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COLLISION SYSTEMS: EXPONENTIAL BOUNDS FOR POINT BILLIARDS AND GEODESIC NETS

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Abstract. A collision system is a finite set of point particles with masses, moving freely with velocities given by the gradient of a convex energy function of the momentum, whose collisions conserve total momentum and total energy and are otherwise unconstrained. Stationary geodesic networks near a line of multiplicity n, point billiards with momentum- and energy-conserving collisions, and relativistic point particles are collision systems. We prove that the number of collisions of a collision system is at most exponential in its total mass, with an explicit base, whenever the energy is close enough to the Newtonian one, and that it can be exponential for every energy on a line. For geodesic networks this replaces the previously reported bounds, of order n^2 from below and n^{n^2} from above, by exponential bounds on both sides; for point billiards it gives explicit exponential bounds. The exponential growth rates remain open.

1. Introduction

How many times can n particles collide? For hard balls in Euclidean space the question was raised by Sinai, shown to have a finite answer by Vaserstein [18] and Gal'perin [11], and made quantitative by Burago, Ferleger and Kononenko [2], whose bound $(32n^{3/2})^{n^2}$ for equal balls was improved to $n^{O_d(n)}$ in each fixed dimension d by Burdzy [5]; Burago and Ivanov [3] constructed examples with $2^{\lfloor n/2 \rfloor}$ collisions in dimension three. The same question arises in two other settings, with the same shape of answer so far. The first is geometric: an integral stationary geodesic network in $\mathbb{R}^d$ that converges weakly to a line of multiplicity n has, on a compact set, a bounded number of singular points, by the partial regularity theorem of Bosch [4]; that paper reports the estimate $(10n)^{n^2}$ in the plane, while n lines in general position give $\binom{n}{2}$ singular points, and nothing better was known on either side. The second is the linear point-billiard framework of Féjoz, Knauf and Montgomery [9], for which a uniform finite collision bound is known. Gasiorek [10] gives an explicit bound for three equal-mass particles in the plane. Here we allow any momentum- and energy-conserving outcome for

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an entire coincident cluster, including collisions of three or more particles and flights of coincident particles.

This paper treats these problems in one framework. A *collision system* is a finite set of point particles with masses, moving in a Euclidean space with velocities $\operatorname{grad} h(p_i)$ given by a convex *energy* h of the momentum, whose collisions conserve total momentum $\sum m_i p_i$ and total energy $\sum m_i h(p_i)$ and are otherwise unconstrained. The particles may have arbitrary positive masses, and our bounds specialize to identical particles. The Newtonian energy $\frac{1}{2}|p|^2$ gives point billiards. The framework applies to geometry because a stationary geodesic network near a line of multiplicity n is a collision system too: reading the axial coordinate as time, stationarity at a vertex is exactly the conservation of the transverse part p of the unit tangent and of its longitudinal defect $1-\sqrt{1-|p|^2}$, so the n strands are particles with the energy $h(p)=1-\sqrt{1-|p|^2}$, which is Newtonian to second order. Relativistic point particles, with the energy $\sqrt{1+|p|^2}$, are a third instance.

Two parameters appear in the bounds, and they should be distinguished. The number n of particles is the natural parameter when all particles are alike, and it is the parameter in the questions above. For particles of different masses the right parameter is the ratio of the total mass to the smallest mass, which we call M. Multiplying all masses by a constant changes nothing, so we normalize the smallest mass to 1; then every mass is at least 1 and $M=m_1+\cdots+m_n\geq n$, with equality exactly when all masses are equal. Thus $M=n$ for n particles of unit mass, for n strands of weight one and for n balls of equal mass, and a bound in terms of M is a bound in terms of n for them. Our upper bounds are proved for arbitrary masses and are stated in terms of M; our lower bounds are constructions with unit masses and are stated in terms of n.

We prove two theorems about collision systems, stated in Section 1.2: an exponential upper bound $\frac{M}{2}\,8^{M-2}$ on the number of collisions of a system of total mass M, in any dimension, for energies close enough to the Newtonian one, closeness being measured by the *pinching* of h, a uniform bound on its normalized Hessian; and an exponential lower bound, a single construction that produces, for every energy on a line and every $n\geq 5$, systems of n particles of unit mass with at least $2^{\lfloor (n-2)/3\rfloor}$ collisions at which at least three particles meet. Each application is then a verification that the system in question is a collision system with a specific energy. Table 1 summarizes what the framework yields against what was known.

For geodesic networks the gap between n^2 and n^{n^2} is thus closed to a gap between two exponentials, $2^{n/3}$ and 8^n, and the upper bound no longer depends on the dimension. For point billiards we obtain bounds for arbitrary particle counts under the cluster collision rule above. The exponential growth rates remain open, as does their dependence on the dimension and the optimal dependence on the mass ratio for a fixed number of particles.

Problem		Known	This paper
Geodesic networks $C(d,n)$	upper	$d^{3(n-1)/2}(5n)^{n^2}$ [4]	$\frac{n}{2}\,8^{n-2}$
	lower	$\binom{n}{2}$	$2^{\lfloor (n-2)/3 \rfloor}$
Point billiards, total mass M	upper	finite [9, 2]	$\frac{M}{2}\,8^{M-2}$
	lower	$\binom{n}{2}$	$2^{\lfloor (n-2)/3 \rfloor}$
Relativistic particles, slow	upper	—	$\frac{M}{2}\,8^{M-2}$

TABLE 1. Bounds on the number of collisions (singular points, for networks); Theorems A, B and C. Lower bounds are for $n \geq 2$, with equal masses, and are realised in the plane, respectively on a line; upper bounds hold in every dimension. The earlier network estimate was reported in [4]; point-billiard finiteness refers to the conventions of [9], and an explicit planar three-particle bound appears in [10].

Uniform upper bounds outside the near-Newtonian regime are not addressed here.

The paper is organized as follows. The rest of this introduction defines collision systems, states the two general theorems and the three applications, and describes the method. Section 2 contains the examples of energies and a rigidity proposition for systems in which all particles meet at one point, Section 3 the proof of the exponential upper bound, Section 4 the construction for the lower bound, and Section 5 the applications. Three longer computations are in the appendices.

1.1. **Collision systems.** Throughout, X is a finite-dimensional real inner-product space.

Definition 1.1. An *energy* on X is a pair (Ω, h), where $\Omega \subset X$ is open, convex and contains 0, and $h\colon \Omega \to \mathbb{R}$ is smooth with

$$h(0) = 0, \qquad \operatorname{grad} h(0) = 0, \qquad D^2 h(p) \geq I \quad (p \in \Omega),$$

and such that the set of velocities $V = \operatorname{grad} h(\Omega)$ is convex. Its *pinching* is the supremum $\rho(h) \in [1, \infty]$ of the largest eigenvalue of $D^2 h(p)$ over $p \in \Omega$. When Ω is clear from the context, we write h for the energy. A point $p \in \Omega$ is a *momentum*; $\operatorname{grad} h(p)$ is its *velocity* and $h(p)$ its *energy*. Since $D^2 h \geq I$, the map $\operatorname{grad} h\colon \Omega \to V$ is a diffeomorphism onto the open set V.

Subtracting a constant and a linear function normalizes $h(0)$ and $\operatorname{grad} h(0)$; the linear term amounts to a Galilean change of frame. If the Hessian has a positive uniform lower bound aI, multiplying h by $1/a$ gives $D^2 h \geq I$ and amounts to a change of the unit of time. When a is the optimal lower bound, the pinching is the ratio of the uniform upper and lower Hessian bounds. This is a global bound over Ω, not the pointwise condition number. Strict convexity alone need not provide a positive uniform lower bound on the whole domain. The *Newtonian energy* $h(p) = \frac{1}{2}\|p\|^2$

on $\Omega = X$ has pinching 1, and an energy of pinching close to 1 is an approximation of it.

Definition 1.2. Let (Ω, h) be an energy on X, let $m_1, \dots, m_n > 0$ be *masses*, and let $J \subset \mathbb{R}$ be an interval. A *collision system* with these masses and energy h on J is a list of continuous piecewise affine functions $y_1, \dots, y_n \colon J \to X$, each with finitely many break points, such that:

(a) on every open interval of J that contains no break point of any y_i (a *flight*) there are momenta $p_1, \dots, p_n \in \Omega$, constant on the flight, with $y_i' = \operatorname{grad} h(p_i)$;

(b) for every $t \in J$ and $y \in X$, the set $S = \{i : y_i(t) = y\}$ satisfies

$$\sum_{i \in S} m_i p_i(t_+) = \sum_{i \in S} m_i p_i(t_-), \qquad \sum_{i \in S} m_i h(p_i(t_+)) = \sum_{i \in S} m_i h(p_i(t_-)),$$

where $p_i(t_\pm)$ are the momenta on the flights just after and just before t (at an endpoint of J the condition is void).

The index i is the *label* of the particle y_i, $m_i p_i$ is its momentum and $m_i h(p_i)$ its energy, and $m_1 + \dots + m_n$ is the *total mass*. If $|S| \geq 2$ and the functions y_i, $i \in S$, do not all coincide on a neighbourhood of t in J, the point (t, y) is a *collision point*; a time at which some momentum changes is an *event time*.

Multiplying all masses by a positive constant changes nothing, so we normalize the smallest mass to 1; the total mass is then the sum of the masses divided by the smallest one. At a collision of two particles of equal mass on a line, the unordered pair of momenta is preserved. A collision of three or more particles, or of two particles in higher dimension, is a relation rather than a map: every outcome compatible with the two conservation laws is allowed, and coincident particles may travel together. For the Newtonian energy, this extends the linear point-billiard framework of Féjoz, Knauf and Montgomery [9]; their stated conventions exclude flights inside collision subspaces and select one collision subspace at a multiple meeting.

1.2. **The general theorems.**

Theorem 1.3 (Upper bound)**.** *For every $M \geq 1$ there is $\rho_M > 1$, nonincreasing in M, such that every collision system, on any interval, with masses at least 1, total mass M, and an energy of pinching less than ρ_M, has at most $\lfloor \frac{M}{2}\, 8^{M-2} \rfloor$ collision points.*

For an integer $n \geq 1$ put

$$U_n = \left\lfloor \frac{n}{2}\, 8^{\,n-2} \right\rfloor,$$

so that $U_1 = 0$ and $U_2 = 1$. The Newtonian energy has pinching 1, so a collision system with masses at least 1, total mass M and the Newtonian energy has at most $\lfloor \frac{M}{2} 8^{M-2} \rfloor$ collision points, with no further hypothesis.

Theorem 1.3 is perturbative: the tolerance ρ_M tends to 1 as $M \to \infty$, and for a fixed energy of pinching $\rho > 1$ the theorem says nothing once M is large. The Newtonian energy, of pinching exactly 1, is covered for all M; the other two applications are covered because in them the pinching can be chosen after the total mass. The restriction cannot be removed by the method of Section 3: the pinching enters an angular error term, and the proof needs that term smaller than a margin that is exponentially small in the total mass.

Theorem 1.4 (Lower bound)**.** *Let (Ω, h) be an energy on $\mathbb{R}$, let $L \geq 0$ be an integer, and let $\delta > 0$ be such that $[-2\delta, 2\delta] \subset h'(\Omega)$. Then there is a diagram with energy h* (Definition 4.1) *that has $3L+2$ particles and $6L+4$ rays, all of whose edges have slope less than 2δ in absolute value, and at least $2^L + L - 1$ of whose vertices have three incoming and three outgoing edges whose slopes are not all equal. Consequently there is a collision system of $3L+2$ particles of unit mass on $\mathbb{R}$ with energy h that has at least $2^L + L - 1$ collision points at which at least three particles meet.*

A diagram is a drawing in the plane of a collision system on a line: its edges form particle graphs, and its nontrivial vertices represent collisions. The lower bound is one construction for all energies; it uses the nonvanishing of the energy's second derivative at the relevant momenta and does not require quantitative global convexity estimates. An energy on X whose gradient maps a line ℓ through 0 into itself, as every radial energy does, restricts to an energy on $\ell \cong \mathbb{R}$, and a collision system on ℓ is a collision system in X; so the lower bound applies in every dimension.

The two theorems are proved in Sections 3 and 4. For the Newtonian energy they put the lower and upper exponential growth rates of the maximal collision count between $2^{1/3} = 1.2599\ldots$ and 8. For other line energies the lower bound always holds, while the upper bound requires a pinching tolerance that depends on n.

Further AI-assisted exploration by the authors, summarized in supplementary material, has reduced the upper exponential base to just below 4 for unit-mass systems on a line. A separate weighted construction raises the lower rate to $3^{1/4} \approx 1.3161$ when noninteger masses arbitrarily close to one are allowed; it does not improve the unit-mass or integral-network lower bound.

1.3. **Applications.** Each application consists of a verification that the system in question is a collision system with a specific energy, followed by an appeal to the general theorems. The verifications are in Section 5.

Geodesic networks near a multiple line.

Definition 1.5. Let $d \geq 2$. An *integral stationary geodesic network* in $\mathbb{R}^d$ is a locally finite union X of straight segments, rays and lines, called its *pieces*, each carrying a positive integer *weight*, such that at every point

$p \in X$ the weighted sum of the unit vectors pointing from p along the pieces that contain p is zero. (At a point in the interior of a piece the two unit vectors along that piece cancel, so the condition only restricts endpoints of pieces.) The *weighted length measure* of X is the sum, over the pieces, of the weight times one-dimensional Hausdorff measure on the piece.

If X coincides near a point $p \in X$ with a single straight line through p, then p is called *regular*; otherwise p is called *singular*. Thus a transverse crossing of two pieces is singular, while a point where a piece of weight two continues as two coincident pieces of weight one is regular.

Definition 1.6. Let $d \geq 2$ and $n \geq 1$. The constant $C(d,n) \in \mathbb{N} \cup \{0, \infty\}$ is the least M with the following property. Whenever $X_1, X_2, \ldots$ are integral stationary geodesic networks in $\mathbb{R}^d$ whose weighted length measures converge weakly on compact sets to n times length measure on a line, and $K \subset \mathbb{R}^d$ is compact, there is i_0 such that for every $i \geq i_0$ the network X_i has at most M singular points in K.

Theorem A. *Let $d \geq 2$.*

(a) $C(d,1) = 0$ *and* $C(d,2) = 1$.

(b) *For every $n \geq 2$, with $L = \lfloor (n-2)/3 \rfloor$,*

$$\max\left\{ \binom{n}{2},\ 2^L + L - 1 \right\} \leq C(d,n) \leq \left\lfloor \frac{n}{2}\, 8^{n-2} \right\rfloor.$$

(c) *Consequently*

$$\frac{\log 2}{3} \leq \liminf_{n\to\infty} \frac{\log C(d,n)}{n} \leq \limsup_{n\to\infty} \frac{\log C(d,n)}{n} \leq \log 8.$$

How the framework applies. By the structure theorem of Bosch [4, Theorem 11], a network that is weakly close to a line of multiplicity n is, on a slab around a compact set, the union of n polygonal paths, called *strands*, each a graph over the axial coordinate t with transverse slope less than any prescribed $a > 0$. If a strand has slope v, put

$$p = \frac{v}{\sqrt{1 + \|v\|^2}}, \qquad h(p) = 1 - \sqrt{1 - \|p\|^2},$$

so that its rightward unit tangent is $(1 - h(p), p)$. Stationarity at a vertex says exactly that the sums $\sum p$ and $\sum h(p)$ over the incoming strands equal those over the outgoing strands: reading t as time, the strands form a collision system with unit masses and the energy h, the Euclidean energy of Section 2, which is Newtonian to second order; if all slopes are less than a, its pinching is $(1 + a^2)^{3/2}$ (Lemma 5.6). Its collision points are the singular points of the network. Choosing a after n so that $(1 + a^2)^{3/2} < \rho_n$, Theorem 1.3 gives the upper bound. For the lower bound, a diagram with the Euclidean energy is a balanced net (Lemma 5.9), so Theorem 1.4 produces planar networks with $3L + 2$ strands and $2^L + L - 1$ singular points of degree six, converging to a line of multiplicity $3L + 2$ as the slopes tend to zero.

Comparison with the literature. That $C(d,n)$ is finite is the partial regularity theorem of Bosch [4, Theorem 1], which reports the estimate $d^{3(n-1)/2}(5n)^{n^2}$, bounded above by $(10n)^{n^2}$ in the plane. The closed quantitative estimate is stated there with a qualification; the finiteness theorem is unconditional. Theorem A replaces this by $\frac{n}{2}8^{n-2}$, which is exponential in n rather than of order n^{n^2}, and which does not depend on d; the lower bound, realised by networks in a plane, shows that the growth is exponential. By the structure theorem of Allard and Almgren [1], an integral stationary one-dimensional varifold has the network structure of Definition 1.5, and such networks with multiplicity arise as limits in min-max theory on surfaces [6]. For planar geodesic nets whose unbalanced vertices have degree one, Frolov, Gorodkov, Korshunov and Nabutovsky [8] bound the number of balanced vertices by $(25u)^{2u^2}$ in terms of the number u of unbalanced vertices; examples with many balanced vertices are in [13, 14]. For $L \geq 1$, cutting the rays of our planar examples gives nets with $u = 6L + 4$ unbalanced vertices of degree one and at least $2^L + L - 1 \geq 2^{(u-4)/6}$ balanced vertices of degree six, so in that problem too the number of balanced vertices can grow exponentially in u. For $n \leq 28$ the trivial bound $\binom{n}{2}$, given by n lines in general position, is the larger of the two lower bounds in Theorem A.

Point billiards. A *motion* of particles with masses $m_1, \dots, m_n$ in $\mathbb{R}^k$ is a collision system on $\mathbb{R}$ with these masses and the Newtonian energy on $\mathbb{R}^k$: a particle moves with constant velocity until it meets others, and a collision conserves total momentum $\sum m_i y_i'$ and total kinetic energy $\sum \frac{1}{2} m_i \|y_i'\|^2$. A collision point at which at least three particles meet is a *multiple collision.* This extends the binary-subspace model of [9] by allowing all coincident particles to exchange momentum and energy within their cluster, as well as allowing coincident flights.

Theorem B. *Let $k \geq 1$. A motion of point particles in $\mathbb{R}^k$ with masses $m_1, \dots, m_n > 0$ has at most $\lfloor \frac{M}{2}\, 8^{\,M-2} \rfloor$ collision points, where $M = (m_1 + \cdots + m_n)/\min_i m_i$. For every $L \geq 0$ there is a motion of $3L + 2$ particles of equal mass in $\mathbb{R}$ with at least $2^L + L - 1$ collision points at which at least three particles meet.*

How the framework applies. A motion is a collision system with the Newtonian energy, of pinching $1 < \rho_M$, so Theorem 1.3 gives the upper bound with no restriction; the lower bound is Theorem 1.4 for the Newtonian energy on $\mathbb{R}$.

Comparison with the literature. The framework of [9] has a uniform finite collision bound derived from [2], and Gasiorek [10, Theorem 4.3] bounds the collision count for three equal-mass particles in the plane by three. Theorem B gives an explicit exponential bound for arbitrary particle counts with our broader cluster collision rule. Some dependence on the masses beyond their number is unavoidable: three particles of masses $M, 1, M$ on a line, with the heavy ones moving towards each other, collide of the order of $\sqrt{M}$

times, the configuration behind Gal'perin's computation of the digits of π [12, 16, 17]; so there is no bound in terms of the number of particles alone, and the bound in Theorem B depends on the masses only through the total mass divided by the smallest one. The results have the same shape as the state of knowledge for the elastic collisions of n hard balls in Euclidean space: that number is finite [18, 11], for equal balls at most $(32n^{3/2})^{n^2}$ [2] and at most $n^{O_d(n)}$ in each fixed dimension [5], and at least $2^{\lfloor n/2 \rfloor}$ in dimension three [3]. A recent preprint of Qiu [15] reports the same exponential lower bound in the plane. The two models differ in that a collision of hard balls is a map while a collision of three or more point particles is a relation; for equal masses on a line this is the whole difference, since without multiple collisions the particles are n straight lines with at most $\binom{n}{2}$ crossings (Lemma 5.1), so that the exponential growth in Theorem B comes entirely from collisions of three or more particles.

Slow relativistic particles. A *relativistic motion* of particles with masses m_i in $\mathbb{R}^k$ is a list of continuous piecewise linear functions $y_1, \dots, y_n \colon \mathbb{R} \to \mathbb{R}^k$, each with finitely many break points and $\|y_i'\| < 1$, such that at every point the particles present there have the same total momentum $\sum m_i p_i$ and the same total energy $\sum m_i \sqrt{1 + \|p_i\|^2}$ before and after, where $p_i = y_i'/\sqrt{1 - \|y_i'\|^2}$ and the speed of light is 1; collision points are defined as in Definition 1.2.

Theorem C. *For every $M \geq 1$ there is $P_M > 0$, nonincreasing in M, with the following property. If $k \geq 1$ and $y_1, \dots, y_n$ is a relativistic motion in $\mathbb{R}^k$ with masses at least 1 and total mass M, all of whose momenta per unit mass have norm less than P_M, then it has at most $\lfloor \frac{M}{2}\, 8^{M-2} \rfloor$ collision points. For every $L \geq 0$ there is a relativistic motion of $3L + 2$ particles of unit mass in $\mathbb{R}$ with at least $2^L + L - 1$ collision points at which at least three particles meet.*

How the framework applies. The relativistic energy $\sqrt{1 + \|p\|^2} - 1$ has Hessian at most I, so after a change of the unit of time its restriction to momenta of norm less than P is an energy of pinching $(1 + P^2)^{3/2}$, and a relativistic motion with such momenta is a collision system with that energy. Choosing P after M so that the pinching is below ρ_M gives the upper bound; the lower bound is Theorem 1.4 again. We know of no earlier bound for this model. The restriction to slow particles reflects the perturbative nature of Theorem 1.3: $P_M \to 0$ as $M \to \infty$, and the theorem does not supply a uniform bound without a restriction on the speeds. Rigidity (Proposition 2.1), by contrast, holds at all speeds.

1.4. **Method.** The proof of Theorem 1.3 has three steps. Proposition 3.1 shows that the direction of the centered configuration of the particles, in the mass-weighted metric, sweeps a total angle of at most π plus a small error per collision; for exact Newtonian collisions this is the classical observation

that the angle between the configuration vector and its velocity is continuous across collisions and decreasing between them. Theorem 3.2 partitions the particles, at each point of the angular parameter, into clusters whose spread is at most a threshold that grows geometrically with the mass of the cluster, and shows that the clusters at two nearby points of the parameter are nested and separated. Lemma 3.3 packs the clusters along the angular parameter and reduces the count for the whole system to counts for lighter subsystems. An induction on the number of particles gives the bound. Its base is Proposition 2.1, which is a convexity statement: the free action of a particle is a convex function of the vertices of its path, criticality is conservation of momentum and energy, and the action of a path with given ends is least for the straight one. For the Euclidean energy this is the convexity of length, and for the relativistic energy the concavity of proper time.

Theorem 1.4 is obtained by repeating one step, described here for nets. Given a net in the plane whose strands cross the axis $y = 0$ at K points, three new strands with slopes of order η, at heights of order η, are added; each of the K crossings becomes a vertex of degree six, at which the old strand is bent by an angle of order η^2 so that momentum and energy balance. The new net has $2K-1$ axis crossings. At $\eta = 0$ the balance equations are linear in the unknown heights and bends and are solved explicitly; the implicit function theorem gives solutions for small $\eta > 0$.

The longer computations are collected in the appendices: the reduction from weak convergence to strands (Appendix A), the angular estimate (Appendix B) and the cluster estimates (Appendix C).

2. Energies and rigidity

2.1. **Examples of energies.** For the Newtonian energy $h(p) = \frac{1}{2}\|p\|^2$ on $\Omega = X$ (Section 1.1), velocity equals momentum and $V = X$. For the *Euclidean energy*, Ω is the open unit ball $\mathbb{B}$ and

$$h(p) = 1 - \sqrt{1 - \|p\|^2}, \qquad \operatorname{grad} h(p) = \frac{p}{\sqrt{1 - \|p\|^2}},$$
$$D^2h(p) = (1 - \|p\|^2)^{-1/2} I + (1 - \|p\|^2)^{-3/2}\, p \otimes p;$$

the eigenvalues of $D^2h(p)$ are $(1-\|p\|^2)^{-1/2}$ on $p^\perp$ and $(1-\|p\|^2)^{-3/2}$ along p. The velocity map $\operatorname{grad} h\colon \mathbb{B} \to X$ is a bijection with inverse

$$p(v) = \frac{v}{\sqrt{1 + \|v\|^2}}, \qquad \text{and} \qquad 1 - h(p(v)) = \frac{1}{\sqrt{1 + \|v\|^2}}. \tag{2.1}$$

Thus, if v is the slope of a segment in $\mathbb{R} \times X$ and $p = p(v)$, the unit tangent of the segment in the direction of increasing time is $(1 - h(p), p)$: the momentum is the transverse part of the unit tangent, and the energy is one minus its longitudinal part. For $a > 0$, the *Euclidean energy with cap* a is the restriction of h to $\Omega_a = \{p : \|\operatorname{grad} h(p)\| < a\} = \{p : \|p\|^2 < a^2/(1+a^2)\}$,

an open ball; its set of velocities is the ball of radius a, and its pinching is

$$\rho_a = (1 + a^2)^{3/2}. \tag{2.2}$$

A third example comes from special relativity. For a free particle of unit rest mass, with the speed of light equal to 1, the energy is $\sqrt{1 + \|p\|^2}$ as a function of the momentum $p \in X$, and the velocity is $p/\sqrt{1 + \|p\|^2}$. Put $h_{\mathrm{rel}}(p) = \sqrt{1 + \|p\|^2} - 1$; the eigenvalues of $D^2 h_{\mathrm{rel}}(p)$ are $(1 + \|p\|^2)^{-1/2}$ on $p^\perp$ and $(1 + \|p\|^2)^{-3/2}$ along p, so $D^2 h_{\mathrm{rel}} \le I$ and h_{rel} itself is not an energy in the sense of Definition 1.1. For $P > 0$ put $c_P = (1 + P^2)^{3/2}$. On $\Omega_P = \{p : \|p\| < P\}$ one has $c_P^{-1} I \le D^2 h_{\mathrm{rel}} \le I$, so the *relativistic energy with cap* P, $(\Omega_P, c_P h_{\mathrm{rel}})$, is an energy, with a ball as set of velocities and with pinching

$$\rho_P = (1 + P^2)^{3/2}, \tag{2.3}$$

the same value as for the Euclidean energy with cap P. Its velocities are c_P times the physical ones, which is a change of the unit of time.

2.2. **Subsystems.** At a point that is not a collision point, either only one particle is present or all particles present coincide on a neighbourhood of that time. In the latter case they have a common momentum on each side, and conservation of their total momentum makes these momenta equal. Thus a particle changes its momentum only at a collision point. Event times are therefore among the times of collision points, and both are finite in number, since a common finite affine subdivision has only finitely many isolated pairwise meetings. If C is a set of labels and $J' \subset J$ an interval such that no particle with label in C is at the position of a particle with label outside C at any time in J', then the particles with labels in C, restricted to J', form a collision system with the same masses and energy, and its collision points are the collision points of the whole system in J' whose set S lies in C: condition (b) only involves sets of coincident particles, and these lie in C or in its complement.

2.3. **Rigidity.** The following proposition is the base of the induction in the proof of Theorem 1.3. It holds for every energy, with no restriction on the pinching.

Proposition 2.1 (Rigidity)**.** *Let (Ω, h) be an energy on X, and let $y_1, \dots, y_n$ be a collision system with energy h on an interval J such that all particles are at one point y_* at a time $t_* \in J$. Then (t_*, y_*) is the only possible collision point.*

The proof is a convexity argument: the free action of a particle is a convex function of the vertices of its path, criticality is the conservation of momentum and energy, and the action of a path with given ends is least for the straight one. For the Euclidean energy the action is length minus duration, for the Newtonian energy it is $\|d\|^2/2\tau$ for a displacement d in

time τ, and for the relativistic energy it is, up to the unit of time, duration minus proper time.

Proof of Proposition 2.1. The action. Let $V = \operatorname{grad} h(\Omega)$, and for $v \in V$ let $L(v) = p \cdot v - h(p)$, where $p \in \Omega$ is the momentum with $\operatorname{grad} h(p) = v$. Since $\operatorname{grad} h\colon \Omega \to V$ is a diffeomorphism, L is smooth, and differentiating gives $\operatorname{grad} L(v) = p$ and $D^2L(v) = D^2h(p)^{-1}$, which is positive definite; so L is strictly convex on the convex set V. On the open convex cone $C = \{(\tau, d) \in \mathbb{R}\times X : \tau > 0,\ d/\tau \in V\}$ put $\mathcal{A}(\tau, d) = \tau L(d/\tau)$. Then $\mathcal{A}$ is smooth and positively homogeneous of degree one, and since $\partial_d\mathcal{A} = \operatorname{grad} L(v)$ and $\partial_\tau\mathcal{A} = L(v) - v \cdot \operatorname{grad} L(v)$ with $v = d/\tau$,

$$\partial_d\mathcal{A}(\tau, d) = p, \qquad \partial_\tau\mathcal{A}(\tau, d) = -h(p), \qquad \text{where } \operatorname{grad} h(p) = d/\tau. \tag{2.4}$$

For $(\tau, d), (\tau', d') \in C$, convexity of L with the weights $\tau/(\tau+\tau')$ and $\tau'/(\tau+\tau')$ gives

$$\mathcal{A}(\tau, d) + \mathcal{A}(\tau', d') \geq \mathcal{A}(\tau + \tau', d + d'), \tag{2.5}$$

with equality only if $d/\tau = d'/\tau'$, by strict convexity. Together with homogeneity, (2.5) shows that $\mathcal{A}$ is convex on C.

The graph. If J consists of one point there are no collision points. Otherwise choose a nondegenerate compact interval $[A, T] \subset J$ containing t_*, all break points and all collision times, with every such time in the interior of $[A, T]$ unless it is an endpoint of J. Thus restriction preserves a relative neighbourhood of every collision time. Put $z_* = (t_*, y_*)$. Subdivide every trajectory at all these times and at A, T, and use the resulting points as vertices, identifying coincident points. Between consecutive subdivision times the particles form groups that travel together on common segments; take these segments as edges, directed forwards in time, and let M_e be the total mass of the group on the edge e. Each edge e has duration $\tau_e > 0$ and displacement d_e with $d_e/\tau_e = \operatorname{grad} h(p_e) \in V$, where p_e is the common momentum of its group.

Criticality. For a placement x of the vertices with $A < t < T$ (as points of $\mathbb{R} \times X$, the vertices on $t = A$ and $t = T$ being fixed) such that $\operatorname{head}(e) - \operatorname{tail}(e) \in C$ for every edge e, put

$$F(x) = \sum_e M_e\, \mathcal{A}\big(\operatorname{head}(e) - \operatorname{tail}(e)\big).$$

The admissible placements form a convex set, on which F is convex, and the given placement x_0 is admissible. By (2.4), the partial derivative of F at x_0 with respect to a vertex N is

$$\sum_{\operatorname{head}(e)=N} M_e\big(-h(p_e), p_e\big) - \sum_{\operatorname{tail}(e)=N} M_e\big(-h(p_e), p_e\big).$$

The particles at N are the union of the groups of the incoming edges and also the union of the groups of the outgoing edges, so this is the difference between the totals of $(-m_ih(p_i), m_ip_i)$ over the particles at N before and

after N, which vanishes by (b). Hence $F(x) \geq F(x_0)$ for every admissible x.

Comparison. Put $s_i = (A, y_i(A))$ and $e_i = (T, y_i(T))$. For the endpoint cases $t_* = A$ or $t_* = T$, set $\mathcal{A}(0,0) = 0$ and omit the corresponding zero-duration terms below. Let x_1 place every vertex with $A < t < T$ at z_*, and let $x_s = (1-s)x_0 + sx_1$ for $0 \leq s < 1$. In x_s, an edge between two such vertices has head $-$ tail $= (1-s)(\tau_e, d_e) \in C$, and an edge with one end on the slice $t = A$ or $t = T$ has head $-$ tail $= (1-s)(\tau_e, d_e) + s(\tau_e^1, d_e^1)$, where (τ_e^1, d_e^1) is $z_* - s_i$ or $e_i - z_*$ for a particle i on e; when $\tau_e^1 > 0$, this vector lies in C, because d_e^1/τ_e^1 is a time average of velocities in the convex set V. When $\tau_e^1 = 0$, the vector is zero and the interpolated edge remains in C for $s < 1$. Edges joining two fixed boundary vertices are unchanged; they can occur only when t_* is a boundary time. Thus all placements x_s are admissible and $F(x_s) \geq F(x_0)$. As $s \to 1$, an edge of the first kind contributes $(1-s)M_e\mathcal{A}(\tau_e, d_e) \to 0$ by homogeneity, and an edge of the second kind converges. Hence, and by (2.5) applied to the edges traversed by y_i before and after t_*, whose displacements add up to $z_* - s_i$ and $e_i - z_*$, and since M_e is the sum of the masses of the particles traversing e,

$$F(x_0) \leq \lim_{s\to 1} F(x_s) = \sum_{i=1}^{n} m_i \big[\mathcal{A}(z_* - s_i) + \mathcal{A}(e_i - z_*)\big]$$
$$\leq \sum_{i=1}^{n} m_i \sum_{e \text{ traversed by } y_i} \mathcal{A}(\tau_e, d_e) = F(x_0).$$

So equality holds in (2.5) throughout, and every y_i has constant velocity on $[A, t_*]$ and on $[t_*, T]$.

Conclusion. Let (t, y) be a collision point with $t < t_*$ and S its set of particles. Each y_i, $i \in S$, is affine on $[A, t_*]$ with $y_i(t) = y$ and $y_i(t_*) = y_*$, so these functions coincide on $[A, t_*]$, which is a neighbourhood of t in J; this contradicts the definition of a collision point. The same holds for $t > t_*$, and at time t_* all particles are at y_*. □

3. The upper bound

3.1. **The angular estimate.** For $\rho \geq 1$ define

$$\eta(\rho) = \pi \sqrt{\frac{2(\rho-1)\sqrt{\rho} + \rho^{3/2} - 1}{2}}; \tag{3.1}$$

η is continuous and increasing, with $\eta(1) = 0$.

For a collision system with masses $m_1, \ldots, m_n$ we use on X^n the mass-weighted inner product and norm

$$\langle x, y\rangle = \sum_{i=1}^{n} m_i\, x_i \cdot y_i, \qquad \|x\| = \langle x, x\rangle^{1/2}.$$

Proposition 3.1 (Angular budget)**.** *Let $y_1, \dots, y_n$, $n \geq 2$, be a collision system on an interval $[A, T]$ with masses m_i, total mass M, and an energy of pinching $\rho < \infty$, with at most ν event times. Let*

$$q_i = y_i - \frac{1}{M} \sum_{j=1}^{n} m_j y_j, \qquad q = (q_1, \dots, q_n),$$

and assume that q never vanishes. Put $u = q/\|q\|$ and let L be the length of the curve u on the unit sphere of X^n for the weighted norm. Then

$$L \leq \pi + \nu\, \eta(\rho).$$

The proof is in Appendix B. Its idea is the following. The function $K = \sum_i m_i h(p_i) - M h(\mu)$, where μ is the mass-weighted mean momentum, is conserved by (b), and it controls both the norm of the centered momentum $z = (p_i - \mu)_i$ and the norm of the velocity $w = q'$ up to the factor ρ. On a flight, the angle between q and w decreases, and its decrease is the spherical length of u. At an event, the jump of w is orthogonal to q up to an error of order $\rho - 1$, so the angle jumps by at most $\eta(\rho)$. For the Newtonian energy the error vanishes and $L \leq \pi$.

3.2. **Thin clusters and packing.** In this section E is a finite-dimensional real inner-product space, $n \geq 2$, $\mathcal{N} = \{1, \dots, n\}$, and $m_1, \dots, m_n \geq 1$ are masses with total mass $M = m_1 + \cdots + m_n$. A *configuration* is a list $x = (x_1, \dots, x_n) \in E^n$, and E^n carries the mass-weighted inner product $\langle x, y \rangle = \sum_i m_i x_i \cdot y_i$ with norm $\|x\|$. If $\sum_i m_i x_i = 0$, the configuration is called *centered*; if $\|x\| = 1$, it is called *unit*. For a nonempty $S \subset \mathcal{N}$ put $m_S = \sum_{i \in S} m_i$ and

$$\mu_S(x) = \frac{1}{m_S} \sum_{i \in S} m_i x_i, \qquad r_S(x) = \Big(\sum_{i \in S} m_i \|x_i - \mu_S(x)\|^2 \Big)^{1/2}.$$

For unit masses these are the usual mean and spread of a configuration.

Theorem 3.2 (Thin clusters)**.** *Let $\alpha > 0$, $0 < \lambda < 1$ and $c > 0$ satisfy*

$$\big(1 + \sqrt{2} + 2\sqrt{2}\, c\big)^2 < \alpha^2 - 1, \tag{3.2}$$

and put $d = 2\sqrt{2}\, c$ and $\epsilon(s) = \lambda \alpha^{s-M}$ for $1 \leq s \leq M$. Let x be a centered unit configuration. If $S \subset \mathcal{N}$ is nonempty and $r_S(x) \leq \epsilon(m_S)$, call S thin *at x; if S is thin and not contained in a larger thin set, call S* maximal thin*. Then:*

(a) *The maximal thin sets at x form a partition of $\mathcal{N}$ into proper subsets.*
(b) *If $S \neq T$ are maximal thin at x, $i \in S$ and $j \in T$, then $\|x_i - x_j\| > d\, \epsilon(\max(m_S, m_T))$. In particular, a set of indices with a common position is contained in exactly one maximal thin set.*

Let moreover $u \colon J \to E^n$ be a 1-Lipschitz curve of centered unit configurations on an interval J, and for $p \in J$ let S_p be a maximal thin set at $u(p)$, $s_p = m_{S_p}$, and $\mathcal{K}_p = [p - c\epsilon(s_p), p + c\epsilon(s_p)]$, called the core *of p. Then:*

(c) *If* $p, q \in J$, $\mathcal{K}_p \cap \mathcal{K}_q \neq \varnothing$, $S_p \cap S_q \neq \varnothing$ *and* $s_p \geq s_q$, *then* $S_q \subset S_p$.
(d) *If* $p, \tau \in J$, $|\tau - p| \leq 2c\epsilon(s_p)$, $i \in S_p$ *and* $j \notin S_p$, *then* $u_i(\tau) \neq u_j(\tau)$.

The proof is in Appendix C. It consists of elementary estimates based on the identity $\sum_{i\in A} m_i\|v_i - z\|^2 = r_A(v)^2 + m_A\|\mu_A(v) - z\|^2$ and Bernoulli's inequality; (3.2) is the only property of α and c that is used. The thresholds depend on the mass of a cluster, not on the number of its particles; the only place where the masses enter the estimates is through the normalization $m_i \geq 1$.

Lemma 3.3 (Packing)**.** *Let* $n \geq 3$, *let* α, λ, c *and* ϵ *be as in Theorem 3.2, and let* $u\colon [0, L] \to E^n$ *be a* 1*-Lipschitz curve of centered unit configurations. Let* $\mathcal{E}$ *be a finite set of* events*; to each* $e \in \mathcal{E}$ *are assigned a point* $x_e \in [0, L]$ *and a set* $P_e \subset \mathcal{N}$ *of at least two indices with a common position at* $u(x_e)$. *Let* C_e *be the maximal thin set at* $u(x_e)$ *containing* P_e, *let* $s_e = m_{C_e}$, *and put* $J_e = [x_e - c\epsilon(s_e), x_e + c\epsilon(s_e)]$. *Let* $F\colon [2, M-1] \to [0, \infty)$. *Assume that for every* $e \in \mathcal{E}$ *the number of* $f \in \mathcal{E}$ *(including* e*) with*

$$|x_f - x_e| \leq 2c\epsilon(s_e) \qquad \text{and} \qquad C_f \subset C_e$$

is at most $F(s_e)$. *Then*

$$|\mathcal{E}| \leq \frac{M}{2c\lambda}\Big(L + \frac{2c\lambda}{\alpha}\Big) \sup_{2\leq s\leq M-1} \frac{\alpha^{M-s}F(s)}{s}. \tag{3.3}$$

Proof. By Theorem 3.2(a),(b), each C_e is well defined, and $2 \leq s_e \leq M-1$, since C_e contains at least two particles, is proper, and all masses are at least 1. Select events greedily: among the events not yet selected or removed, select one with the largest s_e, and remove every event f with $J_f \cap J_e \neq \varnothing$ and $C_f \cap C_e \neq \varnothing$. If f is removed by e, then $s_e \geq s_f$, so Theorem 3.2(c) gives $C_f \subset C_e$, and $|x_f - x_e| \leq c(\epsilon(s_e) + \epsilon(s_f)) \leq 2c\epsilon(s_e)$. By the hypothesis, each selected e accounts for at most $F(s_e)$ events (itself and those it removes), so with $\mathcal{E}_{\text{sel}}$ the set of selected events,

$$|\mathcal{E}| \leq \sum_{e\in\mathcal{E}_{\text{sel}}} F(s_e). \tag{3.4}$$

If two selected events have overlapping intervals J_e, their sets C_e are disjoint, since otherwise the later one would have been removed. Hence for every real z, $\sum\{s_e : e \in \mathcal{E}_{\text{sel}},\ z \in J_e\} \leq M$. All intervals J_e lie in $[-c\lambda/\alpha, L + c\lambda/\alpha]$, because $\epsilon(s_e) \leq \epsilon(M-1) = \lambda/\alpha$. Integrating over z,

$$\sum_{e\in\mathcal{E}_{\text{sel}}} 2c\, s_e\epsilon(s_e) \leq M\Big(L + \frac{2c\lambda}{\alpha}\Big).$$

Since $F(s_e) = 2cs_e\epsilon(s_e) \cdot F(s_e)/(2cs_e\epsilon(s_e))$ and $\epsilon(s) = \lambda\alpha^{s-M}$, combining with (3.4) gives (3.3). □

3.3. **The induction.**

Proof of Theorem 1.3. Parameters. Put $B = 8$, $\alpha = 7.7$, $c = 1.8$ and $\lambda = 0.99$. Then $1 + \sqrt{2} + 2\sqrt{2}c < 7.51$ and $7.51^2 < 56.5 < 58.29 = \alpha^2 - 1$, so (3.2) holds, and

$$B' := \frac{\alpha\pi}{2c\lambda} + 1 < \frac{24.2}{3.56} + 1 < 7.8 < B.$$

Since the number of collision points is an integer, it suffices to prove the bound $\frac{M}{2}B^{M-2}$ without the floor. For $M \geq 3$ put $Q(M) = M\alpha B^{M-3}/(4c\lambda)$, which increases with M, and choose $\rho_M > 1$ with

$$Q(M)\,\eta(\rho_M) \leq 1 - \frac{B'}{B}, \tag{3.5}$$

which is possible since η in (3.1) is continuous with $\eta(1) = 0$; since Q is increasing, we take ρ_M nonincreasing in M. For $M < 3$ put $\rho_M = \rho_3$.

We prove the assertion by induction on the number n of particles. For $n = 1$ there is no collision point, since a collision point has at least two particles. For $n = 2$, a collision point is a time at which both particles are at one point, so Proposition 2.1 gives at most one, and $1 \leq \frac{M}{2}B^{M-2}$ since $M \geq 2$. Let $n \geq 3$, and assume the assertion for fewer particles.

Setting up. Let $y_1, \dots, y_n$ be a collision system on J with masses $m_i \geq 1$, total mass M, and an energy h of pinching $\rho < \rho_M$; then $M \geq 3$. Let ν be its number of collision points. If all particles are at one point at some time, Proposition 2.1 gives $\nu \leq 1$. Otherwise $q = (y_i - \frac{1}{M}\sum_j m_j y_j)_i$ never vanishes. Choose a compact interval $[A, T] \subset J$ containing all break points and all times of collision points. Proposition 3.1 on $[A, T]$, with at most ν event times, gives for the spherical length L of $u = q/\|q\|$

$$L \leq \pi + \nu\,\eta(\rho). \tag{3.6}$$

Reparametrize u by spherical arclength to obtain a 1-Lipschitz curve $\bar{u}\colon [0, L] \to X^n$ of centered unit configurations; the reparametrization is a monotone continuous map $\tau\colon [A, T] \to [0, L]$ onto $[0, L]$ with $\bar{u}(\tau(t)) = u(t)$.

Events and capacities. Let $\mathcal{E}$ be the set of collision points. For $e \in \mathcal{E}$ at time t_e, put $x_e = \tau(t_e)$ and let P_e be the set of labels of the particles at e; it has at least two elements. Apply Theorem 3.2 to $\bar{u}$, with the masses m_i, the total mass M and the parameters α, λ, c, and let C_e, s_e be as in Lemma 3.3. Put $W(s) = \frac{s}{2}B^{s-2}$. We claim that the number of $f \in \mathcal{E}$ with $|x_f - x_e| \leq 2c\epsilon(s_e)$ and $C_f \subset C_e$ is at most $W(s_e)$. Fix e. By Theorem 3.2(d), on the arclength window $W_e = [x_e - 2c\epsilon(s_e), x_e + 2c\epsilon(s_e)] \cap [0, L]$ no particle with label in C_e is at the position of a particle with label outside C_e. The time interval $\tau^{-1}(W_e)$ is a closed subinterval of $[A, T]$; since the positions in (d) are distinct at its endpoints and only finitely many affine pieces are involved, there is an interval $J' \supset \tau^{-1}(W_e)$, relatively open in J, on which the same holds. By Section 2.2, the particles with labels in C_e, restricted to J', form a collision system of fewer than n particles, with masses at least 1, total mass s_e, and the energy h of pinching $\rho < \rho_M \leq \rho_{s_e}$, since $s_e \leq M$ and

ρ_M is nonincreasing in M; by the induction hypothesis it has at most $W(s_e)$ collision points, and these are the collision points f of the whole system with $t_f \in J'$ and $P_f \subset C_e$. Every f as in the claim is of this kind, since $x_f \in W_e$ and $P_f \subset C_f \subset C_e$. This proves the claim.

Conclusion. Lemma 3.3 with $F = W$ gives

$$\nu = |\mathcal{E}| \le \frac{M}{2c\lambda}\Big(L + \frac{2c\lambda}{\alpha}\Big) \sup_{2\le s\le M-1} \frac{\alpha^{M-s} W(s)}{s}.$$

Since $W(s)/s = \frac{1}{2}B^{s-2}$ and $B > \alpha$, the quantity $\alpha^{M-s}B^{s-2}$ increases with s, so the supremum is at most $\frac{1}{2}\alpha B^{M-3}$, and with (3.6)

$$\begin{aligned}\nu \le \frac{M\alpha B^{M-3}}{4c\lambda}\, L + \frac{M}{2}B^{M-3} &\le \frac{M}{2}B^{M-3}\Big(\frac{\alpha\pi}{2c\lambda} + 1\Big) + Q(M)\,\eta(\rho)\,\nu \\ &= \frac{M}{2}B^{M-3}B' + Q(M)\,\eta(\rho)\,\nu.\end{aligned}$$

As $\eta(\rho) \le \eta(\rho_M)$, (3.5) gives $Q(M)\eta(\rho) \le 1 - B'/B$, hence

$$\frac{B'}{B}\,\nu \le \big(1 - Q(M)\eta(\rho)\big)\nu \le \frac{M}{2}B^{M-3}B',$$

that is, $\nu \le \frac{M}{2}B^{M-2}$. □

4. The lower bound

In this section we prove Theorem 1.4. Here $X = \mathbb{R}$, and we work in the plane $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$, whose points we write as (t, y); we call y the *height*, and the *axis* is the line $y = 0$. If a segment or ray in $\mathbb{R}^2$ is not contained in a line $t = \text{const}$, it is called *non-vertical*, and its *slope* is dy/dt along it. Fix an energy (Ω, h) on $\mathbb{R}$ and $\delta > 0$ with $[-2\delta, 2\delta] \subset h'(\Omega)$; all slopes below lie in $(-2\delta, 2\delta)$.

4.1. **Diagrams.** The set of velocities $h'(\Omega)$ is an open interval containing 0, and we put $\kappa = 1/h''(0) \in (0, 1]$ and

$$\varphi = (h')^{-1}\colon h'(\Omega) \to \Omega, \qquad \psi = h\circ\varphi\colon h'(\Omega) \to \mathbb{R};$$

$\varphi(s)$ is the momentum of velocity s and $\psi(s)$ its energy. Since $h'' \ge 1$, both functions are smooth, and

$$\begin{gathered}\varphi(0) = 0, \quad \varphi'(0) = \kappa, \quad \psi(0) = \psi'(0) = 0, \quad \psi''(0) = \kappa, \\ \psi'(s) = s\,\varphi'(s) \ne 0 \ \text{ for } s \ne 0.\end{gathered} \tag{4.1}$$

For the Newtonian energy $\varphi(s) = s$ and $\psi(s) = s^2/2$; for the Euclidean energy $\varphi(s) = s/\sqrt{1+s^2}$ and $\psi(s) = 1 - 1/\sqrt{1+s^2}$; for the relativistic energy with cap P, the velocities are $(-c_P P/\sqrt{1+P^2}, c_P P/\sqrt{1+P^2})$ and $\kappa = 1/c_P$. Only (4.1) is used below.

Definition 4.1. A *net* is a finite set of segments and rays in the plane, called its *edges*; its *vertices* are the endpoints of its edges. Fix an energy h on $\mathbb{R}$. A *diagram* with energy h is a net whose edges are non-vertical with slopes in $h'(\Omega)$. At a vertex p, if an edge ending at p lies in $\{t \le t(p)\}$ it is called *incoming*, and otherwise *outgoing*. It is required that at every vertex the incoming and the outgoing edges are equal in number, have the same sum of $\varphi(s)$, and have the same sum of $\psi(s)$, where s runs over their slopes. The number of *particles* of the diagram is the number of its rays unbounded towards $t = -\infty$.

Lemma 4.2. *Let an energy be fixed and let a diagram with n particles be given.*

(i) *The diagram is the union of the graphs of n continuous piecewise linear functions on $\mathbb{R}$, which change slope only at vertices.*
(ii) *These functions form a collision system of n particles of unit mass on $\mathbb{R}$ with energy h, and every vertex with at least three incoming edges, at which the slopes of the edges ending there are not all equal, is a collision point at which at least three particles meet.*

Proof. (i) At each vertex pair the incoming edges with the outgoing ones. Following paired edges forwards from a ray unbounded towards $-\infty$ gives such a graph, and every edge lies on exactly one of the n graphs so obtained.

(ii) On an edge of slope s a particle has momentum $\varphi(s)$ and velocity $h'(\varphi(s)) = s$, which is (a) of Definition 1.2. Let p be a point and S the set of graphs through p. The graphs without a vertex at p do not change slope there; for the others, the vertex condition is (b). At a vertex with at least three incoming edges, at least three graphs meet, and if the slopes of the edges ending there are not all equal, these graphs do not all coincide near the vertex. □

4.2. **The construction.** Fix an integer $L \ge 1$. A *level* is an integer ℓ with $1 \le \ell \le L$, and $K_\ell = 2^{\ell-1} + 1$.

Unknowns. Level ℓ has a *scale* $\eta_\ell \ge 0$ and real numbers $Y_\ell(m)$ for $3 \le m \le K_\ell + 2$ and $\xi_\ell(k)$ for $1 \le k \le K_\ell$. We fix

$$Y_\ell(-1) = -1, \qquad Y_\ell(0) = 1, \qquad Y_\ell(1) = -2, \qquad Y_\ell(2) = 2.$$

Let η be the list of scales and v the list of all $Y_\ell(m)$ and $\xi_\ell(k)$; level ℓ contributes $2K_\ell$ numbers to v. When the level is clear we drop the index ℓ.

Columns. A *column* is a segment with ends $E^- = (t^-, b^-)$ and $E^+ = (t^+, b^+)$, where $b^- < 0 < b^+$ and $t^- \ne t^+$. Its *velocity* is $V = (b^+ - b^-)/(t^+ - t^-)$, and its *axis time* is $x = t^- - b^-/V$, the time at which it crosses the axis.

The levels. For $\ell = 1, \dots, L$ in turn, level ℓ has $K = K_\ell$ columns, numbered $1, \dots, K$, and the following points and edges. Write $\eta = \eta_\ell$.

(1) *Level* 1. Column $k \in \{1, 2\}$ has ends $(k-1, -\delta)$ and $(k+1, \delta)$.
(2) *Nodes.* For column k, with ends $E^\pm$, velocity V and axis time x, the *node* $C(k)$ is the point of height $\eta Y(k)$ and time
$$X(k) = x + \frac{\eta Y(k)}{V} + \eta^2 \xi(k).$$
The *lower piece* $[E^-, C(k)]$ and the *upper piece* $[C(k), E^+]$ replace the column. If $\xi(k) = 0$, the node lies on the column.
(3) *Outer points.* $C(-1) = (-1, -\eta)$, $C(0) = (0, \eta)$, $C(K+1) = (3, \eta Y(K+1))$, $C(K+2) = (4, \eta Y(K+2))$. Write $X(m)$ for the time of $C(m)$.
(4) *Links.* The segments $[C(k), C(m)]$ with $m - k \in \{1, 2\}$ and k or m in $\{1, \dots, K\}$. A link $[C(k), C(k+1)]$ between two nodes is *short*.
(5) *Columns of level* $\ell+1$, if $\ell < L$: for odd k the upper piece of column k, with ends $C(k)$ and E^+; for even k the lower piece, with ends E^- and $C(k)$; and each short link, with E^- its end of odd index. The piece of column k gets the number $2k-1$ and the short link $[C(k), C(k+1)]$ the number $2k$. There are $2K_\ell - 1 = K_{\ell+1}$ of them.

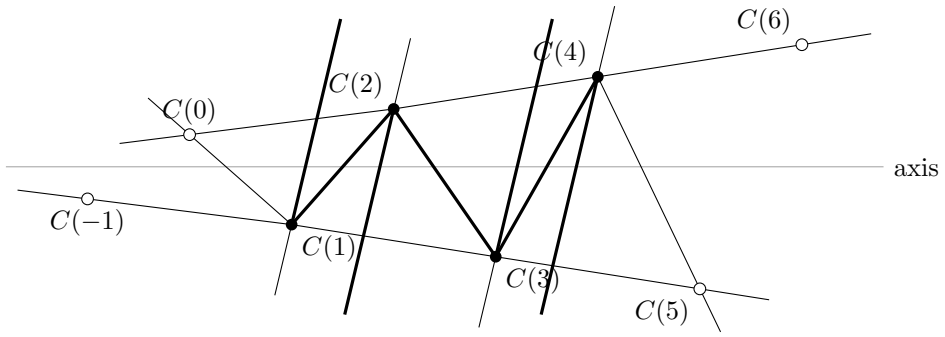

FIGURE 1. One level with $K = 4$ columns, not to scale. Time runs to the right. The lines of larger slope are the columns; they are bent at the nodes (black dots). Each node $C(k)$ is joined by links to $C(k \pm 1)$ and $C(k \pm 2)$; the links form three polygonal lines. Open circles: outer points. Thick: the $2K - 1 = 7$ columns of the next level.

Admissible values. If the following hold for every level, the pair (v, η) is called *admissible*:

(a) $b^- < \eta Y(k) < b^+$ and $b^- < \eta Y(k) + \eta^2 \xi(k) V < b^+$ for every column;
(b) $0 < X(1) < \cdots < X(K) < 3$;
(c) if $\ell < L$, then $\eta_\ell > 0$ and $Y_\ell(k)$ has the sign of $(-1)^k$ for $1 \le k \le K$.

By (c), the segments in step (5) are columns. Since $X(k)-t^- = (\eta Y - b^- + \eta^2\xi V)/V$, with $Y = Y(k)$ and $\xi = \xi(k)$, condition (a) says that $X(k)$ lies strictly between t^- and t^+, and the pieces have slopes

$$s_0 = \frac{V\,(\eta Y - b^-)}{\eta Y - b^- + \eta^2\xi V}, \qquad s_1 = \frac{V\,(b^+ - \eta Y)}{b^+ - \eta Y - \eta^2\xi V} \tag{4.2}$$

(lower and upper piece). The link $[C(k), C(m)]$ has slope $\eta\,\hat{s}(k,m)$, where

$$\hat{s}(k,m) = \frac{Y(m)-Y(k)}{X(m)-X(k)}.$$

These strict inequalities define an open set if the last scale η_L is temporarily allowed to be real. All these quantities are smooth there, so the value $\eta_L = 0$ has an open neighbourhood on which the implicit function theorem can be applied. The constructed diagrams use positive scales.

Equations. Let σ be a piece or a link of level ℓ and N one of its ends. If $\ell < L$ and σ is a column of level $\ell+1$, put $s(N,\sigma) = s(N,\sigma')$, where σ' is the piece of that column that ends at N. Otherwise $s(N,\sigma)$ is the slope of σ.

Six edges of level ℓ end at a node $N = C_\ell(k)$: two pieces and four links. By (a) and (b), three of them lie in $\{t \le X(k)\}$ (incoming, in the sense of Definition 4.1) and three in $\{t \ge X(k)\}$ (outgoing). Put

$$p(N) = \sum_{\text{out}} \varphi(s(N,\sigma)) - \sum_{\text{in}} \varphi(s(N,\sigma)),$$
$$e(N) = \sum_{\text{out}} \psi(s(N,\sigma)) - \sum_{\text{in}} \psi(s(N,\sigma)).$$

Lemma 4.3. *Let (v,η) be admissible with all $\eta_\ell > 0$, all $p(N) = e(N) = 0$, and pairwise distinct nodes. Then there is a diagram with $3L+2$ particles whose vertices are the nodes, each with three incoming and three outgoing edges of slopes $s(N,\sigma)$.*

Proof. Take all pieces and links that are not columns of a later level. Prolong by a ray each of them that ends at an end of a column of level 1 or at an outer point. The vertices are then the nodes. By (a), a piece of a column lies on the same side in time of their common end as the column. So the six edges of the diagram at a node N are three incoming and three outgoing ones, with slopes $s(N,\sigma)$, and the vertex condition is $p(N) = e(N) = 0$. The rays unbounded towards $-\infty$ are one for each column of level 1 and three for each level. □

4.3. **The last level at small scale.** By (4.1) there are smooth functions a, c with $\varphi(s) = s\,a(s)$, $\psi(s) = s^2 c(s)$, $a(0) = \kappa$, $c(0) = \frac{\kappa}{2}$. For a smooth function f put $f^\Delta(u,w) = \int_0^1 f'(u+\tau(w-u))\,d\tau$, so that $f(w) - f(u) = (w-u)f^\Delta(u,w)$ and $f^\Delta(u,u) = f'(u)$.

Consider level L; write $\eta = \eta_L$, $K = K_L$, and let $x_k, V_k, b_k^{\pm}$ belong to its column k. These depend only on the unknowns of the levels below L. By (4.2),

$$s_1 - s_0 = \eta^2\, \xi(k)\, V_k^2\, B_k,$$
$$B_k = \frac{1}{b_k^+ - \eta Y(k) - \eta^2 \xi(k) V_k} + \frac{1}{\eta Y(k) - b_k^- + \eta^2 \xi(k) V_k}\,.$$

If $V_k > 0$, the lower piece is the incoming one, and if $V_k < 0$, the upper piece is. Let $\epsilon_k = \pm 1$ be the sign of V_k. Hence, for $\eta > 0$, $p(C(k)) = \eta P_k$ and $e(C(k)) = \eta^2 Q_k$, where, with $\hat{s}_m = \hat{s}(k,m)$,

$$\begin{aligned} P_k = {} & \sum_{m=k+1,k+2} \hat{s}_m\, a(\eta \hat{s}_m) - \sum_{m=k-1,k-2} \hat{s}_m\, a(\eta \hat{s}_m) \\ & + \eta\, \epsilon_k\, \xi(k)\, V_k^2 B_k\, \varphi^{\Delta}(s_0, s_1), \\ Q_k = {} & \sum_{m=k+1,k+2} \hat{s}_m^2\, c(\eta \hat{s}_m) - \sum_{m=k-1,k-2} \hat{s}_m^2\, c(\eta \hat{s}_m) \\ & + \epsilon_k\, \xi(k)\, V_k^2 B_k\, \psi^{\Delta}(s_0, s_1). \end{aligned}$$

These expressions are smooth also at $\eta = 0$. There $X(m) = x_m$, with $x_{-1} = -1$, $x_0 = 0$, $x_{K+1} = 3$, $x_{K+2} = 4$, and $s_0 = s_1 = V_k$, so

$$\begin{aligned} P_k &= \kappa \Big[\sum_{m=k+1,k+2} \frac{Y(m) - Y(k)}{x_m - x_k} - \sum_{m=k-1,k-2} \frac{Y(k) - Y(m)}{x_k - x_m} \Big], \\ Q_k &= \frac{\kappa}{2} \Big[\sum_{m=k+1,k+2} \hat{s}_m^2 - \sum_{m=k-1,k-2} \hat{s}_m^2 \Big] \\ &\quad + \epsilon_k\, \psi'(V_k)\, V_k^2 \Big(\frac{1}{b_k^+} - \frac{1}{b_k^-} \Big) \xi(k). \end{aligned} \tag{4.3}$$

The energy enters (4.3) only through the positive number κ and the nonzero number $\psi'(V_k)$. For particles, (4.3) says that at scale zero the four links at $C(k)$ conserve momentum among themselves ($P_k = 0$), and the bend $\xi(k)$ of the column accounts for their change of energy ($Q_k = 0$).

Let $\Phi_L(v, \eta)$ be the list of the numbers $p(N), e(N)$ for the nodes of the levels below L, and P_k, Q_k for level L. It is smooth on the admissible set and has as many components as v has entries.

Reference state. Let $\eta_L = 0$ and $0 < x_1 < \cdots < x_K < 3$. Define $Y^0(m) = Y(m)$ for $-1 \le m \le 2$, and for $k = 1, \dots, K$ define $Y^0(k+2)$ by $P_k = 0$ in (4.3):

$$\begin{aligned} Y^0(k+2) = Y^0(k) + (x_{k+2} - x_k) \Big[& \frac{Y^0(k) - Y^0(k-1)}{x_k - x_{k-1}} + \frac{Y^0(k) - Y^0(k-2)}{x_k - x_{k-2}} \\ & - \frac{Y^0(k+1) - Y^0(k)}{x_{k+1} - x_k} \Big]. \end{aligned}$$

Then define $\xi^0(k)$ by $Q_k = 0$ in (4.3); this is possible since $b_k^- < 0 < b_k^+$, $V_k \neq 0$ and $\psi'(V_k) \neq 0$. Let w^0 be the list of the $Y^0(m)$, $3 \leq m \leq K+2$, and the $\xi^0(k)$.

Lemma 4.4. *For $1 \leq m \leq K+2$, $Y^0(m)$ has the sign of $(-1)^m$, and $|Y^0(m)| > |Y^0(m-2)|$.*

Proof. This holds for $m = 1, 2$. Assume it up to $k+1$, where $k \geq 1$. Then $Y^0(k) - Y^0(k-1)$, $Y^0(k) - Y^0(k-2)$ and $Y^0(k) - Y^0(k+1)$ are nonzero and have the sign of $Y^0(k)$. So the bracket above has the sign of $Y^0(k)$. □

Lemma 4.5. *At $\eta_L = 0$, the matrix of partial derivatives of $(P_k, Q_k)_{1 \leq k \leq K}$ with respect to the unknowns of level L is invertible.*

Proof. Group the unknowns as $U_k = (Y(k+2), \xi(k))$. By (4.3), P_k and Q_k involve only $U_{k-4}, \dots, U_k$, so the matrix is block lower triangular. In the diagonal block, $\partial P_k / \partial Y(k+2) = \kappa/(x_{k+2} - x_k) \neq 0$, $\partial P_k / \partial \xi(k) = 0$ and $\partial Q_k / \partial \xi(k) = \epsilon_k \psi'(V_k) V_k^2 (1/b_k^+ - 1/b_k^-) \neq 0$. □

4.4. **Induction over the levels.**

Proposition 4.6. *For every energy, every $\delta > 0$ and every $L \geq 1$ there is an admissible pair (v, η) with $0 < \eta_\ell < \delta$ for all ℓ, such that $\Phi_L(v, \eta) = 0$, the matrix of partial derivatives of Φ_L with respect to v is invertible, and:*

- *(P1) $Y_L(k)$ has the sign of $(-1)^k$ for $1 \leq k \leq K_L$;*
- *(P2) the nodes are pairwise distinct;*
- *(P3) the $2K_L - 1$ segments of step (5) for $\ell = L$ have axis times strictly between 0 and 3, increasing with their number;*
- *(P4) all $|s(N, \sigma)| < 2\delta$, and at every node the values $s(N, \sigma)$ of the two pieces differ from those of the four links.*

Proof. Induction on L. If $L \geq 2$, let (v_0, η_0) be such a pair for $L-1$ levels; if $L = 1$, it is empty. The columns of level L are then determined, with axis times $0 < x_1 < \dots < x_K < 3$ by (P3); if $L = 1$, then $x_k = k$. Let w^0 be the reference state. Write $v = (v', w)$ with w the unknowns of level L, and $t = \eta_L$.

The reference point $(v_0, w^0; \eta_0, 0)$ is admissible: for the levels below L by the induction hypothesis, with (P1) giving (c) for level $L-1$; for level L, (a) reads $b^- < 0 < b^+$ and (b) reads $0 < x_1 < \dots < x_K < 3$.

Solving. At $t = 0$ every column of level L has $s_0 = s_1 = V$, whatever w is. Thus the components of Φ_L of the levels below L do not involve w. For $L \geq 2$, they equal $D\Phi_{L-1}(v')$, where D leaves the earlier rows unchanged and multiplies the momentum and energy rows of level $L-1$ by η_{L-1} and η_{L-1}^2, respectively. These scales are fixed and positive, so D is invertible. Hence Φ_L vanishes at the reference point, and its matrix of partial derivatives with respect to (v', w) is block lower triangular with invertible diagonal blocks, by the induction hypothesis and Lemma 4.5. By the implicit function theorem there are smooth $v'(t)$, $w(t)$ for small $|t|$ with $v'(0) = v_0$, $w(0) = w^0$ and

$\Phi_L(v'(t), w(t); \eta_0, t) = 0$. For small $t > 0$ the pair is admissible and the matrix is still invertible.

Properties for small $t > 0$. As $t \to 0$, the nodes of the levels below L converge to those of (v_0, η_0), which are distinct and, by (c) and (P1), not on the axis; $Y_L(m) \to Y^0(m)$; and $C_L(k) \to (x_k, 0)$.

(P1) follows from Lemma 4.4.

(P2) The nodes of level L converge to distinct points of the axis.

(P3) The piece of column k named in step (5) has an end converging to $(x_k, 0)$, so its axis time converges to x_k. The map $(t, y) \mapsto (t, y/\eta_L)$ fixes axis times and sends the short link $[C(k), C(k+1)]$ to a segment that converges to the one from $(x_k, Y^0(k))$ to $(x_{k+1}, Y^0(k+1))$. By Lemma 4.4 these heights have opposite signs, so the limit of its axis time lies strictly between x_k and x_{k+1}.

(P4) At $t = 0$ the links of level L have slope 0 and the pieces of its column k have slope $V_k \neq 0$. Here V_k is the slope of an edge of level $L - 1$, so $|V_k| < 2\delta$ by (P4) for $L - 1$ levels; if $L = 1$, then $V_k = \delta$. The values $s(N, \sigma)$ at the nodes of the levels below L are those of (v_0, η_0). So (P4) holds at $t = 0$, and it is an open condition. □

4.5. **Conclusion of the proof.**

Proof of Theorem 1.4. For $L = 0$ take two parallel lines of slope δ, each cut at one point. Let $L \geq 1$ and take (v, η) as in Proposition 4.6. All $p(N)$ and $e(N)$ vanish; for level L because $p = \eta_L P_k$ and $e = \eta_L^2 Q_k$. By (P2), Lemma 4.3 gives a diagram with $3L + 2$ particles in which each of the $\sum_{\ell=1}^{L}(2^{\ell-1} + 1) = 2^L + L - 1$ nodes is a vertex with three incoming and three outgoing edges. By (P4) all its edges have slope less than 2δ in absolute value, and at each node the two pieces and the four links do not all have the same slope. The diagram has $2(3L+2)$ rays. The last assertion follows from Lemma 4.2. □

5. Applications

Each application consists of a lemma identifying the system as a collision system with a specific energy, and the deduction of the theorem stated in the introduction.

5.1. **Point billiards.**

Motions of particles in $\mathbb{R}^k$ and multiple collisions were defined in Section 1: a motion is a collision system on $\mathbb{R}$ with the Newtonian energy on $\mathbb{R}^k$, and a multiple collision is a collision point with $|S| \geq 3$.

Proof of Theorem B. Dividing all masses by the smallest one changes neither the motion nor its collision points, and makes the masses at least 1 with total mass M. The Newtonian energy has pinching $1 < \rho_M$, so the upper bound is Theorem 1.3. For the lower bound apply Theorem 1.4 to the Newtonian energy on $\mathbb{R}$, for which $h'(\mathbb{R}) = \mathbb{R}$; a collision system on $\mathbb{R}$

with unit masses and the Newtonian energy is a motion of particles of equal mass, and $\mathbb{R} \subset \mathbb{R}^k$. □

Lemma 5.1 (Two particles)**.** *Let $y_1, \dots, y_n$ be a motion of particles of equal mass in $\mathbb{R}$. Near a collision point (t, y) with $|S| = 2$, the union of the two graphs is the union of two straight lines through (t, y). If the motion has no multiple collision, the union of all graphs is the union of n straight lines, and there are at most $\binom{n}{2}$ collision points.*

Proof. Let $S = \{i, j\}$, and let a, b be the velocities of y_i, y_j before t and a', b' after t. Conservation of $y_i' + y_j'$ and of $y_i'^2 + y_j'^2$ gives $\{a', b'\} = \{a, b\}$, and $a \neq b$, since otherwise the two graphs would coincide near t. In either case the union of the two graphs near (t, y) is the union of the lines through (t, y) with slopes a and b. If there is no multiple collision, exchange the labels i and j after t at every collision point at which $a' = b$; this changes neither the union of the graphs nor the collision points, and the conservation laws do not depend on labels, so the result is a motion in which no velocity changes. Its graphs are n straight lines, and two lines meet at most once. □

5.2. **Slow relativistic particles.** Relativistic motions of particles in $\mathbb{R}^k$ were defined in Section 1; the momentum per unit mass of the particle y_i is $p_i = y_i'/\sqrt{1 - \|y_i'\|^2}$.

Lemma 5.2. *Let $P > 0$. If all p_i of a relativistic motion have norm less than P, then $t \mapsto y_i(c_P t)$, $1 \le i \le n$, is a collision system on $\mathbb{R}$ with the same masses and the relativistic energy with cap P. Its collision points correspond bijectively to the original ones under $(t, y) \mapsto (t/c_P, y)$.*

Proof. On a flight, $\operatorname{grad}(c_P h_{\mathrm{rel}})(p_i) = c_P p_i/\sqrt{1 + \|p_i\|^2} = c_P y_i'$, which is the velocity of $t \mapsto y_i(c_P t)$; so (a) holds with the momenta $p_i \in \Omega_P$. Conservation of $\sum m_i h_{\mathrm{rel}}(p_i)$ is conservation of the total energy, since the total mass of a cluster is conserved; so (b) holds. The change of time unit preserves the number of collision points and their participant sets. □

Proof of Theorem C. Choose $P_M > 0$ with $(1 + P_M^2)^{3/2} < \rho_M$, nonincreasing in M, which is possible since ρ_M is nonincreasing. By Lemma 5.2, the motion is a collision system with masses at least 1, total mass M, and the relativistic energy with cap P_M, whose pinching is less than ρ_M by (2.3); Theorem 1.3 applies. For the second assertion apply Theorem 1.4 to the relativistic energy with cap 1 on $\mathbb{R}$ and $\delta = \frac{1}{4}$, and undo the change of time unit. □

5.3. **Geodesic networks near a multiple line.** Here $Z = \mathbb{R} \times X$; a point of Z is written $z = (t, x)$, with t its *time* and x its *position*. If a segment or ray in Z joins (t_0, x_0) to (t_1, x_1) with $t_0 < t_1$, its *slope* is $v = (x_1 - x_0)/(t_1 - t_0) \in X$ and its *rightward unit tangent* is $(1, v)/\sqrt{1 + \|v\|^2}$.

Definition 5.3. Let $n \geq 1$, $a > 0$ and $A < T$. A *slab network of flow n and slope cap a* on $[A,T] \times X$ is a finite directed graph N embedded in $[A,T] \times X$ such that:

(i) every edge is a nonzero segment, carries a positive integer *multiplicity*, is directed so that time increases strictly along it, and has slope of norm less than a;
(ii) the open edges (edges with their endpoints removed) are pairwise disjoint;
(iii) at every vertex with $A < t < T$, the total multiplicity of the incoming edges equals that of the outgoing edges, and the multiplicity-weighted sum of the rightward unit tangents of the incoming edges equals that of the outgoing edges;
(iv) the vertices on the slices $t = A$ and $t = T$ (the *boundary vertices*) have exactly one edge each, every edge lies on a directed path from a boundary vertex on $t = A$ to one on $t = T$, and the edges at the boundary vertices on $t = A$ have total multiplicity n.

A vertex with $A < t < T$ is *regular* or *singular* as in Definition 1.5, according to whether or not N coincides near it with a single straight line.

By (ii), two edges meet only at vertices, and (iii) is the balance condition of Definition 1.5 at the vertices with $A < t < T$. The total multiplicity crossing a slice $\{t\} \times X$ at a non-vertex time does not change at vertices, so it is n for every such t, and the edges at the boundary vertices on $t = T$ also have total multiplicity n. At a vertex with one incoming and one outgoing edge, (iii) forces equal multiplicities and equal slopes, so such a vertex is regular; hence a vertex is regular exactly when it has one incoming and one outgoing edge, and a singular vertex has at least two edges on one side.

Proposition 5.4 (Transfer)**.** *Let $d \geq 2$, $n \geq 1$, $a > 0$ and $\Lambda \geq 0$, and let $X = \mathbb{R}^{d-1}$. If every slab network of flow n and slope cap a has at most Λ singular vertices, then $C(d,n) \leq \Lambda$.*

The proof, which only unpacks the conclusions of [4, Theorem 11], is in Appendix A.

Lemma 5.5 (Unit paths)**.** *Let N be a slab network of flow n on $[A,T] \times X$. Then there are directed paths $\gamma_1, \dots, \gamma_n$ in N, each from a boundary vertex on $t = A$ to one on $t = T$, such that every edge e of multiplicity m_e lies on exactly m_e of them.*

Proof. Call the multiplicity m_e the remaining multiplicity of e. Start on a boundary vertex on $t = A$ whose edge has positive remaining multiplicity and follow edges forwards; at each vertex, equality of the incoming and outgoing total multiplicities supplies an outgoing edge with positive remaining multiplicity. Since time increases strictly and N is finite, the path reaches the slice $t = T$. Decrease the remaining multiplicity by one along the path. This preserves equality of the incoming and outgoing total multiplicities,

which is all the path decomposition requires. Repeat until the edges on $t = A$ are exhausted, which happens after n paths. If an edge then had positive remaining multiplicity, following incoming edges backwards from it would reach a boundary vertex on $t = A$ with positive remaining multiplicity, which is impossible. □

We call the paths $\gamma_1, \dots, \gamma_n$ the *strands* of N and the index i the *label* of γ_i. At every non-vertex time each strand is at one position in X, and the strands on a common edge are at the same position.

Lemma 5.6 (Networks are collision systems)**.** *Let N be a slab network of flow n and slope cap a on $[A,T] \times X$, with strands $\gamma_1, \dots, \gamma_n$, and let $y_i(t) \in X$ be the position of γ_i at time t. Then $y_1, \dots, y_n$ is a collision system on $[A,T]$ with unit masses and the Euclidean energy with cap a, and its collision points are exactly the singular vertices of N.*

Proof. On an edge of slope v, $\|v\| < a$, each strand has momentum $p(v) \in \Omega_a$ and velocity $v = \operatorname{grad} h(p(v))$, which is (a). For (b), let S be the set of labels at a point (t, y) with $A < t < T$. If (t, y) is not a vertex, the strands in S lie on one edge near t and nothing changes. If it is a vertex, each incoming edge of multiplicity m carries m strands of S, and likewise for outgoing edges, so the sums in (b) are the multiplicity-weighted sums of p and of $h(p)$ over the incoming and the outgoing edges. By (2.1), condition (iii) of Definition 5.3 says that the weighted sums of $(1 - h(p), p)$ agree, and since the total multiplicities agree, so do the weighted sums of p and of $h(p)$. Finally, a point with at least two strands is a vertex or lies on one edge; a point on one edge is not a collision point; a regular vertex is not a collision point, since its strands continue together on one line; and a singular vertex has at least two edges on one side, hence at least two strands, which do not all continue on one line. □

Corollary 5.7. *For every $d \geq 2$ and $n \geq 1$, $C(d,n) \leq U_n$.*

Proof. Choose $\sigma_n > 0$ with $(1 + \sigma_n^2)^{3/2} < \rho_n$, where ρ_n is the constant of Theorem 1.3. By Lemma 5.6, the strands of a slab network of flow n and slope cap σ_n form a collision system of n particles of unit mass, hence of total mass n, with the Euclidean energy with cap σ_n, whose pinching is $(1 + \sigma_n^2)^{3/2} < \rho_n$ by (2.2), and whose collision points are the singular vertices. By Theorem 1.3 there are at most U_n of them. Proposition 5.4 gives $C(d,n) \leq U_n$. □

Balanced nets. The nets of Section 4 lie in a plane with coordinates (t, y). Throughout, we take this plane to be a plane in $\mathbb{R}^d$, $d \geq 2$; the construction is planar, but a balanced net in a plane in $\mathbb{R}^d$ is a network in $\mathbb{R}^d$, so the lower bound it gives holds in every dimension.

Definition 5.8. Let a net be given (Definition 4.1). The *degree* of a vertex p is the number of edges that end at p. If at every vertex p the unit vectors

pointing from p along the edges that end at p sum to zero, the net is called *balanced*.

If each point of a balanced net is given the number of edges through it as its weight, the net is an integral stationary geodesic network in the sense of Definition 1.5: at an endpoint of an edge the balance condition is the vertex condition, and at a point interior to several edges each edge contributes two opposite unit vectors.

Lemma 5.9. *A diagram with the Euclidean energy is a balanced net.*

Proof. An outgoing edge of slope s has unit vector $(1,s)/\sqrt{1+s^2} = (1-\psi(s),\varphi(s))$, and an incoming one the opposite vector. As the two kinds are equal in number, the vertex condition says that these vectors sum to zero. □

Corollary 5.10. *Let $L \geq 0$ be an integer and $\delta > 0$. Then there is a balanced net which is the union of the graphs of $3L+2$ continuous piecewise linear functions on $\mathbb{R}$, has $6L+4$ rays, has at least $2^L + L - 1$ vertices of degree six, and all of whose edges have slope less than 2δ in absolute value.*

Proof. Apply Theorem 1.4 to the Euclidean energy on $\mathbb{R}$, for which $h'(\mathbb{B}) = \mathbb{R}$, and Lemma 5.9. Each of the $2^L + L - 1$ vertices in Theorem 1.4 has degree six. □

Corollary 5.11. *For every $d \geq 2$ and every integer $L \geq 0$, $C(d, 3L+2) \geq 2^L + L - 1$.*

Proof. For $L = 0$ the asserted lower bound is zero and needs no construction. Let $L \geq 1$, put $k = 3L+2$, and fix a plane in $\mathbb{R}^d$ with coordinates (t, y). For $i \geq 1$ let X_i be the union of the edges of the net of Corollary 5.10 for $\delta = 1/i$, as constructed in the proof of Theorem 1.4, placed in this plane, the weight of a point being the number of edges through it. As noted after Definition 5.8, X_i is an integral stationary geodesic network in $\mathbb{R}^d$. By the admissibility conditions (a) and (b) of Section 4.2, the time and height of each node lie strictly between the corresponding coordinates of its column endpoints, so by induction over the levels every node lies in $(0,3) \times (-1/i, 1/i)$; and each of the k graphs passes through a node, since the columns of level 1 are bent at nodes and the links of each level form three polygonal lines through its nodes. As the slopes are less than $2/i$, on every compact set the graphs converge uniformly to the axis and their length measures converge to length measure on the axis; hence the weighted length measures of X_i converge weakly on compact sets to k times length measure on the axis. All nodes lie in the fixed compact set $[0,3] \times [-1,1]$, and by (P4) of Proposition 4.6 each node is a singular point of X_i, since two edges of different slopes end there. □

Lemma 5.12 (Lines in general position)**.** *For every $d \geq 2$ and $n \geq 2$, $C(d,n) \geq \binom{n}{2}$.*

Proof. In a plane in $\mathbb{R}^d$ with coordinates (t, y), let X_i be the union of the lines $y = \frac{1}{i}\, j(t + j)$, $1 \le j \le n$, each of weight one. The lines j and j' meet at the point $(-(j + j'), -jj'/i)$; these $\binom{n}{2}$ points are pairwise distinct, so no three lines are concurrent, and each crossing is a singular point. Crossings of straight lines are balanced, so X_i is an integral stationary geodesic network, and its weighted length measures converge to n times length measure on the t-axis. All crossings lie in the fixed compact set $[-2n, -2] \times [-n^2, 0]$. □

Lemma 5.13 (Monotonicity)**.** *For $2 \le d' \le d$ and $n \ge 1$, $C(d, n) \ge C(d', n)$ and $C(d, n + 1) \ge C(d, n)$.*

Proof. Embed $\mathbb{R}^{d'}$ isometrically as a coordinate subspace of $\mathbb{R}^d$. Pieces, weights, local finiteness, balance, weak convergence to a multiple line, and the property of being locally one line are all preserved, so a sequence witnessing a lower bound for $C(d', n)$ witnesses the same lower bound for $C(d, n)$.

For the second inequality, let X_i converge weakly on compact sets to n times length measure on a line ℓ, let e be a unit vector orthogonal to ℓ, and let X_i' be the union of X_i and the line $\ell + e/i$ with weight one. Then X_i' is an integral stationary geodesic network whose weighted length measures converge to $n + 1$ times length measure on ℓ, and every singular point of X_i is a singular point of X_i'. Hence every lower bound witnessed for $C(d, n)$ is witnessed for $C(d, n + 1)$. □

Proof of Theorem A. (a) By Corollary 5.7, $C(d, 1) \le U_1 = 0$ and $C(d, 2) \le U_2 = 1$; the first is an equality, and Lemma 5.12 gives $C(d, 2) \ge 1$.

(b) The upper bound is Corollary 5.7. The lower bound $\binom{n}{2}$ is Lemma 5.12. For the other lower bound let $L = \lfloor (n - 2)/3 \rfloor$, so that $3L + 2 \le n$; by Lemma 5.13 and Corollary 5.11,

$$C(d, n) \ge C(d, 3L + 2) \ge 2^L + L - 1.$$

(c) Taking logarithms in (b) and dividing by n: the lower bound $2^{\lfloor (n-2)/3 \rfloor}$ gives $\liminf \log C(d, n)/n \ge (\log 2)/3$, and the upper bound $\frac{n}{2} 8^{n-2}$ gives $\limsup \log C(d, n)/n \le \log 8$. □

Growth constants. For $d \ge 2$ put

$$\beta_-(d) = \liminf_{n\to\infty} C(d, n)^{1/n}, \qquad \beta_+(d) = \limsup_{n\to\infty} C(d, n)^{1/n}.$$

Thus $\beta_+(d)$ is the least B such that $C(d, n) \le (B + \epsilon)^n$ for all large n, for every $\epsilon > 0$, and $\beta_-(d)$ is the largest B such that $C(d, n) \ge (B - \epsilon)^n$ for all large n, for every $0 < \epsilon < B$.

Corollary 5.14 (Growth constants)**.** *For every $d \ge 2$,*

$$2^{1/3} \le \beta_-(d) \le \beta_+(d) \le 8,$$

and $\beta_-(d)$ and $\beta_+(d)$ are nondecreasing in d. Consequently the limits $\beta_\pm(\infty) = \lim_{d\to\infty} \beta_\pm(d)$ exist and satisfy $2^{1/3} \le \beta_-(\infty) \le \beta_+(\infty) \le 8$.

Proof. The bounds are Theorem A(c). If $d' \le d$, then $C(d,n) \ge C(d',n)$ for every n by Lemma 5.13, so $\beta_\pm(d) \ge \beta_\pm(d')$. A nondecreasing function of d that is bounded above converges. □

Appendix A. Proof of Proposition 5.4

Proof. Let $X_1, X_2, \dots$ be integral stationary geodesic networks in $\mathbb{R}^d$ whose weighted length measures converge weakly on compact sets to n times length measure on a line ℓ, and let $K \subset \mathbb{R}^d$ be compact. After a rigid motion, ℓ is the first coordinate axis, so that $\mathbb{R}^d = \mathbb{R} \times X$.

Choose $\theta \in (0, \pi/2)$ with $\tan\theta < a$. Choose an open bounded interval I, nonempty open intervals J_-, J_+ with closures in I, and radii $0 < r < R_0 < R$, such that every point of K has time strictly between J_- and J_+ and position of norm less than R_0. Write B_ρ for the open ball of radius ρ about 0 in X. By the structure theorem of Bosch [4, Theorem 11] applied to I, r, R, θ, there is i_0 such that for every $i \ge i_0$:

(1) $X_i \cap (I \times B_R) \subset I \times B_r$;
(2) at every regular point of $X_i \cap (I \times B_R)$ the tangent line makes an angle of at most θ with the time axis;
(3) at every singular point of $X_i \cap (I \times B_R)$, the total weight of the pieces entering from the left equals the total weight of the pieces leaving to the right;
(4) for every $t \in I$, the slice $X_i \cap (\{t\} \times B_R)$ consists of at most n points whose weights add up to n.

Fix $i \ge i_0$. Since X_i is locally finite, it has finitely many singular points in the compact set $\overline{J_- \cup J_+} \times \overline{B_R}$, so we may choose $A \in J_-$ and $T \in J_+$ that are not times of singular points. Let G be the part of X_i in $[A,T] \times \overline{B_{R_0}}$, with the vertices of X_i there, the points of X_i on the two slices $t = A$, $t = T$, and the points where distinct pieces cross as vertices, and with collinear overlapping pieces merged.

No point of X_i lies on the lateral boundary $[A,T] \times \partial B_{R_0}$: such a point would lie in $I \times B_R$ but not in $I \times B_r$, contradicting (1). Hence G is a finite graph whose boundary vertices lie on the two slices. By the choice of A and T, each boundary vertex is a regular point of X_i, so it has exactly one edge, and by (4) the edges at the boundary vertices on each slice have total multiplicity n. By (2), every edge may be directed with strictly increasing time, and its slope has norm at most $\tan\theta < a$. By (3), the balance condition of X_i, and the merging convention, condition (iii) of Definition 5.3 holds at every vertex with $A < t < T$. Finally, every edge lies on a boundary-to-boundary path: following incoming edges backwards from any edge cannot stop at an interior vertex, since by (iii) such a vertex with an outgoing edge has an incoming one, and cannot cycle, since time increases strictly; so it reaches the slice $t = A$, and the forward argument reaches $t = T$. Thus G is a slab network of flow n and slope cap a.

By the hypothesis, G has at most Λ singular vertices. Every singular point of X_i in K lies strictly inside the slab and the ball, hence is a singular vertex of G. So X_i has at most Λ singular points in K for every $i \geq i_0$. Since the sequence and K were arbitrary, $C(d,n) \leq \Lambda$. □

Appendix B. Proof of Proposition 3.1

Proof. Put $\delta = \rho - 1$ and $\epsilon = 2\delta\sqrt{\rho} + \rho^{3/2} - 1$, so that $\eta(\rho) = \pi\sqrt{\epsilon/2}$. By the definition of the pinching,

$$I \leq D^2 h \leq \rho I \qquad \text{on } \Omega. \tag{B.1}$$

All momenta lie in Ω, and so do their mass-weighted means, since Ω is convex. Norms and inner products on X^n are the mass-weighted ones. The clusters at an event time are the sets of coincident particles whose momenta change; by (b) each cluster preserves its momentum sum $\sum m_i p_i$ and its energy sum $\sum m_i h(p_i)$.

Conserved quantities. Let $\mu = \frac{1}{M}\sum_i m_i p_i$ and $K = \sum_i m_i h(p_i) - M h(\mu)$. Both are unchanged at events, since each cluster preserves its momentum sum and its energy sum. Taylor's formula from μ and (B.1) give, for $x \in \Omega$,

$$\tfrac{1}{2}\|x-\mu\|^2 \leq h(x) - h(\mu) - \operatorname{grad} h(\mu)\cdot(x-\mu) \leq \tfrac{\rho}{2}\|x-\mu\|^2.$$

Multiplying by m_i and summing over $x = p_i$, the linear terms cancel, and with $z_i = p_i - \mu$, $z = (z_1, \dots, z_n)$ and $R = \sqrt{2K}$,

$$\frac{R}{\sqrt{\rho}} \leq \|z\| \leq R. \tag{B.2}$$

If $K = 0$, then $z = 0$, all momenta equal μ, q is constant, and $L = 0$. Assume $K > 0$.

The velocity of q. On a flight, $q' = w$ with $w_i = \operatorname{grad} h(p_i) - \bar{v}$, where $\bar{v} = \frac{1}{M}\sum_j m_j \operatorname{grad} h(p_j)$. Since $\sum_i m_i z_i = 0$ and, by (B.1), $(\operatorname{grad} h(a) - \operatorname{grad} h(b))\cdot(a-b) \geq \|a-b\|^2$,

$$\begin{aligned}\langle z, w\rangle &= \sum_i m_i\, z_i \cdot \operatorname{grad} h(p_i) \\ &= \sum_i m_i\,(p_i-\mu)\cdot\big(\operatorname{grad} h(p_i) - \operatorname{grad} h(\mu)\big) \geq \|z\|^2;\end{aligned}$$

and since $\bar{v}$ is the mass-weighted mean of the $\operatorname{grad} h(p_i)$, while $\|\operatorname{grad} h(a) - \operatorname{grad} h(b)\| \leq \rho\|a-b\|$,

$$\begin{aligned}\|w\|^2 &= \sum_i m_i \|\operatorname{grad} h(p_i) - \bar{v}\|^2 \\ &\leq \sum_i m_i \|\operatorname{grad} h(p_i) - \operatorname{grad} h(\mu)\|^2 \leq \rho^2\|z\|^2.\end{aligned}$$

Hence $\|z\| \leq \|w\| \leq \rho\|z\|$ and, by (B.2),

$$\frac{R}{\sqrt{\rho}} \leq \|w\| \leq \rho R. \tag{B.3}$$

One event. Let Δ denote the jump of a quantity at an event. Since q is constant on each cluster and the momentum jump $\sum m_i \Delta p_i$ vanishes over each cluster, $\langle q, \Delta p\rangle = 0$. Write $r(p) = \operatorname{grad} h(p) - p$; by (B.1), r is δ-Lipschitz on Ω. As u is centered, the jump of $\bar v$ contributes nothing to $\langle u, \Delta w\rangle$, and as the mean momentum is unchanged, $\Delta p = \Delta z$; so

$$|\langle u, \Delta w\rangle| = |\langle u, \Delta \operatorname{grad} h(p)\rangle| = |\langle u, \Delta r(p)\rangle| \le \|\Delta r(p)\| \le \delta \|\Delta p\| \le 2\delta R,$$

using $\|\Delta p\| = \|\Delta z\| \le 2R$ at the end. Let w^-, w^+ be the velocities before and after the event, $a = \|w^-\|$, $b = \|w^+\|$, and $\alpha^-, \alpha^+ \in [0,\pi]$ the angles between u and w^-, w^+, respectively. By (B.3),

$$|\cos\alpha^+ - \cos\alpha^-| \le \frac{|\langle u, \Delta w\rangle|}{b} + |\langle u, w^-\rangle| \frac{|a-b|}{ab} \le 2\delta\sqrt{\rho} + \rho^{3/2} - 1 = \epsilon.$$

For $\alpha, \gamma \in [0,\pi]$ with $d = |\alpha - \gamma|/2$ and $c = (\alpha+\gamma)/2 \in [d, \pi - d]$ one has $|\cos\alpha - \cos\gamma| = 2\sin c \sin d \ge 2\sin^2 d \ge 2(|\alpha-\gamma|/\pi)^2$. Therefore

$$|\alpha^+ - \alpha^-| \le \pi\sqrt{\epsilon/2} = \eta(\rho). \tag{B.4}$$

One flight. Let $\alpha(t) \in [0,\pi]$ be the angle between $q(t)$ and the constant w. Where $0 < \alpha < \pi$,

$$\frac{d}{dt}\cos\alpha = \frac{\|w\|}{\|q\|}\sin^2\alpha, \qquad \|u'\| = \frac{\|w\|}{\|q\|}\sin\alpha,$$

so $\|u'\| = -\alpha'$; this also holds almost everywhere on the set where $\alpha \in \{0,\pi\}$, since there $u' = 0$ and $\alpha' = 0$. The length of u over the flight is therefore the decrease of α over the flight.

Summation. Summing the decreases over all flights telescopes to the initial value of α, at most π, minus the final one, plus the jumps at events, each at most $\eta(\rho)$ by (B.4). Hence $L \le \pi + \nu\eta(\rho)$. □

Appendix C. Proof of Theorem 3.2

Proof. Put $K = 1 + \sqrt{2} + d = 1 + \sqrt{2} + 2\sqrt{2}\,c$, so that (3.2) reads

$$1 + K^2 < \alpha^2. \tag{C.1}$$

Four identities. For nonempty $A \subset \mathcal{N}$, $v, w \in E^n$ and $z \in E$:

$$\sum_{i\in A} m_i \|v_i - z\|^2 = r_A(v)^2 + m_A\,\|\mu_A(v) - z\|^2, \tag{C.2}$$

$$\|v_i - v_j\| \le \sqrt{2}\, r_A(v) \qquad (i,j \in A), \tag{C.3}$$

$$|r_A(v) - r_A(w)| \le \|v - w\|, \tag{C.4}$$

$$\|(v_i - v_j) - (w_i - w_j)\| \le \sqrt{2}\,\|v - w\|. \tag{C.5}$$

The first is the bias–variance decomposition; the second follows from it with $z = \mu_A(v)$, since $m_i, m_j \ge 1$ give $\|v_i - \mu_A\|^2 + \|v_j - \mu_A\|^2 \le r_A(v)^2$, and in the same way $\|v_i - \mu_A(v)\| \le r_A(v)$ for $i \in A$; the third holds because r_A is

the norm of an orthogonal projection of v for the weighted inner product; the fourth is Cauchy–Schwarz together with $m_i, m_j \geq 1$.

The growth estimate. Let $A \subset \mathcal{N}$ be nonempty, let $v \in E^n$ and $\epsilon > 0$ with $r_A(v) \leq \epsilon$, and let $T \subset \mathcal{N} \setminus A$ be nonempty with $\|v_j - \mu_A(v)\| \leq K\epsilon$ for every $j \in T$. Then

$$r_{A\cup T}(v) \leq \alpha^{m_T}\epsilon. \tag{C.6}$$

Indeed, (C.2) for the set $A \cup T$ with $z = \mu_A(v)$, and then for the set A with the same z, give

$$\begin{aligned} r_{A\cup T}(v)^2 &\leq \sum_{i\in A\cup T} m_i\|v_i - \mu_A(v)\|^2 \\ &= r_A(v)^2 + \sum_{j\in T} m_j\|v_j - \mu_A(v)\|^2 \leq (1 + m_T K^2)\,\epsilon^2, \end{aligned}$$

and since $m_T \geq 1$, Bernoulli's inequality and (C.1) give $1 + m_T K^2 \leq (1 + K^2)^{m_T} < \alpha^{2m_T}$. Applied with $\epsilon = \epsilon(m_A)$, (C.6) says that $A \cup T$ is thin, since $\alpha^{m_T}\epsilon(m_A) = \epsilon(m_{A\cup T})$.

(a). Singletons are thin. The set $\mathcal{N}$ is not, because $r_{\mathcal{N}}(x) = \|x\| = 1 > \lambda = \epsilon(M)$. Let A, B be thin with $A \cap B \neq \varnothing$ and $m_B \geq m_A$. If $A \subset B$ there is nothing to prove. Otherwise choose $i_0 \in A \cap B$; for $i \in A \setminus B$, (C.3) gives

$$\|x_i - \mu_B(x)\| \leq \|x_i - x_{i_0}\| + \|x_{i_0} - \mu_B(x)\| \leq \sqrt{2}\,\epsilon(m_A) + \epsilon(m_B) \leq K\epsilon(m_B),$$

so $A \cup B$ is thin by (C.6). So the union of two overlapping thin sets is thin. Hence two maximal thin sets that overlap coincide, and every index lies in a maximal thin set: the maximal thin sets partition $\mathcal{N}$, and each is proper.

(b). Let $S \neq T$ be maximal thin, and relabel so that $m_S \leq m_T$. Suppose, for a contradiction, that some $i_0 \in S$, $j_0 \in T$ satisfy $\|x_{i_0} - x_{j_0}\| \leq d\epsilon(m_T)$. For every $i \in S$,

$$\begin{aligned} \|x_i - \mu_T(x)\| &\leq \|x_i - x_{i_0}\| + \|x_{i_0} - x_{j_0}\| + \|x_{j_0} - \mu_T(x)\| \\ &\leq (\sqrt{2} + d + 1)\,\epsilon(m_T) = K\epsilon(m_T), \end{aligned}$$

so $S \cup T$ is thin by (C.6), contradicting the maximality of T. The last assertion of (b) follows: indices with a common position violate the strict inequality, so they cannot lie in distinct maximal thin sets.

(c). Write $S = S_p$, $T = S_q$, $s = s_p$ and $t = s_q \leq s$. Since the cores overlap, $|p - q| \leq c(\epsilon(s) + \epsilon(t)) \leq 2c\,\epsilon(s)$, and by (C.4),

$$r_T(u(p)) \leq r_T(u(q)) + |p - q| \leq \epsilon(t) + 2c\,\epsilon(s) \leq (1 + 2c)\,\epsilon(s). \tag{C.7}$$

Suppose, for a contradiction, that $T \not\subset S$. Choose $i_0 \in S \cap T$. For $j \in T \setminus S$, (C.3) and (C.7) give

$$\begin{aligned} \|u_j(p) - \mu_S(u(p))\| &\leq \|u_j(p) - u_{i_0}(p)\| + \|u_{i_0}(p) - \mu_S(u(p))\| \\ &\leq \sqrt{2}\,(1 + 2c)\,\epsilon(s) + \epsilon(s) = K\epsilon(s), \end{aligned}$$

so $S \cup T$ is thin at $u(p)$ by (C.6). It strictly contains S, contradicting the maximality of S. Hence $T \subset S$.

(d). By (b), $\|u_i(p) - u_j(p)\| > d\epsilon(s_p)$. By (C.5), the difference $u_i - u_j$ changes by at most $\sqrt{2}|\tau - p| \leq 2\sqrt{2}c\epsilon(s_p) = d\epsilon(s_p)$ between p and τ. Hence $u_i(\tau) \neq u_j(\tau)$. □

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References

[1] W. K. Allard and F. J. Almgren, Jr., *The structure of stationary one dimensional varifolds with positive density*, Invent. Math. **34** (1976), 83–97.
[2] D. Burago, S. Ferleger, and A. Kononenko, *Uniform estimates on the number of collisions in semi-dispersing billiards*, Ann. of Math. (2) **147** (1998), 695–708.
[3] D. Burago and S. Ivanov, *Examples of exponentially many collisions in a hard ball system*, Ergodic Theory Dynam. Systems **41** (2021), no. 9, 2754–2769; arXiv:1809.02800.
[4] H. Bosch, *A partial ϵ-regularity theorem for integral stationary geodesic networks in $\mathbb{R}^n$*, arXiv:2607.23872v1 (2026).
[5] K. Burdzy, *An improved upper bound on the number of billiard ball collisions*, Comm. Math. Phys. **391** (2022), no. 1, 107–117; arXiv:2107.10362.
[6] O. Chodosh and C. Mantoulidis, *The p-widths of a surface*, Publ. Math. Inst. Hautes Études Sci. **137** (2023), 245–342.
[7] J. Liu, G. Gao, Z. Sun, B. Wu, S. Liu, J. Jiang, H. Ju, L. Chen, R. Cheng, X. Zhang, and B. Dong, *Danus: Orchestrating mathematical reasoning agents with fact-graph memory*, arXiv:2607.06447 (2026).
[8] I. Frolov, D. Gorodkov, D. Korshunov, and A. Nabutovsky, *Geodesic nets on the Euclidean plane and closed geodesic nets on Riemannian surfaces*, arXiv:2606.14080v2 (2026).
[9] J. Féjoz, A. Knauf, and R. Montgomery, *Lagrangian relations and linear point billiards*, Nonlinearity **30** (2017), no. 4, 1326–1355.
[10] S. Gasiorek, *Counting collisions in an N-billiard system using angles between collision subspaces*, SIGMA **16** (2020), 119, 13 pp., doi:10.3842/SIGMA.2020.119.
[11] G. A. Gal'perin, *Systems of locally interacting and repelling particles that are moving in space* (Russian), Trudy Moskov. Mat. Obshch. **43** (1981), 142–196.
[12] G. Galperin, *Playing pool with π (the number π from a billiard point of view)*, Regul. Chaotic Dyn. **8** (2003), 375–394.
[13] A. Nabutovsky and F. Parsch, *Geodesic nets: some examples and open problems*, arXiv:1904.00483 (2019).

[14] F. Parsch and H. Zhang, *Towards constructing geodesic nets with four boundary vertices and an increasing number of balanced vertices*, arXiv:2511.07810 (2025).
[15] H. Qiu, *Example of exponentially many collisions of identical hard disks in the plane*, arXiv:2610.14282 (2026).
[16] Ya. G. Sinai, *Billiard trajectories in a polyhedral angle* (Russian), Uspekhi Mat. Nauk **33** (1978), no. 1(199), 229–230.
[17] S. Tabachnikov, *Geometry and billiards*, Student Mathematical Library **30**, Amer. Math. Soc., Providence, RI, 2005.
[18] L. N. Vaserstein, *On systems of particles with finite-range and/or repulsive interactions*, Comm. Math. Phys. **69** (1979), 31–56.

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