z_N U_{i,N}^2\|_{p/2} \leq C \sum_i R_{i,N}^{-2} \leq C$, and (18.55) gives $\|\mathcal{M}_N(s+h) - \mathcal{M}_N(s)\|_p \leq C_p h^{1/2}$, and the same holds for $\mathcal{M}_N^{\leq J}$.

We fix $p > 4$. By Kolmogorov's criterion, each of the processes $U_{i,N}$, I_N , $\Theta_N^{\leq J}$, $\Theta_N^{> J}$, $\mathcal{M}_N^{\leq J}$, $\mathcal{M}_N$ has a Hölder seminorm of order $\alpha \in (0, 1/4 - 1/p)$ on $[0, K]$ whose p th moment is bounded uniformly in N (and in J). Since their initial values are tight, the Arzelà–Ascoli theorem implies that each of them is tight in $C([0, K])$. Further, $I_N \rightarrow 0$ in probability in $C([0, K])$: its values at the points of a finite grid tend to zero in L^p , and its Hölder seminorm is bounded in L^p .

Step 3: coordinates with large index. Fix $\delta \in (0, K)$ and $s \in [\delta, K]$. By Itô's isometry and (18.24),

$$\mathbb{E}|\Theta_N^{> J}(s)|^2 \leq C \sum_{i>J} \int_0^s \frac{k_{i,N}(s-v)^2}{1+R_{i,N}v} dv.$$

We split the integral at $v = s/2$ and abbreviate $R = R_{i,N}$. For $v \leq s/2$, we have $k_{i,N}(s-v)^2 \leq C_\delta R^{-2}$ by (18.22), and $\int_0^{s/2} (1+Rv)^{-1} dv \leq R^{-1} \log(1+RK)$. For $v \geq s/2$, we have $(1+Rv)^{-1} \leq 2(R\delta)^{-1}$ and

$$\int_0^{s/2} k_{i,N}(u)^2 du \leq C \int_0^K \frac{u + N^{-2/3}}{(1+Ru)^2} du \leq C \left(\frac{\log(1+RK)}{R^2} + \frac{N^{-2/3}}{R} \right).$$

Together with (18.20), this gives

$$\mathbb{E}|\Theta_N^{> J}(s)|^2 \leq C_\delta \sum_{i>J} \left(\frac{\log(2+R_{i,N})}{R_{i,N}^3} + \frac{N^{-2/3}}{R_{i,N}^2} \right) \leq C_\delta \left(\frac{\log(2+J)}{J} + N^{-2/3} \right). \quad (18.59)$$

We now show that

$$\lim_{J \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{P} \left(\sup_{s \in [0, K]} |\Theta_N^{> J}(s)| > \eta \right) = 0 \quad \text{for every } \eta > 0. \quad (18.60)$$

Let $[\cdot]_\alpha$ denote the Hölder seminorm of Step 2. For $M, \delta, h > 0$, we have

$$\begin{aligned} \mathbb{P} \left(\sup_{[0, K]} |\Theta_N^{> J}| > \eta \right) &\leq \mathbb{P}([\Theta_N^{> J}]_\alpha > M) + \mathbf{1}_{\{M\delta^\alpha \geq \eta/2\}} + \mathbf{1}_{\{Mh^\alpha \geq \eta/2\}} \\ &\quad + \sum_j \mathbb{P}(|\Theta_N^{> J}(s_j)| > \eta/2), \end{aligned}$$

where s_j ranges over the grid $(\delta + h\mathbb{Z}) \cap [\delta, K]$; here we used $\Theta_N^{> J}(0) = 0$ for $s \leq \delta$. We choose M so that the first term is small uniformly in N and J , then δ and h so that the indicators vanish, and then J so that, by (18.59) and Chebyshev's inequality, the sum is

small for all large N . This proves (18.60). For the weighted norm, Doob's inequality [123, Chapter II, (1.7)], (18.24), and (18.20) give

$$\mathbb{E} \sup_{s \leq K} |\mathcal{M}_N^{>J}(s)|^2 \leq C \sum_{i>J} \frac{1}{R_{i,N}^2} \int_0^K \frac{dv}{1 + R_{i,N}v} \leq C \frac{\log(2+J)}{J}. \quad (18.61)$$

Finally, we bound the weighted norm of the coordinates with large index, uniformly in time. Fix $i > J$ and abbreviate $R = R_{i,N}$. We cover $[\delta, K]$ by at most $KR + 1$ intervals $[a, b_a]$, where $b_a = \min\{a + R^{-1}, K\}$. On such an interval, we have by (18.10) that

$$\sup_{a \leq s \leq b_a} |U_{i,N}(s)| \leq |U_{i,N}(a)| + \sqrt{2} \sup_{a \leq s \leq b_a} \left| \int_a^s e^{2R(v-a)} \sqrt{Z_N(v)} dB_i^N(v) \right|,$$

since $e^{-2R(s-a)} \leq 1$. By the Burkholder–Davis–Gundy inequality and (18.24), the L^{2p} norm of the supremum on the right is at most $C_p(e^4 R^{-1} \sup_{\delta \leq v \leq K} \|Z_N(v)\|_p)^{1/2} \leq C_{p,\delta} R^{-1/2}$, and $\|U_{i,N}(a)^2\|_p \leq C_p(1 + R\delta)^{-1} \leq C_{p,\delta} R^{-1}$. Combining these bounds, we find that the p th moment of $\sup_{[a, b_a]} U_{i,N}^2$ is at most $(C_{p,\delta} R^{-1})^p$. Bounding the maximum over the intervals by the ℓ^p norm, we find $\mathbb{E} \sup_{[\delta, K]} U_{i,N}^2 \leq C_{p,\delta} (KR + 1)^{1/p} R^{-1}$. Summing over $i > J$ and using (18.20), we obtain for every fixed $p > 2$

$$\mathbb{E} \sup_{\delta \leq s \leq K} \sum_{i>J} \frac{U_{i,N}(s)^2}{R_{i,N}} \leq C_{p,\delta} \sum_{i>J} R_{i,N}^{-2+1/p} \leq C_{p,\delta} J^{-1/3+2/(3p)}. \quad (18.62)$$

Step 4: limit points and their equations. Let $\mathcal{X}$ be the countable product of copies of $C([0, \infty))$, with the topology of locally uniform convergence in each factor; it is a Polish space. Consider the $\mathcal{X}$ -valued random variable

$$\mathfrak{X}_N = \left((U_{i,N})_{i \geq 1}, Z_N - \psi_N, \|U_N\|_{R_N}^2, (\Theta_N^{\leq J})_{J \geq 1}, (\mathcal{M}_N^{\leq J})_{J \geq 1}, (B_i^N)_{i \geq 1} \right).$$

Every factor is tight by Step 2, so $(\mathfrak{X}_N)$ is tight. By Prokhorov's theorem [30, Theorem 5.1] and Skorokhod's representation theorem [30, Theorem 6.7], every subsequence has a further subsequence along which, on some probability space, $\mathfrak{X}_N \rightarrow \mathfrak{X}$ almost surely, where

$$\mathfrak{X} = \left((U_i)_{i \geq 1}, V, W, (\Theta^{\leq J})_{J \geq 1}, (\mathcal{M}^{\leq J})_{J \geq 1}, (B_i)_{i \geq 1} \right).$$

We fix such a subsequence and work on this probability space. Let $(\mathcal{G}_s)$ be the usual augmentation of the filtration generated by $\mathfrak{X}$, and let $Z = \psi_R + V$ on $(0, \infty)$. Lemma 18.5 implies that $Z_N \rightarrow Z$ locally uniformly on $(0, \infty)$, almost surely. Note that $Z \geq 0$, and that $V(0) = 0$ because $Z_N(0) - \psi_N(0) = I_N(0) \rightarrow 0$.

The Brownian motions. For finite N , every component of $\mathfrak{X}_N(v)$, $v \leq s$, is $\mathcal{F}_s^N$ -measurable; this includes the coordinates, the amplitude, the weighted norm, the partial sums $\Theta_N^{\leq J}$ and $\mathcal{M}_N^{\leq J}$, and the Brownian motions. The increments of $(B_i^N)_i$ after time s

are therefore independent of $(\mathfrak{X}_N(v))_{v \leq s}$. For bounded continuous functions F of finitely many coordinates of $(\mathfrak{X}_N(v))_{v \leq s}$ and G of finitely many increments after s , we have $\mathbb{E}[FG] = \mathbb{E}[F] \mathbb{E}[G]$, and this identity passes to the limit. Let $\mathcal{H}_s^0$ be the σ -field generated by $\mathfrak{X}(v)$, $0 \leq v \leq s$. By a monotone class argument [123, Chapter 0, Section 2], the increments of B after s are independent of $\mathcal{H}_s^0$. Since the law of $(B_i^N)_{i \geq 1}$ does not depend on N , the B_i are independent standard Brownian motions. It remains to replace $\mathcal{H}_s^0$ by its usual augmentation $\mathcal{G}_s$ [123, Chapter I, Section 4]. Let Y be a bounded $\mathcal{G}_s$ -measurable random variable, let $t_1, \dots, t_m > s$ and $d \geq 1$, let G be a bounded continuous function on $\mathbb{R}^{dm}$, and for $r \in [s, \min_j t_j]$ write $G(r) = G((B_i(t_j) - B_i(r))_{i \leq d, j \leq m})$. Let $r_k \downarrow s$ with $r_k < \min_j t_j$. For each k , the σ -field $\mathcal{G}_s$ is contained in the completion of $\mathcal{H}_{r_k}^0$, so Y is almost surely equal to an $\mathcal{H}_{r_k}^0$ -measurable random variable, and $\mathbb{E}[Y G(r_k)] = \mathbb{E}[Y] \mathbb{E}[G(r_k)]$. Letting $k \rightarrow \infty$, and using the continuity of B and dominated convergence, we obtain $\mathbb{E}[Y G(s)] = \mathbb{E}[Y] \mathbb{E}[G(s)]$. By a monotone class argument, this identity extends to bounded measurable functions of the increments. We conclude that the increments of B after s are independent of $\mathcal{G}_s$, and that $(B_i)_{i \geq 1}$ is a sequence of independent $(\mathcal{G}_s)$ -Brownian motions.

Moments. By Fatou's lemma, the bounds (18.24) pass to the limit and give (18.52). The increment bounds of Step 2 also pass to the limit; for instance, $\|V(s+h) - V(s)\|_p \leq C_p h^{1/4}$.

Stochastic integrals. We identify the stochastic integrals of the limit point through the joint laws of the random variables $\mathfrak{X}_N$, using Lemma 18.11. For $x \in \mathcal{X}$, we write $u_i, \zeta, \theta^J, m^J$, and b_i for the components of x in the positions of $U_{i,N}, Z_N - \psi_N, \Theta_N^{\leq J}, \mathcal{M}_N^{\leq J}$, and B_i^N in $\mathfrak{X}_N$, and for $v > 0$ we let

$$q_x(v) = (\max\{\psi_R(v) + \zeta(v), 0\})^{1/2}. \quad (18.63)$$

For each $v > 0$, the map $x \mapsto q_x(v)$ is continuous on $\mathcal{X}$. Since $Z = \psi_R + V \geq 0$, we have $q_{\mathfrak{X}}(v) = \sqrt{Z(v)}$. For finite N , $q_{\mathfrak{X}_N}(v)$ is the square root of the positive part of $Z_N(v) + \psi_R(v) - \psi_N(v)$. The expression inside the positive part may be negative.

Fix $K > 0$ and $s \in (0, K]$. We treat three families of stochastic integrals at once. Each family is given by a constant c , a finite set I of indices, deterministic weights $\kappa_{i,N}$ and κ_i on $[0, s]$, and a continuous function f on $\mathcal{X}$:

1. the coordinates: for fixed i , we take $c = \sqrt{2}$, $I = \{i\}$, $\kappa_{i,N}(v) = e^{-2R_{i,N}(s-v)}$, $\kappa_i(v) = e^{-2R_i(s-v)}$, and $f(x) = u_i(s)$;
2. the rescaled squared norm: for fixed J , we take $c = \sqrt{8}$, $I = \{1, \dots, J\}$, $\kappa_{i,N}(v) = k_{i,N}(s-v)$, $\kappa_i(v) = k_i(s-v)$, and $f(x) = \theta^J(s)$;
3. the weighted norm: for fixed J , we take $c = \sqrt{8}$, $I = \{1, \dots, J\}$, $\kappa_{i,N} = R_{i,N}^{-1}$, $\kappa_i = R_i^{-1}$, and $f(x) = m^J(s)$.

For the coordinates, we set $\beta_{i,N}(v) = \kappa_{i,N}(v) \sqrt{Z_N(v)}$ and $\beta_i(v; x) = \kappa_i(v) q_x(v)$. For the two other families, we set $\beta_{i,N}(v) = \kappa_{i,N}(v) \sqrt{Z_N(v)} U_{i,N}(v)$ and $\beta_i(v; x) =$

$\kappa_i(v) q_x(v) u_i(v)$. By (18.10) and the definitions of $\Theta_N^{\leq J}$ and $\mathcal{M}_N^{\leq J}$ in Step 1, we have

$$f(\mathfrak{X}_N) = \iota_N + c \sum_{i \in I} \int_0^s \beta_{i,N}(v) dB_i^N(v), \quad (18.64)$$

where $\iota_N = e^{-2R_{i,N}S} N^{-1/6} g_i$ for the coordinates and $\iota_N = 0$ for the two other families. Here and below, we evaluate expectations of functions of $\mathfrak{X}_N$ on the original probability space, with the filtration $(\mathcal{F}_v^N)$. This is legitimate, since they depend only on the law of $\mathfrak{X}_N$. We show that $f(\mathfrak{X}) = \Xi$ almost surely, where

$$\Xi = c \sum_{i \in I} \int_0^s \beta_i(v; \mathfrak{X}) dB_i(v). \quad (18.65)$$

We choose a Borel version of Ξ as a function of $\mathfrak{X}$: elementary adapted integral approximations depend measurably on its paths and Brownian coordinates, and an almost surely convergent subsequence of their probability limits supplies such a version. Set this function to zero on the exceptional set where the chosen approximations do not converge.

For each $i \in I$, the process $\beta_i(\cdot; \mathfrak{X})$ is continuous and $(\mathcal{G}_v)$ -adapted on $(0, s]$; with the value zero at $v = 0$, it is progressively measurable. We have $\mathbb{E}Z(v) \leq C_1 v^{-1/2}$ and $\mathbb{E}[Z(v)U_i(v)^2] \leq C_1$ by (18.52), and the weights κ_i are bounded on $[0, s]$ by (18.5) and (18.35). By Tonelli's theorem, we find $\mathbb{E} \int_0^s \beta_i(v; \mathfrak{X})^2 dv < \infty$. Then (18.65) is a sum of square-integrable Itô integrals with respect to the $(\mathcal{G}_v)$ -Brownian motions B_i [123, Chapter IV, Section 2].

The approximating functions are left Riemann sums with the limiting weights. For $0 < \delta < s$ and an integer $n \geq K$, let $v_j = \delta + j(s - \delta)/n$ for $0 \leq j \leq n$, let $h = (s - \delta)/n$, so that $h \leq 1$, and define

$$S_{\delta,n}(x) = c \sum_{i \in I} \sum_{j=0}^{n-1} \beta_i(v_j; x) (b_i(v_{j+1}) - b_i(v_j)).$$

This is a continuous function on $\mathcal{X}$, since it is a finite sum of products of continuous functions of the values of x at the times $v_j \geq \delta$. The same function $S_{\delta,n}$ is evaluated at $\mathfrak{X}_N$ and at $\mathfrak{X}$. A sum formed with ψ_N , $R_{i,N}$, and $k_{i,N}$ would change with N , and Lemma 18.11 would not apply to it. Let $\beta_i^n(v; x) = \beta_i(v_j; x)$ for $v \in (v_j, v_{j+1}]$. The value $\beta_i(v_j; \mathfrak{X}_N)$ is $\mathcal{F}_{v_j}^N$ -measurable, $\beta_i(v_j; \mathfrak{X})$ is $\mathcal{G}_{v_j}$ -measurable, and both are square integrable by the bounds below. Then

$$S_{\delta,n}(\mathfrak{X}_N) = c \sum_{i \in I} \int_\delta^s \beta_i^n(v; \mathfrak{X}_N) dB_i^N(v), \quad S_{\delta,n}(\mathfrak{X}) = c \sum_{i \in I} \int_\delta^s \beta_i^n(v; \mathfrak{X}) dB_i(v).$$

We now bound $f(\mathfrak{X}_N) - S_{\delta,n}(\mathfrak{X}_N)$ and $\Xi - S_{\delta,n}(\mathfrak{X})$ in L^2 . In the following four estimates, C depends only on c_0 , C_0 , K , and the coordinate i or the number J , and C_δ depends

also on δ . The estimates for finite N hold for all N larger than a threshold depending on the same parameters. We use Itô's isometry and the orthogonality of stochastic integrals with respect to different Brownian motions throughout.

First, we bound the contribution of the entrance. We have

$$\begin{aligned} \mathbb{E} \left| c \sum_{i \in I} \int_0^\delta \beta_{i,N}(v) dB_i^N(v) \right|^2 &= c^2 \sum_{i \in I} \int_0^\delta \mathbb{E} \beta_{i,N}(v)^2 dv \\ &\leq \begin{cases} C\delta^{1/2} & \text{for the coordinates,} \\ CJ\delta & \text{for the two other families.} \end{cases} \end{aligned} \quad (18.66)$$

For the coordinates, we used $\kappa_{i,N} \leq 1$ and $\mathbb{E} Z_N(v) = \psi_N(v) \leq Cv^{-1/2}$, from (18.21). For the two other families, we used $\mathbb{E}[Z_N(v)U_{i,N}(v)^2] \leq C$, from (18.24), and $\kappa_{i,N} \leq C$, from (18.22) and $R_{i,N} \geq c_0$. In the same way, using (18.52) and (18.35), we find that the integral over $[0, \delta]$ in (18.65) satisfies the same bound. These are bounds for second moments, and both are at most $C\delta^{1/2}$.

Second, we show that the coefficients $\beta_i(\cdot; \mathfrak{X}_N)$ and $\beta_i(\cdot; \mathfrak{X})$ have a common modulus of continuity in L^2 : for $v, v' \in [\delta, s]$ with $|v - v'| \leq h \leq 1$,

$$\mathbb{E} |\beta_i(v; \mathfrak{X}_N) - \beta_i(v'; \mathfrak{X}_N)|^2 \leq C_\delta h^{1/4}, \quad \mathbb{E} |\beta_i(v; \mathfrak{X}) - \beta_i(v'; \mathfrak{X})|^2 \leq C_\delta h^{1/4}. \quad (18.67)$$

Let $A_N(v) = Z_N(v) + \psi_R(v) - \psi_N(v)$, so that $q_{\mathfrak{X}_N}(v) = (\max\{A_N(v), 0\})^{1/2}$, and let $A(v) = Z(v)$. By (18.24) and the uniform convergence $\psi_N \rightarrow \psi_R$ on $[\delta, K]$ from Lemma 18.5, we have $\|A_N(v)\|_2 \leq C_\delta$ and $\|U_{i,N}(v)\|_4 \leq C$. The increment $A_N(v) - A_N(v')$ is the sum of $\psi_R(v) - \psi_R(v')$, which is at most $C\delta^{-3/2}h$ in absolute value by (18.34), and of the increment of $Z_N - \psi_N = I_N + \Theta_N$. Then Step 2 implies that $\|A_N(v) - A_N(v')\|_2 \leq C_\delta h^{1/4}$ and $\|U_{i,N}(v) - U_{i,N}(v')\|_4 \leq Ch^{1/4}$. Since $A_N(v) \rightarrow A(v)$ and $U_{i,N}(v) \rightarrow U_i(v)$ almost surely for each v , Fatou's lemma shows that these four bounds also hold for A and U_i . We now argue in the same way in the finite model and in the limit, and drop N , $\mathfrak{X}_N$, and $\mathfrak{X}$ from the notation. For real a and b , we have the elementary inequality

$$|(\max\{a, 0\})^{1/2} - (\max\{b, 0\})^{1/2}|^2 \leq |a - b|. \quad (18.68)$$

It gives

$$\begin{aligned} \mathbb{E} |q(v) - q(v')|^2 &\leq \|A(v) - A(v')\|_2 \leq C_\delta h^{1/4}, \quad \|q(v)\|_4^2 \leq \|A(v)\|_2 \leq C_\delta, \\ \|q(v) - q(v')\|_4^2 &\leq \|A(v) - A(v')\|_2 \leq C_\delta h^{1/4}. \end{aligned}$$

Writing $q(v)U_i(v) - q(v')U_i(v') = (q(v) - q(v'))U_i(v) + q(v')(U_i(v) - U_i(v'))$ and using Minkowski's and Hölder's inequalities, we find

$$\begin{aligned} \|q(v)U_i(v) - q(v')U_i(v')\|_2 &\leq \|q(v) - q(v')\|_4 \|U_i(v)\|_4 + \|q(v')\|_4 \|U_i(v) - U_i(v')\|_4 \\ &\leq C_\delta h^{1/8}, \end{aligned}$$

and $\|q(v)U_i(v)\|_2 \leq \|q(v)\|_4 \|U_i(v)\|_4 \leq C_\delta$. Finally, the weights satisfy $|\kappa_i| \leq C$ and $|\kappa_i(v) - \kappa_i(v')| \leq Ch^{1/2}$ on $[0, s]$. For the coordinates, this follows from $|\partial_v e^{-2R_i(s-v)}| \leq 2R_i$ and $h \leq h^{1/2}$. For the rescaled squared norm, we have $|k'_i(u)| \leq Cu^{-1/2}$ by (18.35), and hence $|k_i(u+h) - k_i(u)| \leq 2C\sqrt{h}$ for all $u \geq 0$, including $u = 0$. The weights R_i^{-1} are constant. We write $\beta_i(v) - \beta_i(v')$ as $\kappa_i(v)$ times the increment of q or qU_i , plus $\kappa_i(v) - \kappa_i(v')$ times the value of q or qU_i at v' , and we obtain (18.67).

Third, we replace the coefficients of the finite model by the common coefficients. For $z \geq 0$ and real d, y, κ , and κ' ,

$$|\kappa' \sqrt{z} y - \kappa(\max\{z+d, 0\})^{1/2} y|^2 \leq 2|\kappa' - \kappa|^2 z y^2 + 2\kappa^2 |d| y^2. \quad (18.69)$$

To see this, we write the difference as $(\kappa' - \kappa)\sqrt{z} y + \kappa(\sqrt{z} - (\max\{z+d, 0\})^{1/2})y$ and use (18.68) with $a = z$ and $b = z+d$. No sign condition on $z+d$ is needed. We apply (18.69) with $z = Z_N(v)$, $d = \psi_R(v) - \psi_N(v)$, $\kappa' = \kappa_{i,N}(v)$, $\kappa = \kappa_i(v)$, and $y = 1$ or $y = U_{i,N}(v)$, so that the left side is $|\beta_{i,N}(v) - \beta_i(v; \mathfrak{X}_N)|^2$. For $v \geq \delta$, (18.24) gives $\mathbb{E}[Z_N(v)y^2] \leq C_\delta$ and $\mathbb{E}y^2 \leq C$, and we obtain

$$\begin{aligned} \sup_{v \in [\delta, s]} \mathbb{E} |\beta_{i,N}(v) - \beta_i(v; \mathfrak{X}_N)|^2 &\leq a_{N,\delta}, \\ a_{N,\delta} &= C_\delta \left(\max_{i \in I} \sup_{[\delta, s]} |\kappa_{i,N} - \kappa_i|^2 + \sup_{[\delta, s]} |\psi_N - \psi_R| \right). \end{aligned} \quad (18.70)$$

We have $a_{N,\delta} \rightarrow 0$ as $N \rightarrow \infty$. Indeed, the mean functions converge uniformly on $[\delta, K]$ by Lemma 18.5; $|e^{-2R_{i,N}(s-v)} - e^{-2R_i(s-v)}| \leq 2K|R_{i,N} - R_i|$, and $R_{i,N} \rightarrow R_i$ by (18.4); the kernels $k_{i,N}$ converge to k_i uniformly on $[0, K]$ by Lemma 18.5; and $R_{i,N}^{-1} \rightarrow R_i^{-1}$.

Fourth, using (18.67), we have

$$\mathbb{E} \left| c \sum_{i \in I} \int_\delta^s (\beta_i(v; \mathfrak{X}_N) - \beta_i^n(v; \mathfrak{X}_N)) dB_i^N(v) \right|^2 \leq c^2 |I| (s - \delta) C_\delta h^{1/4}, \quad (18.71)$$

and the same bound holds with $\mathfrak{X}$ and B_i in place of $\mathfrak{X}_N$ and B_i^N .

We now combine these estimates. By (18.64), the difference $f(\mathfrak{X}_N) - S_{\delta,n}(\mathfrak{X}_N)$ is the sum of ι_N , of the integral over $[0, \delta]$ in (18.64), of $c \sum_{i \in I} \int_\delta^s (\beta_{i,N}(v) - \beta_i(v; \mathfrak{X}_N)) dB_i^N(v)$, and of the integral in (18.71). Similarly, $\Xi - S_{\delta,n}(\mathfrak{X})$ is the sum of the integral over $[0, \delta]$ in (18.65) and of the limiting version of the integral in (18.71). Using $(a_1 + \dots + a_k)^2 \leq k(a_1^2 + \dots + a_k^2)$, $\mathbb{E}\iota_N^2 \leq N^{-1/3}$, (18.66), (18.70), and (18.71), we obtain

$$\begin{aligned} \mathbb{E} |f(\mathfrak{X}_N) - S_{\delta,n}(\mathfrak{X}_N)|^2 &\leq 4N^{-1/3} + C\delta^{1/2} + Ca_{N,\delta} + C_\delta h^{1/4}, \\ \mathbb{E} |\Xi - S_{\delta,n}(\mathfrak{X})|^2 &\leq C\delta^{1/2} + C_\delta h^{1/4}. \end{aligned} \quad (18.72)$$

Given $\varepsilon > 0$, we choose δ so that $C\delta^{1/2} \leq \varepsilon/4$ in (18.72), then n so large that $C_\delta h^{1/4} \leq \varepsilon/4$, and then N_ε so that $4N^{-1/3} + Ca_{N,\delta} \leq \varepsilon/2$ for $N \geq N_\varepsilon$. The cutoff δ and the mesh h are

fixed before $N \rightarrow \infty$. Then $S_{\delta,n}$ satisfies (18.54) along our subsequence, with $\mathfrak{Y}_N = \mathfrak{X}_N$ and $\mathfrak{Y} = \mathfrak{X}$, and Lemma 18.11 implies that $f(\mathfrak{X}) = \Xi$ almost surely.

The coordinates. Fix i and $s > 0$. For the coordinates, we have $f(\mathfrak{X}) = U_i(s)$, and we have shown that

$$U_i(s) = \sqrt{2} \int_0^s e^{-2R_i(s-v)} \sqrt{Z(v)} dB_i(v) \quad \text{almost surely.}$$

Moreover, $U_i(0) = \lim_N N^{-1/6} g_i = 0$. The right side has the continuous version $\sqrt{2} e^{-2R_i s} \int_0^s e^{2R_i v} \sqrt{Z(v)} dB_i(v)$, since $\mathbb{E} \int_0^K e^{4R_i v} Z(v) dv < \infty$ by (18.52). Both sides are continuous in s and agree almost surely at all rational times, and hence at all times. This proves the first line of (18.51), almost surely for all $s \geq 0$ and $i \geq 1$ simultaneously.

The rescaled squared norm. Fix J and $s > 0$. For the rescaled squared norm, we have $f(\mathfrak{X}) = \Theta^{\leq J}(s)$, and we have shown that

$$\Theta^{\leq J}(s) = \sqrt{8} \sum_{i \leq J} \int_0^s k_i(s-v) \sqrt{Z(v)} U_i(v) dB_i(v) \quad \text{almost surely.}$$

As a function of s , the right side is a stochastic convolution, not a martingale. The continuous process $\Theta^{\leq J}$, which vanishes at $s = 0$, is a version of it, and we use this version. Since $V - \Theta^{\leq J} = \lim_N (I_N + \Theta_N^{>J})$ almost surely in $C([0, K])$, where on the probability space of the representation $I_N + \Theta_N^{>J}$ stands for the continuous function $Z_N - \psi_N - \Theta_N^{\leq J}$ of $\mathfrak{X}_N$, the bound (18.60) and $I_N \rightarrow 0$ give

$$\mathbb{P} \left(\sup_{[0, K]} |V - \Theta^{\leq J}| > \eta \right) \leq \liminf_{N \rightarrow \infty} \mathbb{P} \left(\sup_{[0, K]} |I_N + \Theta_N^{>J}| > \eta \right) \rightarrow 0 \quad (J \rightarrow \infty).$$

This proves the second line of (18.51), with convergence of the series in probability in $C([0, K])$.

The weighted norm. Fix J and $s > 0$. For the weighted norm, we have $f(\mathfrak{X}) = \mathcal{M}^{\leq J}(s)$, and we have shown that

$$\mathcal{M}^{\leq J}(s) = \sqrt{8} \sum_{i \leq J} \int_0^s R_i^{-1} \sqrt{Z(v)} U_i(v) dB_i(v) \quad \text{almost surely.}$$

As a function of s , the right side is a square-integrable martingale, and the continuous process $\mathcal{M}^{\leq J}$, which vanishes at $s = 0$, is a version of it. For finite N , Step 1 gives $\|U_N(s)\|_{R_N}^2 = \|U_N(0)\|_{R_N}^2 + \mathcal{M}_N^{\leq J}(s) + \mathcal{M}_N^{>J}(s)$, and $\|U_N(0)\|_{R_N}^2 \rightarrow 2$ in probability. On the probability space of the representation, the value at zero of the component $\|U_N\|_{R_N}^2$ of $\mathfrak{X}_N$ converges almost surely to $\mathcal{W}(0)$, so that $\mathcal{W}(0) = 2$, and $\mathcal{M}_N^{>J}$ stands for the continuous function $\|U_N\|_{R_N}^2 - \|U_N(0)\|_{R_N}^2 - \mathcal{M}_N^{\leq J}$ of $\mathfrak{X}_N$. Then $\mathcal{W} - 2 - \mathcal{M}^{\leq J} = \lim_N \mathcal{M}_N^{>J}$ uniformly on $[0, K]$, almost surely, and Fatou's lemma and (18.61) give

$$\mathbb{E} \sup_{s \leq K} |\mathcal{W}(s) - 2 - \mathcal{M}^{\leq J}(s)|^2 \leq C \frac{\log(2+J)}{J}.$$

In particular, the series in the third line of (18.51) converges in probability in $C([0, K])$, with sum $\mathcal{W} - 2$. The process $\mathcal{W}$ is continuous on $[0, \infty)$ with $\mathcal{W}(0) = 2$, and it equals the right side of the third line of (18.51). It remains to identify $\mathcal{W}(s)$ with $\|U(s)\|_R^2$ for $s > 0$. Fix $0 < \delta < K$. For fixed J , the process $\sum_{i \leq J} U_{i,N}^2 / R_{i,N}$ converges uniformly on $[\delta, K]$ to $\sum_{i \leq J} U_i^2 / R_i$, so

$$\sup_{[\delta, K]} \left(\mathcal{W} - \sum_{i \leq J} \frac{U_i^2}{R_i} \right) = \lim_{N \rightarrow \infty} \sup_{[\delta, K]} \sum_{i > J} \frac{U_{i,N}^2}{R_{i,N}}$$

almost surely. Using Fatou's lemma and (18.62), we see that the expectation of the left side is at most $C_{p,\delta} J^{-1/3+2/(3p)}$. The left side is nonnegative and nonincreasing in J , and therefore it tends to zero almost surely as $J \rightarrow \infty$. This shows that, almost surely, $U(s) \in E_R$ and $\|U(s)\|_R^2 = \mathcal{W}(s)$ for all $s \in [\delta, K]$, and the series $\sum_i U_i^2 / R_i$ converges uniformly on $[\delta, K]$. Since each U_i is continuous, we deduce from the uniform convergence that $s \mapsto U(s)$ is continuous from $[\delta, K]$ into E_R . As δ and K are arbitrary, we find that U is continuous from $(0, \infty)$ into E_R , and $\|U(s)\|_R^2 = \mathcal{W}(s) \rightarrow 2$ as $s \downarrow 0$.

Step 5: positivity of Z . Fix $0 < \delta < K$. On $[\delta, K]$, ψ_R has a bounded derivative by (18.34), and V satisfies $\|V(s+h) - V(s)\|_p \leq C_p h^{1/4}$ for every p . Taking $p = 16$ in Kolmogorov's criterion, we find that the Hölder seminorm $[Z]_{1/8}$ of Z on $[\delta, K]$ is almost surely finite. We also have $\mathbb{E}Z(s)^{-16} \leq C$ for $s \in [\delta, K]$, by (18.52). Let $h \in (0, 1)$, and let s_j range over the grid $(\delta + h\mathbb{Z}) \cap [\delta, K]$, which has at most $K/h + 1$ points. If $\min_{[\delta, K]} Z = 0$ and $[Z]_{1/8} \leq M$, then $Z(s_j) \leq Mh^{1/8}$ for some j . By Markov's inequality,

$$\begin{aligned} \mathbb{P} \left(\min_{[\delta, K]} Z = 0, [Z]_{1/8} \leq M \right) &\leq \sum_j \mathbb{P}(Z(s_j) \leq Mh^{1/8}) \\ &\leq \left(\frac{K}{h} + 1 \right) C (Mh^{1/8})^{16} \leq C(K+1)M^{16}h. \end{aligned}$$

Letting $h \downarrow 0$ and then $M \rightarrow \infty$, we find that $\min_{[\delta, K]} Z > 0$ almost surely. Since the parameters δ and K are arbitrary, $Z(s) > 0$ for all $s > 0$, almost surely. Finally, $\sqrt{s} Z(s) = \sqrt{s} \psi_R(s) + \sqrt{s} V(s) \rightarrow \psi_*$ by (18.29) and $V(0) = 0$.

Step 6: uniqueness and convergence. By Steps 4 and 5, (U, V) is a global entrance solution in the sense of Definition 18.7, with $\psi = \psi_R$, on the filtered space $(\mathcal{G}_s)$. The hypotheses of Lemma 18.8 hold for $(R, \psi_R, (k_i))$ by (18.5), Lemma 18.5, and Lemma 18.6. By Lemma 18.8(c), on every interval $[0, K]$, the joint law of (U, V, B) is the image of $\mathbb{W}$ under $b \mapsto (\mathfrak{S}(b), b)$, and so it does not depend on the subsequence. The other components of $\mathfrak{X}$ are determined by (U, V, B) : $\mathcal{W} = \|U\|_R^2$, and $\Theta^{\leq J}$ and $\mathcal{M}^{\leq J}$ are stochastic integrals of functionals of (U, V, B) , whose laws are determined by the law of (U, V, B) by the argument of Step 6 of the proof of Lemma 18.8. It follows that all subsequential limits of $(\mathfrak{X}_N)$ have the same law, and the whole sequence converges in law.

This proves the existence and uniqueness in (a), the properties in (b), and the first claim in (c). Finally, on $[s_0, K]$ the limit amplitude Z is bounded below by a positive random constant, and $\psi_N \rightarrow \psi_R$ uniformly on $[s_0, K]$. The map $(u_1, \dots, u_J, v) \mapsto (u_i / \sqrt{\psi_R + v})_{i \leq J}$ is continuous on the set of paths with $\min_{[s_0, K]}(\psi_R + v) > 0$. Using (18.8) and the continuous mapping theorem (applied on the Skorokhod space, where the convergence is almost sure), we obtain (18.53). $\square$

18.6 The limiting Gibbs law

We recall the countable conditioned Gaussian laws of Section 5.4; at the critical temperature, this construction is due to Du and Huang [62, Section 2.2]. Let $v = (v_i)_{i \geq 1}$ be a variance sequence, that is, a sequence of positive numbers with $\sum_i v_i^2 < \infty$, and suppose that $\sum_i v_i = \infty$. Let $g = (g_i)$ be independent standard Gaussian variables. The series $T_v = \sum_i v_i (g_i^2 - 1)$ converges in L^2 and almost surely, and for $J \geq 0$ the tail $\sum_{i > J} v_i (g_i^2 - 1)$ has a bounded continuous density $p_{v, > J}$, which is strictly positive on $\mathbb{R}$ (Lemma 5.2); we write $p_v = p_{v, > 0}$. The conditioned law μ_v (Definition 5.8) is the unique probability measure on $\mathbb{R}^{\mathbb{N}}$ under which, for every $J \geq 1$, the standardized coordinates $z_i = y_i / \sqrt{v_i}$, $i \leq J$, have the density

$$\frac{p_{v, > J}(-\sum_{i \leq J} v_i (z_i^2 - 1))}{p_v(0)} \prod_{i \leq J} \frac{e^{-z_i^2/2}}{\sqrt{2\pi}} \quad (18.73)$$

on $\mathbb{R}^J$ (Lemmas 5.7 and 5.9). Under μ_v , the centered radial series $\sum_i (y_i^2 - v_i)$ converges and vanishes almost surely (Proposition 5.10). In other words, μ_v is the law of a centered Gaussian sequence with variances v_i conditioned on the renormalized radial constraint $\sum_i (y_i^2 - v_i) = 0$. For a finite variance sequence with N coordinates, μ_v is the law of a centered Gaussian vector with variances v_i conditioned on $|y|^2 = \sum_i v_i$, that is, the Bingham law (5.10); the analogous marginal-density formula holds for $J \leq N - 3$, with the density of the finite sum over the remaining coordinates. We write z_i for the standardized coordinates, which are denoted Z_i in Section 5, to avoid a clash with the amplitude Z .

For a critical sequence with limit R , we set

$$v_i^R = \frac{1}{2R_i}, \quad v_{i,N} = \frac{1}{2R_{i,N}} \quad (i \leq N), \quad \rho_R = \mu_{v^R}. \quad (18.74)$$

We call ρ_R the limiting Gibbs law. In the critical window, $\rho_\sigma = \rho_{R^\sigma}$ is the law of a Gaussian sequence with variances ϱ_σ^{-1} (for $i = 1$) and $(\Delta_i + \varrho_\sigma)^{-1}$ (for $i \geq 2$), conditioned on the renormalized radial constraint.

Corollary 18.12 (The limiting Gibbs law). *Let $(r_{i,N})$ be a critical sequence with limit R , and use the notation (18.74). Then the following hold.*

1. The law ρ_R is well defined. The density p_{v^R} is continuous and strictly positive on $\mathbb{R}$; in particular, $p_{v^R}(0) > 0$. The marginals of ρ_R are given by (18.73), and $\sum_i (y_i^2 - v_i^R) = 0$ ρ_R -almost surely.
2. Let x have law π_A , as in (18.1), and let $y^N = (N^{-1/3}x_i)_{i \leq N}$. Then y^N has law μ_{v_N} , with $v_N = (v_{i,N})_{i \leq N}$. For every $J \geq 1$, the law of $(y_i^N)_{i \leq J}$ converges in total variation to the law of $(y_i)_{i \leq J}$ under ρ_R .
3. There exists N_0 such that the following holds for every $\mu \in \{\rho_R\} \cup \{\mu_{v_N} : N \geq N_0\}$, with standardized coordinates $z_i = y_i / \sqrt{v_i}$ (where $v = v^R$ or $v = v_N$, respectively). For every $q > 0$, $\sup_i \mathbb{E}_\mu |z_i|^q \leq C_{q,R}$. For every $p \geq 1$ and every real sequence $(w_i) \in \ell^2$, the series $\sum_i w_i (z_i^2 - 1)$ converges in $L^p(\mu)$, and

$$\left\| \sum_i w_i (z_i^2 - 1) \right\|_{L^p(\mu)} \leq C_{p,R} \left(\sum_i w_i^2 \right)^{1/2}. \quad (18.75)$$

The constants depend on q , p , and the sequence $(r_{i,N})$, but not on N , i , or w . More generally, for every $m \geq 3$ there exists $N_0(m)$ such that, for μ as above with $N \geq N_0(m)$, the law of $(z_i)_{i \leq m}$ under μ has density at most $C_{m,R}$ with respect to the standard Gaussian law on $\mathbb{R}^m$.

4. We have $\rho_R(E_R) = 1$, and $\mathbb{E}_{\rho_R} \|y\|_R^2 \leq C_{2,R} \sum_i (2R_i^2)^{-1} < \infty$.

Proof. Note that v^R is a variance sequence, by (18.5), with $\sum_i (v_i^R)^2 = \frac{1}{4} \sum_i R_i^{-2} < \infty$ and $\sum_i v_i^R = \infty$. Part (a) follows from Lemmas 5.2, 5.7, and 5.9 and Proposition 5.10; for the positivity of p_{v^R} on all of $\mathbb{R}$, we use that $\sum_i v_i^R = \infty$.

For (b), Proposition 5.6 shows that $\pi_A = \mu_q$ in the eigencoordinates, with $q_i = (z - \alpha_i)^{-1} = (2r_{i,N})^{-1}$, and that $N^{-1/3}x$ has law $\mu_{N^{-2/3}q} = \mu_{v_N}$. Using (18.4), we have $\sum_i (v_{i,N} - v_i^R)^2 = \frac{1}{4} \sum_i |R_{i,N}^{-1} - R_i^{-1}|^2 \rightarrow 0$. The second part of Proposition 5.17, with $v^{(N)} = v_N$ and $A = \{1, \dots, J\}$, implies the convergence of $(y_i^N)_{i \leq J}$ in total variation.

For (c), let $B = \{1, \dots, m\}$ with $m \geq 3$. The first part of Proposition 5.17 gives the bound $M(v^R, B)$ on the density of $(z_i)_{i \leq m}$ under ρ_R with respect to the standard Gaussian law, the bound $\sup_i \mathbb{E}_{\rho_R} |z_i|^q \leq M(v^R, B) \mathbb{E}|G|^q$ for $q \geq 1$, and (18.75) with $C_{p,R} = 2C_p M(v^R, B)^{1/p}$, where G is a standard Gaussian variable and C_p is the constant of Lemma 5.14. By the second part of Proposition 5.17, the same bounds hold for μ_{v_N} , $N \geq N_0(m)$, with $M(v^R, B)$ replaced by $2M(v^R, B)$. We take $N_0 = N_0(3)$ for the moment bounds and the moment bounds for quadratic statistics; for $q < 1$, the moment bound follows from the case $q = 1$.

Finally, the first part of Proposition 5.17 with $u_i = v_i^R / R_i$ gives $\mathbb{E}_{\rho_R} \|y\|_R^2 = \mathbb{E}_{\rho_R} \sum_i u_i z_i^2 \leq M(v^R, B) \sum_i (2R_i^2)^{-1}$, which proves (d). $\square$

The critical profile

The goal of this section is to prove Theorem 1.3, which describes the mixing profile in the critical window $c_N = 1 + \sigma N^{-1/3}$. The proof has a deterministic part and a probabilistic part. The deterministic part is Theorem 19.2. Consider a sequence of half-precisions that satisfies the critical Weyl bounds and a lower bound on the first gap, and whose scaled inverses converge. We show that the total variation distance $D_N(N^{2/3}s)$ then converges, locally uniformly in $s > 0$, to the total variation distance $F_R(s)$ between the time- s law of the normalized critical limit process of Section 18 and the limiting Gibbs law. The limit is continuous, positive, and strictly decreasing, and Corollary 19.3 transfers the convergence to the mixing times at every fixed accuracy. The probabilistic part is Theorem 19.4. We transfer the deterministic theorem to the GOE model with a localized form of the continuous mapping theorem (Lemma 19.23), applied to Borel maps that send the spectral data to the finite profiles and mixing times. The limiting curves are measurable functions of a single Airy point process, the profile is random, and from the limit of the ratio of two mixing times we deduce that there is no cutoff. Finally, in Corollary 19.11 we bound the mixing times above and below by multiples of $N^{2/3} \log(1/\varepsilon)$, for every accuracy $\varepsilon \in (0, 1/2)$, and in Corollary 19.5 we deduce the lower bound $F_R(s) \geq c_1 e^{-C_1 s}$.

The proofs rely on several earlier results. Section 10 provides the representation of the dynamics through the unnormalized process (Lemma 10.2) and its moments (Lemma 10.10), and Section 7 provides the heat estimates of Lemma 7.14. Section 8 contains the functional inequalities of Corollaries 8.9 and 8.13 and Proposition 8.21 and the density bounds of Lemma 8.12. Section 12 bounds moments of the density of the time- t law and gives exponential relaxation from positive times (Theorem 12.13 and Proposition 12.14). Section 18 constructs the critical limit process; finitely many rescaled coordinates converge in law to its normalization (Theorem 18.9), and the rescaled Gibbs measures converge to the conditioned Gaussian law of Corollary 18.12. Section 4 establishes the spectral geometry of the GOE in the critical window (Lemma 4.17). Table 19.1 records the precise statements we use. Conversely, Theorem 19.2 is used again in Sections 20 and 21, where the critical profile is studied far from the critical temperature.

19.1 Conditions, notation, and main results

The critical spectral conditions

Throughout this section $b = N^{2/3}$. We work with the Gibbs measure written through half-precisions: for a vector $r = (r_1, \dots, r_N)$ we set

$$\pi_r(dx) = \frac{1}{Z_r} \exp\left(-\sum_{i=1}^N r_i x_i^2\right) \nu_N(dx), \quad (19.1)$$

and we let $(P_t^r)_{t \geq 0}$ be the semigroup of the Langevin diffusion with generator $L_r = \Delta_{S_N} - \nabla_{S_N}(\sum_i r_i x_i^2) \cdot \nabla_{S_N}$. For the model with matrix $A = cW_N$, written in an eigenbasis, the half-precisions are $r_i = (z - c\lambda_i)/2$, where z is the Lagrange multiplier (see (3.9)); since adding a multiple of $|x|^2$ to the potential changes neither the generator nor the Gibbs measure, π_r and P_t^r are then the Gibbs measure $\pi_{N,c}$ and the semigroup P_t of Section 3. When r is fixed we drop it from the notation and write $\pi_N = \pi_r$, $P_t = P_t^r$, $D_N(t) = \|\nu_N P_t - \pi_N\|_{TV}$, and $t_N(\varepsilon) = \inf\{t \geq 0 : D_N(t) \leq \varepsilon\}$.

Recall from Definition 4.16 that, for constants $0 < c_0 \leq C_0$ and $\delta > 0$, a vector $r = (r_1, \dots, r_N)$ with $0 < r_1 \leq r_2 \leq \dots \leq r_N$ satisfies the *critical spectral conditions with constants* (c_0, C_0, δ) if

$$\sum_{i=1}^N \frac{1}{2r_i} = N, \quad (19.2)$$

$$c_0 \left(\frac{i}{N}\right)^{2/3} \leq r_i \leq C_0 \left(\frac{i}{N}\right)^{2/3} \quad (1 \leq i \leq N), \quad (19.3)$$

$$r_2 - r_1 \geq \delta/b. \quad (19.4)$$

These are the three conditions in (4.36). As in Definition 18.1, (19.2) is the trace condition and (19.3) are the Weyl bounds; we call (19.4) the first-gap bound. The trace condition and the Weyl bounds suffice for the moment estimates of Section 10 and for the lower bound of Lemma 19.10; the first-gap bound enters through the functional inequalities of Proposition 8.21 and the density estimates of Section 12.

As in Definition 18.1, sequences $r_{\cdot, N} = (r_{i, N})_{i \leq N}$ of such vectors are indexed by an infinite set $\mathcal{N}$ of integers, and limits as $N \rightarrow \infty$ are taken along $\mathcal{N}$. We write $R_{i, N} = br_{i, N}$ and set $R_{i, N}^{-1} = 0$ for $i > N$. A sequence $(r_{\cdot, N})_{N \in \mathcal{N}}$ converges to a sequence $R = (R_i)_{i \geq 1}$ of positive numbers if

$$\lim_{N \rightarrow \infty} \sum_{i \geq 1} (R_{i, N}^{-1} - R_i^{-1})^2 = 0 \quad \text{and} \quad \lim_{i \rightarrow \infty} \frac{R_i}{i^{2/3}} = \gamma \quad \text{for some } \gamma \in (0, \infty). \quad (19.5)$$

A sequence that satisfies the critical spectral conditions with fixed constants for every $N \in \mathcal{N}$ and converges to R is a critical sequence with limit R in the sense of Definition 18.1,

which in addition satisfies the first-gap bound. For such a sequence, R is nondecreasing and $R_2 - R_1 \geq \delta$, since $R_{i,N} \rightarrow R_i$ for each i . We denote by $\mathcal{P}$ the set of sequences $R = (R_i)_{i \geq 1}$ with

$$0 < R_1 < R_2 \leq R_3 \leq \cdots \quad \text{and} \quad \lim_{i \rightarrow \infty} R_i / i^{2/3} \in (0, \infty). \quad (19.6)$$

Every $R \in \mathcal{P}$ satisfies $\inf_i R_i / i^{2/3} > 0$ and $\sup_i R_i / i^{2/3} < \infty$. The next lemma shows that every $R \in \mathcal{P}$ is the limit of a sequence satisfying the critical spectral conditions; we use it to define the limiting objects for every $R \in \mathcal{P}$, and in Section 19.8 to prove that they depend measurably on R .

Lemma 19.1 (Finite approximations of a limiting sequence). *Identify $R \in \mathcal{P}$ with the sequence $u = (R_i^{-1})_{i \geq 1} \in \ell^2$. Then $\mathcal{P}$ is a Borel subset of ℓ^2 , and there exist Borel maps $R \mapsto r^{(N)}(R) \in \mathbb{R}^N$, $N \geq 2$, with the following property. For every $R \in \mathcal{P}$ there exists $N_1(R)$ such that, for $N \geq N_1(R)$, the vector $r^{(N)}(R)$ satisfies the critical spectral conditions with constants*

$$c_0 = \min \left(1, \inf_i \frac{R_i}{i^{2/3}} \right), \quad C_0 = \max \left(2, \sup_i \frac{R_i}{i^{2/3}} \right), \quad \delta = R_2 - R_1, \quad (19.7)$$

and the sequence $(r^{(N)}(R))_N$ converges to R .

Proof. The set $\mathcal{P}$ is Borel, because it is defined by countably many conditions on the coordinates u_i (positivity, monotonicity, and existence of $\lim_i i^{2/3} u_i$ in $(0, \infty)$). Fix $R \in \mathcal{P}$, let $c_R = \inf_i R_i / i^{2/3}$, and let $M = \lfloor N^{1/2} \rfloor$. Define

$$\Theta_N = \left(2N^{1/3} - \sum_{i \leq M} R_i^{-1} \right)^{-1} \sum_{M < i \leq N} i^{-2/3}$$

whenever the bracket is positive, let $\hat{R}_{i,N} = R_i$ for $i \leq M$ and $\hat{R}_{i,N} = \Theta_N i^{2/3}$ for $M < i \leq N$, let $(R_{i,N})_{i \leq N}$ be the nondecreasing rearrangement of $(\hat{R}_{i,N})_{i \leq N}$, and set $r_i^{(N)}(R) = R_{i,N} / b$. When the bracket is not positive, set $r^{(N)}(R) = (1/2, \dots, 1/2)$. These maps are Borel, since Θ_N is a continuous function of finitely many coordinates of u on the open set where the bracket is positive, and rearrangement is continuous.

We now verify the critical spectral conditions for large N . Recall that $\sum_{i \leq n} i^{-2/3} = 3n^{1/3} + O(1)$, which gives $\sum_{M < i \leq N} i^{-2/3} = 3N^{1/3} + O(N^{1/6})$. Further, $\sum_{i \leq M} R_i^{-1} \leq c_R^{-1} \sum_{i \leq M} i^{-2/3} = O(N^{1/6})$. Then $\Theta_N \rightarrow 3/2$, and $\Theta_N \in [1, 2]$ for $N \geq N_1(R)$. By the definition of Θ_N ,

$$\sum_{i \leq N} \frac{1}{2r_i^{(N)}(R)} = \frac{b}{2} \left(\sum_{i \leq M} R_i^{-1} + \Theta_N^{-1} \sum_{M < i \leq N} i^{-2/3} \right) = \frac{b}{2} \cdot 2N^{1/3} = N,$$

which is the trace condition. Before rearrangement we have $c_0 i^{2/3} \leq \hat{R}_{i,N} \leq C_0 i^{2/3}$ for every i , with the constants (19.7). These bounds survive the rearrangement: at most $i - 1$

indices j satisfy $\hat{R}_{j,N} < c_0 i^{2/3}$ (all of them have $j < i$), so the i -th smallest value is at least $c_0 i^{2/3}$, and at least i indices satisfy $\hat{R}_{j,N} \leq C_0 i^{2/3}$ (namely $j \leq i$), which bounds the i -th smallest value by $C_0 i^{2/3}$. Dividing by $b = N^{2/3}$ gives the Weyl bounds. For large N we have $\Theta_N(M+1)^{2/3} \geq (M+1)^{2/3} > R_2$. Then the two smallest entries are R_1 and R_2 , and the first-gap bound holds with $\delta = R_2 - R_1$.

Finally, let $\hat{u}_N$ be the vector $(\hat{R}_{i,N}^{-1})_{i \leq N}$, padded with zeros. Then

$$\|\hat{u}_N - u\|_2^2 = \sum_{M < i \leq N} (\Theta_N^{-1} i^{-2/3} - R_i^{-1})^2 + \sum_{i > N} R_i^{-2} \leq 2(1 + c_R^{-2}) \sum_{i > M} i^{-4/3},$$

which tends to zero. The vector $(R_{i,N}^{-1})_{i \geq 1}$ is the nonincreasing rearrangement of $\hat{u}_N$, and u is nonincreasing. Rearrangement does not increase the ℓ^2 distance to u : for $a \geq b$ and $c \geq d$,

$$(a - d)^2 + (b - c)^2 - (a - c)^2 - (b - d)^2 = 2(a - b)(c - d) \geq 0.$$

Successively exchanging inverted pairs therefore shows that sorting the first N entries of $\hat{u}_N$ can only decrease their squared distance to $(u_i)_{i \leq N}$; the entries after N remain zero. Consequently the inverse vectors converge to u in ℓ^2 . The second condition in (19.5) is part of the definition of $\mathcal{P}$. $\square$

Notation

We fix the following notation for a vector r satisfying the critical spectral conditions.

- We write $q_i = (2r_i)^{-1}$ for the variances of the associated conditioned Gaussian law. We have $\sum_i q_i = N$ by (19.2), and π_N is the law of independent centered Gaussians with variances q_i conditioned on $|x|^2 = N$ (Section 5).
- For $t > 0$ the law $\nu_N P_t$ has a positive density of class C^2 in space and C^1 in time with respect to π_N , by the regularity argument of Lemma 12.7. We write $f_{N,s} = d(\nu_N P_{bs})/d\pi_N$ for $s > 0$. By reversibility, $f_{N,s} = P_{bs} f_{N,0}$, where $f_{N,0} = d\nu_N/d\pi_N$. Poincaré inequalities stated on smooth functions extend to H^1 by smooth approximation on the compact weighted spheres; the integration by parts arguments below require only this C^2 spatial regularity.
- For a probability measure μ on S_N and $J \leq N$, we write $\mu^{\leq J}$ for the law of the rescaled first J coordinates $(N^{-1/3}x_i)_{i \leq J}$ under μ . For a probability measure μ on $\mathbb{R}^N$ (with the product Borel σ -field), $\mu^{\leq J}$ is the law of the first J coordinates. We set $D_N^{\leq J}(t) = \left\| (\nu_N P_t)^{\leq J} - \pi_N^{\leq J} \right\|_{\text{TV}}$. Total variation distances do not change under the rescaling.
- We let Y be the unnormalized process of Lemma 10.2 with half-precisions r_i and $Y(0) = g$, a standard Gaussian vector, so that $X = \sqrt{N}Y/|Y|$ is the Langevin diffusion

started from ν_N . We write $\mathcal{A}_t = |Y(t)|^2$, as in Section 10, and consider the scaled processes

$$U_{i,N}(s) = N^{-1/6}Y_i(bs), \quad Z_N(s) = b^{-1}\mathcal{A}_{bs}, \quad N^{-1/3}X(bs) = \frac{U_N(s)}{\sqrt{Z_N(s)}}. \quad (19.8)$$

The processes $B_i^{(N)}(s) = b^{-1/2}B_i(bs)$ are independent Brownian motions, and the equation $dY_i = -2r_i Y_i dt + \sqrt{2/N} |Y| dB_i$ becomes

$$dU_{i,N} = -2R_{i,N}U_{i,N} ds + \sqrt{2Z_N} dB_i^{(N)}. \quad (19.9)$$

Constants denoted C, C' may change from line to line, and small positive constants always carry a subscript, since the letter c alone denotes the rescaled inverse temperature. Unless stated otherwise, constants depend only on (c_0, C_0, δ) and on the parameters indicated in their subscripts, and never on N, s , or the conditioning values. A supremum of nonnegative errors over the empty set is defined to be zero.

The critical limit process

Let $R \in \mathcal{P}$, and let $(r_{\cdot,N})$ be a sequence that satisfies the critical spectral conditions with fixed constants for every $N \in \mathcal{N}$ and converges to R (such sequences exist by Lemma 19.1). We use the following objects from Section 18. They depend only on R , and not on the approximating sequence.

- *The critical limit process* (Theorem 18.9). This is a pair (U, Z) of processes, where $U = (U_i)_{i \geq 1}$ is a sequence of continuous processes with $U_i(0) = 0$, and Z is continuous on $(0, \infty)$ with $Z(s) > 0$ for every $s > 0$. Almost surely, $s^{1/2}Z(s)$ converges to a positive constant as $s \downarrow 0$. For every J and $0 < s_0 < s_1$, the processes $(N^{-1/3}X_i(bs))_{i \leq J}$, $s \in [s_0, s_1]$, converge in distribution in $C([s_0, s_1], \mathbb{R}^J)$ to $(U_i(s)/\sqrt{Z(s)})_{i \leq J}$, by (18.53). Recall from Theorem 18.9 that

$$\xi(s) = (U_i(s)/\sqrt{Z(s)})_{i \geq 1}, \quad s > 0, \quad (19.10)$$

is the normalized critical limit process. We write μ_s^∞ for its law on $\mathbb{R}^\mathbb{N}$.

- *The limiting Gibbs law* (Corollary 18.12). The probability measure ρ_R on $\mathbb{R}^\mathbb{N}$ is the Gaussian sequence with variances $(2R_i)^{-1}$ conditioned on the vanishing of its centered radial series; at the critical temperature this law is due to Du and Huang [62, Section 2.2]. For each J , the marginal $\rho_R^{\leq J}$ has a density with respect to Lebesgue measure on $\mathbb{R}^J$, and $\pi_N^{\leq J} \rightarrow \rho_R^{\leq J}$ in total variation.

We define the *critical mixing profile* by

$$F_R(s) = \|\mu_s^\infty - \rho_R\|_{\text{TV}}, \quad s > 0, \quad (19.11)$$

the total variation distance on $\mathbb{R}^{\mathbb{N}}$ between the law of the normalized critical limit process at time s and the limiting Gibbs law.

Table 19.1 lists the results from other sections that the proofs use, with the form in which we use them. The constants in the middle column depend only on the constants of the critical spectral conditions and on the indicated parameters.

Main results

Theorem 19.2 (The critical profile along a deterministic sequence). *Let $R \in \mathcal{P}$, and let $(r_{\cdot, N})_{N \in \mathcal{N}}$ satisfy the critical spectral conditions with constants (c_0, C_0, δ) for every $N \in \mathcal{N}$ and converge to R . Then*

$$\lim_{N \rightarrow \infty} \sup_{s \in [s_0, s_1]} |D_N(bs) - F_R(s)| = 0 \quad \text{for every } 0 < s_0 < s_1 < \infty. \quad (19.12)$$

The function F_R is continuous and strictly decreasing on $(0, \infty)$, it satisfies $F_R(s) > 0$ for every $s > 0$, and

$$\lim_{s \downarrow 0} F_R(s) = 1, \quad \lim_{s \rightarrow \infty} F_R(s) = 0.$$

Thus F_R is a continuous strictly decreasing bijection from $(0, \infty)$ onto $(0, 1)$, and it has a continuous inverse $F_R^{-1}: (0, 1) \rightarrow (0, \infty)$.

Corollary 19.3 (Mixing times in the critical window). *Under the hypotheses of Theorem 19.2, for every compact interval $I \subset (0, 1)$,*

$$\lim_{N \rightarrow \infty} \sup_{\varepsilon \in I} |N^{-2/3} t_N(\varepsilon) - F_R^{-1}(\varepsilon)| = 0. \quad (19.13)$$

In particular, $F_R^{-1}(\varepsilon)/F_R^{-1}(1 - \varepsilon) > 1$ for every $\varepsilon \in (0, 1/2)$.

We now turn to the GOE model, and we recall the limiting spectral objects of Section 4. Let $\chi_1 < \chi_2 < \dots$ be the reflected GOE Airy point process, the limit of $(N^{2/3}(2 - \lambda_i))_{i \leq k}$ for each fixed k [121, Theorem 1.1], and let $\Delta_i = \chi_i - \chi_1$. Let Ξ be the regularized inverse-gap statistic of Lemma 4.5, constructed in [102, Theorem 6.1], and let

$$\mathcal{S}(x) = \Xi + \frac{1}{x} + \sum_{i \geq 2} \left(\frac{1}{x + \Delta_i} - \frac{1}{\Delta_i} \right), \quad x > 0, \quad (19.14)$$

as in (4.29). By Theorem 4.15 (see also [62, Proposition 1.4]), almost surely $\mathcal{S}$ is continuous and strictly decreasing, with limits $+\infty$ at 0 and $-\infty$ at ∞ . For each $\sigma \in \mathbb{R}$, there exists a unique $\varrho_\sigma > 0$ with $\mathcal{S}(\varrho_\sigma) = \sigma$. The limiting scaled half-precisions are those of (4.30):

$$R_1^\sigma = \frac{\varrho_\sigma}{2}, \quad R_i^\sigma = \frac{\Delta_i + \varrho_\sigma}{2} \quad (i \geq 2). \quad (19.15)$$

Result	What is used	Hypotheses
Lemma 10.2	$X = \sqrt{N}Y/ Y $ is the Langevin diffusion started from the law of $\sqrt{N}Y(0)/ Y(0) $; with $Y(0) = g$ this law is ν_N	any real r ; $Y(0) \neq 0$ almost surely, independent of the Brownian motions
Lemma 7.14	$\bar{h}(t) \asymp (1+t)^{-1/2}$ and $h_i(t) \leq C(1+t)^{1/2}(1+r_it)^{-1}$ for $0 \leq t \leq Kb$	(19.2), (19.3)
Lemma 10.10	$\mathbb{E}[(N\bar{h}(t)/\mathcal{A}_t)^p] \leq C_{K,p}$ and $\ Y_i(t)^2\ _p \leq C_{K,p}h_i(t)$ for $t \leq Kb$, $N \geq N_0$	(19.2), (19.3)
Lemma 8.12	density of $\sum_i q_i G_i^2$ at its mean is at least $c_{\delta'} m_2^{-1/2}$; with one term removed, the density is at most $C_{\delta'} m_2^{-1/2}$, where $m_2 = \sum_i q_i^2$	three weights $\geq \delta' \max_i q_i$
Corollary 8.9	Poincaré constant at most $1/t$	see Lemma 19.13
Corollary 8.13	$C_{LS} \leq C_{\delta'}/t + C_{\delta'} C_P$	see Lemma 19.13
Proposition 8.21	$C_P(\pi_N) \leq K_P b$	critical spectral conditions
Theorem 12.13	$\sup_{t \geq s_0 b} \ d(\nu_N P_t)/d\pi_N\ _{L^p(\pi_N)} \leq C_{p,s_0}$, $N \geq N_0$	critical spectral conditions
Theorem 18.9 and Corollary 18.12	the critical limit process, convergence of its normalized coordinates, the limiting Gibbs law, and the convergence of $\pi_N^{\leq J}$, as listed in Section 19.1	critical spectral conditions, convergence to R
Proposition 5.17; Corollary 18.12(c)	for $m \geq 3$ and $N \geq N_0(m)$, the law of $(\sqrt{2R_{i,N}} N^{-1/3} x_i)_{i \leq m}$ under π_N has a density with respect to the standard Gaussian law on $\mathbb{R}^m$ that is bounded uniformly in N , by domination and continuity in the variance sequence	critical spectral conditions, convergence to R
Lemma 5.9	marginal densities of a Gaussian law conditioned on a sphere	three remaining coordinates
Lemma 4.17	GOE: critical spectral conditions on events of probability close to one; joint convergence in distribution of the spectral data	$ \sigma \leq M$

Table 19.1. Results from other sections used in Section 19. Here $h_i(t)$ and $\bar{h}(t) = N^{-1} \sum_i h_i(t)$ are the heat quantities of Lemma 7.14, $\mathcal{A}_t = |Y(t)|^2$, and G_i are independent standard Gaussians.

Almost surely the χ_i are distinct [121, Proposition 3.5] and $\chi_i/(3\pi i/2)^{2/3} \rightarrow 1$ [62, Proposition 1.4]. These facts show that $R^\sigma = (R_i^\sigma)_{i \geq 1}$ belongs to $\mathcal{P}$, with $R_i^\sigma/i^{2/3} \rightarrow (3\pi/2)^{2/3}/2$. We write $\rho_\sigma = \rho_{R^\sigma}$ and $F_\sigma = F_{R^\sigma}$ on this event, and $F_\sigma(s) = e^{-s}$ on its complement, which has probability zero. By Lemma 19.22 below, F_σ and F_σ^{-1} are random elements of $C((0, \infty))$ and $C((0, 1))$, where these spaces carry the topology of uniform convergence on compact subsets, and they are measurable functions of $(\Xi, (\Delta_i^{-1})_{i \geq 2})$. Finally, we use the augmented spectral data $(\Xi_N, \mathbf{b}_N)$ of Lemma 4.5: with $\Delta_{i,N} = N^{2/3}(\lambda_1 - \lambda_i)$,

$$\Xi_N = \sum_{i=2}^N \Delta_{i,N}^{-1} - N^{1/3}, \quad \mathbf{b}_N = (\Delta_{2,N}^{-1}, \dots, \Delta_{N,N}^{-1}, 0, 0, \dots) \in \ell^2. \quad (19.16)$$

Theorem 19.4 (Critical profiles of the GOE model). *Fix $m \geq 1$ and $\sigma_1, \dots, \sigma_m \in \mathbb{R}$, and let $c_{N,j} = 1 + \sigma_j N^{-1/3}$. For each j , let $D_{N,j}$ and $t_{N,j}$ be the total variation distance and the mixing time of the Langevin dynamics with $c = c_{N,j}$ and the same matrix W_N , started from v_N independently of W_N . Then, as $N \rightarrow \infty$,*

$$(\Xi_N, \mathbf{b}_N, (D_{N,j}(N^{2/3} \cdot), N^{-2/3} t_{N,j}(\cdot))_{j \leq m}) \implies (\Xi, (\Delta_i^{-1})_{i \geq 2}, (F_{\sigma_j}, F_{\sigma_j}^{-1})_{j \leq m}) \quad (19.17)$$

in distribution in $\mathbb{R} \times \ell^2 \times (C((0, \infty)) \times C((0, 1)))^m$. Moreover, for every j and every $\varepsilon \in (0, 1/2)$, with the finite- N quotient defined to be 1 on the event $\{t_{N,j}(1 - \varepsilon) = 0\}$,

$$\frac{t_{N,j}(\varepsilon)}{t_{N,j}(1 - \varepsilon)} \implies \frac{F_{\sigma_j}^{-1}(\varepsilon)}{F_{\sigma_j}^{-1}(1 - \varepsilon)}, \quad (19.18)$$

and the limit is strictly greater than 1 almost surely. In particular, for each fixed σ there is no cutoff.

We note that the limit in (19.17) is a function of a single Airy point process for all j . The convergence is stated in distribution: the theorem does not specify a joint coupling across matrix sizes with the limiting Airy point process. Its proof applies Lemma 19.23 to the joint law of the spectral data, and it uses, at each confidence level, the critical spectral conditions with fixed constants.

Corollary 19.5 (Exponential lower bound for the critical profile). *1. Under the hypotheses of Theorem 19.2, there exist $c_1, C_1 > 0$, depending only on (c_0, C_0, δ) , such that $F_R(s) \geq c_1 e^{-C_1 s}$ for every $s > 0$.*

2. For every $\sigma \in \mathbb{R}$, almost surely there exist $c_1, C_1 > 0$ such that $F_\sigma(s) \geq c_1 e^{-C_1 s}$ for every $s > 0$.

Outline of the proofs

To prove Theorem 19.2, we first establish pointwise convergence at every positive time (Proposition 19.19). We combine two bounds.

1. The coordinates after the first J contribute little to the total variation distance: for $s_0 > 0$, we have $0 \leq D_N(bs) - D_N^{\leq J}(bs) \leq CJ^{-1/3}$ for every $s \geq s_0$ (Lemma 19.15), where C depends on s_0 but not on s or J . The density $f_{N,s_0/2}$ is bounded in $L^2(\pi_N)$ by Theorem 12.13, and finite semigroup smoothing bounds the energy of $f_{N,s}$ for $s \geq s_0$ (Lemma 19.12). We combine this bound with a Poincaré inequality for the conditional law of the remaining coordinates, deduced from the logarithmic Sobolev inequality of Lemma 19.14. Its constant is of order $bJ^{-2/3}$ uniformly over all conditioning values. Here J is fixed before $N \rightarrow \infty$.
2. For fixed J and $s > 0$, the first J rescaled coordinates converge in total variation to their limiting law (Lemma 19.17). Starting from the weak convergence of Theorem 18.9, we integrate by parts under π_N along fields that move the first J coordinates and compensate in the remaining ones. Gaussian domination of the Gibbs measure bounds the fields and their divergences (Lemma 19.16). Together with the energy estimate, this gives a translation bound uniform in N . Mollification and Scheffé's lemma then give total variation convergence at the chosen time.

The Gibbs marginals converge in total variation by Corollary 18.12, and total variation on $\mathbb{R}^N$ is the limit of the distances between finite heads (Lemma 19.7). Taking $N \rightarrow \infty$ and then $J \rightarrow \infty$ therefore proves Proposition 19.19.

Finite semigroup smoothing also gives $\|\nu_N P_{bt} - \nu_N P_{bs}\|_{TV} \leq L_{s_0}|t - s|$ for $s, t \geq s_0$, uniformly in N . Weak convergence of finite heads and Lemma 19.7 pass this estimate to the limiting laws, so F_R is locally Lipschitz (Lemma 19.18). Monotonicity and a finite grid then upgrade pointwise convergence to locally uniform convergence. The finite exponential lower bound (19.26) gives positivity of F_R . If F_R were constant on an interval, correlations of the positive and negative parts of the finite density minus one would tend to zero there, while their constant spectral coefficients would tend to a positive number. The uniform finite spectral gap and a Chebyshev-polynomial argument exclude this (Lemmas 19.9 and 19.20). The degree is fixed before $N \rightarrow \infty$ and tends to infinity afterwards. Table 19.2 records the other parameter choices.

The lower bounds of Section 19.3 do not use the profile. We derive them from a finite-dimensional estimate at a short time (Lemma 19.10) and a lower bound for reversible semigroups (Lemma 19.6). In Corollary 19.5 we pass the finite lower bound to F_R through the pointwise convergence of Proposition 19.19. Besides Theorem 19.2, the proof of Theorem 19.4 needs measurable dependence of F_R on R , which follows from Lemma 19.1 and Theorem 19.2. We then transfer the joint law of the spectral data to profiles and mixing times with the localized continuous mapping theorem Lemma 19.23, using the

Order	Parameter	Depends on
1	times $s \geq s_0$, for an anchor $s_0 > 0$	–
2	number $J \geq J_0$ of retained coordinates	R (through J_0)
3	mollifier scale $\alpha > 0$	chosen after J
4	dimension $N \geq N_0(J, s_0)$	J, s_0 , the sequence $(r_{\cdot, N})$
5	limit $N \rightarrow \infty$, then $\alpha \downarrow 0$, then $J \rightarrow \infty$	–

Table 19.2. Order of the choices in the proofs of Lemmas 19.14, 19.15, and 19.17 and Proposition 19.19: the conditional logarithmic Sobolev inequality, the bound on the remaining coordinates, total variation convergence of the first coordinates, and pointwise convergence of the full profile. The density is bounded in $L^2(\pi_N)$ at the time $s_0/2$. For convergence of the first coordinates alone, any fixed $J \geq 1$ is allowed.

critical spectral conditions with fixed constants on events of probability close to one (Lemma 4.17).

19.2 General facts about Markov semigroups and densities

In this section we collect a reversible lower bound, facts about total variation, and a criterion for vanishing spectral correlations. Throughout, $\|\mu - \rho\|_{\text{TV}} = \sup_B |\mu(B) - \rho(B)|$, which equals $(1/2) \int |f - g| d\theta$ if $\mu = f\theta$ and $\rho = g\theta$.

Lemma 19.6 (Reversible lower bound from two times). *Let $(P_t)_{t \geq 0}$ be a Markov semigroup with reversible probability measure π , and let μ be a probability measure. Let $\ell > 0$ and suppose that μP_ℓ has a density h_ℓ with respect to π such that $\|h_\ell - 1\|_{L^4(\pi)} \leq M$, and that $\|\mu P_{2\ell} - \pi\|_{\text{TV}} \geq d > 0$. Then $4d^2 \leq M^2$, and*

$$\|\mu P_t - \pi\|_{\text{TV}} \geq \frac{4d^3}{M^2} \exp\left(-\frac{3}{2} \log(1/\vartheta) \frac{t}{\ell}\right) \quad (t \geq 0), \quad \vartheta = \min\left(\frac{1}{2}, \frac{4d^2}{M^2}\right). \quad (19.19)$$

Proof. All norms are in $L^p(\pi)$. Let $f = h_\ell - 1$. By reversibility, $\mu P_{\ell+u} = (P_u h_\ell)\pi$ for $u \geq 0$, so that

$$\|\mu P_{\ell+u} - \pi\|_{\text{TV}} = \frac{1}{2} \|P_u f\|_1.$$

We set $a_j = \|P_j \ell f\|_2^2$ for $j \geq 0$. Using that P_ℓ is selfadjoint and the Cauchy–Schwarz inequality, we find

$$a_j = \langle P_{(j-1)\ell} f, P_{(j+1)\ell} f \rangle \leq \sqrt{a_{j-1} a_{j+1}} \quad (j \geq 1).$$

Further, $a_0 \leq \|f\|_4^2 \leq M^2$ and $a_1 \geq \|P_\ell f\|_1^2 = 4 \|\mu P_{2\ell} - \pi\|_{\text{TV}}^2 \geq 4d^2$. From the last display, we see that $a_{j+1} = 0$ would force $a_j = 0$. Then all a_j are positive, and the ratios a_{j+1}/a_j

are nondecreasing in j . As P_ℓ is a contraction on $L^2(\pi)$, we have $4d^2 \leq a_1 \leq a_0 \leq M^2$, and $a_{j+1}/a_j \geq a_1/a_0 \geq \vartheta$. We conclude that $a_j \geq 4d^2\vartheta^{j-1}$ for $j \geq 1$.

Recall that $\|g\|_2 \leq \|g\|_1^{1/3} \|g\|_4^{2/3}$ by Hölder's inequality, and note that $\|P_u f\|_4 \leq \|f\|_4 \leq M$ by Jensen's inequality. We obtain $\|P_u f\|_1 \geq \|P_u f\|_2^3 / M^2$. For $u \geq \ell$ let $j = \lceil u/\ell \rceil$, so that $j\ell \geq u$ and $j-1 < u/\ell$. The contraction property of $P_{j\ell-u}$ gives $\|P_u f\|_2^2 \geq a_j$, and we find that

$$\|\mu P_{\ell+u} - \pi\|_{\text{TV}} \geq \frac{a_j^{3/2}}{2M^2} \geq \frac{4d^3}{M^2} \vartheta^{3(j-1)/2} \geq \frac{4d^3}{M^2} \exp\left(-\frac{3}{2} \log(1/\vartheta) \frac{u}{\ell}\right).$$

For $t \geq 2\ell$ we apply this with $u = t - \ell \leq t$. For $t \leq 2\ell$, the distance is at least its value at time 2ℓ , because it is nonincreasing in time, and this value is at least $d \geq 4d^3/M^2$. This proves (19.19). $\square$

The argument uses only the logarithmic convexity of $t \mapsto \|P_t f\|_2^2$ for a reversible semigroup, a consequence of the spectral theorem.

Lemma 19.7 (Total variation on a countable product). *Let μ and ρ be probability measures on $\mathbb{R}^{\mathbb{N}}$ with its product Borel σ -field. Then $J \mapsto \|\mu^{\leq J} - \rho^{\leq J}\|_{\text{TV}}$ is nondecreasing and converges to $\|\mu - \rho\|_{\text{TV}}$ as $J \rightarrow \infty$.*

Proof. Let $\theta = (\mu + \rho)/2$ and $g = d\mu/d\theta - d\rho/d\theta$. Then $|g| \leq 2$ and $\|\mu - \rho\|_{\text{TV}} = (1/2) \int |g| d\theta$. Let $\mathcal{F}_J$ be the σ -field generated by the first J coordinates. The conditional expectation $g_J = \mathbb{E}_\theta[g \mid \mathcal{F}_J]$ is the difference of the densities of $\mu^{\leq J}$ and $\rho^{\leq J}$ with respect to $\theta^{\leq J}$, evaluated at the first J coordinates. Hence $\|\mu^{\leq J} - \rho^{\leq J}\|_{\text{TV}} = (1/2) \int |g_J| d\theta$. Conditional Jensen gives $|g_J| \leq \mathbb{E}_\theta[|g_{J+1}| \mid \mathcal{F}_J]$, and monotonicity follows. The σ -fields $\mathcal{F}_J$ generate the product σ -field, and the martingale convergence theorem [154, Section 14.2] implies that $g_J \rightarrow g$ in $L^1(\theta)$. This proves the convergence. $\square$

Lemma 19.8 (Total variation convergence from a translation bound). *Let $J \geq 1$, and let μ_n , $n \geq 1$, and μ be Borel probability measures on $\mathbb{R}^J$ such that μ_n converges weakly to μ . For $w \in \mathbb{R}^J$, let $\tau_w(x) = x + w$, and let $\tau_w \mu$ be the image of μ under τ_w .*

1. *If ρ_n , $n \geq 1$, and ρ are Borel probability measures on $\mathbb{R}^J$ such that ρ_n converges weakly to ρ , then $\|\mu - \rho\|_{\text{TV}} \leq \liminf_{n \rightarrow \infty} \|\mu_n - \rho_n\|_{\text{TV}}$.*
2. *Suppose that there exist $B < \infty$ and n_0 such that*

$$\|\tau_w \mu_n - \mu_n\|_{\text{TV}} \leq B|w| \quad \text{for all } n \geq n_0 \text{ and } w \in \mathbb{R}^J. \quad (19.20)$$

Then μ is absolutely continuous with respect to Lebesgue measure, and $\|\mu_n - \mu\|_{\text{TV}} \rightarrow 0$. In particular, if μ_n has the density p_n and μ has the density p , then $p_n \rightarrow p$ in $L^1(\mathbb{R}^J)$.

The second part involves no integration in time: it concerns one sequence of measures on $\mathbb{R}^J$ with J fixed. The limit μ is the given weak limit, and its absolute continuity is part of the conclusion.

Proof. For the first part, we recall that, for probability measures on $\mathbb{R}^J$, the total variation distance is the supremum of $\mu(\phi) - \rho(\phi)$ over continuous functions ϕ with values in $[0, 1]$. Indeed, let E be a Borel set. Since the finite measure $\mu + \rho$ is regular [30, Theorem 1.1], there exist a closed set $F \subset E$ and an open set $O \supset E$ such that $(\mu + \rho)(O \setminus F)$ is arbitrarily small. A continuous function with values in $[0, 1]$ that equals 1 on F and 0 off O then approximates $\mathbf{1}_E$ in $L^1(\mu + \rho)$. For each such ϕ , the difference $\mu_n(\phi) - \rho_n(\phi)$ converges to $\mu(\phi) - \rho(\phi)$, and $\mu_n(\phi) - \rho_n(\phi) \leq \|\mu_n - \rho_n\|_{\text{TV}}$. Taking the supremum over ϕ proves the first part.

For the second part, the maps τ_w are continuous, and $\tau_w \mu_n$ converges weakly to $\tau_w \mu$. Then the first part and (19.20) imply that

$$\|\tau_w \mu - \mu\|_{\text{TV}} \leq B|w| \quad (w \in \mathbb{R}^J). \quad (19.21)$$

Let η be a smooth probability density supported in the unit ball, and let $\eta_\alpha(x) = \alpha^{-J} \eta(x/\alpha)$ for $\alpha > 0$. For a Borel probability measure σ on $\mathbb{R}^J$, the convolution $\sigma * \eta_\alpha$ has the density

$$q_{\sigma, \alpha}(x) = \int \eta_\alpha(x - y) \sigma(dy),$$

which has integral one by Tonelli's theorem. For a Borel function ϕ with values in $[0, 1]$, we have $(\sigma * \eta_\alpha)(\phi) = \int \eta_\alpha(w) (\tau_w \sigma)(\phi) dw$, and then

$$|\sigma(\phi) - (\sigma * \eta_\alpha)(\phi)| \leq \int \eta_\alpha(w) \|\sigma - \tau_w \sigma\|_{\text{TV}} dw.$$

Since $|w| \leq \alpha$ on the support of η_α , the bounds (19.20) and (19.21) imply that

$$\|\mu_n - \mu_n * \eta_\alpha\|_{\text{TV}} \leq B\alpha \quad (n \geq n_0), \quad \|\mu - \mu * \eta_\alpha\|_{\text{TV}} \leq B\alpha. \quad (19.22)$$

Now fix $\alpha > 0$. For every x , the function $\eta_\alpha(x - \cdot)$ is bounded and continuous, and by weak convergence we have $q_{\mu_n, \alpha}(x) \rightarrow q_{\mu, \alpha}(x)$. These functions are probability densities. Writing $|q - q'| = q + q' - 2 \min(q, q')$ and applying dominated convergence to $\min(q_{\mu_n, \alpha}, q_{\mu, \alpha}) \leq q_{\mu, \alpha}$, we find that $q_{\mu_n, \alpha} \rightarrow q_{\mu, \alpha}$ in $L^1(\mathbb{R}^J)$; this is Scheffé's lemma [129, pp. 435–437], [154, Section 5.10]. Using the triangle inequality and (19.22), we obtain

$$\limsup_{n \rightarrow \infty} \|\mu_n - \mu\|_{\text{TV}} \leq 2B\alpha + \lim_{n \rightarrow \infty} \frac{1}{2} \|q_{\mu_n, \alpha} - q_{\mu, \alpha}\|_{L^1(\mathbb{R}^J)} = 2B\alpha.$$

As α is arbitrary, $\|\mu_n - \mu\|_{\text{TV}} \rightarrow 0$. Finally, let E be a Borel set of Lebesgue measure zero. Then $(\mu * \eta_\alpha)(E) = 0$, and (19.22) gives $\mu(E) \leq B\alpha$ for every $\alpha > 0$. Thus $\mu(E) = 0$, and μ is absolutely continuous. The statement about densities follows from Lemma 3.6. $\square$

Lemma 19.9 (Vanishing spectral correlations). *Let $\lambda_0 > 0$, $M < \infty$, and $0 \leq t_1 < t_2$. For each $n \geq 1$, let c_n and $w_{n,j}$, $j \geq 1$, be real numbers with $\sum_j |w_{n,j}| \leq M$, let $\lambda_{n,j} \geq \lambda_0$ for $j \geq 1$, and let*

$$h_n(t) = c_n + \sum_{j \geq 1} w_{n,j} e^{-\lambda_{n,j} t} \quad (t \geq 0).$$

If $c_n \rightarrow c$ and $h_n(t) \rightarrow 0$ for every $t \in [t_1, t_2]$, then $c = 0$.

The coefficients $w_{n,j}$ may have either sign, and we assume no convergence of the $\lambda_{n,j}$ or of the $w_{n,j}$ as $n \rightarrow \infty$. The proof uses both the lower bound λ_0 , which is uniform in n , and the bound on $\sum_j |w_{n,j}|$.

Proof. We use the Chebyshev polynomials T_m , defined by $T_0 = 1$, $T_1(x) = x$, and $T_{m+2}(x) = 2xT_{m+1}(x) - T_m(x)$ [117, (18.9.1) and Table 18.9.1]. The identity $T_m(\cos \vartheta) = \cos(m\vartheta)$ [117, (18.5.1)] shows that $|T_m(x)| \leq 1$ for $x \in [-1, 1]$. We also need the lower bound

$$T_m(x) \geq 1 + m^2(x - 1) \quad (x \geq 1, m \geq 0). \quad (19.23)$$

To prove it, fix $x \geq 1$, and let $d = x - 1$ and $\Delta_m = T_{m+1}(x) - T_m(x)$. We show by induction that $T_m(x) \geq 1 + m^2d$ and $\Delta_m \geq (2m + 1)d$. For $m = 0$, we have $T_0 = 1$ and $\Delta_0 = d$. If both bounds hold for m , then $T_{m+1}(x) = T_m(x) + \Delta_m \geq 1 + (m + 1)^2d$, and the recursion gives $\Delta_{m+1} = \Delta_m + 2dT_{m+1}(x) \geq (2m + 1)d + 2d = (2m + 3)d$, since $T_{m+1}(x) \geq 1$. This proves (19.23).

Fix an integer $m \geq 1$, and let $\tau = (t_2 - t_1)/m$ and $\kappa = e^{-\lambda_0\tau} \in (0, 1)$. We define the polynomial

$$p_m(z) = \frac{T_m(2z/\kappa - 1)}{T_m(2/\kappa - 1)} = \sum_{k=0}^m a_k z^k.$$

The denominator is at least 1 by (19.23), and $p_m(1) = 1$. For $z \in [0, \kappa]$ we have $2z/\kappa - 1 \in [-1, 1]$, and (19.23) gives

$$|p_m(z)| \leq \frac{1}{1 + m^2(2/\kappa - 2)} \leq \frac{1}{1 + 2m\lambda_0(t_2 - t_1)} \quad (0 \leq z \leq \kappa). \quad (19.24)$$

In the last inequality, we used $2/\kappa - 2 = 2(e^{\lambda_0\tau} - 1) \geq 2\lambda_0\tau$. Since $\sum_j |w_{n,j}| < \infty$, we may exchange the finite sum over k with the sum over j , and we find

$$\sum_{k=0}^m a_k h_n(t_1 + k\tau) = c_n p_m(1) + \sum_{j \geq 1} w_{n,j} e^{-\lambda_{n,j}t_1} p_m(e^{-\lambda_{n,j}\tau}).$$

Here $p_m(1) = 1$, $e^{-\lambda_{n,j}t_1} \leq 1$, and $e^{-\lambda_{n,j}\tau} \in (0, \kappa]$. Using (19.24) and $\sum_j |w_{n,j}| \leq M$, we obtain

$$|c_n| \leq \sum_{k=0}^m |a_k| |h_n(t_1 + k\tau)| + \frac{M}{1 + 2m\lambda_0(t_2 - t_1)}.$$

The times $t_1 + k\tau$, $0 \leq k \leq m$, lie in $[t_1, t_2]$, and the coefficients a_k do not depend on n . Letting $n \rightarrow \infty$, we find that $|c| \leq M/(1 + 2m\lambda_0(t_2 - t_1))$. Letting $m \rightarrow \infty$, we conclude that $c = 0$. $\square$

We do not claim that the coefficients a_k are bounded uniformly in m . This is why the degree m is fixed before the limit $n \rightarrow \infty$.

19.3 Lower bounds on the critical time scale

In this section we prove that the total variation distance stays above $3/4$ up to a time of order $b = N^{2/3}$ (Lemma 19.10), and then that it decays no faster than exponentially on the scale b (Corollary 19.11). The first estimate needs only the trace condition and the Weyl bounds.

Lemma 19.10 (Lower bound at short critical times). *There exist $s_* \in (0, 1]$ and N_0 , depending only on c_0 and C_0 , such that the following holds. If $N \geq N_0$ and r satisfies (19.2) and (19.3), then*

$$D_N(s_*b) \geq 3/4.$$

Proof. We find an event that has probability at most $1/8$ under π_N and probability at least $7/8$ under $\nu_N P_{s_*b}$.

Step 1: the stationary law. By the Weyl bounds, the variances $q_i = (2r_i)^{-1}$ satisfy

$$\frac{b}{2C_0 i^{2/3}} \leq q_i \leq \frac{b}{2c_0 i^{2/3}}. \quad (19.25)$$

In particular, $q_1, q_2, q_3 \geq \delta' \max_i q_i$ with $\delta' = c_0/(3^{2/3}C_0)$, and $m_2 = \sum_i q_i^2$ satisfies $b^2/(4C_0^2) \leq m_2 \leq b^2/c_0^2$, since $\sum_i i^{-4/3} \leq 4$. Let $G_1, \dots, G_N$ be independent standard Gaussians, let φ be the density of $\sum_i q_i G_i^2$, and let φ_{-1} be the density of $\sum_{i \geq 2} q_i G_i^2$. Recall that π_N is the Gaussian law with variances q_i conditioned on $|x|^2 = N = \sum_i q_i$. By Lemma 5.9, which applies since $N \geq 4$ (we take $N_0 \geq 4$), the first coordinate has the density

$$\pi_N(x_1 \in dx) = \frac{\varphi_{-1}(N - x^2)}{\varphi(N)} \frac{e^{-x^2/(2q_1)}}{\sqrt{2\pi q_1}} dx, \quad |x| < \sqrt{N}.$$

Indeed, in the notation of Lemma 5.9 with $A = \{1\}$ and $x = \sqrt{q_1}z$, the density of the centered sum $\sum_i q_i(G_i^2 - 1)$ at 0 is $\varphi(N)$, and the density of $\sum_{i \geq 2} q_i(G_i^2 - 1)$ at $-q_1(z^2 - 1)$ is $\varphi_{-1}(N - q_1 z^2)$. By Lemma 8.12 and the bounds on m_2 , there exist $c_3, C_3 > 0$, depending only on c_0 and C_0 , such that $\varphi(N) \geq c_3/b$ and $\sup \varphi_{-1} \leq C_3/b$. Inserting these bounds into the last display, we obtain, for every $a \in (0, 1)$,

$$\pi_N(x_1^2 \leq ab) \leq \frac{C_3}{c_3} \int_{-\sqrt{ab}}^{\sqrt{ab}} \frac{dx}{\sqrt{2\pi q_1}} \leq C_1 \sqrt{a},$$

where we used $q_1 \geq b/(2C_0)$. We fix $a \in (0, 1)$ with $C_1 \sqrt{a} \leq 1/8$.

Step 2: the time- t law. By Lemma 10.2, $X(t) = \sqrt{N}Y(t)/|Y(t)|$ has law $\nu_N P_t$. Using the Cauchy–Schwarz inequality and Lemma 10.10 with $K = 1$ and $p = 2$, we have, for $N \geq N_0$ and $0 \leq t \leq b$,

$$\mathbb{E}[X_1(t)^2] = N \mathbb{E}\left[\frac{Y_1(t)^2}{\mathcal{A}_t}\right] \leq N \|Y_1(t)^2\|_2 \|\mathcal{A}_t^{-1}\|_2 \leq C \frac{h_1(t)}{\bar{h}(t)} \leq C_2(1+t).$$

In the last step we used Lemma 7.14: $h_1(t) \leq C(1+t)^{1/2}$ and $\bar{h}(t) \geq c_4(1+t)^{-1/2}$, where $\bar{h}(t) = N^{-1} \sum_i h_i(t)$ and c_4 depends only on c_0 and C_0 . We now fix $s_* \in (0, 1]$ with $2C_2s_* \leq a/8$, and we increase N_0 so that $s_*b \geq 1$ for $N \geq N_0$. By Markov's inequality, we have

$$\nu_N P_{s_*b}(x_1^2 > ab) \leq \frac{C_2(1+s_*b)}{ab} \leq \frac{2C_2s_*}{a} \leq \frac{1}{8}.$$

Step 3: conclusion. The event $\{x_1^2 \leq ab\}$ has probability at least $7/8$ under $\nu_N P_{s_*b}$ and at most $1/8$ under π_N . Hence $D_N(s_*b) \geq 3/4$. The choices of a , s_* , and N_0 depend only on c_0 and C_0 . $\square$

Corollary 19.11 (Mixing times at every accuracy). *1. There exist $c_1, C_1, c_2, C_2 > 0$ and N_0 , depending only on (c_0, C_0, δ) , such that, if $N \geq N_0$ and r satisfies the critical spectral conditions with constants (c_0, C_0, δ) , then*

$$D_N(t) \geq c_1 e^{-C_1 t/b} \quad (t \geq 0), \quad (19.26)$$

and, for every $\varepsilon \in (0, 1/2)$,

$$c_2 b \log(1/\varepsilon) \leq t_N(\varepsilon) \leq C_2 b \log(1/\varepsilon). \quad (19.27)$$

2. For every $\eta > 0$ and $M < \infty$, there exist $C_{\eta, M}$, N_0 , and events $\mathcal{A}_N$ with $\liminf_{N \rightarrow \infty} \mathbb{P}(\mathcal{A}_N) \geq 1 - \eta$ such that, on $\mathcal{A}_N$ and for $N \geq N_0$, the mixing times of the GOE model with $c = 1 + \sigma N^{-1/3}$ satisfy

$$C_{\eta, M}^{-1} N^{2/3} \log(1/\varepsilon) \leq t_N(\varepsilon) \leq C_{\eta, M} N^{2/3} \log(1/\varepsilon)$$

simultaneously for all $\sigma \in [-M, M]$ and all $\varepsilon \in (0, 1/2)$.

The bounds hold for all $\varepsilon \in (0, 1/2)$ at once, so they apply to accuracies that depend on N .

Proof. We begin with the lower bound (19.26). Let s_* be as in Lemma 19.10 and $\ell = s_*b/2$. By Theorem 12.13 with $p = 4$ and $s_0 = s_*/2$, the density h_ℓ of $\nu_N P_\ell$ with respect to π_N satisfies $\|h_\ell - 1\|_{L^4(\pi_N)} \leq \|h_\ell\|_{L^4(\pi_N)} + 1 \leq M_4$, where M_4 depends only on (c_0, C_0, δ) . Further, $D_N(2\ell) \geq 3/4$ by Lemma 19.10. Then Lemma 19.6 with $d = 3/4$ implies (19.26), with $M = M_4$, $c_1 = 27/(16M_4^2)$, and $C_1 = 3 \log(1/\vartheta)/s_*$, where $\vartheta = \min(1/2, 9/(4M_4^2))$.

For the lower bound in (19.27), let $t \geq 0$ with $D_N(t) \leq \varepsilon$. Then $c_1 e^{-C_1 t/b} \leq \varepsilon$ by (19.26), that is, $t \geq (b/C_1) \log(c_1/\varepsilon)$. Taking the infimum over such t , we have $t_N(\varepsilon) \geq (b/C_1) \log(c_1/\varepsilon)$. For $\varepsilon \leq c_1^2$ we have $\log(c_1/\varepsilon) \geq \log(1/\varepsilon)/2$, and then $t_N(\varepsilon) \geq (b/2C_1) \log(1/\varepsilon)$. For $c_1^2 < \varepsilon < 1/2$, we note that $D_N(t) \geq D_N(s_*b) \geq 3/4$ for $t \leq s_*b$. Then $t_N(\varepsilon) \geq s_*b \geq s_*b \log(1/\varepsilon)/\log(c_1^{-2})$; here $c_1 < 1$, because $M_4^2 \geq 9/4$ by Lemma 19.6. The two cases together prove the lower bound, with $c_2 = \min(1/(2C_1), s_*/\log(c_1^{-2}))$.

For the upper bound, we use the Poincaré inequality and the L^2 bound on the density at time b . Let K_P be as in Table 19.1, so that $C_P(\pi_N) \leq K_P b$ by the first part of Proposition 8.21, and let $C_{2,1}$ be the constant of Theorem 12.13 with $p = 2$ and $s_0 = 1$, so that $\|f_{N,1}\|_{L^2(\pi_N)} \leq C_{2,1}$. We increase N_0 so that both bounds hold for $N \geq N_0$; the constants K_P , $C_{2,1}$, and N_0 depend only on (c_0, C_0, δ) . The density $f_{N,1} = d(\nu_N P_b)/d\pi_N$ is of class C^2 , and $\pi_N(f_{N,1}) = 1$. By reversibility and the semigroup property, $\nu_N P_{b+u}$ has the density $P_u f_{N,1}$ for $u \geq 0$. Let $g = f_{N,1} - 1$ and $V(u) = \|P_u g\|_{L^2(\pi_N)}^2$. Since g is of class C^2 and $\pi_N(P_u g) = \pi_N(g) = 0$, the integration by parts formula (3.4) and the Poincaré inequality give

$$V'(u) = -2 \int |\nabla_{S_N} P_u g|^2 d\pi_N \leq -\frac{2}{K_P b} V(u).$$

Then $e^{2u/(K_P b)} V(u)$ is nonincreasing, and $\|P_u g\|_{L^2(\pi_N)} \leq e^{-u/(K_P b)} \|g\|_{L^2(\pi_N)}$; this is the usual exponential decay in L^2 implied by a Poincaré inequality [12, Section 4.2]. Then Lemma 3.6 and the Cauchy–Schwarz inequality imply that

$$D_N(b+u) = \frac{1}{2} \|P_u g\|_{L^1(\pi_N)} \leq \frac{1}{2} \|f_{N,1} - 1\|_{L^2(\pi_N)} e^{-u/(K_P b)} \leq C_5 e^{-u/(K_P b)} \quad (u \geq 0), \quad (19.28)$$

where $C_5 = (C_{2,1} + 1)/2$. In the last step we used $\|1\|_{L^2(\pi_N)} = 1$ and the triangle inequality. Note that $C_{2,1} \geq \|f_{N,1}\|_{L^1(\pi_N)} = 1$, so that $C_5 \geq 1$. For $\varepsilon \in (0, 1/2)$, let $u_\varepsilon = K_P b \log(C_5/\varepsilon) \geq 0$. Then $D_N(b+u_\varepsilon) \leq \varepsilon$ by (19.28), and $t_N(\varepsilon) \leq b + K_P b \log(C_5/\varepsilon)$. For $\varepsilon < 1/2$ we have $1 \leq \log(1/\varepsilon)/\log 2$ and $\log(C_5/\varepsilon) \leq (1 + \log C_5/\log 2) \log(1/\varepsilon)$, and the upper bound in (19.27) follows, with $C_2 = 1/\log 2 + K_P(1 + \log C_5/\log 2)$.

For the second part, Lemma 4.17 provides, for given η and M , constants (c_0, C_0, δ) and events $\mathcal{A}_N$ with $\liminf_N \mathbb{P}(\mathcal{A}_N) \geq 1 - \eta$ on which, for every $\sigma \in [-M, M]$, the half-precisions of the model in increasing order satisfy the critical spectral conditions with these constants. The constants in the first part depend only on (c_0, C_0, δ) , so the first part, applied for each σ on $\mathcal{A}_N$, proves the claim. $\square$

19.4 The remaining coordinates

In this section we show that, at every time $s \geq s_0 > 0$, the total variation distance is determined by the first J coordinates up to an error $O(J^{-1/3})$, with a constant that depends on s_0 but not on s . There are two inputs. The first is a bound on the energy of the density $f_{N,s}$, which follows from the L^2 bound on the density at time $s_0/2$ and the smoothing of the semigroup. The second is a Poincaré inequality for the conditional law of the remaining coordinates, which holds for every conditioning value. For $t \geq 0$ let $Q_{N,t} = P_{bt}$, whose generator is bL_r , and let

$$\mathcal{E}_N(u) = b \int_{S_N} |\nabla_{S_N} u|^2 d\pi_N$$

be its Dirichlet form, defined on the closure of the smooth functions in the norm $(\|u\|_{L^2(\pi_N)}^2 + \mathcal{E}_N(u))^{1/2}$. There is no factor 1/2 in $\mathcal{E}_N$, in accordance with (3.4). By reversibility and the semigroup property, $f_{N,s} = Q_{N,s-t}f_{N,t}$ for $s \geq t > 0$; indeed, both sides have the same integral against every bounded measurable function.

Lemma 19.12 (Finite semigroup smoothing). *Let $N \geq 2$ and $r \in \mathbb{R}^N$. For every $u \in L^2(\pi_r)$, $t > 0$, and $h \geq 0$, the function $Q_{N,t}u$ belongs to the domain of $\mathcal{E}_N$, and*

$$\|(Q_{N,h} - I)Q_{N,t}u\|_{L^2(\pi_r)} \leq \frac{h}{et} \|u\|_{L^2(\pi_r)}, \quad \mathcal{E}_N(Q_{N,t}u) \leq \frac{1}{2t} \|u\|_{L^2(\pi_r)}^2. \quad (19.29)$$

These bounds do not depend on N or on r .

Proof. By the elliptic spectral theory recalled after (8.1) [130, Section 2, p. 30], there exists an orthonormal basis $(e_j)_{j \geq 0}$ of $L^2(\pi_r)$ consisting of smooth eigenfunctions of $-bL_r$, with eigenvalues $\lambda_j \geq 0$ and $e_0 = 1$. We write $u = \sum_j u_j e_j$. Then $Q_{N,t}u = \sum_j e^{-t\lambda_j} u_j e_j$, and Parseval's identity [37, Corollary 5.10] gives

$$\|(Q_{N,h} - I)Q_{N,t}u\|_{L^2(\pi_r)}^2 = \sum_j (1 - e^{-h\lambda_j})^2 e^{-2t\lambda_j} u_j^2 \leq \left(\sup_{\lambda \geq 0} h\lambda e^{-t\lambda} \right)^2 \sum_j u_j^2.$$

Here we used $1 - e^{-h\lambda} \leq h\lambda$. Since $\sup_{\lambda \geq 0} \lambda e^{-t\lambda} = 1/(et)$, this proves the first bound. For the second bound, let $S_n = \sum_{j \leq n} e^{-t\lambda_j} u_j e_j$, which is smooth. By (8.1) and orthonormality, we have $\mathcal{E}_N(S_n - S_m) = \sum_{m < j \leq n} \lambda_j e^{-2t\lambda_j} u_j^2$ for $m < n$. Since $\lambda e^{-2t\lambda} \leq 1/(2et)$ for $\lambda \geq 0$, the partial sums S_n converge to $Q_{N,t}u$ in the norm of the domain of $\mathcal{E}_N$, and

$$\mathcal{E}_N(Q_{N,t}u) = \sum_j \lambda_j e^{-2t\lambda_j} u_j^2 \leq \frac{1}{2et} \sum_j u_j^2 \leq \frac{1}{2t} \|u\|_{L^2(\pi_r)}^2.$$

□

We apply Lemma 19.12 to the densities $f_{N,s}$. Fix $s_0 > 0$. By Theorem 12.13 with $p = 2$, there exist $M_{s_0} \geq 1$ and N_{s_0} , depending only on s_0 and (c_0, C_0, δ) , such that $\|f_{N,s}\|_{L^2(\pi_N)} \leq M_{s_0}$ for every $s \geq s_0/2$, every $N \geq N_{s_0}$, and every r that satisfies the critical spectral conditions with constants (c_0, C_0, δ) . Let $s \geq s_0$ and $N \geq N_{s_0}$. Applying Lemma 19.12 with $u = f_{N,s_0/2}$ and $t = s - s_0/2 \geq s_0/2$, we have

$$\mathcal{E}_N(f_{N,s}) \leq \frac{M_{s_0}^2}{2(s - s_0/2)} \leq \frac{M_{s_0}^2}{s_0} \quad (s \geq s_0, N \geq N_{s_0}). \quad (19.30)$$

We use the density at time $s_0/2$, and not at time s_0 , so that the semigroup acts for a time at least $s_0/2$; no upper bound on s is needed. Similarly, let $s_0 \leq s \leq s'$. The first bound in (19.29), applied with $u = f_{N,s_0/2}$, the time $s - s_0/2 \geq s_0/2$, and $h = s' - s$, gives

$\|f_{N,s'} - f_{N,s}\|_{L^2(\pi_N)} \leq 2M_{s_0}(s' - s)/(es_0)$. Since π_N is a probability measure, the $L^1(\pi_N)$ norm is at most the $L^2(\pi_N)$ norm, and Lemma 3.6 gives

$$\|\nu_N P_{bt} - \nu_N P_{bs}\|_{TV} \leq L_{s_0}|t - s|, \quad L_{s_0} = \frac{M_{s_0}}{es_0} \quad (s, t \geq s_0, N \geq N_{s_0}). \quad (19.31)$$

The functional inequalities of Section 8 are stated on the sphere S_N of squared radius N . The conditional laws below live on spheres of arbitrary radius, so we first record the rescaled form. Logarithmic Sobolev and Poincaré constants are taken with respect to the tangent gradient of the sphere on which the measure lives.

Lemma 19.13 (Spheres of arbitrary radius). *Fix $\delta' \in (0, 1]$. Let $n \geq 3$, $S > 0$, and let π be the probability measure on $\{w \in \mathbb{R}^n : |w|^2 = S\}$ with density proportional to $\exp(-\sum_i \omega_i w_i^2)$ with respect to the uniform measure, where $\omega_i > 0$ and $\sum_i (2\omega_i)^{-1} = S$. Let $t = \min_i \omega_i$. Suppose that $St \geq 4$, that at least 260 indices satisfy $\omega_i \leq 2t$, and that three distinct weights $(2\omega_i)^{-1}$ are at least $\delta' \max_j (2\omega_j)^{-1}$. Then*

$$C_P(\pi) \leq 1/t, \quad C_{LS}(\pi) \leq C_{\delta'}/t,$$

where $C_{\delta'}$ depends only on δ' .

Proof. This is the scaling recorded in Remark 8.10; we include the short argument. Let $w = \sqrt{S/n} w'$, so that w' lies on the sphere of squared radius n . The density becomes proportional to $\exp(-\sum_i \omega'_i w_i'^2)$ with $\omega'_i = (S/n)\omega_i$, and $\sum_i (2\omega'_i)^{-1} = (n/S) \sum_i (2\omega_i)^{-1} = n$. The minimum is $t' = (S/n)t$, and $nt' = St \geq 4$. The hypotheses on the ratios ω_i/t and on the weights are unchanged. For a function g on the sphere of squared radius S , let $g'(w') = g(\sqrt{S/n} w')$. Then $|\nabla g'|^2 = (S/n)|\nabla g|^2$ at corresponding points, while variances and entropies are unchanged. We obtain $C_P(\pi) = (S/n)C_P(\pi')$ and $C_{LS}(\pi) = (S/n)C_{LS}(\pi')$, where π' is the image measure. Corollary 8.9 with $L = 2$ and $k = 260$ shows that $C_P(\pi') \leq 1/t'$, and then $C_P(\pi) \leq 1/t$. Similarly, Corollary 8.13 implies that $C_{LS}(\pi') \leq C'_{\delta'}/t' + C'_{\delta'}C_P(\pi') \leq 2C'_{\delta'}/t'$, where $C'_{\delta'}$ is the constant of that corollary. We conclude that $C_{LS}(\pi) \leq C_{\delta'}/t$ with $C_{\delta'} = 2C'_{\delta'}$. $\square$

For $h \in \mathbb{R}^J$ with $|h|^2 < N$, we write $\pi_N(\cdot \mid x_{\leq J} = h)$ for the conditional law of $(x_i)_{i>J}$ under π_N given $x_{\leq J} = h$. Conditioned on its first J coordinates, the uniform measure ν_N is the uniform measure on the sphere $\{z \in \mathbb{R}^{N-J} : |z|^2 = N - |h|^2\}$, and the factor $\exp(-\sum_{i \leq J} r_i h_i^2)$ is constant on it. Thus $\pi_N(\cdot \mid x_{\leq J} = h)$ is the measure on this sphere with density proportional to $\exp(-\sum_{i>J} r_i z_i^2)$.

Lemma 19.14 (Logarithmic Sobolev inequality for the remaining coordinates). *Let $(r_{\cdot, N})$ satisfy the critical spectral conditions with fixed constants for every $N \in \mathcal{N}$ and converge to R , and let γ be as in (19.5). There exist an absolute constant C and an integer J_0 , depending on R ,*

such that for every $J \geq J_0$ there exists $N_0(J)$ with the following property: for all $N \geq N_0(J)$ and all $h \in \mathbb{R}^J$ with $|h|^2 < N$,

$$C_{\text{LS}}(\pi_N(\cdot \mid x_{\leq J} = h)) \leq \frac{C b}{\gamma J^{2/3}}. \quad (19.32)$$

The order of quantifiers is $\exists J_0 \forall J \geq J_0 \exists N_0(J) \forall N \geq N_0(J) \forall h$; no uniformity in J is claimed for J growing with N . The bound holds for every conditioning value, including those for which the remaining sphere is very small.

Proof. The proof has four steps. We first describe the conditional law through a common shift of the half-precisions, which reduces the claim to a lower bound on its smallest coefficient. In Steps 2 and 3 we identify this lower bound as $N \rightarrow \infty$ and then as $J \rightarrow \infty$. In Step 4 we apply Lemma 19.13. Throughout the proof, $k = 260$.

Step 1: a common shift. Fix J and $N > J + k$, and let $S = N - |h|^2 \in (0, N]$. Adding a constant to all coefficients r_i , $i > J$, does not change the conditional law on the sphere $|z|^2 = S$. The function $\alpha \mapsto \sum_{i>J} [2(r_i + \alpha)]^{-1}$ decreases continuously from $+\infty$ to 0 on $(-r_{J+1}, \infty)$, because the r_i are nondecreasing. Then there exists a unique $\alpha(S) > -r_{J+1}$ with

$$\sum_{i>J} \frac{1}{2(r_i + \alpha(S))} = S,$$

and $\alpha(S)$ is nonincreasing in S . With $\omega_i = r_i + \alpha(S)$, the conditional law is the measure of Lemma 19.13 with $n = N - J$, and its smallest coefficient is $t(S) = r_{J+1} + \alpha(S)$. Since $S \leq N$, we have $\alpha(S) \geq \alpha(N)$. At $S = N$, the trace condition implies that $\sum_{i>J} (2r_i)^{-1} < N$, and then $\alpha(N) < 0$. We write $\alpha(N) = -d_{N,J}/b$ with $0 < d_{N,J} < R_{J+1,N}$. Multiplying the defining equation by $2/b$ and subtracting the trace condition $\sum_i R_{i,N}^{-1} = 2N^{1/3}$, we find that $d_{N,J}$ is the unique root in $(0, R_{J+1,N})$ of

$$G_{N,J}(d) = A_{N,J}, \quad G_{N,J}(d) = \sum_{i>J} \frac{d}{R_{i,N}(R_{i,N} - d)}, \quad A_{N,J} = \sum_{i \leq J} \frac{1}{R_{i,N}}. \quad (19.33)$$

We have established that, for every $S \in (0, N]$,

$$t(S) \geq t(N) = \frac{R_{J+1,N} - d_{N,J}}{b}. \quad (19.34)$$

Step 2: the limit $N \rightarrow \infty$ at fixed J . Let $G_J(d) = \sum_{i>J} d/(R_i(R_i - d))$ for $0 \leq d < R_{J+1}$ and $A_J = \sum_{i \leq J} R_i^{-1}$. The function G_J is continuous and strictly increasing, with $G_J(0) = 0$ and $G_J(d) \rightarrow \infty$ as $d \uparrow R_{J+1}$, and there exists a unique $d_J \in (0, R_{J+1})$ with $G_J(d_J) = A_J$. We claim that $d_{N,J} \rightarrow d_J$. Note that $R_{i,N} \rightarrow R_i$ for each i by (19.5), and $A_{N,J} \rightarrow A_J$. Fix $0 < D < R_{J+1}$ and u_0 with $R_{J+1}^{-1} < u_0 < D^{-1}$. For large N we have $R_{J+1,N}^{-1} < u_0$, and then

all $u = R_{i,N}^{-1}$ and $u = R_i^{-1}$ with $i > J$ lie in $[0, u_0]$. The summand is $g_d(u) = du^2/(1 - du)$, and for $d \in [0, D]$ and $u \in [0, u_0]$,

$$|g'_d(u)| = \frac{du(2 - du)}{(1 - du)^2} \leq \frac{2Du}{(1 - Du_0)^2}.$$

Integrating this bound, we have $|g_d(u) - g_d(u')| \leq C_D(u + u')|u - u'|$, and the Cauchy–Schwarz inequality together with (19.5) shows that $G_{N,J} \rightarrow G_J$ uniformly on $[0, D]$. Given $0 < \zeta < \min\{d_J, R_{J+1} - d_J\}$, we have $G_J(d_J - \zeta) < A_J < G_J(d_J + \zeta)$. For large N , uniform convergence gives $G_{N,J}(d_J - \zeta) < A_{N,J} < G_{N,J}(d_J + \zeta)$. Since $G_{N,J}$ is increasing on $[0, R_{J+1,N})$, we obtain $|d_{N,J} - d_J| < \zeta$.

Step 3: the limit $J \rightarrow \infty$. We show that

$$\lim_{J \rightarrow \infty} \frac{d_J}{\gamma J^{2/3}} = x_*, \quad \text{where } x_* \in (0, 1) \text{ solves } \sqrt{x_*} \operatorname{artanh} \sqrt{x_*} = 1. \quad (19.35)$$

Numerically $x_* = 0.69481\dots$; we only use $x_* < 1$. Let $\mathcal{G}(x) = 3\sqrt{x} \operatorname{artanh} \sqrt{x}$ for $x \in (0, 1)$, a continuous increasing function from 0 to ∞ . We first show that, uniformly for x in compact subsets of $(0, 1)$,

$$\frac{\gamma}{J^{1/3}} G_J(\gamma J^{2/3} x) \longrightarrow \int_1^\infty \frac{x du}{u^{2/3}(u^{2/3} - x)} = \mathcal{G}(x). \quad (19.36)$$

To evaluate the integral, we substitute $u = v^3$ and obtain

$$\int_1^\infty \frac{x du}{u^{2/3}(u^{2/3} - x)} = 3x \int_1^\infty \frac{dv}{v^2 - x} = \frac{3\sqrt{x}}{2} \log \frac{1 + \sqrt{x}}{1 - \sqrt{x}} = \mathcal{G}(x).$$

Fix $0 < x_1 < x_2 < 1$ and let $\tau_i = R_i/(\gamma J^{2/3})$. The left side of (19.36) equals $J^{-1} \sum_{i>J} x/(\tau_i(\tau_i - x))$. Let $\zeta_J = \sup_{i>J} |R_i/(\gamma i^{2/3}) - 1|$, which tends to zero. Then $\tau_i = (i/J)^{2/3}(1 + O(\zeta_J))$ uniformly in $i > J$. For large J , both τ_i and $(i/J)^{2/3}$ are at least $\tau_0 = (1 + x_2)/2$ for all $i > J$. For $\tau \geq \tau_0$ and $x \leq x_2$ we have $\tau - x \geq c_2\tau$ with $c_2 = (1 - x_2)/(1 + x_2)$, and the derivative of $\tau \mapsto x/(\tau(\tau - x))$ is then at most $2/(c_2^2\tau^3)$ in absolute value. As $|\tau_i - (i/J)^{2/3}| \leq C\zeta_J(i/J)^{2/3}$, replacing τ_i by $(i/J)^{2/3}$ changes the sum by at most

$$\frac{C}{J} \sum_{i>J} \zeta_J \left(\frac{i}{J}\right)^{-4/3} \leq C'\zeta_J.$$

The function $\phi_x(u) = x/(u^{2/3}(u^{2/3} - x))$ is decreasing on $[1, \infty)$, so that

$$\int_{1+1/J}^\infty \phi_x(u) du \leq \frac{1}{J} \sum_{i>J} \phi_x(i/J) \leq \int_1^\infty \phi_x(u) du,$$

and $\int_1^{1+1/J} \phi_x \leq \phi_x(1)/J \leq 1/(J(1-x_2))$. This proves (19.36). Next, since $R_i \sim \gamma i^{2/3}$ and $\sum_{i \leq J} i^{-2/3} \sim 3J^{1/3}$, we have $\gamma A_J/J^{1/3} \rightarrow 3$. Let $0 < \zeta < \min\{x_*, 1-x_*\}$. The function $\mathcal{G}$ is increasing with $\mathcal{G}(x_*) = 3$, so $\mathcal{G}(x_* - \zeta) < 3 < \mathcal{G}(x_* + \zeta)$. For large J , $\gamma J^{2/3}(x_* + \zeta) < R_{J+1}$ because $R_{J+1}/(\gamma J^{2/3}) \rightarrow 1$, and we deduce from (19.36) and $\gamma A_J/J^{1/3} \rightarrow 3$ that $G_J(\gamma J^{2/3}(x_* - \zeta)) < A_J < G_J(\gamma J^{2/3}(x_* + \zeta))$. As G_J is increasing, it follows that $d_J/(\gamma J^{2/3}) \in (x_* - \zeta, x_* + \zeta)$, which proves (19.35).

From (19.35) and $R_{J+1}/(\gamma J^{2/3}) \rightarrow 1$, we obtain $(R_{J+1} - d_J)/(\gamma J^{2/3}) \rightarrow 1 - x_* > 0$. Further, $R_{J+k} - R_{J+1} = o(J^{2/3})$ as $J \rightarrow \infty$, because R_{J+k} and R_{J+1} are both $\gamma J^{2/3}(1 + o(1))$. We choose J_0 such that, for all $J \geq J_0$,

$$R_{J+1} - d_J \geq \frac{1-x_*}{2} \gamma J^{2/3}, \quad R_{J+k} - R_{J+1} \leq \frac{1-x_*}{8} \gamma J^{2/3}. \quad (19.37)$$

Then, for $J \geq J_0$, Step 2 and the convergence $R_{i,N} \rightarrow R_i$ provide $N_0(J)$ such that, for $N \geq N_0(J)$,

$$R_{J+1,N} - d_{N,J} \geq \frac{1-x_*}{4} \gamma J^{2/3}, \quad R_{J+k,N} - R_{J+1,N} \leq \frac{1-x_*}{4} \gamma J^{2/3}. \quad (19.38)$$

Step 4: conclusion. Let $J \geq J_0$, $N \geq N_0(J)$, and $S \in (0, N]$. Combining (19.34) and (19.38), we find that $bt(S) \geq (1-x_*)\gamma J^{2/3}/4 \geq R_{J+k,N} - R_{J+1,N}$. Then, for $J < i \leq J+k$,

$$\omega_i = t(S) + r_i - r_{J+1} \leq t(S) + \frac{R_{J+k,N} - R_{J+1,N}}{b} \leq 2t(S).$$

Thus at least $k = 260$ coefficients are at most $Lt(S)$ with $L = 2$, and $k - 2 = 258 \geq 64L^2$. Using the defining equation of $\alpha(S)$, we have

$$St(S) = \frac{1}{2} \sum_{i>J} \frac{t(S)}{\omega_i} \geq \frac{1}{2} \sum_{i=J+1}^{J+k} \frac{t(S)}{2t(S)} = \frac{k}{4} \geq 4.$$

The three weights $(2\omega_i)^{-1}$, $i = J+1, J+2, J+3$, are at least $(4t(S))^{-1} = \max_j (2\omega_j)^{-1}/2$. Now Lemma 19.13 with $\delta' = 1/2$ implies that

$$C_{LS}(\pi_N(\cdot \mid x_{\leq J} = h)) \leq \frac{C_{1/2}}{t(S)} \leq \frac{4C_{1/2}}{1-x_*} \cdot \frac{b}{\gamma J^{2/3}},$$

which is (19.32). □

Lemma 19.15 (The remaining coordinates at positive times). *Under the hypotheses of Lemma 19.14, let $s_0 > 0$. There exists C , depending only on s_0 , γ , and (c_0, C_0, δ) , such that for every $J \geq J_0$ there exists $N_0(J, s_0)$ with*

$$0 \leq D_N(bs) - D_N^{\leq J}(bs) \leq CJ^{-1/3} \quad \text{for all } N \in \mathcal{N} \text{ with } N \geq N_0(J, s_0) \text{ and all } s \geq s_0. \quad (19.39)$$

As in Lemma 19.14, J is fixed before $N \rightarrow \infty$, and no uniformity is claimed for J growing with N . The bound holds for all $s \geq s_0$ at once.

Proof. Let $J \geq J_0$, and let C' be the absolute constant in (19.32). We choose $N_0(J, s_0) \geq \max\{N_0(J), N_{s_0}, J + 3\}$, with $N_0(J)$ as in Lemma 19.14 and N_{s_0} as in (19.30). Fix $N \geq N_0(J, s_0)$ and $s \geq s_0$, and let $\mu = \nu_N P_{bs}$ and $f = f_{N,s}$, which is of class C^2 .

Step 1: a conditional Poincaré inequality. A logarithmic Sobolev inequality implies a Poincaré inequality with half its constant [12, Section 5.1]. In the convention (3.5), this follows by applying the logarithmic Sobolev inequality to $1 + \zeta g$ for a smooth function g and letting $\zeta \rightarrow 0$: we have $\text{Ent}((1 + \zeta g)^2) = 2\zeta^2 \text{Var}(g) + o(\zeta^2)$, while the energy of $1 + \zeta g$ is ζ^2 times the energy of g . Let $h \in \mathbb{R}^J$ with $|h|^2 < N$. By (19.32), for every smooth function g on the sphere $\{z \in \mathbb{R}^{N-J} : |z|^2 = N - |h|^2\}$,

$$\text{Var}_{\pi_N(\cdot | x_{\leq J} = h)}(g) \leq \frac{C'b}{2\gamma J^{2/3}} \int |\nabla g|^2 d\pi_N(\cdot | x_{\leq J} = h),$$

where ∇ is the tangent gradient of this sphere. This holds for every conditioning value h , including those for which the sphere is very small.

Step 2: the conditional variance. We write $x = (h, z)$ with $h \in \mathbb{R}^J$ and $z \in \mathbb{R}^{N-J}$, we let π_H be the law of h under π_N , and we let $f_H(h) = \mathbb{E}_{\pi_N}[f | x_{\leq J} = h]$, viewed also as a function on S_N . The set $\{|h|^2 = N\}$ has π_H -measure zero, since $N \geq J + 3$ (Lemma 5.9). The tangent space of the fiber $\{(h, z) : |z|^2 = N - |h|^2\}$ at (h, z) consists of the vectors $(0, w)$ with $w \cdot z = 0$, which are tangent to S_N at (h, z) . The fiber gradient $\nabla_{\text{fib}} f$ is the orthogonal projection of $\nabla_{S_N} f$ onto this subspace, and then $|\nabla_{\text{fib}} f| \leq |\nabla_{S_N} f|$. Applying Step 1 to the restriction of f to each fiber, whose tangent gradient is $\nabla_{\text{fib}} f$, and integrating over π_H , we find

$$\|f - f_H\|_{L^2(\pi_N)}^2 = \int \text{Var}_{\pi_N(\cdot | x_{\leq J} = h)}(f) \pi_H(dh) \leq \frac{C'b}{2\gamma J^{2/3}} \int |\nabla_{S_N} f|^2 d\pi_N = \frac{C'}{2\gamma J^{2/3}} \mathcal{E}_N(f). \quad (19.40)$$

Step 3: comparison of distances. Let $\tilde{\mu} = f_H \pi_N$. Since f_H is nonnegative and $\int f_H d\pi_N = 1$, this is a probability measure. It has the same law of the first J coordinates as μ , and the conditional law of π_N given the first J coordinates. Then $\|\tilde{\mu} - \pi_N\|_{\text{TV}} = D_N^{\leq J}(bs)$, since total variation distances do not change under the rescaling of the coordinates. Using Lemma 3.6, the triangle inequality, and the fact that projection does not increase total variation, we have

$$0 \leq D_N(bs) - D_N^{\leq J}(bs) \leq \|\mu - \tilde{\mu}\|_{\text{TV}} = \frac{1}{2} \|f - f_H\|_{L^1(\pi_N)} \leq \frac{1}{2} \|f - f_H\|_{L^2(\pi_N)}.$$

Inserting (19.40) and (19.30), we obtain (19.39) with $C = (M_{s_0}/2)(C'/(2\gamma s_0))^{1/2}$. $\square$

19.5 Total variation of finitely many coordinates

In this section we prove convergence of the first J rescaled coordinates in total variation at every positive time (Lemma 19.17). The proof uses bounds under π_N for vector fields that translate the first J coordinates, the energy bound (19.30), and mollification at the chosen time.

Let ∇_{S_N} and div_{S_N} be the tangent gradient and divergence on S_N . For a smooth tangent vector field V , we write $\operatorname{div}_{\pi_N} V = \operatorname{div}_{S_N} V - V \cdot \nabla(\sum_i r_i x_i^2)$, so that $\int (V\phi) d\pi_N = -\int \phi \operatorname{div}_{\pi_N} V d\pi_N$ for smooth ϕ .

Lemma 19.16 (Stationary bounds for the translation fields). *Let $(r_{\cdot,N})$ satisfy the critical spectral conditions with constants (c_0, C_0, δ) for every $N \in \mathcal{N}$ and converge to R , and fix $J \geq 1$. For $N > J$ and $a \leq J$, let $\Sigma_1(x) = \sum_{i>J} q_i x_i^2$, let V_a be the vector field on $\{x \in S_N : x_{>J} \neq 0\}$ with components*

$$(V_a)_a = 1, \quad (V_a)_i = 0 \quad (i \leq J, i \neq a), \quad (V_a)_i = -\frac{x_a q_i x_i}{\Sigma_1} \quad (i > J),$$

and let $\tilde{V}_a = N^{1/3} V_a$. There exist C_J and N_J such that, for all $N \in \mathcal{N}$ with $N \geq N_J$ and all $a \leq J$,

$$\|\operatorname{div}_{\pi_N} \tilde{V}_a\|_{L^2(\pi_N)} \leq C_J, \quad \int |\tilde{V}_a|^2 d\pi_N \leq C_J b. \quad (19.41)$$

The constants C_J and N_J depend on J and on the sequence $(r_{\cdot,N})$, but not on N .

The bounds (19.41) are integrals under the Gibbs measure π_N ; they do not involve the law of the diffusion at positive times. The set $\{x_{>J} = 0\}$ has π_N -measure zero, and the fields are defined π_N -almost everywhere.

Proof. We first compute the divergences of the fields, and we then bound them using moments of the coordinates under π_N .

Step 1: the divergence. Note that $x \cdot V_a = x_a - x_a \Sigma_1 / \Sigma_1 = 0$. Then V_a is tangent to every sphere centered at the origin, and V_a is homogeneous of degree zero. For such a field the spherical divergence equals the Euclidean divergence, since the Euclidean divergence equals the spherical divergence plus $\langle \partial_n V_a, n \rangle$ for the unit normal $n = x/|x|$, and $\partial_n V_a = 0$ by homogeneity. Let $\Sigma_2(x) = \sum_{i>J} q_i^2 x_i^2$. Differentiating, we find

$$\operatorname{div} V_a = \sum_{i>J} \partial_i (V_a)_i = -x_a \frac{\sum_{i>J} q_i}{\Sigma_1} + 2x_a \frac{\Sigma_2}{\Sigma_1^2}.$$

Using $2r_i q_i = 1$, we find that on S_N $-V_a \cdot \nabla(\sum_i r_i x_i^2) = -2r_a x_a + x_a \sum_{i>J} x_i^2 / \Sigma_1 = -2r_a x_a + x_a(N - |x_{\leq J}|^2) / \Sigma_1$. Using $\sum_i q_i = N$, we obtain

$$\operatorname{div}_{\pi_N} V_a = -2r_a x_a + x_a \left\{ \frac{\sum_{i \leq J} q_i - |x_{\leq J}|^2}{\Sigma_1} + \frac{2\Sigma_2}{\Sigma_1^2} \right\}, \quad |V_a|^2 = 1 + \frac{x_a^2 \Sigma_2}{\Sigma_1^2}. \quad (19.42)$$

Step 2: the rescaled fields. Let $y = N^{-1/3}x$. Then $\tilde{V}_a$ acts on functions of $y_{\leq J}$ as $\partial/\partial y_a$: if $\Phi(x) = \phi(N^{-1/3}x_{\leq J})$, then $\tilde{V}_a\Phi = (\partial_a\phi)(y_{\leq J})$. We write

$$\sigma_1(y) = \sum_{i>J} \frac{y_i^2}{R_{i,N}}, \quad \sigma_2(y) = \sum_{i>J} \frac{y_i^2}{R_{i,N}^2}, \quad H(y) = \sum_{i\leq J} y_i^2,$$

and we recall the sum $A_{N,J} = \sum_{i\leq J} R_{i,N}^{-1}$ of (19.33). Recall that $q_i = b/(2R_{i,N})$ and $N^{2/3} = b$. Then $\Sigma_1 = b^2\sigma_1/2$, $\Sigma_2 = b^3\sigma_2/4$, $\sum_{i\leq J} q_i = bA_{N,J}/2$, and $|x_{\leq J}|^2 = bH$. Substituting into (19.42), we obtain

$$\operatorname{div}_{\pi_N} \tilde{V}_a = -2R_{a,N}y_a + 2y_a \left\{ \frac{A_{N,J}/2 - H}{\sigma_1} + \frac{\sigma_2}{\sigma_1^2} \right\}, \quad \frac{|\tilde{V}_a|^2}{b} = 1 + \frac{y_a^2\sigma_2}{\sigma_1^2}. \quad (19.43)$$

For instance, $N^{1/3}r_ax_a = br_ay_a = R_{a,N}y_a$, and the term with Σ_2 becomes $N^{1/3} \cdot 2x_a\Sigma_2/\Sigma_1^2 = 2by_a \cdot (b^3\sigma_2/4)(b^2\sigma_1/2)^{-2} = 2y_a\sigma_2/\sigma_1^2$. The other terms are converted in the same way, and the identity for $|\tilde{V}_a|^2$ follows from $x_a^2 = by_a^2$.

Step 3: moments under π_N . By Corollary 18.12(b), y has law μ_{v_N} under π_N , the conditioned Gaussian law with variances $v_{i,N} = (2R_{i,N})^{-1}$. Let $z_i = \sqrt{2R_{i,N}}y_i$ be the standardized coordinates. By Corollary 18.12(c), whose proof rests on the domination in Proposition 5.17, there exist N_0 and B_1 such that $\sup_i \int |z_i|^{16} d\pi_N \leq B_1$ for $N \geq N_0$. By the same result, for every $m \geq 3$ there exist $N_0(m)$ and $\mathfrak{M}_m$ such that, for $N \geq N_0(m)$, the law of $(z_i)_{i\leq m}$ under π_N has density at most $\mathfrak{M}_m$ with respect to the standard Gaussian law on $\mathbb{R}^m$. Since $R_{i,N} \geq c_0$ by the Weyl bounds, we have

$$\sup_i \int |y_i|^{16} d\pi_N \leq B_0, \quad \int H^8 d\pi_N \leq J^7 \sum_{i\leq J} \int |y_i|^{16} d\pi_N \leq J^8 B_0, \quad B_0 = \frac{B_1}{(2c_0)^8}.$$

In the second bound we used $(\sum_{i\leq J} y_i^2)^8 \leq J^7 \sum_{i\leq J} y_i^{16}$, which follows from Hölder's inequality. For the inverse moments of σ_1 , let $I = \{J+1, \dots, J+19\}$ and $N \geq J+19$. By the Weyl bounds, $R_{i,N} \leq D_J = C_0(J+19)^{2/3}$ for $i \in I$, and then

$$\sigma_1 \geq \sum_{i\in I} \frac{y_i^2}{R_{i,N}} = \sum_{i\in I} \frac{z_i^2}{2R_{i,N}^2} \geq \frac{1}{2D_J^2} \sum_{i\in I} z_i^2.$$

For $N \geq N_0(J+19)$, the law of $(z_i)_{i\in I}$ has density at most $\mathfrak{M}_{J+19}$ with respect to the standard Gaussian law on $\mathbb{R}^I$, since it is a marginal of the law of $(z_i)_{i\leq J+19}$. In particular, $\sigma_1 > 0$ π_N -almost everywhere, and

$$\int \sigma_1^{-8} d\pi_N \leq B_J, \quad B_J = (2D_J^2)^8 \mathfrak{M}_{J+19} \mathbb{E} \left(\sum_{i=1}^{19} G_i^2 \right)^{-8}, \quad (19.44)$$

where $G_1, \dots, G_{19}$ are independent standard Gaussian variables. The last expectation is finite: by integration in polar coordinates in $\mathbb{R}^{19}$ [134, Chapter 2, Section 6, (6.3), p. 66], it equals

$$\int_0^\infty \varrho^2 e^{-\varrho^2/2} d\varrho \Big/ \int_0^\infty \varrho^{18} e^{-\varrho^2/2} d\varrho,$$

since $\varrho^{-16}\varrho^{18} = \varrho^2$. All integrals in this step are under the Gibbs measure π_N , and no independence of the coordinates under π_N is used. Bounds on positive moments alone would not give (19.44); the inverse moment is finite because the law of a block of $19 > 16$ coordinates is dominated by a Gaussian law.

Step 4: bounds on the fields. Let $N_J = \max\{N_0, N_0(J+19), J+19\}$, let $N \geq N_J$ and $a \leq J$, and let

$$E_a = 1 + y_a^8 + H^8 + \sigma_1^{-8}.$$

The Weyl bounds (19.3) imply that $\sigma_2 \leq c_0^{-1}\sigma_1$, $R_{a,N} \leq C_0 J^{2/3}$, and $A_{N,J} \leq J/c_0$. By the inequality between arithmetic and geometric means and the bound $u^k \leq 1 + u^8$ for $u \geq 0$ and $0 \leq k \leq 8$, each of the quantities y_a^2 , $y_a^2\sigma_1^{-1}$, $y_a^2\sigma_1^{-2}$, and $y_a^2 H^2 \sigma_1^{-2}$ is at most E_a . Using (19.43) and $(u_1 + u_2 + u_3 + u_4)^2 \leq 4(u_1^2 + u_2^2 + u_3^2 + u_4^2)$, we find

$$|\operatorname{div}_{\pi_N} \tilde{V}_a|^2 \leq 16R_{a,N}^2 y_a^2 + \frac{4A_{N,J}^2 y_a^2}{\sigma_1^2} + \frac{16y_a^2 H^2}{\sigma_1^2} + \frac{16y_a^2 \sigma_2^2}{\sigma_1^4} \leq L_J E_a,$$

where $L_J = 16C_0^2 J^{4/3} + 4J^2/c_0^2 + 16 + 16/c_0^2$. In the last inequality, we used $\sigma_2 \leq c_0^{-1}\sigma_1$ for the last term. Similarly, $|\tilde{V}_a|^2/b = 1 + y_a^2 \sigma_2/\sigma_1^2 \leq 1 + c_0^{-1} y_a^2 \sigma_1^{-1} \leq (1 + c_0^{-1})E_a$. By Step 3 and $y_a^8 \leq 1 + y_a^{16}$, we have

$$\int E_a d\pi_N \leq E_J, \quad E_J = 2 + B_0 + J^8 B_0 + B_J.$$

This proves (19.41) with $C_J = \max\{(L_J E_J)^{1/2}, (1 + c_0^{-1})E_J\}$. The fields and their divergences are smooth off the set $\{x_{>J} = 0\}$, which has π_N -measure zero. $\square$

Lemma 19.17 (Total variation convergence of the first coordinates). *Let $(r_{\cdot,N})$ satisfy the critical spectral conditions with constants (c_0, C_0, δ) for every $N \in \mathcal{N}$ and converge to R . Then, for every $J \geq 1$ and every $s > 0$,*

$$\lim_{N \rightarrow \infty} \left\| (\nu_N P_{bs})^{\leq J} - (\mu_s^\infty)^{\leq J} \right\|_{\text{TV}} = 0. \quad (19.45)$$

Proof. Fix $J \geq 1$ and $s > 0$, and choose $s_0 \in (0, s]$. Let $N \in \mathcal{N}$ with $N \geq \max\{N_J, N_{s_0}, J+3\}$, with N_J as in Lemma 19.16 and N_{s_0} as in (19.30). Write $f = f_{N,s}$, which is of class C^2 , and $\mu_{N,s}^J = (\nu_N P_{bs})^{\leq J}$. We establish a translation bound whose constant is independent of N .

Step 1: integration by parts. Let $\phi \in C_c^\infty(\mathbb{R}^J)$ and $\Phi(x) = \phi(N^{-1/3}x_{\leq J})$, and let $a \leq J$. We show that

$$\int f(\partial_a \phi)(y_{\leq J}) d\pi_N = - \int \Phi (f \operatorname{div}_{\pi_N} \tilde{V}_a + \tilde{V}_a \cdot \nabla_{S_N} f) d\pi_N, \quad y = N^{-1/3}x. \quad (19.46)$$

Both terms on the right are integrable, because Φ is bounded and each term is a product of two functions in $L^2(\pi_N)$, by (19.41). No division by f occurs. For $\eta \in (0, 1)$, let $\Omega_\eta = \{x \in S_N : |x_{>J}| > \eta\}$. The field V_a is smooth on Ω_η , and by the divergence theorem [134, Chapter 2, Section 4, (4.21)],

$$\int_{\Omega_\eta} (V_a \Phi) f d\pi_N = - \int_{\Omega_\eta} \Phi (f \operatorname{div}_{\pi_N} V_a + V_a f) d\pi_N + \mathcal{B}_\eta,$$

where the boundary term $\mathcal{B}_\eta$ is the integral of $\Phi f (V_a \cdot n_\eta)$ over $\partial\Omega_\eta$ against the density of π_N times the surface measure of $\partial\Omega_\eta$, and n_η is the outward unit normal. Since f is of class C^2 on the compact manifold S_N , f and $\nabla_{S_N} f$ are bounded. Near $\{x_{>J} = 0\}$ we have $\Sigma_1 \asymp |x_{>J}|^2$. Then $|V_a| = O(|x_{>J}|^{-1})$ and $|\operatorname{div}_{\pi_N} V_a| = O(|x_{>J}|^{-2})$, while $\partial\Omega_\eta$ is the product of spheres $\{|x_{>J}| = \eta\} \times \{|x_{\leq J}|^2 = N - \eta^2\}$, of area $O(\eta^{N-J-1})$. The term $\mathcal{B}_\eta$ is therefore $O(\eta^{N-J-2})$, which tends to zero for $N - J > 2$, and $|x_{>J}|^{-2}$ is integrable near $\{x_{>J} = 0\}$ for the same reason. Letting $\eta \downarrow 0$, multiplying by $N^{1/3}$, and using $\tilde{V}_a \Phi = (\partial_a \phi)(y_{\leq J})$, we obtain (19.46).

Step 2: a translation bound. By (19.46), the Cauchy–Schwarz inequality, and (19.41), we have

$$\left| \int \partial_a \phi d\mu_{N,s}^J \right| \leq \|\phi\|_\infty \left(C_J \|f\|_{L^2(\pi_N)} + (C_J b)^{1/2} \left(\int |\nabla_{S_N} f|^2 d\pi_N \right)^{1/2} \right) \leq C_{J,s_0} \|\phi\|_\infty, \quad (19.47)$$

where $C_{J,s_0} = M_{s_0} (C_J + (C_J/s_0)^{1/2})$. In the last inequality, we used $\|f\|_{L^2(\pi_N)} \leq M_{s_0}$ and $b \int |\nabla_{S_N} f|^2 d\pi_N = \mathcal{E}_N(f) \leq M_{s_0}^2/s_0$, by (19.30); the factors $b^{1/2}$ and $b^{-1/2}$ cancel. Let $w \in \mathbb{R}^J$. Since $\phi(y+w) - \phi(y) = \int_0^1 \sum_{a \leq J} w_a (\partial_a \phi)(y + \theta w) d\theta$, applying (19.47) to the functions $\phi(\cdot + \theta w)$, $\theta \in [0, 1]$, we obtain

$$\left| \int \phi(y+w) d\mu_{N,s}^J(dy) - \int \phi(y) d\mu_{N,s}^J(dy) \right| \leq C_{J,s_0} \|\phi\|_\infty \sum_{a \leq J} |w_a| \leq C_{J,s_0} J^{1/2} |w| \|\phi\|_\infty.$$

For two probability measures on $\mathbb{R}^J$, twice their total variation distance is the supremum of the differences of their integrals of functions $\phi \in C_c^\infty(\mathbb{R}^J)$ with $|\phi| \leq 1$. Indeed, let μ and ν be the two measures. Since finite Borel measures on $\mathbb{R}^J$ are regular [30, Theorem 1.1] and closed bounded sets are compact, every Borel function with values in $[-1, 1]$ is a limit in $L^1(\mu + \nu)$ of continuous compactly supported functions with values in $[-1, 1]$. Each of these is a uniform limit of its mollifications, which are smooth, compactly

supported, and bounded by one in absolute value [37, Proposition 4.21]. Taking the supremum over such ϕ , we find

$$\left\| \tau_w \mu_{N,s}^J - \mu_{N,s}^J \right\|_{\text{TV}} \leq \frac{1}{2} C_{J,s_0} J^{1/2} |w| \quad (w \in \mathbb{R}^J, s \geq s_0), \quad (19.48)$$

with τ_w as in Lemma 19.8.

Step 3: passage to total variation. By (18.53), $\mu_{N,s}^J$ converges weakly to $(\mu_s^\infty)^{\leq J}$. The translation estimate (19.48) has a constant independent of N , so Lemma 19.8(b) applies to this sequence at the fixed time s . It gives (19.45). Equivalently, at a fixed mollifier scale the weak convergence gives convergence of the mollified densities in L^1 by Scheffé's lemma; the translation estimate bounds both mollification errors by a constant times the scale, which is then sent to zero. $\square$

19.6 Continuity and pointwise convergence

We now combine the estimates of the last two sections. For $s > 0$ and $J \geq 1$ let

$$F_R^{(J)}(s) = \left\| (\mu_s^\infty)^{\leq J} - \rho_R^{\leq J} \right\|_{\text{TV}}.$$

By Lemma 19.7,

$$F_R(s) = \sup_{J \geq 1} F_R^{(J)}(s) = \lim_{J \rightarrow \infty} F_R^{(J)}(s). \quad (19.49)$$

Each $F_R^{(J)}$ is lower semicontinuous on $(0, \infty)$, since $s \mapsto (\mu_s^\infty)^{\leq J}$ is weakly continuous and total variation is a supremum of differences of integrals of continuous functions with values in $[0, 1]$. As a supremum of such functions, F_R is lower semicontinuous, and in particular measurable.

Lemma 19.18 (Continuity of the limiting laws). *Under the hypotheses of Theorem 19.2, for every $s_0 > 0$ and all $s, t \geq s_0$,*

$$\left\| \mu_s^\infty - \mu_t^\infty \right\|_{\text{TV}} \leq L_{s_0} |s - t|, \quad |F_R(s) - F_R(t)| \leq L_{s_0} |s - t|, \quad (19.50)$$

where L_{s_0} is the constant of (19.31). In particular, F_R is continuous on $(0, \infty)$.

Proof. Fix $J \geq 1$. Projecting (19.31) gives

$$\left\| (\nu_N P_{bs})^{\leq J} - (\nu_N P_{bt})^{\leq J} \right\|_{\text{TV}} \leq L_{s_0} |s - t|.$$

By (18.53), both heads converge weakly to the corresponding heads of μ_s^∞ and μ_t^∞ . Lower semicontinuity of total variation, Lemma 19.8(a), passes the same bound to those heads. Its right side does not depend on J , so Lemma 19.7 gives the first inequality in (19.50). The second follows from the triangle inequality. This argument uses only weak convergence of the heads and finite semigroup smoothing. $\square$

Proposition 19.19 (Pointwise convergence of the profile). *Let $R \in \mathcal{P}$, and let $(r_{\cdot,N})$ satisfy the critical spectral conditions with constants (c_0, C_0, δ) for every $N \in \mathcal{N}$ and converge to R . Then*

$$\lim_{N \rightarrow \infty} D_N(bs) = F_R(s) \quad (s > 0). \quad (19.51)$$

Proof. Fix $s > 0$ and $J \geq J_0$, with J_0 as in Lemma 19.14. For all sufficiently large N , Lemma 19.15 gives

$$0 \leq D_N(bs) - D_N^{\leq J}(bs) \leq C_s J^{-1/3}.$$

By the triangle inequality,

$$|D_N^{\leq J}(bs) - F_R^{(J)}(s)| \leq \left\| (v_N P_{bs})^{\leq J} - (\mu_s^\infty)^{\leq J} \right\|_{\text{TV}} + \left\| \pi_N^{\leq J} - \rho_R^{\leq J} \right\|_{\text{TV}},$$

which tends to zero at fixed J by Lemma 19.17 and Corollary 18.12. Since $F_R^{(J)}(s) \leq F_R(s)$, we obtain

$$\limsup_{N \rightarrow \infty} |D_N(bs) - F_R(s)| \leq C_s J^{-1/3} + F_R(s) - F_R^{(J)}(s).$$

Letting $J \rightarrow \infty$ and using (19.49) proves the result. The dimension limit is taken before the number of retained coordinates tends to infinity. $\square$

19.7 Proofs of the deterministic results

The proof of strict decrease in Theorem 19.2 uses the following estimate for the finite-dimensional semigroups $Q_{N,t} = P_{bt}$, with K_P as in Table 19.1.

Lemma 19.20 (Finite semigroup correlations). *Fix $s > 0$. There exist $M \geq 1$ and N_0 , depending only on s and (c_0, C_0, δ) , with the following property. Let $N \geq N_0$, let r satisfy the critical spectral conditions with constants (c_0, C_0, δ) , write $f_{N,s} - 1 = g_+ - g_-$ with $g_+ = \max(f_{N,s} - 1, 0)$ and $g_- = \max(1 - f_{N,s}, 0)$, and let*

$$h_N(t) = \langle g_+, Q_{N,t} g_- \rangle_{L^2(\pi_N)} \quad (t \geq 0).$$

Then there exist real numbers $w_{N,j}$ and $\lambda_{N,j} \geq 1/K_P$, $j \geq 1$, with $\sum_j |w_{N,j}| \leq M$, such that

$$h_N(t) = D_N(bs)^2 + \sum_{j \geq 1} w_{N,j} e^{-\lambda_{N,j} t} \quad (t \geq 0). \quad (19.52)$$

Moreover, for every $u \geq 0$,

$$0 \leq h_N(2u) \leq (M^4 + 1)^{1/3} [D_N(bs) - D_N(b(s+u))]^{2/3}. \quad (19.53)$$

The coefficients $w_{N,j}$ may have either sign. The bounds are uniform over all r that satisfy the critical spectral conditions with constants (c_0, C_0, δ) ; the lower bound on $\lambda_{N,j}$ comes from the Poincaré inequality on the full sphere.

Proof. By Theorem 12.13 with $p = 4$ and $s_0 = s$, there exist $M \geq 1$ and N_0 , depending only on s and (c_0, C_0, δ) , such that $\|f_{N,s}\|_{L^4(\pi_N)} \leq M$ for $N \geq N_0$. We increase N_0 so that also $C_P(\pi_N) \leq K_P b$ for $N \geq N_0$. Let $f = f_{N,s}$ and $d_N = D_N(bs)$.

Step 1: the decrease of the distance. Since $f \geq 0$ and $\int f d\pi_N = 1$, we have $\int g_+ d\pi_N = \int g_- d\pi_N = d_N$ by Lemma 3.6. Further, $0 \leq g_+ \leq f$ and $0 \leq g_- \leq 1$. For $u \geq 0$, the functions $Q_{N,u}g_{\pm}$ are nonnegative and have integral d_N , and $Q_{N,u}g_+ - Q_{N,u}g_- = Q_{N,u}(f - 1) = f_{N,s+u} - 1$. Using $|a - a'| = a + a' - 2\min(a, a')$ and Lemma 3.6, we find

$$d_N - D_N(b(s+u)) = \int \min(Q_{N,u}g_+, Q_{N,u}g_-) d\pi_N. \quad (19.54)$$

Step 2: the correlation and the decrease of the distance. For $a, a' \geq 0$, we have

$$aa' \leq \min(a, a')^{2/3} (a^4 + a'^4)^{1/3}.$$

Indeed, if $a \leq a'$, then $(aa')^3 = a^3a'^3 \leq a^2a'^4 \leq a^2(a^4 + a'^4)$, and the other case is symmetric. Since $Q_{N,u}$ is selfadjoint on $L^2(\pi_N)$ and $Q_{N,2u} = Q_{N,u}Q_{N,u}$, we have $h_N(2u) = \int Q_{N,u}g_+ Q_{N,u}g_- d\pi_N \geq 0$. Hölder's inequality with exponents $3/2$ and 3 gives

$$h_N(2u) \leq \left(\int \min(Q_{N,u}g_+, Q_{N,u}g_-) d\pi_N \right)^{2/3} \left(\int ((Q_{N,u}g_+)^4 + (Q_{N,u}g_-)^4) d\pi_N \right)^{1/3}.$$

By Jensen's inequality for the Markov operator $Q_{N,u}$ and the invariance of π_N , we have $\int (Q_{N,u}g_+)^4 d\pi_N \leq \int g_+^4 d\pi_N \leq M^4$ and $\int (Q_{N,u}g_-)^4 d\pi_N \leq \int g_-^4 d\pi_N \leq 1$. Inserting (19.54), we obtain (19.53).

Step 3: the spectral expansion. Let $(e_j)_{j \geq 0}$ be an orthonormal basis of $L^2(\pi_N)$ of smooth eigenfunctions of $-bL_r$, with eigenvalues $\lambda_{N,j} \geq 0$ and $e_0 = 1$, as in the proof of Lemma 19.12. For $j \geq 1$, e_j is orthogonal to e_0 , and then $\text{Var}_{\pi_N}(e_j) = 1$. The Poincaré inequality gives $1 \leq K_P b \int |\nabla_{S_N} e_j|^2 d\pi_N = K_P \lambda_{N,j}$, that is, $\lambda_{N,j} \geq 1/K_P$. Let $\alpha_j = \langle g_+, e_j \rangle$ and $\beta_j = \langle g_-, e_j \rangle$, so that $\alpha_0 = \beta_0 = d_N$. Expanding $Q_{N,t}g_- = \sum_j \beta_j e^{-\lambda_{N,j}t} e_j$ in $L^2(\pi_N)$, we have

$$h_N(t) = \sum_{j \geq 0} \alpha_j \beta_j e^{-\lambda_{N,j}t} = d_N^2 + \sum_{j \geq 1} w_{N,j} e^{-\lambda_{N,j}t}, \quad w_{N,j} = \alpha_j \beta_j.$$

By the Cauchy–Schwarz inequality and Parseval's identity, we have

$$\sum_{j \geq 1} |w_{N,j}| \leq \|g_+\|_{L^2(\pi_N)} \|g_-\|_{L^2(\pi_N)} \leq \|f\|_{L^2(\pi_N)} \leq M.$$

In the second inequality, we used $0 \leq g_+ \leq f$ and $0 \leq g_- \leq 1$. In particular, the series converges absolutely for every $t \geq 0$, and this proves (19.52). $\square$

Proof of Theorem 19.2. The proof uses pointwise convergence, finite spectral correlations, and the continuity estimate already established.

Step 1: continuity and monotonicity. By Lemma 19.18, F_R is continuous. The convergence (19.51) holds by Proposition 19.19. Since every function $s \mapsto D_N(bs)$ is nonincreasing, its pointwise limit F_R is nonincreasing.

Step 2: positivity. By (19.26), there exist $c_1, C_1 > 0$ and N_0 , depending only on (c_0, C_0, δ) , such that $D_N(bs) \geq c_1 e^{-C_1 s}$ for all $s \geq 0$ and $N \in \mathcal{N}$ with $N \geq N_0$. This bound was proved in Section 19.3 without Theorem 19.2. Letting $N \rightarrow \infty$ and using (19.51), we obtain

$$F_R(s) \geq c_1 e^{-C_1 s} > 0 \quad (s > 0). \quad (19.55)$$

Step 3: strict decrease. Suppose that $0 < s_1 < s_2$ and $F_R(s_1) = F_R(s_2)$. Since F_R is nonincreasing, it is constant on $[s_1, s_2]$. Let $\tau = s_2 - s_1$, and let h_N , M , and N_0 be as in Lemma 19.20 with $s = s_1$. For $t \in [\tau, 2\tau]$, we have $s_1 + t/2 \leq s_2$. Since D_N is nonincreasing, (19.53) with $u = t/2$ implies that

$$0 \leq h_N(t) \leq (M^4 + 1)^{1/3} [D_N(bs_1) - D_N(bs_2)]^{2/3} \quad (N \geq N_0).$$

By (19.51), the right side tends to $(M^4 + 1)^{1/3} [F_R(s_1) - F_R(s_2)]^{2/3} = 0$. Thus $h_N(t) \rightarrow 0$ for every $t \in [\tau, 2\tau]$, while the constant term $D_N(bs_1)^2$ in (19.52) tends to $F_R(s_1)^2$. We apply Lemma 19.9 along the increasing enumeration of $\{N \in \mathcal{N} : N \geq N_0\}$, with $t_1 = \tau$, $t_2 = 2\tau$, $\lambda_0 = 1/K_P$, the coefficients $w_{N,j}$ and rates $\lambda_{N,j}$ of (19.52), and the bound M on $\sum_j |w_{N,j}|$. It gives $F_R(s_1)^2 = 0$, which contradicts (19.55). We conclude that $F_R(s_2) < F_R(s_1)$ whenever $0 < s_1 < s_2$. Here M bounds the norm of f_{N,s_1} in $L^4(\pi_N)$. It should not be confused with the bound M_4 in the proof of (19.26), which concerns the centered density at the time $s_* b/2$.

Step 4: uniform convergence on compact sets and endpoints. Let $0 < s_0 < s_1$ and $\zeta > 0$. Since F_R is nonincreasing and uniformly continuous on $[s_0, s_1]$, there exist $s_0 = u_0 < u_1 < \dots < u_k = s_1$ with $F_R(u_{l-1}) - F_R(u_l) < \zeta$ for all l . For $s \in [u_{l-1}, u_l]$, the monotonicity of D_N and of F_R implies that

$$D_N(bu_l) - F_R(u_l) - \zeta \leq D_N(bs) - F_R(s) \leq D_N(bu_{l-1}) - F_R(u_{l-1}) + \zeta.$$

By (19.51) at the finitely many points u_l , we find that $\limsup_{N \rightarrow \infty} \sup_{s \in [s_0, s_1]} |D_N(bs) - F_R(s)| \leq \zeta$. As ζ is arbitrary, this proves (19.12).

For the limit at zero, Theorem 18.9 gives $U_1(s) \rightarrow 0$ and $Z(s) \rightarrow \infty$ almost surely, and hence $\xi_1(s) \rightarrow 0$ almost surely. For $a > 0$, let $B_a = \{y : |y_1| < a\}$. Then $F_R(s) \geq \mu_s^\infty(B_a) - \rho_R(B_a)$, so $\liminf_{s \downarrow 0} F_R(s) \geq 1 - \rho_R(B_a)$. Since $\rho_R^{\leq 1}$ has a density, letting $a \downarrow 0$ gives $F_R(0+) = 1$.

For the limit at infinity, the finite L^2 decay estimate (19.28) gives $D_N(bs) \leq C_5 e^{-(s-1)/K_P}$ for all large N and all $s \geq 1$. Passing to the limit in (19.51) yields the same bound for $F_R(s)$, so $F_R(\infty) = 0$. This completes the proof. $\square$

Proof of Corollary 19.3. The assertion for an empty interval is immediate. Let $I = [\varepsilon_1, \varepsilon_2] \subset (0, 1)$ be nonempty, and let $[u_1, u_2] = F_R^{-1}(I)$, a compact subinterval of $(0, \infty)$ because F_R^{-1} is continuous. Fix $0 < h < u_1/2$. The function

$$\varepsilon \mapsto \min \left\{ F_R(F_R^{-1}(\varepsilon) - h) - \varepsilon, \varepsilon - F_R(F_R^{-1}(\varepsilon) + h) \right\}$$

is continuous and positive on I , since F_R is continuous and strictly decreasing; let $\zeta > 0$ be its minimum. By (19.12) on $[u_1 - h, u_2 + h]$, for large N we have $|D_N(bs) - F_R(s)| < \zeta$ for all s in this interval. Then, for every $\varepsilon \in I$, $D_N(b(F_R^{-1}(\varepsilon) - h)) > \varepsilon$ and $D_N(b(F_R^{-1}(\varepsilon) + h)) < \varepsilon$. Since D_N is nonincreasing, the first inequality implies that $t_N(\varepsilon) \geq b(F_R^{-1}(\varepsilon) - h)$, and the second implies that $t_N(\varepsilon) \leq b(F_R^{-1}(\varepsilon) + h)$. Thus $\sup_{\varepsilon \in I} |N^{-2/3}t_N(\varepsilon) - F_R^{-1}(\varepsilon)| \leq h$ for large N . As h is arbitrary, this proves (19.13). For $\varepsilon \in (0, 1/2)$ we have $\varepsilon < 1 - \varepsilon$, and since F_R^{-1} is strictly decreasing, $F_R^{-1}(\varepsilon) > F_R^{-1}(1 - \varepsilon) > 0$. $\square$

Proof of the first part of Corollary 19.5. By (19.26), with c_1, C_1 and N_0 depending only on (c_0, C_0, δ) , we have $D_N(bs) \geq c_1 e^{-C_1 s}$ for all $s \geq 0$ and $N \in \mathcal{N}$ with $N \geq N_0$. Letting $N \rightarrow \infty$ and using (19.51) of Proposition 19.19, we obtain $F_R(s) \geq c_1 e^{-C_1 s}$ for every $s > 0$. The finite bound is proved using only Lemmas 19.6 and 19.10 and Theorem 12.13, and its passage to the limit uses only Proposition 19.19. Neither argument uses positivity or strict decrease of the limiting profile. $\square$

19.8 The GOE critical window

To apply Theorem 19.2 to the GOE model, we need to know that the finite profiles and the limiting profile are measurable functions of the spectral data. We first treat a fixed dimension.

Lemma 19.21 (Profiles in fixed dimension). *Let $N \geq 2$ and $r \in \mathbb{R}^N$, and let $D^r(t) = \|\nu_N P_t^r - \pi_r\|_{\text{TV}}$ and $t^r(\varepsilon) = \inf\{t \geq 0 : D^r(t) \leq \varepsilon\}$.*

1. *The function $t \mapsto D^r(t)$ is continuous and nonincreasing on $[0, \infty)$ and tends to zero at infinity. If the r_i are not all equal, it is positive and strictly decreasing; otherwise it vanishes identically.*
2. *The function $\varepsilon \mapsto t^r(\varepsilon)$ is continuous on $(0, 1)$.*
3. *The maps $r \mapsto D^r \in C([0, \infty))$ and $r \mapsto t^r \in C((0, 1))$ are Borel.*

Proof. The density $\omega = d\pi_r/d\nu_N$ is bounded above and below, and ν_N satisfies a Poincaré inequality with some constant C_ν , since the Laplacian on S_N has a spectral gap. For smooth g , we have

$$\text{Var}_{\pi_r}(g) \leq \int (g - \nu_N(g))^2 d\pi_r \leq \sup \omega C_\nu \int |\nabla_{S_N} g|^2 d\nu_N \leq \frac{\sup \omega}{\inf \omega} C_\nu \int |\nabla_{S_N} g|^2 d\pi_r,$$

so π_r satisfies a Poincaré inequality with some constant C_P ; this is the perturbation argument of Holley and Stroock [84, Lemma 5.1, pp. 1183–1184]. The semigroup (P_t^r) is a strongly continuous semigroup of selfadjoint Markov operators on $L^2(\pi_r)$, and $\|P_t^r g - \pi_r(g)\|_2 \leq e^{-t/C_P} \|g - \pi_r(g)\|_2$.

For the first part, let $f_0 = dv_N/d\pi_r$, which is bounded. Then $D^r(t) = \|P_t^r f_0 - 1\|_{L^1(\pi_r)}/2$ by reversibility. The continuity of D^r follows from the strong continuity in $L^2(\pi_r)$, its monotonicity from the contraction property, and the bound $D^r(t) \leq e^{-t/C_P} \|f_0 - 1\|_2/2$ from the spectral gap. If the r_i are all equal, then $\pi_r = \nu_N$ and $D^r \equiv 0$. Otherwise $f_0 \neq 1$ and $D^r(0) > 0$. If $D^r(t_0) = 0$ for some $t_0 > 0$, then $P_{t_0}^r(f_0 - 1) = 0$, and the injectivity of $P_{t_0}^r = e^{-t_0 A}$ forces $f_0 = 1$, a contradiction. Thus $D^r > 0$. For $0 \leq s < s'$, write $P_s^r f_0 - 1 = g_+ - g_-$, where $g_{\pm}$ are continuous and nonnegative. Their integrals both equal $D^r(s) > 0$. The finite diffusion sends every nonzero nonnegative continuous function to a strictly positive function at every positive time. Indeed, its heat equation is uniformly parabolic in each relatively compact coordinate chart of the sphere. The strong maximum principle implies that a zero at a positive time forces zeros at earlier positive times throughout that chart; connectedness propagates this through a chain of overlapping charts. This would contradict preservation of the positive integral. Applying this to g_+ and g_- at time $u = s' - s > 0$, we obtain

$$D^r(s') = D^r(s) - \int \min\{P_u^r g_+, P_u^r g_-\} d\pi_r < D^r(s).$$

For the second part, if $D^r \equiv 0$ then $t^r \equiv 0$. Otherwise D^r is a continuous strictly decreasing bijection from $[0, \infty)$ onto $(0, D^r(0)]$. Then $t^r(\varepsilon) = (D^r)^{-1}(\varepsilon)$ for $\varepsilon \in (0, D^r(0)]$, which is continuous and tends to zero as $\varepsilon \uparrow D^r(0)$, and $t^r(\varepsilon) = 0$ for $\varepsilon \geq D^r(0)$.

For the third part, fix $t \geq 0$. By Lemma 10.2, $\nu_N P_t^r$ is the law of $\sqrt{N} Y^r(t)/|Y^r(t)|$, where Y^r solves $dY_i = -2r_i Y_i dt + \sqrt{2/N} |Y| dB_i$ with $Y(0) = g$. We couple all these solutions by using the same random Gaussian vector g , independent of the same Brownian motions (B_i) . If $r^{(n)} \rightarrow r$, Itô's isometry and Gronwall's lemma, applied to the Lipschitz equation, give $\mathbb{E}|Y^{r^{(n)}}(t) - Y^r(t)|^2 \rightarrow 0$, where the expectation is over g and the Brownian motions. As $|Y^r(t)| > 0$ almost surely, it follows that $\nu_N P_t^{r^{(n)}} \rightarrow \nu_N P_t^r$ weakly. The measures π_r are also weakly continuous in r . Total variation on S_N is the supremum of $\mu(\phi) - \pi(\phi)$ over a countable dense set of continuous $\phi: S_N \rightarrow [0, 1]$ (Lemma 3.6), and therefore $r \mapsto D^r(t)$ is lower semicontinuous, hence Borel. As D^r is continuous in t and the Borel σ -field of $C([0, \infty))$ is generated by evaluations at rational times, the map $r \mapsto D^r$ is Borel. Finally, for $\varepsilon \in (0, 1)$ and $s \geq 0$, we have $t^r(\varepsilon) > s$ if and only if $D^r(s) > \varepsilon$, because D^r is continuous and nonincreasing. We conclude that $r \mapsto t^r(\varepsilon)$ is Borel for every ε , and by continuity in ε , $r \mapsto t^r$ is Borel. $\square$

Lemma 19.22 (Measurable dependence of the critical profile). *Let the spaces $C((0, \infty))$ and $C((0, 1))$ carry the topology of uniform convergence on compact subsets.*

1. The maps $R \mapsto F_R \in C((0, \infty))$ and $R \mapsto F_R^{-1} \in C((0, 1))$ are Borel on $\mathcal{P}$, where $\mathcal{P}$ is viewed as a Borel subset of ℓ^2 as in Lemma 19.1.
2. For every $\sigma \in \mathbb{R}$, there exists a Borel map $\Phi^\sigma: \mathbb{R} \times \ell^2 \rightarrow C((0, \infty)) \times C((0, 1))$ such that, almost surely, $(F_\sigma, F_\sigma^{-1}) = \Phi^\sigma(\Xi, (\Delta_i^{-1})_{i \geq 2})$. In particular, F_σ and F_σ^{-1} are random elements of these spaces, and they are measurable functions of $(\Xi, (\Delta_i^{-1})_{i \geq 2})$.

Proof. For $R \in \mathcal{P}$ let $r^{(N)}(R)$ be the vectors of Lemma 19.1, and define

$$\Phi_N(R) = \left(D^{r^{(N)}(R)}(N^{2/3} \cdot), N^{-2/3} t^{r^{(N)}(R)}(\cdot) \right) \in C((0, \infty)) \times C((0, 1)).$$

Lemmas 19.1 and 19.21 show that Φ_N is Borel on $\mathcal{P}$; restriction to $(0, \infty)$ and rescaling of time are continuous operations. Fix $R \in \mathcal{P}$. By Lemma 19.1, the vectors $r^{(N)}(R)$ with $N \geq N_1(R)$ satisfy the hypotheses of Theorem 19.2 with $\mathcal{N} = \{N \geq N_1(R)\}$. Then Theorem 19.2 and Corollary 19.3 imply that $\Phi_N(R) \rightarrow (F_R, F_R^{-1})$, since discarding the finitely many dimensions $N < N_1(R)$ does not affect the limit. A pointwise limit of Borel maps into a metric space is Borel, and the first part follows.

For the second part, let $(\xi, \mathbf{b}) \in \mathbb{R} \times \ell^2$ with $b_i > 0$ for all i , where $\mathbf{b} = (b_i)_{i \geq 2}$. For $x > 0$, let $\mathcal{S}_{\xi, \mathbf{b}}(x) = \xi + x^{-1} - \sum_{i \geq 2} x b_i^2 / (1 + x b_i)$; this is (19.14) with $b_i = \Delta_i^{-1}$, and for fixed x it is a continuous function of $(\xi, \mathbf{b})$, since the series is bounded by $x \|\mathbf{b}\|_2^2$ and is Lipschitz in $\mathbf{b}$ on bounded sets. Let

$$\varrho(\xi, \mathbf{b}) = \sup \{x \in \mathbb{Q} \cap (0, \infty) : \mathcal{S}_{\xi, \mathbf{b}}(x) > \sigma\},$$

a Borel function with values in $[0, \infty]$, and on the Borel set where $0 < \varrho < \infty$ let $u(\xi, \mathbf{b}) \in \ell^2$ have coordinates $u_1 = 2/\varrho$ and $u_i = 2b_i/(1 + \varrho b_i)$ for $i \geq 2$. The map u is Borel, because the Borel σ -field of ℓ^2 is generated by the coordinates. Let G be the Borel set of $(\xi, \mathbf{b})$ for which $u(\xi, \mathbf{b})$ is defined and is the inverse sequence of an element R of $\mathcal{P}$. We set $\Phi^\sigma(\xi, \mathbf{b}) = (F_R, F_R^{-1})$ on G , which is Borel by the first part, and $\Phi^\sigma(\xi, \mathbf{b}) = (s \mapsto e^{-s}, \varepsilon \mapsto \log(1/\varepsilon))$ off G . On the event of probability one where $\mathcal{S}$ is continuous and strictly decreasing from $+\infty$ to $-\infty$, the supremum over rationals equals the root, so that $\varrho(\Xi, (\Delta_i^{-1})) = \varrho_\sigma$ and $u(\Xi, (\Delta_i^{-1})) = ((R_i^\sigma)^{-1})_{i \geq 1}$ by (19.15). Since also $R^\sigma \in \mathcal{P}$ almost surely, the convention for F_σ in Section 19.1 gives $(F_\sigma, F_\sigma^{-1}) = \Phi^\sigma(\Xi, (\Delta_i^{-1}))$ almost surely. $\square$

The proof of Theorem 19.4 transfers Theorem 19.2 and Corollary 19.3 from deterministic sequences to the joint law of the spectral data. The maps that send the spectral data to the finite profiles depend on N , and they are continuous only along sequences that satisfy the critical spectral conditions with fixed constants. We use the following localized form of the continuous mapping theorem; compare [30, Theorem 2.7].

Lemma 19.23 (Convergence for varying measurable maps). *Let E and G be Polish spaces, let μ_n , $n \geq 1$, and μ be Borel probability measures on E such that μ_n converges weakly to μ , and*

let $\mathcal{D} \subset E$ be a Borel set with $\mu(\mathcal{D}) = 1$. Let $f_n, f: E \rightarrow G$ be Borel maps. Suppose that for every $\eta > 0$ there exist Borel sets $A_n^\eta \subset E$, $n \geq 1$, with the following two properties.

1. $\limsup_{n \rightarrow \infty} \mu_n(E \setminus A_n^\eta) \leq \eta$.
2. For every strictly increasing sequence $(n_k)_{k \geq 1}$ of integers, every sequence of points $x_k \in A_{n_k}^\eta$, and every $x \in \mathcal{D}$ with $x_k \rightarrow x$, we have $f_{n_k}(x_k) \rightarrow f(x)$.

Then the image of μ_n under the map $x \mapsto (x, f_n(x))$ converges weakly to the image of μ under $x \mapsto (x, f(x))$, as probability measures on $E \times G$.

Equivalently, if X_n and X are random elements of E with laws μ_n and μ , then $(X_n, f_n(X_n))$ converges in distribution to $(X, f(X))$; the random elements need not be defined on a common probability space. The sets A_n^η may depend on η , but $\mathcal{D}$ and f do not, and the points x_k in the second hypothesis need not lie in $\mathcal{D}$. No continuity of f_n or of f is assumed. If $f_n = f$ for all n and f is continuous at every point of $\mathcal{D}$, then the hypotheses hold with $A_n^\eta = E$, and the lemma reduces to the continuous mapping theorem.

Proof. Let ϕ be a bounded continuous function on $E \times G$, let $B = \sup |\phi|$, and let $h(x) = \phi(x, f(x))$, a Borel function on E with $|h| \leq B$. Fix $\zeta > 0$, and let $\theta = \min\{1/2, \zeta/(8 + 20B)\}$.

Step 1: a continuous approximation. We claim that there exist a compact set $K \subset \mathcal{D}$ with $\mu(K) > 1 - \theta$ and a continuous function $H: E \rightarrow [-B, B]$ with $H = h$ on K . Indeed, μ is a Radon measure, since E is Polish [34, Theorem 7.1.7]. By the continuous approximation in Lusin's theorem [34, Theorem 7.1.13], which applies to real functions on a metric space, there exists a continuous function $H_0: E \rightarrow \mathbb{R}$ with $\mu(H_0 \neq h) < \theta/2$. Let $H = \max(-B, \min(H_0, B))$. Then H is continuous, $H = h$ wherever $H_0 = h$, and the Borel set $\mathcal{D} \cap \{H = h\}$ has measure greater than $1 - \theta/2$. By inner regularity, it contains a compact set K with $\mu(K) > 1 - \theta$.

Step 2: a uniform estimate near K . Let $A_n = A_n^\theta$. We claim that there exist an integer $k_0 \geq 1$ and n_0 such that

$$|\phi(y, f_n(y)) - H(y)| \leq 2\theta \quad \text{whenever } n \geq n_0, y \in A_n, \text{ and } d_E(y, K) < 1/k_0. \quad (19.56)$$

Otherwise, we can choose recursively $n_k > n_{k-1}$, $y_k \in A_{n_k}$, and $z_k \in K$ such that $d_E(y_k, z_k) < 1/k$ and $|\phi(y_k, f_{n_k}(y_k)) - H(y_k)| > 2\theta$. Since K is compact, a subsequence z_{k_l} converges to some $z \in K \subset \mathcal{D}$, and then $y_{k_l} \rightarrow z$. The integers n_{k_l} are strictly increasing, and by the second hypothesis, $f_{n_{k_l}}(y_{k_l}) \rightarrow f(z)$. By the continuity of ϕ and H , the differences $|\phi(y_{k_l}, f_{n_{k_l}}(y_{k_l})) - H(y_{k_l})|$ converge to $|h(z) - H(z)| = 0$, which contradicts the lower bound 2θ . This proves (19.56). We have not used any continuity of f or of the f_n .

Step 3: integration. Let $U = \{y \in E : d_E(y, K) < 1/k_0\}$, an open set containing K . Since $\mu(E \setminus U) \leq \mu(E \setminus K) < \theta$ and $E \setminus U$ is closed, the portmanteau theorem [30, Theorem 2.1] implies that $\mu_n(E \setminus U) < 2\theta$ for large n . The first hypothesis with $\eta = \theta$ implies that also $\mu_n(E \setminus A_n) < 2\theta$ for large n , and then $\mu_n(E \setminus (A_n \cap U)) < 4\theta$. For a probability measure P , Borel functions u, v with $|u|, |v| \leq B$, and a Borel set S on which $|u - v| \leq c$, splitting the integral over S and its complement gives

$$\left| \int u dP - \int v dP \right| \leq c + 2B P(E \setminus S).$$

We apply this with $P = \mu_n$, $u(x) = \phi(x, f_n(x))$, $v = H$, $S = A_n \cap U$, and $c = 2\theta$, using (19.56), and with $P = \mu$, $u = H$, $v = h$, $S = K$, and $c = 0$. We obtain, for large n ,

$$\left| \int \phi(x, f_n(x)) \mu_n(dx) - \int H d\mu_n \right| \leq 2\theta + 8B\theta, \quad \left| \int H d\mu - \int h d\mu \right| \leq 2B\theta.$$

Since H is bounded and continuous, $|\int H d\mu_n - \int H d\mu| < \theta$ for large n . Combining the three bounds, we find that

$$\left| \int \phi(x, f_n(x)) \mu_n(dx) - \int \phi(x, f(x)) \mu(dx) \right| < (3 + 10B)\theta < \zeta$$

for large n . The order of the choices is ϕ , ζ , θ , then K and H , then the sets A_n^θ , then k_0 and n_0 . As ζ and ϕ are arbitrary, this proves the lemma. $\square$

Proof of Theorem 19.4. We apply Lemma 19.23 to the spectral data and to the maps that send them to the finite profiles and mixing times.

Step 1: spaces and maps. Let $\mathcal{Y} = C((0, \infty)) \times C((0, 1))$, where both spaces carry the topology of uniform convergence on compact subsets. This space is Polish. For instance, $d_+(g, g') = \sum_{k \geq 1} 2^{-k} \min\{1, \sup_{s \in [1/k, k]} |g(s) - g'(s)|\}$ is a complete metric for this topology on $C((0, \infty))$. The piecewise linear functions with rational breakpoints and rational values on some $[1/k, k]$, extended to be constant outside this interval, form a countable dense set. For $C((0, 1))$ we use the intervals $[1/k, 1 - 1/k]$, $k \geq 3$, in the same way. Let

$$E = \mathbb{R} \times \ell^2 \times (\ell^2)^m, \quad G = \mathcal{Y}^m,$$

which are Polish spaces. Let $M = 1 + \max_{j \leq m} |\sigma_j|$, and let $N > (2M)^3$; then $c_{N,j} > 0$ for all j , and discarding finitely many N does not affect convergence in distribution. Let $r_N^{(j)}$ be the half-precisions of the model with $c = c_{N,j}$, in increasing order, $R_{i,N}^{(j)} = br_{i,N}^{(j)}$, and $\mathbf{v}_{N,\sigma_j} = ((R_{i,N}^{(j)})^{-1})_{i \leq N}$, padded with zeros, as in Lemma 4.17. We set

$$X_N = (\Xi_N, \mathbf{b}_N, (\mathbf{v}_{N,\sigma_j})_{j \leq m}) \in E.$$

For $v \in \ell^2$ with $v_i > 0$ for $i \leq N$, let $r(v) \in \mathbb{R}^N$ have the coordinates $r_i(v) = 1/(bv_i)$, and let

$$\Psi_N(v) = (s \mapsto D^{r(v)}(bs), \varepsilon \mapsto b^{-1}t^{r(v)}(\varepsilon)) \in \mathcal{Y},$$

in the notation of Lemma 19.21. For the other $v \in \ell^2$, let $\Psi_N(v) = y_*$, where $y_* = (s \mapsto e^{-s}, \varepsilon \mapsto \log(1/\varepsilon))$. The map Ψ_N is Borel: the set where $v_i > 0$ for $i \leq N$ is open, $r(v)$ is continuous on it, the maps $r \mapsto D^r$ and $r \mapsto t^r$ are Borel by Lemma 19.21, and rescaling of time and restriction to $(0, \infty)$ are continuous. We define the Borel map $f_N: E \rightarrow G$ by $f_N(\xi, \mathbf{b}, (v^{(j)})_{j \leq m}) = (\Psi_N(v^{(j)}))_{j \leq m}$. By the reduction to eigencoordinates in Section 3.1 and (3.9), we have $D_{N,j} = D_N^{(j)}$ and $t_{N,j} = t_N^{(j)}$, and $r(\mathbf{v}_{N,\sigma_j}) = r_N^{(j)}$. Thus

$$f_N(X_N) = (D_{N,j}(N^{2/3} \cdot), N^{-2/3}t_{N,j}(\cdot))_{j \leq m}. \quad (19.57)$$

All m curves are computed from the same matrix W_N .

Step 2: the limit. Let $\mathcal{U} = \{(R_i^{-1})_{i \geq 1} : R \in \mathcal{P}\}$, a Borel subset of ℓ^2 by Lemma 19.1. For $v = (R_i^{-1})_{i \geq 1} \in \mathcal{U}$, let $\Theta(v) = (F_R, F_R^{-1})$, and for $v \in \ell^2 \setminus \mathcal{U}$, let $\Theta(v) = y_*$. This map is Borel by the first part of Lemma 19.22. Let

$$\mathcal{D} = \mathbb{R} \times \ell^2 \times \mathcal{U}^m, \quad f(\xi, \mathbf{b}, (v^{(j)})_{j \leq m}) = (\Theta(v^{(j)}))_{j \leq m}.$$

By the third part of Lemma 4.17, from which we omit the roots q_{N,σ_j} , the data X_N converge in distribution to

$$X = (\Xi, \mathbf{b}, (\mathbf{v}_{\sigma_j})_{j \leq m}), \quad \mathbf{b} = (\Delta_i^{-1})_{i \geq 2}, \quad \mathbf{v}_{\sigma_j} = ((R_i^{\sigma_j})^{-1})_{i \geq 1},$$

and all these limits are functions of the same Airy point process. Since $R^{\sigma_j} \in \mathcal{P}$ almost surely for each j , the law μ of X satisfies $\mu(\mathcal{D}) = 1$. By the convention for F_σ in Section 19.1 and the second part of Lemma 19.22, we have

$$f(X) = (F_{\sigma_j}, F_{\sigma_j}^{-1})_{j \leq m} = (\Phi^{\sigma_j}(\Xi, \mathbf{b}))_{j \leq m} \quad \text{almost surely.} \quad (19.58)$$

The inverse vectors $\mathbf{v}_{\sigma_j}$ in X are functions of $(\Xi, \mathbf{b})$, and they add no randomness.

Step 3: sets of good data. Fix $\eta > 0$, and let $\eta_0 = \min\{\eta/2, 1/2\}$. By the second part of Lemma 4.17, with confidence η_0 and with M as above, there exist constants $0 < c_\eta \leq C_\eta$ and $\delta_\eta > 0$, depending only on η and M , and events $\mathcal{B}_N^\eta$ with $\liminf_N \mathbb{P}(\mathcal{B}_N^\eta) \geq 1 - \eta_0$, with the following property. On $\mathcal{B}_N^\eta$, the half-precisions satisfy the critical spectral conditions with constants $(c_\eta, C_\eta, \delta_\eta)$ simultaneously for every $\sigma \in [-M, M]$. Let A_N^η be the set of $x = (\xi, \mathbf{b}, (v^{(j)})_{j \leq m}) \in E$ such that, for every $j \leq m$, we have $v_i^{(j)} > 0$ for $i \leq N$ and $v_i^{(j)} = 0$ for $i > N$, and the vector $r(v^{(j)})$ is nondecreasing and satisfies the critical spectral conditions with constants $(c_\eta, C_\eta, \delta_\eta)$. This is a Borel set: the conditions on the

coordinates $i > N$ are countably many closed conditions, and the others are finitely many continuous equalities and inequalities on the open set where $v_i^{(j)} > 0$ for $i \leq N$. On $\mathcal{B}_N^\eta$, we have $X_N \in A_N^\eta$. With μ_N the law of X_N , it follows that

$$\limsup_{N \rightarrow \infty} \mu_N(E \setminus A_N^\eta) \leq \limsup_{N \rightarrow \infty} \mathbb{P}((\mathcal{B}_N^\eta)^c) \leq \eta_0 \leq \eta.$$

The sets A_N^η are subsets of the space E of spectral data, and they place no condition on ξ or $\mathbf{b}$.

Step 4: the deterministic theorem. Fix $\eta > 0$, a strictly increasing sequence $(N_k)_{k \geq 1}$, points $x_k \in A_{N_k}^\eta$, and $x \in \mathcal{D}$ with $x_k \rightarrow x$ in E . Let $v_k^{(j)}$ and $v^{(j)}$ be the j -th inverse vectors of x_k and x , and let $R^{(j)} \in \mathcal{P}$ be the sequence with inverse sequence $v^{(j)}$. For each j , the vectors $r(v_k^{(j)}) \in \mathbb{R}^{N_k}$ satisfy the critical spectral conditions with the constants $(c_\eta, C_\eta, \delta_\eta)$ for every k , and their scaled inverse vectors, padded with zeros, are the vectors $v_k^{(j)}$. Thus

$$\sum_{i \geq 1} \left((N_k^{2/3} r_i(v_k^{(j)}))^{-1} - (R_i^{(j)})^{-1} \right)^2 = \|v_k^{(j)} - v^{(j)}\|_{\ell^2}^2 \rightarrow 0,$$

with the convention that the finite inverse vector vanishes for $i > N_k$, and the second condition in (19.5) holds because $R^{(j)} \in \mathcal{P}$. We may therefore apply Theorem 19.2 and Corollary 19.3 with $\mathcal{N} = \{N_k : k \geq 1\}$, and we obtain $\Psi_{N_k}(v_k^{(j)}) \rightarrow (F_{R^{(j)}}, F_{R^{(j)}}^{-1})$ in $\mathcal{Y}$ for every $j \leq m$. That is, $f_{N_k}(x_k) \rightarrow f(x)$ in G . This step uses Theorem 19.2 and Corollary 19.3 for general critical sequences, and not only for the sequences of Lemma 19.1: the vectors $r(v_k^{(j)})$ are arbitrary vectors that satisfy the critical spectral conditions with fixed constants.

Step 5: conclusion. By Steps 2–4, Lemma 19.23 applies to the laws μ_N and μ , the set $\mathcal{D}$, the maps f_N and f , and the sets A_N^η . We obtain that $(X_N, f_N(X_N))$ converges in distribution to $(X, f(X))$ in $E \times G$. We apply the continuous projection $((\xi, \mathbf{b}, (v^{(j)})_{j \leq m}), y) \mapsto (\xi, \mathbf{b}, y)$ and the continuous mapping theorem [30, Theorem 2.7], and we use (19.57) and (19.58). This proves (19.17), jointly for all $j \leq m$ and jointly with the spectral data.

The ratio. Fix j and $\varepsilon \in (0, 1/2)$. The map $g \mapsto g(\varepsilon)/g(1-\varepsilon)$ on $C((0, 1))$, defined to be 1 when $g(1-\varepsilon) = 0$, is continuous at every g with $g(1-\varepsilon) > 0$. The finite- N quotient in (19.18) is this map applied to $N^{-2/3}t_{N,j}$, with the convention of Theorem 19.4 on $\{t_{N,j}(1-\varepsilon) = 0\}$. Since $N^{-2/3}t_{N,j}(1-\varepsilon)$ converges in distribution to $F_{\sigma_j}^{-1}(1-\varepsilon)$, which is positive almost surely, we have $\mathbb{P}(t_{N,j}(1-\varepsilon) = 0) \rightarrow 0$, and the convention does not affect the limit. We deduce (19.18) from (19.17) and the continuous mapping theorem [30, Theorem 2.7]; note that $t_{N,j}(\varepsilon)/t_{N,j}(1-\varepsilon)$ does not change when both times are multiplied by $N^{-2/3}$. The limit is strictly greater than 1 almost surely by Corollary 19.3.

If there were cutoff in the sense of Definition 3.7, the ratio would converge in probability to 1, hence in distribution to the point mass at 1, which is not the law of the limit in (19.18). We conclude that there is no cutoff. $\square$

Proof of the second part of Corollary 19.5. Almost surely $R^\sigma \in \mathcal{P}$. For such an outcome, Lemma 19.1 provides a sequence satisfying the critical spectral conditions with constants (19.7) computed from R^σ and converging to R^σ . The first part of the corollary then shows that $F_\sigma(s) \geq c_1 e^{-C_1 s}$, with c_1, C_1 depending on these constants. $\square$

Proof of Theorem 1.3. We use the notation $F_\sigma = F_{R^\sigma}$ of (19.15). The joint convergence in distribution of the profiles and the mixing times, on a common disorder, was proved in Theorem 19.4. Almost surely $R^\sigma \in \mathcal{P}$. Using Theorem 19.2 for the sequence of Lemma 19.1 that converges to R^σ , we find that F_σ is continuous and strictly decreasing, with $F_\sigma(0+) = 1$ and $F_\sigma(\infty) = 0$. We obtain the lower bound $F_\sigma(s) \geq c_1 e^{-C_1 s}$, with random constants, from Corollary 19.5. The convergence of the ratio of mixing times and the absence of cutoff were also shown in Theorem 19.4. Finally, we obtain the bounds $C_\eta^{-1} N^{2/3} \log(1/\varepsilon) \leq t_N(\varepsilon) \leq C_\eta N^{2/3} \log(1/\varepsilon)$ for all $\varepsilon \in (0, 1/2)$, on events $\mathcal{A}_N$ with $\liminf_N \mathbb{P}(\mathcal{A}_N) \geq 1 - \eta$ and uniformly for σ in a compact set, from the second part of Corollary 19.11. $\square$

The critical profile far above the critical temperature

In this section and the next, we prove Theorem 1.4, which describes the critical profile F_σ as $\sigma \rightarrow -\infty$ and as $\sigma \rightarrow +\infty$. This section treats the limit $\sigma = -S \rightarrow -\infty$. The main result is Theorem 20.4. It states that for every realization of the reflected GOE Airy point process (χ_i) , the limit of $N^{2/3}(2 - \lambda_i)$, outside a null set, we have $F_{-S}(\tau_S^h + u/S^2) \rightarrow \Psi(u)$ locally uniformly in u , where

$$\tau_S^h = \frac{\log S^3 - \frac{3}{2} \log \log S^3 - \log(16\sqrt{\pi})}{4S^2}, \quad (20.1)$$

and that the inverse functions converge correspondingly.

The proof combines the convergence theorem of the critical window, which holds along deterministic sequences of half-precisions (Theorem 19.2, Corollary 19.3, and Lemma 18.5), with four finite-dimensional estimates at high temperature whose constants do not depend on the dimension: the comparison of the time- t law with the Bingham comparison law (Theorem 14.10), the likelihood ratio estimate for Gaussian laws conditioned on a common sphere (Theorem 6.6), the decay of the relative errors of the covariance evolution (Lemma 7.12), and the formula for the information (Lemma 7.3). We also use the properties of the GOE Airy point process in Theorem 4.15 and Lemma 4.5, the convergence criterion for positive measures in Lemma 7.15, and Lemmas 3.9 and 3.11. The finite approximations of the limiting scaled half-precisions constructed in Lemma 20.1 are used again in Section 21.

20.1 The order of limits

The proof first approximates F_{-S} by finite-dimensional diffusions at each fixed S , taking the dimension N to infinity. It then lets $S \rightarrow \infty$.

This order of limits is different from the one in case (3) of Theorem 1.1, and neither statement implies the other. If we set $\delta_N = SN^{-1/3}$ in case (3), then $M_N = N\delta_N^3 = S^3$, and the center and window of case (3), divided by $N^{2/3}$, converge to τ_S^h and S^{-2} . However, the theorem behind case (3) assumes $M_N \rightarrow \infty$, and it gives no information when M_N stays fixed while $N \rightarrow \infty$. Conversely, the critical theorem holds at fixed σ and gives no information as $\sigma \rightarrow -\infty$.

The link between the two regimes is that several estimates of Parts II and III hold at every finite N , with constants that depend only on the constant in two-sided bounds

on the spacings $p_i - p_1$ of the precisions, and not on N or on $M = Nd^{3/2}$, as soon as M exceeds a fixed threshold. We apply them to finite sequences of precisions that approximate the limiting scaled half-precisions of the critical window and have a fixed large value of M , let $N \rightarrow \infty$ with M fixed, and let $M \rightarrow \infty$ only afterwards. All comparisons with Gaussian laws conditioned on the sphere are made at finite N , for the diffusion started from ν_N ; in particular, the limiting Gibbs law in the critical window is never compared with an unconditioned Gaussian product law.

20.2 Notation and inputs

The limiting equation. Recall from Section 4.5 the GOE Airy point process $\chi_1 < \chi_2 < \dots$, its gaps $\Delta_i = \chi_i - \chi_1$, with $\Delta_1 = 0$, the regularized inverse-gap statistic Ξ , the function $\mathcal{S}$ of (4.29), its roots ϱ_σ , and the limiting scaled half-precisions $R_i^\sigma = (\Delta_i + \varrho_\sigma)/2$ of (4.30). In this section and the next, we regard Ξ and $(\Delta_i)_{i \geq 2}$ as deterministic. We fix $\Xi \in \mathbb{R}$ and $0 < \Delta_2 < \Delta_3 < \dots$, set $\Delta_1 = 0$ and $t_i = (3\pi i/2)^{2/3}$, and assume that

$$\frac{\Delta_i}{t_i} \rightarrow 1 \quad (i \rightarrow \infty), \quad (20.2)$$

$$\sum_{i \geq 1} \frac{1}{(x + \Delta_i)^2} \sim \frac{1}{2\sqrt{x}} \quad (x \rightarrow \infty), \quad (20.3)$$

$$\varrho_{-S} = S^2 + O(S^{3/4}) \quad (S \rightarrow \infty). \quad (20.4)$$

For these deterministic data, we define

$$\mathcal{S}(x) = \Xi + \frac{1}{x} + \sum_{i \geq 2} \left(\frac{1}{x + \Delta_i} - \frac{1}{\Delta_i} \right) \quad (x > 0). \quad (20.5)$$

This is the first expression in (4.29); its equality there with a regularized limit of partial sums is used only for the GOE Airy data, and not below. We show that (20.5) is meaningful under (20.2). Since $\Delta_i/(i-1)^{2/3}$ is positive for $i \geq 2$ and converges to $(3\pi/2)^{2/3}$ by (20.2), there exist $0 < c_\Delta \leq 1 \leq C_\Delta$, depending only on (Δ_i) , such that

$$c_\Delta(i-1)^{2/3} \leq \Delta_i \leq C_\Delta(i-1)^{2/3} \quad (i \geq 2). \quad (20.6)$$

In particular, $\sum_{i \geq 2} \Delta_i^{-2} < \infty$. For x in a compact subset of $(0, \infty)$, the bounds $|(x + \Delta_i)^{-1} - \Delta_i^{-1}| \leq x\Delta_i^{-2}$ and $|\partial_x(x + \Delta_i)^{-1}| \leq \Delta_i^{-2}$ show that the series in (20.5) and its termwise derivative converge uniformly there. We conclude that $\mathcal{S}$ is continuously differentiable on $(0, \infty)$, with

$$-\mathcal{S}'(x) = \sum_{i \geq 1} \frac{1}{(x + \Delta_i)^2} > 0, \quad \mathcal{S}(x) - \mathcal{S}(y) = \sum_{i \geq 1} \left(\frac{1}{x + \Delta_i} - \frac{1}{y + \Delta_i} \right) \quad (x, y > 0). \quad (20.7)$$

As $x \downarrow 0$, the term $1/x$ tends to infinity while the rest of the series stays bounded, and thus $\mathcal{S}(x) \rightarrow +\infty$. By (20.3), the integral of $-\mathcal{S}'$ over $(1, \infty)$ diverges, and therefore $\mathcal{S}(x) \rightarrow -\infty$ as $x \rightarrow \infty$. Then $\mathcal{S}$ is a decreasing bijection of $(0, \infty)$ onto $\mathbb{R}$, and $\varrho_\sigma = \mathcal{S}^{-1}(\sigma)$ and $R_i^\sigma = (\Delta_i + \varrho_\sigma)/2$ are defined for every $\sigma \in \mathbb{R}$.

By Lemmas 4.2 and 4.5 and Theorem 4.15, the GOE Airy point process satisfies (20.2)–(20.4) almost surely. Indeed, the points are distinct and Ξ is finite almost surely; (20.2) follows from $|\chi_k - t_k| \leq C t_k^{3/8}$ in the first part of Theorem 4.15; (20.3) is the case $k = 2$ of (4.32), since $C_2 = 1/2$; and (20.4) is the pathwise outer-root bound of (4.33). Every statement of this section and the next that is made under (20.2)–(20.4) therefore holds almost surely for the GOE Airy point process.

The critical theorem along deterministic sequences. We use the results of Sections 18 and 19 for deterministic sequences of half-precisions, in the terminology of Section 19.1. Let $\mathcal{P}$ be the class (19.6) of sequences $R = (R_i)_{i \geq 1}$ with $0 < R_1 < R_2 \leq R_3 \leq \dots$ and $\lim_i R_i/i^{2/3} \in (0, \infty)$. Let $\mathcal{N}$ be an infinite set of integers, and for $N \in \mathcal{N}$ let $0 < r_{1,N} \leq \dots \leq r_{N,N}$ satisfy the critical spectral conditions (4.36) with constants (c_0, C_0, δ) that do not depend on N . Let $R_{i,N} = N^{2/3}r_{i,N}$, and suppose that $(r_{\cdot,N})$ converges to $R \in \mathcal{P}$ in the sense of (19.5), that is,

$$\sum_{i \geq 1} (R_{i,N}^{-1} \mathbf{1}_{\{i \leq N\}} - R_i^{-1})^2 \longrightarrow 0, \quad \lim_{i \rightarrow \infty} i^{-2/3} R_i \in (0, \infty). \quad (20.8)$$

The second condition, an exact Weyl asymptotic for the limiting scaled half-precisions, is part of the hypotheses of Theorem 19.2 and Lemma 18.5; two-sided bounds $R_i \asymp i^{2/3}$ alone do not suffice. Let $P_t^{(N)}$ be the semigroup generated by $\Delta_{S_N} - \nabla_{S_N}(\sum_i r_{i,N} x_i^2) \cdot \nabla_{S_N}$, let $\pi^{(N)}$ be its invariant law, and let $D^{(N)}(t) = \|\nu_N P_t^{(N)} - \pi^{(N)}\|_{\text{TV}}$ and $t^{(N)}(\varepsilon) = \inf\{t \geq 0 : D^{(N)}(t) \leq \varepsilon\}$. Then Theorem 19.2 and Corollary 19.3 state that, as $N \rightarrow \infty$ along $\mathcal{N}$,

$$D^{(N)}(N^{2/3}s) \longrightarrow F_R(s), \quad N^{-2/3}t^{(N)}(\varepsilon) \longrightarrow F_R^{-1}(\varepsilon), \quad (20.9)$$

locally uniformly in $s \in (0, \infty)$ and in $\varepsilon \in (0, 1)$, respectively. Here F_R is the critical mixing profile (19.11), which is continuous and strictly decreasing on $(0, \infty)$, with $F_R(0+) = 1$ and $F_R(\infty) = 0$, and F_R^{-1} is its inverse. We set $F_R(0) = 1$.

We also need the mean of the covariance evolution. With $p_{i,N} = 2r_{i,N}$, let

$$H_N = -2 \operatorname{diag}(p_{1,N}, \dots, p_{N,N}) + \frac{2}{N} \mathbf{1} \mathbf{1}^\top, \quad h_N(t) = e^{tH_N} \mathbf{1}, \quad \bar{h}_N(t) = \frac{1}{N} \sum_{i=1}^N h_{N,i}(t),$$

be the heat matrix, the heat vector, and its mean, as in (7.3), and let $\psi_N(s) = N^{1/3} \bar{h}_N(N^{2/3}s)$. By Lemma 18.5, every derivative of ψ_N converges uniformly on compact subsets of $(0, \infty)$ to the corresponding derivative of a positive completely monotone function ψ_R , which satisfies

$$\int_0^\infty e^{-ps} \psi_R(s) ds = \left(\frac{p}{4} \sum_{i \geq 1} \frac{1}{R_i(R_i + p/4)} \right)^{-1} \quad (p > 0). \quad (20.10)$$

For $\sigma \in \mathbb{R}$, we write F_σ for the critical mixing profile F_{R^σ} , with limiting scaled half-precisions $R^\sigma = (R_i^\sigma)_{i \geq 1}$ computed from Ξ and (Δ_i) . Under (20.2), R^σ belongs to $\mathcal{P}$, with $i^{-2/3} R_i^\sigma \rightarrow (3\pi/2)^{2/3}/2$. When Ξ and (Δ_i) come from the GOE Airy point process, F_σ is the critical mixing profile of Theorem 1.3.

Finite sequences of precisions. Let $0 < p_1 \leq p_2 \leq \dots \leq p_N$ satisfy the trace condition (7.1), $\sum_i p_i^{-1} = N$. As in Section 7, we write $d = p_1$, $q_i = 1/p_i$, and $\vartheta = N / \sum_i q_i^2$ for the weight of the zero eigenvalue. We let H , $h(t)$, $\bar{h}(t)$, $v_i(t)$, $e_i(t)$, and $\mathcal{I}_N(t) = \sum_i e_i(t)^2/2$ be the heat matrix, the heat vector and its mean, the variances, the relative errors, and the information of (7.3) and (7.4). We also write $M = Nd^{3/2}$. The matrix $A = -\text{diag}(p_1, \dots, p_N)$ has Lagrange multiplier 0 and precisions p_i , and we write $P_t = P_t^A$, which is generated by $\Delta_{S_N} - \nabla_{S_N}(\sum_i p_i x_i^2/2) \cdot \nabla_{S_N}$, and $\pi = \pi_A$. By Proposition 5.6, $\pi = \mu_q$ is the Gaussian law with variances q_i conditioned on S_N , and $\mu_{v(t)}$ is the Bingham comparison law of A . We recall that $\mu_{v(t)}$ need not coincide with the time- t law and that $v_i(t)$ need not equal its second moments; the estimates below compare these objects. We say that p satisfies the precision-spacing bounds with constant $K \geq 1$ if

$$K^{-1} \left(\frac{i-1}{N} \right)^{2/3} \leq p_i - p_1 \leq K \left(\frac{i-1}{N} \right)^{2/3} \quad (2 \leq i \leq N). \quad (20.11)$$

We use the following four results.

1. By Theorem 14.10, for all $K \geq 1$, $d_0 \in (0, 1)$, and $C_T > 0$, there exist $C_{cp} \geq 1$ and $M_0 \geq 4$, depending only on K , d_0 , and C_T , such that if p satisfies the precision-spacing bounds with constant K , $0 < d \leq d_0$, and $M \geq M_0$, then

$$\|\nu_N P_t - \mu_{v(t)}\|_{\text{TV}} \leq \min \{1, C_{cp} e^{-(d-2/N)t}\} \quad (0 \leq t \leq C_T d^{-1} \log M). \quad (20.12)$$

These are the hypotheses (14.33) of that theorem. The estimate holds for every N ; in particular, it applies when M stays fixed as $N \rightarrow \infty$.

2. By Lemma 7.12, for every $K \geq 1$ there exists C_{re} , depending only on K , such that if p satisfies the precision-spacing bounds with constant K , $d \leq 1$, and $M \geq 1$, which are the hypotheses (7.30), then

$$|e_i(d^{-1} + \tau)| \leq C_{re} \frac{d}{p_i} (1 + d\tau) e^{-2d\tau} \quad (\tau \geq 0, 1 \leq i \leq N). \quad (20.13)$$

3. By Theorem 6.6, applied with $v = q$ and $\tilde{v} = v(t)$, which have the same sum N , there exist a universal constant $\eta_{\text{lik}} > 0$ and, for every $B < \infty$, a constant C_B with the following property. Let $\eta_q = q_1 / (2 \sum_i q_i^2)^{1/2}$, which is the quantity η_v of (5.1) for $v = q$, since $q_1 = \max_i q_i$. If $\max_i |e_i(t)| \leq 1/4$, $\eta_q \leq \eta_{\text{lik}}$, and $\mathcal{I}_N(t) \leq B$, then

$$\left| \|\mu_{v(t)} - \pi\|_{\text{TV}} - \left(2\Phi(\sqrt{\mathcal{I}_N(t)}/2) - 1 \right) \right| \leq C_B \left(\eta_q + \max_i |e_i(t)| \right). \quad (20.14)$$

4. By Lemma 7.3, we have $\mathcal{I}_N(t) = N\bar{h}''(2t)/(8\bar{h}(t)^2)$ for $t \geq 0$, and there exists a finite positive measure ρ on $[2d, \infty)$ such that

$$\bar{h}(t) = \vartheta + \int_{[2d, \infty)} e^{-\lambda t} \rho(d\lambda) \quad (t \geq 0). \quad (20.15)$$

20.3 Finite approximations of the limiting scaled half-precisions

Let $\xi > 0$. The precisions $N^{-2/3}(\Delta_i + \xi)$, $1 \leq i \leq N$, which correspond to the limiting scaled half-precisions $(\Delta_i + \xi)/2$, do not satisfy the trace condition $\sum_i p_i^{-1} = N$. Indeed, by (20.17) below, $N^{-1} \sum_{i \leq N} N^{2/3}(\Delta_i + \xi)^{-1} \rightarrow (12/\pi^2)^{1/3} \neq 1$. The reason is that the gaps Δ_i grow like $i^{2/3}$ for every i , while the eigenvalues of the GOE follow the semicircle law in the bulk of the spectrum. We keep the first half of these precisions and multiply the second half by a factor that restores the trace exactly. This changes no fixed coordinate. We check that the resulting sequences satisfy both the hypotheses of the critical theorem and the precision-spacing bounds (20.11), with a constant that does not depend on the shift ξ .

Lemma 20.1 (Finite approximations of the limiting scaled half-precisions). *Assume (20.2). There exists $K \geq 1$, depending only on (Δ_i) , with the following property. For every $\xi > 0$, there exists $N_0(\xi)$ such that for every $N \geq N_0(\xi)$, with $n = \lfloor N/2 \rfloor$, there exist $\gamma_N \in (1, 3]$ and a sequence*

$$p_{i,N} = \begin{cases} N^{-2/3}(\Delta_i + \xi), & 1 \leq i \leq n, \\ \gamma_N N^{-2/3}(\Delta_i + \xi), & n < i \leq N, \end{cases} \quad (20.16)$$

such that the following hold.

1. We have $\sum_i p_{i,N}^{-1} = N$ and $0 < p_{1,N} < p_{2,N} < \dots < p_{N,N}$. The smallest precision is $p_{1,N} = \xi N^{-2/3}$, and $M = N p_{1,N}^{3/2} = \xi^{3/2}$.
2. The sequence $(p_{i,N})$ satisfies the precision-spacing bounds (20.11) with constant K , and $\max_i p_{i,N} \leq K$. Moreover, $N^{2/3}(p_{i,N} - p_{1,N}) = \Delta_i$ for $2 \leq i \leq n$, and $\Delta_i \leq N^{2/3}(p_{i,N} - p_{1,N}) \leq 5\Delta_i$ for $n < i \leq N$.
3. The half-precisions $r_{i,N} = p_{i,N}/2$ satisfy the critical spectral conditions (4.36) with constants (c_0, C_0, δ) , where $\delta = \Delta_2/2$ and c_0, C_0 depend only on (Δ_i) and ξ . With $R_i = (\Delta_i + \xi)/2$, we have $R = (R_i) \in \mathcal{P}$ and $i^{-2/3}R_i \rightarrow (3\pi/2)^{2/3}/2$, and $(r_{i,N})$ converges to R in the sense of (20.8).
4. The conclusions (20.9) and (20.10) hold for these sequences with $R = R^{\mathcal{S}(\xi)}$, and $F_R = F_{\mathcal{S}(\xi)}$.

Proof. Fix $\xi > 0$, and let c_Δ, C_Δ be as in (20.6).

Step 1: partial traces. For $0 < a \leq 1$, let $\Sigma_N(a) = N^{-1/3} \sum_{1 \leq i \leq aN} (\Delta_i + \xi)^{-1}$. We claim that

$$\Sigma_N(a) \longrightarrow \left(\frac{12a}{\pi^2} \right)^{1/3} \quad (N \rightarrow \infty). \quad (20.17)$$

Fix $\varepsilon \in (0, 1)$. By (20.2), there exists J such that $(1 - \varepsilon)t_i \leq \Delta_i + \xi \leq (1 + \varepsilon)t_i$ for $i > J$. The terms with $i \leq J$ contribute at most $JN^{-1/3}/\xi$ to $\Sigma_N(a)$. The function $i \mapsto i^{-2/3}$ is decreasing, and comparison with an integral gives $\sum_{i \leq aN} i^{-2/3} = 3(aN)^{1/3} + O(1)$, so

$$N^{-1/3} \sum_{J < i \leq aN} \frac{1}{t_i} \longrightarrow \left(\frac{3\pi}{2} \right)^{-2/3} 3a^{1/3} = \left(\frac{12a}{\pi^2} \right)^{1/3}.$$

Combining these facts, we find that the limit points of $\Sigma_N(a)$ lie between $(1 + \varepsilon)^{-1}(12a/\pi^2)^{1/3}$ and $(1 - \varepsilon)^{-1}(12a/\pi^2)^{1/3}$, and ε is arbitrary. This proves (20.17).

Let $\Sigma_N^- = \Sigma_N(1/2)$, the sum over $i \leq n$, and $\Sigma_N^+ = \Sigma_N(1) - \Sigma_N(1/2)$, the sum over $n < i \leq N$. Then (20.17) implies that

$$\Sigma_N^- \longrightarrow \left(\frac{6}{\pi^2} \right)^{1/3} < 1, \quad \Sigma_N^- + \Sigma_N^+ \longrightarrow \left(\frac{12}{\pi^2} \right)^{1/3} > 1,$$

where we used $6 < \pi^2 < 12$. Let $\gamma_N = \Sigma_N^+ / (1 - \Sigma_N^-)$, which is defined once $\Sigma_N^- < 1$. Then γ_N converges to

$$\gamma_* = \frac{(12/\pi^2)^{1/3} - (6/\pi^2)^{1/3}}{1 - (6/\pi^2)^{1/3}} \approx 1.440,$$

and $\gamma_* > 1$ because $(12/\pi^2)^{1/3} > 1$. We choose $N_0(\xi)$ such that for $N \geq N_0(\xi)$ we have $1 < \gamma_N \leq 3$, $n \geq 3$, $\Delta_n \geq \xi$, and $\xi N^{-2/3} \leq 1$. This is possible since $\Delta_n \rightarrow \infty$.

Step 2: the trace and the order. Let $N \geq N_0(\xi)$ and define $p_{i,N}$ by (20.16). By the definitions of $\Sigma_N^\pm$ and γ_N ,

$$\sum_{i=1}^N p_{i,N}^{-1} = N \left(\Sigma_N^- + \frac{\Sigma_N^+}{\gamma_N} \right) = N.$$

The sequence is strictly increasing on $\{1, \dots, n\}$ and on $\{n+1, \dots, N\}$ because (Δ_i) is, and $p_{n+1,N} = \gamma_N N^{-2/3} (\Delta_{n+1} + \xi) > p_{n,N}$ because $\gamma_N > 1$. This proves the first part.

Step 3: the precision-spacing bounds. For $2 \leq i \leq n$, we have $N^{2/3}(p_{i,N} - p_{1,N}) = \Delta_i$. For $n < i \leq N$, we have

$$N^{2/3}(p_{i,N} - p_{1,N}) = \gamma_N \Delta_i + (\gamma_N - 1)\xi \in [\Delta_i, (2\gamma_N - 1)\Delta_i] \subset [\Delta_i, 5\Delta_i],$$

where we used $\gamma_N > 1$, $\xi \leq \Delta_n \leq \Delta_i$, and $\gamma_N \leq 3$. Together with (20.6), this gives

$$c_\Delta \left(\frac{i-1}{N} \right)^{2/3} \leq p_{i,N} - p_{1,N} \leq 5C_\Delta \left(\frac{i-1}{N} \right)^{2/3} \quad (2 \leq i \leq N),$$

and $\max_i p_{i,N} = p_{N,N} \leq \xi N^{-2/3} + 5C_\Delta \leq 1 + 5C_\Delta$. This shows that (20.11) and the bound $\max_i p_{i,N} \leq K$ hold with $K = \max(c_\Delta^{-1}, 1 + 5C_\Delta)$, which depends only on (Δ_i) . This proves the second part.

Step 4: the critical conditions. Let $r_{i,N} = p_{i,N}/2$. Then $\sum_i (2r_{i,N})^{-1} = N$, and $r_{2,N} - r_{1,N} = \Delta_2/(2N^{2/3})$ because $2 \leq n$. For $i = 1$, we have $r_{1,N} = (\xi/2)(1/N)^{2/3}$. For $i \geq 2$, we have $(i-1)^{2/3} \geq 2^{-2/3}i^{2/3}$, and Step 3 gives

$$\frac{c_\Delta}{2^{5/3}} \left(\frac{i}{N}\right)^{2/3} \leq \frac{p_{i,N} - p_{1,N}}{2} \leq r_{i,N} \leq \frac{\xi N^{-2/3} + 5C_\Delta((i-1)/N)^{2/3}}{2} \leq \frac{\xi + 5C_\Delta}{2} \left(\frac{i}{N}\right)^{2/3}.$$

These bounds show that (4.36) holds with $c_0 = \min(\xi/2, 2^{-5/3}c_\Delta)$, $C_0 = (\xi + 5C_\Delta)/2$, and $\delta = \Delta_2/2$.

Let $R_{i,N} = N^{2/3}r_{i,N}$ and $R_i = (\Delta_i + \xi)/2$. Then $R_{i,N} = R_i$ for $i \leq n$, and $R_{i,N} = \gamma_N R_i > R_i$ for $n < i \leq N$. It follows that $0 \leq R_i^{-1} - R_{i,N}^{-1} \mathbf{1}_{\{i \leq N\}} \leq R_i^{-1}$ for every $i > n$. Since this difference vanishes for $i \leq n$, we have

$$\sum_{i \geq 1} (R_{i,N}^{-1} \mathbf{1}_{\{i \leq N\}} - R_i^{-1})^2 \leq \sum_{i > n} R_i^{-2} \leq 4 \sum_{i > n} \Delta_i^{-2} \rightarrow 0.$$

Finally, $i^{-2/3}R_i = i^{-2/3}(\Delta_i + \xi)/2 \rightarrow (3\pi/2)^{2/3}/2$ by (20.2), and $0 < R_1 < R_2 \leq R_3 \leq \dots$, so that $R \in \mathcal{P}$. This proves the third part. Since $\varrho_{\mathcal{S}(\xi)} = \xi$, the limiting scaled half-precisions are $R = R^{\mathcal{S}(\xi)}$, and the fourth part follows from (20.9) and (20.10). $\square$

The constant K in the precision-spacing bounds does not depend on ξ , while $N_0(\xi)$ and the constants c_0, C_0 of the critical conditions do. Only K enters the estimates (20.12) and (20.13), and only the limit $N \rightarrow \infty$ at fixed ξ uses the critical conditions.

20.4 The information of the limiting covariance evolution

For $\alpha > 0$, let $p_{i,N}$ be the sequences of Lemma 20.1 with $\xi = \alpha^2$, so that $d_N = p_{1,N} = \alpha^2 N^{-2/3}$ and $M = \alpha^3$. Let $\bar{h}_N$, ψ_N , and $\mathcal{I}_N$ be the quantities of Section 20.2 for $p_{i,N}$, let $R_i = (\Delta_i + \alpha^2)/2$, and write $\psi = \psi_R$ for the limit in (20.10). We measure time in units of $1/d_N$: the time $N^{2/3}s$ corresponds to $w = \alpha^2 s = d_N N^{2/3}s$. Accordingly, we define for $w > 0$

$$f_\alpha(w) = \frac{1}{\alpha} \psi\left(\frac{w}{\alpha^2}\right), \quad \omega_\alpha = \left(\alpha \sum_{i \geq 1} \frac{1}{(\Delta_i + \alpha^2)^2}\right)^{-1}, \quad j_\alpha(w) = f_\alpha(w) - \omega_\alpha. \quad (20.18)$$

By (20.21) below, for each fixed $\alpha > 0$ and $w_0 > 0$, we have $0 \leq j_\alpha(w) \leq e^{-2(w-w_0)} j_\alpha(w_0)$ for $w \geq w_0$, and hence $f_\alpha(w) \rightarrow \omega_\alpha$ as $w \rightarrow \infty$. Lemma 20.3 below gives a relative asymptotic for $j_\alpha(w)$ in the joint regime $\alpha \rightarrow \infty$, $w \rightarrow \infty$, $w = o(\alpha^2)$. Note that (20.3) implies that $\omega_\alpha \rightarrow 2$ as $\alpha \rightarrow \infty$.

Lemma 20.2 (The limiting information). *Assume (20.2), and fix $\alpha > 0$.*

1. *For every $s > 0$, as $N \rightarrow \infty$,*

$$\mathcal{I}_N(N^{2/3}s) \longrightarrow \mathcal{I}_\alpha(s) = \frac{\psi''(2s)}{8\psi(s)^2} = \frac{\alpha^3 j''_\alpha(2\alpha^2 s)}{8f_\alpha(\alpha^2 s)^2}. \quad (20.19)$$

2. *For every $z > 0$,*

$$\int_0^\infty e^{-zw} f_\alpha(w) dw = \frac{\alpha}{2[\mathcal{S}(\alpha^2) - \mathcal{S}(\alpha^2(1+z/2))]} \quad (20.20)$$

3. *There exists a positive Borel measure $\mathbf{n}_\alpha$ on $[0, \infty)$ such that $\int (3+2x)^{-1} \mathbf{n}_\alpha(dx) < \infty$ and*

$$j_\alpha(w) = e^{-2w} \int_{[0, \infty)} e^{-2wx} \mathbf{n}_\alpha(dx) \quad (w > 0). \quad (20.21)$$

4. *For every real $z > -2$ with $z \neq 0$,*

$$\int_{[0, \infty)} \frac{\mathbf{n}_\alpha(dx)}{z+2+2x} = \frac{\alpha}{2[\mathcal{S}(\alpha^2) - \mathcal{S}(\alpha^2(1+z/2))]} - \frac{\omega_\alpha}{z}. \quad (20.22)$$

The right side extends continuously to $z = 0$, and the identity holds there as well.

The measure $\mathbf{n}_\alpha$ need not have finite total mass. The point of (20.22) is that it holds for negative z , where the Laplace transform of f_α itself diverges because f_α tends to $\omega_\alpha > 0$. In Lemma 20.3 below, we use it at z close to -2 , and this is how the decay rate e^{-2w} enters.

Proof. The first part follows from the formula for the information and Lemma 18.5, and the second is a computation with (20.10). In the third part, the fact that $\mathbf{n}_\alpha$ lives on $[0, \infty)$ comes from the lower bound $2d_N$ on the nonzero frequencies of the covariance evolution at finite N . The fourth part follows from the second and the third by analytic continuation.

Step 1: the information. Recall from Lemma 7.3 that $\mathcal{I}_N(t) = N\bar{h}_N''(2t)/(8\bar{h}_N(t)^2)$. Using $\psi_N''(s) = N^{1/3}N^{4/3}\bar{h}_N''(N^{2/3}s)$, we find

$$\mathcal{I}_N(N^{2/3}s) = \frac{N \cdot N^{-5/3}\psi_N''(2s)}{8N^{-2/3}\psi_N(s)^2} = \frac{\psi_N''(2s)}{8\psi_N(s)^2}.$$

The sequences of Lemma 20.1 satisfy the hypotheses of Lemma 18.5, which gives the convergence of ψ_N and ψ_N'' ; moreover $\psi > 0$. This gives the first expression in (20.19). The second follows from $\psi(s) = \alpha f_\alpha(\alpha^2 s)$, $\psi''(2s) = \alpha^5 f_\alpha''(2\alpha^2 s)$, and $f_\alpha'' = j_\alpha''$.

Step 2: the Laplace transform. For $R_i = (\Delta_i + \alpha^2)/2$ and $p > 0$, we have

$$\frac{p}{4} \cdot \frac{1}{R_i(R_i + p/4)} = \frac{p}{(\Delta_i + \alpha^2)(\Delta_i + \alpha^2 + p/2)} = 2 \left(\frac{1}{\Delta_i + \alpha^2} - \frac{1}{\Delta_i + \alpha^2 + p/2} \right).$$

Summing over i and using (20.7), we find that the right side of (20.10) equals $(2[\mathcal{S}(\alpha^2) - \mathcal{S}(\alpha^2 + p/2)])^{-1}$. The substitution $w = \alpha^2 s$ gives $\int_0^\infty e^{-zw} f_\alpha(w) dw = \alpha \int_0^\infty e^{-z\alpha^2 s} \psi(s) ds$, and taking $p = \alpha^2 z$ proves (20.20).

Step 3: the positive measure. Let $N \geq N_0(\alpha^2)$, and let ϑ_N and ρ_N be the weight of the zero eigenvalue and the measure in (20.15) for the sequence $p_{i,N}$; the measure is supported on $[2d_N, \infty)$. Let $\mathbf{n}_{\alpha,N}$ be the image of $\alpha^{-1} N^{1/3} \rho_N$ under the map $\lambda \mapsto \lambda/(2d_N) - 1$, which maps $[2d_N, \infty)$ to $[0, \infty)$, and define

$$f_{\alpha,N}(w) = \frac{1}{\alpha} \psi_N\left(\frac{w}{\alpha^2}\right), \quad \omega_{\alpha,N} = \frac{N^{1/3} \vartheta_N}{\alpha}, \quad j_{\alpha,N} = f_{\alpha,N} - \omega_{\alpha,N}.$$

Inserting $N^{2/3} w / \alpha^2 = w / d_N$ into (20.15), we have

$$j_{\alpha,N}(w) = \int_{[0,\infty)} e^{-(2+2x)w} \mathbf{n}_{\alpha,N}(dx) \quad (w \geq 0). \quad (20.23)$$

With $P_{i,N} = N^{2/3} p_{i,N} = 2R_{i,N}$ and $q_i = p_{i,N}^{-1}$, we have $\sum_i q_i^2 = N^{4/3} \sum_i P_{i,N}^{-2}$, and therefore $\omega_{\alpha,N} = (\alpha \sum_i P_{i,N}^{-2})^{-1}$. By the ℓ^2 convergence in the third part of Lemma 20.1, we have $\sum_i P_{i,N}^{-2} \rightarrow \sum_i (\Delta_i + \alpha^2)^{-2}$ and $\omega_{\alpha,N} \rightarrow \omega_\alpha$. Also $f_{\alpha,N}(w) \rightarrow f_\alpha(w)$ for every $w > 0$, by Lemma 18.5. We conclude that $j_{\alpha,N}(w) \rightarrow j_\alpha(w)$ for every $w > 0$. Note that j_α is continuous on $(0, \infty)$, since ψ is.

Fix $\tau > 0$, and let $m_N(dx) = e^{-(2+2x)\tau} \mathbf{n}_{\alpha,N}(dx)$, a finite measure on $[0, \infty)$. Using (20.23), we find that for every $t \geq 0$,

$$\int e^{-(2+2x)t} m_N(dx) = j_{\alpha,N}(\tau + t) \longrightarrow j_\alpha(\tau + t). \quad (20.24)$$

Taking $t = 0$, we see that the total masses $m_N([0, \infty)) = j_{\alpha,N}(\tau)$ are bounded. The family (m_N) is also tight. Indeed, for $L > 0$ and $t > 0$, we have $1 - e^{-(2+2x)t} \geq 1 - e^{-2Lt}$ for $x \geq L$, so

$$(1 - e^{-2Lt}) m_N([L, \infty)) \leq \int (1 - e^{-(2+2x)t}) m_N(dx) = j_{\alpha,N}(\tau) - j_{\alpha,N}(\tau + t).$$

Letting $N \rightarrow \infty$, then $L \rightarrow \infty$, and then $t \downarrow 0$, and using the continuity of j_α at τ , we find that $\lim_{L \rightarrow \infty} \limsup_N m_N([L, \infty)) = 0$. Let $b_\tau = j_\alpha(\tau)$, the limit of the total masses. If $b_\tau = 0$, then m_N converges weakly to the zero measure. If $b_\tau > 0$, then for all large N the probability measures $m_N / m_N([0, \infty))$ are tight, and by Prokhorov's theorem [30, Theorem 5.1] a subsequence of them converges weakly to a probability measure;

multiplying it by b_τ , we find that a subsequence of (m_N) converges weakly to a finite measure $m^{(\tau)}$ on $[0, \infty)$ of mass b_τ . The function $x \mapsto e^{-(2+2x)t}$ is bounded and continuous, and (20.24) implies that $\int e^{-(2+2x)t} m^{(\tau)}(dx) = j_\alpha(\tau + t)$ for every $t \geq 0$. This determines $m^{(\tau)}$. Indeed, the value at $t = 0$ is its total mass. If the mass is positive, we divide by it and cancel the factor e^{-2t} ; the image of the resulting probability measure under $x \mapsto 2x$ has the Laplace transform $t \mapsto e^{2t} j_\alpha(\tau + t) / j_\alpha(\tau)$, and a probability measure on $[0, \infty)$ is determined by its Laplace transform [71, Section XIII.1, Theorem 1]. The case of zero mass is immediate. For $\tau < \tau'$, the measures $e^{-(2+2x)(\tau'-\tau)} m^{(\tau)}(dx)$ and $m^{(\tau')}$ have the same Laplace transform $t \mapsto j_\alpha(\tau' + t)$, and they are equal by the same argument. We define $n_\alpha(dx) = e^{(2+2x)\tau} m^{(\tau)}(dx)$, which does not depend on τ . For $w > 0$ and $\tau < w$, we have

$$\int e^{-(2+2x)w} n_\alpha(dx) = \int e^{-(2+2x)(w-\tau)} m^{(\tau)}(dx) = j_\alpha(w),$$

which is (20.21). Note that $j_\alpha \geq 0$. Using Tonelli's theorem and (20.20) with $z = 1$, we find

$$\int \frac{n_\alpha(dx)}{3+2x} = \int_0^\infty e^{-w} j_\alpha(w) dw \leq \int_0^\infty e^{-w} f_\alpha(w) dw < \infty.$$

Step 4: real resolvents. Let $G(z) = \int (z + 2 + 2x)^{-1} n_\alpha(dx)$ for $z \in \mathbb{C} \setminus (-\infty, -2]$. For z in a compact subset $\mathcal{K}$ of this domain, there exists $c_{\mathcal{K}} > 0$ with $|z + 2 + 2x| \geq c_{\mathcal{K}}(3 + 2x)$ for all $x \geq 0$, since the ratio $|z + 2 + 2x| / (3 + 2x)$ is positive and continuous in $(z, x) \in \mathcal{K} \times [0, \infty]$, with value 1 at $x = \infty$. By Step 3 and dominated convergence, G is holomorphic on $\mathbb{C} \setminus (-\infty, -2]$. For real $z > 0$, Tonelli's theorem, (20.21), and (20.20) give

$$G(z) = \int_0^\infty e^{-zw} j_\alpha(w) dw = \frac{\alpha}{2\mathcal{D}(z)} - \frac{\omega_\alpha}{z}, \quad \mathcal{D}(z) = \mathcal{S}(\alpha^2) - \mathcal{S}(\alpha^2(1 + z/2)).$$

The function $\mathcal{S}$ extends holomorphically to $\mathbb{C} \setminus (-\infty, 0]$, since the difference series in (20.5) converges uniformly on compact subsets there by (20.6). Then $\mathcal{D}$ is holomorphic on $\mathbb{C} \setminus (-\infty, -2]$. On the real interval $(-2, \infty)$, $\mathcal{D}$ vanishes only at $z = 0$, because $\mathcal{S}$ is strictly decreasing on $(0, \infty)$, and this zero is simple, with

$$\mathcal{D}'(0) = -\frac{\alpha^2}{2} \mathcal{S}'(\alpha^2) = \frac{\alpha^2}{2} \sum_{i \geq 1} \frac{1}{(\Delta_i + \alpha^2)^2} = \frac{\alpha}{2\omega_\alpha} > 0.$$

It follows that $\alpha / (2\mathcal{D}(z))$ has a simple pole at 0 with residue $\alpha / (2\mathcal{D}'(0)) = \omega_\alpha$, and the right side of (20.22) has a removable singularity at 0. Each $z_0 \in (-2, \infty)$ is the center of an open disc in $\mathbb{C} \setminus (-\infty, -2]$ on which this right side is holomorphic, and the union Ω of these discs is a connected open set containing $(-2, \infty)$. Both sides of (20.22) are holomorphic on Ω and agree on $(0, \infty)$, so they agree on Ω by the identity theorem. This proves (20.22), including the value at $z = 0$. $\square$

The next lemma computes the decay of j_α and of its second derivative in the regime where $w \rightarrow \infty$ but $w/\alpha^2 \rightarrow 0$, which contains the times relevant for the profile. Its

proof reads the asymptotics off the real squared resolvents of the measures $\mathbf{n}_\alpha$, using Lemma 7.15, without differentiating an asymptotic formula.

Lemma 20.3 (Decay of the transient part). *Assume (20.2) and (20.3). Let $\alpha_n \rightarrow \infty$ and $w_n \rightarrow \infty$ with $w_n/\alpha_n^2 \rightarrow 0$. Then*

$$j_{\alpha_n}(w_n) = (1 + o(1)) \frac{e^{-2w_n w_n^{-3/2}}}{4\sqrt{2\pi}}, \quad (20.25)$$

$$j_{\alpha_n}''(2w_n) = (1 + o(1)) \frac{e^{-4w_n w_n^{-3/2}}}{4\sqrt{\pi}}. \quad (20.26)$$

Moreover, for every compact $K \subset \mathbb{R}$, these relations hold uniformly in $u \in K$ for $w = w_\alpha^* + u$ as $\alpha \rightarrow \infty$, where

$$w_\alpha^* = \frac{1}{4} \left(\log \alpha^3 - \frac{3}{2} \log \log \alpha^3 - \log(16\sqrt{\pi}) \right). \quad (20.27)$$

Proof. We abbreviate $\alpha = \alpha_n$ and $w = w_n$. Let ν_n be $w^{3/2}$ times the image of $\mathbf{n}_\alpha$ under $x \mapsto wx$, a positive measure on $[0, \infty)$. For $s > 0$, let $z = -2 + 2s/w > -2$. Using $z + 2 + 2x = 2(s + wx)/w$ and differentiating (20.22) in z , we obtain

$$\int \frac{\nu_n(dy)}{(s + y)^2} = \frac{4}{\sqrt{w}} \int \frac{\mathbf{n}_\alpha(dx)}{(z + 2 + 2x)^2} = -\frac{\alpha^3 \mathcal{S}'(es)}{\sqrt{w} [\mathcal{S}(\alpha^2) - \mathcal{S}(es)]^2} - \frac{4\omega_\alpha}{\sqrt{w} z^2}, \quad (20.28)$$

where $e = \alpha^2/w$, so that $\alpha^2(1 + z/2) = es$. Here the left side of (20.22) is differentiated under the integral, which is justified because it is holomorphic, and the second term on the right requires $z \neq 0$, that is, $w \neq s$. We note that the left side of (20.28) is finite.

We now let $n \rightarrow \infty$ at fixed $s > 0$. Then $e \rightarrow \infty$, and $z \rightarrow -2$. Using (20.7) and (20.3), we find $-\mathcal{S}'(x) = (1 + o(1))/(2\sqrt{x})$ as $x \rightarrow \infty$. Then $-\mathcal{S}'(es) = (1 + o(1))/(2\sqrt{es})$, and integrating $\mathcal{S}'$ over $[es, \alpha^2]$, we find

$$\mathcal{S}(\alpha^2) - \mathcal{S}(es) = -(1 + o(1))(\alpha - \sqrt{es}) = -(1 + o(1))\alpha,$$

where we used $\sqrt{es}/\alpha = \sqrt{s/w} \rightarrow 0$. Also $\omega_\alpha \rightarrow 2$ and $z^2 \rightarrow 4$. Inserting these into (20.28), we find

$$\int \frac{\nu_n(dy)}{(s + y)^2} = (1 + o(1)) \frac{\alpha^3}{2\sqrt{es}\sqrt{w}\alpha^2} + O(w^{-1/2}) \rightarrow \frac{1}{2\sqrt{s}},$$

since $\sqrt{es}\sqrt{w} = \alpha\sqrt{s}$. Let $\mu(dy) = \pi^{-1}\sqrt{y}dy$ on $[0, \infty)$. The substitution $y = sr^2$ gives

$$\int_0^\infty \frac{\sqrt{y}dy}{\pi(s + y)^2} = \frac{2}{\pi\sqrt{s}} \int_0^\infty \frac{r^2 dr}{(1 + r^2)^2} = \frac{2}{\pi\sqrt{s}} \cdot \frac{\pi}{4} = \frac{1}{2\sqrt{s}}.$$

We have shown that the squared Stieltjes transforms of ν_n converge to that of μ at every $s > 0$; these are the hypotheses of Lemma 7.15 for ν_n and μ .

Using (20.21), differentiation under the integral, and the substitution $y = wx$, we have

$$j_\alpha(w) = e^{-2w} w^{-3/2} \int e^{-2y} \nu_n(dy), \quad j_\alpha''(2w) = 4e^{-4w} w^{-3/2} \int \left(1 + \frac{y}{w}\right)^2 e^{-4y} \nu_n(dy).$$

The differentiation is justified for $w > 0$ because $(2 + 2x)^2 e^{-(2+2x)v} \leq C_v e^{-(2+2x)v/2}$ and $\int e^{-(2+2x)v/2} \mathbf{n}_\alpha(dx) = j_\alpha(v/2) < \infty$. The tests $\varphi(y) = e^{-2y}$, $\varphi(y) = e^{-4y}$, and $\varphi_n(y) = (1 + y/w_n)^2 e^{-4y}$ are continuous, $\varphi_n \rightarrow e^{-4y}$ uniformly on compact intervals, and $(1 + y)^2 |\varphi_n(y)| \leq (1 + y)^4 e^{-4y}$ once $w_n \geq 1$, which tends to zero as $y \rightarrow \infty$. By Lemma 7.15, we obtain

$$\begin{aligned} \int e^{-2y} \nu_n(dy) &\longrightarrow \frac{1}{\pi} \int_0^\infty \sqrt{y} e^{-2y} dy = \frac{1}{4\sqrt{2\pi}}, \\ \int \varphi_n d\nu_n &\longrightarrow \frac{1}{\pi} \int_0^\infty \sqrt{y} e^{-4y} dy = \frac{1}{16\sqrt{\pi}}, \end{aligned}$$

where we used $\int_0^\infty \sqrt{y} e^{-ay} dy = \sqrt{\pi} a^{-3/2}/2$. This proves (20.25) and (20.26).

For the last claim, $w = w_\alpha^* + u$ satisfies $w \rightarrow \infty$ and $w/\alpha^2 \rightarrow 0$ uniformly in $u \in K$. If the relations were not uniform in $u \in K$, there would exist $\varepsilon > 0$, $\alpha_n \rightarrow \infty$, and $u_n \in K$ such that one of the relative errors at $w_n = w_{\alpha_n}^* + u_n$ is at least ε for every n . This contradicts the first part applied to the sequences α_n and w_n . $\square$

20.5 The profile

We now prove the hot half of Theorem 1.4. Recall (20.1) and (20.27). The function Ψ and its inverse $u_\varepsilon = \Psi^{-1}(\varepsilon)$ are those of (3.16).

Theorem 20.4 (The critical profile far above the critical temperature). *Assume (20.2)–(20.4); in particular, the following holds almost surely for the GOE Airy point process. For every compact $K \subset \mathbb{R}$,*

$$\sup_{u \in K} \left| F_{\mathcal{J}(\alpha^2)} \left(\frac{w_\alpha^* + u}{\alpha^2} \right) - \Psi(u) \right| \longrightarrow 0 \quad (\alpha \rightarrow \infty) \quad (20.29)$$

and

$$\sup_{u \in K} \left| F_{-S} \left(\tau_S^h + \frac{u}{S^2} \right) - \Psi(u) \right| \longrightarrow 0 \quad (S \rightarrow \infty). \quad (20.30)$$

For every compact $J \subset (0, 1)$,

$$\sup_{\varepsilon \in J} \left| S^2 (F_{-S}^{-1}(\varepsilon) - \tau_S^h) - u_\varepsilon \right| \longrightarrow 0 \quad (S \rightarrow \infty). \quad (20.31)$$

Moreover, $S^2 F_{-S}^{-1}(\varepsilon) / \log S \rightarrow 3/4$ for every $\varepsilon \in (0, 1)$.

In this section and the next, a supremum of nonnegative errors over the empty set is defined to be zero. In (20.29) and (20.30), the parameters α and S are large enough that $\log \alpha^3$ and $\log \log S^3$ are defined and all the times are positive.

Proof. The assertions for empty K or J are immediate under this convention, so we consider nonempty compact sets. The proof has four steps: the asymptotics of the limiting information, the comparison at fixed α and finite N , the limit $N \rightarrow \infty$ followed by $\alpha \rightarrow \infty$, and the change of parameter from α to S .

Step 1: the limiting information at the center. Let $w = w_\alpha^* + u$ and $s = w/\alpha^2$, and write $\ell = \log \alpha^3$. Lemma 20.3 implies that $j_\alpha(w) \rightarrow 0$, and since $\omega_\alpha \rightarrow 2$, we have $f_\alpha(w) \rightarrow 2$. Inserting this and (20.26) into (20.19), we obtain

$$\mathcal{I}_\alpha(s) = (1 + o(1)) \frac{\alpha^3}{8 \cdot 4 \cdot 4\sqrt{\pi}} w^{-3/2} e^{-4w} = (1 + o(1)) \frac{\alpha^3}{128\sqrt{\pi}} w^{-3/2} e^{-4w}.$$

We have $e^{-4w} = 16\sqrt{\pi} \ell^{3/2} \alpha^{-3} e^{-4u}$ and $w^{-3/2} = (1 + o(1)) 8\ell^{-3/2}$ by (20.27). Hence

$$\mathcal{I}_\alpha\left(\frac{w_\alpha^* + u}{\alpha^2}\right) \rightarrow e^{-4u} \quad (\alpha \rightarrow \infty), \quad (20.32)$$

uniformly in $u \in K$, by the uniformity in Lemma 20.3.

Step 2: the comparison at finite N . The constants are chosen in the order of Table 20.1. Let K_{sh} be the constant K of Lemma 20.1 in the precision-spacing bounds, which depends only on (Δ_i) . Let C_{cp} and M_0 be the constants of (20.12) for the constant K_{sh} , $d_0 = 1/2$, and $C_T = 1$, and let C_{re} be the constant of (20.13) for K_{sh} . Let $B = 1 + 2 \max_{u \in K} e^{-4u}$, and let C_B and η_{lik} be the constants of (20.14). By (20.3),

$$\frac{\alpha^{-2}}{(2 \sum_{i \geq 1} (\Delta_i + \alpha^2)^{-2})^{1/2}} = (1 + o(1)) \alpha^{-3/2} \quad (\alpha \rightarrow \infty). \quad (20.33)$$

Using (20.32) and (20.33), we choose $\alpha_0 \geq 1$ such that for all $\alpha \geq \alpha_0$ and $u \in K$, with $w = w_\alpha^* + u$:

$$\begin{aligned} \alpha^3 &\geq M_0, & 1 \leq w \leq \log \alpha^3, & & C_{\text{re}} e^2 w e^{-2w} \leq \frac{1}{4}, & & \mathcal{I}_\alpha\left(\frac{w}{\alpha^2}\right) \leq \frac{B}{2}, \\ & & & & & & \frac{\alpha^{-2}}{(2 \sum_{i \geq 1} (\Delta_i + \alpha^2)^{-2})^{1/2}} \leq \min\left(\frac{\eta_{\text{lik}}}{2}, \frac{2}{\alpha^{3/2}}\right). \end{aligned} \quad (20.34)$$

Fix $\alpha \geq \alpha_0$ and $u \in K$, let $w = w_\alpha^* + u$ and $s = w/\alpha^2$, and let $p_{i,N}$ be the sequences of Lemma 20.1 with $\xi = \alpha^2$, for $N \geq N_0(\alpha^2)$ so large that $d_N = \alpha^2 N^{-2/3} \leq 1/2$. Let $t = N^{2/3} s = w/d_N$. We bound the three errors in turn.

Order	Choice	Depends on
1	$\Xi, (\Delta_i)$ satisfying (20.2)–(20.4); constant K_{sh} in (20.11)	—
2	$d_0 = 1/2, C_T = 1$; constants C_{cp}, M_0 in (20.12)	K_{sh}
3	constant C_{re} in (20.13)	K_{sh}
4	compact $K; B$; constants C_B, η_{lik} in (20.14)	K
5	$\alpha \geq \alpha_0$ satisfying (20.34)	all of the above
6	$u \in K$, and then $N \rightarrow \infty$	α
7	$\alpha \rightarrow \infty$, and then $S \rightarrow \infty$ through $\alpha^2 = \varrho_{-S}$	—

Table 20.1. The order of choices in the proof of Theorem 20.4. No constant in rows 1–5 depends on N , and the constants in rows 1–4 do not depend on α .

First, $p_{i,N}$ satisfies the precision-spacing bounds with constant K_{sh} , $d_N \leq 1/2$, and $M = \alpha^3 \geq M_0$. Since $w \leq \log \alpha^3$, we have $t \leq d_N^{-1} \log M$, and (20.12) implies that

$$\|\nu_N P_t^{(N)} - \mu_{v(t)}\|_{\text{TV}} \leq C_{\text{cp}} e^{-(d_N - 2/N)t} = C_{\text{cp}} e^{-w} e^{2sN^{-1/3}}.$$

Second, since $w \geq 1$, we may write $t = d_N^{-1} + \tau$ with $d_N \tau = w - 1 \geq 0$. Since $d_N \leq p_{i,N}$, (20.13) gives

$$\max_i |e_i(t)| \leq C_{\text{re}} w e^{-2(w-1)} = C_{\text{re}} e^2 w e^{-2w} \leq \frac{1}{4}.$$

Third, with $q_i = p_{i,N}^{-1}$ and $P_{i,N} = N^{2/3} p_{i,N}$, we have

$$\eta_q = \frac{q_1}{(2 \sum_i q_i^2)^{1/2}} = \frac{\alpha^{-2}}{(2 \sum_i P_{i,N}^{-2})^{1/2}},$$

which converges as $N \rightarrow \infty$ to the left side of (20.33), by the ℓ^2 convergence in Lemma 20.1. Also $\mathcal{I}_N(t) \rightarrow \mathcal{I}_\alpha(s) \leq B/2$, by (20.19). Then there exists $N_1(\alpha, u)$ such that for $N \geq N_1(\alpha, u)$ we have $\eta_q \leq \eta_{\text{lik}}$, $\eta_q \leq 3\alpha^{-3/2}$, and $\mathcal{I}_N(t) \leq B$. Then (20.14) applies, and, since $\pi^{(N)} = \mu_q$, the triangle inequality gives

$$\left| D^{(N)}(t) - \left(2\Phi(\sqrt{\mathcal{I}_N(t)}/2) - 1 \right) \right| \leq C_{\text{cp}} e^{-w} e^{2sN^{-1/3}} + C_B (3\alpha^{-3/2} + C_{\text{re}} e^2 w e^{-2w}) \quad (20.35)$$

for $N \geq N_1(\alpha, u)$.

Step 3: the limits $N \rightarrow \infty$ and $\alpha \rightarrow \infty$. Let $N \rightarrow \infty$ in (20.35), at fixed α and u . By (20.9) and Lemma 20.1, $D^{(N)}(N^{2/3}s) \rightarrow F_{\mathcal{J}(\alpha^2)}(s)$, and $\mathcal{I}_N(t) \rightarrow \mathcal{I}_\alpha(s)$. We obtain, for all $\alpha \geq \alpha_0$ and $u \in K$, with $w = w_\alpha^* + u$,

$$\left| F_{\mathcal{J}(\alpha^2)}\left(\frac{w}{\alpha^2}\right) - \left(2\Phi(\sqrt{\mathcal{I}_\alpha(w/\alpha^2)}/2) - 1 \right) \right| \leq C_{\text{cp}} e^{-w} + C_B (3\alpha^{-3/2} + C_{\text{re}} e^2 w e^{-2w}). \quad (20.36)$$

The right side tends to zero as $\alpha \rightarrow \infty$, uniformly in $u \in K$, because $w \geq w_\alpha^* + \min K \rightarrow \infty$. By (20.32) and the uniform continuity of $I \mapsto 2\Phi(\sqrt{I}/2) - 1$ on $[0, B]$, we have $2\Phi(\sqrt{\mathcal{I}_\alpha(w/\alpha^2)}/2) - 1 \rightarrow 2\Phi(e^{-2u}/2) - 1 = \Psi(u)$ uniformly in $u \in K$. This proves (20.29).

Step 4: the parameter S . Let $\alpha_S = \varrho_{-S}^{1/2}$, so that $\mathcal{S}(\alpha_S^2) = -S$ and $F_{\mathcal{S}(\alpha_S^2)} = F_{-S}$. We have $\alpha_S^2/S^2 = 1 + O(S^{-5/4})$ by (20.4). In particular, $\alpha_S \rightarrow \infty$. Using the definitions (20.1) and (20.27), we have

$$\alpha_S^2 \left(\tau_S^h + \frac{u}{S^2} \right) - w_{\alpha_S}^* - u = \left(\frac{\alpha_S^2}{S^2} - 1 \right) (S^2 \tau_S^h + u) - \frac{1}{4} \log \frac{\alpha_S^3}{S^3} + \frac{3}{8} \log \frac{\log \alpha_S^3}{\log S^3}.$$

The first term is $O(S^{-5/4} \log S)$ uniformly in $u \in K$, since $S^2 \tau_S^h = O(\log S)$; the second is $O(S^{-5/4})$; and the third is $O(S^{-5/4}/\log S)$. We conclude that $\tau_S^h + u/S^2 = (w_{\alpha_S}^* + u + \varepsilon_S(u))/\alpha_S^2$ with $\sup_{u \in K} |\varepsilon_S(u)| \rightarrow 0$. Let $K' = K + [-1, 1]$. For S large, $|\varepsilon_S| \leq 1$, and

$$\begin{aligned} \sup_{u \in K} \left| F_{-S} \left(\tau_S^h + \frac{u}{S^2} \right) - \Psi(u) \right| \\ \leq \sup_{u' \in K'} \left| F_{\mathcal{S}(\alpha_S^2)} \left(\frac{w_{\alpha_S}^* + u'}{\alpha_S^2} \right) - \Psi(u') \right| + \sup_{u \in K} |\Psi(u + \varepsilon_S(u)) - \Psi(u)|. \end{aligned}$$

The first term tends to zero by (20.29) with K' in place of K , and the second by the uniform continuity of Ψ on K' . This proves (20.30).

Finally, we apply the deterministic version of Lemma 3.11 along an arbitrary sequence $S_n \rightarrow \infty$, with $D_n = F_{-S_n}$, $b_n = \tau_{S_n}^h$, $w_n = S_n^{-2}$, and $V = \Psi$. Its hypotheses hold: F_{-S} is nonincreasing with $F_{-S}(\infty) = 0$ by (20.9), Ψ has the required properties by Lemma 3.9, $b_n/w_n = S_n^2 \tau_{S_n}^h \rightarrow \infty$, and the pointwise convergence is (20.30). As F_{-S} is continuous and strictly decreasing, we have $F_{-S}^{-1}(\varepsilon) = \inf\{s \geq 0 : F_{-S}(s) \leq \varepsilon\}$. The second part of Lemma 3.11 gives (20.31). The last claim follows from (20.31) and $S^2 \tau_S^h / \log S \rightarrow 3/4$. $\square$

The critical profile far below the critical temperature

This section treats the limit $\sigma = S \rightarrow +\infty$ of the critical mixing profile, which is the second half of Theorem 1.4. The main result is Theorem 21.2. It states that for every realization of the reflected GOE Airy point process (χ_i) , the limit of $N^{2/3}(2 - \lambda_i)$, outside a null set, we have $F_S(\tau_S^c + 2u/\Delta_2) \rightarrow \mathcal{V}(u)$ locally uniformly in u , where

$$\tau_S^c = \frac{\log((S - \Xi)\Delta_2)}{2\Delta_2}, \quad (21.1)$$

and that the inverse functions converge correspondingly.

We use the notation and the conditions (20.2)–(20.4) of Section 20.2, the finite approximations of Lemma 20.1, and the critical theorem in the form (20.9). The other inputs are the uniform version of the low-temperature profile theorem (Corollary 17.4), the identity for the top fraction in Lemma 4.5, the properties of $\mathcal{V}$ in Lemma 3.9, and Lemma 3.11.

The order of limits is the same as in Section 20: first $N \rightarrow \infty$ at fixed S , then $S \rightarrow \infty$. If we set $\epsilon_N = SN^{-1/3}$ in case (4) of Theorem 1.1, then $N\epsilon_N^3 = S^3$ stays bounded, and the profile parameter $m_N = \theta_N Na_2$ converges in distribution to the finite limit $(S - \Xi)\Delta_2/2$. The theorem behind case (4), on the other hand, requires $m_N \rightarrow \infty$. What makes the passage possible is that the low-temperature theorem holds uniformly over the vectors of scaled eigenvalue gaps that satisfy the scaled-gap bounds and the separation condition of Definition 17.1 with fixed constants, in the iterated sense $\lim_{L \rightarrow \infty} \limsup_{N \rightarrow \infty} \sup$ of (21.4) below. This statement gives no bound at a fixed finite value of m and fixed N , and none is needed.

21.1 Finite approximations at a positive shift

In the form (3.8) of the generator, the low-temperature theorems are stated in terms of the scaled eigenvalue gaps $a_i = (p_i - p_1)/2$ and the top fraction θ_N of (3.14). We compute these quantities for the finite approximations of Lemma 20.1 with shift $\xi = \varrho_S$.

Lemma 21.1 (Scaled eigenvalue gaps and top fraction of the approximations). *Assume (20.2) and (20.3). Fix $S \in \mathbb{R}$, write $Q_S = S - \Xi$, and let $p_{i,N}$, $N \geq N_0(\varrho_S)$, be the sequences of Lemma 20.1 with $\xi = \varrho_S$. Let*

$$a_{i,N} = \frac{p_{i,N} - p_{1,N}}{2}, \quad \theta_N = 1 - \frac{1}{N} \sum_{i=2}^N \frac{1}{2a_{i,N}}, \quad m_N = \theta_N Na_{2,N}.$$

1. The diffusion generated by $\Delta_{S_N} - \nabla_{S_N}(\sum_i a_{i,N} x_i^2) \cdot \nabla_{S_N}$ is the diffusion with half-precisions $p_{i,N}/2$ of (20.9). Moreover, $0 = a_{1,N} < a_{2,N} < \dots < a_{N,N}$, $N^{2/3}a_{2,N} = \Delta_2/2$, $a_{3,N}/a_{2,N} = \Delta_3/\Delta_2$, and

$$\frac{c_\Delta}{2} \left(\frac{i-1}{N} \right)^{2/3} \leq a_{i,N} \leq \frac{5C_\Delta}{2} \left(\frac{i-1}{N} \right)^{2/3} \quad (2 \leq i \leq N), \quad (21.2)$$

with c_Δ, C_Δ as in (20.6), which do not depend on S .

2. As $N \rightarrow \infty$,

$$N^{1/3}\theta_N \longrightarrow Q_S. \quad (21.3)$$

3. If $Q_S > 0$, then, as $N \rightarrow \infty$,

$$m_N \longrightarrow \frac{Q_S \Delta_2}{2}, \quad N\theta_N^3 \longrightarrow Q_S^3, \quad \frac{\log(2m_N)}{4N^{2/3}a_{2,N}} \longrightarrow \tau_S^c.$$

4. As $S \rightarrow \infty$, we have $q_S \rightarrow 0$, and $0 \leq Q_S^{-1} - q_S \leq Q_S^{-3} \sum_{i \geq 2} \Delta_i^{-2}$ whenever $Q_S > 0$.

Proof. Let $\xi = q_S$. For the first part, on S_N we have $\sum_i p_{i,N} x_i^2 / 2 = \sum_i a_{i,N} x_i^2 + p_{1,N} N / 2$, and the constant does not change the generator, by (3.7). The remaining claims follow from the first two parts of Lemma 20.1. Indeed, since the indices 2 and 3 are at most n , we have $N^{2/3}a_{2,N} = \Delta_2/2$ and $N^{2/3}a_{3,N} = \Delta_3/2$; and $\Delta_i/2 \leq N^{2/3}a_{i,N} \leq 5\Delta_i/2$ for all $2 \leq i \leq N$, which gives (21.2) by (20.6).

For the second part, let $P_{i,N} = N^{2/3}p_{i,N}$ and $B_{i,N} = P_{i,N} - \xi = 2N^{2/3}a_{i,N}$. The trace condition $\sum_i p_{i,N}^{-1} = N$ reads $\sum_i P_{i,N}^{-1} = N^{1/3}$, and $P_{1,N} = \xi$. Hence

$$N^{1/3}\theta_N = N^{1/3} - \sum_{i=2}^N \frac{1}{B_{i,N}} = \frac{1}{\xi} - \sum_{i=2}^N \left(\frac{1}{B_{i,N}} - \frac{1}{P_{i,N}} \right) = \frac{1}{\xi} - \sum_{i=2}^N \frac{\xi}{B_{i,N}(B_{i,N} + \xi)}.$$

For each fixed $i \geq 2$, we have $B_{i,N} = \Delta_i$ once $i \leq n$. By the first part, $B_{i,N} \geq \Delta_i$, and the summands are bounded by $\xi \Delta_i^{-2}$, which is summable. By dominated convergence,

$$N^{1/3}\theta_N \longrightarrow \frac{1}{\xi} - \sum_{i \geq 2} \frac{\xi}{\Delta_i(\Delta_i + \xi)} = \mathcal{S}(\xi) - \Xi = S - \Xi,$$

where we used (20.5) and $(\xi + \Delta_i)^{-1} - \Delta_i^{-1} = -\xi / (\Delta_i(\Delta_i + \xi))$.

The third part follows from $m_N = N^{1/3}\theta_N \cdot N^{2/3}a_{2,N}$, $N\theta_N^3 = (N^{1/3}\theta_N)^3$, and

$$\frac{\log(2m_N)}{4N^{2/3}a_{2,N}} = \frac{\log(2m_N)}{2\Delta_2} \longrightarrow \frac{\log(Q_S \Delta_2)}{2\Delta_2} = \tau_S^c.$$

For the fourth part, $q_S \rightarrow 0$ as $S \rightarrow \infty$ because $\mathcal{S}$ is a decreasing bijection of $(0, \infty)$ onto $\mathbb{R}$. The computation in the second part shows that $Q_S = \xi^{-1} - \sum_{i \geq 2} \xi / (\Delta_i(\Delta_i + \xi))$. Multiplying by ξ , we obtain

$$Q_S \xi = 1 - \xi^2 \sum_{i \geq 2} \frac{1}{\Delta_i(\Delta_i + \xi)}.$$

The right side is at most one. If $Q_S > 0$, this gives $\xi \leq Q_S^{-1}$ and $0 \leq Q_S^{-1} - \xi = Q_S^{-1} \xi^2 \sum_{i \geq 2} (\Delta_i (\Delta_i + \xi))^{-1} \leq Q_S^{-3} \sum_{i \geq 2} \Delta_i^{-2}$. $\square$

The fourth part shows that the smallest scaled precision q_S is close to $1/(S - \Xi)$ for large S . The second part shows that the top fraction of the approximations is of order $N^{-1/3}$, with $N^{1/3} \theta_N \rightarrow S - \Xi$; this agrees with the top fraction of the model in the critical window, by Lemma 4.5.

21.2 The profile

We use the low-temperature profile theorem in the following uniform form. Fix $0 < c_* \leq C_*$ and $\varepsilon_* > 0$, and let $\mathcal{A}_N = \mathcal{A}_N(c_*, C_*, \varepsilon_*)$ be the set of vectors of Definition 17.1, that is, of vectors satisfying (17.3): the vectors $0 = a_1 < a_2 \leq \dots \leq a_N$ such that

$$c_* \left(\frac{i-1}{N} \right)^{2/3} \leq a_i \leq C_* \left(\frac{i-1}{N} \right)^{2/3} \quad (2 \leq i \leq N), \quad \frac{a_3}{a_2} \geq 1 + \varepsilon_*.$$

For $a \in \mathcal{A}_N$, let $\theta(a) = 1 - N^{-1} \sum_{i \geq 2} (2a_i)^{-1}$ and $m(a) = \theta(a) N a_2$. When $\theta(a) > 0$, define

$$t_a(u) = \frac{\log(2m(a))}{4a_2} + \frac{u}{a_2} \quad (u \in \mathbb{R}),$$

and let $D^a(t)$ be the total variation distance between $\nu_N P_t^a$ and the invariant law of P_t^a , where P_t^a is generated by $\Delta_{S_N} - \nabla_{S_N} (\sum_i a_i x_i^2) \cdot \nabla_{S_N}$. For a nonempty compact interval $I \subset \mathbb{R}$ and L large enough that $\log(2L) + 4 \min I > 0$, so that all the times below are positive, let

$$E_I(L) = \limsup_{N \rightarrow \infty} \sup \left\{ \sup_{u \in I} |D^a(t_a(u)) - \mathcal{V}(u)| : a \in \mathcal{A}_N, 0 < \theta(a) \leq \frac{1}{L}, m(a) \geq L \right\}, \quad (21.4)$$

where the supremum over the empty set is zero. In the notation of Corollary 17.4, $E_I(L) = \limsup_N \mathcal{E}_N(L)$, and (17.7) states that $E_I(L) \rightarrow 0$ as $L \rightarrow \infty$. This is stronger than Theorem 17.3 applied to a single sequence with $\theta \rightarrow 0$ and $m \rightarrow \infty$: it bounds the distance, for all large N and uniformly over $\mathcal{A}_N$, at every fixed large value of m .

Theorem 21.2 (The critical profile far below the critical temperature). *Assume (20.2)–(20.4); in particular, the following holds almost surely for the GOE Airy point process. For every compact $I \subset \mathbb{R}$,*

$$\sup_{u \in I} \left| F_S \left(\tau_S^c + \frac{2u}{\Delta_2} \right) - \mathcal{V}(u) \right| \rightarrow 0 \quad (S \rightarrow \infty). \quad (21.5)$$

For every compact $J \subset (0, 1)$,

$$\sup_{\varepsilon \in J} \left| \frac{\Delta_2}{2} (F_S^{-1}(\varepsilon) - \tau_S^c) - \mathcal{V}^{-1}(\varepsilon) \right| \rightarrow 0 \quad (S \rightarrow \infty). \quad (21.6)$$

Moreover, $F_S^{-1}(\varepsilon) / \log S \rightarrow 1 / (2\Delta_2)$ for every $\varepsilon \in (0, 1)$.

Here S is large enough that $S > \Xi$ and that all the times in (21.5) are positive. The center and the window depend on the realization through Ξ and Δ_2 .

Proof. The assertions for empty I or J are immediate under the convention of Section 20. For the profile assertion we may enlarge a nonempty compact I to its compact interval hull, since a supremum over I is no larger than the supremum over that hull. We therefore fix a nonempty compact interval I . For the quantile assertion we likewise enlarge a nonempty compact $J \subset (0, 1)$ to its compact interval hull, which is still contained in $(0, 1)$. By Lemma 21.1, for every S and every $N \geq N_0(\varrho_S)$, the sequence $(a_{i,N})_{i \leq N}$ belongs to $\mathcal{A}_N(c_*, C_*, \varepsilon_*)$ with

$$c_* = \frac{c_\Delta}{2}, \quad C_* = \frac{5C_\Delta}{2}, \quad \varepsilon_* = \frac{\Delta_3}{\Delta_2} - 1 > 0.$$

These constants depend only on (Δ_i) , and not on S ; only the dimension $N_0(\varrho_S)$ from which the sequences are defined depends on S . Let $D^{(N)}$ be the distance for the corresponding diffusion, which is the same for the gaps $a_{i,N}$ and for the half-precisions $p_{i,N}/2$, by the first part of Lemma 21.1. For $u \in \mathbb{R}$, let

$$t_N(u) = \frac{\log(2m_N)}{4a_{2,N}} + \frac{u}{a_{2,N}}$$

be the time $t_a(u)$ for $a = (a_{i,N})$.

Step 1: the bound at fixed S . Fix L with $\log(2L) + 4 \min I > 0$, and let S be such that $Q_S = S - \Xi > 0$, $Q_S \Delta_2 / 2 > L$, and $\tau_S^c + 2 \min I / \Delta_2 > 0$. The second and third parts of Lemma 21.1 give $N^{1/3} \theta_N \rightarrow Q_S > 0$ and $m_N \rightarrow Q_S \Delta_2 / 2 > L$. Then for all large N we have $0 < \theta_N \leq 1/L$ and $m_N \geq L$, so that $(a_{i,N})$ belongs to the set in (21.4). Therefore

$$\limsup_{N \rightarrow \infty} \sup_{u \in I} |D^{(N)}(t_N(u)) - \mathcal{V}(u)| \leq E_I(L). \quad (21.7)$$

We note that at fixed S the parameter m_N converges to a finite limit. For this reason Theorem 17.3 itself does not apply, while the uniform bound (21.4) does.

Step 2: identification of the left side. Using the first and third parts of Lemma 21.1, we find

$$\frac{t_N(u)}{N^{2/3}} = \frac{\log(2m_N)}{2\Delta_2} + \frac{2u}{\Delta_2} \longrightarrow \tau_S^c + \frac{2u}{\Delta_2},$$

uniformly in $u \in I$. The limits lie in the compact interval $[\tau_S^c + 2 \min I / \Delta_2, \tau_S^c + 2 \max I / \Delta_2] \subset (0, \infty)$, and for large N all the times $t_N(u) / N^{2/3}$, $u \in I$, lie in a fixed

compact subset $\mathcal{K}_S$ of $(0, \infty)$. By Lemma 20.1, (20.9) holds with $F_R = F_S$, since $\xi = \varrho_S$. Thus

$$\begin{aligned} \sup_{u \in I} \left| D^{(N)}(t_N(u)) - F_S\left(\tau_S^c + \frac{2u}{\Delta_2}\right) \right| &\leq \sup_{s \in \mathcal{K}_S} |D^{(N)}(N^{2/3}s) - F_S(s)| \\ &\quad + \sup_{u \in I} \left| F_S\left(\frac{t_N(u)}{N^{2/3}}\right) - F_S\left(\tau_S^c + \frac{2u}{\Delta_2}\right) \right|, \end{aligned}$$

and both terms tend to zero, the second by the uniform continuity of F_S on $\mathcal{K}_S$. Combining this with (21.7), we obtain

$$\sup_{u \in I} \left| F_S\left(\tau_S^c + \frac{2u}{\Delta_2}\right) - \mathcal{V}(u) \right| \leq E_I(L)$$

for every S satisfying the three conditions of Step 1.

Step 3: the limits $S \rightarrow \infty$ and $L \rightarrow \infty$. Since $Q_S \rightarrow \infty$ and $\tau_S^c \rightarrow \infty$ as $S \rightarrow \infty$, the three conditions of Step 1 hold for all large S . By Step 2, the left side of (21.5) has $\limsup_{S \rightarrow \infty}$ at most $E_I(L)$, for every L with $\log(2L) + 4 \min I > 0$. Letting $L \rightarrow \infty$ and using (17.7), we obtain (21.5). Equivalently, after discarding finitely many dimensions at each fixed S , Steps 1 and 3 verify the hypotheses of the second statement (17.8) of Corollary 17.4 for the family $a^{(N,S)} = (a_{i,N})$, with $m_S = Q_S \Delta_2 / 2$, and Step 2 identifies its left side.

Finally, we apply the deterministic version of Lemma 3.11 along an arbitrary sequence $S_n \rightarrow \infty$, with $D_n = F_{S_n}$, $b_n = \tau_{S_n}^c$, $w_n = 2/\Delta_2$, and $V = \mathcal{V}$. Its hypotheses hold: F_S is nonincreasing with $F_S(\infty) = 0$ by (20.9), $\mathcal{V}$ has the required properties by Lemma 3.9, $b_n/w_n = \log(Q_{S_n} \Delta_2)/4 \rightarrow \infty$, and the pointwise convergence is (21.5). As F_S is continuous and strictly decreasing, we have $F_S^{-1}(\varepsilon) = \inf\{s \geq 0 : F_S(s) \leq \varepsilon\}$, and the second part of Lemma 3.11 gives (21.6). The last claim follows from (21.6) and $\tau_S^c / \log S \rightarrow 1/(2\Delta_2)$, which holds because $Q_S/S \rightarrow 1$. $\square$

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