

Biperfectness of the theta basis of a maximal unipotent subgroup

Qiyue Tang

Abstract

We prove that the theta basis of the coordinate ring of a maximal unipotent subgroup of a complex semisimple group is biperfect. This answers affirmatively Question 6.5 in Kamnitzer’s ICM 2022 survey.

AI use and scholarly credit

The results were obtained in a 65-minute Codex session using GPT-6-Astra at high reasoning effort. The author and Codex then reviewed and refined the arguments and exposition over several rounds. The manuscript subsequently passed an AI review using Rethlas-OSV [17, 18, 19].

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1 Introduction

The search for distinguished bases in representation theory connects the action of Lie algebra generators, tensor product multiplicities, and positivity. For a complex semisimple group, these questions can be studied together in the coordinate ring of a maximal unipotent subgroup. The dual canonical basis of Lusztig and Kashiwara, Lusztig’s dual semicanonical basis in simply-laced type, and the Mirković–Vilonen basis arise from different constructions. Berenstein and Kazhdan introduced perfect bases to describe the leading action of Lie algebra generators; see [1, §1.1]. In this normalization, the leading action of each positive Chevalley generator is a single basis vector, with a prescribed coefficient and a remainder of smaller nilpotence degree. The left and right multiplication actions on the unipotent subgroup lead to biperfectness. The common bicrystal of biperfect bases records their leading operators even when the bases themselves differ; see [1, §§1–2] and [12, §1].

Cluster algebras provide a second approach to distinguished bases. Fomin and Zelevinsky introduced them in connection with canonical bases and total positivity [8]. Mutation produces overlapping coordinate systems and their cluster monomials. Extending those monomials to a basis becomes essential when the cluster monomials fail to span the algebra. The cluster structures on double Bruhat cells [2] and on unipotent coordinate rings place this question inside representation theory. In the latter setting, the dual canonical

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and, in simply-laced type, dual semicanonical bases contain cluster monomials; Kamnitzer reviews these results in [12, §6.1].

The theta construction grew from the duality conjectures of Fock and Goncharov [7] and the mirror-symmetry construction of Gross, Hacking, and Keel [9]. These predict bases indexed by integral tropical points of a dual space. Gross, Hacking, Keel, and Kontsevich [10] construct the functions by broken lines: piecewise linear paths whose attached monomials change when they cross walls of a scattering diagram. Consistency of the diagram makes the construction compatible with mutation. Their theorem supplies a theta basis for a middle algebra between the cluster algebra and its upper cluster algebra, together with nonnegative multiplication constants. Under their geometric hypotheses the middle and upper algebras coincide. They also treat partial compactifications obtained by allowing frozen variables to vanish, which select the theta functions regular along the boundary.

Quantum positivity developed alongside this construction. Ben Davison proved positivity for skew-symmetric quantum cluster algebras using purity in cohomological Donaldson–Thomas theory [4]. Davison and Mandel then combined Donaldson–Thomas theory with scattering diagrams to construct quantum theta functions [5]. Their Laurent coefficients and multiplication constants are nonnegative Laurent polynomials in the quantum parameter, and their classical limits recover the GHKK theta functions. These results explain the compatibility between scattering constructions and positivity, while biproductness asks for a specific leading term under a Lie algebra action.

On the representation-theoretic side, Baumann, Kamnitzer, and Knutson construct the Mirković–Vilonen basis of the unipotent coordinate ring and prove that it is biproduct [1, Theorem 6.2]. Their construction uses the geometric Satake correspondence, and the Chevalley action is expressed through intersection multiplicities. Kamnitzer’s ICM 2022 report [12] compares this basis with the dual canonical and dual semicanonical bases, then introduces the theta basis in §6.3. Theorem 6.4 there states that the theta-basis construction applies to the unipotent coordinate ring and that its elements are pointed in each cluster. The resulting tropical parametrization is naturally identified with the bicrystal underlying biproduct bases. This motivates Question 6.5: is the theta basis a biproduct basis? The report separately formulates compatibility of the Mirković–Vilonen basis with clusters and its equality with the theta basis as Conjectures 6.6 and 6.9.

Subsequent work develops both the crystal and scattering aspects of this question. Fei constructs crystal structures and their algebraic lifts for upper cluster algebras associated with quivers with potential [6]. In the version of June 2026, Theorem 9.24 proves BK-biproductness of the generic basis for the weak upper normal crystal, and Conjecture 9.25 proposes the corresponding property for theta and triangular bases. Qin’s freezing theorem [16] identifies a selected part of a pointed theta expansion with a theta function for a seed with additional frozen vertices. Cheung, Magee, Mandel, and Muller [3] prove valuative independence for theta functions of positive scattering diagrams and derive applications to boundary extensions. These results address the control of leading terms that is needed to relate theta functions to crystal operators.

Let G be a connected complex semisimple group and N its maximal unipotent subgroup for a chosen positive root system. Fix a pinning, and let $\mathcal{I}$ index the simple roots. For $i \in \mathcal{I}$, let e_i be the positive Chevalley generator and put $x_i(t) = \exp(te_i)$, for $t \in \mathbb{C}$. On $\mathbb{C}[N]$ define

$$D_i^R f(n) = \left. \frac{d}{dt} \right|_{t=0} f(nx_i(t)), \quad D_i^L f(n) = \left. \frac{d}{dt} \right|_{t=0} f(x_i(t)n). \quad (1)$$

The multiplication maps are algebraic, so substituting $nx_i(t)$ or $x_i(t)n$ into a regular function gives an element of $\mathbb{C}[N][t]$. Differentiating more times than its t -degree gives zero.

Thus both operators are locally nilpotent derivations. For such a derivation D and $f \neq 0$, set

$$\varepsilon_D(f) = \max\{r \in \mathbb{N} : D^r f \neq 0\}, \quad \mathbb{N} = \{0, 1, 2, \dots\}.$$

We use the weight decomposition for the action of the maximal torus of the pinning $(h \cdot f)(n) = f(h^{-1}nh)$.

A basis is *good for D* if each $\ker D^r$, $r \geq 0$, is spanned by a subset of that basis, with D^0 the identity. It is *perfect for D* if it is good and, for every basis element b with $\varepsilon_D(b) > 0$, there is a basis element $\tilde{e}_D(b)$ such that

$$Db = \varepsilon_D(b)\tilde{e}_D(b) + r_b, \quad r_b \in \ker D^{\varepsilon_D(b)-1}. \quad (2)$$

The nonzero arrows $b \mapsto \tilde{e}_D(b)$ are required to be injective. A weight basis of $\mathbb{C}[N]$ is *biperfect* if it contains 1 and is perfect for every D_i^L and D_i^R . These conventions give the normalization in [12, §§1.2, 1.4].

Write $\mathcal{B}_\theta(\mathbb{C}[N])$ for the regular theta basis in this cluster structure.

Theorem 1.1. *The basis $\mathcal{B}_\theta(\mathbb{C}[N])$ is biperfect.*

This answers Question 6.5 of [12] affirmatively. The proof uses coordinates in which one Chevalley derivation is a monomial times a partial derivative in a frozen variable. The highest degree part of a theta function in that variable is another theta function, obtained by freezing certain mutable vertices. Linear independence of these highest parts gives the kernel filtration. Frozen monomial translation then identifies the leading derivative with a unique theta function and gives the coefficient in (2). The prefix and suffix minor seeds realize the two multiplication actions. Their relevant frozen rows have opposite signs, so we also compare the theta functions for a seed and its opposite.

The freezing operation is supplied by Qin [16]. The comparison with the opposite seed uses reflection of scattering diagrams and the Donaldson–Thomas twist of Kimura, Qin, and Wei [13]. The required minor seeds and their reachability are given by Qin [15], using the cluster structures of Goodearl and Yakimov [11]. We retain the symmetrizing lattice data in the reflection argument, so these inputs apply to all finite semisimple root systems.

2 Theta functions and freezing

We work with classical cluster seeds of geometric type. A seed s consists of algebraically independent variables x_j , indexed by a finite set $I = I_{\text{mut}} \sqcup I_{\text{fr}}$, and an integer exchange matrix

$$B = (b_{jk})_{j \in I, k \in I_{\text{mut}}}$$

with skew-symmetrizable principal part. The indices in I_{mut} are mutable and those in I_{fr} are frozen. All frozen variables are inverted in this section. The ambient field is $K = \mathbb{C}(x_j : j \in I)$. For $k \in I_{\text{mut}}$, mutation replaces x_k by

$$x'_k = \frac{\prod_{j \in I} x_j^{[b_{jk}]_+} + \prod_{j \in I} x_j^{[-b_{jk}]_+}}{x_k}, \quad [a]_+ = \max(a, 0),$$

leaving the other variables fixed. The new matrix has entries

$$b'_{ij} = \begin{cases} -b_{ij}, & i = k \text{ or } j = k, \\ b_{ij} + [b_{ik}]_+[b_{kj}]_+ - [-b_{ik}]_+[-b_{kj}]_+, & i \neq k, j \neq k, \end{cases} \quad i \in I, \quad j \in I_{\text{mut}}.$$

This matrix change preserves rank: add $[b_{ik}]_+$ times row k to row $i \neq k$, then add $[-b_{kj}]_+$ times column k to column $j \neq k$, and finally negate row k and column k . Here the coefficients refer to the matrix before mutation; its kk -entry is zero, so the indicated operations give the displayed formula. The upper cluster algebra is the intersection in K of the Laurent polynomial rings of all mutation-equivalent seeds.

Let $M^\circ = \mathbb{Z}^I$ be the exponent lattice, with basis $\mathbf{f}_j$. For $g = (g_j)_{j \in I} \in M^\circ$, write

$$x^g = \prod_{j \in I} x_j^{g_j}.$$

We use the sign convention of [14, Appendix A]. Throughout the cluster arguments, B has full column rank.

We specify the lattices used in the scattering construction. Choose positive integers d_j , $j \in I$, such that $b_{jk}/d_j = -b_{kj}/d_k$ on the principal part. Let $\mathbf{N} = \bigoplus_{j \in I} \mathbb{Z} \mathbf{e}_j$ and $\mathbf{N}^\circ = \bigoplus_{j \in I} \mathbb{Z} d_j \mathbf{e}_j$. If $\mathbf{e}_j^*$ is the basis dual to $\mathbf{e}_j$, identify

$$M^\circ = \text{Hom}(\mathbf{N}^\circ, \mathbb{Z}), \quad \mathbf{f}_j = d_j^{-1} \mathbf{e}_j^*, \quad \langle d_j \mathbf{e}_j, \mathbf{f}_k \rangle = \delta_{jk}.$$

Here δ_{jk} is 1 for $j = k$ and 0 otherwise. The skew form ω on $\mathbf{N}_\mathbb{Q} = \mathbf{N} \otimes_\mathbb{Z} \mathbb{Q}$ is specified whenever the first index is mutable by $\omega(\mathbf{e}_k, \mathbf{e}_j) = b_{jk}/d_j$. Extend it skew-symmetrically on the remaining pairs. Its restriction defines

$$p^* : \bigoplus_{k \in I_{\text{mut}}} \mathbb{Z} \mathbf{e}_k \longrightarrow M^\circ, \quad p^* \mathbf{e}_k = \sum_{j \in I} b_{jk} \mathbf{f}_j.$$

A cluster scattering diagram has walls supported on rational codimension-one cones in $M_\mathbb{R}^\circ = M^\circ \otimes_\mathbb{Z} \mathbb{R}$. For a normal vector n , its orthogonal hyperplane is $n^\perp = \{m : \langle n, m \rangle = 0\}$, and a wall with normal n has support in $n^\perp$. A wall with normal n , a primitive nonnegative integral combination of mutable $\mathbf{e}_k$, carries a formal function $f = 1 + \sum_{\ell \geq 1} a_\ell z^{\ell p^* n}$. Here z^m denotes the formal character with exponent $m \in M^\circ$, and $z^m z^{m'} = z^{m+m'}$; in the initial chart it is x^m . If n_γ is the primitive normal in $\mathbf{N}^\circ$, with sign chosen so that $\langle n_\gamma, \gamma' \rangle < 0$ at a transverse crossing by a path γ , its wall-crossing automorphism acts by

$$z^m \longmapsto z^m f^{\langle n_\gamma, m \rangle}. \quad (3)$$

Write $p_{\gamma, \mathfrak{d}}$ for the automorphism in (3) at the wall supported on $\mathfrak{d}$. The initial walls are $(\mathbf{e}_k^\perp, 1 + z^{p^* \mathbf{e}_k})$, $k \in I_{\text{mut}}$. A wall with normal n is incoming when its support contains $p^* n$, and outgoing otherwise. An exponent increment $B\nu$, with $\nu \in \mathbb{N}^{I_{\text{mut}}}$, has order $\sum_k \nu_k$; full column rank makes ν unique. For every integer $r \geq 0$, only finitely many walls have a nontrivial crossing automorphism after discarding increments of order greater than r . Consistency means that the ordered product of the crossing automorphisms depends only on the endpoints of a generic path, which crosses walls transversely and avoids their boundaries and intersections. The cluster diagram is determined, up to equality of these products, by its initial walls and the requirement that the additional walls be outgoing [10, Theorem 1.12].

A broken line is a piecewise linear path from an unbounded initial segment to a generic point Q , with an attached monomial cz^m on each segment and velocity $-m$. At a wall it chooses a term of the wall-crossing image of its previous monomial. Its initial monomial is z^g , and g is its label. Summing the final monomials for label g gives the theta function $\theta_{g, Q}^s$. These sums are interpreted in the completion defined by the order on $B\mathbb{N}^{I_{\text{mut}}}$. For label zero the theta function is 1. At a generic point in the positive chamber

$$C^+ = \{m \in M_\mathbb{R}^\circ : m_k \geq 0 \text{ for } k \in I_{\text{mut}}\},$$

write $\theta_g^s = \theta_{g,Q}^s$. Frozen coordinates are unrestricted in this chamber. A Laurent-finite theta function has the pointed expansion

$$\theta_g^s = x^g \sum_{\nu \in \mathbb{N}^{I_{\text{mut}}}} c_{g,\nu} x^{B\nu}, \quad c_{g,0} = 1, \quad (4)$$

with finitely many nonzero coefficients. Theta functions are compatible with mutation and with scattering transport.

A seed is *injective-reachable* if a mutation-equivalent seed has mutable g -vectors equal to the negative mutable unit vectors after projection to $\mathbb{Z}^{I_{\text{mut}}}$, up to permutation. Here a g -vector is the exponent of the leading monomial in a pointed expansion. We denote such a seed by $s[1]$. Every mutation-equivalent seed is then injective-reachable. Moreover, after the indicated permutation, $s[1]$ and s have the same principal exchange matrix and symmetrizers; see [13, §2.12 and §3.1]. For a full-rank injective-reachable seed, all theta functions in (4) are Laurent polynomials and form a basis of the upper cluster algebra [14, Theorem A.1.5]. Theta functions with labels in a reachable cluster chamber are the corresponding cluster monomials times frozen Laurent monomials [14, Theorems A.1.3–A.1.4].

For a subset $F \subset I_{\text{mut}}$, *freezing* F makes those vertices frozen, without changing the variables. The new exchange matrix B' is obtained by deleting the columns in F . On a pointed expansion define

$$\mathfrak{f}_{F,g} \left(x^g \sum_{\nu} c_{\nu} x^{B\nu} \right) = x^g \sum_{\nu: \nu_k=0 \ (k \in F)} c_{\nu} x^{B\nu}. \quad (5)$$

Full column rank makes this definition unambiguous. Qin's freezing theorem [16, Theorem 5.2] states that this is the theta function with label g for the frozen seed. Denote it by θ'_g . The frozen-factor property is

$$x^u \theta'_g = \theta'_{g+u} \quad \text{if } u \in \mathbb{Z}^I \text{ is supported on } I_{\text{fr}} \cup F. \quad (6)$$

The F -polynomial of a pointed theta function is $\sum_{\nu} c_{g,\nu} \prod_k t_k^{\nu_k}$, in independent variables t_k indexed by the mutable vertices. Substituting $t_k = x^{B_k}$, where B_k is column k of the exchange matrix, gives the factor after x^g in (4). For the frozen seed it depends only on the remaining mutable coordinates of g and on the remaining principal exchange matrix; see [13, Assumption 6.1 and the theta example] and [16, §6.4]. This identity includes negative frozen exponents.

3 A frozen variable and a perfect filtration

Fix a seed s and write $\theta_g = \theta_g^s$ throughout this section. For a finite Laurent polynomial f in the variables x_j , let $\deg_{x_a} f$ be the largest exponent of x_a with nonzero coefficient when the other variables are treated as elements of $\mathbb{C}(x_j : j \neq a)$. Set $\deg_{x_a} 0 = -\infty$. Its part of degree d is the sum of all terms with x_a -exponent d .

Lemma 3.1. *Suppose B' has full column rank and a family of Laurent polynomials, indexed by all $g \in \mathbb{Z}^I$, has the form*

$$q_g = x^g \left(1 + \sum_{0 \neq \nu \in \mathbb{N}^{I_{\text{mut}} \setminus F}} a_{g,\nu} x^{B'\nu} \right).$$

Then the q_g are linearly independent over $\mathbb{C}$.

Proof. Choose a real linear functional ℓ on $\mathbb{R}^I$ taking the value 1 on each column of B' ; its existence follows from column independence. In a nonzero finite relation $\sum_g a_g q_g = 0$, choose an index g with $a_g \neq 0$ and minimal $\ell(g)$. A nonleading exponent from any summand with nonzero coefficient has strictly larger ℓ -value than its own label, and hence cannot equal g . The leading exponents of different summands are distinct. The coefficient of x^g is therefore a_g , a contradiction. If B' has no columns, the functions are the distinct monomials x^g . $\square$

Proposition 3.2. *Suppose all the theta functions of a full-rank seed are Laurent finite. Let $A \subset K$ be a $\mathbb{C}$ -subalgebra with basis $\{\theta_g : g \in S\}$, for a subset $S \subset \mathbb{Z}^I$. Choose $a \in I_{\text{fr}}$ such that $b_{ak} \leq 0$ for all $k \in I_{\text{mut}}$, and put*

$$F = \{k \in I_{\text{mut}} : b_{ak} < 0\}.$$

Let $u \in \mathbb{Z}^I$ satisfy $u_a = 0$ and $\text{supp}(u) \cap I_{\text{mut}} \subset F$, where $\text{supp}(u) = \{j : u_j \neq 0\}$. Assume that $D = x^u \partial / \partial x_a$ preserves A and is locally nilpotent on A . Then the basis is good for D , and

$$\varepsilon_D(\theta_g) = g_a \quad (g \in S).$$

For $g_a > 0$, the label $g + u - \mathbf{f}_a$ lies in S , and

$$D\theta_g = g_a \theta_{g+u-\mathbf{f}_a} + r_g, \quad r_g \in \ker D^{g_a-1}. \quad (7)$$

The map $g \mapsto g + u - \mathbf{f}_a$ is injective.

Proof. Every element of A is a finite Laurent polynomial, since it has a finite expansion in the given basis. Put $K_0 = \mathbb{C}(x_j : j \neq a)$. If a nonzero $f \in A$ had smallest x_a -exponent $r < 0$, with coefficient $a_r \in K_0^\times$, then the coefficient of x_a^{r-m} in $D^m f$ would be

$$(x^u)^m a_r r(r-1) \cdots (r-m+1) \neq 0 \quad (m \geq 0),$$

where the empty product is 1. This contradicts local nilpotence. Thus $A \subset K_0[x_a]$, and $\varepsilon_D(f) = \deg_{x_a} f$ for every nonzero $f \in A$.

In (4), the exponent of x_a is $g_a + \sum_k b_{ak} \nu_k$. Its maximal value is g_a , attained precisely when $\nu_k = 0$ for every $k \in F$. Hence the highest degree part of θ_g is $\theta'_g = \mathbf{f}_{F,g}(\theta_g)$. It is nonzero and has degree g_a . The matrix B' has full column rank because its columns are a subset of those of B . Lemma 3.1 shows that all the θ'_g , $g \in \mathbb{Z}^I$, are linearly independent. In a nonzero finite sum $\sum_{g \in S} a_g \theta_g$, let d be the largest g_a with $a_g \neq 0$. Its part of degree d is the nonzero sum $\sum_{g_a=d} a_g \theta'_g$. Consequently, for every integer $r \geq 0$,

$$\ker(D^r|_A) = \text{span}_{\mathbb{C}}\{\theta_g : g \in S, g_a < r\}.$$

This proves goodness and the formula for ε_D , including $g_a \geq 0$ on S .

Let $g_a > 0$. The part of degree $g_a - 1$ in $D\theta_g$ is

$$g_a x^{u-\mathbf{f}_a} \theta'_g = g_a \theta'_{g+u-\mathbf{f}_a},$$

by (6). Expand $D\theta_g$ in the basis of A . No basis element of degree greater than $g_a - 1$ can occur, by the preceding independence. Comparing the parts of degree $g_a - 1$ and using independence for all integer labels forces the label $g + u - \mathbf{f}_a$ to belong to S , with coefficient g_a . Every remaining basis element has degree at most $g_a - 2$. This gives (7). The target has degree $g_a - 1$, since $u_a = 0$. Two distinct possible targets would have a nonzero difference of degree $g_a - 1$, whereas (7) would put that difference in $\ker D^{g_a-1}$; hence the target is unique. If $g_a = 1$, the remainder is zero because $A \subset K_0[x_a]$. If $g_a = 0$, the derivative is zero. Finally, translation by $u - \mathbf{f}_a$ is injective. $\square$

4 Opposite seeds

Let s^{op} have the same variables and frozen vertices as s , with the exchange skew form negated. Its extended matrix is $-B$. We identify the two initial rational function fields by the identity on each x_j . Write $\Theta_s = \{\theta_g^s : g \in M^\circ\}$, and similarly for s^{op} .

Lemma 4.1. *Let Q be a generic point of the positive chamber, and let ν be the involution of the Laurent ring defined by $\nu(z^m) = z^{-m}$. Reflection of scattering diagrams gives*

$$\nu(\theta_{m,Q}^{s^{\text{op}}}) = \theta_{-m,-Q}^s. \quad (8)$$

This identity holds for skew-symmetrizable classical seeds, with their original symmetrizing data.

Proof. Retain the lattices $\mathbb{N}^\circ \subset \mathbb{N}$, the integers d_j , and the pairing defined in Section 2. Negating ω replaces p^* by $-p^*$.

Send each wall $(\mathfrak{d}, f)$ to $(-\mathfrak{d}, \nu(f))$ and each transverse path γ to $-\gamma$. The oriented primitive normal becomes $-n_\gamma$, and $\langle -n_\gamma, -m \rangle = \langle n_\gamma, m \rangle$. Formula (3) consequently gives

$$p_{-\gamma, -\mathfrak{d}} \nu = \nu p_{\gamma, \mathfrak{d}}.$$

The reflected diagram is consistent. Its initial walls are $(e_k^\perp, 1 + z^{-p^* e_k})$, with the same primitive normal $d_k e_k$ up to orientation. The incoming condition is preserved because both $p^* e_k$ and the support are negated. Uniqueness of the consistent completion therefore identifies it with the opposite cluster diagram.

Reflecting each segment of a broken line and negating its monomial exponent preserves the rule that the velocity is the negative exponent. The preceding wall identity preserves the possible bending terms and their coefficients. It gives a bijection of broken lines with label and endpoint (m, Q) and $(-m, -Q)$, proving (8). The map on formal series goes from the completion for the original positive monoid to the completion for its negative image. $\square$

We now use the twist to compare the two diagrams in the same positive chamber. Write $C^- = -C^+$. For chambers with generic points Q_1, Q_2 , denote their scattering transport by $p_{Q_2 \leftarrow Q_1}^s$.

Proposition 4.2. *For a full-rank injective-reachable classical seed,*

$$\Theta_{s^{\text{op}}} = \Theta_s$$

as sets of rational functions in the initial field.

Proof. All theta functions are Laurent finite in C^+ by [14, Theorem A.1.5]. The negative chamber is the chamber of the reachable seed $s[1]$; see [14, Appendix A.1]. Mutation compatibility and the same theorem for this seed give Laurent finiteness in its chamber chart. The linear identification of this chart with the chamber in $M_{\mathbb{R}}^\circ$ changes exponents by an invertible integral map, so it preserves finite support. Lemma 4.1 then gives Laurent finiteness for the opposite seed in C^+ .

Consider the composite

$$\text{DT}_s = p_{Q \leftarrow -Q}^s \nu. \quad (9)$$

Chamber monomials give

$$\text{DT}_s(x_j) = \theta_{-f_j, -Q}^s \quad (j \in I), \quad (10)$$

Indeed, $f_j \in C^+$ for every j , including frozen j , so the positive-chamber monomial theorem gives $\theta_{f_j, Q}^{s^{\text{op}}} = x_j$. Reflection gives $\theta_{-f_j, -Q}^s = x_j^{-1}$. Transporting this last identity from $-Q$ to Q yields (10).

Relabel the mutable vertices of $s[1]$ by the permutation in injective reachability. Let H be the integer matrix with rows indexed by I_{fr} and columns indexed by I_{mut} , whose j -th column records the frozen components of the j -th mutable g -vector of $s[1]$. The lattice identification ψ^M from its monomial lattice to that of s , and the variation var^M from the latter lattice to the former, have matrices

$$\psi^M = \begin{pmatrix} -\text{Id}_{I_{\text{mut}}} & 0 \\ H & \text{Id}_{I_{\text{fr}}} \end{pmatrix}, \quad \text{var}^M = \begin{pmatrix} \text{Id}_{I_{\text{mut}}} & 0 \\ -H & -\text{Id}_{I_{\text{fr}}} \end{pmatrix}. \quad (11)$$

These are the maps constructed in [13, Proposition 5.1], and their product is $-\text{Id}_{M^\circ}$. Write var for the map of Laurent rings that substitutes the monomials specified by var^M . Composing with the mutation pullback μ^* , which expresses functions in the $s[1]$ chart as rational functions in the s chart, gives a twist automorphism $T = \mu^* \text{var}$.

For mutable j , $T(x_j)$ is one cluster variable of $s[1]$ times a frozen Laurent monomial. It is a theta function, and its full label is $\psi^M \text{var}^M \mathbf{f}_j = -\mathbf{f}_j$. For frozen j , it is $x_j^{-1} = \theta_{-\mathbf{f}_j, Q}^s$. Thus T and DT_s agree on every field generator by (10). In particular DT_s is the invertible twist in [13, Theorem 5.2]. The theta F -polynomials satisfy the mutation and variation hypotheses there, so [13, Theorem 6.2(2)] gives

$$\text{DT}_s(\Theta_s) = \Theta_s. \quad (12)$$

Combining reflection with transport yields, for every $m \in M^\circ$,

$$\text{DT}_s(\theta_{m, Q}^{s^{\text{op}}}) = p_{Q \leftarrow -Q}^s \theta_{-m, -Q}^s = \theta_{-m, Q}^s.$$

As m ranges over M° , the right side ranges over Θ_s . Together with (12) and injectivity of DT_s , this proves the asserted equality. $\square$

Lemma 4.3. *Let $R \subset K$ be a $\mathbb{C}$ -subalgebra and suppose a subset $\mathcal{B} \subset \Theta_s$ is a basis of R . Then $\mathcal{B} = \Theta_s \cap R$. Under the assumptions of Proposition 4.2, it also equals $\Theta_{s^{\text{op}}} \cap R$.*

Proof. An element of $\Theta_s \cap R$ has a finite expansion in $\mathcal{B}$. Linear independence of Θ_s forces that element to belong to $\mathcal{B}$. The second assertion follows from Proposition 4.2. $\square$

5 Minor seeds and Chevalley derivations

The simply connected covering of G restricts to an isomorphism on N : its kernel is finite and central, its restriction to the maximal unipotent subgroup is surjective, and a complex unipotent group has no nontrivial finite subgroup. This isomorphism identifies the pinned root subgroups and both actions in (1). We may therefore assume G simply connected. Let W be its Weyl group, with simple reflections s_i , simple roots α_i , simple coroots $\alpha_i^\vee$, and fundamental weights ω_i , $i \in \mathcal{I}$. Let w_{long} be the longest element. In the irreducible representation of highest weight ω_i , choose a highest weight vector v_{ω_i} and the functional ϕ_{ω_i} which is 1 on that vector and zero on all other weight spaces.

Let f_i be the negative Chevalley generator, normalized by $[e_i, f_i] = \alpha_i^\vee$, and put $y_i(t) = \exp(t f_i)$. Use the Weyl lifts $\bar{s}_i = x_i(-1)y_i(1)x_i(-1)$. Their products along reduced words define $\bar{w}$ for $w \in W$. The generalized minors are

$$\Delta_{u\omega_i, v\omega_i}(n) = \phi_{\omega_i}(\bar{u}^{-1} n \bar{v} v_{\omega_i}) \quad (u, v \in W, n \in N). \quad (13)$$

These are the pinned normalization of [2].

Fix a reduced word $(j_1, \dots, j_L)$ for w_{long} , and define $w_k = s_{j_1} \cdots s_{j_k}$, with $w_0 = 1$. For $1 \leq k \leq L$, let $p(k)$ and $s(k)$ be the previous and next positions of j_k in this word. Set $p(k) = -\infty$ or $s(k) = +\infty$ when the corresponding position is absent. The two coordinate families are

$$X_k = \Delta_{\omega_{j_k}, w_k \omega_{j_k}}, \quad Y_k = \Delta_{w_{k-1} \omega_{j_k}, w_{\text{long}} \omega_{j_k}}. \quad (14)$$

The X_k form a full-rank injective-reachable seed with the last occurrence of each letter frozen. The Y_k , indexed by their starting position in (14), form a mutation-equivalent seed with the first occurrence of each letter frozen. Denote their extended exchange matrices by $B^X = (b_{rk}^X)$ and $B^Y = (b_{rk}^Y)$. In Qin's sign convention their endpoint rows are

$$b_{L,k}^X = -\delta_{k,p(L)}, \quad b_{1,k}^Y = \delta_{k,s(1)} \quad (15)$$

for mutable columns k ; a nonexistent predecessor or successor gives the zero row.

These inputs come from [15, Theorems 8.1.2, 8.2.2 and Lemma 9.1.3]. The interval minors there become (14) under the pinned transpose from the lower to the upper unipotent subgroup. The integral identification in [15, Theorem 9.2.3], together with [11, §6.3, (6.14), and Theorem 7.3(a)], specializes to the ordinary coordinate ring and the normalized classical minors when the quantum deformation parameter q is set to 1. The rescaling denoted by cor in [15] is a power of q , so its specialization is 1. The compatible skew matrix Λ in [15, Theorem 8.1.2(1)] satisfies

$$(B^X)^T \Lambda = (\text{diag}(\lambda_k : k \in I_{\text{mut}}) \ 0), \quad \lambda_k \in \mathbb{Z} \setminus \{0\},$$

with mutable columns ordered first on the right. The right side has rank $|I_{\text{mut}}|$, so B^X has full column rank. Matrix mutation preserves that rank, and hence gives the same conclusion for the suffix seed. Injective reachability is preserved by mutation as recalled in Section 2. To verify the endpoint rows, write $C_{ab} = \langle \alpha_a^\vee, \alpha_b \rangle$ for the Cartan matrix, with $a, b \in \mathcal{I}$. The formulas in [15, Remarks 8.1.4 and A.3.1] are

$$b_{rk}^X = \begin{cases} 1 & r = p(k), \\ -1 & r = s(k), \\ C_{j_r j_k} & r < k < s(r) < s(k), \\ -C_{j_r j_k} & k < r < s(k) < s(r), \\ 0 & \text{otherwise,} \end{cases} \quad b_{rk}^Y = \begin{cases} 1 & r = p(k), \\ -1 & r = s(k), \\ C_{j_r j_k} & p(r) < p(k) < r < k, \\ -C_{j_r j_k} & p(k) < p(r) < k < r, \\ 0 & \text{otherwise.} \end{cases}$$

Here r is any row and k is mutable in the respective seed. For the suffix formula, Qin's index is first sent to its starting position: within the occurrences of each fixed letter, this permutation reverses their order. This gives exactly the indexing of Y_k in (14). At $r = L$, the prefix case $r = p(k)$ and the first inclined case are impossible since $k \leq L$. The second inclined case would require $L < s(k) < +\infty$, also impossible. Only $L = s(k)$, equivalently $k = p(L)$, remains. At $r = 1$, the suffix case $r = s(k)$ and the second inclined case are impossible. The first inclined case would require $-\infty < p(k) < 1$; every finite predecessor is at least 1. Only $1 = p(k)$, equivalently $k = s(1)$, remains. This proves (15).

We use the regular theta basis supplied by the theta-basis theorem for this coordinate ring, as stated in [12, Theorem 6.4]. Its elements are the regular members of the localized theta collection: this identification also follows from Lemma 4.3. Thus it is the same function basis in every reduced-word chart above.

Lemma 5.1. *For $i \in \mathcal{I}$, define $i^* \in \mathcal{I}$ by $w_{\text{long}} \alpha_{i^*} = -\alpha_i$. Choose a reduced word ending in i^* . In its prefix coordinates,*

$$D_i^R = X_{p(L)} \frac{\partial}{\partial X_L},$$

where $X_{p(L)}$ means 1 if $p(L) = -\infty$.

Proof. For each k , the positive roots sent to negative roots by w_k^{-1} are precisely $\{w_{r-1}\alpha_{j_r} : 1 \leq r \leq k\}$. The roots in this inversion sequence are distinct. Its last root is $w_{L-1}\alpha_{i^*} = -w_{\text{long}}\alpha_{i^*} = \alpha_i$. Consequently $w_k^{-1}\alpha_i$ is positive for every $k < L$. Conjugating e_i by $\bar{w}_k$ gives a positive-root operator, which kills $v_{\omega_{j_k}}$. Differentiating (13) on the right therefore gives $D_i^R X_k = 0$ for $k < L$.

The vector $\bar{w}_{L-1}v_{\omega_{i^*}}$ is killed by e_i and has $\alpha_i^\vee$ -weight

$$\langle \alpha_i^\vee, w_{L-1}\omega_{i^*} \rangle = \langle \alpha_{i^*}^\vee, \omega_{i^*} \rangle = 1.$$

Its i -string has length one. For a highest vector v of this string, $e_i v = f_i^2 v = 0$ and $e_i f_i v = v$. The normalization of the lift can be checked directly:

$$x_i(-1)y_i(1)x_i(-1)v = x_i(-1)(v + f_i v) = f_i v.$$

Thus $\bar{s}_i v = f_i v$ with coefficient $+1$. Since $w_{\text{long}} = s_i w_{L-1}$ is length additive, the braid lifts give $\bar{w}_{\text{long}} = \bar{s}_i \bar{w}_{L-1}$. Hence

$$D_i^R X_L(n) = \phi_{\omega_{i^*}}(n \bar{w}_{L-1} v_{\omega_{i^*}}).$$

If $p(L)$ exists, the reflections between positions $p(L)$ and L have labels different from i^* , and their lifts fix $v_{\omega_{i^*}}$. The last expression is $X_{p(L)}(n)$. If there is no such position, it is $\phi_{\omega_{i^*}}(n v_{\omega_{i^*}}) = 1$. The derivation is determined by its values on the coordinate generators. $\square$

Lemma 5.2. *Choose a reduced word beginning with $i \in \mathcal{I}$. In its suffix coordinates,*

$$D_i^L = Y_{s(1)} \frac{\partial}{\partial Y_1},$$

where $Y_{s(1)}$ means 1 if $s(1) = +\infty$.

Proof. For $k > 1$, the root $w_{k-1}^{-1}\alpha_i$ is negative because the inversion set already contains α_i . In the derivative

$$D_i^L Y_k(n) = \phi_{\omega_{j_k}}(\bar{w}_{k-1}^{-1} e_i n \bar{w}_{\text{long}} v_{\omega_{j_k}}),$$

conjugate e_i by $\bar{w}_{k-1}$. The resulting negative-root operator has zero highest weight component after application to any vector in the representation. Thus $D_i^L Y_k = 0$.

For $k = 1$, the functionals $\phi_{\omega_i} e_i$ and $\phi_{\omega_i} \bar{s}_i^{-1}$ are supported on the same extremal weight space of weight $s_i \omega_i$. They both take the value 1 on $f_i v_{\omega_i}$, by the length-one i -string calculation. They are therefore equal. If $s(1)$ exists, all letters in positions $2, \dots, s(1) - 1$ differ from i . Their inverse lifts fix ϕ_{ω_i} . Using the order of factors in $\bar{w}_{s(1)-1}^{-1}$ gives

$$\phi_{\omega_i} \bar{w}_{s(1)-1}^{-1} = \phi_{\omega_i} \bar{s}_i^{-1}.$$

It follows that $D_i^L Y_1 = Y_{s(1)}$. If no next occurrence exists, the same calculation gives $\phi_{\omega_i} \bar{w}_{\text{long}}^{-1} = \phi_{\omega_i} \bar{s}_i^{-1}$. The derivative is then the diagonal extremal minor

$$\Delta_{w_{\text{long}}\omega_i, w_{\text{long}}\omega_i}(n) = 1.$$

Indeed, the action of $n \in N$ on an extremal weight vector changes it only by terms of strictly higher weight, which the functional defining the same extremal minor annihilates. This proves the formula on every coordinate generator. $\square$

Example 5.3. For $G = \mathrm{SL}_3$, write

$$n = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

For the word $(1, 2, 1)$, the two coordinate lists are

$$(X_1, X_2, X_3) = (a, ac - b, b), \quad (Y_1, Y_2, Y_3) = (b, ac - b, c).$$

The right derivation D_2^R has values $0, 0, a$ on the X -coordinates, while D_1^L has values $c, 0, 0$ on the Y -coordinates. The exchange relation is $ac = b + (ac - b)$. In the prefix convention its frozen row at X_3 is -1 , whereas the suffix row at Y_1 is $+1$.

6 Proof of biperfectness

Proof of Theorem 1.1. Fix $i \in \mathcal{I}$. Choose a reduced word ending in i^* and use its prefix seed. The coordinate X_L is frozen, and its row is nonpositive by (15). If $p(L)$ exists, it is mutable and is the unique negative entry in that row. Lemma 5.1 and Proposition 3.2, applied to $A = \mathbb{C}[N]$, therefore prove goodness and (2) for D_i^R , with the required coefficient and injective arrows. If $p(L)$ is absent, the monomial is 1, the row is zero, and the same proposition applies with $F = \emptyset$.

For D_i^L , choose a reduced word beginning with i and use its suffix seed. The coordinate Y_1 is frozen. Its row is nonnegative, and its only possible nonzero mutable entry is at $s(1)$. Proposition 4.2 and Lemma 4.3 identify the regular theta basis for this seed with that for its opposite, in the same rational function field with the same coordinate functions. The opposite row at Y_1 is nonpositive. By Lemma 5.2, the monomial in the derivation is $Y_{s(1)}$, or 1 when no successor exists. Proposition 3.2 now gives goodness, (2), and injective arrows for D_i^L .

All the coordinate choices belong to the same cluster structure, so these statements concern the same basis $\mathcal{B}_\theta(\mathbb{C}[N])$. It contains $1 = \theta_0$. The minor $\Delta_{u\omega_i, v\omega_i}$ has weight $v\omega_i - u\omega_i$: in the torus action, h^{-1} acts on the extremal functional and h on the extremal vector in (13). Each column of the exchange matrix has total weight zero, by [15, Theorem 8.1.2(1)]. Every monomial in a pointed expansion therefore has the weight of its leading monomial. Thus the theta basis is a weight basis and is biperfect. $\square$

References

- [1] P. Baumann, J. Kamnitzer, and A. Knutson, *The Mirković–Vilonen basis and Duistermaat–Heckman measures*, with an appendix by A. Dranowski, J. Kamnitzer, and C. Morton-Ferguson; [arXiv:1905.08460v2](#).
- [2] A. Berenstein, S. Fomin, and A. Zelevinsky, *Cluster algebras III: Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), 1–52; [arXiv:math/0305434v3](#).
- [3] M.-W. Cheung, T. Magee, T. Mandel, and G. Muller, *Valuative independence and cluster theta reciprocity*, [arXiv:2505.09585v2](#).
- [4] B. Davison, *Positivity for quantum cluster algebras*, Ann. of Math. (2) **187** (2018), 157–219; [doi:10.4007/annals.2018.187.1.3](#).

- [5] B. Davison and T. Mandel, *Strong positivity for quantum theta bases of quantum cluster algebras*, Invent. Math. **226** (2021), 725–843; [arXiv:1910.12915v3](#).
- [6] J. Fei, *Crystal structure of upper cluster algebras*, [arXiv:2309.08326v3](#).
- [7] V. Fock and A. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. **103** (2006), 1–211.
- [8] S. Fomin and A. Zelevinsky, *Cluster algebras I: Foundations*, J. Amer. Math. Soc. **15** (2002), 497–529.
- [9] M. Gross, P. Hacking, and S. Keel, *Mirror symmetry for log Calabi–Yau surfaces I*, Publ. Math. Inst. Hautes Études Sci. **122** (2015), 65–168.
- [10] M. Gross, P. Hacking, S. Keel, and M. Kontsevich, *Canonical bases for cluster algebras*, J. Amer. Math. Soc. **31** (2018), 497–608; [arXiv:1411.1394v2](#).
- [11] K. R. Goodearl and M. T. Yakimov, *Integral quantum cluster structures*, [arXiv:2003.04434v1](#).
- [12] J. Kamnitzer, *Perfect bases in representation theory: three mountains and their springs*, in *Proceedings of the International Congress of Mathematicians 2022*, Vol. 4, EMS Press, 2023; [arXiv:2201.02289v2](#).
- [13] Y. Kimura, F. Qin, and Q. Wei, *Twist automorphisms and Poisson structures*, SIGMA **19** (2023), 105; [published version](#).
- [14] F. Qin, *Bases for upper cluster algebras and tropical points*, [arXiv:1902.09507v4](#).
- [15] F. Qin, *Dual canonical bases and quantum cluster algebras*, [arXiv:2003.13674v3](#).
- [16] F. Qin, *Applications of the freezing operators on cluster algebras*, [arXiv:2407.03186v1](#).
- [17] H. Ju, G. Gao, J. Jiang, B. Wu, Z. Sun, S. Liu, L. Chen, Yutong Wang, Yuefeng Wang, Z. Wang, W. He, P. Wu, L. Xiao, R. Liu, B. Dai, and B. Dong, *Automated conjecture resolution with formal verification*, [arXiv:2604.03789v2](#), 2026.
- [18] S. Liu, *Rethlas-OSV: Desktop math agents for mathematicians*, 2026, updated October 20, 2026; [official guide](#).
- [19] F. Nie, H. Ye, S. Liu, Z. Wang, S. Ermon, and J. Zou, *One-sided verification under limited expert review*, [arXiv:2610.18089v2](#), 2026.

ZHILI COLLEGE, TSINGHUA UNIVERSITY, BEIJING 100084, CHINA
 Email address: tangqy24@mails.tsinghua.edu.cn