

FINITE LENGTH DATA DO NOT UNIVERSALLY DETERMINE FLAT THREE-TORI

EXPLICIT CONSTRUCTIONS, PROOFS, AND A CAUTION FOR COSMIC TOPOLOGY

ABSTRACT. A flat three-torus is determined by a Euclidean lattice up to orthogonal changes of coordinates, or equivalently by a positive-definite Gram matrix up to integral change of basis. A finite list of short lattice lengths is a much weaker datum. This paper gives an explicit and fully elementary construction showing that the weakness can persist over a large range of scales. For every real parameter $A \geq 2$, we exhibit two positive-definite Gram matrices $G_+(A)$ and $G_-(A)$ whose associated flat tori have different volumes and hence are not isometric, while their length multisets agree below a radius

$$R_A = \sqrt{2A^2 - 2A^{-2} - 4A^{-4}}.$$

The shortest nonzero lattice length is exactly one in both examples, so R_A is not made large by a uniform rescaling; it tends to infinity with A . The proof isolates the coordinate support on which the two quadratic forms differ and supplies a sharp enough lower bound to exclude that support from every shorter vector. A second, isotropic example gives a compact rational pair whose short length multisets agree below $\sqrt{2}$ and differ at that threshold. We then place the construction alongside the theory of ternary quadratic forms and explain what it does—and does not—say about cosmic topology. In particular, finite agreement of direct-lattice lengths is not equivalent to equality of CMB covariance data, nor does it establish an observational degeneracy for any specified cosmological model.

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1. INTRODUCTION

The global shape of a spatially flat universe is not fixed by the local statement that its spatial curvature vanishes. Even when the universal cover is Euclidean three-space, a spatial slice may be a quotient by a discrete group of isometries. The simplest compact example is a flat three-torus, obtained by identifying opposite faces of a parallelepiped. Mathematically, its geometry is encoded by a lattice in $\mathbb{R}^3$; observationally, its global identifications can in principle affect repeated images, the cosmic microwave background (CMB), and other large-scale correlations. These questions form part of the broader subject of cosmic topology [3, 4, 7, 5].

A useful mathematical question is whether a finite amount of short-distance information can uniquely identify the torus. For a lattice Λ , one may record the lengths of all lattice vectors below a cutoff R , with multiplicities. This data is called here the *truncated length multiset*. It is closely related to the first part of the theta series of the corresponding quadratic form. It is not the same thing as the full theta series, and it is not the same thing as the complete spectrum of every physical operator on a compact space.

The distinction between finite data and complete data is the point of this paper. We prove an explicit nonuniqueness statement, rather than proposing a generic heuristic. Our principal construction changes one off-diagonal entry of a Gram matrix from $+\varepsilon$ to $-\varepsilon$. The changed term appears only when two selected integer coordinates are both nonzero. When those coordinates are nonzero, the vector is forced to be long; when one of them vanishes, the two quadratic forms agree exactly. By increasing the two corresponding diagonal scales while keeping the shortest direction fixed, we push all potentially distinguishing vectors to arbitrarily large length.

Main result in plain language. There are non-isometric flat three-tori with the same short closed-geodesic length data up to a cutoff that is arbitrarily large compared with the shortest closed-geodesic length. The result is explicit, parameterized, and proved by direct estimates.

1.1. What is and is not claimed. The result is deliberately narrow. It does not say that the full length spectrum fails to determine a flat three-torus. In dimension three, Schiemann proved a strong uniqueness theorem for ternary positive-definite quadratic forms from their full theta series [8]. Our pair is consistent with that theorem: the two forms eventually differ, and we explain why their full length multisets cannot agree.

Nor does the construction prove that two different cosmic-topology models produce identical CMB maps, identical angular power spectra, or identical likelihoods. A direct-lattice length multiset is only one geometric input. CMB predictions also depend on the reciprocal lattice, the orientation of the fundamental domain relative to the sky, cosmological transfer functions, initial conditions, observational masks, and noise. The cosmological discussion therefore gives a careful interpretation of the mathematical example, not a claim of a new observational loophole.

1.2. Contributions and structure. The paper makes four modest, concrete contributions.

- (1) It defines the comparison data carefully, including multiplicities and normalization by the shortest nonzero lattice length.
- (2) It proves a parameterized construction with a common finite length multiset whose cutoff grows without bound after fixing the systole to one.
- (3) It gives a second rational example in which the first discrepancy can be located exactly at a small threshold, making the mechanism easier to inspect by hand.
- (4) It relates the example to theta series and to cosmic-topology observables while keeping the logical scope of each inference explicit.

Section 2 establishes notation. Section 3 gives the main construction and proof. Sections 5–6 provide an arithmetic companion example and the relation to classical uniqueness results. Sections 7 and 8 discuss reproducibility and cosmological interpretation. The appendices collect detailed estimates and a finite-search verification protocol.

2. LATTICES, GRAM MATRICES, AND FLAT TORI

2.1. From a lattice basis to a metric. Let B be an invertible real 3×3 matrix whose columns are basis vectors b_1, b_2, b_3 in Euclidean space. The lattice generated by those vectors is

$$\Lambda = B\mathbb{Z}^3 = \{Bn : n \in \mathbb{Z}^3\}.$$

Its Gram matrix is

$$G = B^\top B,$$

which is symmetric positive definite. The squared norm of the lattice vector Bn is

$$\|Bn\|^2 = n^\top G n =: Q_G(n).$$

Conversely, every real symmetric positive-definite matrix G is the Gram matrix of some basis: for example, choose a Cholesky factorization $G = B^\top B$.

The quotient $\mathbb{R}^3/\Lambda$ inherits a flat Riemannian metric. Using the integer coordinate model, we can represent the same torus as $\mathbb{R}^3/\mathbb{Z}^3$ with constant metric tensor G . This is the convenient representation for calculations: the integer vector n labels a deck translation, and its length is $\sqrt{Q_G(n)}$.

Definition 2.1. For a symmetric positive-definite matrix G , let

$$\mathbb{T}_G = (\mathbb{R}^3, G)/\mathbb{Z}^3$$

be the associated flat three-torus. Its covolume, equivalently the Riemannian volume of a fundamental domain, is

$$\text{vol}(\mathbb{T}_G) = \sqrt{\det G}.$$

Its systole is the length of its shortest noncontractible closed geodesic,

$$\text{sys}(\mathbb{T}_G) = \min_{n \in \mathbb{Z}^3 \setminus \{0\}} \sqrt{n^\top G n}.$$

The volume formula follows from $|\det B| = \sqrt{\det(B^\top B)}$. The systole is positive because G is positive definite and the integer lattice is discrete.

2.2. Integral changes of basis and isometry. A change of lattice basis replaces B by BU , where $U \in \mathrm{GL}(3, \mathbb{Z})$. The Gram matrix then becomes

$$G \longmapsto U^\top G U.$$

Since $|\det U| = 1$, this transformation preserves determinant and volume. Conversely, an isometry between flat tori induces a lattice isometry and hence an integral equivalence of their Gram forms. Thus a difference in volume is a simple obstruction to isometry.

Proposition 2.2. *If G_1 and G_2 are Gram matrices of isometric flat tori, then $\det G_1 = \det G_2$. In particular, unequal determinants imply non-isometry.*

Proof. An isometry carries one lattice to the other by an orthogonal linear map after lifting to the universal cover. If B_i are basis matrices, then after an orthogonal transformation and an integral basis change they describe the same lattice. The resulting Gram matrices are integrally equivalent, $G_2 = U^\top G_1 U$ with $U \in \mathrm{GL}(3, \mathbb{Z})$. Consequently

$$\det G_2 = (\det U)^2 \det G_1 = \det G_1.$$

□

2.3. Length data with multiplicity. A torus contains infinitely many closed geodesics, including repetitions of shorter geodesics. To avoid ambiguity, we compare the lengths of deck translations indexed by all nonzero integer vectors. The vectors n and $-n$ are both counted; this is a convenient multiplicity convention and makes no difference to equality of the two data sets being compared.

Definition 2.3. For G positive definite and $R \geq 0$, define its truncated length multiset by

$$\mathcal{L}_G(R) = \left\{ \sqrt{n^\top G n} : n \in \mathbb{Z}^3 \setminus \{0\}, n^\top G n \leq R^2 \right\},$$

where each vector contributes one entry, so repeated values occur with their natural multiplicities. Thus this definition uses a closed cutoff: vectors of length exactly R are included. The main theorem compares these multisets only at radii $R < R_A$, so no convention issue occurs at the boundary. In the isotropic example, we explicitly note what happens when the cutoff itself is included.

The length spectrum is the multiset $\mathcal{L}_G(\infty)$. If one instead records the number of vectors with squared length exactly t , one obtains the representation number

$$r_G(t) = \#\{n \in \mathbb{Z}^3 : n^\top G n = t\},$$

whenever t is in the value set. For rational or integral forms, these numbers are the coefficients of a theta series after choosing a common scale.

2.4. A global count eventually detects the volume. Define the cumulative lattice-point count, including the zero vector, by

$$N_G(R) = \#\{n \in \mathbb{Z}^3 : n^\top G n \leq R^2\}.$$

For each fixed positive-definite G , the standard lattice-point estimate for a dilated ellipsoid gives

$$N_G(R) = \frac{4\pi}{3\sqrt{\det G}} R^3 + O_G(R^2), \quad R \rightarrow \infty. \quad (1)$$

Indeed, the ellipsoid $\{x : x^\top G x \leq R^2\}$ has volume $4\pi R^3/(3\sqrt{\det G})$; comparing it with the union of unit cubes centered at its lattice points leaves only a bounded-width boundary layer, whose volume is $O_G(R^2)$ (see the general lattice-point framework in [1]). If $\det G_1 \neq \det G_2$, the leading coefficients in (1) differ. Hence the full length multisets cannot coincide: their cumulative counts differ for all sufficiently large R . This applies to the anisotropic family for every real $A \geq 2$, not only to integer parameters for which an integral theta-series normalization is available.

2.5. Why finite data are different from the full spectrum. An arbitrary finite list of measurements generally gives fewer constraints than a full infinite sequence. For lattices, however, this observation alone is not a proof: infinitely many integer vectors could create unexpected short relations, and a matrix pair that differs algebraically might still have matching length multiplicities for more reasons than expected. The construction below addresses both issues directly. It gives a map between all candidate short vectors, not merely a numerical coincidence among a few listed lengths.

3. THE ANISOTROPIC CONSTRUCTION

3.1. The family. Fix $A \geq 2$ and set

$$\varepsilon = A^{-2}.$$

Consider the pair of symmetric matrices

$$G_\pm(A) = \begin{pmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon & A^2 & \pm\varepsilon \\ \varepsilon & \pm\varepsilon & A^2 \end{pmatrix}. \quad (2)$$

Their quadratic forms are

$$Q_\pm(n_1, n_2, n_3) = n_1^2 + A^2(n_2^2 + n_3^2) + 2\varepsilon n_1 n_2 + 2\varepsilon n_1 n_3 \pm 2\varepsilon n_2 n_3. \quad (3)$$

The matrices agree in every entry except the $(2, 3)$ and $(3, 2)$ entries. This design is purposeful: the only changed term is proportional to $n_2 n_3$, so the forms agree at each integer vector with $n_2 n_3 = 0$.

Theorem 3.1 (Finite length-spectrum agreement with fixed systole). *For every $A \geq 2$, the matrices $G_+(A)$ and $G_-(A)$ in (2) are positive definite and define non-isometric flat three-tori. Both tori have systole one. Set*

$$R_A = \sqrt{2A^2 - 2A^{-2} - 4A^{-4}}. \quad (4)$$

Then

$$\mathcal{L}_{G_+(A)}(\mathbf{R}) = \mathcal{L}_{G_-(A)}(\mathbf{R}) \quad \text{for every } \mathbf{0} < \mathbf{R} < \mathbf{R}_A.$$

In particular, $R_A / \text{sys}(\mathbb{T}_{G_\pm(A)}) \rightarrow \infty$ as $A \rightarrow \infty$.

Proof roadmap. The proof has four logically separate parts. First, Sylvester's criterion shows both matrices are positive definite. Second, their determinants differ, so their volumes differ and the tori are non-isometric. Third, a completed-square estimate shows that any vector with both n_2 and n_3 nonzero has length at least R_A . Finally, the two forms agree exactly whenever $n_2 n_3 = 0$. Thus every vector below R_A has the same squared length for both forms, with the same integer label. A separate estimate shows the shortest nonzero vectors are $\pm(1, 0, 0)$, giving systole one.

3.2. Positive definiteness. The leading principal minors of $G_{\pm}(A)$ are

$$\Delta_1 = 1, \quad \Delta_2 = A^2 - \varepsilon^2,$$

and direct expansion gives

$$\det G_{\pm}(A) = A^4 - \varepsilon^2 - 2A^2\varepsilon^2 \pm 2\varepsilon^3. \quad (5)$$

For $A \geq 2$, one has $0 < \varepsilon \leq 1/4$. Therefore

$$\begin{aligned} \Delta_2 &\geq 4 - \frac{1}{16} > 0, \\ \det G_-(A) &\geq 16 - \frac{1}{16} - \frac{1}{2} - \frac{1}{32} > 0. \end{aligned}$$

The determinant of G_+ is larger than that of G_- , so it is positive as well. All leading principal minors are positive, and Sylvester's criterion proves positive definiteness.

Remark 3.2. The lower bound just used is intentionally simple rather than optimized. Since $A \geq 2$, monotonicity of the positive powers and the estimate $\varepsilon \leq 1/4$ already leave a wide positive margin. No asymptotic argument is needed to establish positive definiteness.

3.3. Different volumes and non-isometry. Subtracting the determinants in (5) yields

$$\det G_+(A) - \det G_-(A) = 4\varepsilon^3 > 0. \quad (6)$$

Consequently

$$\text{vol}(\mathbb{T}_{G_+(A)}) = \sqrt{\det G_+(A)} \neq \sqrt{\det G_-(A)} = \text{vol}(\mathbb{T}_{G_-(A)}).$$

Proposition 2.2 then implies that the tori are not isometric. This conclusion is invariant under an integral change of basis; it does not depend on the chosen coordinate representation.

It is important to separate this step from the length comparison. Determinant inequality certifies non-isometry, but it does not by itself establish agreement of short lengths. The latter requires the lower bound proved next.

3.4. Lower bound for mixed-coordinate vectors. Write $n = (n_1, n_2, n_3) \in \mathbb{Z}^3$. Completing the square in (3) gives

$$\begin{aligned} Q_{\pm}(n) &= \left(n_1 + \varepsilon(n_2 + n_3)\right)^2 \\ &\quad + (A^2 - \varepsilon^2)(n_2^2 + n_3^2) + 2(\pm\varepsilon - \varepsilon^2)n_2 n_3. \end{aligned} \quad (7)$$

The last two terms form the quadratic form with matrix

$$M_{\pm} = \begin{pmatrix} A^2 - \varepsilon^2 & \pm\varepsilon - \varepsilon^2 \\ \pm\varepsilon - \varepsilon^2 & A^2 - \varepsilon^2 \end{pmatrix}.$$

Its eigenvalues are $A^2 - \varepsilon^2 \pm (\pm\varepsilon - \varepsilon^2)$, with the sign order determined by the absolute value of the off-diagonal entry. In particular, its smallest eigenvalue is

$$\lambda_{\min}(M_+) = A^2 - \varepsilon, \quad \lambda_{\min}(M_-) = A^2 - \varepsilon - 2\varepsilon^2.$$

It follows for either sign that

$$(n_2, n_3)M_{\pm}(n_2, n_3)^{\top} \geq (A^2 - \varepsilon - 2\varepsilon^2)(n_2^2 + n_3^2). \quad (8)$$

If $n_2n_3 \neq 0$, then $n_2^2 + n_3^2 \geq 2$. Combining (7) and (8) gives

$$Q_{\pm}(n) \geq 2A^2 - 2\varepsilon - 4\varepsilon^2 = R_A^2 \quad \text{whenever } n_2n_3 \neq 0. \quad (9)$$

This is the pivotal estimate. It does not require n_1 to be bounded and does not assume that n_1 is small compared with A . The completed square controls all possible cancellations involving the first coordinate.

3.5. Equality of truncated length multisets. The difference of the quadratic forms is especially transparent:

$$Q_+(n) - Q_-(n) = 4\varepsilon n_2 n_3. \quad (10)$$

Thus

$$n_2 n_3 = 0 \implies Q_+(n) = Q_-(n).$$

Fix any $R < R_A$. If $Q_{\pm}(n) < R^2$, estimate (9) shows that $n_2 n_3$ cannot be nonzero. Hence every vector contributing to either truncated length multiset lies in the union of the coordinate planes $n_2 = 0$ and $n_3 = 0$, where the quadratic forms agree pointwise. Moreover, if a vector lies below the cutoff for one form, the same vector has the same value under the other form, so it lies below the cutoff there as well. Multiplicities are preserved because the correspondence is the identity on integer vectors.

This proves the finite length-spectrum equality in Theorem 3.1. Notice that no pairing argument or rearrangement of values is needed in this anisotropic family: the same labelled vector has the same length throughout the common range.

3.6. The systole remains one. We verify that the construction is not just a uniform rescaling. If $n_2 = n_3 = 0$, then $Q_{\pm}(n) = n_1^2$, whose smallest positive value is one, attained by $n = \pm(1, 0, 0)$. If exactly one of n_2, n_3 is nonzero, assume for definiteness that $n_3 = 0$. Then

$$Q_{\pm}(n_1, n_2, 0) = (n_1 + \varepsilon n_2)^2 + (A^2 - \varepsilon^2)n_2^2.$$

For $n_2 \neq 0$ this is at least $A^2 - \varepsilon^2 > 1$. The same estimate holds with indices 2 and 3 interchanged. If $n_2 n_3 \neq 0$, equation (9) gives $Q_{\pm}(n) \geq R_A^2 > 1$ for $A \geq 2$. To see the last strict inequality at the smallest allowed value, observe

$$R_A^2 \geq 8 - \frac{1}{2} - \frac{1}{8} > 1,$$

and the expression increases for $A \geq 2$. Therefore the only shortest nonzero vectors are $\pm e_1$, and

$$\text{sys}(\mathbb{T}_{G_+(A)}) = \text{sys}(\mathbb{T}_{G_-(A)}) = 1. \quad (11)$$

3.7. The cutoff grows without rescaling. Substituting $\varepsilon = A^{-2}$ into (4) gives

$$R_A^2 = 2A^2 - 2A^{-2} - 4A^{-4}.$$

Consequently $R_A \sim \sqrt{2}A$ as $A \rightarrow \infty$. By (11), the normalized cutoff is exactly $R_A/\text{sys} = R_A$, and hence

$$\frac{R_A}{\text{sys}(\mathbb{T}_{G_{\pm}(A)})} \longrightarrow \infty. \quad (12)$$

This is the main strength of the example. One can make the initial interval of exact agreement arbitrarily large when measured in units of the shortest closed geodesic, while retaining non-isometry.

4. GEOMETRIC PICTURE AND INTERPRETATION OF THE MAIN PROOF

The construction can be understood through coordinate support. The first coordinate represents a short direction; the second and third coordinates represent two long directions of comparable scale. The $+/-$ choice changes only the mutual inner product of the two long basis vectors. A lattice vector using at most one of those long directions does not see that change. A vector using both must pay the length cost of at least one unit in each of those directions, placing it beyond the derived cutoff.

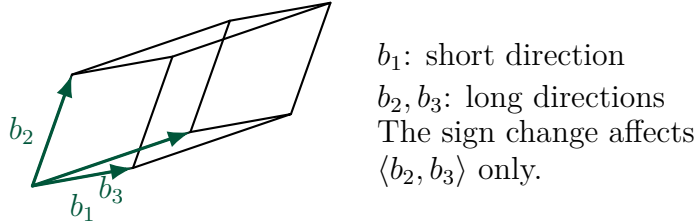


FIGURE 1. A schematic fundamental parallelepiped. The illustration is qualitative: the proof is entirely matrix-based and does not rely on the drawing being to scale. The changed inner product couples coordinates n_2 and n_3 , so only vectors using both long directions can distinguish the two forms.

There are two distinct geometric statements here. First, G_+ and G_- represent different global flat geometries because their volumes differ. Second, their low-length vector data agree because those vectors are confined to a common union of coordinate planes. The second statement is not a topological identification: both underlying spaces are topologically $\mathbb{T}^3$, while their Riemannian metrics are different.

The construction does not say that all finite geometric information is useless. A cutoff can be chosen large enough to detect the difference for any fixed A . Rather, no single rule based only on the short length multiset can be assumed to distinguish every possible flat three-torus without additional hypotheses or a cutoff adapted to the unknown lattice parameters.

5. AN ISOTROPIC RATIONAL EXAMPLE

The anisotropic family is best for proving an arbitrarily large relative cutoff, but its parameter and estimate obscure the basic mechanism. We now give a smaller example with rational Gram matrices and an exactly located first discrepancy.

Set $\delta = 1/6$ and define

$$H_+ = \begin{pmatrix} 1 & \delta & \delta \\ \delta & 1 & \delta \\ \delta & \delta & 1 \end{pmatrix}, \quad H_- = \begin{pmatrix} 1 & \delta & \delta \\ \delta & 1 & -\delta \\ \delta & -\delta & 1 \end{pmatrix}. \quad (13)$$

Their quadratic forms are

$$P_+(a, b, c) = a^2 + b^2 + c^2 + \frac{1}{3}(ab + ac + bc),$$

$$P_-(a, b, c) = a^2 + b^2 + c^2 + \frac{1}{3}(ab + ac - bc).$$

The forms agree on the coordinate planes $c = 0$ and $b = 0$. On $a = 0$, changing b to $-b$ maps one restricted form to the other. Thus, on each coordinate plane, the multiset of lengths is the same, although not every labelled vector is fixed.

Proposition 5.1. *The matrices H_+ and H_- are positive definite and define non-isometric flat tori. Their truncated length multisets agree for every $R < \sqrt{2}$, and differ at $R = \sqrt{2}$ under the convention that lengths equal to the cutoff are included.*

Proof. The eigenvalues of H_+ are $1 - \delta$ (with multiplicity two) and $1 + 2\delta$, hence are positive. The eigenvalues of H_- are $1 + \delta$ (with multiplicity two) and $1 - 2\delta$, hence are also positive. The determinants are

$$\det H_+ = (1 - \delta)^2(1 + 2\delta) = 1 - 3\delta^2 + 2\delta^3 = \frac{25}{27}, \quad (14)$$

$$\det H_- = (1 + \delta)^2(1 - 2\delta) = 1 - 3\delta^2 - 2\delta^3 = \frac{49}{54}. \quad (15)$$

They differ, so the tori are not isometric.

Partition vectors with at least one zero coordinate by their exact support. For support of size one the squared lengths agree, and for support of size two the restricted forms are either identical or related by the sign change $b \mapsto -b$ on the (b, c) -plane. This gives a multiplicity-preserving correspondence without double-counting vectors on the axes. If all three are nonzero, then

$$P_{\pm}(a, b, c) \geq (1 - 2\delta)(a^2 + b^2 + c^2) \geq 3 \left(1 - \frac{1}{3}\right) = 2.$$

The first inequality follows from $|ab| + |ac| + |bc| \leq a^2 + b^2 + c^2$ and the fact that the absolute value of each off-diagonal coefficient in the quadratic polynomial is $1/3$. Hence no vector with all three coordinates nonzero has length below $\sqrt{2}$. Below that threshold, the length multiset consists only of coordinate-plane vectors, whose lengths match as just described.

At the threshold, $P_-(1, -1, -1) = 2$, and the same is true for its negative. These are two additional vectors in H_- 's length multiset. No vector of H_+ has squared length

2: if all three coordinates are nonzero, its smallest possible squared length is at least $(1 - \delta) \cdot 3 = 5/2 > 2$; if exactly two are nonzero and both have absolute value one, the value is $2 \pm 1/3$, while if one has absolute value at least two then $a^2 + b^2 \geq 5$ and the value is at least $(1 - \delta)5 > 2$. An axis vector has squared length an integer square, not 2. Thus the multiplicities differ at length $\sqrt{2}$. $\square$

5.1. What this example adds. The isotropic example shows the same support-based mechanism in a more symmetric setting. Unlike the anisotropic family, however, its guaranteed common interval is bounded in the chosen normalization. The purpose of including both examples is not to repeat one theorem twice: the first establishes an unbounded normalized cutoff, while the second makes it easy to see how the first distinguishing vector can sit exactly at a threshold.

At $\delta = 1/6$, multiplying the squared lengths by three produces the integer-valued forms

$$F_+(a, b, c) = 3a^2 + 3b^2 + 3c^2 + ab + ac + bc,$$

$$F_-(a, b, c) = 3a^2 + 3b^2 + 3c^2 + ab + ac - bc.$$

Their theta series have equal coefficients for exponents below 6 and differ at exponent 6. The scaling changes all lengths by the same factor and does not change the logical comparison. This gives an elementary finite-prefix example in the language of quadratic forms.

6. THETA SERIES AND THE FULL LENGTH SPECTRUM

6.1. From length multisets to theta series. For a positive-definite quadratic form Q taking integer values on $\mathbb{Z}^3$, its theta series is

$$\Theta_Q(q) = \sum_{n \in \mathbb{Z}^3} q^{Q(n)} = \sum_{m=0}^{\infty} r_Q(m) q^m,$$

where $r_Q(m)$ counts the integer vectors with $Q(n) = m$. If Q takes rational values, multiplication by a common denominator puts the series in integer powers of q . For a real Gram matrix with irrational entries, it is often more natural to speak directly of the multiset of lengths or the corresponding exponential sum, rather than an integer-power modular form.

The coefficient sequence is a complete summary of the lengths with multiplicity, but the phrase *complete* is crucial. Equality of the first N coefficients says nothing by itself about all later coefficients unless a separate theorem supplies a bound for the form class under study.

6.2. Schiemann's theorem and its proper scope. Schiemann proved that a positive-definite ternary quadratic form is determined up to integral equivalence by its full theta series [8]. The result is a strong dimension-three uniqueness theorem. The paper also gives a finite bound, expressed in terms of the third successive minimum of a reduced form, beyond which equality of coefficients forces equality of the forms; the exact hypotheses and normalization in that theorem matter and should be read from the original source when applying it.

Our construction is compatible with this uniqueness result. It does not supply two forms with the same full theta series. Rather, it supplies a pair whose initial data agree over a finite interval while their determinants differ. The difference must eventually appear. For the rational isotropic example above, it appears explicitly at the squared length 2 in the Q normalization, or exponent 6 after multiplication by three.

For the anisotropic family, if A is an integer then the entries of $G_{\pm}(A)$ are rational. The form

$$\tilde{Q}_{\pm}(n) = A^4 Q_{\pm}(n)$$

has integer coefficients and integral values on $\mathbb{Z}^3$. The two full theta series cannot be equal: otherwise the ternary uniqueness theorem would force integral equivalence of the forms, contradicting their unequal determinants. This observation does not identify the very first differing coefficient for general A ; the main theorem only supplies a lower bound on where differences can begin.

6.3. Why a finite prefix can fail to identify the form. The key logic is simple but often blurred in informal discussions. A finite vector-length prefix is a finite set or multiset of numerical values. The matrix itself has six independent real entries before basis equivalence, and constraints from a finite prefix depend on which integer vectors happen to enter the cutoff. In the present construction, the scale separation makes this incidence pattern transparent: the short vectors live on two coordinate planes, and the changed coefficient is invisible on those planes.

This is stronger than merely saying that a finite list contains fewer values than an infinite one. It identifies a mechanism and quantifies a cutoff. It is also weaker than saying that the metric cannot be reconstructed from observations at any finite precision. If the whole metric is known a priori to belong to a restricted parameter family, or if continuous measurements supply more information than a finite length list, uniqueness may be restored. The theorem is a counterexample to unrestricted finite-length identification, not a universal no-go theorem for metric reconstruction.

7. COMPUTATIONAL AUDIT AND REPRODUCIBILITY

The principal results admit exact symbolic proofs. Computation is useful as a check, but it is not used as a substitute for the inequalities above. This section specifies what can be checked independently and what any numerical experiment should report.

7.1. Symbolic checks. For the anisotropic family, a computer algebra system can verify the determinant identity

$$\det G_{\pm} = A^4 - \varepsilon^2 - 2A^2\varepsilon^2 \pm 2\varepsilon^3.$$

It can also verify the completed-square identity (7). These are polynomial identities, so a symbolic verification is definitive for the displayed algebra, though it does not on its own verify the logical implication from the identity to the cutoff estimate. That implication is the separate eigenvalue argument in (8)–(9).

7.2. Finite lattice enumeration. A bounded search over integer vectors can check small instances of the theorem. For each selected A , choose an integer bound N large enough to include all vectors whose computed squared norm is below a chosen cutoff. Then enumerate

$$\{n \in \mathbb{Z}^3 \setminus \{0\} : |n_i| \leq N, Q_{\pm}(n) < R^2\}.$$

The observed value multisets should match for every $R < R_A$. Because a coordinate bound is not itself a proof that no omitted vector is short, numerical experiments should be described as spot checks only; the analytical lower bound covers all integers without a search limit.

Quantity	Exact expression	Role in the proof
Parameter	$\varepsilon = A^{-2}$	couples the directions weakly
Determinant gap	$4\varepsilon^3$	proves distinct volume
Systole	1	fixes global length normalization
Support condition	$n_2 n_3 \neq 0$	locates changed term
Squared cutoff	$2A^2 - 2\varepsilon - 4\varepsilon^2$	excludes changed support
Asymptotic cutoff	$R_A \sim \sqrt{2}A$	gives unbounded relative range

TABLE 1. The proof certificate for the main family. Each row corresponds to a separate part of the argument; none is inferred solely from floating-point output.

7.3. Floating-point caution. For small determinant gaps or lengths close to the cutoff, floating-point comparisons can incorrectly merge two values or assign a vector to the wrong side of the threshold. The main family has determinant difference $4A^{-6}$, which becomes small for large A . A numerical demonstration should therefore use rational or symbolic arithmetic where possible, or state a precision tolerance and avoid claiming exact equality based on rounded decimals. The mathematical proof itself is exact and remains valid for every real $A \geq 2$.

7.4. A reproducible checking protocol. A minimal independent audit can proceed as follows:

- (1) Define G_+ and G_- by (2); do not transcribe the off-diagonal entries from a rounded decimal table.
- (2) Expand both determinants symbolically and subtract them. The result must simplify to $4\varepsilon^3$.
- (3) Expand the right-hand side of (7); verify that its coefficients match $Q_{\pm}$ exactly.
- (4) Diagonalize the two-dimensional block $M_{\pm}$ symbolically, and check the lower eigenvalue in each sign case.
- (5) Confirm the systole estimate separately from the truncated-spectrum estimate.
- (6) For chosen numerical values, compare counts with multiplicity, not just sets of distinct decimal lengths.

This protocol is intentionally short. It makes the dependency chain clear and reduces the risk that a numerical illustration is mistaken for the theorem.

8. COSMIC TOPOLOGY: WHAT GEOMETRIC NONUNIQUENESS MEANS

8.1. The cosmological setting. In a spatially flat FLRW model with compact spatial sections, one may write locally

$$ds^2 = -c^2 dt^2 + a^2(t) d\mathbf{x}^\top G d\mathbf{x},$$

where $a(t)$ is the scale factor and G describes the constant shape of the flat spatial torus. In this convention the comoving length of the translation labelled by n is $\sqrt{n^\top G n}$, while its physical length at cosmic time t is multiplied by $a(t)$.

The local field equations determine curvature and expansion only after the usual dynamical inputs are specified; they do not by themselves fix the global identifications. This is the general starting point of cosmic-topology studies [3, 4]. Searches using CMB temperature and polarization data provide important constraints on compact spatial topologies, but the interpretation depends on the class of spaces considered and the signal statistic being used [6, 7, 5].

8.2. Matched-circle geometry. In the universal cover, a spatial identification by a translation of comoving length d produces a translated copy of the last-scattering sphere. If both spheres have comoving radius χ_{rec} and their centers are separated by $d < 2\chi_{\text{rec}}$, the two spheres intersect in a circle. If α is the angular radius of that circle as seen from either center, elementary Euclidean geometry gives

$$\cos \alpha = \frac{d}{2\chi_{\text{rec}}}. \quad (16)$$

Here d is a comoving translation length and, in the torus model, has the form $d = \sqrt{n^\top G n}$ for a nonzero integer vector n . Equation (16) is a geometric relation; it does not state that a matched circle will necessarily be statistically detectable in real data.

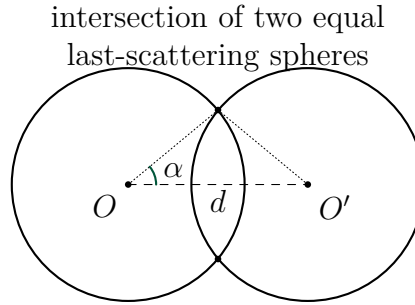


FIGURE 2. The elementary geometry behind matched circles. The diagram shows the cross-section through the sphere centers; the three-dimensional spheres intersect in a circle. The angular radius satisfies $\cos \alpha = d/(2\chi_{\text{rec}})$ when $d < 2\chi_{\text{rec}}$.

The construction in Theorem 3.1 says that two non-isometric tori can share all direct-lattice translation lengths below a large normalized cutoff. If the comoving last-scattering distance sampled only that common interval, then the geometric list of candidate translations short enough to intersect the last-scattering sphere would be shared. This is a conditional statement about a simplified geometric list, not a forecast for matched-circle

data: actual observability also depends on circle radius, orientation, masks, foregrounds, noise, and whether temperature or polarization correlations survive the full forward model.

8.3. Direct and reciprocal lattices. It is especially important not to identify the direct-lattice length multiset with the CMB covariance. If $G = B^\top B$ and the direct lattice is $B\mathbb{Z}^3$, then the reciprocal lattice is generated (up to the chosen 2π convention) by $B^{-\top}$. The squared norm of a reciprocal vector indexed by $m \in \mathbb{Z}^3$ is

$$\|2\pi B^{-\top} m\|^2 = (2\pi)^2 m^\top G^{-1} m.$$

Thus reciprocal-space information involves G^{-1} , not just the short-vector list for G . Furthermore, CMB multipoles depend on how primordial modes evolve through the cosmological transfer functions and project onto the sky. Equality of a finite direct length multiset therefore does not imply equality of reciprocal-lattice mode data or equality of CMB statistics.

This distinction matters for claims about observational identifiability. A proper model comparison must specify the observable, noise and masking assumptions, the cosmological parameter set, and the likelihood or test statistic. Published analyses have tested matched-circle and likelihood signals for several classes of topologies and reported constraints or nondetections under their stated assumptions; those results should be read with their model scope intact [6, 7, 5]. The present mathematical example does not revise those constraints.

8.4. A bounded inference, not a cosmological claim. The rigorous inference is:

A finite truncation of the direct-lattice length multiset is not, without additional restrictions, a universal identifier of a flat three-torus, even when lengths are normalized by the systole.

The following stronger statements do *not* follow from the theorem:

- that two such tori have identical full Laplace spectra;
- that their full CMB covariance matrices or sky maps are identical;
- that either torus is compatible with a particular observed CMB map;
- that the present universe has a compact toroidal topology.

The paper therefore provides a mathematical caution about finite geometric data, not a new fit to cosmological observations. Any future application should build the full forward model and compute the observables explicitly.

9. RELATION TO ESTABLISHED MATHEMATICAL LITERATURE

9.1. Geometry of numbers. The basic language of lattices, Gram matrices, successive minima, and covolume belongs to the classical geometry of numbers; standard treatments include Cassels [1] and Conway–Sloane [2]. In the present proof, the main estimate is a two-dimensional eigenvalue bound after completing a square. It is not a deep geometry-of-numbers theorem; its value lies in the design of a pair for which the estimate controls every integer vector.

9.2. Spectral geometry and isospectrality. There is a broad literature on the extent to which spectral data determine geometry. The key distinctions are between the length spectrum of closed geodesics, the spectrum of the Laplace operator, and a theta series associated with a lattice. They are related in important settings, but they are not interchangeable by definition. The present article makes claims only about the multiset of lattice lengths arising from translations of a flat torus.

Schiemann’s theorem is directly relevant to the full theta series of positive-definite ternary forms [8]. It says that the complete theta series determines the ternary form up to integral equivalence. Our pair instead exhibits a long common prefix for a family of forms with different determinants. This avoids a conflict with the uniqueness theorem and illustrates why its hypotheses and the word “full” cannot be omitted.

9.3. Cosmic topology. Reviews by Lachièze-Rey and Luminet [3] and by Levin [4] explain how nontrivial spatial topology may leave signatures in astronomical observations. The Planck collaboration has analysed the background geometry and topology using CMB temperature and polarization data, including matched circles and likelihood methods for specified classes of spaces [6, 7]. More recent work considers a broader range of orientable Euclidean manifolds and emphasizes that the absence of an easily detectable circle pair does not by itself eliminate every compact topology [5].

These references motivate the interpretation in Section 8; they are not evidence that the particular matrices in this paper describe our universe. That would require a separate data analysis.

10. LIMITATIONS AND OPEN QUESTIONS

10.1. The cutoff is a guaranteed bound. The value R_A in Theorem 3.1 is a sufficient common cutoff obtained from a uniform eigenvalue estimate. It is not claimed to be the exact first radius where the two length multisets differ for every A . The bound is intentionally robust: it works for every integer vector, irrespective of the value of n_1 . A sharper analysis could seek the shortest vector with $n_2 n_3 \neq 0$ for each sign and parameter, then compare multiplicities at the resulting candidate values.

10.2. The examples are special families. The matrices have a transparent block pattern and use a single sign change. That is a feature of the proof, but it is also a restriction. The theorem does not classify all pairs of flat three-tori with the same finite length data. It does not characterize all possible finite length prefixes, nor does it supply a minimal sufficient number of lengths for a given parameter class.

10.3. Finite precision is a different question. The definition of $\mathcal{L}_G(R)$ uses exact lengths. In practical observations, values are noisy, rounded, and inferred through a measurement model. Exact agreement below a radius is therefore different from approximate agreement within a tolerance. A future quantitative version could bound the distance between truncated length distributions under perturbations of G , and could ask how stable the result remains when the Gram matrices are known only to finite precision.

10.4. Possible extensions. Several natural follow-ups are mathematically specific.

- (1) Determine an exact first-discrepancy formula in the anisotropic family, rather than a safe cutoff.
- (2) Study whether analogous constructions can be made in dimensions two or four with comparable normalization. In dimension three, full ternary theta-series uniqueness imposes a restriction on how far a finite-prefix example can be extended for a fixed integral form.
- (3) Formulate a stability theorem for noisy length lists, allowing bounded perturbations of the metric matrix and a finite observation radius.
- (4) For cosmological use, compare the full reciprocal-lattice mode structure and predicted CMB covariance rather than only the direct-lattice lengths.

Each question requires additional work and is not asserted as a consequence of the present results.

11. CONCLUSION

We have constructed two explicit families of non-isometric flat three-tori whose short translation-length data agree exactly. The central family fixes the systole to one and produces a common cutoff

$$R_A = \sqrt{2A^2 - 2A^{-2} - 4A^{-4}} \sim \sqrt{2}A,$$

which grows without bound. The determinant difference proves non-isometry, while a completed-square argument shows that every vector capable of seeing the changed matrix entry lies beyond the cutoff. A second rational example illustrates how the first new length can enter at a precisely identified threshold.

The conclusion is a precise limitation on finite length data: in the unrestricted class of flat three-tori, there is no universal bounded cutoff—measured only in units of the systole—that identifies the metric from the truncated translation-length multiset. This should be understood alongside, rather than in opposition to, uniqueness results for full theta series. In cosmology, it supplies a reason to distinguish a finite list of candidate topological lengths from the complete signal model used for observations. It does not establish observational indistinguishability of any two universes.

APPENDIX A. DETAILED QUADRATIC-FORM ESTIMATES

This appendix records the algebra behind the main inequality in a form that can be checked line by line.

A.1. The block eigenvalues. Let $x = n_2$ and $y = n_3$. Define

$$M_\sigma = \begin{pmatrix} A^2 - \varepsilon^2 & \sigma\varepsilon - \varepsilon^2 \\ \sigma\varepsilon - \varepsilon^2 & A^2 - \varepsilon^2 \end{pmatrix}, \quad \sigma \in \{+1, -1\}.$$

A symmetric matrix of the form $\begin{pmatrix} d & c \\ c & d \end{pmatrix}$ has eigenvectors $(1, 1)$ and $(1, -1)$ with eigenvalues $d + c$ and $d - c$. Hence its smallest eigenvalue is $d - |c|$. For $\sigma = +1$, the assumption

$0 < \varepsilon \leq 1/4$ implies $|\varepsilon - \varepsilon^2| = \varepsilon - \varepsilon^2$, and therefore

$$\lambda_{\min}(M_+) = A^2 - \varepsilon^2 - (\varepsilon - \varepsilon^2) = A^2 - \varepsilon.$$

For $\sigma = -1$, $|\varepsilon - \varepsilon^2| = \varepsilon + \varepsilon^2$, giving

$$\lambda_{\min}(M_-) = A^2 - \varepsilon^2 - (\varepsilon + \varepsilon^2) = A^2 - \varepsilon - 2\varepsilon^2.$$

Since $A^2 - \varepsilon \geq A^2 - \varepsilon - 2\varepsilon^2$, the latter is a valid common lower eigenvalue bound.

A.2. Why no restriction on the first coordinate is needed. A less careful estimate might try to bound each cross term separately, producing an expression with $|n_1 n_2|$ and $|n_1 n_3|$. Such estimates can appear to fail when $|n_1|$ is large. Completing the square avoids this problem exactly:

$$n_1^2 + 2\varepsilon n_1(n_2 + n_3) = \left(n_1 + \varepsilon(n_2 + n_3)\right)^2 - \varepsilon^2(n_2 + n_3)^2.$$

The negative correction is absorbed into the modified block $M_{\pm}$. The remaining square is nonnegative, uniformly for every real n_1, n_2, n_3 . The integer assumption is used only after the quadratic-form bound, when $n_2 n_3 \neq 0$ implies $n_2^2 + n_3^2 \geq 2$.

A.3. Systole bound in detail. The systole proof can also be written as a case split on the support of n .

- (1) If $n_2 = n_3 = 0$ and $n \neq 0$, then $Q_{\pm}(n) = n_1^2 \geq 1$.
- (2) If $n_2 \neq 0$ and $n_3 = 0$, then $Q_{\pm}(n) = (n_1 + \varepsilon n_2)^2 + (A^2 - \varepsilon^2)n_2^2 \geq A^2 - \varepsilon^2 > 1$.
- (3) If $n_2 = 0$ and $n_3 \neq 0$, the same bound holds by symmetry.
- (4) If $n_2 n_3 \neq 0$, then $Q_{\pm}(n) \geq R_A^2 > 1$ by (9).

The lower bound $R_A^2 > 1$ follows immediately from $A \geq 2$ and $\varepsilon \leq 1/4$. Equality at one is therefore achieved only by $n = \pm e_1$.

APPENDIX B. EXACT CHECKS FOR THE ISOTROPIC EXAMPLE

For H_+ , the matrix is $I + \delta(J - I)$, with eigenvalues $1 + 2\delta$ and $1 - \delta$ (twice). For H_- the off-diagonal sign pattern has two eigenvalues $1 + \delta$ and one eigenvalue $1 - 2\delta$. These statements can be verified by direct multiplication with the vectors $(1, 1, 1)$ and $(1, -1, -1)$ and their orthogonal complements.

The determinant expansions are

$$\det H_+ = 1 - 3\delta^2 + 2\delta^3, \quad \det H_- = 1 - 3\delta^2 - 2\delta^3.$$

At $\delta = 1/6$, these become $100/108 = 25/27$ and $98/108 = 49/54$. Their difference is $2/108 = 1/54$, which is also $4\delta^3$.

For an all-nonzero integer vector, the elementary inequality

$$|ab| + |ac| + |bc| \leq a^2 + b^2 + c^2$$

follows by adding $2|ab| \leq a^2 + b^2$, $2|ac| \leq a^2 + c^2$, and $2|bc| \leq b^2 + c^2$, then dividing by two. Consequently

$$P_{\pm}(a, b, c) \geq \left(1 - \frac{1}{3}\right)(a^2 + b^2 + c^2) \geq 2.$$

This proves the short-cutoff statement without needing to list any sign patterns. To establish the exact discrepancy at squared length two, the eigenvalue lower bound for H_+ gives

$$P_+(a, b, c) \geq \frac{5}{6}(a^2 + b^2 + c^2) \geq \frac{5}{2}$$

when all three coordinates are nonzero. When only two coordinates are nonzero, the value has the form $a^2 + b^2 \pm ab/3$. For $|a| = |b| = 1$ it is $5/3$ or $7/3$; if $a^2 + b^2 \geq 5$, it is at least $(5/6) \cdot 5 = 25/6$. On a coordinate axis the squared length is an integer square. Therefore H_+ contributes no vector of squared length two. In H_- , the two vectors $(1, -1, -1)$ and $(-1, 1, 1)$ both have squared length two, so the multiplicities differ at the threshold.

APPENDIX C. FINITE-SEARCH PSEUDOCODE

The following pseudocode is intended as a reproducibility aid for small integer A , not as part of the proof. It uses exact rational arithmetic when A is an integer.

```
input: integer A >= 2; rational cutoff R < sqrt(2*A^2 - 2/A^2 - 4/A^4)
eps := 1/A^2
Gplus := [[1,eps,eps],[eps,A^2,eps],[eps,eps,A^2]]
Gminus := [[1,eps,eps],[eps,A^2,-eps],[eps,-eps,A^2]]
choose N large enough for the illustrative finite search
for n1,n2,n3 in integers with -N <= ni <= N:
  if (n1,n2,n3) != (0,0,0):
    qplus := transpose(n) * Gplus * n
    qminus := transpose(n) * Gminus * n
    if qplus < R^2: record (qplus,n)
    if qminus < R^2: record (qminus,n)
compare the two multisets of recorded rational values
```

For exact arithmetic, all entries are rational when A is an integer, and the quadratic values can be compared without decimal rounding. A finite search with arbitrary N remains only a check: the proof of the theorem covers every integer vector and is independent of how N is chosen.

APPENDIX D. TERMINOLOGICAL NOTE: THREE DIFFERENT SPECTRA

The words “spectrum” and “length spectrum” are used for several related objects. The present paper concerns only the multiset of norms $\{\sqrt{n^T G n} : n \in \mathbb{Z}^3 \setminus \{0\}\}$ with multiplicity. The full theta series records the same squared lengths as exponents when an appropriate integral normalization is available. The Laplace spectrum records eigenvalues of a differential operator on functions on the torus; for a flat torus these can be expressed through the reciprocal lattice and therefore involve G^{-1} in the present coordinate convention. Although these data are connected, one should specify which object is being compared before drawing a uniqueness conclusion.

In particular, the main theorem should be quoted as a statement about finite truncations of the direct-lattice translation-length multiset. It is not, without additional proof, a statement about the Laplace spectrum or an astronomical observable.

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