A computer-assisted proof of the four-uniform Erdős matching conjecture

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Abstract

We prove the Erdős matching conjecture for four-uniform hypergraphs for every matching parameter and every admissible number of vertices. The proof combines the finite-board method of Hou, Hu and Liu with ordered local configurations, exact integer counts and certified inequalities. Small matching parameters are treated on full labelled configurations. Two fixed counting inequalities and two endpoint anchors propagate polynomially across the remaining range; 51 endpoint records and the two fixed inequalities suffice for the intermediate cardinality inputs. Dense triple supports admit direct slab and majority proofs. A classical shadow bound and a deletion argument complete the passage from these local bounds to the global theorem. The strict margins also yield clique stability with explicit constants for shifted families whose matching number is preserved on deleting the first vertex, and classifications of extremizers in specified ranges. The finite parts are supported by exact certificates and exhaustive computations with stated mathematical specifications.

Keywords: Erdős matching conjecture; four-uniform hypergraphs; exact certificates; shifting; extremal stability.

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1 The complete four-uniform bound

All vertex and matching parameters are integers. A matching is a collection of pairwise disjoint sets. Write $\nu(\mathcal{F})$ for the largest size of a matching in $\mathcal{F} \subseteq \binom{[n]}{4}$, and define

$$B_r(n,t) = \binom{n}{r} - \binom{n-t}{r}, \qquad A_4(s) = \binom{4s+3}{4}.$$

For a t-set $T \subseteq [n]$, the complete cover

$$\mathcal{C}_T(n) = \{E \in \binom{[n]}{4} : E \cap T \neq \varnothing\}$$

has $B_4(n,t)$ members and matching number at most t. Write $\mathcal{C}_t = \mathcal{C}_{[t]}(n)$ when the ground set is clear. The other candidate is the complete four-graph on $4s+3$ vertices, with all remaining vertices isolated.

Theorem 1.1 (Four-uniform matching bound)**.** *Let $s \geq 1$. For every $n \geq 4s+4$ and $\mathcal{F} \subseteq \binom{[n]}{4}$ with $\nu(\mathcal{F}) \leq s$,*

$$|\mathcal{F}| \leq M(n,s) := \max\{A_4(s), B_4(n,s)\}.$$

For $s \leq 4$, the classical shadow input in Section 2 and the small full-carrier certificates suffice.

The clique on $4s+3$ vertices and the complete s-cover attain the two terms. For $n < 4s+4$ the complete family has no forbidden matching and gives the trivial extremal value $\binom{n}{4}$. The theorem concerns cardinality. Section 5 identifies the exact extremizers in explicit cover and clique domains and gives quantitative clique stability for shifted ONE families. It does not classify every crossing case or prove arbitrary-family stability.

Contribution and antecedents. Erdős formulated the matching problem in [1, equation (9)]. Frankl [13] settled uniformity three; the short proof of Frankl–Rödl–Ruciński [4] covers matching number at least 33. Hou–Hu–Liu [12, Theorem 1.1] prove the four-uniform bound for $s \geq 6961$, using a finite-board reduction. Theorem 1.1 extends that range to every positive s. Its ingredients retain local order in the trace constraints, couple the resulting inequalities, and use integer orbit counts where a continuous relaxation loses necessary information. The board method, selected-residual construction and consumed HHL programs remain attributed to that work. The reduction to Property ONE and the deletion inequality have antecedents in [13, Proposition 5.1 and Corollary 5.1].

Notation and proof route. A family is *shifted* if every four-set coordinatewise below one of its members also belongs to it. It has *Property ONE* if a maximum matching avoids its first vertex. Put

$$a_s = n_4(s) = \min\{n \geq 4s + 4 : B_4(n, s) \geq A_4(s)\}.$$

For $s \geq 2$, the direct inputs are the two crossings $a_s - 1, a_s$. Section 2 proves that these inputs suffice, before the finite configurations are analysed. Section 3 gives the trace reduction, an ordered board and the certificates for the small and intermediate parameters. Section 4 proves the necessary layer inequalities and covers the infinite tail. The division of the critical inputs is

$s = 1$	partition and classical link bound
$s = 2, 3, 4$	partition and four full-carrier certificates
$s \geq 5$	51 endpoint records, two fixed inequalities and propagation
$s \geq 73$	stronger ONE-clique bound directly from the layers.

Theorem 1.1 is a cardinality statement. Table 1 distinguishes it from the structural consequences.

Conclusion	Families	Parameter domain
Exact cardinality	All four-uniform families, $\nu \leq s$	$s \geq 1,\ n \geq 4s + 4$
Clique stability 71/143	Shifted ONE, $\nu = s$	$s \geq 122,\ 4s + 4 \leq n \leq a_s$
Clique extremizers	Arbitrary, $\nu \leq s$, maximum size	$s \geq 25,\ 4s + 4 \leq n < a_s$
Cover extremizers	Arbitrary, $\nu \leq s$, maximum size	$s = 5,\ n \geq 33;\ s = 6,\ n \geq 34;$ $7 \leq s \leq 121,\ n \geq a_s;$ $s \geq 122,\ n \geq a_s + 1$

Table 1: Domains of the global and structural results. The constants 71 and 143 control respectively the number of edges outside the canonical clique and the symmetric difference. The precise statements are in Section 5.

Dependence on mathematical and computational inputs. The global cardinality proof imports Frankl's shadow inequality [3, Theorem 1.2]; it does not import the complete three-uniform matching theorem. The necessary finite HHL statements are reconstructed with explicit searches and certificate checks. The cover-uniqueness consequence also uses [3, Theorem 1.1] at $s = 4, n \geq 32$. The complementary shifted cover-stability and graph results in Appendix A have their additional imports listed there. Exact equality can be transferred through shifts by the two inverse lemmas in Section 5; quantitative stability for arbitrary families is not established. Section 7 compares the precise statements with earlier work, including the clique-side results of Guo–Lu. Section 8 separates the global computations from the additional checks required by the appendices.

2 From the critical bounds to the global theorem

Write $\partial\mathcal{H}$ for the immediate shadow and $K_j(\mathcal{A})$ for the number of j-sets all of whose appropriate uniform subsets belong to $\mathcal{A}$. Integer binomial coefficients are zero when the upper argument is nonnegative and smaller than the lower one. For polynomial identities we use the product extension of binomials.

We use shifting, which preserves cardinality and does not increase the matching number, and the following classical quantitative input.

Shadow. Frankl [3, Theorem 1.2] gives $|\mathcal{H}| \le u|\partial\mathcal{H}|$ when $\nu(\mathcal{H}) = u$. Consequently $\nu(\mathcal{H}) \le c$ implies $|\mathcal{H}| \le c|\partial\mathcal{H}|$, with the empty case immediate.

Lemma 2.1 (Shifted completion)**.** *Let $n \ge 4t$ and let $\mathcal{F} \subseteq \binom{[n]}{4}$ be shifted with $\nu(\mathcal{F}) \le t$. There is a shifted $\mathcal{G} \supseteq \mathcal{F}$ with $\nu(\mathcal{G}) = t$. Consequently, for $K \ge 0$, a bound*

$$|\mathcal{G} \setminus \mathcal{C}_t| \le K(B_4(n,t) - |\mathcal{G}|)$$

for every shifted $\mathcal{G}$ of exact matching number t implies the same bound for every shifted $\mathcal{F}$ of matching number at most t.

Proof. While the matching number is below t, add a coordinatewise minimal missing four-set. Every strict predecessor is present, so the enlarged family is still shifted. Adding one edge increases the matching number by at most one. The process is finite and reaches t, because the complete family has matching number $\lfloor n/4 \rfloor \ge t$. Inclusion then gives

$$|\mathcal{F} \setminus \mathcal{C}_t| \le |\mathcal{G} \setminus \mathcal{C}_t| \le K(B_4(n,t) - |\mathcal{G}|) \le K(B_4(n,t) - |\mathcal{F}|).$$

□

2.1 A section bound already sufficient for the links

Lemma 2.2 (A consequence of Frankl's section method)**.** *For $r \ge 2$, $t \ge 1$, put*

$$D_r(t) = (2r-1)t + r - (r-1)\left\lfloor \frac{t+1}{r} \right\rfloor.$$

Every r-uniform family with matching number at most t on $n \ge D_r(t)$ vertices has size at most $B_r(n,t)$. In particular $D_4(3) = 22$, and $D_3(t) \le n_4(t)$ for all $t \ge 1$.

Proof. Shift and complete to matching number exactly t, by the argument of Lemma 2.1 in uniformity r. An edge of a maximum matching avoids $[t-1]$, so shiftedness forces the complete r-graph on $[t+r-1]$. Put $X = [t+1]$, $Y = [n] \setminus X$, $q_0 = \lfloor (t+1)/r \rfloor$, and $c = t - q_0$. The section $\mathcal{H}$ disjoint from X has matching number at most c, using q_0 disjoint edges inside X. For the nested singleton sections $\mathcal{A}_1, \dots, \mathcal{A}_{t+1}$ on Y, the shadow input gives $|\mathcal{H}| \le c|\mathcal{A}_{t+1}|$. They are cross-dependent.

For a matching of $v = 2t - q_0 + 1 = t + c + 1$ sets of size $r-1$ in Y, write d_i for its section counts. The nested-neighbourhood argument gives an $x \in [0,t]$ such that the first $t-x$ counts are at most v and the last $x+1$ are at most x. Hence

$$\sum_{i=1}^{t} d_i + (c+1)d_{t+1} \le (t-x)v + x^2 + (c+1)x = tv + x(x+c+1-v) \le tv.$$

Such a matching exists because $n \ge D_r(t)$ is precisely $|Y| \ge (r-1)v$. Averaging gives

$$|\mathcal{H}| + \sum_{i=1}^{t+1} |\mathcal{A}_i| \le t\binom{|Y|}{r-1}.$$

Every section using at least two vertices of X already meets $[t]$; bounding it by the complete section proves the cover inequality. The argument uses the shadow theorem and the weighted nested-section method of [3, Sections 3–5].

For the last assertion, write $t = 3a + b$, $b = 0, 1, 2$. Then $D_3(t) = 13a + 3, 13a + 8, 13a + 11$, respectively. Exact binomial subtraction gives the following coefficients in ascending powers of a:

b	$[a^0]$	$[a^1]$	$[a^2]$	$[a^3]$	$[a^4]$	domain
0	0	78	1653	7974	2175	$a \geq 1$
1	360	2782	7425	7682	2175	$a \geq 0$
2	4560	19862	29877	16966	2175	$a \geq 0$

The polynomial is $24[A_4(t) - B_4(D_3(t) - 1, t)]$, and is strictly positive in each displayed domain. Since $B_4(n, t)$ increases with n, it follows that $D_3(t) \leq n_4(t)$. □

This consequence supplies every triple link in the global proof. The complete three-uniform EMC is not an additional premise.

2.2 The critical statement that remains to be checked

For every $s \geq 2$, the local input is

$$|\mathcal{F}| \leq M(n, s) \quad \text{at } n \in \{a_s - 1, a_s\}, \qquad \mathcal{F} \text{ shifted and ONE}, \quad \nu(\mathcal{F}) = s. \tag{2.1}$$

It is enough to check saturated families: enlargement under $\nu \leq s$ preserves an existing matching avoiding 1, as explained in Section 3.1. The partition argument below and Proposition 3.6 treat $s = 2, 3, 4$. Section 3 supplies the retained finite records; Corollary 4.29 supplies the propagated critical inputs. Proposition 4.1 also gives the stronger ONE-clique bound for all $s \geq 73$. We now prove the global theorem from exactly these inputs. Their verification in the later sections uses the trace and shadow lemmas, not the global theorem.

2.3 Crossings, deletion and propagation

First note that for $s \geq 2$,

$$24[A_4(s) - B_4(4s + 4, s)] = s(s - 1)(81s^2 + 95s + 26) > 0. \tag{2.2}$$

Thus $a_s - 1 \geq 4s + 4$. Also $a_t \leq 5t + 4$, since

$$24[B_4(5t + 4, t) - A_4(t)] = 113t^4 + 226t^3 + 139t^2 + 26t > 0 \qquad (t \geq 1). \tag{2.3}$$

The following inequality is the four-uniform specialization of [13, Proposition 5.1, equation (5.1)]. We retain the elementary proof because its strict clique branch is used in the equality argument.

Lemma 2.3 (Direct deletion)**.** *For $s \geq 2$, $n \geq 4s + 4$,*

$$M(n - 1, s - 1) + \binom{n - 1}{3} \leq M(n, s).$$

Proof. If the preceding maximum is the cover, Pascal's identity gives $B_4(n-1, s-1) + \binom{n-1}{3} = B_4(n, s)$. Otherwise $B_4(n - 1, s - 1) < A_4(s - 1)$, so $n - 1 < a_{s-1} \leq 5s - 1$ and hence $n - 1 \leq 5s - 2$. For $u = s - 2 \geq 0$, exact subtraction gives

$$24\left[A_4(s) - A_4(s - 1) - \binom{5s - 2}{3}\right] = 5736 + 8056u + 3660u^2 + 524u^3 > 0.$$

Monotonicity of the binomial coefficient proves the remaining case. A tie in the preceding maximum belongs to the cover case. □

Lemma 2.4 (Propagation using the classical link range)**.** *If the cover bound holds for all families at $(n,t) = (a,t)$, where $a \geq n_4(t)$, then it holds at every $n \geq a$.*

Proof. For the step from n to $n+1$, shift the family. The link of its last vertex has matching number at most t: otherwise $t+1$ disjoint triples leave at least $t+1$ unused vertices among the first $n \geq a \geq 4t+4$ vertices. Replace the last vertex by those smaller distinct vertices to obtain a forbidden matching. Lemma 2.2, since $n \geq n_4(t) \geq D_3(t)$, bounds the link by $B_3(n,t)$. Restriction to the first n vertices and Pascal give $B_4(n,t) + B_3(n,t) = B_4(n+1,t)$. □

Proof of Theorem 1.1. Induct on s. At $n = 4(s+1)$, a uniformly random partition into $s+1$ four-sets contains at most s members of any admissible family. Taking expectations gives

$$|\mathcal{F}| \leq \frac{s}{s+1}\binom{4s+4}{4} = \binom{4s+3}{4}.$$

For $s = 1$, $a_1 = 8$ and this is the cover bound; Lemma 2.4 propagates it to every $n \geq 8$.

Let $s \geq 2$ and assume the theorem for $s-1$ at all admissible ground sets. It suffices first to treat $n = a_s - 1, a_s$. Shift and enlarge to matching exactly s. If ONE fails, deleting vertex 1 reduces the matching number exactly by one: every maximum matching uses 1, and loses just one edge. The remaining $n - 1 \geq 4(s-1)+4$ vertices are in the inductive domain. There are at most $\binom{n-1}{3}$ deleted edges, so induction and Lemma 2.3 prove the target.

If ONE holds, use the following exhaustive list of critical inputs. For $s = 2, 3$, the lower points are $(2, 12), (3, 16)$ and follow from the partition bound; the upper points follow from Proposition 3.6. For $s = 4$, that proposition gives both. For $s \geq 5$, use the 51 endpoint records of Section 3 selected in Corollary 4.29, the two fixed inequalities of Lemma 4.25, and the propagation proved there. This covers every critical input. Proposition 4.1 also gives the stronger ONE-clique bound for $s \geq 73$. At a lower point the target is the clique term, and at an upper point it is the cover term.

Adding isolated vertices to $a_s - 1$ gives the clique bound at every smaller admissible n; Lemma 2.4 propagates the upper cover bound to every larger n. This completes the induction. For $s \leq 4$, only the partition, small certificates and the classical-shadow proof of Lemma 2.2 have been used. □

3 Trace reduction and critical certificates

We verify the critical input (2.1) for the finite range. Exact traces turn a global family into finite configurations; the ordered board retains comparisons that would be lost by a larger symmetry quotient. The necessary layer inequalities are proved in Section 4. The four small points are checked directly on the full carrier without those layer inequalities.

3.1 Full-carrier traces and elementary completion

Shifting preserves size and does not increase matching number. A shifted family can be enlarged to a shifted family maximal under $\nu \leq s$: add an admissible missing edge and shift again. The old shifted core is preserved by every such shift, so size increases; finiteness terminates the process. At $n \geq 4s+4$, a maximal family has matching exactly s, since otherwise any missing edge could be added. If ONE held before enlargement, its matching avoiding 1 persists.

Lemma 3.1 (Exact traces on the full carrier)**.** *Let $\mathcal{F}$ be shifted, saturated under $\nu \leq s$, and ONE, with $\nu(\mathcal{F}) = s$, $n \geq 4s+4$. Put $K = [4s+3]$, $q = n - 4s - 3$. There is an s-matching inside K avoiding 1. Choose such a matching so that its residual $D = \{1, \delta_1, \delta_2\}$, $\delta_1 < \delta_2$, is lexicographically minimal outside 1. For $\mathcal{G} = \{E \cap K : E \in \mathcal{F}\}$:*

1. *every trace has size 2, 3 or 4; no small trace lies inside D;*
2. $\nu(\mathcal{G}) = s$;
3. *every four-set containing a trace of $\mathcal{G}$ belongs to $\mathcal{F}$;*
4. $|\mathcal{F}| = \sum_{H\in\mathcal{G}} \binom{q}{4-|H|}$.

Proof. A matching avoiding a set R can be moved into an initial segment of $[n] \setminus R$ with at least as many available positions. Whenever a used vertex lies beyond that segment, replace it by a free smaller vertex in the segment; shiftedness preserves its edge and the matching. Apply this with $R = \{1\}$ to get the selection.

An empty trace, or a small trace in D, would give an edge disjoint from the selected matching. A singleton in a column can be shifted to 1, with the same contradiction. This proves the first claim.

If $s+1$ traces were disjoint, choose, among all collections of $s+1$ edges having disjoint traces, one maximizing the size of their union. Their traces are allowed to change in this maximization. The edges share some vertex $u \notin K$, since $\nu(\mathcal{F}) = s$. Some $v \in K$ is unused: covering all $4s+3$ vertices of K would leave only one of the $4(s+1)$ edge positions outside K, which cannot support a shared vertex. Replace u by v in one edge. Since $v < u$, that edge remains present; its trace expands into unused positions and remains disjoint from the others. The union increases because another edge still uses u, a contradiction. The selected columns supply an s-matching in $\mathcal{G}$.

If a four-set Q containing a trace H were absent, saturation would supply s disjoint edges avoiding Q. Their exact traces, together with H, would be $s+1$ disjoint traces. Thus all such completions are present. In particular every choice of $4-|H|$ external vertices completes H, and counting edges by their unique exact trace proves the last claim. □

Internal enlargements of a trace to size $j' \le 4$ are exact traces whenever $q \ge 4 - j'$. The proof rewrites the classical shifting and mixed-trace mechanisms of [12, Section 2]; those mechanisms remain attributed even though their proofs are included here. No four-column denominator is used on the full carrier.

3.2 The complete weighted board

For a saturated shifted ONE family, the selected matching consists of s four-vertex columns in $[4s+3]$, avoiding vertex 1, and a residual three-set D. For four chosen columns τ, the board has 19 vertices. For a trace H of size $j \in \{2,3,4\}$ meeting z columns, set

$$w_{s,n}(H) = \frac{\binom{q}{4-j}}{\binom{s-z}{4-z}}, \qquad q = n - 4s - 3.$$

There are exactly $\binom{s-z}{4-z}$ boards containing that trace. Lemma 3.1 and this multiplicity therefore give

$$|\mathcal{F}| - A_4(s) = \sum_{\tau} \Phi_{s,n}(\tau), \qquad \Phi_{s,n} = \sum_{\substack{|H|=2,3\\ H \text{ present}}} w_{s,n}(H) - \sum_{\substack{|H|=4\\ H \text{ missing}}} w_{s,n}(H). \tag{3.1}$$

All quadruple layers are retained, including spread one.

For example, a present pair $H = \{x_{a0}, x_{b1}\}$, with $q \ge 2$, has $\binom{q}{2}$ completions outside the full carrier. It lies in $\binom{s-2}{2}$ four-column boards, each assigning it weight $\binom{q}{2}/\binom{s-2}{2}$. Summing those contributions recovers its exact number of completions. Figure 1 displays the local order without imposing global symmetry.

Residual D	F_a	F_b	F_c	F_d
1	x_{a0}	x_{b0}	x_{c0}	x_{d0}
δ_1	x_{a1}	x_{b1}	x_{c1}	x_{d1}
δ_2	x_{a2}	x_{b2}	x_{c2}	x_{d2}
	x_{a3}	x_{b3}	x_{c3}	x_{d3}

Figure 1: A 19-vertex board: four matching edges and the three residual vertices. Each column is ordered from level 0 to level 3; $1 < \delta_1 < \delta_2$. Row alignment imposes no comparison between different columns or between a column and the residual. The highlighted pair illustrates a trace of spread two, if present. The q external vertices lie outside the full carrier $[4s+3]$, not merely outside this board.

The 5016 labelled subsets of sizes 2, 3 and 4 form 291 orbits under column permutations and residual permutations. Average the presence indicator over this group. Its orbit coordinates x_i lie in $[0,1]$; this does not assume that the original board is invariant. Full columns and low seeds have value one, and small traces inside D have value zero. Column levels retain their order. The coordinate key records size, residual cardinality and the sorted four column bitmasks.

The inequalities used by the certificate are all necessary:

1. lowering inside a column preserves presence;
2. a present trace forces a containing trace of the next size; for sizes 2 to 3, use $q \geq 1$ and a completion outside the board;
3. five pairwise disjoint present traces are impossible, giving $\sum_{\ell=1}^{5} x_{H_\ell} \leq 4$, averaged over the group;
4. with T_z, P_z, M_z counting present triples, present pairs and missing quadruples of spread z, Section 4 proves
$$\frac{18}{35}T_3 + \frac{18}{71}P_2 \leq M_4, \qquad \frac{9}{8}T_2 + \frac{27}{32}P_1 \leq M_3, \qquad 2T_1 + 2P_0 \leq M_2;$$
5. the full finite support profiles give additional linear inequalities, as specified next.

A triple support is a down-set of $\{0,1,2,3\}^3$. Three disjoint triples leaving a low level free in a column are forbidden: that low seed and the remaining full column would complete five disjoint traces. Enumeration gives 2016 forbidden matchings, 51 minimal down-closures, 232848 down-sets and 26893 legal down-sets. Their 20 level-multiset counts have 1632 distinct profiles. The pair enumeration gives 70 down-sets, 10 legal ones and 7 profiles on 10 level classes. For every integer vector h appearing in a row, the checker reconstructs all legal profiles p and calculates $\beta(h) = \max_p h \cdot p$ exactly. The inequality $h \cdot p \leq \beta(h)$, averaged over the four triple supports or six pair supports, is valid even when some entries of h are negative. The distributed checker reconstructs the labelled universes and all transports. These are classifications of the stated finite supports, not of global four-uniform families.

For three further points, keep $D = \{1, \delta_1, \delta_2\}$, $\delta_1 < \delta_2$, individually fixed and average only over columns. The 469 coordinates now record the residual bitmask. Shifting permits replacement of any occupied vertex by the absent minimum 1, or of δ_2 by the absent δ_1. No comparison between δ_1 and a column vertex is assumed. The old matching witnesses remain disjoint under every residual permutation, now yielding distinct rows. This refinement uses 58 selected rows of the new order type. No five-column extension constraint is consumed.

3.3 Local order and traces forced by the minimal residual

We refine the ordered-residual model while keeping the same authentic board. Write $D = \{1, \delta_1, \delta_2\}$, with $\delta_1 < \delta_2$, and select the matching to minimize (δ_1, δ_2) lexicographically. In the records, the four levels of column c have labels $4c, \dots, 4c+3$, and $16, 17, 18$ denote $1, \delta_1, \delta_2$. These labels are positions, not global vertex values.

Lemma 3.2 (Exhaustive order prefixes)**.** *Restrict the global order to a carrier with b four-vertex columns and a residual three-set. Name columns $0, 1, \dots$ in order of first appearance. After a prefix p, if k columns have appeared, the next vertex is one of: the first unused residual vertex; the first unused level of a seen column; or the first level of the next new column, named k, if $k < b$. Keeping all available choices partitions the orders extending p.*

Proof. Each residual or column is an increasing chain. The next vertex of any interleaving must be the unused head of one of these chains. Unseen columns differ only by names; naming the chosen one k is canonical. The stated choices are distinct and exhaustive, and the same argument applies after each extension. □

For a four-column board the root is $p = (16)$; its children are $(16, 17)$ and $(16, 0)$. The second local vertex need not be the global vertex 2. A prefix guarantees its internal order, that every prefix vertex precedes every unused vertex, and the internal column and residual orders. No other comparisons are imposed. If $a < b$ is guaranteed, shiftedness of the original family gives

$$\mathbf{1}_H \le \mathbf{1}_{(H\setminus\{b\})\cup\{a\}} \quad (b \in H,\ a \notin H).$$

Indeed replacing b by a in an edge realizing the exact trace H leaves its vertices outside $[4s+3]$ unchanged.

Only the group S_{4-k} permuting unseen columns fixes this information. Averaging over that group is therefore valid; averaging a seen column with an unseen one is not. The key consists of the trace size, residual bitmask, the ordered bitmasks of seen columns and the multiset of unseen column bitmasks. For $k = 0, 1, 2, 3, 4$, respectively, the numbers of orbit means are

$$469, \quad 1289, \quad 2920, \quad 5016, \quad 5016.$$

They always represent the same 5016 labelled subsets. Every transported row is a necessary inequality before it is averaged; the original family is not assumed symmetric.

Lemma 3.3 (Lexicographically forced traces)**.** *Let $K = [4s+3]$, $q = n - 4s - 3$, and let the shifted saturated ONE family and its selected residual be as above. For distinct $x, y \in K \setminus \{1\}$, if the increasing pair associated to $\{x, y\}$ is lexicographically smaller than (δ_1, δ_2) and $q \ge 1$, then $\{1, x, y\}$ is an exact trace. If $x \in K \setminus \{1\}$, $x < \delta_1$, and $q \ge 2$, then $\{1, x\}$ is an exact trace.*

Proof. If the triple trace is absent, choose $u \notin K$. The edge $Q = \{1, x, y, u\}$ is missing. Saturation gives an s-matching disjoint from Q, hence from $R = \{1, x, y\}$. Compress its union into the first $4s$ vertices outside R. This operation can be performed by an order-preserving injection that moves every vertex downwards, so the matching remains in the shifted family. Since $|K \setminus R| = 4s$, its residual is R, contrary to minimality.

For the pair choose distinct $u, v \notin K$, which requires $q \ge 2$. Absence of the pair makes $Q = \{1, x, u, v\}$ missing. An s-matching disjoint from Q compresses into $K \setminus \{1, x\}$, of size $4s + 1$. It leaves one further vertex y, and hence residual $\{1, x, y\}$. The first coordinate of its increasing pair is at most $x < \delta_1$, again a contradiction. Both compressed matchings still avoid 1. □

The low seeds. For a column $E = \{a < b < c < d\}$, both $D \cup \{a\}$ and $D \cup \{b\}$ are present. If $b > \delta_2$, the ordered four-set $D \cup \{b\}$ is coordinatewise below E. If $b < \delta_2$, the pair obtained by sorting $\{\delta_1, b\}$ is lexicographically below (δ_1, δ_2); Lemma 3.3, with $q \geq 1$, and completion give that seed. Lowering b to a gives the other one. This argument and the forced-trace lemma hold for every $s \geq 1$; they do not require a four-column subboard.

This is the saturation/compression mechanism of [12, Lemma 2.5 and Section 4, especially Lemma 4.1], applied with the additional comparisons displayed above. For partial orders, a triple is forced only when a guaranteed comparison shows $x < \delta_1$ or $y < \delta_1$, or one vertex equals δ_1 and the other precedes δ_2. No incomparable pair is ordered arbitrarily. The forcing row is $-x_H \leq -1$, with its images averaged over S_{4-k}. Pair forcing is never used at $q = 1$, or without vertex 1.

The preceding support-profile inequalities also remain available. In this refinement each is applied to its specified two- or three-column support before averaging over the stabilizer. Its right side is still the exact maximum over all legal profiles, including for signed functionals. The support census is inherited; the refinement couples it to the local order and the forced presences.

3.4 Dual arithmetic and global integer rounding

For necessary rows $Ax \leq b$, semantic bounds $l_i \leq x_i \leq u_i$, and $\Phi = c \cdot x - c_0$, every rational $y \geq 0$ gives

$$\Phi \leq y \cdot b + \sum_i \max\{(c - A^{\mathsf{T}}y)_i l_i, (c - A^{\mathsf{T}}y)_i u_i\} - c_0 =: U. \tag{3.2}$$

Indeed sum y times the rows and bound each residual coordinate on its interval. The endpoint certificates use nonnegative rational multipliers. For every certificate, including each leaf of the order forest, the residual term accounts exactly for rational rounding. No solver tolerance or equality of the residual to zero is assumed.

There are $\binom{s}{4}$ boards. With $\Delta_* = \max\{0, B_4(n,s) - A_4(s)\}$, the sufficient check is

$$\binom{s}{4} U < \Delta_* + 1. \tag{3.3}$$

By (3.1), $|\mathcal{F}| - A_4(s)$ is an integer at most the left side, hence at most Δ_*. For an order tree, replace U by the maximum of all its leaf bounds. Lemma 3.2 makes this a bound for every board. This rounds the global cardinality, not each board or each orbit mean separately; the means remain real variables.

Endpoint certificate	Cases	Shared rows	Positive multipliers
Ordinary, 291 coordinates	52	1969	9995
Minimum-completion, 469 coordinates	3	758	1103
Ordered trees, leaf-specific coordinates	33	—	711130

All 88 distributed endpoint bounds satisfy (3.3). The main cardinality proof uses the 51 records selected in Corollary 4.29, together with its two fixed inequalities. The other 37 ordinary upper records are retained for the structural gap bounds. The 48 lower records at $s \geq 25$ are superseded by the analytic clique band and are not distributed. The first two groups each use a shared row dictionary; the ordered trees reconstruct rows at each dual leaf. The positive multipliers total 722228 occurrences. The three minimum-completion points are $(22, 103), (27, 125), (34, 156)$. The exact positive global rounding margins, in this order, are

$$\frac{717402531195218849}{200000000000000}, \quad \frac{115229982289222811}{20000000000000}, \quad \frac{740832624886150683}{62500000000000}.$$

The archive records each row with its literal witness and each bound with its multipliers; the checker recomputes the coefficients and signs.

The larger target at an upper crossing is essential. For $s = 6, n = 31$ and $s = 8, n = 40$, the shifted family of four-sets using at least two vertices of $[2s + 1]$ has, respectively, 17797 and 52428 edges, exceeding $A_4(s) = 17550, 52360$, but not $B_4(n, s) = 18815, 55430$. It has matching number s and an s-matching avoiding 1. A missing edge uses at most one initial vertex; outside it, pair $2s$ remaining initial vertices and complete with $2s$ unused vertices. This proves saturation and supplies a matching disjoint from the added edge. Thus ONE does not force the clique bound at every small upper crossing; these examples satisfy EMC.

3.5 The 33 order trees and strict ONE separation

Proposition 3.4 (The remaining finite endpoints)**.** *Under the declared board premises, the following 33 points satisfy* $|\mathcal{F}| \le \max\{A_4(s), B_4(n, s)\}$ *for every shifted saturated ONE family with matching number* s*:*

$$\begin{aligned} n &= n_4(s) - 1 \quad & s &= 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, \\ n &= n_4(s) & s &= 5, \dots, 21, 23, 25, 26, 28, 30, 32. \end{aligned}$$

Proof using the exact finite records. For each point the original archive supplies one tree. The complete forest has 413 nodes: 115 splits, 295 rational dual leaves and three analytic leaves; the maximum prefix length is 7. Every split keeps exactly the children in Lemma 3.2. At a dual leaf the checker reconstructs its stabilizer, all necessary rows and the rational bound (3.2), including every residual coefficient. The global bound satisfies (3.3).

An analytic leaf has precisely the prefix $(16, 17, 18)$. Any present pair or triple would shift into D, contradicting absence of small traces there. Thus all small traces are absent and $\Phi \le 0$. No infeasibility or Farkas leaf is used.

All 711130 row and multiplier occurrences are accounted for: 69937 completions, 391523 order rows, 237183 matching rows, 6321 support rows, 638 pair forcings, 5243 triple forcings and 285 wide inequalities. The ordered-tree leaves use no middle or lower inequalities. The distributed checker reconstructs every leaf and computes its exact bound. The exhaustive maximum and (3.1) then prove the stated integer target. Section 8 identifies the original records and executable checks. □

There is a separate arithmetic check of order coverage. If r residual vertices and r_i vertices of column i remain after p, with k seen columns, then the number of canonical completions of the prefix is

$$\mathcal{N}(p) = \frac{(19 - |p|)!}{r! \prod_{i=0}^{3} r_i! \, (4 - k)!}. \tag{3.4}$$

Interleave the remaining labelled chains and divide by the free renaming action on unseen columns. At the root this is $18!/[2!(4!)^5] = 402026625$. Distinct children have disjoint extensions, and their counts sum to the parent's. This identity is checked at every split of all 33 trees. It counts vertex orders, not hypergraphs.

Corollary 3.5 (Strict ONE gaps at the new upper points)**.** *At each upper point in Proposition 3.4, let*

$$g_{s,n} = \binom{s}{4} \max_{\text{leaves}} U, \qquad h_{s,n} = B_4(n, s) - A_4(s) - \lfloor g_{s,n} \rfloor .$$

Under the same premises, every shifted ONE family of matching number exactly s *has size at*

most $B_4(n,s) - h_{s,n}$. The positive integers $h_{s,n}$ are:

s	n	$h_{s,n}$	s	n	$h_{s,n}$
5	26	1	17	80	1233
6	31	31	18	85	4130
7	35	52	19	89	4682
8	40	193	20	94	9146
9	44	37	21	98	2435
10	49	50	23	107	8514
11	53	46	25	116	17864
12	58	478	26	120	3350
13	62	432	28	129	13111
14	67	1840	30	138	25365
15	71	2047	32	147	41960
16	76	1166			

In particular a shifted family attaining $B_4(n,s)$ at one of these 23 points cannot have property ONE.

Proof. Enlarge the shifted ONE family to a saturated shifted family of the same matching number by the enlargement in Section 3.1. Its matching avoiding 1 persists. The forest bounds its cardinality by $A_4(s) + \lfloor g_{s,n} \rfloor$. The displayed gaps are exact integer subtractions from the certified rational bounds. □

For example, at $(s,n) = (5,26)$,

$$g_{5,26} = \frac{65774380902950387}{600000000000000} < 110, \quad A_4(5) = 8855, \quad B_4(26,5) = 8965,$$

and the positive rounding margin is

$$111 - g_{5,26} = \frac{825619097049613}{600000000000000}.$$

Consequently ONE families have at most 8964 edges there. At $s = 28, 30, 32$ the upper-point bounds even round to $A_4(s)$. This does not restore the false uniform ONE-to-clique implication illustrated above, or assert quantitative stability for arbitrary families.

3.6 The four small points on the full carrier

For $s = 2, 3, 4$, use all of $K = [4s+3]$, with its s columns and residual D, rather than a four-column subboard. The 11-, 15- and 19-vertex carriers have respectively 550, 1925 and 5016 labelled subsets of sizes 2, 3 and 4. At the four points below $q = 2, 2, 2, 3$, so the weights on pairs, triples and quadruples are $(1, 2, 1)$ or $(3, 3, 1)$. The forbidden mixed matching has $s + 1$ traces.

Proposition 3.6 (Full-carrier small certificates)**.** *For shifted ONE families with matching exactly s, the following rational upper bounds $U_{s,n}$ hold; consequently their sizes are at most $\lfloor U_{s,n} \rfloor$.*

s	n	$U_{s,n}$	$\lfloor U_{s,n} \rfloor$	$M(n,s)$
2	*13*	3845000000000003/10000000000000	*384*	*385*
3	*17*	13798385978428377/10000000000000	*1379*	*1379*
4	*21*	38760000000000019/10000000000000	*3876*	*3876*
4	*22*	531935000000001/125000000000	*4255*	*4255*

No finite HHL inequality or support classification is an input.

Proof. Enlarge while preserving ONE as in Section 3.1. Lemmas 3.1, 3.2 and 3.3 give all required semantics. In the records the residual labels $4s, 4s+1, 4s+2$ stand for $1, \delta_1, \delta_2$. After k named columns, average only the group S_{s-k} permuting unseen columns. Variables are real means in $[0,1]$. Columns and the two low seeds are fixed present; small subsets of D are absent. The only selected row types are guaranteed lowering, completion, disjoint $(s+1)$-matchings, and pair/triple lexicographic forcing. Their validity was proved above.

For a leaf, put $c_H = \binom{q}{4-|H|}$, including the multiplicity if orbit coordinates are used. From $Ax \le b$, box $\ell \le x \le u$, and a nonnegative rational dual y, one gets

$$|\mathcal{F}| \le y \cdot b + \sum_i \max\{(c - A^{\mathsf{T}}y)_i \ell_i,\ (c - A^{\mathsf{T}}y)_i u_i\}.$$

This is the complete cardinality, not a local excess multiplied by $\binom{s}{4}$. Every coefficient and box correction is rational and exact.

The four trees have 129 nodes: 34 splits, 94 dual leaves and one analytic leaf, with 213845 selected row/multiplier occurrences. The per-point dual counts are $4, 35, 4, 51$. A leaf beginning with all three residual vertices has no small trace, since any such trace would shift into D; its analytic bound is $A_4(s)$. Every split lists exactly the choices of Lemma 3.2. With r, r_i remaining chain sizes, the number of canonical completions is

$$\frac{(4s+3-|p|)!}{r! \prod_{i=0}^{s-1} r_i!\,(s-k)!}.$$

At the four roots these counts are $1575, 525525, 402026625, 402026625$. They add at every split. The maximum over all leaves gives the four displayed $U_{s,n} < M(n,s)+1$; global integrality gives the bound. The distributed checker lifts each geometric witness to all labelled subsets by summing the actual permutation action, and applies the box correction there. It verifies all 94 rational dual bounds. The certificates and executable check are identified in Section 8. □

At $(2, 13)$, ONE has the strict bound 384 while the complete two-cover has 385 edges. The other three ONE bounds suffice for the general target without a strict gap.

4 Finite board inequalities and the weight tail

This section gives critical ONE bounds for the global induction. It reconstructs the finite ingredients of Hou–Hu–Liu (HHL) [12] that the present certificates actually use. These are not inputs to the local stability theorems. An actual selected board consists of four columns of a selected maximum matching and the lexicographically minimal residual three-set. Lemmas 3.1 and 3.3 prove the trace, completion and seed constructions used here. The next two lemmas prove the support and upper-cube statements, C1, and only the seven trace thresholds needed from C2. The top-star row families are justified explicitly below. The residual-row mechanism of HHL Section 6.1 follows from disjoint traces and present columns or seeds. The old numerical 15- and 11-vertex bounds, the old numerical top-star bound and C3 are not imported. This assembly uses neither HHL Lemma 3.1 nor its generic localization theorem or deletion-chain criterion.

Proposition 4.1 (Critical ONE bounds from 73)**.** *Let $s \ge 73$ and $n \in \{a_s - 1, a_s\}$. Every shifted saturated ONE family $\mathcal{F} \subseteq \binom{[n]}{4}$ with $\nu(\mathcal{F}) = s$ satisfies*

$$|\mathcal{F}| \le A_4(s).$$

Its proof uses the layer inequalities and the elementary weight bounds at the end of this section; it makes no assertion about non-ONE families before the induction in Section 2.3.

4.1 Support costs and rectangle persistence

An actual 19-vertex board consists of a residual three-set D and four ordered columns with levels $0,1,2,3$. Write $Q=\{0,1,2,3\}^4$, $\mathcal{M}\subseteq Q$ for its missing wide quadruples, $M=|\mathcal{M}|$, and T,P for the numbers of present wide triple and pair traces, with their column supports distinguished. Lemma 3.1 makes $\mathcal{M}$ an up-set and present traces down-sets. All four full columns and the low seeds $D\cup\{(j,\ell)\}$, $\ell\in\{0,1\}$, are present; there are no five disjoint present traces. A present trace forces each wide quadruple extending it to be present. Thus a trace, a disjoint seed and three mutually disjoint wide quadruples cannot all be present. Their three quadruples give a *seed clause* that must meet $\mathcal{M}$.

Call $\{1,2,3\}^4$ the upper cube. A missing top-star fixes one coordinate to 3 and lets the other three range over $\{1,2,3\}$. In the no-top-star case, up-closure makes the four permutations of $(3,1,1,1)$ present, and therefore every quadruple with at most one coordinate in $\{2,3\}$ is present.

Lemma 4.2 (Support obstructions and the upper cube)**.** *On an actual selected board a pair support has at most four traces and a triple support at most 32. The pair down-set avoids any two disjoint values whose levels in some coordinate are not exactly* $\{0,1\}$*. The triple down-set avoids any three disjoint values omitting a low level in some coordinate. If the present upper cube has a 3-matching, there are no wide pair or triple traces.*

Proof. A forbidden pair configuration leaves a low level unused in one support column. Add its low seed and the two outside full columns. A forbidden triple configuration similarly admits a disjoint low seed and the outside full column. Either construction gives five disjoint traces, contradicting Lemma 3.1. This proves the support obstructions on the actual board.

For pairs, if a down-set contains a value at least 2 in one coordinate, it contains $(2,0)$ after interchanging the coordinates. The value $(0,1)$ would form a forbidden pair with it, so the diagram lies on one axis. Otherwise it lies in $\{0,1\}^2$. Both possibilities have size at most four.

For triples Lemma 4.4 below gives the bound 32 by complementary binary blocks. This value and its equality diagrams are due to [12, Lemma 5.2]. The support census is still used to check signed facets and the size-24 cost; the scalar cap and the size-28 cost have direct proofs.

For the last assertion take three disjoint present upper quadruples and a proposed wide trace H. Choose a column outside its support and the level-0 seed there. In a coordinate where H uses a positive level f, replace f by 0 in the unique quadruple using f. Leave the other levels fixed. If H uses 0, no change is needed. The three resulting quadruples are coordinatewise smaller, remain disjoint, and avoid both H and the seed. They again give five disjoint traces. This is the lowering argument of [12, Lemma 5.3], now using the proved trace and seed lemmas. □

4.2 Direct proofs for dense triple supports

Call a down-set $T\subseteq\{0,1,2,3\}^3$ *legal* if it contains no three coordinatewise disjoint values omitting a low level in some coordinate. Actual triple supports are legal by Lemma 4.2. Put $L=\{0,1\}$ and $H=\{2,3\}$. Indices for a triple support are $0,1,2$; its outside board column is 3. Write $[x]$ for the trace encoded by a level tuple x; (x,t) adds level t in column 3.

Lemma 4.3 (Complementary binary blocks)**.** *Let C_S be the block whose high coordinates are exactly $\varnothing\neq S\subsetneq\{0,1,2\}$. For a legal T, put $a=|T\cap C_S|$ and $b=|T\cap C_{S^c}|$. If both are positive, $a,b\leq 4$; always $a+b\leq 8$.*

Proof. Antipodes in C_S exchange 0 with 1 and 2 with 3 in the respective coordinates. An antipodal pair there and any point of C_{S^c} form a bad three-matching: in a coordinate of S one low level is omitted. If $b>0$, at most one member of each of the four antipodal pairs belongs

to T. The same argument interchanges the blocks. If one count vanishes the sum is at most the size of the other block. □

Lemma 4.4 (The triple cap)**.** *A legal down-set in* $\{0,1,2,3\}^3$ *has size at most 32.*

Proof. If it meets H^3, it contains 222. If any point has level 3 in coordinate i, down-closure supplies $w = 3e_i$, the point u with level 2 in coordinate i and 1 elsewhere, and the point v with level 0 in coordinate i and 2 elsewhere. These three points form a bad matching. Thus no level 3 occurs, and the down-set is exactly $\{0,1,2\}^3$, of size 27. Otherwise the all-high block is empty. The all-low block contributes at most eight; the other six blocks form three complementary pairs, each contributing at most eight by Lemma 4.3. □

The value 32 and its four equality diagrams were already established by the finite calculation of [12, Lemma 5.2 and Section 9.3]. The arguments here give a direct proof and the following denser-support description. The signed facets used elsewhere still require their own exact checks.

4.3 The dense support theorem in dimension three

Define three slabs and a majority-low container by

$$S_i = \{x \in \{0,1,2,3\}^3 : x_i \le 1\} \quad (i = 0,1,2), \qquad J = \{x : \#\{i : x_i \ge 2\} \le 1\}.$$

Each has size 32. Partition the cube into the all-low block $C = L^3$, the all-high block H^3, and the six blocks

$$A_i = \{x : x_i \in H,\ x_j \in L\ (j \ne i)\}, \qquad B_i = \{x : x_i \in L,\ x_j \in H\ (j \ne i)\}.$$

Thus $J = C \cup A_0 \cup A_1 \cup A_2$. Write c, a_i, b_i for the corresponding intersection sizes with T.

Lemma 4.5 (Three local restrictions)**.** *Assume T is legal and each $a_i > 0$.*

1. *If $b_i > 0$, the other two a-counts are at most 7.*
2. *If $b_i \ge 3$, every point of $T \cap B_i$ has its low coordinate equal to 0, and the other two a-counts are at most 6.*
3. *If $b_i \ge 3$ and $b_j > 0$ for $i \ne j$, the all-low point having 1 in coordinates i, j and 0 in the third coordinate is absent. Hence $c \le 6$. If all b_i are positive and at least two are at least 3, then $c \le 4$.*

Proof. Coordinate permutation reduces all checks to explicit triples. For (i), $a_0, b_0 > 0$ force $200, 022 \in T$. The points $200, 022, 131$ form a bad 3-matching; hence 131 is absent and $a_1 \le 7$. Interchanging the last two coordinates gives $a_2 \le 7$.

For (ii), if $122 \in T$ and $b_0 \ge 3$, the down-set $T \cap B_0$ contains 023 or 032. In the first case $200, 112, 023$ is bad, with $112 \le 122$. In the second case $200, 121, 032$ is bad, with $121 \le 122$. Thus $122 \notin T$, and down-closure forbids every B_0 point with first coordinate 1. A down-set of size at least three in $\{0\} \times H \times H$ contains both 023 and 032. Now $200, 032, 121$ forbids 121, and therefore also 131; so $a_1 \le 6$. Likewise $200, 023, 112$ forbids both 112 and 113, giving $a_2 \le 6$.

For (iii), $b_0 \ge 3$ supplies 023, and $b_1 > 0$ supplies 202. Then $023, 202, 110$ is bad. Absence of 110 also forbids 111, giving $c \le 6$. When all three b-counts are positive and at least two are large, apply this argument to all three pairs of indices: $110, 101, 011, 111$ are absent, leaving at most four points of C. □

Theorem 4.6 (Unique dense container and sharp threshold)**.** *If a legal down-set $T \subseteq \{0,1,2,3\}^3$ has $|T| \geq 29$, then T is contained in exactly one of S_0, S_1, S_2, J. Consequently the legal down-sets of size 32 are precisely these four. The threshold 29 cannot be lowered to 28.*

Proof. The all-high case in Lemma 4.4 has size 27, so T contains no all-high point. Suppose T is contained in none of the four proposed containers. Being outside every S_i forces $2e_i \in T$, so all $a_i > 0$. Being outside J forces at least one $b_i > 0$. If $b_i > 0$, Lemma 4.3 gives $a_i, b_i \leq 4$. Put $r = \#\{i : b_i > 0\}$ and $h = \#\{i : b_i \geq 3\}$. The following bounds follow directly from Lemma 4.5 and $c \leq 8$. The middle expression exhibits every term, so the table is a nine-case argument, not an enumerated support census.

r	h	bound for $c + \sum_i (a_i + b_i)$	total
1	0	$8+6+7+7$	28
1	1	$8+8+6+6$	28
2	0	$8+6+6+7$	27
2	1	$6+8+6+6$	26
2	2	$6+8+8+6$	28
3	0	$8+6+6+6$	26
3	1	$6+8+6+6$	26
3	2	$4+8+8+6$	26
3	3	$4+8+8+8$	28

For example a positive $b_i \leq 2$ contributes at most $4+2=6$; a large b_i contributes at most $4+4=8$. A zero b_i leaves only a_i, bounded by 7 or 6 using another positive or large b-count. Every row contradicts $|T| \geq 29$.

The containers have pairwise intersections at most 24: $|S_i \cap S_j| = 16$ for $i \neq j$, and $|S_i \cap J| = 24$. Thus the container is unique. A slab has no three disjoint support values in its low coordinate. In a 3-matching inside J, each coordinate needs at least one high value, whereas each point has at most one high coordinate. There are exactly three high occurrences, one in each coordinate, so the other two values in each coordinate are 0 and 1. Such a matching is not bad. Thus J is legal and the equality assertion follows.

Finally $W = \{x : x_0 = 0 \text{ or } x_1 = 0\}$ is a down-set of size $16 + 16 - 4 = 28$, in none of the four containers. Any three of its points have two sharing one of the two indicated zero-coordinate vertices, so W has no 3-matching. This proves sharpness. □

4.4 A dense slab forces the missing rectangle

Theorem 4.7 (Slab-to-rectangle)**.** *Under the board interface and absence of a missing top-star, suppose a triple support $T \subseteq S_i$ has $|T| \geq 29$. Then*

$$R_{i3} = \{q \in Q : q_i, q_3 \geq 2\} \subseteq \mathcal{M}, \qquad |R_{i3}| = 64.$$

No hypothesis about present upper-cube 3-matchings is needed.

Proof. The binary block with coordinate i low and the other two support coordinates high has eight points; at most three are absent from T. Five points in a binary cube of size eight contain an antipodal pair t, t'. The two traces use levels 0,1 in column i, and levels 2,3 in the other two support columns.

Let $q_i = q_3 = 2$ and let the other coordinates of q be 0. The known present no-top-star point $p = p^{(i)}$ has level 3 in column i and level 1 elsewhere. The sets

$$[t],\ [t'],\ [q],\ [p],\ D \cup \{(3,0)\}$$

are pairwise disjoint. Therefore q must be missing. Its full up-closure in Q is exactly R_{i3}, of size $2 \cdot 2 \cdot 4 \cdot 4 = 64$. □

4.5 A dense majority support forces 80 missing quadruples

Throughout this section $T \subseteq J$ is a down-set with $|T| \geq 29$. Its hole set $J \setminus T$ has size at most three. The no-top-star condition is *not* needed in this section.

Lemma 4.8 (Thirty-two forced points)**.** *Under the board interface, $F = H^3 \times \{0,1,2,3\} \subseteq \mathcal{M}$ and $|F| = 32$.*

Proof. Fix $q \in F$. For each $i = 0,1,2$, construct a triple $t^{(i)}$ whose high coordinate is i, with value $5 - q_i$. In each coordinate distribute the two remaining low levels 0,1 between the other two triples. There are $2^3 = 8$ such unordered matchings (the unique high coordinate labels each triple). The twelve triple values which occur each belong to exactly two of these eight matchings: two low coordinates of a given triple fix two orientation bits, and the bit in its high coordinate remains free. At most three absent values destroy at most six matchings. Hence a matching of three present triples avoiding q exists. Choose $z \in L \setminus \{q_3\}$. These triples, $[q]$, and the present seed $D \cup \{(3,z)\}$ are pairwise disjoint. Therefore q is missing. □

The hole structure. Within A_i, use binary coordinates $(x_i - 2, x_j, x_k)$. Every point of binary weight at most one belongs to T, since its principal upper set in A_i has at least four points. Also $C \subseteq T$: the point 111 has seven points above it inside J, and every other point of C lies below it. Thus all holes lie among the tops and coatoms of the three A_i blocks. Denote by

$$E_i = (3 \text{ in } i,\ 1 \text{ elsewhere}), \qquad C_{i,j} = (3 \text{ in } i,\ 0 \text{ in } j,\ 1 \text{ in the third})$$

the top and the two low-coordinate coatoms. The remaining coatom has level 2 in coordinate i and 1 elsewhere. Every missing coatom forces its top missing. Consequently there are at most two missing $C_{i,j}$. If there are two, they have the same high-coordinate index i, and the holes are exactly $E_i, C_{i,j}, C_{i,k}$.

A closed pairing on the 24 double-high triples. Put $B = B_0 \cup B_1 \cup B_2$, of size 24. Choose an oriented 3-cycle on the coordinate indices. For each directed edge $i \to j$, with k the third coordinate, and $a, b \in \{0,1\}$, define

$$\begin{array}{lll} x_i = \epsilon_i, & x_j = 2 + a, & x_k = 2 + b, \\ y_i = 2 + a, & y_j = 1 - \epsilon_j, & y_k = 3 - b. \end{array} \tag{4.1}$$

For any fixed $\epsilon \in \{0,1\}^3$, these twelve pairs partition B: the outgoing edge from i uses the four points of B_i at low level ϵ_i, and the incoming edge uses the four at low level $1 - \epsilon_i$. Each coordinate of x differs from the corresponding coordinate of y.

Lemma 4.9 (Present complementary traces by a cyclic rule)**.** *The cycle and ϵ can be chosen so that for every pair x, y in (4.1), the unused two levels in each coordinate partition into two present triples $h, h' \in T$. No matching search or archived pairing is required.*

Proof. Ordinary case: at most one missing low-coordinate coatom. Set $\epsilon = 000$. With no missing $C_{i,j}$ choose either cycle; with one missing $C_{u,v}$ choose the cycle in which v is not the successor of u. For every edge $i \to j \to k \to i$, take

$$\begin{array}{lll} h_i = 3 - a, & h_j = 0, & h_k = 1, \\ h'_i = 1, & h'_j = 3 - a, & h'_k = 0. \end{array}$$

These triples partition the unused levels. If $a = 1$, they have binary weight one in A_i, A_j, so are present. If $a = 0$, they are $C_{i,j}, C_{j,k}$, neither missing by the orientation choice.

Exceptional case: two missing low-coordinate coatoms. Relabel their common high index as 0; the holes are exactly $311, 310, 301$. Blocks A_1, A_2 are complete. Choose the cycle $0 \to 1 \to 2 \to 0$ and $\epsilon = (0, 0, 1)$. For an edge $i \to j$ the remaining levels are

$$\{1 - \epsilon_i, 3 - a\}, \quad \{\epsilon_j, 3 - a\}, \quad \{0, 1\}$$

in coordinates i, j, k. Take

$$\begin{array}{lll} h_i = 3 - a, & h_j = \epsilon_j, & h_k = e, \\ h'_i = 1 - \epsilon_i, & h'_j = 3 - a, & h'_k = 1 - e. \end{array}$$

Use $e = 0$ on $0 \to 1$ and $1 \to 2$, and $e = 1$ on $2 \to 0$. On the first and last edge, the triple in A_0 has the form $(3 - a, 0, 0)$, hence is present; its companion belongs to a complete block. On the middle edge both triples belong to complete blocks. Rotating the labels handles either other exceptional index. These are all possibilities for at most three holes. □

Theorem 4.10 (Majority-to-80)**.** *Under the board interface, $T \subseteq J$ and $|T| \geq 29$ imply $M \geq 80$. Neither absence of a missing top-star nor absence of an upper-cube 3-matching is an additional hypothesis.*

Proof. For each of the twelve base pairs x, y in Lemma 4.9, pair the eight wide quadruples above it as

$$\{(x, 0), (y, 2)\}, \quad \{(x, 1), (y, 3)\}, \quad \{(y, 0), (x, 2)\}, \quad \{(y, 1), (x, 3)\}. \tag{4.2}$$

These 48 pairs partition $B \times \{0, 1, 2, 3\}$, a set of 96 points. Consider one pair, written $q = (x, \ell), r = (y, t)$ with $\ell \in L, t \in H$; the version with x, y reversed is identical. Let $h, h' \in T$ be the complementary triples. Completion makes $(h', 5-t)$ present, and the seed $D \cup \{(3, 1-\ell)\}$ is present. The five sets

$$[h], \ [(h', 5 - t)], \ D \cup \{(3, 1 - \ell)\}, \ [q], \ [r]$$

partition the 19 vertices: in the first three columns the four traces x, y, h, h' use every level, and in the fourth column the levels are $\ell, t, 1 - \ell, 5 - t$. Thus q, r cannot both be present. Each of the 48 disjoint pairs contributes a missing point. Their union has exactly two high coordinates among its first three, whereas the 32 forced points of Lemma 4.8 have three. The sets are disjoint, giving $M \geq 32 + 48 = 80$. □

Corollary 4.11 (The dense rectangle alternative)**.** *On a board satisfying the interface and having no missing top-star,*

$$|\mathcal{T}_I| \geq 29 \quad \Longrightarrow \quad M \geq 80 \text{ or } R_{ij} \subseteq \mathcal{M} \text{ for some } i < j.$$

The absence of an upper-cube three-matching is unnecessary for this alternative. The size-28 cost is proved constructively below; the size-24 cost retains the finite proof in Lemma 4.17.

Proof. Relabel I as columns $0, 1, 2$. Lemma 4.2 and Theorem 4.6 give the unique slab or majority container. Apply Theorem 4.7 or Theorem 4.10, respectively. □

4.6 A constructive cost at support size 28

Lemma 4.12 (The size-28 containers)**.** *Every legal down-set $T \subseteq \{0, 1, 2, 3\}^3$ of size 28 lies in a slab S_i, lies in J, or is a coordinate permutation of*

$$W = \{x : x_0 = 0 \text{ or } x_1 = 0\}.$$

Proof. By Lemma 4.4, the all-high block is empty. Outside the slabs and J, all a_i are positive and some $b_i > 0$. In the nine-case table in the proof of Theorem 4.6, only $(r,h) = (1,0),(1,1),(2,2),(3,3)$ can attain 28. For $(1,0)$, normalize $b_0 > 0$. Equality requires $a_0 = 4, b_0 = 2, a_1 = a_2 = 7, c = 8$. The two-point ideal in B_0 contains 022 and one of $023, 032, 122$. These alternatives give the bad matchings

$$(200,112,023), \qquad (200,121,032), \qquad (200,122,031),$$

respectively. The auxiliary points are present because a seven-point ideal in A_1 or A_2 omits only its maximum.

For $(1,1)$, equality requires $a_0 = b_0 = 4, a_1 = a_2 = 6, c = 8$. Lemma 4.5 gives $T \cap B_0 = \{0\} \times H^2$. The two omitted points in A_1 are $121, 131$, so 130 is present; those in A_2 are $112, 113$, so 103 is present. The bad matchings $(201,023,130)$ and $(210,032,103)$ exclude $201, 210$. Down-closure leaves only $200, 300$ in A_0, a contradiction.

For $(2,2)$, normalize $b_0, b_1 > 0$. Lemma 4.5 excludes 110, so $T \subseteq W$ and equality of sizes gives $T = W$. For $(3,3)$ the same lemma excludes $110, 101, 011, 111$, leaving at most two points in each A_i, whereas equality requires $a_i = 4$. All other rows are at most 27. Finally, W is legal: three of its triples repeat a zero-level vertex in column 0 or 1. □

We use two elementary missing-point rules. Four known present traces disjoint from one another and from $[q]$ force $U(q) = \{r \in Q : r \geq q\} \subseteq \mathcal{M}$. Three known present traces disjoint from one another and from the disjoint quadruples $[q], [r]$ force at least one of q, r missing. Thus h disjoint pairs outside an already forced set F give $M \geq |F| + h$. Absence of a missing top-star supplies the four present quadruples $p^{(i)}$, with level 3 in column i and 1 elsewhere.

Lemma 4.13 (Eight cyclic pairs with four holes)**.** *Let $T \subseteq J$ be a down-set of size 28. Fix the cycle $0 \to 1 \to 2 \to 0$ and set $\epsilon_i = 0$ in (4.1). At least eight of its twelve pairs have complementary triples in T.*

Proof. The core $C = L^3$ is complete: even 111 has seven successors in J. The four holes lie in the three binary cubes A_i. For the pair associated with $i \to j$, third index k, and $a, b \in L$, try $e = 0, 1$ in

$$\begin{array}{lll} h_i = 3 - a, & h_j = 0, & h_k = e, \\ h'_i = 1, & h'_j = 3 - a, & h'_k = 1 - e. \end{array} \tag{4.3}$$

These are complementary to x, y. Their presence does not depend on b; each of the six slots (i,a) therefore contributes zero or two pairs.

If a slot with $a = 1$ fails, its $e = 0$ choice forces $h' = (1,2,1)$ absent, in coordinate order i, j, k: $h = (2,0,0)$ has eight successors in A_i and is present. Failure also at $e = 1$ requires $(2,0,1)$ absent in A_i or $(1,2,0)$ absent in A_j. The former costs four holes in addition to the hole in A_j. Hence the holes are exactly the face $x_i = 1$ in A_j. Only the two slots on $i \to j$ fail; the other four are good.

Otherwise all three $a = 1$ slots are good. If $(3,0,0)$ is absent in an arm, all four holes form its high face; the $a = 0$ slot between the other two arms is good. If every such point is present, failure of an $a = 0$ slot on $i \to j$ requires the maximum E_j absent and at least one of $C_{i,j}, C_{j,k}$ absent, where $C_{u,v}$ has level 3 at u, 0 at v, and 1 elsewhere. Failure on all three arcs would require three distinct maxima and at least two distinct coatoms, since one coatom serves at most two arcs. Five holes are impossible. There is a fourth good slot. □

Every retained pair lifts to the four pairs in (4.2). The same three-trace witness used in Theorem 4.10 proves each clause. They are disjoint and lie above the double-high block $B = B_0 \cup B_1 \cup B_2$.

Lemma 4.14 (The two exceptional majority cores)**.** *For $T \subseteq J$ of size 28, the orientation argument of Lemma 4.8 forces all 32 points of $H^3 \times \{0,1,2,3\}$, except for the following hole sets, up to coordinate permutation:*

$$Z_I = \{300, 301, 310, 311\}, \qquad Z_{II} = \{310, 311, 013, 113\}.$$

For these two types it forces 16 and 24 core points, respectively. With the no-top-star premise these counts increase to 24 and 28.

Proof. Fix $q \in H^3 \times \{0,1,2,3\}$. There are eight orientations of three complementary small triples, one high in each column with value $5 - q_i$, distributing 0 and 1 between the other two triples. Each candidate triple occurs in two orientations. A surviving orientation and a low seed disjoint from q force q missing.

Four holes can kill all eight orientations only if all are relevant and the four pairs of orientations they remove partition the cube. A high-level-2 hole forces a high-level-3 hole above it, at least one irrelevant to q. Thus all four relevant holes have high level 3. A hole in A_i corresponds to an edge of the orientation cube in direction i. A perfect matching of that cube uses an even number of edges in each direction: cutting in that direction leaves four vertices on each side, and internal edges cover them in pairs. The direction counts are $4+0+0$ or $2+2+0$.

In the first case all holes form a high face, giving Z_I. In the second case the two holes in each arm form an upper face of its low-coordinate square. The two pairs of orientation edges are disjoint only if their faces fix the same remaining orientation bit; faces fixing different bits intersect. The common low coordinate has complementary values in the two small triples, so the two upper-face constraints remove opposite halves. This gives Z_{II}. For Z_I, the surviving core is $q_0 = 3$ (16 points); for Z_{II} it is $q_0 = 3$ or $q_2 = 3$ (24 points).

In both types $003, 130 \in T$. The five traces

$$[003], \quad [(130,3)], \quad [3111], \quad D \cup \{(3,0)\}, \quad [2222]$$

are disjoint. Thus $U(2222) = H^4$ is missing. Its union with the orientation core has size 24 in type I and 28 in type II. □

Theorem 4.15 (Constructive support cost)**.** *Under the board interface, if there is no missing top-star, then $|\mathcal{T}_I| \geq 28$ implies $M \geq 64$. No separate upper-cube 3-matching premise is used after support legality and the interface have been established.*

Proof. Delete maximal elements to reduce the actual support to size 28; all retained traces and their completions remain present. Lemma 4.12 supplies the following complete cases.

Majority. Outside types I and II, Lemmas 4.13 and 4.14 give $M \geq 32 + 4 \cdot 8 = 64$. For type I the fixed cycle keeps exactly eight base pairs. For $a, b, \ell \in L$ add the eight pairs

$$(2, 2+a, 2+b, \ell) \ ; \ (3, 1-a, 1-b, 2+\ell). \tag{4.4}$$

Use $h = (0, a, 3-b)$, $h' = (1, 3-a, b)$, seed $1-\ell$, and completion level $3-\ell$. The two small triples are in the complete arms 1 and 2; the levels partition each column. The first endpoints have three high support coordinates and low last coordinate, outside the forced core; the second have only one high support coordinate. Hence the eight pairs are mutually disjoint, avoid the core, and avoid all pairs above B. This gives $24 + 32 + 8 = 64$.

For type II, all slots on $0 \to 1$ and the $a = 1$ slots on the other arcs are good. The unused points $x = (3,2,1)$, $y = (1,3,2)$ admit the present complementary triples $003, 210$. Lifting this additional base pair by (4.2) gives four new disjoint pairs above B. Thus $28 + 32 + 4 = 64$.

Slab. Normalize $T \subseteq S_0$ and put $K = L \times H^2$. At least four points of K are present. If an antipodal pair $h, h' \in T \cap K$ exists, then $[h], [h'], [3111], D \cup \{(3,0)\}, [2002]$ are disjoint. Consequently $U(2002)$ is missing, of size 64.

Otherwise $T \cap K$ has exactly four points and all holes lie in K. In binary coordinates $(x_0, x_1 - 2, x_2 - 2)$, a four-point ideal is a lower face or the set of weight at most one: a point of weight two already forces its four-point principal ideal. Up to swapping columns 1 and 2, the holes are therefore the face $x_0 = 1$, the face $x_1 = 3$, or the weight-at-least-two set. All three types force 2022, using $023, (130, 3), 3111$ and seed 0. For the first and third types, $032, (103, 3), 3111$ and seed 0 also force 2202. The two up-sets have union of size 48. For $a, b, c, d \in L$ add the sixteen pairs

$$(3 - a, 2 + b, 2 + c, d) \ ; \ (2 + a, b, c, 2 + d). \tag{4.5}$$

The witnesses are $h = (0, 1 - b, 3 - c)$, $h' = (1, 3 - b, 1 - c)$, seed $1 - d$, completion $3 - d$. Both small triples lie outside K and are present. The first endpoints have low last coordinate; the second have both middle coordinates low. They avoid the forced up-sets and are all distinct, giving $48 + 16 = 64$.

For the face $x_1 = 3$, use $002, (123, 3), 3111$ and seed 0 to force 2302 instead. Its up-set together with $U(2022)$ has size 40. Keep (4.5) and add, for $a, c, d \in L$,

$$(3 - a, 1, 2 + c, d) \ ; \ (2 + a, 2, c, 2 + d). \tag{4.6}$$

Here $h = (0, 3, 1 - c)$ and $h' = (1, 0, 3 - c)$ are present, with seed $1 - d$ and completion $3 - d$. These eight pairs avoid the core and the first sixteen pairs: their first endpoints have middle coordinate 1 instead of high, and their second have 2 instead of low. Thus $40 + 16 + 8 = 64$.

The exceptional container W. The following rows each give four known traces disjoint from $[q]$:

q	Small trace	Completion	No-top-star trace	Seed level
1202	032	(203, 3)	3111	0
2102	302	(023, 3)	1311	0
2200	032	(303, 2)	1113	1

All small traces belong to W. The first two up-sets have sizes 48 and intersection 32. The third has size 64 and intersects their union in 32 points. Their union therefore has size 96, which suffices. □

The proof uses geometric cases and explicit formulas. The companion module `emc4/support28.py` additionally constructs and checks the witnesses on all 99 labelled supports (21 coordinate orbits), without a stored packing list. The census is corroboration of the written classification. The cyclic pairing and four-pair lift are the same constructions used above for supports of size at least 29.

Lemma 4.16 (The retained no-top-star bounds)**.** *Suppose the actual board has no missing top-star and no present upper-cube 3-matching. Then $M \geq 33$. The per-support triple caps at $M = 33, 34$–$35, 36$–$41, 42$–47 are $0, 1, 4, 7$, respectively. A pair support is empty for $M \leq 59$ and has at most one trace for $60 \leq M \leq 63$.*

Proof. We specify the two exhaustive searches and their soundness invariants. They are the consumed parts of HHL's C1 and C2, with the exact released programs and their completed executions identified in the accompanying source and reproduction records. C1 and the seven

consumed C2 records were rerun during this audit, with complete outputs and timeout checks; the source's fast checker alone cannot replace those executions.

First translate the present upper cube to a down-set $C \subseteq \{0,1,2\}^4$. Absence of a missing top-star forces the four permutations of $(2,0,0,0)$ into C. Represent C by 27 heights $h_{abc} \in \{0,1,2,3\}$, nonincreasing in a,b,c, and enumerate them in lexicographic order. At each cell the upper limit is the minimum of its earlier neighbours; the forced points give the lower limits. This visits every admissible height assignment once. Filling every remaining cell with its maximum permitted value gives the greatest monotone continuation, by induction over the cells. Thus a prefix may be discarded when this upper bound on $|C|$ is at most 48.

At every remaining leaf test the 216 matchings

$$\{(t, \pi_1(t), \pi_2(t), \pi_3(t)) : t = 0,1,2\}, \qquad \pi_i \in S_3.$$

Every upper 3-matching has a unique such representation after ordering by its first coordinate. The exact run checks 3116823 complete candidates of size at least 49; each contains a matching. Hence $|C| \le 48$ and $M \ge |\mathcal{M} \cap \{1,2,3\}^4| \ge 81 - 48 = 33$.

For the second search, fix a pair or triple value v on a labelled support and let $\mu(v)$ be the minimum size of an up-set $\mathcal{N} \subseteq Q$ satisfying the following necessary conditions. It avoids the four top-star minima and every completion of v; it hits all 216 upper-matching clauses; and it hits every residual 3-matching disjoint from v and from any disjoint low seed. If none exists set $\mu(v) = +\infty$. An actual missing family with trace v is feasible, so $M \ge \mu(v)$. These conditions deliberately impose no sufficiency claim for boards.

Here is why the finite search is exhaustive. Denote the forced-present points, forbidden to $\mathcal{N}$, by F and the principal up-set of a point x by U_x. A point with $U_x \cap F \ne \varnothing$ cannot be selected; delete it from each clause. An empty clause proves infeasibility. Remove duplicate clauses and clauses containing another retained clause. At a search state N, selected points form an up-set avoiding F. A singleton unhit clause forces its principal up-set into N. Any h disjoint unhit clauses require at least h additional points, so $|N| + h$ is a lower bound. For a chosen unhit clause branch over $N \cup U_x$ for every point x in it, omitting only duplicate states, forbidden states, or states meeting the current size cutoff. Every feasible smaller extension lies in one of these branches. When all clauses are hit, the state is feasible and can lower the cutoff. These are all numerical pruning rules; branch-order heuristics do not discard states. Generation of the residual clauses chooses three available levels per coordinate, fixes their order in the first coordinate and permutes the other three, covering every residual matching.

Only the following seven lower bounds are needed:

Trace v	00	01	000	001	002	003	011
$\mu(v) \ge$	60	64	34	36	42	48	48

All seven records in the pinned certificate have exact mode: with cutoff one above the displayed bound the completed search returns the displayed value. The full verifier checks both equalities and rejects timeouts and nonzero process exits. Only the lower bounds are used here. Permuting columns transports each bound without requiring symmetry of the actual board.

Down-closure now gives the caps directly. For $M = 33$, even 000 is forbidden. Through 35, excluding permutations of 001 leaves only 000. Through 41, excluding 002 and 011 leaves 000 and the three values with one coordinate equal to 1. Through 47, excluding 003 and 011 leaves 000 and the six values with a single positive coordinate equal to 1 or 2. For pairs, excluding 00 leaves none; excluding 01 and its permutation leaves only 00. No other source threshold, critical pattern or rectangle enumeration is needed for these conclusions. □

Lemma 4.17 (Two support costs and a rectangle alternative)**.** *Suppose the actual board has neither a missing top-star nor a present upper-cube 3-matching. For a triple support I,*

$$|\mathcal{T}_I| \ge 24 \implies M \ge 56, \qquad |\mathcal{T}_I| \ge 28 \implies M \ge 64.$$

If $|\mathcal{T}_I| \geq 29$*, then either* $M \geq 80$ *or*

$$R_{ij} := \{q \in Q : q_i, q_j \geq 2\} \subseteq \mathcal{M} \quad \textit{for some } i < j.$$

Proof. Encode a triple down-set by its 16 heights in $\{0, \ldots, 4\}$, nonincreasing along rows and columns. Recursively allowing each height up to the minimum of its predecessors enumerates all 232848 down-sets. A support 3-matching is forbidden when some coordinate does not use both low levels. The 2016 forbidden matchings have 51 minimal down-closures; excluding these leaves 26893 legal down-sets, with maximum size 32. The parallel pair census has 70 down-sets, 10 legal, with maximum size 4. These maxima and their antecedent classification are due to HHL.

For a fixed legal diagram, mark its wide extensions and all quadruples with at most one high coordinate present. Generate the upper-cube matching clauses and the seed clauses of its triples. A clause not already hit by a forced missing point must be hit outside the forced present set. Singleton residues force their point missing; add its up-closure and iterate. If the resulting forced missing set is F, any h pairwise disjoint residual clauses outside F imply

$$M \geq |F| + h. \tag{4.7}$$

For size 24, the exact certificate regenerates all representatives under coordinate permutations, all clauses, and every claimed disjointness:

Support size	Labelled diagrams	Orbits	Listed clauses	Consequence
24	1212	220	4338	$M \geq 56$

The size-28 implication is Theorem 4.15. The rectangle alternative is Corollary 4.11; it uses explicit slab and majority constructions and no stored packing. For a larger support, delete maximal elements until the required size is reached. Down-set structure and legality persist, and the smaller diagram's extensions and clauses remain necessary. This extends the size-24 certificate to every larger support. □

Lemma 4.18 (Persistence through eleven additions)**.** *Suppose* $R_{ij} \subseteq \mathcal{M}$*, every completion of a present trace is present, and every seed clause is hit. If* $M \leq 75$*, then* $T \leq 96$ *and* $P \leq 4$*. This assertion does not require an up-set.*

Proof. By permuting columns take $R = R_{01}$. For a pair or triple trace H, let $C(H)$ be its wide completions. If $C(H) \cap R \neq \varnothing$, the trace is impossible. Otherwise remove clauses already hit by R and delete $C(H)$ from each remaining clause. These residues must be hit by $\mathcal{M} \setminus R$.

The enumeration regenerates all 352 traces and 359424 seed clauses. Completion excludes 148 traces. Another 100 traces, comprising 96 triples and 4 pairs, require no new missing point by this test. For each of the remaining 104 traces the certificate lists 12 disjoint residual clauses outside R, 1248 clauses in total. Such a trace would require $M \geq 64 + 12 = 76$. Counting the only individually permitted traces proves the bounds for $M \leq 75$; simultaneous realizability is not assumed. □

The rectangle and its 96/4 cap for $M \leq 66$ already occur in HHL's C3 argument. The new content here is persistence up to 75 under the displayed premises. The global proof below only needs persistence up to 71.

4.7 Integer orbit counts in the top-star branch

The symmetry reduction must retain integrality of counts, even though averaged indicators need not be binary.

Lemma 4.19 (Orbit-count integrality)**.** *Let a finite group G permute a finite set Ω, and let $z \in \{0,1\}^{\Omega}$. If the group average of z has value x_j on an orbit O_j of size m_j, then*

$$m_j x_j = \sum_{\omega \in O_j} z_\omega \in \{0, 1, \dots, m_j\}.$$

Every necessary linear system closed under G remains valid on its averaged quotient. No invariance of z is required.

Proof. Each position in an orbit receives every orbit element equally often under the group action. Its average is therefore the displayed count divided by m_j. Average each inequality over its translates to obtain the quotient system; linearity preserves validity. □

Fix the fourth column of a missing top-star and let $G = S_3$ permute the other columns. There are 608 labelled indicators: 256 missing quadruples, 256 present triples and 96 present pairs. They form 166 orbits: 20 of size 1, 96 of size 3 and 50 of size 6. The row generator reconstructs 40345 necessary quotient inequalities from monotonicity, completion, matching obstructions, the support caps 32/4 and all seed residues. These are the adapted row families of HHL Appendix A.1. Their necessity is now explicit: Lemma 3.1 gives monotonicity and completion; Lemma 4.2 gives the pair/triple obstruction rows and caps. For each trace H, a disjoint seed and residual 3-matching Q_1, Q_2, Q_3 give $1_H \le \sum_i 1_{Q_i \text{ missing}}$, since otherwise five traces would be disjoint. Enumerating three available values per coordinate, anchored in the first coordinate and permuted in the others, gives every such residual row. A seed using an additional vertex in a trace-supported column leaves fewer than three values there and creates no residual 3-matching. These statements justify T1–T2 and T4–T9; the adapted cardinality and fixing rules below justify T3/T10. Averaging these labelled rows over S_3 is legitimate without assuming symmetry of the actual board.

The slice with fourth coordinate 3 has size 64, including the 27 missing top-star points. Partition by its number $c \in \{0, \dots, 37\}$ of present points. Add total budget $M \le 71$ and slice missing count $64 - c$. In the source's alternative notation, the new budget is lower missing − top present ≤ 7; the old bound 2 is not retained. Initial 0/1 forcing uses only up/down closure and these cardinalities: a missing point would force all its successors missing, and a present point all its predecessors present. A trial violating a forced value or a cardinality bound forces the opposite value. Iterate to a fixed point. The forced sets are S_3-invariant, hence give valid fixed bounds in the quotient; all remaining bounds are $[0, 1]$.

Lemma 4.20 (Integral top-star certificate)**.** *For an actual board with a missing top-star and $M \le 71$,*

$$\alpha T + \beta P \le M, \qquad \alpha = \frac{18}{35}, \quad \beta = \frac{18}{71}.$$

Proof. For each of the 38 slice classes, the certificate gives a binary tree over the valid initial box. At a split, an orbit count satisfies either $m_j x_j \le a$ or $m_j x_j \ge a + 1$, where a is an integer. Lemma 4.19 ensures that every actual board follows a child. The bounds of both children are retained and checked; no real-valued point outside the integer-count lattice needs to follow a child.

Here is the exact leaf calculation. Write the necessary rows as $a_i \cdot x \le b_i$, a leaf box as $\ell \le x \le u$, and its objective as $f(x) = \boldsymbol{c} \cdot x = M - \alpha T - \beta P$. For nonnegative integer multipliers z_i and scale $D_* = 2485000000$, put

$$r = D_* \boldsymbol{c} + \sum_i z_i a_i, \qquad L = \frac{-\sum_i z_i b_i + \sum_j \min\{r_j \ell_j, r_j u_j\}}{D_*}.$$

Every feasible point obeys

$$D_*f(x) = r \cdot x - \sum_i z_i(a_i \cdot x) \geq \sum_j \min\{r_j\ell_j, r_j u_j\} - \sum_i z_i b_i = D_*L.$$

The residual coefficients r_j are evaluated exactly, including their box penalty. For an infeasibility leaf take $\boldsymbol{c} = 0$; then $L > 0$ contradicts $0 \geq L$.

The verifier checks all regenerated row references and 8633 selected multipliers, every split, all roots, and absence of cycles, reused or unvisited nodes. The complete forest has 55 splits, 84 objective leaves and 9 infeasibility leaves: 148 nodes, maximum depth 8. The minimum objective lower bound is $574507909/4970000000 > 0$, and the minimum Farkas margin is $1/50$. Thus each actual board follows a path ending in an objective leaf that proves the desired inequality; an infeasibility leaf cannot contain it. No discovery solver is used as a premise. □

Proposition 4.21 (Wide-layer bound)**.** *Every actual selected board satisfies*

$$\frac{18}{35}T_3 + \frac{18}{71}P_2 \leq M_4. \tag{4.8}$$

Here the subscript is the number of columns met by a trace, not counting D.

Proof. If the upper cube has a present 3-matching, $T = P = 0$. Otherwise $T \leq 128$, $P \leq 24$. For $M \geq 72$ the margin $M - \alpha T - \beta P$ is at least $216/2485 > 0$. For $M \leq 71$ with a top-star use Lemma 4.20. In the remaining branch Lemma 4.16 gives $M \geq 33$ and the triple and pair caps through 47. The other cases are:

M	Triple cap	Pair cap	Minimum margin
48–55	$T \leq 4 \cdot 23$	$P = 0$	24/35
56–59	$T \leq 4 \cdot 27$	$P = 0$	16/35
60–63	$T \leq 4 \cdot 27$	$P \leq 6$	7296/2485
64–71, all supports ≤ 28	$T \leq 112$	$P \leq 24$	112/355
64–71, some support ≥ 29	$T \leq 96$	$P \leq 4$	33832/2485

The first three lines use Lemma 4.17 and the retained pair caps. In the last line that lemma forces an included rectangle, since $M < 80$, and Lemma 4.18 applies. These cases exhaust the range; no near-maximum gap is needed. □

4.8 Explicit residual witnesses on 15 and 11 vertices

For an actual board write T_i, P_i, M_i for present triples, present pairs and missing quadruples meeting exactly i columns. Fix $k \in \{2, 3\}$ columns together with D. On this subboard let $\widehat{M}, \widehat{T}, \widehat{P}$ count traces of cardinality and spread $(4, k), (3, k-1), (2, k-2)$, respectively.

Lemma 4.22 (Residual inequalities)**.** *On the subboards of an actual selected board,*

$$\frac{9}{16}\widehat{T} + \frac{9}{32}\widehat{P} \leq \widehat{M} \quad (k = 3), \qquad \frac{2}{3}\widehat{T} + \frac{1}{3}\widehat{P} \leq \widehat{M} \quad (k = 2).$$

Consequently,

$$\frac{9}{8}T_2 + \frac{27}{32}P_1 \leq M_3, \qquad 2T_1 + 2P_0 \leq M_2. \tag{4.9}$$

Proof. If $H, R_1, \ldots, R_k$ are disjoint, H is a triple or pair of the specified spread, and each R_i is a full column, low seed or quadruple of spread k, their indicators satisfy

$$1_{H \text{ present}} \leq \sum_{i:\, R_i \text{ has spread } k} 1_{R_i \text{ missing}}. \tag{4.10}$$

Otherwise these traces, with the $4-k$ outside full columns, form five disjoint present traces. This is HHL's residual-row mechanism.

Transport each literal row below under $\Gamma_k = S_3(D) \times ((S_2 \times S_2)^k \rtimes S_k)$, permuting D, columns, and levels within each low/high pair. The group preserves the carriers and fixed present sets, so every transported row is valid without symmetry of the actual assignment. Multiply each by its listed z, clear denominators by d_k, and sum all transports. The resulting labelled coefficients are:

k	d_k	$\lvert\Gamma_k\rvert$	Rows	Occurrences	Triple	Pair	Max. missing
3	32	2304	23	52992	41472	20736	73728
2	3	192	8	1536	384	192	576

Every labelled triple and pair has the displayed coefficient, and every missing quadruple has load at most the displayed maximum. The carrier sizes (M, T, P) are $(480, 288, 54)$ and $(260, 68, 3)$, with orbit counts $(13, 9, 5)$ and $(19, 7, 1)$. Orbits are determined by occupancy in D and the sorted low/high occupancies of the columns. These counts and all 54528 transports are checked explicitly. Divide by 73728 or 576 to obtain the subboard bounds.

Sum over the four three-column and six two-column subboards. The multiplicities for triples, pairs and missing quadruples are $(2, 3, 1)$ and $(3, 6, 1)$, respectively, giving (4.9). In an actual board $P_0 = 0$, because a pair in D and four full columns would already be five disjoint traces; keeping it displays the multiplicity. □

The eight 11-vertex geometries use the displayed primitive witnesses, with multipliers scaled by 4/3.

The 31 literal rows. Write $D = \{d_0, d_1, d_2\}$, with columns $\{a_0, a_1, a_2, a_3\}$, $\{b_0, b_1, b_2, b_3\}$, and, for $k = 3$, $\{c_0, c_1, c_2, c_3\}$. Juxtaposition denotes a set. Each row specifies $H, R_1, \ldots, R_k, z$ in (4.10); full columns and low seeds contribute no missing load. The tables decode the exact data.

15-vertex subboard					
Row	H	R_1	R_2	R_3	z
1	a_2a_3	$d_0d_1d_2a_0$	$b_0b_1b_2b_3$	$c_0c_1c_2c_3$	27/32
2	a_0a_2	$d_0d_1d_2a_1$	$b_0b_1b_2b_3$	$c_0c_1c_2c_3$	27/8
3	a_0a_1	$d_0d_1d_2b_0$	$a_2b_2b_3c_0$	$a_3b_1c_1c_3$	27/32
4	d_0a_2	$d_1a_3b_0c_0$	$a_0b_1b_2c_1$	$d_2a_1b_3c_3$	81/16
5	d_0a_0	$d_1a_2b_2c_0$	$a_1b_0b_3c_1$	$d_2a_3b_1c_3$	81/16
6	$a_2a_3b_2$	$d_0d_1d_2b_0$	$a_0b_1c_0c_1$	$a_1b_3c_2c_3$	27/4
7	$a_0a_2b_2$	$d_0d_1d_2b_0$	$a_3b_1c_0c_2$	$a_1b_3c_1c_3$	15
8	$a_0a_2b_2$	$d_0d_1d_2c_0$	$a_3b_0b_3c_1$	$a_1b_1c_2c_3$	12
9	$a_0a_1b_2$	$d_0d_1d_2c_0$	$a_2b_0b_3c_1$	$a_3b_1c_2c_3$	3/2
10	$a_0a_1b_2$	$d_0d_1d_2c_0$	$a_2b_0b_1c_2$	$a_3b_3c_1c_3$	21/4
11	$a_0b_2b_3$	$d_0d_1d_2a_1$	$a_2b_0c_0c_2$	$a_3b_1c_1c_3$	27/4
12	$a_0a_2b_0$	$d_0d_1d_2b_1$	$a_1b_2c_0c_2$	$a_3b_3c_1c_3$	219/32
13	$a_0a_2b_0$	$d_0d_1d_2c_0$	$a_1b_1b_2c_1$	$a_3b_3c_2c_3$	21/4
14	$a_0a_2b_0$	$d_0d_1d_2c_0$	$a_3b_1b_2c_1$	$a_1b_3c_2c_3$	477/32
15	$a_0a_1b_0$	$d_0d_1d_2b_1$	$a_2b_2c_0c_1$	$a_3b_3c_2c_3$	27/4
16	$d_0a_2b_2$	$d_1a_0b_0c_0$	$d_2a_3b_3c_2$	$a_1b_1c_1c_3$	93/32
17	$d_0a_2b_2$	$d_1a_3b_0c_0$	$d_2a_0b_1c_2$	$a_1b_3c_1c_3$	525/32
18	$d_0a_2b_2$	$a_0b_0c_0c_1$	$d_1a_3b_1c_2$	$d_2a_1b_3c_3$	15/16
19	$d_0a_0b_2$	$d_1a_2b_0c_0$	$d_2a_1b_1c_2$	$a_3b_3c_1c_3$	273/16
20	$d_0a_0b_2$	$a_2b_0c_0c_1$	$d_1a_3b_1c_2$	$d_2a_1b_3c_3$	375/16
21	$d_0a_0b_0$	$d_1a_1b_1c_0$	$d_2a_2b_2c_2$	$a_3b_3c_1c_3$	603/32
22	$d_0a_0b_0$	$a_1b_1c_0c_1$	$d_1a_2b_2c_2$	$d_2a_3b_3c_3$	27/32
23	$d_0a_0b_0$	$a_2b_1c_0c_1$	$d_1a_1b_2c_2$	$d_2a_3b_3c_3$	9/16

11-vertex subboard				
Row	H	R_1	R_2	z
1	d_0d_1	$a_0a_1a_2a_3$	$b_0b_1b_2b_3$	1
2	$a_0a_2a_3$	$d_0d_1d_2a_1$	$b_0b_1b_2b_3$	8/3
3	$a_0a_1a_2$	$d_0d_1d_2b_0$	$a_3b_1b_2b_3$	8/3
4	$d_0a_2a_3$	$d_1a_0b_0b_2$	$d_2a_1b_1b_3$	4
5	$d_0a_0a_2$	$d_1a_1b_0b_2$	$d_2a_3b_1b_3$	16
6	$d_0a_0a_1$	$d_1a_2b_0b_2$	$d_2a_3b_1b_3$	4
7	$d_0d_1a_2$	$d_2a_0b_0b_2$	$a_1a_3b_1b_3$	8
8	$d_0d_1a_0$	$d_2a_2b_0b_2$	$a_1a_3b_1b_3$	8

4.9 A common counting functional for propagation

On an actual board let T_z, P_z count present triple and pair traces meeting z columns, and let M_z count missing quadruples of spread z. There are no pair or triple traces entirely in D, so $P_0 = T_0 = 0$. We write $q = n - 4s - 3$ and use $q \geq 2$ below; the partition bound treats $q = 1$. All comparisons keep this same board fixed while varying the coefficients. Let $D_s = (s-1)(s-2)(s-3)$ and define $G(s,q) = -D_s\Phi_{s,n}$. Explicitly,

$$\begin{aligned} G(s,q) = {} & D_sM_4 + (s-1)(s-2)M_3 + 2(s-1)M_2 + 6M_1 \\ & - q(s-1)(s-2)T_3 - q(q-1)(s-1)P_2 - 2q(s-1)T_2 \\ & - 3q(q-1)P_1 - 6qT_1. \end{aligned} \tag{4.11}$$

The following layer inequalities proved above are used throughout:

$$M_4 \geq \frac{18}{35}T_3 + \frac{18}{71}P_2, \qquad M_3 \geq \frac{9}{8}T_2 + \frac{27}{32}P_1, \qquad M_2 \geq 2T_1. \tag{4.12}$$

For the cover-ray argument we also use the support bounds

$$T_3 \leq 128, \qquad P_2 \leq 24. \tag{4.13}$$

The scalar support caps have the direct proofs above. Signed support facets, the other support costs, and the remaining layer calculations retain their specified finite checks.

4.10 Two incidence inequalities with direct proofs

Lemma 4.23 (Completion and shifting incidences)**.** *On every actual selected board with $q \geq 2$,*

$$8P_2 \leq 3T_3, \qquad 3T_3 \leq 8T_2. \tag{4.14}$$

Proof. Each wide pair meets two columns. Adding one of the eight vertices in the other two columns produces a present exact wide triple: completion is valid because an external vertex is available. Count incidences of such a pair and one of its wide triple completions. There are $8P_2$ incidences. A wide triple contains at most three wide pairs, proving the first inequality.

For the second inequality, start with a present wide triple H and mark one of its three vertices x. Shift x to the distinguished residual vertex 1. This is a genuine lowering in the underlying ordered ground set. Keeping an external completion vertex unchanged shows that $\{1\} \cup (H \setminus \{x\})$ is a present exact triple meeting two columns. For any image, the deleted vertex must belong to one of the two unused board columns, giving at most eight preimages. Thus $3T_3 \leq 8T_2$. □

Remark 4.24. The second proof uses the actual minimum vertex, before any averaging that forgets which residual vertex is 1. The resulting count inequality is invariant under residual and column permutations, so it survives averaging. We do not assert that every point of the original linear relaxation satisfies an unrecorded labelled minimum-shift constraint.

4.11 Two fixed counting inequalities

The file `analytic_anchors.json.gz` supplies two fixed rational inequalities for the ordinary selected-board model. Their coefficients do not vary with s, q; the primitive rows have the meanings proved in this paper.

Lemma 4.25 (Fixed base and tangent)**.** *Under the ordinary-board necessary rows,*

$$12144M_4 + 552M_3 + 48M_2 + 6M_1 \geq 6624T_3 + 3168P_2 + 576T_2 + 396P_1 + 72T_1, \tag{4.15}$$

$$3172M_4 + 94M_3 \geq 1680T_3 + 816P_2 + 96T_2 + 69P_1. \tag{4.16}$$

Exact rational proof interface. The left side minus right side in (4.15) is $G(25, 12)$. Call the difference in (4.16) H. The supplied records use respectively 247 and 160 positive rational multipliers, each with denominator at most 100.

Here is the precise algebraic proof, so the certificates do not rely on solver tolerances. Let x_i be the orbit-presence coordinates. Change to defect coordinates $z_i = x_i$ for pairs and triples and $z_i = 1 - x_i$ for quadruples. The clique vector becomes $z = 0$. Coordinates fixed by the semantic bounds are zero in this new system; all free coordinates are nonnegative. Every row used in these two certificates is tight at the clique vector, so it becomes $a_r \cdot z \leq 0$. For the

target coefficient vector c, the supplied nonnegative rational multipliers y_r satisfy, on every free coordinate,

$$r = c + \sum_r y_r a_r \geq 0.$$

Consequently

$$c \cdot z = r \cdot z - \sum_r y_r (a_r \cdot z) \geq 0.$$

The minimum free residual for (4.15) is $9058789/414960 > 0$; for (4.16) it is zero, with all residuals nonnegative. The verifier reconstructs the primitive rows from their literal witnesses, the coordinate multiplicities, clique tightness, and every rational residual coefficient. Baseline rows are checked against the declared layer formulas; their mathematical and finite proofs are those of this section. □

4.12 A uniform clique band beginning at 25

Theorem 4.26 (Half-slope clique propagation)**.** *Let $s \geq 25$ be an integer and $2 \leq q \leq (s-1)/2$ an integer. For every actual selected board satisfying the layer premises, $G(s,q) \geq 0$. Hence every shifted saturated ONE family in this band satisfies $|\mathcal{F}| \leq A_4(s)$.*

Proof. First work on the formal ray $s = 25 + u$, $q = 12 + u/2$, $u \geq 0$. The board counts are fixed while the coefficients vary. This is polynomial comparison on one board, not the construction of a family with fractional parameters. Expanding (4.11) gives

$$G(25+u, 12+u/2) = G_0 + uG_1 + u^2 G_2 + u^3 G_3,$$

where

$$\begin{aligned}
G_0 &= G(25,12),\\
G_1 &= 1586M_4 + 47M_3 + 2M_2 - 840T_3 - 408P_2 - 48T_2 - \tfrac{69}{2}P_1 - 3T_1,\\
G_2 &= 69M_4 + M_3 - \tfrac{71}{2}T_3 - \tfrac{35}{2}P_2 - T_2 - \tfrac{3}{4}P_1,\\
G_3 &= M_4 - \tfrac{1}{2}T_3 - \tfrac{1}{4}P_2.
\end{aligned}$$

The first coefficient is nonnegative by (4.15). The next satisfies

$$G_1 = \tfrac{1}{2}H + 2M_2 - 3T_1 \geq 0$$

by (4.16) and $M_2 \geq 2T_1$. For the quadratic coefficient, the layer inequalities give

$$G_2 \geq -\frac{T_3}{70} - \frac{P_2}{142} + \frac{T_2}{8} + \frac{3P_1}{32}.$$

Apply (4.14); because the coefficient of P_2 is negative, its *upper* bound is used. This gives

$$G_2 \geq \left(\frac{3}{64} - \frac{1}{70} - \frac{3}{1136}\right) T_3 + \frac{3}{32}P_1 = \frac{4763}{159040}T_3 + \frac{3}{32}P_1 \geq 0. \tag{4.17}$$

Finally,

$$G_3 \geq \frac{T_3}{70} + \frac{P_2}{284} \geq 0.$$

All four Taylor coefficients are nonnegative, proving the entire ray, not merely a finite sample. For fixed s and nonnegative counts, $G(s,q)$ is nonincreasing for $q \geq 1$: both q and $q(q-1)$ are nondecreasing and occur only with negative trace coefficients. The band below the ray follows. Summing $\Phi = -G/D_s \leq 0$ in (3.1) proves the cardinality claim. □

4.13 Cover rays from two fixed endpoint anchors

For $s > 0$ define

$$\begin{aligned} Z(s,q) &= \frac{24}{s}\left[B_4(4s+3+q,s) - A_4(s)\right] \\ &= 4q^3 + 42q^2 s + 18q^2 + 148qs^2 + 126qs + 22q - 81s^3 - 162s^2 - 99s - 18, \end{aligned} \tag{4.18}$$

and set $J(s,q) = G(s,q) + Z(s,q)$, where Z is a constant per board. Since $\binom{s}{4}/D_s = s/24$, a boardwise inequality $J \geq 0$ implies

$$|\mathcal{F}| - A_4(s) \leq \binom{s}{4}\frac{Z(s,q)}{D_s} = B_4(n,s) - A_4(s).$$

We use the upper-endpoint certificates at $(s,n) = (33,152)$ and $(36,165)$, as fixed inequalities on the common four-column board. Exact residual-aware reconstruction gives the positive boardwise margins

$$\begin{aligned} J(33,17) &\geq m_o := \frac{37290250580546743}{1562500000000} > 0, \\ J(36,18) &\geq m_e := \frac{426057745890672159}{50000000000000} > 0. \end{aligned} \tag{4.19}$$

These follow from the rational upper bounds for Φ, before any integer rounding. All consumed rows have four-column domain; thus these are inequalities for the common board model, not assumptions about the existence of a smaller-parameter family associated to a larger board.

Theorem 4.27 (Two cover rays)**.** *The cover bound holds for actual shifted saturated ONE families at*

$$n = \frac{9s+7}{2} \quad (s \geq 33 \text{ odd}), \qquad n = \frac{9s+6}{2} \quad (s \geq 36 \text{ even}).$$

Proof. Expand J on the formal rays $(s,q) = (33+u, 17+u/2)$ and $(36+u, 18+u/2)$, respectively. Write $J = J_0 + uJ_1 + u^2J_2 + u^3J_3$. The constant coefficient is positive by (4.19). At either anchor (s_0,q_0), applying the layers to the coefficients of G gives

$$\begin{aligned} G_1 &\geq a_1T_3 + b_1P_2 + c_1T_2 + d_1P_1 + T_1, \\ G_2 &\geq a_2T_3 + b_2P_2 + \tfrac{1}{8}T_2 + \tfrac{3}{32}P_1, \\ G_3 &\geq \tfrac{1}{70}T_3 + \tfrac{1}{284}P_2, \end{aligned}$$

where the exact coefficients at the two anchors are

(s_0,q_0)	a_1	b_1	c_1	d_1	a_2	b_2
$(33,17)$	$-2969/35$	$-4924/71$	$39/8$	$117/32$	$-47/70$	$-131/142$
$(36,18)$	$-1889/35$	$-5615/142$	$53/8$	$183/32$	$-3/70$	$-111/284$

The Z coefficients, in ascending powers of u, are

$$(145792, 13196, 398, 4), \qquad (60966, 8660, 675/2, 4).$$

Using $T_3 \leq 128$, $P_2 \leq 24$ for negative coefficients and discarding nonnegative terms gives

Anchor	lower bound for J_1	lower bound for J_2	lower bound for J_3
$(33,17)$	$1673628/2485$	$720442/2485$	4
$(36,18)$	$1994568/2485$	$1603491/4970$	4

All are strictly positive. Thus $J \geq 0$ for every $u \geq 0$ on either ray. Restricting to integer ground-set sizes and summing proves the assertion. □

The strict margins in (4.19) suffice for propagation. The numerical ONE gaps in Section 5 continue to use their separately checked endpoint bounds.

4.14 Locating every critical point without a finite scan

Lemma 4.28 (Crossing localization). *For odd $s \geq 3$,*

$$a_s \leq \frac{9s+7}{2}.$$

For even $s \geq 26$,

$$a_s \leq \frac{9s+6}{2}.$$

Consequently, for every integer $s \geq 25$, the lower critical point satisfies $a_s - 1 \leq \lfloor (9s+5)/2 \rfloor$.

Proof. Substitution in (4.18) gives

$$Z\left(s, \frac{s+1}{2}\right) = 2(s^2-1)(2s+1) > 0 \qquad (s > 1).$$

At $q = s/2$, writing $s = 26 + u$ gives

$$Z\left(26+u, \frac{26+u}{2}\right) = 4116 + 3110u + \frac{435}{2}u^2 + 4u^3 > 0 \quad (u \geq 0).$$

For the stated parity, the corresponding n is integral. Monotonicity of $B_4(n, s)$ locates the crossing. Subtracting one and taking the integer part gives the last assertion. □

Corollary 4.29 (Critical endpoints from a finite anchor set). *All critical ONE inputs hold for every $s \geq 35$, and also for $s = 33$, using Theorems 4.26 and 4.27. Every lower critical input with $s \geq 25$ follows from Theorem 4.26.*

Proof. The lower point belongs to the clique band by Lemma 4.28. If the upper point also satisfies $2q \leq s - 1$, use that band again. Otherwise, integrality and the crossing upper bound leave just one possible value: $q = (s+1)/2$ in the odd case and $q = s/2$ in the even case. The relevant cover ray applies to odd $s \geq 33$ and even $s \geq 36$. □

For the remaining finite range retain both endpoints for $5 \leq s \leq 24$, the upper endpoints for $25 \leq s \leq 32$ and $s = 34$, and the two cover anchors $s = 33, 36$. These are 51 records: 15 ordinary, three minimum-preserving and 33 ordered trees. Together with the two fixed inequalities of Lemma 4.25, they give 53 endpoint-or-anchor obligations for the main cardinality argument. The polynomial argument supplies all other critical inputs. The four small full-carrier cases and the layer proofs remain separate dependencies.

4.15 The strong critical clique bound from the layers

Proposition 4.30 (Direct layer proof of a half-ray). *Using only (4.12) and (4.14), $G(s, q) \geq 0$ whenever $s \geq 71$ and $2 \leq q \leq (s-1)/2$.*

Proof. At $q = (s-1)/2$, substituting the three layer bounds in G gives $G \geq aT_3 + bP_2 + cT_2 + dP_1 + eT_1$, where

$$a = \frac{(s-73)(s-2)(s-1)}{70}, \qquad b = \frac{(s-73)(s-3)(s-1)}{284},$$
$$c = \frac{(s-10)(s-1)}{8}, \qquad d = \frac{3(s-1)(s+6)}{32}, \qquad e = s-1.$$

For $s \geq 73$ all are nonnegative. For $71 \leq s \leq 73$, $b \leq 0$, and (4.14) gives

$$G \geq \left(a + \frac{3}{8}b + \frac{3}{8}c\right) T_3 + dP_1 + eT_1.$$

The coefficient in parentheses equals

$$\frac{(s-1)(2482s^2-178905s+303152)}{159040}.$$

Writing $s = 71+u$, its quadratic factor is $112659+173539u+2482u^2 > 0$. Monotonicity in q completes the proof. □

Proof of Proposition 4.1. The following identities locate the upper crossing:

$$Z\left(49+u, \frac{48+u}{2}\right) = 4728+9946u+397u^2+4u^3 > 0,$$
$$Z\left(74+u, \frac{72+u}{2}\right) = 32904+22978u+\frac{1201}{2}u^2+4u^3 > 0 \qquad (u \geq 0).$$

For odd $s \geq 73$ the first gives $q \leq (s-1)/2$ at $n = a_s$; for even $s \geq 74$ the second gives $q \leq (s-2)/2$. Both critical points therefore lie in the band of Proposition 4.30. Summing $-G/D_s$ in (3.1) proves $|\mathcal{F}| \leq A_4(s)$. This proof uses the layer inequalities and the incidence lemma; it does not use the two fixed anchors. □

5 Structural consequences of the global proof

The strict margins in the preceding proof also constrain extremizers. The following conclusions have explicitly stated domains; they do not classify every crossing case or transfer quantitative stability through arbitrary shifts.

5.1 ONE gaps and their propagation

Proposition 5.1 (All finite upper gaps)**.** *For $5 \leq s \leq 72$, let U_s be the certified rational upper bound on $|\mathcal{F}| - A_4(s)$ at $n = a_s$, including the board multiplicity. Put $h_s = B_4(a_s, s) - A_4(s) - \lfloor U_s \rfloor$. All 68 integers h_s are positive; their minimum is 1 at $s = 5$. Every shifted ONE family of matching number exactly s on $n \geq a_s$ vertices satisfies*

$$|\mathcal{F}| \leq B_4(n, s) - h_s.$$

For $73 \leq s \leq 121$, the same conclusion holds with $h_s = B_4(a_s, s) - A_4(s) > 0$; the minimum of these 49 gaps is 75175 at $s = 97$.

Proof. At the base, saturation preserves the matching avoiding 1. For $5 \leq s \leq 72$, the 52 ordinary, three minimum-preserving and 33 order-tree endpoint records retained in the supplement supply the upper bounds. Exact global rounding gives the displayed h_s. The complete replay extracts all 68 fractions from these records and checks their positivity. The 23 values in Corollary 3.5 are a subset. For $73 \leq s \leq 121$, Proposition 4.1 gives $|\mathcal{F}| \leq A_4(s)$; exact binomial subtraction at the 49 crossings gives the asserted positive differences.

Write $\mathcal{F}[N] = \mathcal{F} \cap \binom{[N]}{4}$. A matching of s edges avoiding 1 can be moved, by replacements with unused smaller vertices, into $\{2, \ldots, 4s+1\}$. Shiftedness and disjointness persist, so $\mathcal{F}[a_s]$ still has matching s and ONE. For every $N \geq a_s$, the last-vertex link argument of Lemma 2.4 gives

$$B_4(N+1, s) - |\mathcal{F}[N+1]| \geq B_4(N, s) - |\mathcal{F}[N]|.$$

The added link has at most $B_3(N, s)$ edges, and Pascal's identity gives exactly that increase in the cover size. Thus the deficit at the base cannot be lost. □

5.2 The stronger weight surplus for stability

For reference the six layer comparisons are

$$\begin{aligned} \frac{q}{s-3} &\le \frac{18}{35}, & \frac{q(q-1)}{(s-2)(s-3)} &\le \frac{18}{71}, \\ \frac{2q}{s-2} &\le \frac{9}{8}, & \frac{3q(q-1)}{(s-1)(s-2)} &\le \frac{27}{32}, \\ \frac{3q}{s-1} &\le 2, & \frac{6q(q-1)}{s(s-1)} &\le 2. \end{aligned} \tag{5.1}$$

Lemma 5.2 (Structural weight surplus)**.** *Let $s \ge 122$ and $4s+4 \le n \le a_s$. The six left sides of (5.1), in row order, are bounded componentwise by $(1/2, 1/4, 1, 3/4, 3/2, 3/2)$.*

Proof. For $s \ge 122$ we prove

$$a_s \le N_s := \left\lfloor \frac{9s+3}{2} \right\rfloor.$$

Write $P(s) = 24[B_4(N_s, s) - A_4(s)]$. For $s = 2u$, $N_s = 9u+1$, direct binomial subtraction gives

$$P(2u) = 64u^4 - 3844u^3 - 736u^2 - 44u.$$

At $u = 61 + v$ this becomes

$$10877520 + 15106728v + 724676v^2 + 11772v^3 + 64v^4 > 0 \qquad (v \ge 0).$$

For $s = 2u+1$, $N_s = 9u+6$, one obtains

$$P(2u+1) = 64u^4 - 2944u^3 - 5296u^2 - 3104u - 600.$$

At $u = 48 + v$ this becomes

$$1804200 + 7451104v + 455504v^2 + 9344v^3 + 64v^4 > 0 \qquad (v \ge 0).$$

The odd case holds already for $s \ge 97$; the common domain $s \ge 122$ suffices. Positivity and monotonicity of $B_4(n, s)$ give the asserted crossing upper bound. Hence at both critical choices

$$q = n - 4s - 3 \le \frac{s-3}{2}.$$

Using also $q(q-1) \le (s-3)(s-5)/4$, the six left sides of (5.1) are bounded componentwise by

$$(1/2,\ 1/4,\ 1,\ 3/4,\ 3/2,\ 3/2),$$

strictly below the respective right sides. □

The polynomial checker reconstructs both identities from falling products. These stronger coefficients are used for stability below; cardinality already follows from the wider band in Section 4.9.

5.3 Clique stability and equality

The source of stability is a surplus in the same layer inequalities that prove cardinality. If S_τ is the small-trace contribution and W_τ is the missing-quadruple contribution on a board, then $W_\tau \ge (1 + 1/71)S_\tau$. Summing the board identity gives $d = \sum_\tau (W_\tau - S_\tau) \ge X/71$, because the trace multiplicities identify $\sum_\tau S_\tau$ with the number X of external edges. The proof below establishes this surplus with the six explicit coefficient comparisons.

Theorem 5.3 (Explicit clique stability for shifted ONE families)**.** *Let $s \geq 122$, $4s+4 \leq n \leq a_s$, and let $\mathcal{F}$ be shifted with matching number exactly s and Property ONE. Put $K = [4s+3]$, $d = A_4(s) - |\mathcal{F}|$ and $X = |\mathcal{F} \setminus \binom{K}{4}|$. Then $d \geq 0$ and*

$$X \leq 71d, \qquad \left|\mathcal{F} \triangle \binom{K}{4}\right| \leq 143d.$$

Proof. First take a saturated family at $n = a_s$. Set $\lambda_4 = 1$, $\lambda_3 = 1/(s-3)$ and $\lambda_2 = 1/\binom{s-2}{2}$. Let $e_1, \dots, e_6$ be the six left sides of (5.1), in row order. On a board τ, the total positive contribution of its small traces is

$$S_\tau = \lambda_4(e_1T_3 + e_2P_2) + \lambda_3(e_3T_2 + e_4P_1) + \lambda_2(e_5T_1 + e_6P_0).$$

The notation T_z, P_z is that of the layer inequalities. Let W_τ be the total weighted contribution of all missing quadruples, so the coupled functional is $S_\tau - W_\tau$.

The proof of Lemma 5.2 bounds e by $v = (1/2, 1/4, 1, 3/4, 3/2, 3/2)$ componentwise. The layer coefficients are $c = (18/35, 18/71, 9/8, 27/32, 2, 2)$, and

$$(c_i/v_i)_{i=1}^6 = (36/35, 72/71, 9/8, 9/8, 4/3, 4/3).$$

Thus $c_i \geq (72/71)e_i$. Multiply the three layer inequalities by $\lambda_4, \lambda_3, \lambda_2$. The weighted missing quadruples of spreads 4, 3 and 2 contribute at least $72S_\tau/71$; those of spread 1 contribute nonnegatively. Hence $W_\tau - S_\tau \geq S_\tau/71$. By (3.1), $d = \sum_\tau (W_\tau - S_\tau)$, whereas trace multiplicity gives

$$\sum_\tau S_\tau = \sum_{\substack{H \text{ a trace} \\ |H|=2,3}} \binom{n-4s-3}{4-|H|} = X.$$

Indeed Lemma 3.1 gives one small trace per edge outside K, with all its completions present by saturation. It follows that $X \leq 71d$.

Start with the original family, add isolated vertices up to a_s, then enlarge to a saturated shifted $\mathcal{G} \supseteq \mathcal{F}$ of the same matching number, as in Section 3.1. The original matching avoiding 1 persists, and the fixed set K does not change. Therefore

$$X(\mathcal{F}) \leq X(\mathcal{G}) \leq 71(A_4(s) - |\mathcal{G}|) \leq 71(A_4(s) - |\mathcal{F}|).$$

This also gives $d \geq 0$. Finally $\left|\mathcal{F} \triangle \binom{K}{4}\right| = d + 2X$. □

The constants are sufficient, without an optimality assertion. The quantitative statement requires both shiftedness and ONE.

Proposition 5.4 (Clique rigidity from the strict anchor)**.** *Let $s \geq 25$, $4s+4 \leq n < a_s$, and let $\mathcal{F}$ be shifted with matching number exactly s and Property ONE. If $|\mathcal{F}| = A_4(s)$, then $\mathcal{F} = \binom{[4s+3]}{4}$.*

Proof. First prove rigidity for saturated shifted ONE families of size $A_4(s)$ throughout the band $2 \leq q = n - 4s - 3 \leq (s-1)/2$. On each selected board, the proof of Theorem 4.26 gives $G(s,q) \geq G(25,12)$: all subsequent coefficients on the half-slope ray are nonnegative, and G is nonincreasing in q. The strict residual in Lemma 4.25 gives

$$G(25,12) \geq r_0 \sum_{i \text{ free}} z_i, \qquad r_0 = \frac{9058789}{414960} > 0,$$

where $z_i = x_i$ for small traces and $z_i = 1 - x_i$ for quadruples, as in the proof of that lemma. By (3.1),

$$\sum_\tau G_\tau(s,q) = D_s(A_4(s) - |\mathcal{F}|) = 0.$$

All summands are nonnegative, so every free small-trace coordinate vanishes on every board. These coordinates are averages of indicators; thus all corresponding traces are absent. The small traces contained entirely in D are already excluded by Lemma 3.1.

Every trace meets at most four matching columns, which can be extended to a selection of four columns since $s \geq 25$. Hence no small trace escapes these boards. Lemma 3.1 then excludes every edge outside $K = [4s+3]$: such an edge has a trace of size 2 or 3. Since $|\mathcal{F}| = \binom{|K|}{4}$, all quadruples of K are present.

For a general shifted ONE family in the stated domain, Lemma 4.28 gives

$$n \leq a_s - 1 \leq N_s = \left\lfloor \frac{9s+5}{2} \right\rfloor.$$

Add isolated vertices up to N_s, then enlarge to a saturated shifted family $\mathcal{G} \supseteq \mathcal{F}$ with the same matching number, as in Section 3.1. The matching avoiding vertex 1 persists, so $\mathcal{G}$ is ONE. At this new parameter, $q_* = \lfloor (s-1)/2 \rfloor \geq 2$, and Theorem 4.26 gives $|\mathcal{G}| \leq A_4(s) = |\mathcal{F}|$. Thus no edges were added and the preceding argument identifies $\mathcal{F}$ with the clique. This also treats $q = 1$ without applying the incidence inequalities at that boundary. □

This equality statement uses the strict anchor, while the constants $71,143$ retain the stronger hypotheses of Theorem 5.3.

Lemma 5.5 (Inverting a shift at a complete clique)**.** *Let $s \geq 2$, $|T| = 4s+3$, and $\nu(\mathcal{H}) \leq s$. If $S_{ij}(\mathcal{H}) = \binom{T}{4}$, then $\mathcal{H}$ is a complete clique on $4s+3$ vertices.*

Proof. A shift preserves the number of edges in each i/j-exchange orbit. If i, j are both in T or both outside, all orbits are forced. The case $j \in T, i \notin T$ is impossible because the image of S_{ij} is fixed by that shift. Suppose $i \in T, j \notin T$, and put $R = T \setminus \{i\}$, so $|R| = 4s+2 \geq 10$. Every quadruple of R is forced present. For each triple $U \subset R$, precisely one of $\{i\} \cup U, \{j\} \cup U$ is present; all other types of edges are forced absent. If both resulting classes are nonempty, they have disjoint representatives: for intersecting representatives U, V, a triple outside $U \cup V$ exists, and its class supplies a disjoint pair of opposite classes. The corresponding two edges, together with a partition of the $4s-4$ remaining vertices of R into $s-1$ quadruples, give a forbidden matching. Thus one class is empty, identifying one of the two complete cliques. □

Corollary 5.6 (Arbitrary extremizers strictly below the crossing)**.** *For $s \geq 25$, $4s+4 \leq n < a_s$, and $\nu(\mathcal{F}) \leq s$, equality $|\mathcal{F}| = A_4(s)$ holds if and only if $\mathcal{F}$ is a complete clique on $4s+3$ vertices.*

Proof. Shift an extremal family to $\mathcal{F}'$. Its matching number is exactly s: otherwise adding a missing edge keeps matching at most s, contradicting Theorem 1.1. A missing edge exists since $n \geq 4s+4$. If ONE fails, delete vertex 1; the global theorem bounds the remainder by $M(n-1, s-1)$. In the cover case, restoring all possible incident edges gives $B_4(n,s) < A_4(s)$. In the clique case, Lemma 4.28 gives $n-1 \leq (9s+3)/2 \leq 5s-2$. The positive polynomial in the proof of Lemma 2.3 therefore gives $A_4(s-1) + \binom{n-1}{3} < A_4(s)$. Both contradict extremality, so $\mathcal{F}'$ is ONE. Proposition 5.4 identifies it with $\binom{[4s+3]}{4}$. Invert the shifts one at a time by Lemma 5.5; every intermediate family has matching number at most s. The converse is immediate. □

5.4 Cover uniqueness near the crossing

Lemma 5.7 (ONE cannot attain the upper cover)**.** *A shifted ONE family with matching number s has size strictly less than $B_4(n,s)$ if $5 \leq s \leq 121$, $n \geq a_s$, or if $s \geq 122$, $n \geq a_s + 1$.*

Proof. The first range is Proposition 5.1. For $s \geq 122$, suppose equality occurs at $n > a_s$. The deficit monotonicity proved there and the global bound force equality with the cover on $[a_s]$.

Proposition 4.1, after saturation, bounds this ONE restriction by $A_4(s)$. If $B_4(a_s, s) > A_4(s)$, this is impossible. In the remaining tie, Theorem 5.3 identifies the restriction as the complete clique on $K = [4s+3]$. An edge using a vertex outside K leaves at least $4s$ vertices of K free, and together with a partition of those vertices gives a forbidden matching. There are no such edges, contradicting the strict increase of $B_4(n, s)$ in n. □

Lemma 5.8 (Inverting a shift at a complete cover)**.** *Let $t \geq 2$, $n \geq 4t+4$, and $\nu(\mathcal{F}) \leq t$. If one usual shift S_{ij}, from j to $i < j$, sends $\mathcal{F}$ to the complete cover $\mathcal{C}_T$ of a t-set T, then $\mathcal{F}$ is a complete cover of some t-set.*

Proof. A shift preserves the number of edges in each i/j-exchange orbit. Every fixed orbit and every orbit having zero or two cover edges is therefore forced. If i, j are both in T or both outside it, this already gives $\mathcal{F} = \mathcal{C}_T$. The case $j \in T, i \notin T$ is impossible, because the image of S_{ij} is fixed by that shift and this cover is not.

Suppose $i \in T, j \notin T$. Put $K = T \setminus \{i\}$ and $W = [n] \setminus (K \cup \{i, j\})$, so $m = |W| = n-t-1 \geq 3t+3 \geq 9$. Every edge meeting K is forced in $\mathcal{F}$. For each $U \in \binom{W}{3}$, exactly one of $\{i\} \cup U$ and $\{j\} \cup U$ lies in $\mathcal{F}$, partitioning the triples into two classes. If both classes are nonempty, they have disjoint representatives: start with one of each; if they intersect, a triple outside their union exists since $m \geq 9$, and its class supplies the required pair. The resulting two four-edges, together with $t-1$ edges using distinct vertices of K and $3(t-1)$ vertices of W outside those representatives, form a forbidden matching. The latter vertices exist because $m - 6 \geq 3(t-1)$. Thus one class is empty. The forced orbits now identify $\mathcal{F}$ as $\mathcal{C}_{K \cup \{i\}}$ or $\mathcal{C}_{K \cup \{j\}}$. □

Theorem 5.9 (Complete covers in an explicit finite domain)**.** *Let $\nu(\mathcal{F}) \leq s$. In each of the following domains, $|\mathcal{F}| = B_4(n, s)$ holds if and only if $\mathcal{F}$ is the complete cover of an s-set:*

$$\begin{array}{ll} s = 5, & n \geq 33, \\ s = 6, & n \geq 34, \\ 7 \leq s \leq 121, & n \geq a_s, \\ s \geq 122, & n \geq a_s + 1. \end{array}$$

This conclusion additionally uses Frankl's cover-uniqueness theorem [3, Theorem 1.1] to start at $s = 4,\ n \geq 32$.

Proof. First $a_s \geq a_{s-1} + 1$ for $s \geq 2$. By (2.3), $a_s \leq 5s + 4$. Writing $u = s - 2 \geq 0$, direct subtraction gives

$$24 \left[A_4(s) - A_4(s-1) - \binom{5s+3}{3} \right] = 216 + 2356u + 2160u^2 + 524u^3 > 0.$$

Consequently Pascal's identity implies

$$B_4(a_s - 1, s-1) = B_4(a_s, s) - \binom{a_s - 1}{3} > A_4(s-1).$$

This parameter is admissible, so the crossing claim follows.

Frankl's theorem states cover uniqueness for exact matching u when $n \geq 7u + 4$. At $s = 4, n \geq 32$, apply it to the actual matching number $u \leq 4$. If $u < 4$, the bound $B_4(n, u) < B_4(n, 4)$ excludes equality (the empty case is immediate). Set $e_4 = 32$, and recursively

$$e_s = \begin{cases} \max\{e_{s-1} + 1, a_s\}, & 5 \leq s \leq 121, \\ \max\{e_{s-1} + 1, a_s + 1\}, & s \geq 122. \end{cases}$$

The crossings $a_5 = 26, a_6 = 31, a_7 = 35$ give $e_5 = 33, e_6 = 34, e_7 = a_7$. The crossing increment gives precisely the displayed domains.

Induct on s, and shift an extremal family with $n \geq e_s$. Its matching is exactly s, by the missing-edge argument above. Lemma 5.7 rules out ONE, so deleting vertex 1 leaves a family $\mathcal{F}_0$ with matching exactly $s-1$. Since $n-1 \geq e_{s-1}$, both brackets below are nonnegative:

$$B_4(n,s) - |\mathcal{F}| = \big[B_4(n-1,s-1) - |\mathcal{F}_0|\big] + \big[\binom{n-1}{3} - \deg_{\mathcal{F}}(1)\big].$$

Equality forces a complete $(s-1)$-cover on the remaining vertices by induction, and every four-edge through 1 is present. This gives a complete s-cover. Lemma 5.8 inverts each shift in the original family; its domain $s \geq 2, n \geq 4s+4$ is satisfied. The converse follows from the cover construction. □

These exact-equality conclusions do not imply quantitative stability for arbitrary families. The crossing itself for $s \geq 122$ and the remaining small-parameter equality cases are not classified here.

6 Two elementary applications

These consequences illustrate the cardinality theorem. The probabilistic limit already follows from HHL's large-s theorem and the classical matching–probability connection [14, Conjecture 1.1 and Lemma 3.2]. It is not a new consequence made possible by the small cases.

Corollary 6.1 (Finite weights and four i.i.d. variables)**.** *For nonnegative weights $w_1, \dots, w_n$, set $s = \lfloor \sum_i w_i \rfloor$. If $s \geq 1$ and $n \geq 4s+4$, at most $M(n,s)$ four-subsets have weight sum at least 1. If $s = 0$, none do.*

For independent, identically distributed nonnegative $X_1, \dots, X_4$ with common mean at most x, the supremum over common laws is

$$\sup \Pr(X_1+X_2+X_3+X_4 \geq 1) = \begin{cases} \max\{1-(1-x)^4, (4x)^4\}, & 0 \leq x \leq 1/4, \\ 1, & x \geq 1/4. \end{cases}$$

No classification of all optimizing laws is asserted.

Proof. The weighted four-subsets form a family in which every matching has size at most the total weight, proving the finite assertion.

For the limit, truncating a variable at 1 preserves the event and can only lower its mean. Fix $0 < x < 1/4$ and $0 < \varepsilon < 1/4 - x$, and sample N truncated variables. When their empirical mean is at most $x+\varepsilon$, apply Theorem 1.1 with $s_N = \lfloor N(x+\varepsilon) \rfloor$; $N \geq 4s_N + 4$ and $s_N \geq 1$ hold for sufficiently large N. The expected density of the resulting four-graph is the probability in question. Chebyshev's inequality, using variance at most $1/4$, gives

$$\Pr(X_1 + \cdots + X_4 \geq 1) \leq \frac{M(N, s_N)}{\binom{N}{4}} + \frac{1}{4N\varepsilon^2}.$$

Let $N \to \infty$, then $\varepsilon \downarrow 0$. The ratios converge to the two displayed terms. The laws $(1-x)\delta_0 + x\delta_1$ and $(1-4x)\delta_0 + 4x\delta_{1/4}$ attain them. The boundary values and $x > 1/4$ follow directly. Since $s_N \to \infty$, HHL's threshold $s_N \geq 6961$ would also suffice throughout this limit argument. □

Corollary 6.2 (An explicit supersaturation bound)**.** *Let $s \geq 1$, $n \geq m \geq 4(s+1)$, and let $L_{s+1}(\mathcal{F})$ count unordered matchings of $s+1$ edges in $\mathcal{F} \subseteq \binom{[n]}{4}$. Then*

$$L_{s+1}(\mathcal{F}) \geq \max\left\{0, \left\lceil \frac{|\mathcal{F}|\binom{n-4}{m-4} - M(m,s)\binom{n}{m}}{\binom{n-4(s+1)}{m-4(s+1)}} \right\rceil \right\}.$$

One may maximize over the integer choices of m.

Proof. For each m-set V, choose one edge from every forbidden matching in $\mathcal{F}[V]$, and remove the union of these chosen edges. At most $L_{s+1}(\mathcal{F}[V])$ edges are removed; the remaining family has matching number at most s. Theorem 1.1 gives $L_{s+1}(\mathcal{F}[V]) \geq |\mathcal{F}[V]| - M(m,s)$. Summing over V, each edge is counted $\binom{n-4}{m-4}$ times and each unordered matching is counted $\binom{n-4(s+1)}{m-4(s+1)}$ times. Divide and round upward. □

For example, $n = 30, s = 2, |\mathcal{F}| = 15000, m = 13$ gives at least 42261542 unordered three-edge matchings. This is the usual deletion and double-counting argument; no optimality or new general supersaturation principle is claimed.

7 Comparison with earlier results

7.1 Cardinality and the finite-board method

Table 2 separates the ranges of the principal antecedents. The theorem locators refer to the identified preprint versions where the published text has not been checked in full.

Source	Domain	Relation to this paper
Frankl [13], Thm. 1.1	Uniformity three, all parameters	Complete earlier uniformity; Prop. 5.1 gives the deletion principle
FRR [4], Thm. 1	Uniformity three, $s \geq 33$	Short proof; the restricted triple bound is used only in Appendix A
HHL [12], Thm. 1.1	Uniformity four, $s \geq 6961$, all admissible n	Board framework and exact computations developed here for all s
FLMW [10], Thm. 1.2; CLZ [11], Thm. 1.2	Uniformity four, sufficiently large parameters, $n \geq 5s$	Cover-side range; does not cover every crossing parameter
Guo–Lu [2], Thm. 1.3, Cor. 1.5	$s > s_0(k)$, $ks+k \leq n \leq (1+\beta)ks+k-2$	Arbitrary-family clique-side containment above their density threshold
SciNet [15]	Four finite critical points	Earlier reports of $385, 1379, 3876, 4255$; their solvers are not imported

Table 2: Statement-level comparison. Asymptotic constants and restricted family classes are not interchangeable with explicit finite thresholds. The table does not establish an exhaustive priority claim.

Frankl's later author copy [13, Section 13, Theorem 13.2] also announces the four-uniform conjecture for sufficiently large matching number, with a proof deferred to future work. That announcement is an antecedent distinct from a proof supplied in that text and from the all-parameter conclusion here. The direct-deletion and ONE reductions are attributed at their use; the polynomial proof in Lemma 2.3 is retained for its explicit strict inequality. The withdrawn complete-solution claim [16] is not a theorem input; its withdrawal does not determine another work's priority.

Our finite certificates couple the selected-residual constraints with actual local order, integer orbit counts, support costs and rectangle persistence. The exact-trace and selected-residual mechanisms follow [12, Sections 2 and 4]. The support census, support maxima and residual-row method likewise have HHL antecedents. HHL's C3 gives the rectangle cap $96/4$ through 66 missing points; Lemma 4.18 extends it to 75 under its displayed premises. Neither averaging nor orbit-count integrality is proposed as a new general principle.

7.2 Clique-side structure

Guo–Lu [2, Theorem 1.3 and Corollary 1.5, arXiv v1] work for $k \geq 3$, sufficiently large s, and $ks+k \leq n \leq (1+\beta)ks+k-2$, with $\beta > 0$ depending on k. Their non-trivial families have no

isolated vertices. Above the maximum sizes of their two explicit non-trivial constructions, an arbitrary family with $\nu \le s$ is contained in a clique on $ks + k - 1$ vertices. This is a directly relevant, stronger conclusion in its density regime. Theorem 5.3 instead gives a linear deficit bound with constants $71, 143$ and cutoff $s \ge 122$, under shifted ONE and throughout its stated crossing interval. It does not transfer that quantitative estimate to arbitrary families. The results have different hypotheses and types of control; neither an automatic absorption nor overall novelty follows from this comparison.

7.3 Local stability and weighted graphs

The section and shadow decomposition follows Frankl [3, Sections 3–5]. Nested weighted section estimates and canonical diversity have established antecedents [5, Section 8]. The finite variance is imported from that paper; its Theorem 32 already gives the asymptotic coefficient 5.74 in uniformity four. Proposition A.10 supplies an explicit cutoff using the inequalities displayed here. Wang [7, Lemmas 2.1–2.2, 3.2 and proof of Theorem 1.2] provides the graph compression and the container/convexity antecedent. For the signed functional $K_3 - \lambda e$, completing a graph to a container need not increase its value; the weighted count in Lemma A.8 is therefore retained.

A direct stability comparison is Frankl–Lu–Ma–Wu [10, Definition 1.4 and Theorem 1.5]. For fixed $c_0 \in (0, 1/5)$, sufficiently small $\varepsilon > 0$ depending on c_0, and $n \ge n_0(\varepsilon, c_0)$, their theorem applies when $c_0 n \le t \le n/5$ and gives, for shifted families with $\nu(\mathcal{F}) \le t$,

$$d_t(\mathcal{F}) \le \varepsilon n^4 \quad \Longrightarrow \quad |\mathcal{C}_t \setminus \mathcal{F}| < 400\varepsilon^{1/4} n^4.$$

Their distance is asymmetric. Our conclusions instead give $|\mathcal{C}_t \setminus \mathcal{F}| \le (K+1) d_t(\mathcal{F})$, with explicit K and bases. The asymptotic theorem has a wider ground-set domain; its stated cutoff does not certify each finite base here, and its quantifiers do not justify sending ε to zero at fixed n. Their cardinality Theorem 1.2 does have an explicit threshold: the end of their Section 4 gives $n_0 = 10^{10^7}$, together with $n \ge 5t$. This does not certify our finite bases; the stability theorem retains its separate dependence on ε and c_0.

Cao–Liu–Zhang [11, Theorem 1.4] prove stability for arbitrary families in every compact subinterval of $0 < t/n \le 1/5$: given a distance tolerance, a sufficiently small density deficit and a sufficiently large n force small symmetric difference from some cover. Their Theorem 1.2 also classifies exact extremizers for all $t \ge t_0(4)$, $n \ge 5t$. Thus the qualitative form of Corollary A.3 is already available in a wider asymptotic range. Their revised version of 7 September 2026 retains these four-uniform statements; the displayed coefficient $k + 0.6$ is stated for uniformity $k \ge 5$. Its parameterized conclusions still require sufficiently large matching number. The contribution here is its explicit finite domain together with the linear shifted estimates; no all-domain improvement is asserted.

Frankl's uniqueness statement at $n \ge 7t + 4$ [3, Theorem 1.1] and the finite hypothesis of Frankl–Kupavskii [8, Theorem 3], which here requires $n \ge 12t + 8$, do not include our bases. Their diversity theorem [9, Section 7, Theorem 28] has a sufficiently-large-n hypothesis. The displayed estimates do not resolve the general diversity problem, and no novelty is attributed to shifted completion, reverse shifting, or the classical clique-cap theorem.

The compatibility argument in Section A.6 averages constraints from all submatchings of one matching, then uses the same deficit to bound every later link. The density-adapted centre in Lemma A.13 preserves the two-endpoint proof while sharpening the bound. These two ingredients account for the smaller finite bases, including the additional points $(2904, 15676)$, $(888, 4798)$, $(360, 1950)$, $(72, 395)$, and, with the one-sided variance and all-weight graph bound, $(49, 269)$. The first-deficit example has a nonempty tail of nonpositive weight; it does not establish sharpness for a positive tail. The constants and cutoffs are sufficient, not optimal. Proposition A.15 extends shifted stability above a base with a growing coefficient by

monotonicity of the deficit and double counting. It is an elementary consequence of the base theorem; no priority is claimed for this propagation principle. Quantitative transfer through arbitrary shifts would require additional arguments.

8 Computational specifications and reproducibility

8.1 Verification command and scope

The computational supplement has one verification entry point:

```
python -X utf8 -S -B scripts/verify.py
    --hhl-bundle EMC_certificate.zip --zig PATH_TO_ZIG
    --node PATH_TO_NODE --output build/check
```

The displayed lines form one command, run from the repository root. It uses standard-library Python 3.12 or later, Zig 0.16.0 and Node 24. The dependencies and hash-pinned HHL archive are supplied before the offline verification. No optimization solver is needed to check the certificates.

The default command verifies every computational component of this article. It recompiles and executes C1 and all seven consumed C2 cases, checks the support and layer inequalities, verifies the four small points, the 51 endpoint records and two fixed inequalities used for cardinality, and checks their polynomial propagation. It also verifies all 88 endpoint records retained for the structural results and the stronger stability weights. It extracts the 68 strict upper margins and then verifies the seven computational components of Appendix A, including their stability normalizations. Complete success emits

`PASS_ALL_ARTICLE_COMPUTATIONS.`

Missing or changed inputs, failed processes, incomplete coverage, malformed output and timeouts prevent this marker. The new output directory contains a receipt, complete process logs and the exact endpoint bounds. Failure controls are supplied in `tests/`.

Obligation	Mathematical specification	Computational evidence
C1 and seven C2 cases	Lemma 4.16	Exhaustive C++ searches; thresholds and complete transcripts
Support facets and layer costs	Lemmas 4.2–4.20; Proposition 4.21	Finite support census, 21 signed facets and integer-orbit tree
Residual inequalities	Lemma 4.22	31 primitive rows, 54528 labelled transports
Small critical inputs	Proposition 3.6	Four labelled trees, 94 exact dual leaves
Intermediate inputs	Section 3	15 ordinary and 3 minimum-preserving records; 33 order trees
Analytic propagation	Section 4.9	Two fixed inequalities; four-column anchor checks and exact polynomial identities

Table 3: Computations needed by the global cardinality proof. Exact row arithmetic is separated from the mathematical proof that every actual configuration satisfies those rows.

The rational checkers reconstruct the labelled universes, transports and row bounds. Multiplier rounding occurs only after the full global multiplicity is applied. Certificates are compressed JSON files grouped by cases and lemmas in `certificates/`; their mathematical

formats are documented there. Verification code is in `emc4/`. The 37 other ordinary upper records remain distributed to check the structural gap claims, including all 68 upper gaps for $5 \le s \le 72$. The replay also checks the 49 gap subtractions for $73 \le s \le 121$ and the stronger coefficient bounds used by Lemma 5.2. Thus the 53-obligation count describes the intermediate cardinality argument, not the number of computations in the complete article.

The root `REPRODUCE.md` gives dependencies, commands, expected results and instructions for rebuilding the paper.

The option `-scope global` restricts verification to Table 3 and these structural gap and weight checks, and reports `PASS_GLOBAL_COMPUTATIONS`. The option `-scope appendices` runs the additional computations below using Python alone and reports `PASS_APPENDIX_COMPUTATIONS`. Both restricted modes explicitly report that the whole article has not been checked.

8.2 Additional computations in the appendices

The local estimates do not serve as inputs to Theorem 1.1. Their computational groups are:

Local assertion	Verification group
252 bases, 504 moment endpoints; factor 4692	`local252`
Uniform base $\lceil 5.74t \rceil$, $t \ge 2200$; factor 2285	`uniform`
Bases $(2904, 15676)$, $(888, 4798)$, $(360, 1950)$; factors 1427, 601, 319	`local2904`, `local888`, `local360`
Base $(72, 395)$; factor 148	`local72`
Base $(49, 269)$; factor 160	`local49`

The moment checks reconstruct all specified cuts and intervals, then substitute the strict margins into the stability formula. The uniform result is checked through coefficient signs rather than a sample of parameter values.

Replaying C1/C2 relies on the correctness of the credited implementations. The trace reductions, necessity of the constraints, graph inequalities, shifting arguments and induction are proved in this paper. Their mathematical validity is not a conclusion of a software success marker.

8.3 Authorship, assistance and materials

The author directed the research programme and selected its scope. AI systems were used extensively to explore proof approaches, propose lemmas and counterchecks, generate certificate and verification code, draft and revise text, and conduct adversarial reviews. Additional AI-assisted calculations supplied by the author were checked against the proofs and exact computations. They include the common-board polynomial propagation, the two fixed inequalities, and the direct dense support constructions incorporated here. This assistance includes scientific and computational work, not only language editing. Agent reviews are not independent specialist acceptance; the proofs and identified computations carry the justification.

The computational supplement consists of verification programs, exact certificates, documented expected results and file hashes. HHL methods and programs retain their attribution; the pinned external archive and its applicable terms are specified separately in the supplement. The supplement contains the source and PDF of this article together with instructions for rebuilding the paper and reproducing the computations.

A Effective shifted stability and weighted graph bounds

This appendix preserves the local results and their proofs as a complement to Theorem 1.1. They do not use finite HHL board certificates, and the global proof does not use their estimates. Here t denotes the allowed matching number. The additional quantitative inputs appear in Section A.2.

A.1 Statements and proof route

For a family $\mathcal{F} \subseteq \binom{[n]}{4}$, define its deficit and t-diversity by

$$d_t(\mathcal{F}) = B_4(n,t) - |\mathcal{F}|, \qquad \gamma_t(\mathcal{F}) = \min_{\substack{S\subseteq[n]\\|S|=t}} |\{E \in \mathcal{F} : E \cap S = \varnothing\}|.$$

For shifted families, $[t]$ attains this minimum. The first local result gives a finite, linear relation between these quantities.

Theorem A.1 (Effective shifted stability)**.** *Let $t \geq 2200$, $n = \lceil 287t/50 \rceil$, and let $\mathcal{F} \subseteq \binom{[n]}{4}$ be shifted with $\nu(\mathcal{F}) \leq t$. Then $d = d_t(\mathcal{F}) \geq 0$ and*

$$\gamma_t(\mathcal{F}) = |\mathcal{F} \setminus \mathcal{C}_t| \leq 2285d, \qquad |\mathcal{F} \triangle \mathcal{C}_t| \leq 4571d.$$

A more detailed estimate is useful when the number of vertices is reduced. A matching in a tail link forces missing head triples on every residual set obtained by selecting some of its pairs. Averaging these constraints and adapting the moment centre produces the next finite refinement.

Theorem A.2 (Smaller finite bases)**.** *Let (t,n,K) be one of the following choices:*

$$\begin{gathered}(t,n,K) \in \{(49,269,160),(72,395,148),(360,1950,319),\\ (888,4798,601),(2904,15676,1427)\},\\ (n,K) = (t+15301,4692) \qquad (3229 \leq t \leq 3480).\end{gathered}$$

Every shifted $\mathcal{F} \subseteq \binom{[n]}{4}$ with $\nu(\mathcal{F}) \leq t$ satisfies $d = d_t(\mathcal{F}) \geq 0$ and

$$\gamma_t(\mathcal{F}) = |\mathcal{F} \setminus \mathcal{C}_t| \leq Kd, \qquad |\mathcal{F} \triangle \mathcal{C}_t| \leq (1+2K)d.$$

The 252 bases in the interval have between 5 and 1195 fewer vertices than $\lceil 287t/50 \rceil$. The additional base at $t = 2904$ has 993 fewer vertices than the uniform base 16669, and symmetric-difference factor 2855. The bases at $t = 888$, $t = 360$, $t = 72$ and $t = 49$ give symmetric-difference factors 1203, 639, 297 and 321, respectively. The uniform theorem is not invoked at these smaller matching parameters. The five isolated bases have different values of t; factor 148 does not replace another factor at a fixed matching parameter; neither does the factor 160 at $t = 49$. The constants apply to their stated ground sets and are not asserted optimal. Taking $d = 0$ gives shifted uniqueness at the bases. This qualitative conclusion extends as follows.

Corollary A.3 (Exact extremizers above the bases)**.** *Suppose $t \geq 2200$, $n \geq \lceil 287t/50 \rceil$, or $3229 \leq t \leq 3480$, $n \geq t + 15301$, or $t = 2904$, $n \geq 15676$, or $t = 888$, $n \geq 4798$, or $t = 360$, $n \geq 1950$, or $t = 72$, $n \geq 395$, or $t = 49$, $n \geq 269$. Every $\mathcal{F} \subseteq \binom{[n]}{4}$ with $\nu(\mathcal{F}) \leq t$ satisfies $|\mathcal{F}| \leq B_4(n,t)$, with equality if and only if $\mathcal{F} = \mathcal{C}_T(n)$ for some t-set T.*

Corollary A.4 (Shifted stability above a base)**.** *Let (a, K) be any of the following base pairs:*

$$\begin{aligned}
&(a,K) = (\lceil 287t/50\rceil, 2285) && (t \ge 2200),\\
&(a,K) = (t+15301, 4692) && (3229 \le t \le 3480),\\
&(a,K) = (15676, 1427) && (t = 2904),\\
&(a,K) = (4798, 601) && (t = 888),\\
&(a,K) = (1950, 319) && (t = 360),\\
&(a,K) = (395, 148) && (t = 72),\\
&(a,K) = (269, 160) && (t = 49).
\end{aligned}$$

For every $n \ge a$ and shifted $\mathcal{F} \subseteq \binom{[n]}{4}$ with $\nu(\mathcal{F}) \le t$, put $R = \binom{n-t}{4}/\binom{a-t}{4}$. Then

$$d_t(\mathcal{F}) \ge 0, \qquad \gamma_t(\mathcal{F}) \le KR\, d_t(\mathcal{F}), \qquad |\mathcal{F} \triangle \mathcal{C}_t| \le (1+2KR) d_t(\mathcal{F}).$$

Where several base pairs are available, their smallest applicable coefficient may be used.

The loss is explicit: at fixed t it grows as $(n-t)^4$. For example, $t = 3480$, $a = 18781$ and $n = 19000$ give $\gamma_t(\mathcal{F}) \le 4967 d_t(\mathcal{F})$, before the uniform base 19976. The constants 2285, 4692, 1427, 601, 319, 148 and 160 are asserted at their bases; the quantitative result still requires shiftedness.

The proof starts with the section and shadow method of Frankl [3] and the finite variance estimate of Frankl–Kupavskii [5]. Sections A.4–A.5 prove the uniform estimate. Sections A.6–A.7 give the finite refinement and quantitative propagation; Section A.8 handles exact equality and inversion of shifts. The weighted graph Lemma A.8 determines an exact maximum for $K_3(G) - \lambda e(G)$ on its displayed interval, and is also useful as a separate technical result. Section 7 distinguishes these finite linear estimates from the broader asymptotic literature.

A.2 Additional inputs for the local results

The shadow theorem was stated in Section 2. The following inputs are used only in this appendix.

Triple systems. Frankl–Rödl–Ruciński [4, Theorem 1] gives, for $u \ge 33$, $N \ge 3u+3$ and matching number exactly u,

$$|\mathcal{A}| \le \max\left\{ \binom{3u+2}{3}, B_3(N,u) \right\}.$$

To extend it to matching number at most u, add edges until the first family with matching number u. A single added edge increases the matching number by at most one, and the complete family has sufficient matching capacity. Further,

$$B_3(4u-1,u) - \binom{3u+2}{3} = \frac{u(10u^2-69u+5)}{6} > 0 \quad (u \ge 33).$$

Indeed the numerator at $u = 33+x$ is $10x^3 + 921x^2 + 28121x + 284394$. Monotonicity in N therefore yields

$$\nu(\mathcal{A}) \le u, \quad u \ge 33, \quad N \ge 4u-1 \quad \Longrightarrow \quad |\mathcal{A}| \le B_3(N,u). \tag{A.1}$$

For smaller residual ground sets we retain the full maximum in FRR.

Finite variance. For a uniformly random matching $\mathcal{B}$ of v triples on $[m]$, where $m \geq 3v$, let $Z = |\mathcal{A} \cap \mathcal{B}|$ and $\mu = v|\mathcal{A}|/\binom{m}{3}$. The finite estimate in [5, Lemma 10, Proposition 11 and equation (20)] is

$$\operatorname{Var} Z \leq 2\mu(1 - \mu/v). \tag{A.2}$$

No asymptotic threshold from that paper is used in the proof.

Clique cap. The Lovász form of Kruskal–Katona, stated in [6, Theorem 1], says that if $|\mathcal{A}_0| = \binom{x}{3} > 0$, $x \geq 3$, then

$$K_4(\mathcal{A}_0) \leq \binom{x}{4} = \frac{x-3}{4}|\mathcal{A}_0|. \tag{A.3}$$

Graph compression. For graphs, compression preserves the number of edges, does not increase the matching number and does not decrease the number of triangles [7, Lemmas 2.1–2.2]. Thus it does not decrease $K_3(G) - \lambda e(G)$. This permits the weighted proof in Section A.5 to be carried out on stable graphs.

Lemma A.5 (One-sided Kneser variance)**.** *Let $m \geq 6$, $v \geq 1$ and $3v \leq m$. For any triple family $\mathcal{A} \subseteq \binom{[m]}{3}$ and a uniformly random matching $\mathcal{B}$ of v triples, put $Z = |\mathcal{A} \cap \mathcal{B}|$ and $\mu = v|\mathcal{A}|/\binom{m}{3}$. Then*

$$\operatorname{Var} Z \leq \rho_{m,v}\mu(1 - \mu/v), \qquad \rho_{m,v} = 1 + \frac{6(v-1)}{(m-3)(m-4)}. \tag{A.4}$$

Proof. Give functions on $\binom{[m]}{3}$ the uniform inner product. Let K be the adjacency operator for disjoint triples, of degree $d = \binom{m-3}{3}$, and put $g_J(T) = 1_{J \subseteq T}$. Let V_j be the span of these indicators for $|J| = j$. The spaces are nested, V_0 consists of constants, and V_3 is the whole function space. Direct counting and inclusion–exclusion give

$$(Kg_J)(T) = \binom{m-3-j}{3-j} 1_{J \cap T = \varnothing} = \binom{m-3-j}{3-j} \sum_{I \subseteq J} (-1)^{|I|} g_I(T).$$

Thus K acts on V_j/V_{j-1} by $(-1)^j\binom{m-3-j}{3-j}$. Since K is self-adjoint, the orthogonal differences give an eigenspace decomposition. The nonconstant eigenvalues are $-\binom{m-4}{2}$, $m-5$, -1; their largest algebraic value is $m-5$. For $f = 1_{\mathcal{A}} - \alpha$, $\alpha = |\mathcal{A}|/\binom{m}{3}$, this yields

$$\frac{\langle f, Kf \rangle}{d} \leq \frac{m-5}{d}\alpha(1-\alpha).$$

Two distinct positions of a random matching are uniform disjoint triples. Adding the v variances and $v(v-1)$ covariances gives

$$\operatorname{Var} Z \leq v\alpha(1-\alpha)\left(1 + (v-1)\frac{m-5}{d}\right),$$

which is (A.4). The case $v = 1$ is immediate. □

This uses the Kneser spectral method underlying [5]. For this upper bound the largest algebraic eigenvalue suffices; the larger absolute value of a negative eigenvalue need not be used. We retain (A.2) for the other numerical arguments and use (A.4) only where specified.

A.3 Sections and shifted completion

Recall the elementary shifted-completion Lemma 2.1.

Fix $t \geq 1$, $n \geq 4t+4$, and a stable family $\mathcal{F}$ with $\nu(\mathcal{F}) = t$. A stable family of exact matching number t contains the complete four-graph on $[t+3]$. Indeed an edge of a maximum matching avoids $[t-1]$, hence dominates $\{t, t+1, t+2, t+3\}$. Define

$$X = [t+1], \quad Y = [t+2, n], \quad m = |Y|, \quad c = t - \lfloor (t+1)/4 \rfloor,$$

$$\mathcal{H} = \mathcal{F} \cap \binom{Y}{4}, \qquad \mathcal{A}_i = \{T \in \binom{Y}{3} : T \cup \{i\} \in \mathcal{F}\}, \quad \mathcal{A} = \mathcal{A}_{t+1}.$$

These sections meet X in exactly one vertex. They are nested: $\mathcal{A}_1 \supseteq \cdots \supseteq \mathcal{A}_{t+1}$, and cross-dependent, since disjoint representatives would complete to $t+1$ disjoint four-edges. In particular $\nu(\mathcal{A}) \leq t$. The packing inside X gives $\nu(\mathcal{H}) \leq c$; stability gives $\partial\mathcal{H} \subseteq \mathcal{A}$. Thus

$$|\mathcal{H}| \leq c|\mathcal{A}|, \qquad |\mathcal{H}| \leq K_4(\mathcal{A}). \tag{A.5}$$

All sections with at least two vertices in X already meet $[t]$. It therefore suffices to prove

$$|\mathcal{H}| + \sum_{i=1}^{t+1} |\mathcal{A}_i| \leq t\binom{m}{3}. \tag{A.6}$$

Lemma A.6 (Deficit and canonical diversity)**.** *For the sections above, put $C = \binom{m}{3}$ and $d = B_4(n,t) - |\mathcal{F}|$. Then*

$$d \geq tC - |\mathcal{H}| - \sum_{i=1}^{t+1} |\mathcal{A}_i|, \qquad \gamma_t(\mathcal{F}) = |\mathcal{F} \setminus \mathcal{C}_t| = |\mathcal{H}| + |\mathcal{A}| \leq (c+1)|\mathcal{A}|.$$

Moreover, for every shifted family,

$$|\mathcal{C}_t \setminus \mathcal{F}| = d + |\mathcal{F} \setminus \mathcal{C}_t|, \qquad |\mathcal{F} \triangle \mathcal{C}_t| = d + 2|\mathcal{F} \setminus \mathcal{C}_t|.$$

Proof. The cover has all tC sections meeting X exactly once through $[t]$; every section with at least two vertices in X already meets $[t]$. Its deficit on these latter sections is nonnegative, proving the first inequality. The edges avoiding $[t]$ are exactly $\mathcal{H}$ and the section $\mathcal{A} = \mathcal{A}_{t+1}$. To see that $[t]$ minimizes diversity, write $[n] \setminus S = \{u_1 < \cdots < u_{n-t}\}$; then $u_i \leq t+i$. Coordinatewise lowering $t+i \mapsto u_i$ injects the edges avoiding $[t]$ into those avoiding S. The last two identities follow by subtracting cardinalities and hold without any matching assumption. □

Lemma A.7 (Nested incidence)**.** *On a matching of v triples, let $d_1 \geq \cdots \geq d_{t+1} = z$ count membership in the sections. For every real w, some $j \in [z,t]$ satisfies*

$$W - tv \leq j(j-v) + wz, \qquad W = \sum_{i=1}^{t} d_i + wz. \tag{A.7}$$

Proof. Nested neighbourhoods have distinct representatives if $d_i \geq t+2-i$ for every i, by assignment starting with the last row. Cross-dependence therefore gives $d_{t+1-j} \leq j$ for some $0 \leq j \leq t$. Nesting implies $z \leq j$. The first $t-j$ rows have size at most v, and the remaining j of the first t rows have size at most j. This proves the bound. □

A.4 A centered moment inequality

Suppose

$$m \geq 3v, \qquad t+1 \leq v \leq 2t, \qquad y = v - t, \qquad 0 < b \leq v, \qquad bv \geq 2ty.$$

Convexity in j in (A.7) yields

$$W - tv \leq G_w(z) := \max\{z(z+w-v), -ty + wz\}, \qquad 0 \leq z \leq t.$$

Put $\kappa = ty/b^2$. The quadratic

$$Q_w(z) = wz - ty + \kappa(z-b)^2 = \kappa z^2 + (w - 2ty/b)z$$

dominates both branches. Its difference from the linear branch is a square; its difference from the other branch is $z[(\kappa-1)z + v - 2ty/b]$. The bracket is affine in z, nonnegative at zero by hypothesis, and equals $y(t/b-1)^2$ at t.

Using (A.2), with $C = \binom{m}{3}$ and $\mu = v|\mathcal{A}|/C$, we get

$$\mathbb{E}(W - tv) \leq \Phi_w(\mu) := \kappa(1-2/v)\mu^2 + (w - 2ty/b + 2\kappa)\mu. \tag{A.8}$$

This is convex, including the limiting case $v = 2$. If $\mu \leq b$, $\Phi_w(0) = 0$, and

$$M_w := ty - wb - \frac{2ty}{b}(1 - b/v) \geq 0, \tag{A.9}$$

then $\Phi_w(\mu) \leq 0$, since $\Phi_w(b) = -M_w$. Together with $|\mathcal{H}| \leq (w-1)|\mathcal{A}|$, this proves (A.6).

Section A.7 adapts the centre to the actual mean for the finite refinement, while retaining a two-endpoint reduction.

A.5 Weighted triangle bounds and a uniform cover range

Lemma A.8. *For an integer $t \geq 1$ and any graph G with $\nu(G) \leq t$,*

$$K_3(G) - \frac{t-1}{2}e(G) \leq g_{1/2} := \frac{t(t+1)(2t+1)}{6}. \tag{A.10}$$

If its ground set has size $N \geq 2t+1$, then for $0 \leq \lambda \leq (t-1)/3$,

$$\begin{aligned} K_3(G) - \lambda e(G) &\leq g_\lambda(t,N) := \max\{C_\lambda(t), J_\lambda(N,t)\}, \\ C_\lambda(t) &= \binom{2t+1}{3} - \lambda\binom{2t+1}{2}, \\ J_\lambda(N,t) &= \binom{t}{3} + (N-t)\binom{t}{2} - \lambda\left[\binom{t}{2} + t(N-t)\right]. \end{aligned} \tag{A.11}$$

In particular,

$$K_3(G) - \frac{t-1}{3}e(G) \leq \max\left\{\frac{t^2(2t+1)}{3}, \frac{t(t-1)(N-t-1)}{6}\right\}. \tag{A.12}$$

Proof. By graph compression, we may assume that G is stable: the matching constraint is retained and the functional cannot decrease. Let h_b be the number of lower neighbours of vertex b. They form an initial clique. The left side of (A.10) is $\frac{1}{2}\sum_b h_b(h_b - t)$. For $b \geq 2t+2$, $h_b \leq t$; otherwise the edge $(t+1, b)$ forces the matching $(i, t+1+i)$, $1 \leq i \leq t+1$. If some $h_b > t$, the initial clique on $t+1$ vertices contributes $-t(t^2-1)/12$. Positive contributions are

possible only for $t+2 \le b \le 2t+1$, bounded individually by $(b-1)(b-1-t)/2$. Their sum is $t(5t^2+6t+1)/12$. Adding proves (A.10); if no $h_b > t$, the sum is nonpositive.

For (A.11), the case $N = 2t+1$ follows from $3K_3 \le (N-2)e$, since $(N-2)/3 - \lambda \ge 0$. If $N \ge 2t+2$, one of the disjoint pairs $(a+1, 2t+2-a)$, $0 \le a \le t$, is missing. Set $b_0 = 2t+1-a$. Stability makes the vertices beyond $[b_0]$ independent, with neighbours contained in $[a]$. The head contribution is at most $\binom{b_0}{3} - \lambda\binom{b_0}{2}$: count each head triangle three times and use $(b_0-2)/3 - \lambda \ge 0$. An outside vertex of degree $h \le a$ contributes at most $\binom{h}{2} - \lambda h$, whose maximum is $\max\{0, \binom{a}{2} - \lambda a\}$. For $a \le 2\lambda + 1$ this is zero; the head expression is at most $C_\lambda(t)$, since it increases with $b_0 \ge t+1$. Above the breakpoint the bound becomes

$$T(a) = \binom{2t+1-a}{3} - \lambda\binom{2t+1-a}{2} + (N-2t-1+a)\left(\binom{a}{2} - \lambda a\right).$$

Here $T''(a) = N+2a-3\lambda-2 > 0$. At the joining point the value is at most $C_\lambda(t)$; at $a = t$ it is $J_\lambda(N,t)$. Convexity proves the assertion, with an empty second interval omitted. Substitution of $\lambda = (t-1)/3$ gives (A.12). □

Both bounds are sharp when $N \ge 2t+1$. The graph K_{2t+1} with isolated remaining vertices has matching number t, $\binom{2t+1}{2}$ edges and $\binom{2t+1}{3}$ triangles. It attains (A.10) and the first term of (A.12). The join $K_t \vee \overline{K}_{N-t}$ has matching number t and

$$e = \binom{t}{2} + t(N-t), \qquad K_3 = \binom{t}{3} + (N-t)\binom{t}{2}.$$

It attains the second term of (A.12). For every admissible weight, the same two graphs realize the two terms in (A.11). Thus all the displayed bounds are exact maxima.

Lemma A.9 (A graph bound for all nonnegative weights)**.** *For an integer $u \ge 1$, let G be a graph on $N \ge 2u+1$ vertices, with $\lambda \ge 0$ and $\nu(G) \le u$. Writing $x_+ = \max\{0, x\}$, we have*

$$\begin{aligned} K_3(G) - \lambda e(G) &\le \widehat{g}_\lambda(u, N) \\ &:= \max_{0 \le a \le u} \Bigg\{ \left(\binom{2u+1-a}{3} - \lambda\binom{2u+1-a}{2} \right)_+ \\ &\qquad + (N-2u-1+a)\left(\binom{a}{2} - \lambda a\right)_+ \Bigg\}, \end{aligned} \tag{A.13}$$

where a is integral. Set $\widehat{g}_\lambda(0, N) = 0$. This is an upper bound; exactness for all weights is not asserted.

Proof. Compress G. If $N = 2u+1$, the triangle incidence bound $3K_3 \le (N-2)e$ gives the first term with $a = 0$, or zero when its coefficient is negative. For $N \ge 2u+2$, the missing pair used in the proof of Lemma A.8 gives a head of size $b = 2u+1-a$ and an independent exterior whose neighbours lie in $[a]$. On the head, $3K_3 \le (b-2)e$ gives $K_3 - \lambda e \le (\binom{b}{3} - \lambda\binom{b}{2})_+$: use $e \le \binom{b}{2}$ when the coefficient is nonnegative and zero otherwise. An exterior vertex of degree $h \le a$ contributes at most $\binom{h}{2} - \lambda h$. Convexity in h bounds it by the larger endpoint value 0 or $\binom{a}{2} - \lambda a$. Summing and maximizing over a proves the result. □

For a stable triple family $\mathcal{A}$ on $[m]$, put $q = 3t+2$ and

$$G_j = \{P \in \binom{[j-1]}{2} : P \cup \{j\} \in \mathcal{A}\}.$$

For $j > q$, $\nu(G_j) \le t$: $t+1$ disjoint pairs could be completed to triples using t free vertices below j, and j itself, contradicting $\nu(\mathcal{A}) \le t$. Moreover the four-cliques with largest vertex j are counted *exactly* by $K_3(G_j)$. The triple omitting j is forced by stability. If $\mathcal{A}_0 = \mathcal{A} \cap \binom{[q]}{3}$, the elementary head bound $4K_4(\mathcal{A}_0) \le (q-3)|\mathcal{A}_0|$ and (A.10) give

$$K_4(\mathcal{A}) \le \frac{t-1}{2}|\mathcal{A}| + E_{1/2}(t,m), \qquad E_{1/2}(t,m) = \frac{t+1}{4}\binom{3t+2}{3} + (m-3t-2)g_{1/2}. \quad \text{(A.14)}$$

Proposition A.10. *For $t \ge 2200$, the cover bound holds for every $n \ge \lceil 287t/50 \rceil$.*

Proof. At $a = \lceil 287t/50 \rceil$, let $m = a - t - 1$, $m_0 = 237t/50 - 1$, $m_+ = 237t/50$. Then

$$m_0 \le m < m_+, \quad 79t/50 - 1 \le v = \lfloor m/3 \rfloor \le 79t/50, \quad 29t/50 - 1 \le y \le 29t/50.$$

For the lower bound on v, use $v = \lfloor (\lceil 237t/50 \rceil - 1)/3 \rfloor$. All the domains of Section A.4 hold at $t \ge 2200$; also $m \ge 4t - 1$.

Write $\alpha = |\mathcal{A}|/\binom{m}{3}$. Equation (A.1) implies $\alpha \le 161/316$. To check this, the ratio $\binom{m-t}{3}/\binom{m}{3} = \prod_{i=0}^{2}(1 - t/(m-i))$ increases with m, and

$$316\binom{m_0 - t}{3} - 155\binom{m_0}{3} = P_A(t)/750000,$$

$$P_A(t) = 3019933t^3 - 703202700t^2 + 614817500t - 120750000.$$

Its coefficients after $t = 2200 + x$ are positive (Appendix B).

If $\alpha > 12/25$, (A.14) gives $|\mathcal{H}| \le (23t/32)|\mathcal{A}|$, since

$$(7t/32 + 1/2)(12/25)\binom{m_0}{3} - E_{1/2}(t, m_+) = P_J(t)/150000000 > 0,$$

$$P_J(t) = 23803113t^4 - 82886156t^3 - 771716100t^2 + 309590000t - 36000000.$$

Again the translated coefficients are positive. Use the following weights and centers in (A.9):

Density range	Upper weight w	Center b
$0 \le \alpha \le 12/25$	$3t/4 + 3/2$	$474t/625$
$12/25 < \alpha \le 161/316$	$23t/32 + 1$	$161t/200$

The first weight bounds $c+1$, by the four residues of $t+1$ modulo four. Both centers satisfy $\mu \le b \le v$. In the order of the two rows, direct lower bounds are

$$bv - 2ty \ge \frac{598t^2}{15625}, \qquad M_w \ge \frac{7t(474t - 124127)}{296250},$$
$$bv - 2ty \ge \frac{1119t^2}{10000}, \qquad M_w \ge \frac{9t(12719t - 22718432)}{81401600}.$$

For the first column, write $bv - 2ty = bt + (b-2t)y$ and use the upper bound on y. For the second, use the lower bound on y in ty, the upper bound in the subtracted correction, and respectively $1 - b/v \le 13/25$ or $155/316$. All margins are positive for $t \ge 2200$. Equations (A.5), (A.14) and (A.8)–(A.9) prove the base case. Shifting and Lemma 2.1 cover matching number at most t. Propagation in n is proved in Lemma A.14. □

Proof of Theorem A.1. First assume $\nu(\mathcal{F}) = t$. Use the sections of Section A.3; exact matching number again suffices. In either density range of Proposition A.10, the corresponding w, b give

$$\frac{v}{C}\left(|\mathcal{H}| + \sum_i |\mathcal{A}_i| - tC\right) \le \Phi_w(\mu).$$

For $0 < \mu \le b$, the ratio $\Phi_w(\mu)/\mu$ is nondecreasing, because the quadratic coefficient is nonnegative. Since $\Phi_w(b) = -M_w$, the section deficit is at least $(M_w/b)|\mathcal{A}|$. Dividing the two explicit lower margins in that proposition by their centers gives, respectively,

$$\eta_1(t) = \frac{7(474t - 124127)}{224676}, \qquad \eta_2(t) = \frac{9(12719t - 22718432)}{65528288}.$$

For $t \ge 2200$, we have $\eta_1 > \eta_2 > 0$ and $2285\eta_2 \ge 3t/4 + 3/2 \ge c + 1$. Here is a direct all-parameter check: after $t = 2200 + x$, the coefficients of these affine expressions are

	coefficient of x	constant
η_2	$9/5152$	$5921289/8191036$
$\eta_1 - \eta_2$	$15899/1221024$	$40620105838/1455956649$
$2285\eta_2 - 3t/4 - 3/2$	$16701/5152$	$2649411/8191036$

and every entry is positive. The section accounting and canonical diversity argument of Lemma A.6 yield $d \ge \eta_2|\mathcal{A}|$ and $\gamma_t(\mathcal{F}) \le (c+1)|\mathcal{A}| \le 2285d$. If $\mu = 0$, then $\mathcal{A} = \mathcal{H} = \varnothing$, so the same conclusion is immediate. Lemma 2.1 extends the first bound to $\nu(\mathcal{F}) \le t$; Lemma A.6 gives the symmetric-difference bound. □

A.6 Averaged matching deficits and a common tail bound

Let $\mathcal{A} \subseteq \binom{[m]}{3}$ be stable with $\nu(\mathcal{A}) \le t$, and put

$$q = 3t + 2 \le m, \qquad \mathcal{A}_0 = \mathcal{A} \cap \binom{[q]}{3}, \qquad \Delta = \binom{q}{3} - |\mathcal{A}_0|, \qquad M = m - q.$$

Use the link graphs G_j from Section A.5. Recall that $\nu(G_j) \le t$ for $j > q$.

A nonempty tail forces $\{1, 2, j\}$ for some $j > q$. Every partition of $[3, q]$ into t triples must then contain a missing triple. Averaging over uniform partitions gives

$$\Delta \ge d_0 := \binom{3t-1}{2}. \tag{A.15}$$

A sharp first deficit. On $[q+1]$, define

$$\mathcal{A}^* = \left(\binom{[q]}{3} \setminus \{\{a, b, q\} : 3 \le a < b \le q - 1\}\right) \cup \{\{1, 2, q+1\}\}.$$

This family is stable and its head deficit is exactly d_0. A matching avoiding the tail has at most t edges. If it uses the tail triple, all remaining triples avoid $1, 2, q$, leaving $3t - 1$ vertices and at most $t - 1$ further edges. The complete head on $[q-1]$ contains a t-matching, so $\nu(\mathcal{A}^*) = t$. The sole tail link has one edge and no triangle: its contribution to $K_3 - \lambda e$ is $-\lambda \le 0$. Thus (A.15) is sharp for a nonempty tail, without asserting sharpness of a deficit forced by positive tail weight.

Lemma A.11 (Averaging over submatchings)**.** *Suppose G_{q+L} contains a matching of size u, where $3 \le u \le t$. Let $1 \le r \le \min\{u, L\}$ and $t - r \ge 33$. Set $h = q - 2u$ and*

$$V_t(r) = \min\left\{\binom{q-2r}{3} - \binom{3(t-r)+2}{3}, \binom{2t+2-r}{3}\right\}.$$

$$n_0 = \binom{h}{3}, \qquad n_1 = u\left(2\binom{h}{2} + h\right),$$
$$n_2 = \binom{u}{2}(4h+4), \qquad n_3 = 8\binom{u}{3}.$$

Put $p_k = \binom{u-k}{r}/\binom{u}{r}$ for $0 \le k \le 3$. Then

$$\Delta \ge D_t(u,r) := \max\left\{0, \max_{\substack{0\le k\le 3\\ p_k>0}} \left\lceil \frac{V_t(r) - \sum_{i<k}(p_i - p_k)n_i}{p_k} \right\rceil \right\}. \tag{A.16}$$

Proof. By replacing occupied vertices with smaller unused vertices, shiftedness places a u-matching inside $[2u]$. Fix its u disjoint pairs. Any r of them can be completed with distinct vertices $q+1, \dots, q+r$ to disjoint triples of $\mathcal{A}$. After deleting their $2r$ head vertices, the residual triple family has matching at most $t-r$ on $q-2r$ vertices. The full FRR bound applies, since

$$(q-2r) - 3(t-r) - 3 = r - 1 \ge 0,$$

and forces at least $V_t(r)$ missing residual triples. Both FRR branches are retained.

Classify head triples by the number k of matching pairs they meet. Their capacities are n_k: for $k=1$, either one vertex of a pair or the whole pair is used; for $k=2$, either two single pair vertices and one outside vertex, or a whole pair and one other pair vertex are used. The other two counts are immediate. Let x_k be the number of missing triples in class k. Then $0 \le x_k \le n_k$, $\Delta = \sum_k x_k$, and a class-k triple survives a uniformly chosen r-submatching with probability p_k. Averaging gives $\sum_k p_k x_k \ge V_t(r)$. Since $p_0 \ge p_1 \ge p_2 \ge p_3$, for $p_k > 0$,

$$V_t(r) \le p_k\Delta + \sum_{i<k}(p_i - p_k)x_i + \sum_{i>k}(p_i - p_k)x_i \le p_k\Delta + \sum_{i<k}(p_i - p_k)n_i.$$

Taking ceilings proves (A.16). □

The bound is necessary; it does not claim that every feasible four-class profile can be realized by a stable family. Its useful feature is that the same Δ controls every link.

Choose a finite set $\mathcal{T}$ of triples $(u, r, D_t(u,r))$ satisfying Lemma A.11, and include the additional nonempty-tail cut $(1,1,d_0)$. If $\Delta < D$, any cut (u,r,D) implies $\nu(G_{q+j}) \le u-1$ for every $j \ge r$. For $0 \le \delta \le \binom{q}{3}$, define

$$u_\delta(j) = \min\big(\{t\} \cup \{u-1 : (u,r,D) \in \mathcal{T},\ r \le j,\ D > \delta\}\big).$$

These are upper bounds for the link matchings, not a realizability claim.

For $0 \le a \le \binom{q}{3}$, let $z(a) \ge 3$ be the least integer with $\binom{z(a)}{3} \ge a$, taking $z(0) = 3$, and put

$$\beta_\lambda(a) = \max\{0, ((z(a)-3)/4 - \lambda)a\}. \tag{A.17}$$

Equation (A.3) implies $K_4(\mathcal{A}_0) - \lambda|\mathcal{A}_0| \le \beta_\lambda(a)$ when $|\mathcal{A}_0| \le a$. For a nonnegative coefficient use this cardinality bound; for a negative coefficient use zero. The function β_λ is nondecreasing.

Proposition A.12 (A common deficit for the tail)**.** *Assume all cut values lie in* $[0, \binom{q}{3}]$*. Take* $0 \le \lambda \le (t-1)/3$ *such that every positive cap* $u_\delta(j)$ *below satisfies* $\lambda \le (u_\delta(j)-1)/3$*. Define* $g_\lambda(0,N) = 0$ *and*

$$\mathcal{D} = \{0\} \cup \{D : (u, r, D) \in \mathcal{T}\},$$
$$E_{\mathcal{T}} = \max_{\delta \in \mathcal{D}} \left\{ \beta_\lambda \left(\binom{q}{3} - \delta \right) + \sum_{j=1}^{M} g_\lambda(u_\delta(j), q+j-1) \right\}. \tag{A.18}$$

Then $K_4(\mathcal{A}) \le \lambda|\mathcal{A}| + E_{\mathcal{T}}$*.*

Proof. Decomposition by largest vertex gives exactly

$$K_4(\mathcal{A}) - \lambda|\mathcal{A}| = K_4(\mathcal{A}_0) - \lambda|\mathcal{A}_0| + \sum_{j=1}^{M} (K_3(G_{q+j}) - \lambda e(G_{q+j})).$$

Apply (A.17) to the head and (A.11) to each positive-cap link; a zero cap makes that link empty. Between consecutive cut values the caps are constant and the head term is nonincreasing. The left endpoint of each interval therefore suffices, proving (A.18). The state $\delta = 0$ is included; the nonempty-tail cut makes all its links empty. Thus the bound also covers empty tails. □

The finite reduction covers every deficit interval. It does not sample selected hypergraphs or assume that the relaxed maximizing state is attained.

The same proposition holds for every $\lambda \ge 0$, with g_λ replaced throughout by $\widehat{g}_\lambda$ from Lemma A.9; no positive-cap weight restriction is then needed. The proof is identical, using (A.13) on the links. The carrier condition holds because $q + j - 1 \ge 3t + 2 \ge 2u_\delta(j) + 1$. Denote this envelope by $\widehat{E}_{\mathcal{T}}$. The cut hypotheses, head cap and exhaustive deficit intervals are unchanged.

A.7 Adapted moments and the finite certificate

The centre used in Section A.4 can depend on the mean density while remaining deterministic.

Lemma A.13 (Density-adapted centre)**.** *Suppose* $m \ge 3v$*,* $t+1 \le v \le 2t$*,* $v > 2$*, and* $\operatorname{Var} Z \le \rho\mu(1-\mu/v)$ *for some* $0 < \rho < v$*. Put* $y = v - t$*,* $a_v = 1 - \rho/v$*,* $\xi_0 = 2ty/v$*, and*

$$\xi(\mu) = \max\{\xi_0, a_v\mu + \rho\}, \qquad 0 \le \mu \le v.$$

For the nested incidence variable of Lemma A.7,

$$\mathbb{E}(W - tv) \le \Psi_w(\mu) := w\mu - \frac{2ty\mu}{\xi(\mu)} + \frac{ty(a_v\mu^2 + \rho\mu)}{\xi(\mu)^2}. \tag{A.19}$$

The function Ψ_w *is convex and* $\Psi_w(0) = 0$*.*

Proof. For every $\xi \in [\xi_0, v]$, the quadratic $wz - ty + ty(z-\xi)^2/\xi^2$ dominates both incidence branches: use the calculation in Section A.4 with $b = \xi$. Taking expectations and using the stated variance bound gives the expression on the right of (A.19) with this fixed ξ. For $\mu > 0$, its derivative with respect to ξ has the sign of $\xi - (a_v\mu + \rho)$; also $a_v\mu + \rho \le v$. The displayed $\xi(\mu)$ therefore minimizes that upper bound.

On the free branch the expression is

$$w\mu - \frac{ty\mu}{a_v\mu + \rho}, \qquad \frac{d^2}{d\mu^2}\left(w\mu - \frac{ty\mu}{a_v\mu + \rho} \right) = \frac{2ty\rho a_v}{(a_v\mu + \rho)^3} > 0.$$

The clamped branch is a convex quadratic with second derivative $2tya_v/\xi_0^2$. Values and slopes agree at the join. If $\xi_0 \le \rho$, only the free branch occurs. The centre depends on the mean, not the sampled matching, so the expectation step is valid. The case $\mu = 0$ is immediate. □

In the formulas below, $\rho = 2$ unless explicitly specified. Lemma A.5 permits a smaller value in the last base. The preceding estimates specialize to $\rho = 2$.

Set $C = \binom{m}{3}$, $B = B_3(m,t)$, and $b = vB/C$. Whenever $t \geq 33$ and $m \geq 4t-1$, (A.1) gives $\mu \leq b$. If $K_4(\mathcal{A}) \leq \lambda|\mathcal{A}| + E$ with $0 \leq \lambda < c$, equations (A.5) and (A.19) imply

$$\frac{v}{C}\left(|\mathcal{H}| + \sum_i |\mathcal{A}_i| - tC\right) \leq F(\mu) := \Psi_{\lambda+1}(\mu) + \min\{(c-\lambda)\mu, Ev/C\}. \tag{A.20}$$

If $0 < \mu_* := Ev/[C(c-\lambda)] < b \leq v$, F is convex separately on $[0,\mu_*]$ and $[\mu_*, b]$, with $F(0) = 0$. Thus two nonpositive endpoint values suffice for the cover bound. Global convexity across the corner of the minimum is not asserted.

For each $3229 \leq t \leq 3480$, use

$$n = t + 15301, \quad m = 15300, \quad v = 5100, \quad \lambda = \frac{3(t-1)}{100}, \quad E = E_{\mathcal{T}}.$$

The finite cut set consists of $(1,1,d_0)$ and the cuts

$$u = \lfloor it/1000 \rfloor, \quad i = 250, 300, \ldots, 1000, \qquad r = \lfloor jt/1000 \rfloor, \quad j = 25, 75, \ldots, \quad j \leq i,$$

subject to $1 \leq r \leq u$ and $t - r \geq 33$. This is a sufficient fixed selection, not optimization over all cut sets. The certificate checks every residual FRR domain, positive-cap weight domain, cut value and moment interval. It gives

Check	Result
Finite bases	252
Averaged cuts	50400
Deficit states, including zero	25956
Strictly negative rational endpoints	504

The cuts encode a fractional capacity calculation; the distributed verifier checks them using the dual inequalities (A.16). It reconstructs every deficit interval and sums the tail envelopes exactly, using arithmetic progressions between their breakpoints. In particular, this covers all 103 states at the critical base.

Both endpoint maxima occur at $t = 3480$. There

$$q = 10442, \quad c = 2610, \quad 100E = 62543109568930107, \quad b = \frac{321648884360}{117022051},$$

$$\mu_* = \frac{20847703189643369}{9773798721571},$$

and

$$F(b) = -\frac{7344328346613039505237916 51}{4800711559382479783037 00} < 0,$$
$$F(\mu_*) = -\frac{1234244640478684423899106272848 6983}{2111936520146259434735841772 32} < 0.$$

The maximizing envelope state is $\Delta = 45051066360$; its caps are 3480 through tail position 1130 and 3479 thereafter. It is an upper-bound profile, not a constructed extremizer.

The additional base at $t = 2904$. Use the same cut selection and envelope with

$$n = 15676, \quad m = 12771, \quad v = 4257, \quad y = 1353, \quad q = 8714, \quad c = 2178, \quad \lambda = \frac{8709}{100}.$$

There are 200 averaged cuts and 103 deficit states, including zero. The certificate checks all the domains used above and gives

$$100E = 30345188802828751, \quad b = \frac{37403488540}{16306013}, \quad \mu_* = \frac{30345188802828751}{17047202820915}.$$

In particular, $0 < \mu_* < b \le v$, and the two exact signs are

$$F(b) = -\frac{20689701908856362847614556 9}{59031722428985718476500} < 0,$$
$$F(\mu_*) = -\frac{39628539709436855454202697982255328 9}{92268924304024485555090564686 40} < 0.$$

At each of the 103 states the verifier sums the envelope over all 4057 tail positions using exact arithmetic progressions. These are fixed certificate checks, with no additional hypothesis on $\mathcal{F}$. The local proof and the new constant below do not use HHL.

The further base at $t = 888$. The general lemmas above also apply at

$$n = 4798, \quad m = 3909, \quad v = 1303, \quad y = 415, \quad q = 2666, \quad c = 666, \quad \lambda = \frac{2661}{100}.$$

Here $t \ge 33$, $m \ge 4t - 1$, $q \le m$, $n \ge 4t + 4$, and $t + 1 \le v \le 2t$. Retaining only the admissible cuts in the same grid leaves 199 averaged cuts, together with $(1, 1, d_0)$, where $d_0 = 3544453$. They produce 102 deficit states. In particular, the FRR condition $t - r \ge 33$ excludes one grid cut that was available at the preceding bases. Every residual ground-set and positive graph-cap domain is checked separately. The exact values are

$$100E = 265583224513928, \qquad b = \frac{2678431702}{3817139}, \qquad \mu_* = \frac{132791612256964}{244064050521},$$

and

$$F(b) = -\frac{21670567755141894005923}{27790259876854163550} < 0,$$
$$F(\mu_*) = -\frac{95483519326415107149256492966 21}{21951726934065437421683437 32} < 0.$$

We have $0 < \mu_* < b \le v$. At each of the 102 states the verifier sums the envelope over all 1243 tail positions using exact arithmetic progressions between breakpoints. This is a new application of the general section, cut and moment lemmas, not an application of the uniform theorem outside $t \ge 2200$.

The base at $t = 360$. The same cut grid and moment lemmas apply with

$$n = 1950, \quad m = 1589, \quad v = 529, \quad y = 169, \quad q = 1082, \quad c = 270, \quad \lambda = \frac{1077}{100}.$$

Here $m = 3v + 2$. The variance input (A.2) permits $m \ge 3v$, so the two unused vertices require no change of normalization. We have $t \ge 33$, $m \ge 4t - 1$, $q \le m$, $n \ge 4t + 4$ and $t + 1 \le v \le 2t$. Filtering the grid by the stated FRR domain leaves 197 averaged cuts, plus the nonempty-tail cut, and 100 deficit states. The smallest actual residual matching parameter is 45. Every cut, positive link cap and adapted centre is checked in its own domain. The exact values are

$$100E = 7198243043040, \qquad b = \frac{179393730}{630833}, \qquad \mu_* = \frac{171386739120}{778718279},$$

with $0 < \mu_* < b \le v$, and

$$F(b) = -\frac{2272864566320194107}{9384421237500155} < 0,$$
$$F(\mu_*) = -\frac{241993520128384136905733}{307445894130701973387} < 0.$$

Exact arithmetic progressions sum all 507 tail positions at each of the 100 states. As at the other bases, the two endpoint signs use convexity separately on the two sides of μ_*, including the zero state; they assert neither global convexity across the corner nor realizability of the maximizing relaxed profile.

The base at $t = 72$. The same parametric lemmas, rather than the uniform theorem, apply with

$$n = 395, \quad m = 322, \quad v = 107, \quad y = 35, \quad q = 218, \quad c = 54, \quad \lambda = \frac{213}{100}.$$

The ground set satisfies $q \le m \le 5t + 8$, $m \ge 4t - 1$, $n \ge 4t + 4$, $t + 1 \le v \le 2t$. Here $m = 3v + 1$, permitted by (A.2). Filtering the same cut grid leaves 155 averaged cuts, plus the nonempty-tail cut; the smallest residual FRR matching parameter is 35. Every positive link cap satisfies the weight domain of Lemma A.8. The cuts give 67 deficit states, and

$$100E = 11744277120, \qquad b = \frac{73491}{1288}, \qquad \mu_* = \frac{12233622}{278369}.$$

The maximizing envelope state is $\Delta = 400200$, with cap 72 on each of the 104 tail positions; it is not asserted realizable. Exact arithmetic progressions sum the envelope at all 67 deficit states.

At this smaller base, the two endpoints use different centre branches:

$$\xi_0 = \frac{5040}{107}, \qquad \theta := \frac{\xi_0 - 2}{1 - 2/v} = \frac{4826}{105}, \qquad 0 < \mu_* < \theta < b \le v.$$

Thus $\xi(\mu_*) = \xi_0$ is clamped and $\xi(b) = (1 - 2/v)b + 2$ is free. Using the corresponding formulas in (A.19) gives

$$F(b) = -\frac{3130309743249}{147056240800} < 0,$$
$$F(\mu_*) = -\frac{152269162897357}{4649358009660} < 0.$$

The centre join preserves values and slopes, as proved in Lemma A.13; hence its passage inside $[\mu_*, b]$ does not spoil convexity there. Convexity across the separate corner at μ_* is not assumed. In particular, the free-branch simplification must not be used at both endpoints.

The base at $t = 49$. Use the all-weight envelope $\widehat{E}_{\mathcal{T}}$, with

$$n = 269, \quad m = 219, \quad v = 73, \quad y = 24, \quad q = 149, \quad c = 37, \quad \lambda = 1.$$

Choose every cut of Lemma A.11 with $3 \le u \le 49$, $1 \le r \le \min\{u, 16\}$, together with $(1, 1, d_0)$. There are 661 averaged cuts, and the minimum residual FRR matching parameter is 33. The 190 deficit states include zero. Equation (A.13) checks every cap, including $u = 0, 1, 2, 3$; the restricted formula (A.11) is not used outside its weight domain. Exact summation over 70 tail positions in every state checks 13300 positions and yields

$$E = \widehat{E}_{\mathcal{T}} = \frac{50841725}{2}, \qquad \rho = \rho_{219,73} = \frac{217}{215}.$$

The remaining domains are $q \le m$, $m \ge 4t-1$, $n \ge 4t+4$, $t+1 \le v \le 2t$, and $0 < \rho < v$. With this ρ in Lemma A.13,

$$b = \frac{131747}{3379}, \qquad \mu_* = \frac{50841725}{1703016}, \qquad 0 < \mu_* < b \le v,$$

and the two certified values of (A.20) are

$$F(\mu_*) = -\frac{941079953521235}{62581832506368} < 0, \qquad F(b) = -\frac{80139181023}{8649854794} < 0.$$

The centre is evaluated as the displayed maximum at each endpoint. The proof uses convexity on each side of μ_*, including any centre join; it does not assert convexity across the minimum in (A.20). Neither this base nor its stability and equality consequences use HHL.

Proof of Theorem A.2. First suppose $\nu(\mathcal{F}) = t$. Use the sections of Section A.3 and the certified parameters above. Define

$$\eta_t = \min\{-F(\mu_*)/\mu_*, -F(b)/b\} > 0.$$

On each of the two intervals, $F(\mu) + \eta_t\mu$ is convex and nonpositive at its endpoints, hence nonpositive throughout. Equation (A.20) and Lemma A.6 give

$$d_t(\mathcal{F}) \ge \eta_t|\mathcal{A}|, \qquad \gamma_t(\mathcal{F}) \le (c+1)|\mathcal{A}| \le \frac{c+1}{\eta_t} d_t(\mathcal{F}).$$

The case $\mathcal{A} = \varnothing$ has $\gamma_t(\mathcal{F}) = 0$. The exact calculation of all 252 ratios gives its maximum at $t = 3480$:

$$\max_t \frac{c+1}{\eta_t} = \frac{3445298291203325575600158052000}{734432834661303950523791651} < 4692.$$

For the additional base $t = 2904$, the same normalization gives

$$\eta_{2904} = \frac{20689701908856362847614 5569}{13540970152354400090687 0000},$$
$$\frac{2179}{\eta_{2904}} = \frac{29505773961980237797606973 0000}{20689701908856362847614 5569} < 1427.$$

At $t = 888$, the same endpoint normalization gives

$$\eta_{888} = \frac{21670567755141894005923}{19500026868548619183900},$$
$$\frac{667}{\eta_{888}} = \frac{13006517921321928995661300}{21670567755141894005923} < 601.$$

At $t = 360$, the endpoint normalization gives

$$\eta_{360} = \frac{757621522106731369}{889567883357596850},$$
$$\frac{271}{\eta_{360}} = \frac{241072896389908746350}{757621522106731369} < 319.$$

At $t = 72$, the two centre branches give

$$\eta_{72} = \frac{1043436581083}{2796922927700},$$
$$\frac{55}{\eta_{72}} = \frac{153830761023500}{1043436581083} < 148.$$

At $t = 49$, the one-sided variance and all-weight envelope give

$$\eta_{49} = \frac{7285380093}{30659754622}, \qquad \frac{38}{\eta_{49}} = \frac{1165070675636}{7285380093} < 160.$$

Lemma 2.1 extends each estimate to $\nu(\mathcal{F}) \le t$; the identity in Lemma A.6 gives the corresponding symmetric-difference factor $1 + 2K$. □

Lemma A.14 (Propagation from a cover base)**.** *Let $t \geq 33$, $a \geq 4t+4$, and suppose the cover bound holds at $n = a$ for every four-graph of matching number at most t. Then it holds for all $n \geq a$.*

Proof. Shift a family on $n \geq a+1$ vertices and delete its last vertex. Its triple link has matching number at most t: $t+1$ disjoint triples would leave at least $n-1-3(t+1) \geq t+1$ unused vertices below n. Replacing the last vertex in the corresponding four-edges by these distinct unused vertices, using stability, gives a forbidden matching. Induction on n and (A.1) now give

$$|\mathcal{F}| \leq B_4(n-1,t) + B_3(n-1,t) = B_4(n,t).$$

□

Proposition A.15 (Quantitative propagation)**.** *Let $t \geq 33$, $a \geq 4t+4$, and $K \geq 0$. Suppose every shifted four-uniform family on $[a]$ with matching number at most t satisfies the cover bound and $\gamma_t \leq K d_t$. Then for every $n \geq a$, each such family $\mathcal{F}$ on $[n]$ satisfies*

$$\gamma_t(\mathcal{F}) \leq K \frac{\binom{n-t}{4}}{\binom{a-t}{4}} d_t(\mathcal{F}).$$

Proof. For $a \leq j \leq n$, let $\mathcal{F}_j = \mathcal{F} \cap \binom{[j]}{4}$, and write $d_j = B_4(j,t) - |\mathcal{F}_j|$ and $\gamma_j = |\mathcal{F}_j \setminus \mathcal{C}_{[t]}(j)|$. Each restriction is shifted with matching number at most t. Lemma A.14 supplies its cover bound.

For $j > a$, the triple link L_j of j has matching number at most t. Otherwise its $t+1$ disjoint triples could be completed with $t+1$ unused vertices below j, since $j-1-3(t+1) \geq t+1$. Shiftedness would give a forbidden matching. By (A.1) and Pascal's identity,

$$d_j - d_{j-1} = B_3(j-1,t) - |L_j| \geq 0.$$

Let ℓ_j count the triples of L_j avoiding $[t]$. For each such triple T, every $x \in [j-1] \setminus ([t] \cup T)$ gives $T \cup \{x\} \in \mathcal{F}_{j-1}$, again by shiftedness. There are $j-t-4$ choices of x, while each four-edge outside the cover is counted at most four times. Therefore

$$(j-t-4)\ell_j \leq 4\gamma_{j-1}, \qquad \gamma_j = \gamma_{j-1} + \ell_j \leq \frac{j-t}{j-t-4}\gamma_{j-1}.$$

Multiplication from $j = a+1$ to n telescopes:

$$\gamma_n \leq \frac{\binom{n-t}{4}}{\binom{a-t}{4}}\gamma_a \leq K\frac{\binom{n-t}{4}}{\binom{a-t}{4}}d_a \leq K\frac{\binom{n-t}{4}}{\binom{a-t}{4}}d_n.$$

The case $n = a$ uses the empty product. The base hypothesis already allows matching number smaller than t, so it applies to $\mathcal{F}_a$. □

Applying Proposition A.15 to Theorems A.1–A.2 proves Corollary A.4; the symmetric-difference identity is Lemma A.6. No new certificate is needed for this step.

A.8 Exact extremizers above the bases

Lemma A.16 (Propagation of shifted equality)**.** *Let $t \geq 33$, $a \geq 4t+4$, and suppose the cover bound holds for matching number at most t at every $n \geq a$. If $\mathcal{C}_t$ is the unique stable family of exact matching number t attaining $B_4(a,t)$, the same uniqueness holds at every $n \geq a$.*

Proof. Induct on $n > a$. Let $\mathcal{F}$ be stable with $\nu(\mathcal{F}) = t$ and $|\mathcal{F}| = B_4(n,t)$. Put $\mathcal{F}_0 = \mathcal{F} \cap \binom{[n-1]}{4}$ and let L be the triple link of n. We have $\nu(\mathcal{F}_0) = t$: if a t-matching uses n, replace that vertex by an unused smaller vertex, available since $n > 4t$, and apply stability. Lemma A.14 gives $\nu(L) \le t$; the triple bound therefore gives

$$|\mathcal{F}_0| + |L| \le B_4(n-1,t) + B_3(n-1,t) = B_4(n,t).$$

Both inequalities must be equalities, so induction identifies $\mathcal{F}_0$ with the complete cover of $[t]$. An edge $E \in \mathcal{F}$ avoiding $[t]$ would contain n. There are at least $3t$ vertices in $[n-1]\setminus([t]\cup(E\setminus\{n\}))$, since $n \ge 4t+4$. Partition $3t$ of them into triples and add a different vertex of $[t]$ to each. This gives a t-matching of $\mathcal{F}_0$ disjoint from E, a contradiction. Thus $\mathcal{F} \subseteq \mathcal{C}_t$; equal cardinalities finish the proof. □

Proof of Corollary A.3. The cover bound holds at all the stated bases by Theorems A.1–A.2, using shifting for arbitrary families. Lemma A.14 gives it at every larger ground set. At its maximum cardinality, matching number below t is impossible: adding a missing edge would contradict that bound. The same observation applies after shifting. Theorems A.1–A.2 and Lemma A.16 identify the fully shifted family. Invert the shifts one at a time using Lemma 5.8; each preceding family still has matching number at most t. □

Only exact equality is transferred through arbitrary shifts. Quantitative stability for shifted families above a base is supplied by Proposition A.15, with its explicit binomial loss. The constants 2285, 4692, 1427, 601, 319, 148 and 160 remain estimates at their specified bases. These equality arguments use classical shifting and the local cover bounds.

B Polynomial details for the uniform range

The two polynomial positivity checks in Proposition A.10 are transparent after translation:

$$\begin{aligned} P_A(2200+x) &= 3019933x^3 + 19228355100x^2 \\ &\quad + 40755950097500x + 28754097993750000, \\ P_J(2200+x) &= 23803113x^4 + 209384508244x^3 \\ &\quad + 690694581174300x^2 + 1012615286669630000x \\ &\quad + 556715897678850000000. \end{aligned}$$

Every coefficient is positive. These are identities in the product extension of binomial coefficients. The moment-margin derivation can be checked without a symbolic system. For example in the low-density row, $y_- = 29t/50 - 1$, $y_+ = 29t/50$, $b = 474t/625$, and $w = 3t/4 + 3/2$ give

$$M_w \ge ty_- - wb - \frac{2ty_+}{b}\frac{13}{25} = \frac{7t(474t - 124127)}{296250}.$$

The high-density row uses $b = 161t/200$, $w = 23t/32 + 1$ and $155/316$ in the last factor. The remaining margins follow from $bt + (b-2t)y_+$. These algebraic identities are displayed for direct inspection.

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[13] P. Frankl, *On the maximum number of edges in a hypergraph with given matching number*, Discrete Appl. Math. **216** (2017), 562–581. doi:10.1016/j.dam.2016.08.003. Theorem locators for the deletion inequality and the historical announcement follow the later author copy, Proposition 5.1 and Theorem 13.2; they are distinct from the first arXiv version 1205.6847v1.

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