

Stability and Convergence of a Regularized Immersed Boundary Method under Dynamic Cartesian Refinement. Part II: General Prescribed Motion

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Abstract

We study a regularized immersed boundary discretization of unsteady Stokes flow around a rigid body with prescribed translation and rotation. The interface may change its position and orientation, while remaining uniformly separated from the outer wall. The fluid spaces are conforming, exactly divergence free, and constructed on fixed physical Cartesian grids with a retained coarse space and arbitrarily changing local enrichments. The method uses regularized surface-moment constraints, backward Euler time stepping, and current-space orthogonal projection. We construct a time-dependent coarse lifting that satisfies every discrete rigid-velocity constraint exactly. A moving collar approximation has a small spatial error in H^1 and a small Eulerian time-derivative error in L^2 , although its time derivative need not be accurate in H^1 . This distinction permits a quantitative convergence proof without a restriction coupling the time step to the number of mesh changes. A shifted-interface trace estimate accounts for the motion of the reaction surface during each time step and for physical slip between time nodes. Under stated piecewise regularity and an L^2 interface-reaction assumption, the energy error is bounded by the square roots of the retained coarse size, regularization width, surface-panel size, and maximum time step, together with a specified load-consistency error. The analysis assumes exact integration and does not prove adaptive optimality or cover fluid-driven motion. Manufactured moving-ellipse computations and a Lean algebraic companion test specified implementation and proof components.

Keywords: immersed boundary method; prescribed rigid motion; dynamic Cartesian refinement; moving interface; Stokes equations; regularization; convergence.

1 Introduction and contribution

Regularized immersed boundary (IB) methods couple a fluid velocity on a background grid to an independently represented interface [1]. Part I of this series [2] analyzed a conforming realization under dynamic Cartesian refinement for fixed geometric interfaces, including a

circular disk with prescribed angular velocity. This paper allows the rigid interface itself to translate and rotate. In particular, rotation of a noncircular body changes the constrained velocity space even if the Cartesian mesh is left unchanged.

The required extension is not obtained by flattening a neighborhood of the entire swept interface. Such a neighborhood need not shrink to the instantaneous boundary and may exclude legitimate fluid velocities. Instead, our comparison function is flattened near the current interface and changes continuously in time. Its Eulerian time derivative generally fails to approximate the exact derivative in H^1 . We prove that its error does tend to zero in the bulk L^2 norm, which is sufficient for the energy argument. Orthogonality against the current test space then removes an explicit sum of mesh-change defects, even though the previous comparison function is not in the current constraint space.

There is substantial prior convergence theory for IB methods. Mori established velocity convergence for periodic Stokes flow and a small-amplitude dynamic problem [3]; Liu and Mori analyzed discrete-delta accuracy and stationary Stokes velocity and pressure convergence [4, 5]. Li, Pan, and Ruiz-Álvarez treated stationary Stokes discretizations with prescribed outer boundary velocity [6]. The moving-particle analysis of Milewski and Peskin [7] concerns a different, codimension-two elastic coupling with a fixed physical mollifier width. These results do not constitute a convergence proof for the present moving codimension-one velocity constraints with shrinking regularization and changing Cartesian enrichments.

Prescribed rigid motion in fictitious-domain multiplier formulations is established [8], as are constrained rigid-body IB operators [9]. Adaptive IB implementations include the work of Roma, Peskin, and Berger [10] and the dynamically adaptive DLM-IB framework of Zeng, Bhalla, and Shen [11]. Regularization estimates and adaptive regularized elliptic methods provide neighboring analytical tools [12, 13]; optimal adaptive fictitious-domain convergence is also known in an elliptic setting [14]. The present theorem is for a specified conforming method, not a proof for these fluid solvers or an adaptive-optimality result.

Moving-domain Stokes stability and convergence also predate this work. Eulerian unfitted methods extend previous-time values to newly occupied fluid regions [15]. Burman, Frei, and Massing analyze stabilized Eulerian time stepping for moving-domain Stokes flow [16]; von Wahl, Richter, and Lehrenfeld study a related unfitted mixed method including geometric approximation [17]. Evolving-space theory identifies the transport and compatibility questions of time-dependent function spaces [18, 19]. ALE Stokes error analysis includes optimal convergence estimates [20], and recent Piola-based space-time DG work treats divergence-free moving-domain spaces without spatial/time partition restrictions [21]. Our method stays in physical Eulerian coordinates and imposes regularized surface moments on global background velocities. The claimed contribution is the concrete moving-collar recovery and its use with arbitrarily changing retained-coarse Cartesian spaces, not a first convergence theorem for moving domains. Likewise, stable mesh transfer and projection-change errors are established topics [22–24].

Computational IB studies motivate the interface and motion distinctions. He et al. [25] introduced pressure-boundary treatment to reduce spurious penetration through thin structures in two-phase flow. The prescribed heave and pitch studied by Zheng et al. [26] genuinely change the immersed plate’s geometry, unlike fixed-disk rotation. Their free surface and nonlinear flow are beyond the present Stokes model. Fixed-geometry IB applications include backward-facing-step LES [27] and airfoil flow-transition calculations [28]; flexible-canopy simulations [29] illustrate the additional structural dynamics excluded from this rigid-body series. These method and application connections do not serve as mathematical premises for the proofs.

The analysis makes four structural choices explicit. Velocities are curls of conforming

Cartesian bicubic Hermite streamfunctions. The full coarse space is retained while local enrichments may be added or removed at every step. Interface constraints are exact surface moments of an exact convolution. The continuous target is a sufficiently regular Eulerian Stokes solution with a square-integrable interface reaction and a prescribed interior extension. Under these assumptions we prove feasibility, unconditional driven stability, moving-interface slip control, and an energy-error estimate independent of the number of refinement events. The square-root temporal term accounts for interface displacement and piecewise smooth fields; it is not an assertion that backward Euler cannot converge faster on smoother problems.

2 Prescribed geometry and the Eulerian problem

Let $\Omega = (0, 1)^2$ and let $B_* \subset \mathbb{R}^2$ be a bounded simply connected reference body with connected C^6 boundary. Write

$$B(t) = c(t) + Q(\theta(t))B_*, \quad \Gamma(t) = \partial B(t), \quad F(t) = \Omega \setminus \overline{B(t)}, \quad b(t, x) = \dot{c}(t) + \omega(t)J(x - c(t)), \quad (1)$$

where $Q(\theta)$ is the planar rotation matrix, $J(x_1, x_2) = (-x_2, x_1)$, and $\omega = \dot{\theta}$. Assume $c, \theta \in C^2([0, T])$ and $\text{dist}(\overline{B(t)}, \partial\Omega) \geq d_* > 0$ for every t . All motion and geometry constants below may depend on these fixed data. They do not depend on a mesh history. This is one prescribed rigid body; there is no equation determining c or θ from fluid forces.

All velocities are expressed at physical Eulerian points x . Set

$$U = \{v \in H_0^1(\Omega)^2 : \text{div } v = 0\}, \quad \|v\|_U = \|\nabla v\|_{L^2(\Omega)}, \quad a(v, z) = \nu(\nabla v, \nabla z), \quad \nu > 0. \quad (2)$$

The bulk inner product and norm are denoted by $(\cdot, \cdot)$ and $\|\cdot\|_0$. Define $V(t) = \{v \in U : \gamma_t v = 0\}$, where γ_t is the trace on $\Gamma(t)$. Unlike U , this constrained space changes with time.

We consider a sufficiently regular target velocity satisfying

$$u(t) \in U, \quad \gamma_t u(t) = b(t)|_{\Gamma(t)}, \quad (\partial_t u(t), v) + a(u(t), v) = \langle f(t), v \rangle \quad (v \in V(t)). \quad (3)$$

For the quantitative theorem we assume the stronger reaction identity

$$(\partial_t u(t), v) + a(u(t), v) + \int_{\Gamma(t)} \lambda(t) \cdot \gamma_t v \, ds = \langle f(t), v \rangle \quad (v \in U), \quad (4)$$

with $\int_0^T \|\lambda(t)\|_{L^2(\Gamma(t))}^2 dt < \infty$. The sign of λ is defined by this identity. This is an assumption on the continuous reaction, not a uniform stability assumption on discrete multiplier rows. The pressure has been eliminated from the test space. No pressure or multiplier error estimate is asserted.

Equation (3) is the weak form of Eulerian unsteady Stokes flow, not a material-derivative equation. The transport calculus for evolving Hilbert spaces and the additional incompressibility issues in moving-domain fluid spaces are developed in Alphonse et al. [18], Djurdjevac et al. [19]. We do not invoke those existence results to infer the piecewise smoothness required here. The convergence theorem is conditional on a target solution with the explicitly stated regularity. Its comparison and lifting constructions are proved below.

The background-domain formulation must specify the solid interior. To represent the prescribed rigid velocity there, take $u_0 = b(0)$ in $B(0)$ and, with interior pressure chosen to be zero, take

$$f_B(t, x) = \partial_t b(t, x) = \ddot{c}(t) + \dot{\omega}(t)J(x - c(t)) - \omega(t)J\dot{c}(t). \quad (5)$$

Since $\Delta b = 0$ and $\text{div } b = 0$, this gives the rigid interior solution. The difference from any other sufficiently regular interior solution has zero initial and boundary data. Its energy identity

on $B(t)$ has no boundary transport contribution, since that difference vanishes on $\Gamma(t)$, and implies uniqueness. Thus the exterior restriction is the intended fluid solution. Formula (5) is not the material rigid acceleration: a centrifugal term must not be inserted into this Eulerian Stokes model. The prescribed interior load is a modeling device, not an external force on a freely moving solid.

Choose a nonnegative, even $\rho \in C_c^\infty(\mathbb{R}^2)$, supported in the unit ball, with $\int \rho = 1$, and set $\rho_\varepsilon(x) = \varepsilon^{-2}\rho(x/\varepsilon)$. For the zero extension Ev of $v \in U$, define

$$J_{\varepsilon,t}v = \gamma_t(\rho_\varepsilon * Ev), \quad b_{\varepsilon,t}(v, \mu) = \int_{\Gamma(t)} \mu \cdot J_{\varepsilon,t}v \, ds. \quad (6)$$

This exact-convolution operator retains the interpolation/spreading adjoint pairing of regularized IB methods [1]. Its integrals are not replaced by marker sampling in the analysis. Let $\mathcal{P}$ be an arclength partition of $\Gamma(t)$, let ℓ be its largest panel length, and let $\Pi_{\mathcal{P}}$ be the L^2 projector onto vector-valued panel constants. The inhomogeneous discrete constraint is

$$\Pi_{\mathcal{P}}(J_{\varepsilon,t}u_h - b(t)) = 0. \quad (7)$$

Regularization width, panel partition and Cartesian enrichment may change independently at each time node. A bar denotes the maximum width used in the entire computation, including initialization.

3 Nodal and moving-interface trace estimates

Lemma 3.1 (Uniform trace and moment defect). *For sufficiently small ε , uniformly in $t \in [0, T]$,*

$$\|J_{\varepsilon,t}v\|_{H^{1/2}(\Gamma(t))} \leq C \|v\|_U, \quad (8)$$

$$\|\gamma_t v - J_{\varepsilon,t}v\|_{L^2(\Gamma(t))} \leq C\sqrt{\varepsilon} \|v\|_U. \quad (9)$$

If $\Pi_{\mathcal{P}}J_{\varepsilon,t}v = 0$, then

$$\|\gamma_t v\|_{L^2(\Gamma(t))} \leq C(\sqrt{\varepsilon} + \sqrt{\ell}) \|v\|_U. \quad (10)$$

Proof. Zero extension is bounded $H_0^1(\Omega) \rightarrow H^1(\mathbb{R}^2)$; convolution is uniformly H^1 bounded. The trace theorem yields the first estimate. For $q = Ev - \rho_\varepsilon * Ev$, the first-order mollification estimate gives $\|q\|_{L^2} \leq C\varepsilon \|v\|_U$, whereas $\|q\|_{H^1} \leq C \|v\|_U$. The multiplicative trace inequality on a fixed tubular neighborhood, transported by the rigid map, gives

$$\|\gamma_t q\|_{L^2(\Gamma(t))}^2 \leq C \|q\|_{L^2(\mathbb{R}^2)} \|q\|_{H^1(\mathbb{R}^2)}.$$

For these standard ingredients see Ern and Guermond [30, Theorems 3.18–3.19, Theorem 3.10(iii), and Lemma 12.15] and Heltai and Lei [12, Lemmas 3–4]. The face trace inequality is transferred through smooth charts; all chart constants are uniform because the interfaces are rigid images of one fixed curve. The same argument was used for fixed geometry in Part I [2].

On a panel I with length ℓ_I and average z_I , the fractional Poincare calculation is

$$\int_I |z - z_I|^2 = \frac{1}{2\ell_I} \int_I \int_I |z(s) - z(r)|^2 \, ds \, dr \leq \frac{\ell_I}{2} \int_I \int_I \frac{|z(s) - z(r)|^2}{|s - r|^2} \, ds \, dr.$$

This is the one-dimensional case of Ern and Guermond [30, Lemma 3.26]. Summation gives $\|(I - \Pi_{\mathcal{P}})z\|_{L^2} \leq C\sqrt{\ell} \|z\|_{H^{1/2}}$. Apply it to $J_{\varepsilon,t}v$, whose panel averages vanish, and use the triangle inequality. $\square$

Lemma 3.2 (Trace at a displaced rigid interface). *For $s, t \in [0, T]$ and $v \in U$, let $R_{s,t}(x) = c(s) + Q(\theta(s))Q(\theta(t))^T(x - c(t))$. Then*

$$\|(\gamma_s v) \circ R_{s,t} - \gamma_t v\|_{L^2(\Gamma(t))} \leq C\sqrt{|s-t|} \|v\|_U. \quad (11)$$

Consequently, if v satisfies homogeneous regularized moments at t ,

$$\|\gamma_s v\|_{L^2(\Gamma(s))} \leq C(\sqrt{\varepsilon} + \sqrt{\ell} + \sqrt{|s-t|}) \|v\|_U. \quad (12)$$

Proof. Set $W = Ev \circ R_{s,t} - Ev$ on $\mathbb{R}^2$. Rigid maps preserve measure and have orthogonal derivatives, so $\|W\|_{H^1} \leq 2\|Ev\|_{H^1}$. Differentiating the rigid path between t and s first for smooth Ev gives

$$\|W\|_{L^2(\mathbb{R}^2)} \leq C|s-t| \|Ev\|_{H^1(\mathbb{R}^2)}.$$

Indeed the derivative is a bounded translation plus rotation generator dotted with ∇Ev at the transformed point. Its coefficients are bounded on the inverse images of the bounded support of Ev , uniformly over the prescribed trajectory. Integrate along the path, change variables by the rigid map, and extend by density to H^1 functions. The multiplicative trace estimate from the preceding proof now gives (11). Composition of Sobolev traces with a smooth rigid map is justified by the same density argument. Arc length is preserved, and (12) follows from (10). $\square$

The square root in (11) includes tangential as well as normal displacement; the proof does not assume that material interface points move along normal rays. It gives a consistency cost depending on elapsed physical time, not on the number of refinement events. The related thin-strip estimates in moving-domain Eulerian methods [16] motivate tracking interface motion separately, but the rigid-map trace argument above is supplied explicitly.

The quantity being controlled is the physical trace of a reconstructed velocity. This matters when algebraic boundary equations alone do not expose leakage. The explicit pressure-boundary remedy of He et al. [25] addresses such penetration in a different two-phase finite-difference IB algorithm; it is a computational motivation, not an ingredient of the present trace proof.

4 A moving comparison family on a fixed Cartesian grid

4.1 The retained coarse space

For $h = 1/M$, let $\mathcal{T}_h$ be the uniform square grid on $\Omega = (0, 1)^2$. Let $S_h \subset H_0^2(\Omega)$ be its clamped bicubic Hermite space. On each cell the polynomials belong to $\mathbb{Q}_3$, and the nodal degrees of freedom are the value, ∂_x , ∂_y , and ∂_{xy} . Shared degrees of freedom give C^1 conformity; the boundary degrees of freedom vanish. This is the Cartesian Bogner–Fox–Schmit construction [31]. If a coarse cell K is divided into four equal squares, let $B_h(K)$ be the space of C^1 piecewise bicubics clamped on ∂K , extended by zero outside K . Its four degrees of freedom are the Hermite data at the center of K . For an arbitrary set of enriched cells $\mathcal{A}$, define

$$S_{h,\mathcal{A}} = S_h + \sum_{K \in \mathcal{A}} B_h(K), \quad X_{h,\mathcal{A}} = \text{curl } S_{h,\mathcal{A}}, \quad \text{curl } \psi = (\partial_y \psi, -\partial_x \psi). \quad (13)$$

Thus $X_{h,\mathcal{A}} \subset U$, with exact distributional incompressibility. Every configuration contains the full coarse space $\text{curl } S_h$. The construction specifies a conforming local enrichment; it is not identified with all hanging-node Cartesian implementations.

Write I_h for the coarse Hermite interpolant. The estimates needed below are the standard fixed-grid bounds

$$\|I_h \varphi - \varphi\|_{H^j(\Omega)} \leq Ch^{4-j} \|\varphi\|_{H^4(\Omega)}, \quad j = 1, 2, \quad \varphi \in H^4(\Omega) \cap H_0^2(\Omega). \quad (14)$$

The mixed-derivative nodal values are well defined at this regularity. No H^3 interpolation assertion involving those point values is used.

For each time t , let $Z_{h,\mathcal{A},\varepsilon,\mathcal{P}}(t)$ denote the subspace of $X_{h,\mathcal{A}}$ whose regularized moments on $\Gamma(t)$ vanish for every multiplier in the panel space $\Lambda_{\mathcal{P}}(t)$. All moments in this definition are exact integrals. Let $\bar{\varepsilon}$ bound every regularization width in a run, and define

$$D_h(t) = \bigcup \{K \in \mathcal{T}_h : K \cap \{x : \text{dist}(x, \Gamma(t)) \leq \bar{\varepsilon}\} \neq \emptyset\}, \quad (15)$$

$$K_h(t) = \{v_h \in \text{curl } S_h : v_h = 0 \text{ on } D_h(t)\}. \quad (16)$$

Compact support of the regularizing kernel implies

$$K_h(t) \subset V(t) \cap Z_{h,\mathcal{A},\varepsilon,\mathcal{P}}(t) \quad \text{for every } \mathcal{A}, \mathcal{P}, \quad 0 < \varepsilon \leq \bar{\varepsilon}. \quad (17)$$

Only the velocity is zero on these cells. The streamfunction may take a nonzero constant on the immersed collar, retaining the circulation mode of the multiply connected fluid region. The spaces $K_h(t)$ depend on time. They are not asserted to have a useful fixed intersection over the entire trajectory.

4.2 Regularity and the moving streamfunction trace

For a piecewise smooth field, define

$$\mathcal{M}_t(v) = \|v\|_{W^{3,\infty}(B(t))} + \|v\|_{W^{3,\infty}(F(t))}.$$

The quantitative analysis assumes

$$\begin{aligned} w &\in H^1(0, T; L^2(\Omega)^2), \quad w(t) \in V(t), \\ M_0 &= \text{ess sup}_{t \in (0, T)} \mathcal{M}_t(w(t)) < \infty, \quad m_1(t) = \mathcal{M}_t(\partial_t w(t)) \in L^2(0, T). \end{aligned} \quad (18)$$

Here ∂_t is the Eulerian derivative at a fixed spatial point. Its restrictions to the two phases are meant in the Sobolev sense. It is not assumed to belong to $V(t)$ or even to global $H^1(\Omega)^2$. A moving interface generally produces a jump in this derivative. In particular, piecewise spatial regularity of a material derivative must not be substituted for the Eulerian assumption in (18) without accounting for the additional spatial derivative in the transport term.

There is a unique normalized clamped streamfunction $\psi(t)$ with $\text{curl } \psi(t) = w(t)$. On the square it can be written as

$$\psi(t, x, y) = \int_0^y w_1(t, x, s) ds. \quad (19)$$

The divergence constraint and outer zero trace give $\partial_x \psi = -w_2$, $\partial_y \psi = w_1$, and $\psi \in H_0^2(\Omega)$. The same identities after weak time differentiation show $\psi \in H^1(0, T; H^1(\Omega))$ and $\nabla \psi_t = (-w_{2,t}, w_{1,t})$. The streamfunction and its time derivative are separately $W^{4,\infty}$ in the two phases, with norms bounded by $C\mathcal{M}_t(w)$ and $Cm_1(t)$, respectively.

Because $\nabla \psi = 0$ on the connected interface, there is a scalar $c_\psi(t)$ such that $\psi|_{\Gamma(t)} = c_\psi(t)$. This scalar has an $H^1(0, T)$ representative and satisfies

$$c'_\psi(t) = \psi_t|_{\Gamma(t)}. \quad (20)$$

To justify the moving trace derivative, one may pull back by a smooth fixed-box flow equal to the prescribed rigid motion near the body. Such a flow is generated by the compactly supported solenoidal lift constructed in Lemma 5.1. Composition of $\psi \in H_t^1 H_x^1 \cap L_t^\infty H_x^2$ with this flow has weak derivative $(\psi_t + g \cdot \nabla \psi)$ composed with the flow, in $L_t^2 H_x^1$. This chain rule follows first for smooth functions and then by approximation in $H_t^1 H_x^1 \cap L_t^2 H_x^2$; the $L_t^\infty H_x^2$ bound is used only for uniform estimates. The continuous H^1 trace map on the fixed interface commutes with that weak derivative. Since $\nabla \psi|_{\Gamma(t)} = 0$, the transport term has zero trace, which proves (20).

Let $d(t, x)$ be signed distance to $\Gamma(t)$ in a uniform tubular neighborhood. The smooth collision-free rigid trajectory gives a time-independent tubular radius and uniformly bounded spatial derivatives through the order used below. Near the interface,

$$d_t + b(t, x) \cdot \nabla d = 0. \quad (21)$$

Indeed, rigid transformations preserve signed distance. In particular $\|d_t\|_{W^{4,\infty}}$ is uniformly bounded. With $q = \psi - c_\psi(t)$, normal integration on either side of the interface gives

$$\begin{aligned} |q| &\leq C|d|^2 \mathcal{M}_t(w), & |\nabla q| &\leq C|d| \mathcal{M}_t(w), \\ |q_t| &\leq C|d| m_1(t), & |\nabla q_t| &\leq C m_1(t). \end{aligned} \quad (22)$$

The second line follows from (20); it does not require w_t to vanish on the moving interface.

4.3 A constructive approximation with an Eulerian derivative bound

Lemma 4.1 (Moving common recovery). *Assume (18). For $h + \bar{\varepsilon}$ sufficiently small, put $\delta = 4(\bar{\varepsilon} + \sqrt{2}h)$. There is an explicitly constructed trajectory $z_h(t) = Q_h(t)w(t) \in K_h(t)$ such that*

$$\operatorname{ess\,sup}_t \|z_h(t) - w(t)\|_U \leq C\sqrt{\delta} M_0, \quad (23)$$

$$\operatorname{ess\,sup}_t \|z_h(t) - w(t)\|_0 \leq C\delta^{3/2} M_0, \quad (24)$$

$$\|\partial_t(z_h - w)\|_{L^2(0,T;L^2)} \leq C\sqrt{\delta} (\|m_1\|_{L^2(0,T)} + \sqrt{T}M_0). \quad (25)$$

The constant is independent of all enrichment sets, panel partitions, individual regularization widths, and their numbers of changes. It may depend on the prescribed trajectory, its clearance from the outer boundary, and the fixed geometry. The trajectory z_h belongs to $H^1(0, T; \operatorname{curl} S_h)$.

Proof. Choose $\chi \in C_c^\infty(\mathbb{R})$ equal to one for $|s| \leq 1$ and zero for $|s| \geq 2$. For 2δ smaller than the uniform tubular radius and the outer clearance, define

$$\psi_\delta(t, x) = \psi(t, x) - \chi(d(t, x)/\delta)q(t, x) \quad (26)$$

in the tube, and set $\psi_\delta = \psi$ elsewhere. It equals $c_\psi(t)$ throughout $|d| \leq \delta$, preserves the outer clamped condition, and removes all original interface derivative jumps. Thus $\psi_\delta(t) \in H^4(\Omega) \cap H_0^2(\Omega)$.

The support of $\psi_\delta - \psi$ has area $O(\delta)$. By (22), its derivatives of orders zero, one, and two are bounded by $C\delta^2 \mathcal{M}_t(w)$, $C\delta \mathcal{M}_t(w)$, and $C \mathcal{M}_t(w)$. Fourth derivatives on the transition strip are bounded by $C\delta^{-2} \mathcal{M}_t(w)$. Consequently,

$$\begin{aligned} \|\psi_\delta - \psi\|_{H^1} &\leq C\delta^{3/2} \mathcal{M}_t(w), \\ \|\psi_\delta - \psi\|_{H^2} &\leq C\sqrt{\delta} \mathcal{M}_t(w), \\ \|\psi_\delta\|_{H^4} &\leq C(1 + \delta^{-3/2}) \mathcal{M}_t(w). \end{aligned} \quad (27)$$

For example, four cutoff derivatives multiply $q = O(d^2)$, and three cutoff derivatives multiply $\nabla q = O(d)$; both contributions have size at most $C\delta^{-2}\mathcal{M}_t(w)$ in the strip.

The Eulerian time derivative must include the moving-cutoff term:

$$\partial_t(\psi_\delta - \psi) = -\chi(d/\delta)q_t - \delta^{-1}\chi'(d/\delta)d_tq. \quad (28)$$

The first spatial derivatives of this expression are bounded by $C(m_1(t) + M_0)$ on a strip of area $O(\delta)$, using (22) and the uniform distance bounds. Its zeroth derivative is bounded by $C\delta(m_1(t) + M_0)$. Moreover, the fourth spatial derivatives of $\psi_{\delta,t}$ are bounded by $C\delta^{-3}(m_1(t) + M_0)$ on the transition strip. The worst terms are four spatial cutoff derivatives times $q_t = O(\delta)$, and the fourth derivative of $\delta^{-1}\chi'(d/\delta)d_tq$, whose most singular factor has size $\delta^{-5}q = O(\delta^{-3})$. Away from the strip, $\psi_{\delta,t}$ is either ψ_t or the spatial constant c'_ψ . Therefore

$$\begin{aligned} \|\partial_t(\psi_\delta - \psi)\|_{H^1} &\leq C\sqrt{\delta}(m_1(t) + M_0), \\ \|\psi_{\delta,t}\|_{H^4} &\leq C(1 + \delta^{-5/2})(m_1(t) + M_0). \end{aligned} \quad (29)$$

These identities hold as weak time derivatives as well: the original moving-interface jumps have been removed in an open tube, and the right-hand side of (28) is in $L_t^2 H_x^4$ after adding ψ_t .

Define

$$z_h(t) = Q_h(t)w(t) = \text{curl } I_h\psi_\delta(t). \quad (30)$$

Every coarse cell meeting the $\bar{\varepsilon}$ tube is contained in the $\bar{\varepsilon} + \sqrt{2}h$ tube. The latter lies strictly inside $|d| < \delta$, where ψ_δ is constant. Exact constant reproduction by I_h gives $z_h(t) \in K_h(t)$.

Apply (14) and (27). The velocity H^1 and L^2 errors are bounded, respectively, by

$$C(\sqrt{\delta} + h^2(1 + \delta^{-3/2}))M_0, \quad C(\delta^{3/2} + h^3(1 + \delta^{-3/2}))M_0.$$

Since $\delta \geq 4\sqrt{2}h$, these imply (23)–(24). Only the fixed interpolant I_h commutes with time differentiation. Using its H^1 estimate on $\psi_{\delta,t}$ gives

$$\|\partial_t(z_h - w)\|_0 \leq C(\sqrt{\delta} + h^3(1 + \delta^{-5/2}))(m_1(t) + M_0) \leq C\sqrt{\delta}(m_1(t) + M_0).$$

Integration proves (25). The bounds also give $\psi_\delta \in H_t^1 H_x^4$ for fixed $h, \bar{\varepsilon}$, hence the asserted finite-dimensional H^1 trajectory. Changes in the set of cells in (15) do not introduce jumps in this trajectory. $\square$

In particular, the estimate is for $\partial_t(Q_h(t)w(t) - w(t))$. The generally false identity $\partial_t(Q_h(t)w(t)) = Q_h(t)\partial_t w(t)$ is not used. Nor is convergence of this derivative in the U norm asserted.

Lemma 4.2 (Time modulus across a moving interface). *Under (18), w has a continuous U -valued representative and, for $0 \leq s < t \leq T$,*

$$\|w(t) - w(s)\|_U \leq C\sqrt{t-s}M_0 + C \int_s^t m_1(r) dr \leq C\sqrt{t-s}(M_0 + \|m_1\|_{L^2(0,T)}). \quad (31)$$

Proof. The rigid boundary travels a distance at most $L(t-s)$ during $[s, t]$, with a fixed trajectory-dependent L . Its swept set is contained in the $L(t-s)$ tube of $\Gamma(s)$. For sufficiently small $t-s$, that tube has area at most $C(t-s)$. Outside this tube, a fixed spatial point stays in the same phase throughout the interval. There the weak fundamental theorem of calculus and the piecewise bound on ∇w_t give a gradient difference bounded by $C \int_s^t m_1(r) dr$. On the tube, each endpoint gradient is bounded by CM_0 , so its contribution to the L^2 gradient norm is at most $C\sqrt{t-s}M_0$. This proves the first inequality for almost every pair of times. For larger intervals it follows after enlarging the constant from the uniform U bound. Cauchy–Schwarz proves the second inequality, which also supplies the continuous representative and extends the result to all times. The existing $H_t^1 L_x^2$ representative agrees with it. $\square$

5 A common coarse lifting of general prescribed rigid motion

Using $J(x_1, x_2) = (-x_2, x_1)$ from (1), set $v_b(t) = c'(t)$ and $\omega(t) = \theta'(t)$, and write the prescribed rigid velocity as

$$b(t, x) = v_b(t) + \omega(t)J(x - c(t)). \quad (32)$$

No symmetry of the reference body is needed. The positive clearance of the entire trajectory permits a fixed, smooth, compactly supported cutoff $\chi_* \in C_c^5(\Omega)$ (which may be chosen C^∞) which is one on a neighborhood of the swept closed body $\bigcup_{0 \leq t \leq T} \overline{B(t)}$. Choose $r_g > 0$ such that this neighborhood contains the entire r_g neighborhood of that swept set. Define the fixed origin $x_* = (1/2, 1/2)$, set $X = x - x_*$, and put

$$\begin{aligned} \beta(t) &= v_b(t) - \omega(t)J(c(t) - x_*), \\ p_b(t, x) &= \beta_1(t)X_2 - \beta_2(t)X_1 - \frac{\omega(t)}{2}|X|^2, \\ \Psi_g(t, x) &= \chi_*(x)p_b(t, x), \quad g(t) = \text{curl } \Psi_g(t), \quad g_h(t) = \text{curl } I_h \Psi_g(t). \end{aligned} \quad (33)$$

The sign of the quadratic term follows from the convention $\text{curl } \psi = (\partial_y \psi, -\partial_x \psi)$. Indeed, $\text{curl } p_b = \beta + \omega JX = b$. We fix this additive normalization of the quadratic streamfunction: changing its constant before multiplying by χ_* changes the lift outside the plateau. Both lifts are solenoidal and outer-zero. The discrete lift belongs to the same fixed coarse space $\text{curl } S_h$ at every time.

Lemma 5.1 (Exact moving rigid moments and lifting accuracy). *Suppose $c, \theta \in W^{2,\infty}(0, T)$ and the swept body has positive clearance from $\partial\Omega$. For $\bar{\varepsilon} + \sqrt{2}h < r_g$, the field $g_h(t)$ equals $b(t)$ on every $B(s)$ and on every regularization support centered on $\Gamma(s)$, for all $s, t \in [0, T]$, with width at most $\bar{\varepsilon}$. In particular this reproduction holds when the interface and lifting are evaluated at different times. If the regularizing kernel has unit mass and zero first moment, then*

$$J_{\varepsilon,t} g_h(t)(X) = b(t, X), \quad X \in \Gamma(t), \quad 0 < \varepsilon \leq \bar{\varepsilon}. \quad (34)$$

Consequently $g_h(t) + Z_{h,A,\varepsilon,\mathcal{P}}(t)$ satisfies the prescribed regularized moments for every adaptive configuration and every panel partition. Moreover,

$$\begin{aligned} \|g_h - g\|_{L^\infty(0,T;H^1)} &\leq Ch^2, \\ \|g_h - g\|_{L^\infty(0,T;L^2)} &\leq Ch^3, \\ \|\partial_t(g_h - g)\|_{L^\infty(0,T;L^2)} &\leq Ch^3. \end{aligned} \quad (35)$$

The constants depend on the fixed cutoff and the prescribed trajectory, but not on the regularization, surface partition, or enrichment pattern.

Proof. The swept closed body is compact and lies a positive distance from the outer boundary. A smooth cutoff with the stated plateau follows by mollifying the indicator of a slightly larger neighborhood; the mollification radius is chosen smaller than the gaps between the nested neighborhoods and the outer wall. Thus the cutoff is fixed throughout the run.

On the plateau, $\Psi_g = p_b$, and direct differentiation gives $\text{curl } p_b = b$. Each Cartesian cell intersecting a regularization support lies within distance $\bar{\varepsilon} + \sqrt{2}h$ of the swept body. The streamfunction therefore equals the quadratic polynomial p_b throughout that cell. Bicubic Hermite interpolation reproduces this polynomial exactly. The same argument applies to every cell meeting $B(s)$ for any s . Since the plateau contains the whole swept set, this reasoning is independent of the time t in the polynomial coefficients. It proves the asserted cross-time reproduction by g_h . The averaging of an affine field by a kernel of unit mass and zero first

moment equals that field at the kernel center, proving (34). An even kernel has the required zero first moment.

The cutoff is C^5 and fixed, while the polynomial coefficients of p_b belong to $W^{1,\infty}(0, T)$. Hence $\Psi_g \in W^{1,\infty}(0, T; H^4(\Omega) \cap H_0^2(\Omega))$. Apply (14) with $j = 2$ and $j = 1$, and apply its $j = 1$ version to $\partial_t \Psi_g$. The fixed interpolant commutes with this weak derivative, giving (35). $\square$

The flow generated by g also supplies a smooth identification of the moving phases when a change of coordinates is useful in a proof. It is the identity near the outer wall and is volume preserving because $\operatorname{div} g = 0$. On each body trajectory it agrees with the prescribed rigid velocity. Uniqueness for the flow ordinary differential equation therefore maps $B(0)$ to $B(t)$ and $\Gamma(0)$ to $\Gamma(t)$; the exterior is mapped accordingly. This auxiliary flow justifies the trace chain rule used above. The numerical transfer itself remains a projection on the fixed physical Cartesian background and does not apply this flow or a Piola transform.

If the physical velocity is extended inside the body by b , then $w = u - g$ is zero in $B(t)$ as well as on $\Gamma(t)$. The approximation results also allow a homogeneous interior field, provided the background-domain forcing and initial data are specified consistently. For the unsteady Stokes operator, the rigid extension has $\Delta b = 0$ and interior forcing b_t ; the nonlinear convective acceleration would be additional in a Navier–Stokes formulation.

6 Affine time stepping, stability, and physical slip

Let $0 = t_0 < \dots < t_N = T$, $\tau_n = t_n - t_{n-1}$, and $\tau_{\max} = \max_n \tau_n$. At each node choose any admissible enrichment set $\mathcal{A}^n$, regularization width ε_n and panel partition $\mathcal{P}^n$ on $\Gamma(t_n)$. In the Cartesian spaces constructed above, put

$$Z^n = \{v_h \in X_{h,\mathcal{A}^n} : \Pi_{\mathcal{P}^n} J_{\varepsilon_n, t_n} v_h = 0\}, \quad P^n : L^2(\Omega)^2 \longrightarrow Z^n. \quad (36)$$

Here P^n is the exact bulk orthogonal projection. Let $g_h^n = g_h(t_n)$ be the common coarse lift from Lemma 5.1. Feasibility gives the affine velocity space $g_h^n + Z^n$, and its orthogonal projector is

$$\mathcal{P}_g^n v = g_h^n + P^n(v - g_h^n).$$

With $u_h^0 = \mathcal{P}_g^0 u_0$, the scheme is

$$u_h^n \in g_h^n + Z^n, \quad \left(\frac{u_h^n - \mathcal{P}_g^n u_h^{n-1}}{\tau_n}, v_h \right) + a(u_h^n, v_h) = \langle f^n, v_h \rangle \quad (v_h \in Z^n). \quad (37)$$

The preferred exact-integral load is $f^n = \tau_n^{-1} \int_{t_{n-1}}^{t_n} f(t) dt$. Other choices are allowed in the error theorem through an explicit load-consistency term. All time derivatives and transfers refer to the same physical background domain. No material advection or Piola mapping is built into (37).

Set $y_h^n = u_h^n - g_h^n$. Orthogonality against Z^n gives the equivalent centered scheme

$$y_h^n \in Z^n, \quad \left(\frac{y_h^n - P^n y_h^{n-1}}{\tau_n}, v_h \right) + a(y_h^n, v_h) = \langle F_h^n, v_h \rangle, \quad (38)$$

$$\langle F_h^n, v \rangle = \langle f^n, v \rangle - \left(\frac{g_h^n - g_h^{n-1}}{\tau_n}, v \right) - a(g_h^n, v). \quad (39)$$

The lifting term is its finite difference, not an endpoint replacement by $\partial_t g_h(t_n)$. Even when these differ by a small consistency term, they are not algebraically interchangeable. The

diffusion term is likewise evaluated at g_h^n . This is the time-dependent version of the standard boundary-lifting reduction [32, Chapter 33]; its exact simultaneous feasibility for all regularized constraints was proved in Lemma 5.1.

Theorem 6.1 (Unconditional centered energy stability). *At every step (37) has a unique velocity. Its centered form satisfies*

$$\frac{1}{2}(\|y_h^n\|_0^2 - \|y_h^{n-1}\|_0^2 + \|y_h^n - y_h^{n-1}\|_0^2) + \tau_n a(y_h^n, y_h^n) = \tau_n \langle F_h^n, y_h^n \rangle. \quad (40)$$

Moreover,

$$\|y_h^n - y_h^{n-1}\|_0^2 = \|y_h^n - P^n y_h^{n-1}\|_0^2 + \|(I - P^n)y_h^{n-1}\|_0^2, \quad (41)$$

and

$$\max_n \|y_h^n\|_0^2 + \sum_n \|y_h^n - y_h^{n-1}\|_0^2 + \nu \sum_n \tau_n \|y_h^n\|_U^2 \leq C \left(\|y_h^0\|_0^2 + \nu^{-1} \sum_n \tau_n \|F_h^n\|_{U'}^2 \right). \quad (42)$$

These statements require neither nested spaces nor a bound on the number of mesh or geometry updates. They impose no relation between τ_n and h .

Proof. The matrix of $(v_h, z_h)/\tau_n + a(v_h, z_h)$ on Z^n is positive definite. Thus the centered velocity exists uniquely, regardless of linear dependence among multiplier rows. Test (38) with y_h^n , use $(P^n y_h^{n-1}, y_h^n) = (y_h^{n-1}, y_h^n)$, and polarize. This gives (40). The two vectors in (41) are orthogonal. Summing and applying Young's inequality proves (42). These energy and projection principles are established tools in changing-space Stokes and DLM-IB analysis [22, 23, 33]; the present affine constraints and moving lift specify their application here. $\square$

The load in (39) is uniformly controlled. In particular,

$$\sum_n \tau_n \|\delta_n g_h\|_0^2 \leq \|\partial_t g_h\|_{L^2(0,T;L^2)}^2, \quad \delta_n g_h = (g_h^n - g_h^{n-1})/\tau_n.$$

The lift estimates and the bound on g_h in $L^\infty(0,T;U)$ therefore control the energy by the prescribed motion and fluid data, independently of refinement history. This is driven stability, not decay of the total velocity in a moving-boundary problem.

Corollary 6.2 (Physical slip on the moving curve). *Let $U_h(t) = u_h^n$ for $t \in (t_{n-1}, t_n]$. With the exact swept-region lift,*

$$\left(\sum_n \tau_n \|\gamma_{t_n} u_h^n - b(t_n)\|_{L^2(\Gamma(t_n))}^2 \right)^{1/2} \leq C(\sqrt{\varepsilon} + \sqrt{\ell}) \left(\sum_n \tau_n \|y_h^n\|_U^2 \right)^{1/2}, \quad (43)$$

$$\|\gamma_t U_h - b(t)\|_{L^2(0,T;L^2(\Gamma(t)))} \leq C(\sqrt{\varepsilon} + \sqrt{\ell} + \sqrt{\tau_{\max}}) \left(\sum_n \tau_n \|y_h^n\|_U^2 \right)^{1/2} + C\tau_{\max}. \quad (44)$$

The last constant depends on $\partial_t b$ on the swept region.

Proof. At a node, $g_h^n = b(t_n)$ on $\Gamma(t_n)$ and y_h^n satisfies homogeneous moments, so Lemma 3.1 applies. At any t in the same interval, Lemma 3.2 controls the trace of y_h^n on $\Gamma(t)$. The lift constructed on the entire swept region also satisfies $g_h^n = b(t_n, x)$ on $\Gamma(t)$, not only on its nodal curve. Finally $\|b(t_n) - b(t)\|_{L^2(\Gamma(t))} \leq C\tau_n$. Integrate these bounds in time. $\square$

For completeness, the uncentered kinetic identity is

$$\frac{1}{2}(\|u_h^n\|_0^2 - \|u_h^{n-1}\|_0^2 + \|u_h^n - u_h^{n-1}\|_0^2) + \tau_n a(u_h^n, u_h^n) = \tau_n \langle f^n, u_h^n \rangle + \tau_n \mathcal{W}_h^n, \quad (45)$$

where

$$\mathcal{W}_h^n = (\delta_n u_h, g_h^n) + a(u_h^n, g_h^n) - \langle f^n, g_h^n \rangle.$$

To verify it, test (37) with $u_h^n - g_h^n$, replace the projected difference by the unprojected one in that pairing, and polarize. The work term need not vanish. This is a background-domain balance including the artificial interior. An exterior hydrodynamic force or torque requires the corresponding interior correction; (45) is not a two-way rigid-body energy theorem.

7 Convergence for a moving prescribed rigid body

The estimate in this section is conditional on explicitly stated regularity of the exact solution. It does not assert that arbitrary finite-energy data have that regularity. All transfer operators act in the original Eulerian bulk inner product. No transported mesh or material time derivative is introduced in the numerical scheme.

7.1 Regularity and the comparison trajectory

Use the piecewise norm $\mathcal{M}_t$ defined in Section 4. Let u satisfy the Eulerian weak equation and reaction identity (4), and set $w = u - g$. We assume that

$$w \in H^1(0, T; L^2(\Omega)^2), \quad w(t) \in V(t),$$

$$M_0 := \sup_{0 \leq t \leq T} \mathcal{M}_t(w(t)) < \infty, \quad M_1 := \left(\int_0^T \mathcal{M}_t(\partial_t w(t))^2 dt \right)^{1/2} < \infty. \quad (46)$$

The side restrictions and their derivatives are interpreted in the Sobolev sense on the smooth moving space–time phases. The trace differentiation justified in Lemma 4.1 uses these explicit assumptions. In particular, $\partial_t w$ in (46) is the *Eulerian* derivative. Bounds for a material derivative alone do not imply the displayed spatial regularity. The Eulerian derivative may jump across $\Gamma(t)$ and need not belong to $H^1(\Omega)^2$ or $V(t)$. The trace compatibility is $w|_{\Gamma(t)} = 0$, not $\partial_t w|_{\Gamma(t)} = 0$.

For the reaction in (4), assume

$$\Lambda := \left(\int_0^T \|\lambda(t)\|_{L^2(\Gamma(t))}^2 dt \right)^{1/2} < \infty. \quad (47)$$

Surface norms are defined by the rigid parametrization of $\Gamma(t)$, which preserves arclength. Define also

$$\mathfrak{G} = \|g\|_{W^{1,\infty}(0,T;H^3(\Omega)^2)}.$$

This quantity is finite for the smooth fixed-cutoff lifting constructed in Lemma 5.1 under the stated motion assumptions. The constants below may depend on the fixed body, its clearance from the outer boundary, the prescribed motion, the fixed cutoff, T , and ν . They are independent of the choices and number of local enrichments, surface partitions, and regularization widths.

Write $\bar{\varepsilon} = \max_n \varepsilon_n$, $\bar{\ell} = \max_n \ell_n$, and $\tau_{\max} = \max_n \tau_n$. Lemma 4.1 constructs, on the retained coarse Cartesian grid, a trajectory

$$z_h(t) = Q_h(t)w(t), \quad \eta(t) = w(t) - z_h(t), \quad z_h(t_n) \in Z^n. \quad (48)$$

With $r_h = \sqrt{h + \bar{\varepsilon}}$, its estimates imply

$$\begin{aligned} \sup_t \|\eta(t)\|_U &\leq Cr_h M_0, \\ \|\partial_t \eta\|_{L^2(0,T;L^2(\Omega))} &\leq Cr_h (M_0 + M_1). \end{aligned} \quad (49)$$

These are the two different norms needed by the energy argument. An H^1 -small Eulerian derivative of the comparison error is neither asserted nor used. Lemma 4.2 also gives

$$\|w(t) - w(s)\|_U \leq C\sqrt{|t-s|} M_0 + C \int_{\min(s,t)}^{\max(s,t)} \mathcal{M}_r(\partial_r w(r)) dr. \quad (50)$$

7.2 Consistency on the current constraint space

For any time-dependent quantity q , let $\bar{q}^n = \tau_n^{-1} \int_{t_{n-1}}^{t_n} q(s) ds$. The selected load f^n need not equal $\bar{f}^n$ exactly; measure its error by

$$E_f = \left(\sum_n \tau_n \|f^n - \bar{f}^n\|_{U'}^2 \right)^{1/2}. \quad (51)$$

The primary theoretical choice $f^n = \bar{f}^n$ gives $E_f = 0$. For the centered exact and discrete loads, write

$$\begin{aligned} \langle \mathcal{F}(t), v \rangle &= \langle f(t), v \rangle - (\partial_t g(t), v) - a(g(t), v), \\ \langle \mathcal{F}_h^n, v \rangle &= \langle f^n, v \rangle - (\delta g_h^n, v) - a(g_h(t_n), v), \quad \delta g_h^n = \frac{g_h(t_n) - g_h(t_{n-1})}{\tau_n}. \end{aligned} \quad (52)$$

The centered reaction identity is consequently

$$(\partial_t w(t), v) + a(w(t), v) + \int_{\Gamma(t)} \lambda(t) \cdot \gamma_t v ds = \langle \mathcal{F}(t), v \rangle, \quad v \in U, \quad (53)$$

for almost every t .

Lemma 7.1 (Load and moving-interface residuals). *Under the preceding assumptions,*

$$\left(\sum_n \tau_n \|\mathcal{F}_h^n - \bar{\mathcal{F}}^n\|_{U'}^2 \right)^{1/2} \leq E_f + C(h^2 + \tau_{\max}) \mathfrak{G}. \quad (54)$$

For $v_h \in Z^n$, define

$$L_n(v_h) = \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} \int_{\Gamma(s)} \lambda(s) \cdot \gamma_s v_h ds_\Gamma ds.$$

Then

$$\begin{aligned} |L_n(v_h)| &\leq C(\sqrt{\varepsilon_n} + \sqrt{\ell_n} + \sqrt{\tau_n}) \|\bar{\lambda}\|_\Gamma^n \|v_h\|_U, \\ \|\bar{\lambda}\|_\Gamma^n &:= \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} \|\lambda(s)\|_{L^2(\Gamma(s))} ds. \end{aligned} \quad (55)$$

Finally,

$$\left(\sum_n \tau_n \|w(t_n) - \bar{w}^n\|_U^2 \right)^{1/2} \leq C\sqrt{\tau_{\max}}(M_0 + M_1). \quad (56)$$

Proof. Let $d_h = g_h - g$. Subtraction of the two loads gives the exact identity

$$\langle \mathcal{F}_h^n - \bar{\mathcal{F}}^n, v \rangle = \langle f^n - \bar{f}^n, v \rangle - \langle \partial_t \bar{d}_h^n, v \rangle - a(d_h(t_n), v) - a(g(t_n) - \bar{g}^n, v).$$

The fixed-cutoff Hermite lifting estimate gives

$$\|\partial_t d_h\|_{L^2(0,T;L^2)} + \sup_t \|d_h(t)\|_U \leq Ch^2 \mathfrak{G}.$$

Here the supremum follows also from the one-dimensional H^1 -in-time embedding for the finite family of lifting coefficients. Jensen's inequality controls the averaged derivative. The fundamental theorem of calculus and Cauchy–Schwarz give

$$\sum_n \tau_n \|g(t_n) - \bar{g}^n\|_U^2 \leq \tau_{\max}^2 \|\partial_t g\|_{L^2(0,T;U)}^2.$$

Poincaré's inequality and the triangle inequality in the weighted sequence norm prove (54).

Let $R_{s,n}$ denote the exact rigid map from $\Gamma(t_n)$ to $\Gamma(s)$. It preserves surface measure. By the moving-trace estimate of Lemma 3.2 and the current-space trace defect,

$$\begin{aligned} \|\gamma_s v_h\|_{L^2(\Gamma(s))} &\leq \|\gamma_s v_h \circ R_{s,n} - \gamma_{t_n} v_h\|_{L^2(\Gamma(t_n))} + \|\gamma_{t_n} v_h\|_{L^2(\Gamma(t_n))} \\ &\leq C(\sqrt{|s - t_n|} + \sqrt{\varepsilon_n} + \sqrt{\ell_n}) \|v_h\|_U. \end{aligned}$$

Integration against $\lambda(s)$ proves (55). In particular, the time-averaged reaction is not incorrectly evaluated on the current interface. Jensen's inequality gives $\sum_n \tau_n (\|\lambda\|_{\Gamma}^n)^2 \leq \Lambda^2$.

Apply (50) for $s \in (t_{n-1}, t_n)$, average in s , and then use Cauchy–Schwarz. This bounds $\|w(t_n) - \bar{w}^n\|_U$ by $C\sqrt{\tau_n}(M_0 + \|\mathcal{M}(\partial_t w)\|_{L^2(t_{n-1}, t_n)})$. Summation proves (56), with constants allowed to depend on T . $\square$

7.3 The energy error estimate

Define the explicit residual size

$$\begin{aligned} \mathfrak{R}_h &= \sqrt{h + \bar{\varepsilon}}(M_0 + M_1) + \sqrt{\tau_{\max}}(M_0 + M_1) \\ &\quad + (\sqrt{\bar{\varepsilon}} + \sqrt{\bar{\ell}} + \sqrt{\tau_{\max}})\Lambda + (h^2 + \tau_{\max})\mathfrak{G} + E_f. \end{aligned} \tag{57}$$

Theorem 7.2 (Moving-geometry velocity convergence). *Assume the collision-free rigid motion and Cartesian spaces specified above, sufficiently small $h + \bar{\varepsilon}$ for the collar construction, and (46)–(47). Use the true affine initialization*

$$u_h^0 = g_h(0) + P^0(u(0) - g_h(0))$$

and the scheme (37), equivalently (38). Then

$$\max_{0 \leq n \leq N} \|u_h^n - u(t_n)\|_0 + \left(\sum_{n=1}^N \tau_n \|u_h^n - u(t_n)\|_U^2 \right)^{1/2} \leq C\mathfrak{R}_h. \tag{58}$$

The constant is independent of the number and choices of mesh changes. There is no restriction coupling τ_n to h , ε_n , or a remeshing frequency.

For the piecewise constant reconstruction $U_h(t) = u_h^n$ on $(t_{n-1}, t_n]$, set $U_h(0) = u_h^0$. The same right-hand side, with a different constant, bounds

$$\sup_{0 \leq t \leq T} \|U_h(t) - u(t)\|_0 + \|U_h - u\|_{L^2(0,T;U)}. \tag{59}$$

Consequently, for a fixed exact solution in this class, simultaneous $h, \bar{\varepsilon}, \bar{\ell}, \tau_{\max}, E_f \rightarrow 0$ gives strong convergence in both norms. In particular the rate for exact time-averaged loads is $C_{\text{data}}(\sqrt{h} + \sqrt{\bar{\varepsilon}} + \sqrt{\bar{\ell}} + \sqrt{\tau_{\max}})$.

Proof. Put $z^n = z_h(t_n)$ and $e^n = y_h^n - z^n$, where $y_h^n = u_h^n - g_h(t_n) \in Z^n$. The previous comparison z^{n-1} need not belong to the current space. Nevertheless, orthogonality gives

$$(P^n z^{n-1}, v_h) = (z^{n-1}, v_h) \quad (v_h \in Z^n). \quad (60)$$

Thus subtraction of the comparison produces its ordinary Eulerian time difference, with no additional projection-change term. Average (53) in time and subtract it from the discrete centered equation. Since $\delta w^n = \partial_t \bar{w}^n$ and $\delta z^n = \delta w^n - \delta \eta^n$, the result is

$$\begin{aligned} ((e^n - P^n e^{n-1})/\tau_n, v_h) + a(e^n, v_h) &= \mathcal{R}_n(v_h), \\ \mathcal{R}_n(v_h) &= \langle \mathcal{F}_h^n - \bar{\mathcal{F}}^n, v_h \rangle + L_n(v_h) + (\delta \eta^n, v_h) \\ &\quad + a(\eta(t_n), v_h) - a(w(t_n) - \bar{w}^n, v_h). \end{aligned} \quad (61)$$

This is the usual residual-energy argument for backward Euler; compare the dynamic-space argument in Brenner et al. [22, Section 3.1]. The moving comparison estimates and the moving-interface reaction bound above are the specific ingredients that close it here.

Let $r_n = \sup_{0 \neq v_h \in Z^n} |\mathcal{R}_n(v_h)| / \|v_h\|_U$, with $r_n = 0$ if $Z^n = \{0\}$. From (49), Jensen's inequality, and Lemma 7.1,

$$\left(\sum_n \tau_n r_n^2 \right)^{1/2} \leq C \mathfrak{R}_h. \quad (62)$$

In particular, $\sum_n \tau_n \|\delta \eta^n\|_0^2 \leq \|\partial_t \eta\|_{L^2(0,T;L^2)}^2$; no global H^1 regularity of $\partial_t w$ is used.

Test (61) with e^n and multiply by $2\tau_n$. By (60) and polarization,

$$\|e^n\|_0^2 - \|e^{n-1}\|_0^2 + \|e^n - e^{n-1}\|_0^2 + 2\nu\tau_n \|e^n\|_U^2 = 2\tau_n \mathcal{R}_n(e^n).$$

Young's inequality and summation therefore give, for every k ,

$$\|e^k\|_0^2 + \sum_{n=1}^k \|e^n - e^{n-1}\|_0^2 + \nu \sum_{n=1}^k \tau_n \|e^n\|_U^2 \leq \|e^0\|_0^2 + \nu^{-1} \sum_{n=1}^k \tau_n r_n^2. \quad (63)$$

The affine initialization and $P^0 z^0 = z^0$ give $e^0 = P^0(\eta(0) + g(0) - g_h(0))$. Contractivity bounds its norm by $Cr_h M_0 + Ch^2 \mathfrak{G}$. Combining (63) with (62) proves the desired bounds for e . Finally,

$$u_h^n - u(t_n) = e^n - \eta(t_n) + g_h(t_n) - g(t_n).$$

The spatial recovery and lifting estimates prove (58).

For the reconstructed energy norm, use (50) interval by interval and $g \in H^1(0, T; U)$; the extra error is bounded by $C\sqrt{\tau_{\max}}(M_0 + M_1) + C\tau_{\max} \mathfrak{G}$. For the reconstructed supremum bulk norm, apply the fundamental theorem of calculus separately to w and the smooth prescribed lift:

$$\sup_n \sup_{t \in [t_{n-1}, t_n]} \|w(t_n) - w(t)\|_0 \leq \sqrt{\tau_{\max}} \|\partial_t w\|_{L^2(0,T;L^2)},$$

while the corresponding lift difference is at most $\tau_{\max} \|\partial_t g\|_{L^\infty(0,T;L^2)}$. The first derivative norm is bounded by CM_1 . These two terms are already included in (57), proving (59). $\square$

Corollary 7.3 (Vanishing centered transfer losses). *Under the convergent parameter sequence in Theorem 7.2,*

$$\sum_n \|y_h^n - y_h^{n-1}\|_0^2 \longrightarrow 0.$$

Hence both the accumulated centered projection loss and the accumulated centered time-step loss in the orthogonal decomposition of the energy identity tend to zero.

Proof. Equation (63) implies $\sum_n \|e^n - e^{n-1}\|_0^2 \rightarrow 0$. Moreover, $z_h = w - \eta \in H^1(0, T; L^2)$ and (49) bounds its derivative uniformly. Thus

$$\sum_n \|z_h(t_n) - z_h(t_{n-1})\|_0^2 \leq \tau_{\max} \|\partial_t z_h\|_{L^2(0, T; L^2)}^2 \rightarrow 0.$$

The triangle inequality squared proves the assertion for $y_h = e + z_h$. The two nonnegative orthogonal losses sum to that full increment loss. $\square$

Remark 7.4 (Endpoint forcing and the time rate). The load term (51) permits the independently evaluated endpoint forcing used in a manufactured-solution test. For example, suppose the bulk force has uniformly bounded values on each side of the moving interface, and its sidewise Eulerian time derivative has an L^2 -in-time L^∞ bound. Splitting into the region swept by the interface between s and t , of area at most $C|t - s|$, and its complement gives

$$\|f(t) - f(s)\|_{L^2(\Omega)} \leq C\sqrt{|t - s|} M_{f,0} + C \int_{\min(s,t)}^{\max(s,t)} m_{f,1}(r) dr.$$

The same averaging argument as in (56), followed by Poincaré's inequality, gives $E_f \leq C\sqrt{\tau_{\max}}(M_{f,0} + \|m_{f,1}\|_{L^2(0,T)})$ for $f^n = f(t_n)$. This additional regularity is required only for that choice of load. The half-order time bound in the theorem allows a gradient jump to move with the interface. A first-order energy estimate is not inferred from the formal order of backward Euler.

8 Manufactured moving-ellipse verification

We use a continuous manufactured solution to check the implementation, following the verification principle of Salari and Knupp [34]. The body both translates and changes its orientation; it is not a disk whose rotation leaves the geometric interface fixed. The calculations test the stated Eulerian method and its identities. They do not establish an asymptotic rate, adaptive optimality, or a quadrature-consistency theorem.

8.1 Exact geometry, velocity, forcing, and reaction

Let $T = 0.5$, $\nu = 0.1$, and let the reference body be an ellipse with semiaxes $a_1 = 0.19$ and $a_2 = 0.115$. Prescribe

$$c(t) = (0.5 + 0.04 \sin(2t), 0.5 + 0.03 \sin(3t)), \quad \theta(t) = 0.6 \sin(2t). \quad (64)$$

For $q = Q(\theta(t))^T(x - c(t))$, put

$$\phi(t, x) = q_1^2/a_1^2 + q_2^2/a_2^2 - 1, \quad W(x) = x_1^2(1 - x_1)^2 x_2^2(1 - x_2)^2, \quad \alpha(t) = 0.3 + 0.2 \sin(2t). \quad (65)$$

Use the fixed-origin lift of (33), with a radial cutoff about $x_* = (1/2, 1/2)$ equal to one for $r \leq 0.40$ and zero for $r \geq 0.49$. In the transition, the implemented C^5 cutoff is

$$\chi_*(r) = 1 - 2772 \int_0^{s(r)} z^5(1 - z)^5 dz, \quad s(r) = \frac{r^2 - 0.40^2}{0.49^2 - 0.40^2}.$$

This has the finite derivative bounds needed by the interpolation estimates. Define the exact solution by

$$\psi_w(t, x) = \alpha(t)W(x) \max\{\phi(t, x), 0\}^2, \quad w = \text{curl } \psi_w, \quad u = g + w, \quad p = 0. \quad (66)$$

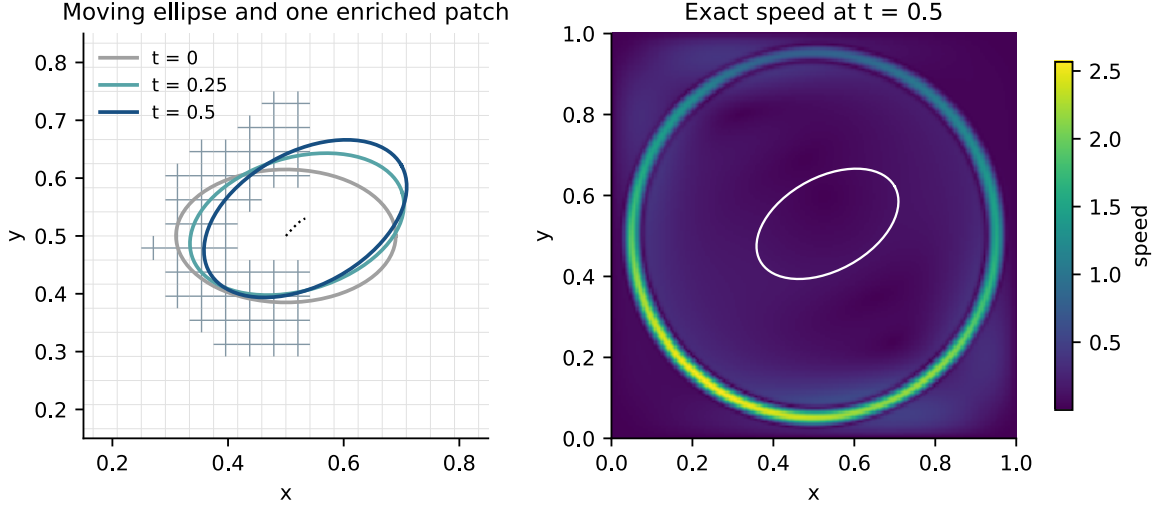


Figure 1: Left: the translating and rotating ellipse at three times, with a representative enriched Cartesian patch at $t = T$ for $M = 24$. Right: speed of the continuous exact velocity. The narrow outer cutoff transition contributes appreciably to the coarse-grid approximation error.

The field is divergence free, is zero on the outer wall, and equals the prescribed rigid velocity in the body. Its exterior interface shear is nonzero: the residual velocity vanishes to first order in the normal distance, rather than on an open collar. All spatial derivatives and the Eulerian time derivative are evaluated analytically from these formulas. The bulk forcing is $f = \partial_t u - \nu \Delta u$ separately in $F(t)$ and $B(t)$. In particular, its interior value is exactly (5). The initial condition is the nonzero exact field at $t = 0$; its discrete value is the constrained affine L^2 projection.

For an independent reaction check, let $n_B = \nabla \phi / |\nabla \phi|$ point from the body into the fluid. The sign convention in (4) gives

$$\lambda = -2\nu\alpha(t)W|\nabla\phi|^2 Jn_B \quad \text{on } \Gamma(t). \quad (67)$$

The bulk weak residual is therefore $-\int_{\Gamma} \lambda \cdot v$. Four smooth solenoidal test velocities with zero outer trace verify this identity with relative discrepancy 2.08×10^{-8} and maximum absolute discrepancy 8.48×10^{-12} . This check uses continuous fields and surface integrals, not the discrete system matrices.

8.2 Discrete realization and diagnostics

The coarse grid has $h = 1/M$. Each enriched coarse cell is bisected in both coordinates and contributes the four Hermite degrees of freedom at its center; the complete coarse space remains present. Dynamic runs alternate one-sided patches near the moving ellipse. Specifically, cell centers satisfy $|\phi| < 1.7h \max\{|\nabla\phi|, 1\}$ and either $x_1 < c_1(t) + h/4$ or $x_1 > c_1(t) - h/4$. A frozen-enrichment control retains its initial Cartesian patch throughout the run, while its interface constraints still move.

We take $\varepsilon = 0.5h$ and partition the ellipse into $P = \lceil |\Gamma|/h \rceil$ equal angular intervals. Their arclengths are nonuniform, with $\ell \leq 2\pi a_1/P$. A product of compact smooth one-dimensional bump kernels is supported in a square of half-width $\varepsilon/\sqrt{2}$, hence inside the prescribed radius- ε ball. Kernel integration is split at Cartesian half-cell boundaries and their reflections about

each kernel center. Positive weights are normalized on each axis and preserve zero first moment. This preserves the rigid moments even though the remaining integrals are approximate.

The fixed lift is reproduced on the whole support region, not merely at panel averages. Indeed, for every time in $[0, T]$, every coarse cell meeting any kernel support lies within radius

$$0.05 + 0.19 + \varepsilon + \sqrt{2}h \leq 0.399518 < 0.40 \quad (M \geq 12).$$

Thus the streamfunction is quadratic throughout each such cell. A separate check at 46,656 support points over 81 time nodes for each of $M = 12, 24, 48$ gives a maximum rigid-velocity interpolation error 2.34×10^{-15} .

Bulk mass and stiffness matrices use polynomial-exact four-point Gauss rules in each coordinate of the Cartesian half-cells. Loads and exact error integrals use six-point rules on uncut half-cells and ten-point nested rules on cells split at the exact ellipse and cutoff-circle intersections. The default kernel and angular panel rules have orders six and four. Sharp traces use ten-point rules on exact ellipse arcs split at Cartesian crossings. A stronger-quadrature run increases these orders to $(10, 16)$ for bulk/cut integration, $(10, 8)$ for kernel/panel integration, and 16 for the sharp trace.

The implemented load is $f(t_n)$, rather than its exact interval average. The centered right-hand side retains the exact difference $-(g_h(t_n) - g_h(t_{n-1}))/\tau_n$ and the endpoint viscous lift. Thus it is algebraically equivalent to the true affine scheme. A separate dense solve of the untransformed affine saddle system agrees to relative coefficient error 2.82×10^{-15} . Endpoint load quadrature is an additional load-consistency term in the theorem. The manufactured force is bounded and piecewise smooth across a moving smooth interface, so its swept-strip time modulus yields a vanishing load defect; we do not set that defect to zero.

Write $u_h^n = g_h(t_n) + y_h^n$. We record final relative errors $E_0 = \|u_h^N - u(T)\|_0 / \|u(T)\|_0$ and $E_1 = \|u_h^N - u(T)\|_U / \|u(T)\|_U$, and the sharp-slip diagnostic

$$D_\Gamma = \left(\sum_n \tau_n \|\gamma_{t_n} u_h^n - b(t_n)\|_{L^2(\Gamma(t_n))}^2 \right)^{1/2}.$$

This last quantity samples the moving interface at time nodes; it is not the continuous-in-time norm on the full moving surface. The reported projection loss is the centered quantity $\sum_n \|y_h^{n-1} - P^n y_h^{n-1}\|_0^2$, not the distance of the physical old velocity to the new affine space.

8.3 Results and scope of the checks

Table 1 and Figure 2 use $N = 80$ equal steps and switch the patch side at every step. The error components are $e_g = \|g_h(T) - g(T)\|_0$ and $e_y = \|y_h^N - w(T)\|_0$. The total error decreases from 49.6038% to 1.9965% over this finite-resolution sequence. The large coarsest-grid error is substantially associated with the narrow lift cutoff: e_g decreases from 0.264 to 0.0104, while e_y decreases from 0.102 to 0.00223. These are not independent orthogonal error components, so their squares should not be added. At the finest grid, the exact residual has norm 0.1506 and omitting it entirely would give 31.53% relative error; the computation is not merely reproducing the lift. The three resolutions do not identify an asymptotic rate or quantify an advantage of adaptive marking.

The four additional runs hold $M = 24$ fixed: frozen enrichment, patch-side switching every eight steps, $N = 160$, and stronger quadrature. The first three $N = 80$ mesh histories have respectively 0, 24, and 80 actual enrichment changes; all have 80 geometric updates. Their final relative errors agree within 1.8×10^{-7} in absolute value. Figure 3 shows that the

Table 1: Moving-ellipse verification with $N = 80$, $T = 0.5$. E_0, E_1 are shown as percentages; the remaining columns are absolute norms.

M	$100E_0$	$100E_1$	D_Γ	e_g	e_y
12	49.6038	78.2956	1.31×10^{-3}	2.64×10^{-1}	1.02×10^{-1}
24	12.6951	36.5950	4.22×10^{-4}	7.11×10^{-2}	2.42×10^{-2}
48	1.9965	10.9414	1.87×10^{-4}	1.04×10^{-2}	2.23×10^{-3}

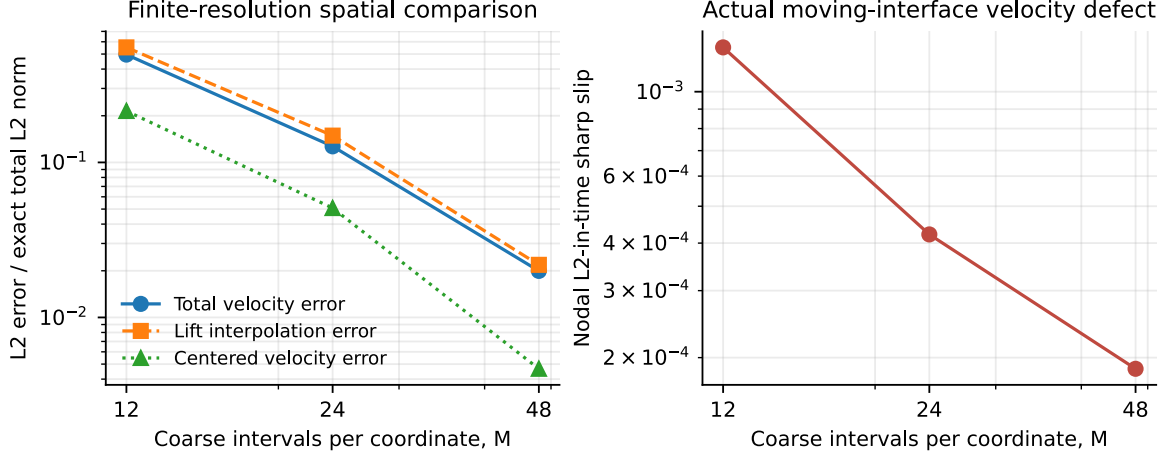


Figure 2: Left: total, lift, and centered velocity errors, all normalized by the exact total L^2 norm at T . Right: nodal sharp-slip diagnostic. The curves are observed errors, not claimed convergence-rate fits.

centered projection loss can be positive even with frozen enrichment because the constraints themselves move. Halving the time step changes the reconstructed final velocity by relative bulk L^2 difference 1.60×10^{-4} ; the corresponding difference after increasing quadrature is 1.15×10^{-5} . The spatial error dominates this temporal comparison, which is not presented as a temporal-rate study.

Across all seven cases, the maximum absolute centered energy residual is 2.48×10^{-15} , the full-velocity work-identity residual is 7.99×10^{-15} , and the affine constraint residual is 3.17×10^{-16} . These algebraic checks use the same assembled quadrature on both sides of each identity. They do not certify the accuracy of that quadrature. Independent central differences of the continuous velocity check its analytic gradient, Laplacian, and time derivative to relative discrepancies 1.24×10^{-9} , 3.91×10^{-7} , and 9.61×10^{-11} , respectively. The archive contains all seven configurations, raw coefficient and history arrays, the exact-field, traction, affine-step and lift checks, and figure/table generators. No full PDE proof or numerical solver correctness claim is inferred from the Lean companion.

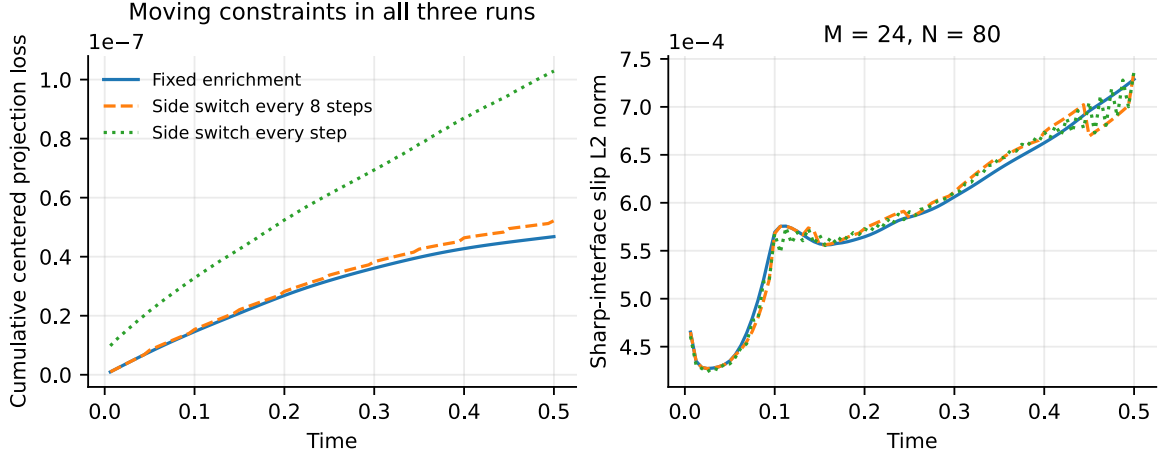


Figure 3: Centered projection loss and sharp slip for three refinement histories at $M = 24$. The geometric interface moves in every run. Nearly overlapping slip curves indicate small sensitivity in this test, not a theorem about arbitrary approximate transfers.

9 Scope, limitations, and the route to coupled motion

The result covers a general smooth prescribed translation and rotation of one planar rigid body, including noncircular shapes, as long as a positive wall separation persists. It permits geometry, regularization, surface panels, and fine-enrichment patches to change at every time node. No time-step restriction counts those changes. The interface and all integrals in the theorem are exact, and the spatial limit refines the retained coarse grid. Holding that coarse grid fixed while only changing local patches is not a convergence sequence covered by the theorem.

The mathematical assumptions are stronger than those of a general weak moving-domain Stokes problem. The quantitative conclusion assumes the stated piecewise spatial regularity, Eulerian time regularity and continuous reaction bound. We do not prove those norms are uniformly bounded for arbitrary data or near a collision. The contribution is a constructive approximation and error argument under those hypotheses, rather than a new general well-posedness theorem. In particular, the half-order time bound reflects a moving interface at which gradients may jump. It does not establish a sharp temporal rate.

A fixed cutoff on the swept region is legitimate for the *lift*: a rigid velocity is affine in space and can be reproduced on every relevant kernel support. The same construction would not be legitimate for a *zero-velocity comparison*: forcing it to vanish on the entire swept region would suppress actual fluid motion. The moving collar of Lemma 4.1 is needed precisely to avoid that loss of approximation. Its constants depend on the prescribed motion and interface geometry, but not on how a user chooses fine patches.

Physical slip between nodes is distinct from nodal constraint feasibility. Even a vanishing nodal moment residual does not directly control the trace on $\Gamma(t)$ for $t \neq t_n$. Lemma 3.2 supplies that missing geometric term. Similarly, exact multiplier-row feasibility does not imply a uniform multiplier inf-sup constant or good finite-precision conditioning. Our proofs concern the uniquely determined constrained velocity and do not establish multiplier convergence.

The numerical realization evaluates convolution, surface constraints and cut-domain forces by quadrature. It therefore adds consistency terms absent from the theorem. The companion reports quadrature sensitivity and algebraic residuals rather than treating those computations as a proof for arbitrary marker kernels, approximate projections, mass lumping, geometry

approximations or conventional MAC coarse–fine stencils. There is also no error-based marking theorem or adaptive optimality assertion. Several neighboring unfitted methods analyze geometric errors that are deliberately excluded here [15, 17].

For fluid-driven motion, the trajectory is an unknown rather than data. The current constants depend on that trajectory, and the velocity theorem alone gives no control of its position or orientation error. A subsequent analysis would need coupled translational and angular momentum equations, consistent hydrodynamic work, correction for the fictitious interior, and a comparison argument for two different body configurations. A collision-free interval with positive mass and inertia is a natural starting point, but no two-way coupling theorem is claimed in this paper. Flexible bodies, multiple interacting bodies, nonlinear Navier–Stokes flow and free surfaces are likewise additional problems.

10 Formal verification and reproducibility

The Lean companion was compiled with Lean 4.19.0 [35] and a pinned Mathlib dependency [36]. It contains 35 checked theorems: the 26 unchanged algebraic results from Part I and nine new statements in `MovingComparison.lean`. The new module formalizes current-space projection against a moving comparison, the equivalent physical affine weak equation, one-step and cumulative error estimates retaining the full error jump, and finite weighted Cauchy estimates. In particular, the previous comparison is not assumed to lie in the current subspace. The cumulative bound is derived from an explicit weak scheme and an explicit residual estimate, rather than asserted from the PDE theorem.

Component	Verified scope
Moving comparison	Actual orthogonal projection; ordinary comparison time difference; no nesting or previous-membership assumption.
Changing affine lift	Equivalence between the physical affine scheme and the centered error equation with reference $g_h^n + z_h^n$.
Error energy	Full squared error jumps and dissipation in one-step and summed estimates, conditional on the stated analytic residual bound.
Weighted sums	Finite weighted Cauchy and discrete trajectory-increment inequalities.
Inherited results	Linear/affine projection, step energy and discrete interface-error composition from Part I.

The explicit axiom audit lists only `propext`, `Classical.choice` and `Quot.sound`; there are no `sorry` proofs or user-declared axioms. Source hashes, commands and actual build logs accompany the release. This does not formalize Sobolev traces, moving geometry, the collar or lift construction, continuous time regularity, or the full PDE convergence proof. Nor does Lean certify the numerical solver.

The repository includes this manuscript, BibTeX records, numerical source and raw results, figure-generation scripts, the Lean project and pinned dependencies, local validation reports, a release manifest, and a frozen Part I v0.3.2 companion. Reproduction commands write new output to `build/` instead of overwriting archived data. A clean-source paper build and citation-order audit are part of the package checks. GitHub Actions configuration is provided, but a hosted CI run is not claimed before uploading the repository.

The detailed AI-use and human-understanding declaration at the end of this manuscript applies to this work. The internal mathematical and implementation reviews are not external peer review.

AI use, human understanding, and verification limits

The research direction, scope, and revision requests were provided by Sida He. OpenAI ChatGPT/Codex agents generated most of the mathematical derivations, manuscript text, numerical implementation, and Lean formalization, and performed automated literature searches and internal checks. This work is therefore disclosed as primarily AI-generated text, rather than merely language editing.

At the preparation date of this version, the author reports that he does not understand the mathematical proofs and has not independently verified them. The intended Hexagon disclosure is *Human understanding: no parts*, with respect to the proofs. No independent human expert verification or external peer review of the proofs is claimed. Internal reviews by additional AI agents do not constitute independent human mathematical review.

The Lean checks described above cover selected algebraic identities and finite-step arguments only. They do not formalize the full PDE convergence theorems or establish their analytic hypotheses. Numerical experiments are implementation checks and illustrations, not proofs. The original proof source, dependency pins, numerical results, and recorded build logs are retained in the companion reproducibility material. This preparation revision does not claim a new Lean compilation or a new numerical experiment. The manuscript is offered for critical scrutiny; its correctness and novelty are not certified by these disclosures or by repository acceptance.

Relationship to companion manuscripts. The other parts of this series are Part I [2], and Part III [37]. They are cited as unpublished companion manuscripts; no public identifier or acceptance is claimed. Each part has its own hypotheses and scope. The series is not asserted to be a chain of special-case theorems.

Availability of reproducibility material. A separate versioned package has been prepared with the Lean source, dependency lockfiles, numerical code, and archived results. No public repository URL or archive identifier has been assigned to that package at the preparation date of this manuscript. The mathematical source submitted to Hexagon does not itself contain the Lean project.

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