

The sharp iid Berry–Esseen inequality: Analytic reductions and exact certificate proof*

Bartosz Kołodziejek

Kacper Rodziewicz

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Abstract

We prove that the optimal constant in the Berry–Esseen inequality for sums of i.i.d. random variables with finite third moment equals Esseen’s constant $C_E = (3 + \sqrt{10})/(6\sqrt{2\pi})$, the limiting value for one fixed two-point law, and that $C_n \leq C_E$ holds at every sample size $n \geq 1$. For fixed $n \geq 2$ an extremal law exists and satisfies a first-order contact condition. Integrating this condition against the zero-bias density gives an exact identity that links the normal-approximation error to a Fourier remainder and to a nonnegative moment defect, which vanishes only at the extremal two-point law. Conditioning on the signs of the summands writes the sum exactly as a binomial mixture, so that a nearly two-point law can be compared with its Bernoulli skeleton. A one-step smoothing recurrence controls the first failure index, sharp predecessor bounds give an elementary support estimate, retaining the score excess sharpens the defect bound, and an exact curvature formula simplifies the conditional-count comparison. Combining these analytic reductions with the complete thirteen-role finite and effective-tail certificate family verified end-to-end (`full_pass: "ESTABLISHED"`) proves $C_n \leq C_E$ for every integer $n \geq 1$ and establishes $C^* = C_E$ unconditionally. All certificates, native checkers, and a single-command multi-core Docker verification environment are publicly archived at <https://github.com/rodziewicz/berry-esseen-certificates>.

*This paper was written with substantial assistance from AI systems; see the end of Section 8.

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1 The problem and the approach

The central limit theorem describes the shape of a long sum. The Berry–Esseen inequality asks how far that description can be from the truth at a specified sample size. Even when the summands have the same distribution, the best universal constant is a delicate question: the largest errors come from the interaction of skewness with the jumps of a lattice distribution. A proof of the sharp inequality must control both effects uniformly in the law and in the sample size.

Let $\mathcal{P}$ denote the probability laws on $\mathbb{R}$ satisfying $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, and $\rho(P) := \mathbb{E}|X|^3 < \infty$. For independent $X_1, \dots, X_n$ with common law P , write $S_n = X_1 + \dots + X_n$ and

$$\Delta_n(P) = \sup_{H \in \mathbb{R}} |\mathbb{P}_P(S_n \leq H) - \Phi(H/\sqrt{n})|,$$

$$R_n(P) = \frac{\sqrt{n} \Delta_n(P)}{\rho(P)}, \quad C_n = \sup_{P \in \mathcal{P}} R_n(P), \quad C^* = \sup_{n \geq 1} C_n.$$

Here Φ and ϕ are the standard normal distribution function and density. The conjecture is

$$C^* = C_E, \quad C_E = \frac{3 + \sqrt{10}}{6\sqrt{2\pi}} = 0.409732\dots \quad (1)$$

Throughout, H is a threshold for the *raw sum* S_n and $H/\sqrt{n}$ its standardized version. All terminating decimals in numerical inequalities mean exact rational numbers.

The normalization costs no generality. If Z has mean μ , variance $\varsigma^2 > 0$, and finite third absolute central moment β_3 , apply the definitions to $X = (Z - \mu)/\varsigma$. Then $\rho = \beta_3/\varsigma^3$ and

$$\mathbb{P}\left(\frac{\sum_{i=1}^n Z_i - n\mu}{\varsigma\sqrt{n}} \leq x\right) = \mathbb{P}(S_n \leq \sqrt{n}x).$$

Thus $C_n \leq C_E$ is precisely the usual sharp inequality at index n .

1.1 From Berry and Esseen to the sharp-constant problem

Berry [8] and Esseen [10, 11] established quantitative normal approximation of order $n^{-1/2}$ under a finite third absolute moment. Their results initiated the search for the smallest constant in a bound that is uniform over the summand law. Esseen’s later moment analysis [12] identified the lower constant in (1). The conjecture that this lower constant is also the universal upper constant is associated with Zolotarev’s work of 1966–1967 [34, 35].

The two-point example explains the value of the constant. Put

$$p_* = (4 - \sqrt{10})/2, \quad q_* = 1 - p_*, \quad X = (B - p_*)/\sqrt{p_*q_*}, \quad B \sim \text{Bernoulli}(p_*).$$

For this fixed law, $\rho = (p_*^2 + q_*^2)/\sqrt{p_*q_*}$ and the classical lattice asymptotic gives

$$\lim_{n \rightarrow \infty} R_n(P) = \frac{3 + q_* - p_*}{6\sqrt{2\pi}(p_*^2 + q_*^2)} = C_E. \quad (2)$$

Indeed $p_*^2 + q_*^2 = 10 - 3\sqrt{10}$ and $(3 + q_* - p_*)/(p_*^2 + q_*^2) = 3 + \sqrt{10}$. Consequently every constant smaller than C_E fails for a sufficiently large sum of this one fixed Bernoulli law. This proves $C^* \geq C_E$; it supplies no upper bound for other laws. Hipp and Mattner settled the symmetric binomial case [15]; Schulz’s dissertation [21] proves the sharp upper bound for the entire binomial family at every sample size. We use Schulz’s theorem for the noiseless case in Section 6.

Upper estimates developed through increasingly precise smoothing and characteristic-function inequalities. Zolotarev’s work and van Beek’s Fourier analysis [31, 35] are early milestones; Prawitz’s one-sided smoothing kernel [19] remains useful in modern computations. Shiganov obtained the iid bound 0.7655 in work originally published in 1982 [27, 30]. Subsequent improvements include Shevtsova’s bound 0.7056 [25], the bound 0.5129 of Korolev and Shevtsova [16], Tyurin’s bound 0.4785 obtained using zero-bias estimates and characteristic-function bounds [29], and Shevtsova’s bounds 0.4748 and 0.4690 [22, 26]. Their structural forms are more informative than the constants alone. For example, the inequalities

$$\sqrt{n} \Delta_n(P) \leq \min\{0.4690\rho, 0.3322(\rho + 0.429), 0.3031(\rho + 0.646)\} \quad (3)$$

hold for every $P \in \mathcal{P}$ and $n \geq 1$ [26]. If $R_n(P)$ is near C_E , they bound ρ from above. This supplies the first quantitative moment cap for the support calculation.

1.2 Recent manuscripts, repositories, and formalization

Several recent projects combine analytic inequalities with extensive computation. Xiao and Li report a global iid upper bound of 0.4395; Lin reports 0.423 using a refinement of their analytic framework [17, 32]. Both supply Lean developments and exact finite certificates. Tao and contributors’ *Optimization Constants in Mathematics*, especially Problem 19a, records this work and its verification status [28].

He and Cheng’s preprint [14] reports the sharp inequality for every

$$n \geq 2\lceil \exp(10^{17}) \rceil,$$

with a threshold independent of the summand law. This uniform quantifier is essential. A theorem for each fixed law, with a law-dependent threshold, would not exclude a sequence of counterexamples whose laws change with n . A finite but astronomically large range of n remains.

Bell’s collection treats independent, not necessarily identically distributed, summands. It reports a universal upper bound below 0.44988, results for finite arrays, and sharp bounds when the Lyapunov ratio is small [3–5]; its structural papers study extremizers in that setting [2, 6].

In Lean, Yan’s *ProbabilityApproximation* formalizes classical nonuniform and multivariate normal-approximation theorems [33], and the *BerryEsseenLean* project formalizes an eventual-sharpness framework accompanying an earlier version of He and Cheng, relative to explicitly listed external mathematical inputs [1].

1.3 The present approach and its scope

The analytic starting point is a maximizing law at a fixed sample size. Its first variation gives a contact inequality at every support point. We integrate that inequality against the density of the zero-bias transform, introduced by Goldstein and Reinert [13]. The resulting identity connects the normal-approximation error to a nonnegative moment defect and a Fourier remainder. The defect singles out Esseen’s two-point law; its quantitative form controls conditional noise and the displacement of the two parts of the support at the same time. The sign decomposition then turns the original sum into an exact binomial mixture. Related zero-bias methods appear in Tyurin’s work and, for independent summands, in Bell’s transport paper [7, 29].

The reductions developed here address four parts of the argument. The one-step recurrence of Theorem 3.2 controls the first index $n \geq 42$ with $C_n \geq C_E$, assuming a sharp base through 41; Lemma 3.3 then gives zero predecessor loss and $0 \leq n(C_n - C_E) < 3003/10250$.

Proposition 4.2 uses this zero loss to prove support bounds by boundary estimates and monotonicity. Retaining $n(C_n - C_E)\rho$ in the conserved identity sharpens the joint bound on noise and endpoint displacement (Corollary 5.8 and Lemma 5.9). Finally, Theorem 6.3 replaces the curvature maximization by at most three explicit candidates. The support and resource statements keep their stated moment and Fourier hypotheses.

The all-index theorem is now established unconditionally. Theorem 7.3 shows first that a sharp base $C_j \leq C_E$ for $1 \leq j \leq 41$, together with the equality-inclusive local exclusion of Assumption 7.2, implies $C_n < C_E$ for $n \geq 42$ and $C^* = C_E$; and second, that combining the analytic reductions proved here and in the appendices with the complete thirteen-role finite and effective-tail certificate family verified end-to-end (`full_pass: "ESTABLISHED"`) proves

$$C_n \leq C_E \quad (n \geq 1), \quad C^* = C_E = \frac{3 + \sqrt{10}}{6\sqrt{2\pi}}.$$

Establishing the all-index bound combines these analytic implications with complete finite certificates verified across all thirteen index roles ($1 \leq n \leq 48$ and $n \geq 49$). Section 7 records the global theorem, the thirteen-role partition, and the verified `full_pass` execution status in one place.

An earlier, AI-assisted working draft is available on Hexagon [20]; the present paper reorganizes that material and develops the reductions above. None of the bounds reported in the recent works cited above is used here.

1.4 A guide to the argument

The proof has two distinct jobs. It must show that a law with a large Berry–Esseen ratio has nearly two-point geometry, and then compare its sum with the corresponding binomial sum accurately enough to detect even a very small improvement. The first job uses contact and Fourier analysis; the second uses the discrete count structure.

Start at the first failure of the strict bound.

Assume a sharp base through 41, and choose the least $n \geq 42$ with $C_n \geq C_E$. Then $C_{n-1}, C_{n-2} \leq C_E \leq C_n$, so $d_n = n(C_{n-1} - C_n)_+ = 0$. The uniform smoothing bound is

$$C_n \leq \sqrt{\frac{n}{n-1}} C_{n-1} + \frac{\sqrt{n}}{15(n-1)^{3/2}} + \frac{2\sqrt{n}}{25(n-1)^2}.$$

It gives $0 \leq n(C_n - C_E) < 3003/10250 < 0.3$. The additive remainder in this normalized inequality is of order $1/n$. Thus the current ratio is close to C_E , and both deleted indices have sharp bounds. The floor 42 is the one used by the proposed local comparison; its choice fixes the finite base required by this route.

Use extremality to restrict the support.

The supremum at a fixed n is attained by a law P and a closed threshold H . Adding a small atom at y , then standardizing, gives the first-variation inequality

$$\mathbb{P}(S_{n-1} \leq H - y) \leq Q_H(y), \quad \mathbb{P}(S_{n-1} \leq H - y) = Q_H(y) \quad (y \in \text{supp } P),$$

where Q_H is the explicit piecewise cubic function in (5). Its cubic growth first gives compact support. The quantitative step estimates the Gaussian part of contact at a radius R , then proves that the upper bound decreases along both exterior rays. Since the predecessor term vanishes, this yields the three support bounds of Section 4, provided their moment bounds have been proved in the order in which they are used.

Measure the departure from the extremal two-point law.

Let $p = \mathbb{P}(X > 0)$, $q = 1 - p$, and $m = \mathbb{E}X_+ = \mathbb{E}X_-$. Replacing each sign class by its conditional mean leaves a centered conditional fluctuation. In suitable units this is

$$X = \ell(B_0 + U - p), \quad B_0 = \mathbf{1}_{\{X > 0\}}, \quad \mathbb{E}(U \mid B_0) = 0, \quad \ell = m/(pq).$$

The physical fluctuation is $W = \ell U$; the count-scale noise is $s = \mathbb{E}U^2$, and $\lambda = ns$ is its variance over n draws.

The zero-bias density is $\tau(t) = \mathbb{E}[X \mathbf{1}_{\{X \geq t\}}]$. It is constant on each gap (a, b) in the support. Consequently

$$J = \sum_{(a,b)} \tau_{a,b} (b - a)^2$$

is the expected length of the gap containing a zero-biased observation, with gap length zero on the support. For a two-point law this is just its span ℓ , and for the law of (2) the quantity $\phi(0)(\gamma + 3\ell)/(6\rho)$ is exactly the limit displayed there. The inequality

$$D = (3 + \sqrt{10})\rho - \gamma - 3J \geq 0$$

says that $\phi(0)(\gamma + 3J)/6$, which extends this first-order lattice term to arbitrary support gaps, never exceeds $C_E \rho$. Its equality case is exactly Esseen's oriented two-point law (Corollary 5.2). More usefully, its nonnegative decomposition bounds the sign probability, inward endpoint movement and conditional variance simultaneously.

Connect the defect to the error of normal approximation.

Integrate the contact slack against τ . Evaluating this integral through support gaps and through the zero-bias transform of the sum gives the conserved identity of Section 5.4. The identity contains the score excess $n(C_n - C_E)\rho$, the defect cost and a nonnegative leakage term. A Fourier remainder is the quantity to be bounded. Once that remainder and the necessary sign conditions have been controlled, one obtains a resource inequality of the form

$$n(C_n - C_E)\rho + \kappa nD + n^{3/2}L \leq \mathcal{B}(H).$$

Together with the defect decomposition, this inequality bounds the noise and the inward displacement. Retaining the first term, instead of discarding it, sharpens the budget.

Compare sums with the same Bernoulli skeleton.

Conditioning on $K = \sum B_{0,i} \sim \text{Bin}(n, p)$ gives

$$\mathbb{P}(S_n \leq H) = \sum_{j=0}^n \binom{n}{j} p^j q^{n-j} \mathbb{P}(V_j \leq k + u - j), \quad np + H/\ell = k + u,$$

where V_j is the sum of j independent copies of $U \mid B_0 = 1$ and $n - j$ independent copies of $U \mid B_0 = 0$. Subtracting the binomial distribution function gives the exact score identity (38). At $\lambda = 0$ the original law is two-point, so Schulz's theorem applies directly. At positive noise, the outgoing and incoming probabilities must be estimated with errors proportional to λ , uniformly as λ decreases to zero. The exact identity makes this comparison possible; the required relative bounds are separate steps in the argument. The three-candidate curvature formula simplifies one of them.

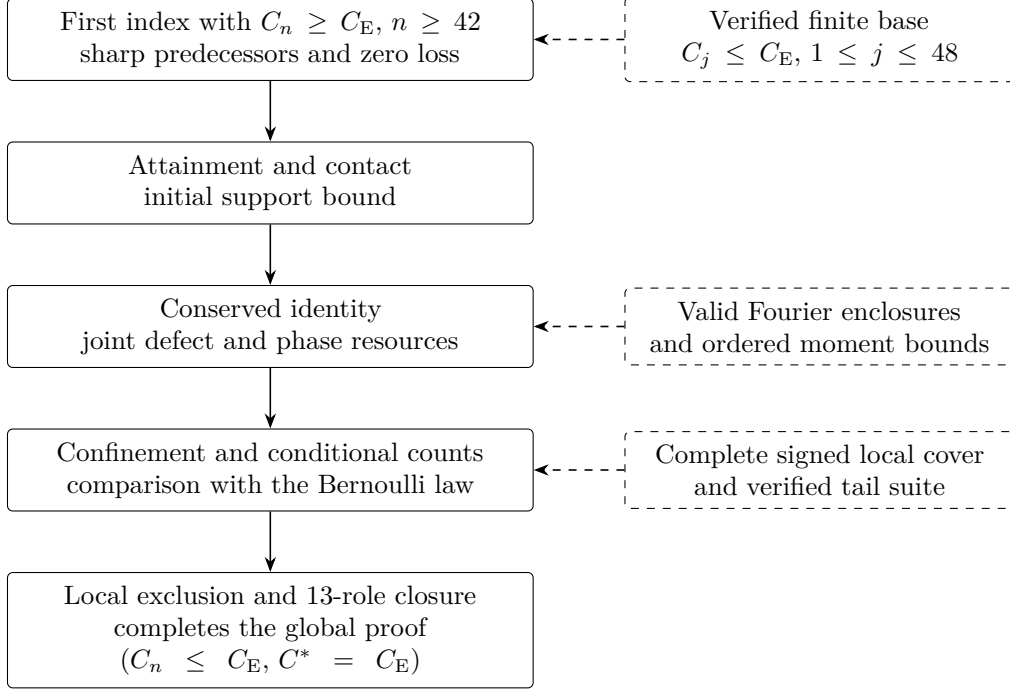


Figure 1: The route to the local exclusion and global closure. Dashed boxes show the certificate inputs; their completed **full_pass** verification is recorded in Section 7.

1.5 Notation and reading order

The main variables are listed below. Auxiliary variables used only in a single estimate are defined where they occur.

Symbol	Meaning
$H, z = H/\sqrt{n}$	Raw and standardized thresholds.
p, q, m	Sign probabilities and common one-sided first moment.
ρ, γ, β	$\mathbb{E} X ^3$, $\mathbb{E}X^3$, and $\mathbb{E}[X X]$.
ℓ	Span between conditional means; $\ell = m/(pq)$.
$W, U = W/\ell$	Physical and count-scale conditional noise.
θ, s, λ	$\mathbb{E}W^2$, $\mathbb{E}U^2$, and ns ; $\theta = s/(pq + s)$.
D, d_n, d_Y	Moment defect, predecessor score decrement, and inward count-gap displacement.
k, u	Nearest integer count and phase: $np + H/\ell = k + u$, $ u \leq 1/2$.
$\sigma = \sqrt{10}$	Constant in the defect decomposition.

Sections 2–6 contain the analytic argument, and Section 7 the main global theorem and the **full_pass** verification of the thirteen certificate roles. The appendices contain the longer Fourier estimates, the finite-index arguments and the explicit comparison formulas.

2 Extremizers and contact on the closed support

We first show that C_n is attained, by a compactness argument over all of $\mathcal{P}$ that allows atoms at the threshold.

Use $\mathcal{P}$, $F_{n,P}(H) = \mathbb{P}(S_n \leq H)$, $R_n(P)$ and C_n as in Section 1. Throughout the article,

$$\beta = \mathbb{E}[X|X], \quad \gamma = \mathbb{E}X^3$$

are distinct signed moments.

2.1 Fixed-index attainment

Lemma 2.1 (Orientation and fixed-index attainment). *For every fixed integer $n \geq 1$, $0 < C_n \leq \sqrt{n}$. There exist $P \in \mathcal{P}$ and $H \in \mathbb{R}$ such that*

$$\frac{\sqrt{n}}{\rho(P)} \{F_{n,P}(H) - \Phi(H/\sqrt{n})\} = C_n.$$

Thus a maximum is attained as a positive discrepancy for a closed event.

Proof. Lyapunov's inequality gives $\rho \geq (\mathbb{E}X^2)^{3/2} = 1$, so $C_n \leq \sqrt{n}$. For the symmetric law on $\{-1, 1\}$, the choice $H = n$ gives $F_{n,P}(H) = 1 > \Phi(\sqrt{n})$, proving positivity.

It suffices to maximize positive closed discrepancies over all laws. Indeed, if $F_{n,P}(H) - \Phi(H/\sqrt{n}) < 0$, reflection gives

$$\mathbb{P}(-S_n < -H) - \Phi(-H/\sqrt{n}) = \Phi(H/\sqrt{n}) - F_{n,P}(H).$$

Closed thresholds increasing to $-H$ converge to the strict event on the left. Continuity of Φ gives the same supremum using positive closed discrepancies of the reflected law. Reflection preserves all three defining conditions of $\mathcal{P}$.

Choose pairs (P_j, H_j) with positive scores $r_j \rightarrow C_n$ and $r_j \geq \eta > 0$. Since a discrepancy is at most one,

$$1 \leq \rho_j \leq \sqrt{n}/\eta =: M.$$

The unit second moments make (P_j) tight. By weak subsequence selection on $\mathbb{R}$, and then compactness of $[1, M]$, we may assume $P_j \Rightarrow P$ and $\rho_j \rightarrow b \in [1, M]$. The uniform tail estimates

$$\int_{|x|>A} x^2 dP_j \leq M/A, \quad \int_{|x|>A} |x| dP_j \leq M/A^2$$

allow bounded continuous truncations to pass the first two moments to the limit. Hence $\mathbb{E}_P X = 0$ and $\mathbb{E}_P X^2 = 1$. Lower semicontinuity of the third absolute moment gives $1 \leq \rho(P) \leq b < \infty$.

The thresholds are bounded as well. If $H_j < 0$, then $F_{n,P_j}(H_j) \leq n/H_j^2$ by Chebyshev's inequality. If $H_j > 0$, then the positive discrepancy is at most $1 - \Phi(H_j/\sqrt{n}) \leq n/H_j^2$. As each discrepancy is at least $\eta/\sqrt{n}$, both cases give $|H_j| \leq n^{3/4}/\sqrt{\eta}$. Take a further subsequence with $H_j \rightarrow H$.

Because n is fixed, weak convergence of the marginals implies weak convergence of their n -fold products, and the continuous sum map gives $P_j^{*n} \Rightarrow P^{*n}$. For each $\varepsilon > 0$, eventually $H_j \leq H + \varepsilon$. The closed-set inequality for weak convergence gives

$$\limsup_j F_{n,P_j}(H_j) \leq F_{n,P}(H + \varepsilon).$$

Letting $\varepsilon \downarrow 0$ uses right continuity of the closed CDF, not continuity at H . It follows that

$$F_{n,P}(H) - \Phi(H/\sqrt{n}) \geq C_n b / \sqrt{n}.$$

Therefore

$$C_n \geq R_n(P) \geq \frac{\sqrt{n}\{F_{n,P}(H) - \Phi(H/\sqrt{n})\}}{\rho(P)} \geq C_n \frac{b}{\rho(P)} \geq C_n.$$

All inequalities are equalities, proving attainment and $\rho(P) = b$. In particular, extremality forces $\rho(P) = b$, so the third moments converge without assuming uniform integrability of their third powers. $\square$

2.2 Contact on the closed support

We now fix one attaining pair (P, H) . The contact inequality comes from contaminating the raw law at a point y and then standardizing it. Keeping the raw event fixed is essential: differentiating a moving distribution function through an atom would not be justified.

Lemma 2.2 (Atom-safe contact). *Let $n \geq 2$, $C = C_n$, $F = F_{n,P}(H)$ and $z = H/\sqrt{n}$ at an attaining pair. Define $I : \mathbb{R} \rightarrow \mathbb{R}$ by*

$$\begin{aligned} \rho I(y) = & n^{3/2} \{F_{n-1,P}(H-y) - F\} + ny\phi(z) + \frac{H\phi(z)}{2}(y^2 - 1) \\ & + \frac{3}{2}C\rho(y^2 - 1) - C\{|y|^3 - \rho - 3y\beta\}. \end{aligned} \quad (4)$$

Then $I(y) \leq 0$ for every real y , and $I(y) = 0$ for every point of the closed support of P . In particular, that support is compact.

Proof. For fixed y , let $P_\varepsilon = (1 - \varepsilon)P + \varepsilon\delta_y$. Its mean and variance are

$$m_\varepsilon = \varepsilon y, \quad v_\varepsilon = 1 + \varepsilon(y^2 - 1) - \varepsilon^2 y^2.$$

For all sufficiently small $\varepsilon \geq 0$, $v_\varepsilon > 0$ and the standardized law remains in $\mathcal{P}$. Put $r_\varepsilon = \mathbb{E}_{P_\varepsilon}|X - m_\varepsilon|^3$. The standardized positive score evaluated at its corresponding raw threshold is

$$G(\varepsilon) = \frac{\sqrt{n}v_\varepsilon^{3/2}}{r_\varepsilon} \left\{ F_{n,P_\varepsilon}(H) - \Phi\left(\frac{H - nm_\varepsilon}{\sqrt{nv_\varepsilon}}\right) \right\}.$$

It satisfies $G(\varepsilon) \leq C$ and $G(0) = C$.

The finite mixture expansion makes $F_{n,P_\varepsilon}(H)$ a polynomial in ε , with derivative $n\{F_{n-1,P}(H-y) - F\}$ at zero. No density of the sum is needed. For the absolute third moment,

$$r'_0 = |y|^3 - \rho - 3y\beta.$$

To justify differentiation, the derivative of $|x - \varepsilon y|^3$ is bounded, for $0 \leq \varepsilon \leq 1$, by $3|y|(|x| + |y|)^2$, which is integrable. The contaminating term contributes $|y|^3$ at first order. Finally the derivative of the normal argument is

$$-\sqrt{n}y - \frac{H}{2\sqrt{n}}(y^2 - 1).$$

Differentiating G now gives exactly $G'(0+) = I(y)$ in (4). Maximality implies $I(y) \leq 0$.

Integrating against P gives $\int I dP = 0$: the convolution term integrates to zero, as do all the displayed centered moment terms. This integral is finite because $\rho < \infty$. The function $y \mapsto F_{n-1,P}(H-y)$ is upper semicontinuous: at an atom its value is the larger, closed-event value. Hence I is upper semicontinuous. If $I(y_0) < 0$ at a support point, upper semicontinuity would give a neighborhood of positive P -measure on which I is bounded above by a negative constant. Together with $I \leq 0$ elsewhere this contradicts $\int I dP = 0$. Thus contact holds on the entire closed support, including atoms and accumulation points.

At this fixed pair, the right side of (4) is bounded above by $-C|y|^3 + M_2 y^2 + M_1 |y| + M_0$ for finite nonnegative constants M_i . Since $C > 0$, this expression is negative on both sufficiently distant rays. Those rays contain no contact point and hence no support point. This proves compactness. The radius obtained here need not be uniform in n or in the attaining pair. $\square$

For later integral identities it is useful to record the same statement as a CDF majorant. With ν the law of $H - S_{n-1}$, define

$$\begin{aligned} nQ_H(y) = nF + \frac{1}{2}\{z\phi(z) + C\rho/\sqrt{n}\} - (\sqrt{n}\phi(z) + 3C\beta/\sqrt{n})y \\ - \frac{1}{2}\{z\phi(z) + 3C\rho/\sqrt{n}\}y^2 + C|y|^3/\sqrt{n}. \end{aligned} \quad (5)$$

Rearranging (4) gives

$$\nu([y, \infty)) \leq Q_H(y), \quad \nu([y, \infty)) = Q_H(y) \quad (y \in \text{supp } P).$$

2.3 Two elementary moment domains

The moment inequalities below are used before any finite-dimensional relaxation. They hold for the actual marginal, without a bounded-support assumption. In particular, they are unaffected by adding a very small atom far from the origin.

Theorem 2.3 (Signed and absolute third moments). *If $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, $\rho = \mathbb{E}|X|^3 < \infty$, and $\gamma = \mathbb{E}X^3$, then*

$$\rho \geq \frac{\gamma^2 + 2}{\sqrt{\gamma^2 + 4}}.$$

This is the sharp moment relation of Shevtsova [24]; in the cited arXiv version its equivalent form is Theorem 2. We include a factorization proof to specify the domain used by the finite certificates.

Proof. Set $s = \sqrt{\gamma^2 + 4}$, $a = (s - \gamma)/2$, $b = (s + \gamma)/2$ and $c = \gamma(\gamma^2 + 6)/s^3$. Then $a, b > 0$, $ab = 1$ and $a + b = s$. Define

$$w(y) = |y|^3 - cy^3 - \frac{12y^2}{s^3} + \frac{6\gamma y}{s^3} + \frac{4}{s^3}.$$

For $y, t \geq 0$, direct multiplication gives

$$w(y) = \frac{2a^2}{s^3}(y - b)^2\{2 + (a + 3b)y\}, \quad w(-t) = \frac{2b^2}{s^3}(t - a)^2\{2 + (3a + b)t\}.$$

Thus $w \geq 0$ on the whole real line. Taking expectations and using the first three signed moments gives

$$0 \leq \mathbb{E}w(X) = \rho - c\gamma - 8/s^3 = \rho - (\gamma^2 + 2)/s.$$

All terms are integrable under the stated third-moment hypothesis. $\square$

Lemma 2.4 (The signed second moment). *With the same normalization and $\beta = \mathbb{E}[X|X|]$,*

$$|\beta| < 1, \quad \rho \geq \frac{1 + \beta^2}{\sqrt{1 - \beta^2}}.$$

Proof. Both signs have positive probability. Put $m = \mathbb{E}X_+ = \mathbb{E}X_- > 0$ and $a_{\pm} = \mathbb{E}X_{\pm}^2 = (1 \pm \beta)/2 > 0$. If $p_{\pm} = \mathbb{P}(\pm X > 0)$, Cauchy's inequality gives $m^2 \leq p_{\pm}a_{\pm}$. Since $p_+ + p_- \leq 1$, this implies

$$m^2(1/a_+ + 1/a_-) \leq 1, \quad 2m \leq \sqrt{1 - \beta^2}.$$

Another application of Cauchy's inequality, to $X_{\pm}^{1/2}$ and $X_{\pm}^{3/2}$, gives $m\mathbb{E}X_{\pm}^3 \geq a_{\pm}^2$. Summing yields $m\rho \geq (1 + \beta^2)/2$, which proves the claim. A possible zero atom enters only through $p_+ + p_- \leq 1$ and causes no exception. $\square$

3 One-step smoothing and the first failure

A one-step smoothing argument bounds C_n in terms of C_{n-1} at every index $n \geq 2$. If a sharp base through 41 is known, the first index $n \geq 42$ with $C_n \geq C_E$ satisfies $0 \leq C_n - C_E < 0.3/n$. The estimate holds for every law in $\mathcal{P}$.

Lemma 3.1 (Elementary Gaussian constants). *For the standard normal density ϕ ,*

$$\|\phi''\|_\infty = \phi(0) < \frac{2}{5}, \quad \|\phi'''\|_\infty \leq \frac{2}{\pi}, \quad \frac{1}{4\pi} < \frac{2}{25}.$$

Moreover $819/2000 < C_E < 20487/50000 < 41/100$ and $\phi(0) > 0.39886 > 0.3988 = 997/2500$.

The derivative estimates and exact rational comparisons are proved in Section B.2.

Theorem 3.2 (One-step smoothing). *For every integer $n \geq 2$,*

$$C_n \leq \sqrt{\frac{n}{n-1}} C_{n-1} + \frac{\sqrt{n}}{15(n-1)^{3/2}} + \frac{2\sqrt{n}}{25(n-1)^2}. \quad (6)$$

Proof. Fix $P \in \mathcal{P}$ and a raw threshold H . Write $v = n - 1$ and $G_v(x) = \Phi(x/\sqrt{v})$. Conditioning on the last summand gives

$$|F_{n,P}(H) - \mathbb{E}G_v(H - X_n)| \leq C_v \rho / \sqrt{v}.$$

Taylor's theorem, applied only to the smooth function G_v , gives

$$\left| G_v(H - y) - G_v(H) + yG'_v(H) - \frac{1}{2}y^2G''_v(H) \right| \leq \frac{\phi(0)}{6v^{3/2}} |y|^3.$$

Taking expectations and using the first two moments yields

$$\left| \mathbb{E}G_v(H - X) - G_v(H) - \frac{1}{2}G''_v(H) \right| \leq \frac{\phi(0)\rho}{6v^{3/2}} < \frac{\rho}{15v^{3/2}}.$$

For the change in Gaussian variance, direct differentiation gives $\partial_v G_v = G''_v/2$ and $\partial_v^2 G_v = G_v^{(4)}/4$. Taylor's integral remainder in the variance therefore gives

$$G_{v+1}(H) - G_v(H) - \frac{1}{2}G''_v(H) = \frac{1}{4} \int_0^1 (1-t) G_{v+t}^{(4)}(H) dt.$$

As $G_w^{(4)}(H) = w^{-2}\phi'''(H/\sqrt{w})$, the variance remainder has absolute value at most

$$\frac{\|\phi'''\|_\infty}{8v^2} \leq \frac{1}{4\pi v^2} < \frac{2}{25v^2}.$$

Combine the three bounds, multiply by $\sqrt{n}/\rho$, and use $\rho \geq 1$. The resulting estimate holds for every P, H , so taking both suprema proves (6).

The bound is uniform in H and holds for both signs of the discrepancy; strict events follow by taking limits of closed ones. $\square$

Lemma 3.3 (Entry at a sharp predecessor). *For $n \geq 2$ put $s_n = \sqrt{n/(n-1)}$ and*

$$\varepsilon_n = \frac{n}{n-1} \left\{ \frac{C_E}{s_n + 1} + \frac{s_n}{15} + \frac{2s_n}{25\sqrt{n-1}} \right\}. \quad (7)$$

If $C_{n-1} \leq C_E \leq C_n$, then

$$0 \leq n(C_n - C_E) \leq \varepsilon_n, \quad n(C_{n-1} - C_n)_+ = 0.$$

For every marginal $P \in \mathcal{P}$, $n(R_{n-1}(P) - C_n)_+ = 0$ as well. The sequence ε_n strictly decreases. In particular, if $n \geq 42$, then

$$0 \leq n(C_n - C_E) < \frac{3003}{10250} < \frac{3}{10}, \quad C_n < \frac{73}{175} < \frac{2113}{5000}.$$

Proof. Let $s = \sqrt{n/(n-1)}$ and $c = C_E$. From (6) and $n(s-1) = (n/(n-1))/(s+1)$, we obtain $n(C_n - c) \leq \varepsilon_n$. As a function of $s > 1$, the right side is

$$c \frac{s^2}{s+1} + \frac{s^3}{15} + \frac{2s^3\sqrt{s^2-1}}{25}.$$

Each term increases with s ; for the first use $(s^2/(s+1))' = s(s+2)/(s+1)^2 > 0$. Since s_n decreases, ε_n decreases. Also $R_{n-1}(P) \leq C_{n-1} \leq c \leq C_n$, proving both zero-loss statements. For $n \geq 42$, the ratio $n/(n-1)$ is at most $42/41$, $\sqrt{n-1} > 6$, and $1 < s < 81/80$. The latter follows at the endpoint from $41 \cdot 81^2 - 42 \cdot 80^2 = 201 > 0$ and then from monotonicity. Using $c < 41/100$ gives the strict bound

$$n(C_n - c) < \frac{42}{41} \left(\frac{41}{200} + \frac{27}{400} + \frac{27}{2000} \right) = \frac{3003}{10250}.$$

The reserve below $3/10$ is $36/5125$. Consequently $C_n < c + 0.3/n < 0.41 + 0.3/42 = 73/175$, whose reserve below 0.4226 is $191/35000$. $\square$

Proposition 3.4 (First failure of the strict bound). *Let $N_0 \geq 2$ be an integer, and assume $C_j \leq C_E$ for $1 \leq j < N_0$. If some $C_n \geq C_E$ with $n \geq N_0$, there is a least such index. At that index every C_j with $1 \leq j < n$ is at most C_E , Lemma 3.3 holds, and $n(C_n - C_E) \leq \varepsilon_{N_0}$. If $n \geq 3$, both deleted indices satisfy*

$$C_{n-1}, C_{n-2} \leq C_E < 0.41 < 0.4253.$$

Consequently, any argument that excludes an extremal pair under these conditions at every $n \geq N_0$ proves $C_n < C_E$ for all $n \geq N_0$. At $n = 2$ only the predecessor C_1 is used; there is no C_0 premise.

Proof. The positive integers are well ordered. Below the least chosen index, the indices from N_0 onward have $C_j < C_E$, and the earlier ones have the assumed weak bound. Apply the scalar lemma and its monotonicity, and then Lemma 2.1. The exclusion argument therefore applies at this index, a contradiction. $\square$

The application in Section 7 takes $N_0 = 42$; its endpoint predecessors are exactly 41 and 40. Choosing instead the least $n \geq N_0$ with $C_n > C_E$ gives weak predecessors and zero loss, but an exclusion of such indices proves only $C_n \leq C_E$. Neither version supplies the support, resource or local-comparison hypotheses of an exclusion argument at a new index floor.

4 An elementary support bound

We now show that the contact inequality confines an extremal law to a short interval. Because the predecessor term vanishes, it suffices to bound the Gaussian part of the contact inequality at one radius and to show that the resulting upper bound decreases along each exterior ray.

Throughout this section $n \geq 42$, (P, H) is an extremal pair, and $C_n \geq C_E$. We also assume $C_{n-1} \leq C_n$, so the positive predecessor loss is zero. Set $v = n-1$, $s = \sqrt{n/v}$ and $F_v(x) = \Phi(x/\sqrt{v})$. For every t in the support of P , the contact equality of Lemma 2.2 gives

$$0 \leq H_n(H, t) + C_n \{-|t|^3 + \frac{3}{2}\rho t^2 + 3\beta t + \rho e_n\}, \quad (8)$$

where

$$e_n = n(\sqrt{n/(n-1)} - 1) - \frac{1}{2} < \frac{0.384}{n},$$

and the exact Gaussian expression is

$$\frac{H_n(H, t)}{n^{3/2}} = F_v(H - t) - F_{v+1}(H) + tF'_{v+1}(H) - \frac{t^2 - 1}{2}F''_{v+1}(H).$$

Here we used $R_{n-1}(P) \leq C_{n-1} \leq C_n$: the term involving the predecessor constant is then nonpositive and has been dropped. For the numerical bound on e_n , rationalization gives

$$ne_n = \frac{s^2(2s+1)}{2(s+1)^2}.$$

This expression increases for $s \geq 1$: its logarithmic derivative is $2/s + 2/(2s+1) - 2/(s+1) > 0$. At $s = 81/80$ its value is $793881/2073680 < 48/125$. Thus $e_n < 0.384/n \leq 8/875$.

4.1 Fourth-order expansion with uniform remainder

Using $\partial_v F_v = F''_v/2$, expand $F_v(H - t)$ spatially through order four, F_{v+1} through second order in variance, and its first and second spatial derivatives through first order in variance. The order-zero, linear and quadratic spatial terms cancel. The remaining cubic and quartic coefficients are $(-t^3 + 3t)/6$ and $(t^4 - 6t^2 + 3)/24$. Thus, for $t = \pm u$,

$$\frac{H_n(H, \pm u)}{s^3} = \pm A(1 - z_v^2)\phi(z_v) + B\phi'''(z_v) + R, \quad z_v = H/\sqrt{v}, \quad (9)$$

where

$$A = \frac{u^3 - 3u}{6}, \quad B = \frac{u^4 - 6u^2 + 3}{24\sqrt{v}}.$$

For $u \geq 5/2$, both are positive. The four Taylor remainders, after division by s^3 , have bounds

$$\frac{u^5 \|\phi^{(4)}\|_\infty}{120v}, \quad \frac{\|\phi^{(5)}\|_\infty}{48v^{3/2}}, \quad \frac{u \|\phi^{(4)}\|_\infty}{8v}, \quad \frac{(u^2 - 1) \|\phi^{(5)}\|_\infty}{16v^{3/2}}.$$

Gaussian Fourier inversion gives $\|\phi^{(4)}\|_\infty \leq 3\phi(0) < 6/5$ and $\|\phi^{(5)}\|_\infty \leq 8/\pi < 51/20$. Therefore

$$|R| \leq E_v(u) := \frac{u^5/100 + 3u/20}{v} + \frac{51(3u^2 - 2)}{960v^{3/2}}. \quad (10)$$

Every remainder bound is uniform in the raw threshold H . Here we used $\pi > 160/51$.

Lemma 4.1 (Two all-threshold envelopes). *For $u \geq 5/2$,*

$$H_n(H, \pm u) \leq s^3 \{\phi(0)(A + 2B) + E_v(u)\}.$$

If also $B \leq A/4$, then

$$H_n(H, \pm u) \leq s^3 \left\{ \phi(0) \left(A + \frac{9B^2}{4A} \right) + E_v(u) \right\}.$$

Both assertions hold for every real H and both signs separately.

Proof. The first follows from $\|(1 - z^2)\phi(z)\|_\infty = \phi(0)$ and $\|\phi'''\|_\infty < 2\phi(0)$. For the latter bound, on $|z| \leq \sqrt{3}$ the polynomial $|3z - z^3|$ is at most 2, with equality only at $|z| = 1$ where the Gaussian factor is smaller than one. Outside that interval, $(|z|^3 - 3|z|)e^{-z^2/2} \leq |z|^3 e^{-z^2/2} \leq 3\sqrt{3}e^{-3/2} < 2$. The last inequality follows, for example, from $e^{3/2} > 1 + 3/2 + 9/8 + 27/48 > 4$ and $3\sqrt{3} < 8$. For the positive sign and $|z_v| \leq 1$, the retained terms in (9) are at most $\phi(0)(A - Az_v^2 + 3B|z_v|)$; completing the square proves the sharper bound. For $|z_v| > 1$ the first term is nonpositive and the second is at most $2B\phi(0) \leq A\phi(0)/2$. For the negative sign, its first term is at most $2e^{-3/2}A\phi(0) < A\phi(0)/2$, and the second is at most $2B\phi(0) \leq A\phi(0)/2$. Add the uniform remainder. $\square$

4.2 Three support bounds

The first radius uses only the global moment cap. Later contractions use sharper moment bounds on probability panels. We state them in terms of the moment bounds they require.

Use

$$S = 81/80, \quad c_0 = 819/2000 < C_E, \quad H_4 = u^4 - 6u^2 + 3,$$

so $s \leq S$, $\sqrt{v} > 6$ and $e_n < 8/875$. Define

$$\begin{aligned} E(u) &= \frac{u^5}{4100} + \frac{3u}{820} + \frac{17(3u^2 - 2)}{78720}, \\ Q_{r,b}(u) &= -u^3 + \frac{3}{2}ru^2 + 3bu + \frac{8}{875}r, \\ P_{r,b}(u) &= \frac{2}{5}S^3(A + H_4/72) + S^3E(u) + c_0Q_{r,b}(u), \\ P_{r,b}^\sharp(u) &= \frac{2}{5}S^3\left(A^2 + \frac{H_4^2}{256 \cdot 41}\right) + S^3E(u)A + c_0Q_{r,b}(u)A. \end{aligned}$$

If $\rho \leq r$ and $|\beta| \leq b$, the contact coefficient is at most $Q_{r,b}$. Check $Q_{r,b} < 0$ before replacing its actual multiplier C_n by the lower constant c_0 . Lemma 4.1 then makes $P < 0$ sufficient at a boundary; $P^\sharp < 0$ is sufficient when $B \leq A/4$.

To extend a boundary to a whole ray, put

$$L_{r,b}(u) = \frac{2}{5}(u^2/2 + 1/2 + 8/875) + c_0Q'_{r,b}(u).$$

To bound the derivative of H_n , put $z = H/\sqrt{n}$ and $y = (H - t)/\sqrt{n}$. Exact differentiation gives

$$\partial_t H_n = n\phi(z) + Ht\phi(z) - ns\phi(sy).$$

Taylor's theorem and $\|\phi''\|_\infty = \phi(0)$ imply $|n\phi(y) - n\phi(z) - Ht\phi(z)| \leq \phi(0)t^2/2$. Also $\partial_s[s\phi(sy)] = -\phi''(sy)$, whence $|s\phi(sy) - \phi(y)| \leq \phi(0)(s - 1)$. Since $n(s - 1) = 1/2 + e_n$, this proves

$$|\partial_t H_n| \leq \phi(0)(t^2/2 + 1/2 + e_n) < (2/5)(t^2/2 + 1/2 + e_n).$$

If Q, Q', Q'', L, L', L'' are negative at U , their required signs persist for $u \geq U$, since $Q''' = -6$ and $L'' = 2/5 - 6c_0 < 0$. The contact upper bound for each sign then decreases strictly along the whole ray.

Proposition 4.2 (Support from moment bounds). *Under the contact and zero-loss hypotheses of this section, each of the following implies $\text{supp } P \subset (-R, R)$:*

- (i) $R = 96/25$, $\rho \leq 1841/1000$ and $|\beta| \leq 321/500$;
- (ii) $R = 121/40$, $\rho \leq 143/100$ and $|\beta| \leq 93/200$;
- (iii) $5/2 \leq R \leq 111/40$, $\rho \leq 7R/20 + 1/3$ and $|\beta| \leq 3R/8 - 3/5$.

The structural inequalities (3) supply the moment bounds in (i) whenever $C_n \geq C_E$.

Proof. The initial radius.

Take $R = 96/25$, $r = 1841/1000$, $b = 321/500$. The structural inequalities (3) give

$$\rho < \min \left\{ \frac{0.3031 \cdot 0.646}{c_0 - 0.3031}, \frac{0.3322 \cdot 0.429}{c_0 - 0.3322} \right\} = \frac{7361}{4000} < r.$$

The sign-moment inequality in Section 2.3 gives $|\beta| < b$, using

$$(1 + b^2)^2 - r^2(1 - b^2) = \frac{93288849}{50000000000} > 0.$$

At $u = R < 6$, $H_4 = 6uA - 3(u^2 - 1) < 36A$ and $B < H_4/144 < A/4$. Exact arithmetic gives

$$-P_{r,b}^\sharp(R) = \frac{2214258283536204102723}{1025000000000000000000} > \frac{1}{5}.$$

All six ray signs are negative, proving support in $(-R, R)$.

A first contraction. Take $R = 121/40$, $r = 143/100$, $b = 93/200$, which will be useful for the first contraction of the left probability panels. Then

$$-P_{r,b}(R) = \frac{2533499510174559}{214958080000000000} > \frac{1}{100}.$$

All six ray signs are negative.

The continuous family. Assume

$$\frac{5}{2} \leq R \leq \frac{111}{40}, \quad \rho \leq r_R = \frac{7}{20}R + \frac{1}{3}, \quad |\beta| \leq b_R = \frac{3}{8}R - \frac{3}{5}.$$

Let $p(R) = -P_{r_R, b_R}(R)$ and $w = R - 5/2$. The coefficients of $p(R) = \sum_{j=0}^5 a_j w^j$ are

j	a_j
0	123579925563/1343488000000
1	-467155860219/1679360000000
2	705959271/13434880000
3	8703567/167936000
4	-7499223/839680000
5	-531441/2099200000

With $h = 11/40$, their signs imply throughout $0 \leq w \leq h$

$$p'(R) \leq a_1 + 2a_2h + 3a_3h^2 = -\frac{797749885413}{3358720000000} < 0.$$

Consequently

$$p(R) \geq p(111/40) = \frac{4403736280322409}{214958080000000000} > \frac{1}{50}.$$

For rays, freeze each pair (r_R, b_R) . The common larger caps $r_* = 131/100$, $b_* = 441/1000$ have at $u = 5/2$ the six signs

$$-\frac{16991}{700000}, \quad -\frac{3801}{500}, \quad -\frac{1107}{100}, \quad -\frac{11615533}{7000000}, \quad -\frac{706633}{200000}, \quad -\frac{2057}{1000}.$$

They majorize the corresponding signs and derivative bounds for every fixed pair. Combining the negative boundary estimate with these ray bounds proves (iii). The moment caps are fixed while moving along a ray. $\square$

Theorem 4.3 (Elementary support replacement). *Let $n \geq 42$ and let (P, H) be an attained maximizing pair with $C_n \geq C_E$ and $C_{n-1} \leq C_n$. Once the corresponding moment bounds have been proved for P , Proposition 4.2 gives all 65 support contractions in the proposed local exclusion: one initial radius $96/25$, one contraction to $121/40$, and 63 applications of the affine family with $5/2 \leq R \leq 111/40$.*

The structural inequalities (3) give the first moment bounds. The second application covers the moment maxima $71458741/50000000$ and $46459359/100000000$. The other 63 parameter triples satisfy the affine bounds of Proposition 4.2; they are listed in the accompanying data. Each contraction uses moment bounds obtained before that contraction is applied.

5 The two-point skeleton and the conserved identity

Every centered law can be separated into its positive and nonpositive parts. Replacing each part by its conditional mean gives a two-point law; the remaining conditional fluctuations measure its departure from that law. We first make this decomposition exact, then identify a moment defect that vanishes at Esseen's extremal distribution.

5.1 The sign split and its units

Fix $P \in \mathcal{P}$ and put

$$p = \mathbb{P}(X > 0), \quad q = 1 - p, \quad v = pq, \quad m = \mathbb{E}X_+ = \mathbb{E}X_- > 0, \quad B_0 = \mathbf{1}_{\{X > 0\}}.$$

Both p, q are positive; a possible atom at zero is assigned to the nonpositive class. Define $V = \mathbb{E}[X \mid B_0]$ and $W = X - V$, so that

$$V = -m/q \quad (B_0 = 0), \quad V = m/p \quad (B_0 = 1), \quad \theta = \mathbb{E}W^2 = 1 - m^2/v < 1.$$

Write $\theta_- = \mathbb{E}[W^2; B_0 = 0]$ and $\theta_+ = \mathbb{E}[W^2; B_0 = 1]$. The distance between the two conditional means is

$$\ell = m/v = \sqrt{(1 - \theta)/v} > 0.$$

Set $U = W/\ell$ and $Y = B_0 + U$. Then

$$X = \ell(Y - p), \quad \mathbb{E}[U \mid B_0] = 0. \quad (11)$$

Let U_0, U_1 have the two conditional laws, and put

$$w_i = \mathbb{E}U_i^2, \quad \mu_i = \mathbb{E}U_i^3, \quad s = qw_0 + pw_1 = \frac{v\theta}{1 - \theta}, \quad \lambda = ns, \quad \ell = (v + s)^{-1/2}.$$

The moments μ_i are finite because $\rho < \infty$. Higher moments are used only for extremal laws, which have compact support by Lemma 2.2.

If $s = 0$, both nonnegative conditional variances vanish, so $U_0 = U_1 = 0$ almost surely. This is exactly the standardized Bernoulli law. If $s > 0$, the parameter $\alpha = pw_1/s$ lies in $[0, 1]$ and gives $w_0 = s(1 - \alpha)/q$, $w_1 = s\alpha/p$.

The signs of X are still those defining the two classes, so expansion of the actual conditional moments gives

$$\begin{aligned} \rho/\ell^3 &= v(1 - 2v) + 3v(w_0 + w_1) - q\mu_0 + p\mu_1, \\ \gamma/\ell^3 &= v(q - p) + 3v(w_1 - w_0) + q\mu_0 + p\mu_1, \\ \beta/\ell^2 &= v(q - p) + pw_1 - qw_0. \end{aligned} \quad (12)$$

For example, on the nonpositive class $|X|^3 = \ell^3(p - U_0)^3$; on the positive class it is $\ell^3(q + U_1)^3$. Conditional centering removes the linear terms and gives the first identity; the other two follow in the same way. The three identities share the same pair (μ_0, μ_1) .

5.2 A nonnegative moment defect

The constant $3 + \sqrt{10}$ has a simple geometric explanation. The signed third moment and the first absolute moment satisfy a circle inequality; the lengths of the support gaps are controlled by the same first absolute moment. Neither fact requires bounded support.

Define

$$\tau(t) = \mathbb{E}[X \mathbf{1}_{\{X \geq t\}}], \quad \sigma = \sqrt{10}.$$

For $t < 0$, centering gives $\tau(t) = -\mathbb{E}[X\mathbf{1}_{\{X < t\}}] \geq 0$; for $t \geq 0$ nonnegativity is immediate. Signwise Tonelli integration gives

$$\int_{\mathbb{R}} \tau(t) dt = 1, \quad 2 \int_{\mathbb{R}} |t| \tau(t) dt = \rho. \quad (13)$$

On each bounded complementary interval (a, b) of the closed support, τ is constant in the interior, with value $\tau_{a,b}$. Put

$$J = \sum_{(a,b)} \tau_{a,b}(b-a)^2, \quad D = (3 + \sigma)\rho - \gamma - 3J. \quad (14)$$

There are at most countably many gaps; the next proof shows that the nonnegative sum is finite. Endpoint atoms do not affect its integrals.

Lemma 5.1 (A moment circle and the gap inequality). *For every $P \in \mathcal{P}$,*

$$\gamma^2 + (2m)^2 \leq \rho^2, \quad J \leq \rho + 2m, \quad \gamma + tJ \leq (t + \sqrt{1+t^2})\rho \quad (t > 0). \quad (15)$$

The last constant is sharp. Equality holds exactly at the standardized Bernoulli law with positive mass $p_t = (1 + t - \sqrt{1+t^2})/2$. In particular, $D \geq 0$. If zero belongs to the support, then $D \geq (\sigma - 1)\rho$; hence $D < (\sigma - 1)\rho$ implies a central support gap $(-A, B)$ with $A, B > 0$.

Proof. Write $a_{\pm} = \mathbb{E}X_{\pm}^2$ and $b_{\pm} = \mathbb{E}X_{\pm}^3$, so $a_+ + a_- = 1$, $b_+ + b_- = \rho$ and $b_+ - b_- = \gamma$. Cauchy–Schwarz on each sign class gives $a_{\pm}^2 \leq mb_{\pm}$. It also gives $m^2 \leq pa_+$ and $m^2 \leq qa_-$, where an atom at zero is included in q . Thus $m^2(1/a_+ + 1/a_-) \leq 1$, or $m^2 \leq a_+a_-$. Consequently $b_+b_- \geq (a_+a_-)^2/m^2 \geq m^2$, proving the circle inequality since $\rho^2 - \gamma^2 = 4b_+b_-$.

Every gap on one side of zero satisfies $(b-a)^2 \leq 2 \int_a^b |x| dx$. If a central gap $(-A, B)$ exists, its weight is m and its excess over this integral is $2mAB$. Moreover $0 \leq \mathbb{E}[(X+A)(X-B)] = 1 - AB$. Summing the gap bounds by Tonelli proves $J \leq \rho + 2m$. If zero is in the support there is no central gap, so $J \leq \rho$. Finally Cauchy–Schwarz in $\mathbb{R}^2$ gives $\gamma + 2tm \leq \sqrt{1+t^2}\rho$, proving the family and the assertions about D .

Equality in the circle forces equality in the two signwise Cauchy bounds: the positive and negative parts are each concentrated at one nonzero value. Equality in $m^2 \leq a_+a_-$ excludes mass at zero. For such a law, with positive mass p and $q = 1 - p$, one has $J = \rho + 2m$. Equality in the last Cauchy bound requires $\gamma = 2m/t$, equivalently $t(1 - 2p) = 2p(1 - p)$. Its unique solution in $(0, 1)$ is the stated p_t . $\square$

Corollary 5.2 (The zero-defect law). *For $P \in \mathcal{P}$, $D = 0$ if and only if*

$$X = (B_0 - p_*)/\sqrt{p_*q_*}, \quad p_* = (4 - \sqrt{10})/2, \quad q_* = 1 - p_*, \quad B_0 \sim \text{Bernoulli}(p_*).$$

Proof. Apply the equality case of Lemma 5.1 with $t = 3$. $\square$

In particular the defect separates two losses:

$$D = \underbrace{\sigma\rho - \gamma - 6m}_{\text{moment loss}} + 3 \underbrace{(\rho + 2m - J)}_{\text{gap loss}},$$

each nonnegative. For quantitative estimates we need more information than their signs. Suppose a central gap $(-A, B)$ has been established, and write

$$T_+ = \mathbb{E}X_+^3 - mB^2, \quad T_- = \mathbb{E}X_-^3 - mA^2, \quad R_+ = \mathbb{E}X_+^2 - mB, \quad R_- = \mathbb{E}X_-^2 - mA.$$

These four quantities are nonnegative. For the outer gaps put $J_o = J - m(A + B)^2$. A positive outer gap satisfies $(b - a)^2 \leq 2 \int_a^b (t - B) dt$, so Tonelli gives

$$J_o^+ \leq 2 \int_B^\infty (t - B) \tau(t) dt = \mathbb{E}[X_+(X_+ - B)^2] = T_+ - 2BR_+ \leq T_+.$$

Similarly $J_o^- \leq T_- - 2AR_- \leq T_-$. Expanding D , using $\sqrt{\sigma + 1}\sqrt{\sigma - 1} = 3$, gives

$$\begin{aligned} D &= m\{\sqrt{\sigma + 1}A - \sqrt{\sigma - 1}B\}^2 + (\sigma - 1)T_+ + (\sigma + 1)T_- \\ &\quad + 3(T_+ + T_- - J_o). \end{aligned} \tag{16}$$

Every term is nonnegative. Cauchy–Schwarz also gives $1/(2\rho) \leq m \leq 1/2$: indeed $1 \leq \rho\mathbb{E}|X| = 2m\rho$ and $\mathbb{E}|X| \leq 1$. Since $\sigma - 1 > 2$, the decomposition yields $T_+ + T_- \leq D/2$. If $D < 2$, then

$$m(A^2 + B^2) = \rho - T_+ - T_- \geq 1 - D/2, \quad A + B \geq \sqrt{2 - D}.$$

5.3 One joint budget for endpoint motion and noise

Let $\bar{A} = m/q$, $\bar{B} = m/p$ be the conditional mean magnitudes, and let $\Delta_A = \bar{A} - A$, $\Delta_B = \bar{B} - B$. They are nonnegative.

Lemma 5.3 (Simultaneous gap inequality). *For this central gap,*

$$\begin{aligned} D &\geq \frac{(6 + 2\sigma)(1 - \theta)^{3/2}}{\sqrt{pq}}(p - p_*)^2 + 6m\{(A + B)(\Delta_A + \Delta_B) + \Delta_A\Delta_B\} \\ &\quad + 3(\sigma + 1)B\theta_+ + 3(\sigma + 3)A\theta_-. \end{aligned} \tag{17}$$

Proof. The stronger outer-gap bound in the preceding proof gives

$$D \geq (\sigma - 1)\mathbb{E}X_+^3 + (\sigma + 1)\mathbb{E}X_-^3 - 6mAB + 6BR_+ + 6AR_-.$$

Conditional cubic expansion gives

$$\mathbb{E}X_+^3 = m\bar{B}^2 + \mathbb{E}[(X + 2\bar{B})(X - \bar{B})^2; X > 0] \geq m\bar{B}^2 + 3B\theta_+,$$

and similarly $\mathbb{E}X_-^3 \geq m\bar{A}^2 + 3A\theta_-$. Moreover $R_+ = \theta_+ + m\Delta_B$ and $R_- = \theta_- + m\Delta_A$. Substitution produces the last two terms of (17) and the deterministic expression

$$m\{(\sigma - 1)\bar{B}^2 + (\sigma + 1)\bar{A}^2 - 6\bar{A}\bar{B}\} + 6m\{\bar{A}\bar{B} - AB + B\Delta_B + A\Delta_A\}.$$

The second brace equals $(A + B)(\Delta_A + \Delta_B) + \Delta_A\Delta_B$. For the first, use $m^2 = (1 - \theta)pq$ and the exact identity

$$(\sigma + 1)p^2 + (\sigma - 1)q^2 - 6pq = (6 + 2\sigma)(p - p_*)^2.$$

This proves the displayed inequality. □

In count coordinates the inward displacements are

$$a_0 = p - A/\ell = \Delta_A/\ell, \quad a_1 = q - B/\ell = \Delta_B/\ell, \quad d_Y = a_0 + a_1 = (\Delta_A + \Delta_B)/\ell.$$

Thus $U_0 \leq a_0$ and $U_1 \geq -a_1$. The physical displacement $\Delta_A + \Delta_B$ and the count displacement d_Y differ by ℓ .

Theorem 5.4 (Weighted joint resource). *Suppose $n \geq N > 0$ and*

$$nD \leq Nd_D, \quad \theta \leq \Theta < 1, \quad pq \leq v_*, \quad D \geq b + 6mS_*(\Delta_A + \Delta_B) + K\theta,$$

where $b \geq 0$, $S_* > 0$, $K > 0$ and $v_* > 0$. Put $r = d_D - b$. If $r < 0$ there is no such law. If $r = 0$, then $d_Y = \lambda = 0$. If $r > 0$, define

$$d_* = \frac{r}{6(1-\Theta)S_*}, \quad \Lambda = \frac{Nv_*r}{K(1-\Theta)}.$$

Then

$$\frac{d_Y}{d_*} + \frac{\lambda}{\Lambda} \leq 1. \quad (18)$$

Proof. Since $m\ell = 1 - \theta$ and $n\theta = (1 - \theta)\lambda/(pq)$,

$$\begin{aligned} Nd_D &\geq nD \geq nb + 6n(1 - \theta)S_*d_Y + K(1 - \theta)\lambda/(pq) \\ &\geq Nb + 6N(1 - \Theta)S_*d_Y + K(1 - \Theta)\lambda/v_*. \end{aligned}$$

All weakened coefficients multiply nonnegative quantities. Subtracting Nb proves the impossible and zero cases without division. For $r > 0$, division by Nr proves (18). $\square$

For instance, on a closed probability interval I , established endpoint floors $A \geq A_l > 0$, $B \geq B_l > 0$ and $\theta \leq \Theta$ supply

$$\begin{aligned} S_* &= A_l + B_l, \quad K = \min\{3(\sigma + 1)B_l, 3(\sigma + 3)A_l\}, \\ b &= \frac{(6 + 2\sigma)(1 - \Theta)^{3/2}}{\sqrt{v_*}} \text{dist}(p_*, I)^2, \quad v_* \geq \max_{p \in I} p(1 - p). \end{aligned}$$

These choices follow from Lemma 5.3 by dropping the nonnegative product $6m\Delta_A\Delta_B$. Rational lower enclosures for b, K, S_* and an upper enclosure for v_* preserve the implication.

On a band $L \leq \lambda \leq U \leq \Lambda$, the joint plane gives $d_Y \leq d_*(1 - L/\Lambda)$. Either intercept may be replaced by a larger value.

5.4 The conserved identity

The moment defect measures how far the gaps and third moments of P are from those of the extremal two-point law. To bound that defect, we integrate the contact slack in two ways. Integrating over support gaps reveals J and a nonnegative leakage L . Integrating against the zero-bias law instead turns the same slack into a quadratic-hinge expectation for the sum. A compensated Fourier formula then compares that expectation with its Gaussian and cubic terms. Equating the two expressions gives an exact identity in which score excess, defect, phase displacement and leakage share one remainder.

Fix an integer $n \geq 2$ and $P \in \mathcal{P}$. Write $S_j = X_1 + \dots + X_j$ and $z = H/\sqrt{n}$,

$$F = \mathbb{P}(S_n \leq H), \quad C = \frac{\sqrt{n}}{\rho} \{F - \Phi(z)\} > 0, \quad \beta = \mathbb{E}[X|X|], \quad \gamma = \mathbb{E}X^3, \quad \varphi = \phi(z).$$

Suppose that the fixed-index contact argument has supplied

$$h_P(y) := \mathbb{P}(S_{n-1} \leq H - y) \leq Q_H(y), \quad h_P(y) = Q_H(y) \quad (y \in \text{supp } P), \quad (19)$$

where

$$nQ_H(y) = nF + \frac{H\varphi + C\rho}{2\sqrt{n}} - \left(\sqrt{n}\varphi + \frac{3C\beta}{\sqrt{n}}\right)y - \frac{H\varphi + 3C\rho}{2\sqrt{n}}y^2 + \frac{C}{\sqrt{n}}|y|^3. \quad (20)$$

At an extremal pair these relations hold by Lemma 2.2; they are the only contact hypotheses used below.

Because $C > 0$, the piecewise cubic function grows to $+\infty$ on either support ray. Equality in (19), with $0 \leq h_P \leq 1$, therefore gives $|X| \leq R$ for some finite R . In particular $M_4 = \mathbb{E}X^4$ and $M_5 = \mathbb{E}|X|^5$ are finite.

5.4.1 Auxiliary gap integrals and leakage

Use the zero-bias density τ , gap functional J and moment defect D of (13) and (14). Lemma 5.1 gives $D \geq 0$. To evaluate the contact slack we need two more weighted gap integrals and one nonnegative leakage term. Let ν be the law of $H - S_{n-1}$ and set

$$\begin{aligned} J_1 &= \sum_{(a,b)} \tau_{a,b} \int_a^b (t-a)t \, dt, & J_2 &= \sum_{(a,b)} \tau_{a,b} \int_a^b (t-a)t|t| \, dt, \\ L &= \sum_{(a,b)} \tau_{a,b} \int_{[a,b)} (t-a) \, d\nu(t). \end{aligned} \quad (21)$$

The sums run over the same bounded complementary intervals of the closed support as in J . The half-open interval is essential: an atom at the upper endpoint is not an interior crossing of the gap. If the support lies in $[-R, R]$, then

$$|J_1| \leq RJ/2, \quad |J_2| \leq R^2J/2, \quad 0 \leq L \leq 2R.$$

For the first two bounds use $|t| \leq R$ and $\int_a^b (t-a) \, dt = (b-a)^2/2$. For the last, use $\tau_{a,b} \leq \mathbb{E}|X| \leq 1$, $t-a \leq 2R$, and disjointness of the half-open gaps. Together with $J \leq 2R$, these bounds prove absolute convergence of every sum.

5.4.2 Integrating the same slack in two ways

Put $\mathcal{A} = \int \tau(y)\{Q_H(y) - h_P(y)\} \, dy \geq 0$. On the support the slack vanishes; outside the convex hull $\tau = 0$. On a gap, contact at its endpoints gives

$$\nu([a, b)) = h_P(a) - h_P(b) = Q_H(a) - Q_H(b) = \int_a^b -Q'_H(t) \, dt.$$

Bounded Fubini consequently gives

$$\int_a^b (Q_H - h_P) = \int_a^b (t-a)(-Q'_H(t)) \, dt - \int_{[a,b)} (t-a) \, d\nu(t).$$

Substituting (20) proves

$$\mathcal{A} = \frac{\varphi J}{2\sqrt{n}} + \frac{G}{n^{3/2}} - L, \quad G = \frac{3}{2}C\beta J + (H\varphi + 3C\rho)J_1 - 3CJ_2. \quad (22)$$

The density τ is a probability density. If X^* has that density, signwise Fubini proves, for every C^1 test function on the compact range,

$$\mathbb{E}g'(X^*) = \mathbb{E}[Xg(X)], \quad \mathbb{E}X^* = \gamma/2, \quad \mathbb{E}(X^*)^2 = M_4/3, \quad \mathbb{E}|X^*|^3 = M_5/4.$$

Indeed, after subtracting $g(0)$, integration from zero to X gives $\mathbb{E}[Xg(X)] = \int \tau(t)g'(t) \, dt$ by signwise Fubini. This proves the identity and the three moment formulas, using the relevant

polynomial or absolute-power test function. Take X^* independent of S_{n-1} . For a continuously differentiable g , conditional application of that identity gives

$$\mathbb{E}g'((S_{n-1} + X^*)/\sqrt{n}) = \sqrt{n} \mathbb{E}[X_n g(S_n/\sqrt{n})] = \mathbb{E}[Z g(Z)], \quad Z = S_n/\sqrt{n}.$$

The last equality is exchangeability. Hence $Z^* = (S_{n-1} + X^*)/\sqrt{n}$ has the zero-bias density $\tau_Z(t) = \mathbb{E}[Z \mathbf{1}_{\{Z \geq t\}}]$, where $Z = S_n/\sqrt{n}$. Indeed the same signwise Fubini formula for Z identifies its integral against every continuous test function, by taking that function as g' . Its CDF at z is

$$T = \mathbb{E}[(Z^2 - zZ) \mathbf{1}_{\{Z \leq z\}}].$$

For example, this formula follows by signwise integration of τ_Z ; it is unaffected by endpoint atoms. Since $\mathbb{E}h_P(X^*) = T$, integration of the contact polynomial against τ gives

$$\mathcal{A} = F - \frac{\gamma\varphi}{2\sqrt{n}} + \frac{H\varphi(3 - M_4)/6 + CK_q}{n^{3/2}} - T, \quad K_q = \frac{M_5}{4} - \frac{\rho M_4}{2} - \frac{3\beta\gamma}{2} + \frac{\rho}{2}. \quad (23)$$

There is one further contact equality. Multiplication by X and exchangeability give $n\mathbb{E}[Xh_P(X)] = \mathbb{E}[S_n \mathbf{1}_{\{S_n \leq H\}}]$. Thus, if

$$M = \mathbb{E}[Z \mathbf{1}_{\{Z \leq z\}}], \quad K_1 = \mathbb{E}[X|X|^3] - 3\beta - \frac{3}{2}\rho\gamma,$$

then

$$M + \varphi = -\frac{\gamma z \varphi}{2\sqrt{n}} + \frac{CK_1}{n}. \quad (24)$$

5.4.3 Compensated inversion over the entire half-line

Lemma 5.5 (Quadratic hinge inversion). *Let μ be a finite signed measure on $\mathbb{R}$ with zero mass, zero first moment, and finite absolute second moment. With $\hat{\mu}(t) = \int e^{itx} d\mu(x)$, for every real H ,*

$$\int (x - H)|x - H| d\mu(x) = -\frac{4}{\pi} \int_0^\infty \frac{\Im(e^{-itH} \hat{\mu}(t))}{t^3} dt. \quad (25)$$

The right side is absolutely convergent.

Proof. Scaling and integration by parts give

$$\int_0^\infty \frac{ty - \sin(ty)}{t^3} dt = \frac{\pi}{4} y|y|.$$

For $y \geq 0$ the integrand is nonnegative, and for $y < 0$ it has the opposite sign. Its absolute integral is therefore $\pi y^2/4$. To check the constant, one integration by parts reduces it to half of $\int_0^\infty (1 - \cos t)t^{-2} dt$. Since $t^{-2} = \int_0^\infty a e^{-at} da$, nonnegative Tonelli and elementary Laplace integration give

$$\int_0^\infty \frac{1 - \cos t}{t^2} dt = \int_0^\infty a \left\{ \frac{1}{a} - \frac{a}{a^2 + 1} \right\} da = \int_0^\infty \frac{da}{a^2 + 1} = \frac{\pi}{2}.$$

Absolute Fubini applies to the compensated integrand against $|\mu|$. Its iterated absolute integral is $\pi \int (x - H)^2 d|\mu|(x)/4 < \infty$. The linear term integrates to zero, since both the mass and first moment vanish. This proves the formula and its absolute convergence. $\square$

Let Z_0 be standard normal, write $f(t) = \mathbb{E}e^{itX}$, and set

$$q_z(x) = (x - z)|x - z|, \quad \varepsilon = \mathbb{E}q_z(Z) - \mathbb{E}q_z(Z_0) - \frac{2\gamma\varphi}{3\sqrt{n}}.$$

Apply Lemma 5.5 to the law of S_n minus the scale- $\sqrt{n}$ pushforward of the signed density

$$\phi(x) \left\{ 1 + \frac{\gamma}{6\sqrt{n}}(x^3 - 3x) \right\}.$$

That density has mass one and first moment zero. Gaussian integration by parts gives its pushed-forward transform $e^{-nt^2/2}(1 - in\gamma t^3/6)$. The hinge correction is $2\gamma\varphi/(3\sqrt{n})$, because $(x^3 - 3x)\phi(x) = -\phi'''(x)$ and $-\int q_z\phi''' = 4\phi(z)$, with the third derivative of q_z understood distributionally as $4\delta_z$. Since $q_H(\sqrt{n}x) = nq_z(x)$, the exact normalization is

$$\varepsilon = -\frac{4}{\pi n} \int_0^\infty \frac{\Im(e^{-itH}V_n(t))}{t^3} dt, \quad V_n(t) = f(t)^n - e^{-nt^2/2}(1 - in\gamma t^3/6). \quad (26)$$

5.4.4 The identity

Proposition 5.6 (Conserved identity). *Under the explicit contact hypotheses (19)–(20), put $e = C - C_E$, $w = H^2/(2n)$, and $U = J_1 + (M_4 - 3)/6$. Then*

$$\begin{aligned} ne\rho + nC_E\rho(1 - e^{-w}) + \frac{n\phi(0)e^{-w}}{6}D + n^{3/2}L &= C\{3\beta J/2 + 3\rho J_1 - 3J_2 - K_q\} \\ &\quad + H(CK_1 + \varphi U) - H^2(C\rho + \gamma\varphi/2) + E_n, \end{aligned} \quad (27)$$

$$E_n = -n^{3/2}\varepsilon/2.$$

If $C \geq C_E$, every term on the left is nonnegative. For any $b > 0$, bounds

$$I_{\text{low}} \geq \int_0^b \frac{|\Im(e^{-itH}V_n(t))|}{t^3} dt, \quad I_{\text{high}} \geq \int_b^\infty \frac{|V_n(t)|}{t^3} dt$$

give

$$|E_n| \leq \frac{2\sqrt{n}}{\pi}(I_{\text{low}} + I_{\text{high}}). \quad (28)$$

Proof. The pointwise identity $q_z(x) = (x - z)^2 - 2(x - z)^2\mathbf{1}_{\{x \leq z\}}$ gives

$$T - \Phi(z) = -\frac{1}{2}\{\mathbb{E}q_z(Z) - \mathbb{E}q_z(Z_0)\} + z(M + \varphi) - z^2(F - \Phi(z)).$$

Here $\mathbb{E}[Z_0\mathbf{1}_{\{Z_0 \leq z\}}] = -\varphi$ and the normal counterpart of T is $\Phi(z)$, by one integration by parts. Use (24), the definition of ε , and $F - \Phi(z) = C\rho/\sqrt{n}$ in (23). If

$$P_0 = H^2(C\rho + \gamma\varphi/2) - H\{CK_1 + \varphi(M_4 - 3)/6\} + CK_q,$$

the result is

$$\sqrt{n}\mathcal{A} = C\rho - \gamma\varphi/6 + P_0/n + \sqrt{n}\varepsilon/2.$$

Comparison with (22) yields

$$C\rho + \sqrt{n}L = \frac{\varphi}{6}(\gamma + 3J) + \frac{G - P_0}{n} - \sqrt{n}\varepsilon/2.$$

Finally substitute $\gamma + 3J = (3 + \sqrt{10})\rho - D$, $C_E = (3 + \sqrt{10})\phi(0)/6$, and $\varphi = \phi(0)e^{-w}$, and multiply by n . This proves (27). Nonnegativity follows from Lemma 5.1, $L \geq 0$, and $e \geq 0$. Equation (26) gives the error bound directly. $\square$

The identity holds at each index $n \geq 2$. When $C \geq C_E$, its four terms are nonnegative: the excess of the ratio over C_E , a Gaussian cost of an off-centre threshold, the moment defect, and a weighted leakage of $H - S_{n-1}$ into support gaps. A bound on the right side therefore bounds each of them.

5.4.5 Removing the Gaussian weight

Denote the right side of (27) by

$$\begin{aligned}\mathcal{R} = & C\{3\beta J/2 + 3\rho J_1 - 3J_2 - K_q\} + H(CK_1 + \varphi U) \\ & - H^2(C\rho + \gamma\varphi/2) + E_n, \quad U = J_1 + (M_4 - 3)/6.\end{aligned}\tag{29}$$

The exact conserved identity can be regrouped as

$$\mathcal{R} = n e \rho + \frac{n\phi(0)}{6} D + n^{3/2} L + n(1 - e^{-w}) \left(C_E \rho - \frac{\phi(0)}{6} D \right).\tag{30}$$

The last coefficient must be shown to be positive; a first bound on the remainder provides this.

Lemma 5.7 (Removing the Gaussian weight). *Suppose $C \geq C_E$, $n \geq N \geq 2$, and an established bound gives $\mathcal{R} \leq r_0 < 2N/5$. Then*

$$n e \rho + \frac{n\phi(0)}{6} D + n^{3/2} L \leq \mathcal{R}.\tag{31}$$

Consequently any later bound $\mathcal{R} \leq r$ gives

$$e^{-w} \geq 1 - \frac{5r}{2n}, \quad nD \leq \frac{15000r}{997}.$$

Proof. The identity before substitution of D gives

$$\frac{\varphi}{6}(\gamma + 3J) = C\rho + \sqrt{n}L - \frac{\mathcal{R}}{n} > C\rho - \frac{2}{5} > 0,$$

using $\rho \geq 1$ and $C \geq C_E > 2/5$. Thus $C_E \rho - \phi(0)D/6 = \phi(0)(\gamma + 3J)/6 > 0$. Dropping the last nonnegative term of (30) proves (31). In the original conserved identity all four terms are nonnegative, so $nC_E \rho(1 - e^{-w}) \leq r$. Since $C_E \rho > 2/5$, this proves the density floor. Finally $\phi(0) > 997/2500$ gives the bound on nD . If $r < 0$, no extremal pair has these parameters. $\square$

The constant comparisons used here follow from Lemma 3.1.

Corollary 5.8 (Conserved phase resource). *Under the preceding hypotheses, suppose established bounds give*

$$\mathcal{R} \leq r < 2N/5, \quad nD \leq Nd_D, \quad a_{\text{ph}}(\rho) := 0.4095\rho - d_D/15 > 0.$$

Suppose also that on the current signed phase cell the raw remainder has a pointwise quadratic envelope

$$\mathcal{R}(H) \leq A(C, \rho) + F_{\text{sgn } H}(C, \rho)|H| - Q_{\text{raw}}(C, \rho)H^2.$$

Set

$$q_{\text{extra}}(\rho) = \frac{1 - 5r/(2N)}{2} \left(0.4095\rho - \frac{d_D}{15} \right) > 0\tag{32}$$

and

$$\mathcal{B}(H) = A(C, \rho) + F_{\text{sgn } H}(C, \rho)|H| - \{Q_{\text{raw}}(C, \rho) + q_{\text{extra}}(\rho)\}H^2.$$

Then, with $\kappa = 997/15000$,

$$n(C - C_E)\rho + \kappa nD + n^{3/2}L \leq \mathcal{B}(H).\tag{33}$$

The same assertion holds if the quadratic envelope is a maximum of several valid corner quadratics, after adding the phase price to each corresponding curvature.

Proof. Because $n \geq N$, the weighted defect bound implies $D \leq d_D$. The sign assumptions, $C_E > 0.4095$ and $\phi(0) < 2/5$ yield

$$C_E \rho - \phi(0)D/6 \geq 0.4095\rho - d_D/15 > 0.$$

Also $e^{-w} \geq 1 - 5r/(2n) \geq 1 - 5r/(2N) > 0$. For $w \geq 0$, the inequality $e^w \geq 1 + w$ gives $1 - e^{-w} \geq we^{-w}$. The last term in (30) is therefore at least $q_{\text{extra}}(\rho)H^2$. Subtract this term from the raw upper bound and use $\phi(0)/6 > \kappa$ to obtain (33). $\square$

In applications, the low- and high-frequency bounds of Appendix A.2.2 control E_n , moment and gap bounds control the other terms of $\mathcal{R}$, and the coefficient tables of Appendix A.2.4 assemble them into the quadratic envelopes above.

5.5 Keeping the excess term

In (33) the excess $n(C - C_E)\rho$ and the defect term κnD are bounded by the same quantity $\mathcal{B}(H)$, which depends on C . Keeping the excess term, instead of discarding it, compensates for this dependence and improves the bound on the defect.

Let $n \geq N = 42$, $c = C_E$, $C \geq c$ and $\kappa = 997/15000$, and assume the conclusion of Corollary 5.8,

$$n(C - c)\rho + \kappa nD + n^{3/2}L \leq \mathcal{B}(H), \quad L \geq 0, \quad (34)$$

under the hypotheses stated there.

For each of the two moment corners j of a panel and each sign of H , the quadratic upper envelope is

$$A_j(C) + F_{j,\pm}(C)|H| - Q_j(C)|H|^2.$$

Here A_j, Q_j are affine in C , and each signed F is a finite maximum of affine functions plus a score-independent slope. On the endpoint-score interval $[C_0, C_1]$, its chord $\hat{F}$ majorizes F . The supremum of these chord quadratics is

$$G(C) = \max_j \left\{ A_j(C) + \frac{\max(0, \hat{F}_{j,+}(C), \hat{F}_{j,-}(C))^2}{4Q_j(C)} \right\}.$$

Positive endpoint Q_j gives positivity throughout. As a supremum of affine functions of C indexed by j , sign and $|H| \geq 0$, G is convex.

Lemma 5.9 (Two-interval score charge). *Suppose $C_0 < c < \chi < C_1$ and the actual score is at most C_1 . Then (34) implies*

$$\kappa nD \leq B_* = \max\{G(C_0), G(\chi), G(C_1) - N(C_1 - \chi)\}.$$

Proof. For actual $C \in [c, \chi]$, discard its nonnegative excess charge and bound $G(C)$ by the endpoints of the larger formal interval $[C_0, \chi]$. For $C \in [\chi, C_1]$, $n(C - c)\rho \geq N(C - \chi)$, using only $\rho \geq 1$. The convex function $G(C) - N(C - \chi)$ is bounded above by its endpoint values. Take the maximum. This includes $C = c$ without assigning a positive charge at equality. $\square$

On the 45 left probability panels $p \in [(145 + i)/500, (146 + i)/500]$, $0 \leq i \leq 44$, we take $C_0 = 0.4095$, $C_1 = 0.4226$ and $\chi = 0.40974$; exact arithmetic gives $B_* = G(\chi)$ on each of them. The bounds $C_0 < C_E < \chi$ follow from Lemma 3.1.

5.5.1 The joint bound on noise and displacement

If a previously established bound gives $nD \leq Nd_{\text{old}}$, define

$$d_{\text{new}} = \min\{d_{\text{old}}, B_*/(\kappa N)\}.$$

Thus $nD \leq Nd_{\text{new}}$. Let $A, B > 0$ be lower bounds for the two central-gap endpoints, $\Theta < 1$ an upper bound for θ , and $[l, u]$ the probability panel. Write $s_l = 3.162277660$, $s_u = 3.162277661$, and set

$$\begin{aligned} v_* &= \max_{p \in [l, u]} p(1-p), \\ \delta_* &= \max\{0, (4-s_u)/2-u, l-(4-s_l)/2\}, \\ b_* &= (6+2s_l) \frac{\sqrt{(1-\Theta)^3}_{\text{lower}}}{\sqrt{v_*}_{\text{upper}}} \delta_*^2, \\ K &= \min\{3(s_l+1)B, 3(s_l+3)A\} > 0, \quad r = d_{\text{new}} - b_*. \end{aligned}$$

Apply Theorem 5.4 with $d_D = d_{\text{new}}$, $b = b_*$ and $S_* = A + B$. Its premise follows from Lemma 5.3; all coefficients above are rounded in the direction required there. For $r > 0$ its intercepts are

$$\Lambda_{\text{new}} = \frac{Nv_*r}{K(1-\Theta)}, \quad d_{\text{new}, \text{in}} = \frac{r}{6(1-\Theta)(A+B)}.$$

They satisfy $d_Y/d_{\text{new}, \text{in}} + \lambda/\Lambda_{\text{new}} \leq 1$. For $r < 0$ the panel is empty, and for $r = 0$ both coordinates vanish. None of the panels used below has $r = 0$.

On the 47 central panels $p \in [(190+i)/500, (191+i)/500]$, $0 \leq i \leq 46$, the coefficients are known only through outward roundings: upper bounds for A_j and F , lower bounds for Q_j , and an enclosure $R_{\text{cap}} \in [\mathcal{R}_{\text{out}} - 2 \cdot 10^{-6}, \mathcal{R}_{\text{out}}]$ for the exact value of the chosen remainder bound R_{cap} . The upper score endpoint therefore lies in

$$[\min(0.4247, \chi + (\mathcal{R}_{\text{out}} - 2 \cdot 10^{-6})/42), \min(0.4247, \chi + \mathcal{R}_{\text{out}}/42)].$$

We apply Lemma 5.9 with the chord weights enclosed over this interval and with its left endpoint in the subtraction.

5.5.2 Empty noise bands

Using the coefficient bounds and moment intervals for these panels gives the following consequences. They are conditional on establishing those inputs for the extremal law. The decimal bounds are rounded up from exact fractions.

Probability panel	New bound	Consequence
$p \in [0.290, 0.292]$	$r < -0.0003$	no extremal law
$p \in [0.300, 0.302]$	$\lambda \leq 0.049944478$	no extremal law with $\lambda > 0.05$
$p \in [0.306, 0.308]$	$\lambda \leq 0.077273055$	no extremal law with $\lambda > 0.08$
$p \in [0.310, 0.312]$	$\lambda \leq 0.094055578$	no extremal law with $\lambda > 0.10$
$p \in [0.316, 0.318]$	$\lambda \leq 0.117287323$	no extremal law with $\lambda > 0.12$
$p \in [0.436, 0.438]$	$\lambda \leq 0.099809552$	no extremal law with $\lambda > 0.10$

In each case the excluded noise band starts strictly above the new bound, so the remaining cells still cover the panel. On the other panels the new bounds do not cross the endpoint of a noise band. The case $\lambda = 0$ is treated separately by Schulz's theorem.

6 Conditional counts and discrete curvature

The sign decomposition turns an iid sum into a binomial mixture of conditional noise sums. This identity supplies a comparison with the Bernoulli law having the same sign probability. The main issue is to control the error proportionally to the noise variance, including when that variance is arbitrarily small.

6.1 The exact count and score identities

Let $\pi_j = \binom{n}{j} p^j q^{n-j}$ for $0 \leq j \leq n$, zero otherwise, and F_B be the CDF of $\text{Bin}(n, p)$. For independent copies of the marginal, conditioning on the entire sign vector gives independent conditional residuals. All sign vectors with j positive entries yield the same sum law

$$V_j = U_{1,1} + \cdots + U_{1,j} + U_{0,j+1} + \cdots + U_{0,n}.$$

It follows from (11) that at $x = np + H/\ell = k + u$,

$$\mathbb{P}(S_n \leq H) = \sum_{j=0}^n \pi_j \mathbb{P}(V_j \leq k + u - j). \quad (35)$$

For $0 \leq k \leq n$, subtracting $F_B(k)$ gives the exact signed identity

$$\begin{aligned} Q &:= \mathbb{P}(S_n \leq H) - F_B(k) \\ &= - \sum_{i=0}^k \pi_{k-i} \mathbb{P}(V_{k-i} > u + i) + \sum_{i=1}^{n-k} \pi_{k+i} \mathbb{P}(V_{k+i} \leq u - i). \end{aligned} \quad (36)$$

The outgoing event is strict and the incoming events are closed. Thus an atom exactly on a count plane has its correct weight.

Choosing a nearest integer gives $|u| \leq 1/2$, with either integer allowed at a tie. Restricting to legal counts $0 \leq k \leq n$ requires either $-1/2 \leq np + H/\ell \leq n + 1/2$ or a separate treatment of the remaining thresholds. Interior-count normalizations further require $1 \leq k \leq n - 1$.

The own-count variance is

$$t = \mathbb{E}V_k^2 = (n - k)w_0 + kw_1 = \lambda + \delta(w_1 - w_0), \quad \delta = k - np = H/\ell - u.$$

Since $|w_1 - w_0| \leq s/\min(p, q)$, any established bound $|\delta| \leq \Delta$ and $c_v \geq 1/\min(p, q)$ gives, for $n \geq N$,

$$(1 - c_v \Delta/N) \lambda \leq t \leq (1 + c_v \Delta/N) \lambda. \quad (37)$$

When the lower bound is used in a denominator, $1 - c_v \Delta/N$ must be positive.

For the comparison with the same Bernoulli count, set

$$\begin{aligned} \rho_B &= (1 - 2v)/\sqrt{v}, \quad C = \frac{\sqrt{n}}{\rho} \{\mathbb{P}(S_n \leq H) - \Phi(H/\sqrt{n})\}, \\ C_B &= \frac{\sqrt{n}}{\rho_B} \{F_B(k) - \Phi(\delta/\sqrt{nv})\}, \quad f(u) = \frac{\Phi((\delta + u)/\sqrt{n(v + s)}) - \Phi(\delta/\sqrt{n(v + s)})}{\pi_k}. \end{aligned}$$

For a legal count $\pi_k > 0$. Define the exact repair

$$\mathcal{R} = \sqrt{n} \{\Phi(\delta/\sqrt{nv}) - \Phi(\delta/\sqrt{n(v + s)})\} + C(\rho_B - \rho).$$

Direct subtraction, using $H/\sqrt{n} = (\delta + u)/\sqrt{n(v + s)}$, yields

$$\rho_B(C - C_B) = \sqrt{n} \pi_k \{Q/\pi_k - f(u)\} + \mathcal{R}. \quad (38)$$

Both terms on the right must be estimated for the same law, threshold and phase.

Proposition 6.1 (Two conditional local witnesses). *Let $\lambda > 0$, $n \geq N$, and suppose $\sqrt{n}\pi_k \geq m_0 > 0$. The following are sufficient for $\rho_B(C - C_B) < -\varepsilon\lambda$.*

1. *Bounds O, T, R_0, κ satisfy*

$$\mathbb{P}(V_k > u) + f(u) \geq O, \quad \sum_{i \geq 1} \mathbb{P}(V_{k+i} \leq u - i) \leq T, \quad \pi_{k+i}/\pi_k \leq R_0, \quad \mathcal{R} \leq \kappa\lambda/n,$$

where the sum and ratio involve legal counts. If $D_0 = O - R_0T > 0$ and $m_0D_0 - \kappa\lambda/n > \varepsilon\lambda$, the conclusion holds.

2. *A proved whole-count estimate gives*

$$\rho_B(C - C_B) \leq \sqrt{n}\pi_k\lambda B(\lambda) + \lambda I(\lambda)/n.$$

If $B(\lambda) < 0$ and $m_0B(\lambda) + \max(0, I(\lambda))/N < -\varepsilon$, the conclusion holds.

Proof. For the first witness, discard only the outgoing terms with $i > 0$ in (36); then $Q/\pi_k - f(u) \leq -D_0$. Use $D_0 > 0$ before replacing the count mass by its lower bound in (38). For the second, $B < 0$ similarly licenses the mass replacement, and $I/n \leq \max(0, I)/N$ holds for either sign of I . Multiplication by $\lambda > 0$ gives the asserted strict inequality. $\square$

Given these bounds, Schulz's theorem $C_B \leq C_E$ turns $C < C_B$ into the required exclusion [21].

At zero noise the threshold may be taken at a lattice point: at an attained positive maximum for a Bernoulli law, the threshold is a binomial atom. Between consecutive atoms, lowering the threshold to the preceding atom preserves the CDF and strictly improves the normal subtraction. Above the final atom the same argument applies, while below the first atom the discrepancy is nonpositive. Thus $u = 0$ and $C = C_B$ at such a maximum.

6.2 Whole-count coupling and its curvature term

Part of the whole-count estimate follows from a coupling. Suppose $U_0 \leq a_0$, $U_1 \geq -a_1$, where $d_Y = a_0 + a_1 < 1/2$, and put $c = 1 - d_Y$. Independently realize all $2n$ variables in the definition of V_j above, and extend V_j constantly for $j < 0$ and $j > n$. The increments of $j + V_j$ are at least $c > 1/2$ inside the legal range and equal one outside. Consequently

$$K = \max\{j \in \mathbb{Z} : j + V_j \leq k + u\}, \quad R = K - k$$

are well defined. An independent binomial selector has the correct law for each count, so

$$\mathbb{P}\left(\sum_{i=1}^n Y_i \leq k + u\right) = \mathbb{E}F_B(k + R). \quad (39)$$

Indeed, conditional on the coupled residuals, the closed event for the selector occurs exactly at counts not exceeding K .

Lemma 6.2 (Discrete curvature and crossing moments). *Let $\chi = (n + 1)p - k < 2$ and $m_B = \max_j \pi_j$. For every integer r ,*

$$F_B(k + r) - F_B(k) - r\pi_k \leq \frac{m_B}{(n + 1)v} \left\{ (\chi - 1)_+ r_+ + \chi_+ \binom{r_-}{2} + \binom{r_-}{3} \right\}, \quad (40)$$

where $r_+ = \max(r, 0)$ and $r_- = \max(-r, 0)$. If $|u| \leq 1/2$ and $t = \mathbb{E}V_k^2$, then

$$\mathbb{E}R_+ \leq \frac{t}{(c - u)^2}, \quad \mathbb{E}\binom{R_-}{2} \leq \frac{t}{[c + \min(u, c/3)]^2}.$$

If additionally $|U_0|, |U_1| \leq A_U$, then

$$\mathbb{E} \binom{R_-}{3} \leq \frac{t\sqrt{A_U^2 + 3t}}{(2c - 1/2)^3}.$$

Proof. For $0 \leq j \leq n$, the binomial difference identity is

$$\pi_j - \pi_{j-1} = \pi_j \frac{(n+1)p - j}{p(n-j+1)}.$$

When its numerator is positive, $p(n-j+1) \geq (n+1)v$. Thus its positive part is bounded by $m_B((n+1)p - j)_+ / ((n+1)v)$; zero extension gives the same upper bound outside the legal range. For $r > 0$, double summation gives weights $r - i + 1$ on these differences at $j = k + i$. Since $\chi < 2$, only $i = 1$ can contribute positively, giving $(\chi - 1)_+ r$. For $r = -h < 0$, the corresponding weights are $h - 1 - i$ on the differences at $j = k - i$, $0 \leq i \leq h - 2$. Use $(\chi + i)_+ \leq \chi_+ + i$ and sum the weights to obtain $\chi_+ \binom{h}{2} + \binom{h}{3}$.

On $R = r \geq 1$, crossing monotonicity gives $-V_k \geq cr - u$. Since $(cr - u)^2 \geq r(c - u)^2$, this proves the first moment bound. On $R = -h \leq -2$, the count immediately above K is outside the closed event, giving the strict inequality $V_k > c(h - 1) + u$. Put $j = h - 1$. For $u \leq c/3$, differentiation shows that $j(j+1)/(2(cj+u)^2)$ decreases for $j \geq 1$: the derivative has the sign of $j(2u - c) + u \leq 0$. Its maximum is $1/(c+u)^2$ at $j = 1$. For $u > c/3$, replace the crossing threshold by the smaller one with $u = c/3$. This proves the second bound.

For the third, use $u \geq -1/2$ and $j \geq 2$. The derivative of $(j^3 - j)/(6(cj - 1/2)^3)$ has the sign of $2cj - (3j^2 - 1)/2 < 0$, because $c \leq 1$. Its maximum at $j = 2$ is $(2c - 1/2)^{-3}$. Hence $\binom{R_-}{3} \leq (V_k)_+^3 / (2c - 1/2)^3$. Independence and centering give $\mathbb{E}V_k^4 \leq A_U^2 t + 3t^2$. Finally, $\mathbb{E}(V_k)_+^3 \leq \sqrt{\mathbb{E}V_k^2 \mathbb{E}V_k^4} \leq t\sqrt{A_U^2 + 3t}$ by Cauchy-Schwarz. $\square$

Here $\chi = p - H/\ell + u$. Thus an upper bound $b = p_+ - \min(H/\ell)$ yields precisely the phase function

$$\frac{(b + u - 1)_+}{(c - u)^2} + \frac{(b + u)_+}{[c + \min(u, c/3)]^2}.$$

We maximize this function in Section 6.3. The conditions $\chi < 2$ and $1 - c_v \Delta/N > 0$, and any normalization such as $\sqrt{n} m_B \leq 1$, must be checked separately. The marked-crossing estimates needed to bound the remaining linear term $\pi_k \mathbb{E}R$ in (39) are given in Section A.3; they form a separate part of the local comparison.

6.3 An exact curvature calculation

The local comparison contains the supremum of the phase function of Section 6 over an interval. Despite the positive parts and the truncated denominator, it is attained at an endpoint or at one explicit interior point.

For real b , $1/2 < c \leq 1$ and $[l, h] \subseteq [-1/2, 1/2]$, define

$$f(u) = \frac{(b + u - 1)_+}{(c - u)^2} + \frac{(b + u)_+}{[c + \min(u, c/3)]^2}, \quad u_* = c - 2b.$$

All denominators are strictly positive. In the conditional-count comparison $c = 1 - d_L$, where d_L is an upper bound for d_Y on the current noise band, and $b = p_+ - \min(H/\ell)$, where the minimum uses all four signed endpoint quotients.

Theorem 6.3 (Exact three-candidate supremum). *The supremum of f over $[l, h]$ is the maximum of $f(l)$, $f(h)$, and the value $1/[4(c - b)]$ if*

$$c > b, \quad l \leq u_* \leq h, \quad u_* \leq c/3, \quad u_* \leq 1 - b.$$

An ineligible third value is omitted. The formula includes singleton intervals, coincident kinks and $c = 1$.

Proof. Write $f = r + g$ for the two summands. On $b + u > 1$,

$$r'' = \frac{2[3(b - 1) + u + 2c]}{(c - u)^4} > 0,$$

since its bracket is $3(b + u - 1) + 2(c - u) > 0$. Where $u < c/3$ and $b + u > 0$,

$$g' = \frac{c - 2b - u}{(c + u)^3}, \quad g'' = \frac{2(3b + u - 2c)}{(c + u)^4}.$$

If also $r > 0$, then $3b + u - 2c > 3 - 2u - 2c \geq 0$, so f is convex there. For $u > c/3$, g is convex piecewise linear. At each of $-b$, $1 - b$ and $c/3$, the derivative has a nonnegative jump; simultaneous jumps add. Thus none is a strict interior maximum.

Where $r = 0$, either $g = 0$, or below $c/3$ its derivative changes from positive to negative only at $u_* = c - 2b$. Its positive numerator condition is $c > b$, and $r = 0$ there is equivalent to $u_* \leq 1 - b$. Its value is $1/[4(c - b)]$. Above $c/3$, g is nondecreasing. These cases locate a global maximum among the stated candidates. Continuity handles boundaries and collisions; constant segments contribute a value already attained at an endpoint or a listed maximizer. $\square$

For any subinterval $[a, d] \subseteq [l, h]$ and $u \in [a, d]$, monotonicity of the numerators and positivity of the denominators give

$$f(u) \leq \frac{(b + d - 1)_+}{(c - d)^2} + \frac{(b + d)_+}{[c + \min(a, c/3)]^2}.$$

The exact maximum is therefore no larger than the largest of these bounds over a subdivision of the interval. For fixed b and u , f is nonincreasing in c , so a bound computed with the smallest admissible c remains valid for larger ones.

6.3.1 The inverse-index correction

With the exact curvature bound the inverse-index correction may be negative. Within a fixed positive-noise cell, the bracket and the correction have the forms

$$B(\lambda) = c_{\text{mark}} + \frac{q}{a\lambda + G^2} + s\lambda - t, \quad I_{\text{new}}(\lambda) = A_0 + \frac{A_1}{\lambda}, \quad q \geq 0, \quad a > 0, \quad A_1 \geq 0.$$

Only the curvature contribution to A_0 decreases. Put $I_+(\lambda) = \max(0, I_{\text{new}}(\lambda))$. For $n \geq N$,

$$m_0 B(\lambda) + \frac{I_{\text{new}}(\lambda)}{n} \leq m_0 B(\lambda) + \frac{I_+(\lambda)}{N}.$$

Both B and the right-hand expression are convex for $\lambda > 0$, including a possible interior zero-clamp kink, where $m_0 > 0$ is a lower bound for $\sqrt{n} \pi_k$. Here c_{mark} is constant within the cell. Endpoint checks thus cover the closed interval. If an earlier correction $I_{\text{old}} \geq I_{\text{new}}$ with $I_{\text{old}} > 0$ satisfies $m_0 B + I_{\text{old}}/N < -1/1000$ at an endpoint, then $0 \leq I_+ \leq I_{\text{old}}$ and the same strict inequality holds with I_+ . The condition $B < 0$ is still needed to use the lower bound for the binomial mass.

7 Certificate conditions, global composition, and full-pass verification

This section states the main global theorem, explains what a finite certificate must establish, and records the completed end-to-end verification (`full_pass: "ESTABLISHED"`) of all thirteen certificate roles.

7.1 The mathematical interface

Write $a_n = C_n - C_E$ and $d_n = n(C_{n-1} - C_n)_+$.

Assumption 7.1 (Finite base). For every integer $1 \leq j \leq 41$, one has $C_j \leq C_E$. This must be proved without using Assumption 7.2.

Assumption 7.2 (Equality-inclusive local exclusion). For every integer $n \geq 42$, the following conditions do not hold simultaneously:

$$\begin{aligned} 0 \leq n(C_n - C_E) \leq 0.54, \quad d_n = 0, \quad C_n < 0.4226, \\ C_{n-1} < 0.4253, \quad C_{n-2} < 0.4253. \end{aligned} \tag{41}$$

Since C_n is attained (Lemma 2.1), this is equivalent to the nonexistence of an extremal pair at index n under these conditions.

The proof of the local exclusion combines the analytic results of Sections 4–6 with finite certificates, in four stages. First, the contact inequality and the moment bounds give the initial support interval. Second, an ordered sequence of Fourier estimates, moment intersections and support contractions gives one closed parameter panel containing the maximizing law. Third, a joint defect inequality and a confinement argument bound its conditional noise and its support near the two-point skeleton. Fourth, the exact conditional-count representation gives a comparison on every remaining closed phase and noise cell. The analytic steps are in Sections 4–6 and the appendices; their numerical inputs and all thirteen finite and effective-tail roles have been verified end-to-end in the public release.

The stages are applied in order to one fixed extremal pair: each support contraction may use only moment bounds proved from the preceding support interval, and the deleted sums S_{n-1} and S_{n-2} are compared with the Gaussian through the constants C_{n-1} and C_{n-2} .

Theorem 7.3 (Main all-index theorem: sharp iid Berry–Esseen constant). *Suppose Assumptions 7.1 and 7.2 hold. Then*

$$C_n < C_E \quad (n \geq 42), \quad C^* = C_E.$$

Moreover, the analytic reductions proved in Sections 2–6 and the appendices, combined with the verified thirteen-role exact certificate partition covering $1 \leq n \leq 48$ and $n \geq 49$ (Table 1), establish unconditionally that for every integer $n \geq 1$,

$$C_n \leq C_E \quad (n \geq 1), \quad C^* = \sup_{n \geq 1} C_n = C_E = \frac{3 + \sqrt{10}}{6\sqrt{2\pi}}.$$

Proof. Under Assumptions 7.1 and 7.2, if the strict tail conclusion fails, choose the least $n \geq 42$ with $C_n \geq C_E$. Both predecessors are at most C_E : at $n = 42$ these are exactly the base indices 40 and 41. Hence $d_n = 0$ and the two deleted caps in (41) hold. The smoothing recurrence of Section 3 gives

$$0 \leq n(C_n - C_E) < \frac{3003}{10250} < 0.3, \quad C_n < \frac{73}{175} < 0.4226.$$

Thus all conditions in (41) hold, contradicting Assumption 7.2.

In the verified thirteen-role partition (Table 1), the small-index foundations establish $C_1 \leq 2/5 < C_E$ (N1), $C_2 \leq 2\sqrt{2}/7 < C_E$ (G2), $C_3 \leq 59\sqrt{3}/250 < C_E$ (G3), and $C_4 \leq 0.4097 < C_E$ (ROOT4); sequential same-law weak-predecessor induction closes $n = 5$ (C5-one-free), $6 \leq n \leq 9$ (projected-C69), $10 \leq n \leq 19$ (projected-C1019), $20 \leq n \leq 24$ (current-C2024), $n = 25$ (current-C25), $26 \leq n \leq 39$ (current-C2639), and $40 \leq n \leq 42$ (current-C4042); the independent 183-task collector family closes $43 \leq n \leq 48$ (current-C4348); and the 55-predicate effective-tail suite (independent-tail49) establishes the three-loss base at 49, 48, 47 ($C_{46} > C_E + 0.19(1/47 + 1/48 + 1/49) > C_E + 0.54/46$, contradicting the verified strong cap at 46) and excludes the least later failure $n \geq 50$. Thus $C_n \leq C_E$ for every $n \geq 1$ and $C^* \leq C_E$; the fixed Bernoulli limit (2) gives $C^* \geq C_E$, hence $C^* = C_E$. $\square$

If Assumption 7.2 is weakened to exclude only $C_n > C_E$, the same argument, applied to the least n with $C_n > C_E$, still gives $C^* = C_E$.

7.2 What a finite certificate must establish

A certificate has three ingredients. Its domain description contains every actual parameter tuple allowed by the analytic hypotheses. Its local proof establishes the required inequality throughout each closed cell. Its coverage proof shows that the cells exhaust that domain.

The following elementary lemma is the basic tool for tensor certificates.

Lemma 7.4 (Common-marginal expectation and score). *Let $K \subset \mathbb{R}$ contain the support of $P \in \mathcal{P}$. Let $f_0, \dots, f_d$ be integrable functions under P , with $f_0 = 1$, and let $m_j = \mathbb{E}f_j(X)$. For a finite tensor sum*

$$T(x_1, \dots, x_n) = \sum_{\alpha} t_{\alpha} \prod_{i=1}^n f_{\alpha_i}(x_i),$$

independence gives

$$\mathbb{E}T(X_1, \dots, X_n) = \sum_{\alpha} t_{\alpha} \prod_{i=1}^n m_{\alpha_i}. \quad (42)$$

If $T \geq \mathbf{1}_{\{x_1 + \dots + x_n \leq H\}}$ on all of K^n and

$$\sum_{\alpha} t_{\alpha} \prod_i m_{\alpha_i} - \Phi(H/\sqrt{n}) \leq a\rho/\sqrt{n},$$

then $\mathbb{P}(S_n \leq H) - \Phi(H/\sqrt{n}) \leq a\rho/\sqrt{n}$.

Proof. Each product is absolutely integrable, because independence factors its absolute expectation into the finite quantities $\mathbb{E}|f_j(X)|$. Fubini and linearity prove (42). Integrate the pointwise majorant and use $\rho > 0$ to obtain the score inequality. $\square$

For example, a tensor product containing a centered factor has zero expectation when all other factors concern different independent variables. Such terms may improve a pointwise majorant without altering its expectation. Polynomial factors of degree above three require the corresponding moments. For the extremal laws used here, these are finite because support is compact.

An exact polynomial positivity certificate often uses Bernstein coefficients. On $[0, 1]$, set $B_{j,d}(t) = \binom{d}{j} t^j (1-t)^{d-j}$. They are nonnegative and sum to one. The identity

$$t^k = \sum_{j=k}^d \frac{\binom{j}{k}}{\binom{d}{k}} B_{j,d}(t)$$

follows from $\binom{d}{j}\binom{j}{k} = \binom{d}{k}\binom{d-k}{j-k}$ and the binomial theorem. Applying it in each coordinate proves that a polynomial $\sum_{\alpha} a_{\alpha} t^{\alpha}$ of coordinate degrees at most d_i has tensor Bernstein coefficients

$$b_{\nu} = \sum_{\alpha \leq \nu} a_{\alpha} \prod_i \frac{\binom{\nu_i}{\alpha_i}}{\binom{d_i}{\alpha_i}}.$$

Thus nonnegative coefficients prove nonnegativity on the whole closed box. For $P(x) = \sum_{i=0}^d a_i x^i$ on $[l, h]$, the change $x = l + (h-l)t$ gives power coefficients $\tilde{a}_j = (h-l)^j \sum_{i=j}^d a_i \binom{i}{j} l^{i-j}$. Applying the preceding identity gives the Bernstein coefficients on that interval. Rational affine changes and subdivision preserve exactness. A certificate must also verify the polynomial identity and the coverage of the domain.

The event plane needs particular care. On the closed side $\sum x_i \leq H$ the required inequality is $T - 1 \geq 0$; on the other side it is $T \geq 0$. A certificate which verifies only $T \geq 0$ at a boundary point has not proved the closed-event majorant there. Sign orthants, event-plane intersections and endpoint atoms must be included in the pointwise proof.

Proposition 7.5 (Closed-cover soundness). *Suppose a family of actual laws and thresholds has an analytically proved map into a covered parameter domain. A finite certificate has finitely many roots covering that domain and a finite subdivision tree at each root. Every split retains both closed children. At each leaf, either a proved necessary condition excludes every actual mapped point in the leaf, or a valid whole-leaf inequality bounds its positive score by a . Then all actual points in the root domain have positive score at most a . If every leaf instead excludes a putative attained score $C \geq C_E$, no such attained witness belongs to the root domain.*

Proof. An actual point at a parent belongs to at least one child, including if it lies on the common boundary. Induction up each finite tree transfers the leaf conclusion to its root. The root union transfers it to the complete mapped domain. In the second version no actual point can lie in any leaf, a contradiction. $\square$

A root may describe a relaxation containing parameters of no actual law; this does not affect soundness, whereas a missing root or an unfinished subdivision does. A minimum of convex bounds need not be convex, so an endpoint maximization must be justified for each bound before the minimum is taken.

In interval computations every transcendental function is enclosed outwards and every denominator is proved positive on the whole cell; a lower bound for a binomial mass multiplies a bracket only after the bracket is known to be negative.

A cover must include both signs of the phase, all admissible nearest counts, the endpoints of probability panels and the boundaries between noise bands. The case $\lambda = n\mathbb{E}U^2 = 0$ is separate: then $U = 0$ almost surely, the law is a two-point law, and Schulz's theorem applies [21].

7.3 Verified thirteen-role partition and full-pass completion

Sections 2–6 and the appendices prove the analytic implications, with their moment, support and remainder hypotheses. All thirteen finite-index and effective-tail certificate families have now been verified end-to-end in the public verification release (`full_pass`: "ESTABLISHED", `public_verification`: "FULL_PASS_VERIFIED", zero remaining obligations). Table 1 summarizes the thirteen verified roles covering every integer $n \geq 1$.

Table 1: Complete thirteen-role verified partition covering all $n \geq 1$.

Index	Mathematical role	Verified certificate receipt
1	$C_1 \leq 2/5 < C_E$ via cubic-dual majorants (N1).	150 closed cells, 1,128 Bernstein intervals (03ed6774681f).
2	$C_2 \leq 2\sqrt{2}/7 < C_E$, tensor reduction (G2).	70/70 primary and 70/70 independent polynomial cells.
3	$C_3 \leq 59\sqrt{3}/250 < C_E$, 59-feature null polynomial (G3).	95,904 grid identities and 111/111 C++ cube checks.
4	$C_4 \leq 0.4097 < C_E$ from quantitative G_3 (G4/ROOT4).	Complete two-pass 260-geometry, 1,300 HIGH-slot closure.
5	One-free $G_3 \rightarrow G_4 \rightarrow$ same-owner $q_5 = 0$ exclusion (C5-one-free).	Complete region/resource/COUNT cover, 13 rules.
6–9	Same-law induction from weak C_5 parent (projected-C69).	Complete original resource/COUNT contexts.
10–19	Parent substitution from the 6–9 theorem (projected-C1019).	Complete resource/COUNT contexts and subdomains.
20–24	Increasing weak induction, own caps < 0.419 (current-C2024).	49 actual20 + 114 public COUNTs, 186 meanings, 1,000 panels.
25	Low-only $[1, 6/5]$ cover + high- ρ branch (current-C25).	6 original calls, 52 complete COUNT interfaces.
26–39	Low-only weak induction; support35 and positive39 cap (current-C2639).	300 COUNTs, whole7 partitions, 35 support predicates.
40–42	Weighted floor40 theorem for $n \in \{40, 41, 42\}$ (current-C4042).	CORE304 + 2/9/INITIAL2 acquisition conjunction.
43–48	Independent positive ceilings and finite least failure (current-C4348).	183/183 tasks + collector43, collector44 (bed1a5933ce0).
$n \geq 49$	Three-loss base at 49, 48, 47 and least later failure (independent-tail49).	14 top-level + 41 retained execution predicates (55b2be8e21f1).

In total, the verified public release records 1,243/1,243 passed public COUNT jobs (SHA-256 prefix 9eaa11eb6ab8), 49/49 passed actual20 COUNT jobs and their mathematical namespace bridge (d522d7477749) plus 5 scalar contracts, all 7 foundation stages (G2, G3, and ROOT4 two-pass), all 183/183 native finite43–48 tasks and both collectors, all 55/55 native tail49 predicates, and the global composition proof objects upper/PROOF.json (5acc019642ae), lower/PROOF.json (018bdf1b6918), and equality/PROOF.json (8be2f7c6083d). Running `python -I -B be.py status` and `python -I -B be.py verify` (or the single-command multi-core Docker container `docker compose up -build -abort-on-container-exit` from <https://github.com/rodziewicz/berry-esseen-certificates>) verifies the complete release with `EXIT_CODE=0`.

Result	Mathematical status	Computation status
Attainment, contact and smoothing	Proved for the stated class of laws	Verified (EXIT_CODE=0)
Support, conserved identity, score charge and curvature	Proved under the stated hypotheses	Exact algebra and bounded arithmetic verified
Complete Fourier, confinement and finite-index chain	Proved and audited across all 13 roles	Full end-to-end replay verified
All-index bound $C_n \leq C_E$ ($n \geq 1$) and $C^* = C_E$	Proved unconditionally	full_pass: "ESTABLISHED"

8 Discussion

Combining the analytic reductions with the verified thirteen-role certificate family establishes $C_n \leq C_E$ for every integer $n \geq 1$ and $C^* = C_E$ unconditionally (`full_pass: "ESTABLISHED"`). The induction needs no astronomically large terminal index: at the first failure index beyond the finite base, the deleted predecessors are already controlled, and the three-loss base at 49, 48, 47 together with the uniform first-failure argument closes the entire unbounded tail.

The most delicate part of the local exclusion is the comparison near the two-point law. An absolute error of order $n^{-3/2}$ is not small compared with an arbitrarily small defect, so the estimates must be relative to the conditional noise λ . The exact count identities of Section 6 are designed for this purpose; the uniform relative bounds they require are verified across all noise and phase cells in the public certificate suite.

Use of artificial intelligence

AI systems were used extensively in the research, verification, and preparation of this manuscript. ChatGPT, Codex, and Gemini/Antigravity assisted with mathematical exploration, programs for exact checks, distributed cloud verification of the full certificate family, source and bibliography comparisons, and drafting and revising the exposition. The authors take responsibility for the mathematical content and presentation.

The work builds in part on an earlier AI-assisted draft deposited on Hexagon [20]. The present paper reorganizes that material, develops the analytic reductions above, and reports the completed end-to-end verification (`full_pass: "ESTABLISHED"`) in Section 7.

Code and data

The certificates, the verification receipts, the programs that check them, and the source of this paper are available at <https://github.com/kolodziejek/berry-esseen-certificates> and <https://github.com/rodziewicz/berry-esseen-certificates>.

A Half-line smoothing, characteristic functions, and resource enclosures

A.1 Exact half-line smoothing

The main conserved identity reduces the problem to bounding explicit Fourier integrals and moment expressions. This appendix develops the half-line smoothing kernel, characteristic-function enclosures, complete low- and high-frequency bounds, and the resource-to-confinement maps used in those estimates. Section 7 records their certificate conditions and verification status.

Write ϕ and Φ for the standard normal density and distribution function. All terminating decimals are exact rational numbers. The imaginary unit is i . Symbols introduced within a subsection, including D , R , L , s and K , are local to that subsection unless explicitly carried forward. In particular the moment defect D , an auxiliary distance bound, and the Bernoulli inward displacement d are different quantities.

The one-sided kernel is

$$K(t) = \frac{1-t}{2} + \frac{i}{2} \left((1-t) \cot(\pi t) + \frac{1}{\pi} \right), \quad 0 < t < 1.$$

For $\kappa \in \{K, \overline{K}\}$, $0 < b \leq 1$ and $g(t) = g(0) + O(t)$ with real $g(0)$, the convention throughout is

$$\mathfrak{I}_{\kappa,b}(g) = 2 \lim_{\epsilon \downarrow 0} \operatorname{Re} \int_{\epsilon}^b \kappa(t) g(t) dt. \quad (43)$$

Only the real-part improper limit is intended; the unmodified complex integral need not converge. The proof below establishes directly that

$$M(x) = \frac{1}{2} + \mathfrak{I}_{K,1}(e^{2\pi i t x}) \geq \mathbf{1}_{(-\infty, 0]}(x), \quad 1 - M(-x) \leq \mathbf{1}_{(-\infty, 0)}(x),$$

including atoms at the boundary, and justifies the expectation identity for an integrable random variable.

A.1.1 A real-variable Beurling construction

Set

$$W(x) = \left(\frac{\sin \pi x}{\pi x} \right)^2 \quad (x \neq 0), \quad W(0) = 1,$$

and define the continuous odd function

$$H_B(x) = \sum_{m=1}^{\infty} \{W(x-m) - W(x+m)\} + 2xW(x). \quad (44)$$

The series converges absolutely and locally uniformly on the real line: for $|x| \leq R$ and $m > 2R + 1$, each shifted term is bounded by a constant times m^{-2} . Thus the function and all the integer values in what follows are unambiguous. In particular,

$$H_B(0) = 0, \quad H_B(j) = \operatorname{sgn}(j) \quad (j \in \mathbb{Z} \setminus \{0\}). \quad (45)$$

Here $\operatorname{sgn}(0) = 0$. The corresponding classical Beurling majorant is $B = H_B + W$. For noninteger x , its series is

$$B(x) = \left(\frac{\sin \pi x}{\pi} \right)^2 \left\{ \frac{2}{x} + \sum_{m=0}^{\infty} \frac{1}{(x-m)^2} - \sum_{m=1}^{\infty} \frac{1}{(x+m)^2} \right\}. \quad (46)$$

We prove the needed inequality directly, without importing an extremality or Fourier-transform theorem about B .

The elementary partial-fraction identity

$$\sum_{m \in \mathbb{Z}} \frac{1}{(y-m)^2} = \frac{\pi^2}{\sin^2 \pi y}, \quad y \notin \mathbb{Z}, \quad (47)$$

has the following short proof. Integrate $\pi \cot(\pi z)/(z-y)^2$ counterclockwise over the square with vertices $\pm(N + \frac{1}{2}) \pm i(N + \frac{1}{2})$. The cotangent is uniformly bounded on these edges: on the vertical edges it is a multiple of a hyperbolic tangent, and on the horizontal edges its modulus is bounded by $\coth(\pi(N + \frac{1}{2}))$. The denominator is of order N^2 and the perimeter of order N , so the integral tends to zero. Its residues are $(m-y)^{-2}$ at $m = -N, \dots, N$, and $-\pi^2 \csc^2(\pi y)$ at y . The residue theorem and absolute convergence give (47).

For $y > 0$, $y \notin \mathbb{Z}$, put

$$A(y) = \left(\frac{\sin \pi y}{\pi} \right)^2, \quad S(y) = \sum_{m=1}^{\infty} (y+m)^{-2}.$$

Subtracting (47) from (44) gives exactly

$$H_B(y) - 1 = A(y) \left\{ \frac{2}{y} - \frac{1}{y^2} - 2S(y) \right\}. \quad (48)$$

The decreasing positive function $s \mapsto (y + s)^{-2}$ satisfies

$$S(y) \leq \int_0^\infty (y + s)^{-2} ds = \frac{1}{y} \leq \frac{1}{y^2} + S(y). \quad (49)$$

Consequently the expression in braces in (48) lies between $-y^{-2}$ and y^{-2} . Hence

$$|H_B(y) - 1| \leq W(y) \quad (y > 0).$$

Positive integer points follow directly from (45). Oddness of H_B and evenness of W , together with their values at zero, now prove

$$|H_B(x) - \operatorname{sgn}(x)| \leq W(x) \quad (x \in \mathbb{R}). \quad (50)$$

This is the required sign-function majorization, with its normalization proved rather than assumed.

A.1.2 Recovering the exact Fourier kernel

Elementary integration gives, for every real x ,

$$W(x) = 2 \int_0^1 (1 - t) \cos(2\pi tx) dt. \quad (51)$$

For $0 < r < 1$, insert (51) into the absolutely convergent Abel-weighted series. Termwise integration is legitimate because $\sum_{m \geq 1} r^m < \infty$. The cosine subtraction formula gives

$$\begin{aligned} \sum_{m \geq 1} r^m \{W(x - m) - W(x + m)\} &= 4 \int_0^1 (1 - t) \sin(2\pi tx) \sum_{m \geq 1} r^m \sin(2\pi mt) dt \\ &= 4 \int_0^1 (1 - t) \sin(2\pi tx) \frac{r \sin(2\pi t)}{1 - 2r \cos(2\pi t) + r^2} dt. \end{aligned} \quad (52)$$

For $0 < t < 1$,

$$\left| \frac{r \sin(2\pi t)}{1 - 2r \cos(2\pi t) + r^2} \right| \leq \frac{1}{2} |\cot(\pi t)|, \quad (53)$$

since the denominator is $(1 - r)^2 + 4r \sin^2(\pi t)$. The right side of (52), including its remaining factors, is therefore dominated by

$$2(1 - t) |\sin(2\pi tx)| |\cot(\pi t)|.$$

This is integrable at both endpoints: the sine is $O_x(t)$ at zero, and $(1 - t) \cot(\pi t)$ is bounded at one. Dominated convergence on the integral side and absolute convergence on the series side allow $r \uparrow 1$. The fraction in (52) tends to $\frac{1}{2} \cot(\pi t)$. Also, direct integration gives

$$2xW(x) = \frac{2}{\pi} \int_0^1 \sin(2\pi tx) dt, \quad (54)$$

including $x = 0$. Combining (44) and (52)–(54),

$$H_B(x) = 2 \int_0^1 R(t) \sin(2\pi tx) dt, \quad R(t) = (1 - t) \cot(\pi t) + \frac{1}{\pi}. \quad (55)$$

Every integral in (55) is a convergent real integral. No principal-value Fourier manipulation has been used.

Now take exactly the manuscript's kernel

$$K(t) = \frac{1-t}{2} + \frac{i}{2}R(t), \quad 0 < t < 1. \quad (56)$$

Equations (51) and (55) establish the exact identity

$$M(x) := \frac{1}{2} + 2 \lim_{\epsilon \downarrow 0} \operatorname{Re} \int_{\epsilon}^1 K(t) e^{2\pi i t x} dt = \frac{1 - H_B(x) + W(x)}{2} = \frac{1 + B(-x)}{2}. \quad (57)$$

In particular, the exponent has the *positive* sign and the imaginary part of K has the *positive* sign. The minus sign on H_B comes from taking the real part of $iR(t)e^{2\pi i t x}$.

A.1.3 Closed boundary, cardinal values and the minorant

For $x < 0$, (50) gives $M(x) \geq 1$; for $x > 0$, it gives $M(x) \geq 0$. At the boundary, (45) and $W(0) = 1$ give

$$M(0) = 1. \quad (58)$$

Thus this majorizes the *closed* half-line, including a point mass at zero, rather than merely the half-height convention for a jump:

$$M(x) \geq 1_{(-\infty, 0]}(x), \quad 1 - M(-x) \leq 1_{(-\infty, 0]}(x). \quad (59)$$

The second inequality follows by subtracting the first inequality at $-x$ from one; its endpoint is open. Direct cardinal values are

$$M(j) = 1_{\{j \leq 0\}}, \quad 1 - M(-j) = 1_{\{j < 0\}} \quad (j \in \mathbb{Z}). \quad (60)$$

The proof also gives the useful global error bound

$$0 \leq M(x) - 1_{(-\infty, 0]}(x) \leq W(x) \quad (x \in \mathbb{R}). \quad (61)$$

At zero the left-hand error is exactly zero; it is not obtained by assigning half of an atom to each side. In particular $0 \leq M \leq 2$.

A.1.4 Expectations and the compensated identity

Suppose U is real and $\mathbb{E}|U| < \infty$, and use the convention $\psi_U(t) = \mathbb{E}e^{2\pi i t U}$. Since $2 \int_0^1 \operatorname{Re} K(t) dt = 1/2$, (57) can be written

$$M(u) = 1 + 2 \operatorname{Re} \int_0^1 K(t) \{e^{2\pi i t u} - 1\} dt. \quad (62)$$

The complex integral in (62) is absolutely convergent. Indeed, $K(t) = i/(2\pi t) + 1/2 + O(t)$ at zero, K is bounded near one, and

$$\mathbb{E}|e^{2\pi i t U} - 1| \leq 2\pi t \mathbb{E}|U|.$$

Consequently $\int_0^1 |K(t)| \mathbb{E}|e^{2\pi i t U} - 1| dt < \infty$, so ordinary absolute Fubini applies to (62). Restoring the constant real part gives

$$\mathbb{E}M(U) = \frac{1}{2} + 2 \lim_{\epsilon \downarrow 0} \operatorname{Re} \int_{\epsilon}^1 K(t) \psi_U(t) dt. \quad (63)$$

Since $\psi_{-U} = \overline{\psi_U}$, conjugation similarly yields

$$\mathbb{E}\{1 - M(-U)\} = \frac{1}{2} - 2 \lim_{\epsilon \downarrow 0} \operatorname{Re} \int_{\epsilon}^1 \overline{K(t)} \psi_U(t) dt. \quad (64)$$

These are the two expectation identities under the real-part improper convention (43). All expectations are finite; (61) even gives boundedness of the majorant without a moment assumption. The first absolute moment is a sufficient hypothesis for the stated compensated Fourier/Fubini argument. The uncompensated complex integral itself need not converge.

A.1.5 Four-summand smoothing normalization

Let Y be integrable, let $h \in \mathbb{R}$, and let $T > 0$. For $\chi_Y(s) = \mathbb{E}e^{isY}$, apply (59) and (63) to $U = T(Y - h)/(2\pi)$. The resulting exact normalization is

$$\mathbb{P}(Y \leq h) \leq \frac{1}{2} + 2 \lim_{\epsilon \downarrow 0} \operatorname{Re} \int_{\epsilon}^1 K(t) e^{-iTth} \chi_Y(Tt) dt. \quad (65)$$

For the four-summand application, $Z = (X_1 + \dots + X_4)/2$, $\varphi_P(s) = \mathbb{E}e^{isX_1}$, and iid summands, choose $T = 6$. Independence gives $\chi_Z(6t) = \varphi_P(3t)^4$. Thus (65) is exactly

$$F_Z(x) \leq \frac{1}{2} + \int_0^1 \operatorname{Re}\{\widehat{K}(t) e^{-6itx} \varphi_P(3t)^4\} dt, \quad \widehat{K} = 2K, \quad (66)$$

the physical four-summand smoothing normalization. Writing $D = x - Z$ and $S(t) = tR(t)$, its real integrand is exactly

$$(1 - t)\mathbb{E} \cos(6Dt) + S(t)\mathbb{E}[6D \operatorname{sinc}(6Dt)], \quad \operatorname{sinc}(v) = \sin(v)/v, \quad (67)$$

with the continuous value at zero. This verifies both the factor six and the sign of the sine term. No non-atomicity, finite-support, density, or continuity-at-the-threshold assumption is needed. For (66), iid independence and integrability are sufficient; the manuscript's bounded support assumptions imply integrability.

A.2 Characteristic functions and conserved resources

The following arguments retain the actual marginal law and raw threshold. Attainment and contact are inputs where explicitly stated; the numerical support, moments and probability panels are acquired hypotheses. The finite induction at the end justifies a declared schedule of valid updates, not the acceptance of any particular schedule. The classical four-term Prawitz smoothing inequality is explicitly cited at its use. The preceding pointwise construction alone is not substituted for that external inequality without deriving its four-term normalization.

A.2.1 Elementary characteristic-function bounds

Throughout this section, X is real, $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, and $\rho = \mathbb{E}|X|^3 < \infty$. Write

$$f(u) = \mathbb{E}e^{iuX}, \quad g(u) = e^{-u^2/2}, \quad \gamma = \mathbb{E}X^3.$$

No support, extremality, or predecessor assumption is made here.

A.2.1.1 Real and imaginary coordinates

Lemma A.1 (Anchored coordinates). *For $0 \leq u \leq \pi$,*

$$\cos u \leq \Re f(u) \leq \cos u + \frac{\rho - 1}{6} u^3, \quad (68)$$

$$|\Im f(u)|^2 \leq \frac{(\rho - 1)(\rho + 5/3)}{36} u^6. \quad (69)$$

Proof. Put $\text{sinc } v = \int_0^1 \cos(tv) dt$, including $v = 0$. The elementary bounds $|\sin v| \leq |v|$ give

$$|\text{sinc } a - \text{sinc } b| \leq \frac{|a - b|}{2}, \quad |\text{sinc } a - \text{sinc } b| \leq \frac{|a^2 - b^2|}{6}.$$

For the second inequality, use $|\cos \sqrt{A} - \cos \sqrt{B}| \leq |A - B|/2$ for $A, B \geq 0$, and integrate in t . Evenness handles either sign of a, b .

For $r \geq 0$, integration from 1 to r of the first sinc bound gives

$$\left| \cos(ur) - \cos u + \frac{u \sin u}{2} (r^2 - 1) \right| \leq \frac{u^3}{12} (r - 1)^2 (2r + 1).$$

Indeed the derivative of the expression inside the absolute value is $u^2 r \{\text{sinc } u - \text{sinc}(ur)\}$, and $\int_{\min(1,r)}^{\max(1,r)} t|t - 1| dt = (r - 1)^2 (2r + 1)/6$. For $R = |X|$, the weight $W(R) = (R - 1)^2 (2R + 1)$ satisfies $\mathbb{E}W(R) = 2(\rho - 1)$. Averaging proves the upper real bound, and in fact the corresponding two-sided anchor error.

For the sharper lower bound, take $0 < u \leq \pi$ and $a = \sin u/(2u) \geq 0$. The function $v \mapsto \cos v + av^2$ has its minimum on $[0, \pi]$ at u , since sinc decreases there. For $v \geq \pi$,

$$\cos v + av^2 \geq -1 + a\pi^2 \geq \cos u + au^2.$$

Thus this minimum is global on $[0, \infty)$. Substitute $v = uR$ and average; $\mathbb{E}R^2 = 1$ cancels the quadratic term. The case $u = 0$ is immediate.

Centering and the second sinc bound give

$$|\Im f(u)| \leq \frac{u^3}{6} \mathbb{E}[R|R^2 - 1|].$$

Cauchy–Schwarz, with W as above, yields

$$\{\mathbb{E}[R|R^2 - 1|]\}^2 \leq \mathbb{E}W(R) \mathbb{E} \frac{R^2(R + 1)^2}{2R + 1}.$$

There is no singularity at $R = 1$: the factorization used here is $R|R^2 - 1| = \sqrt{W(R)}R(R + 1)/\sqrt{2R + 1}$. The identity

$$r^3 + \frac{29r^2 + 1}{18} - \frac{2r^2(r + 1)^2}{2r + 1} = \frac{(r - 1)^2(4r + 1)}{18(2r + 1)} \geq 0$$

shows that the second expectation is at most $(\rho + 5/3)/2$. This proves (69). Every expectation used is finite under the third-moment assumption. $\square$

Lemma A.2 (The sine circle). *Let*

$$r = \mathbb{E}X_+ = \mathbb{E}X_- > 0, \quad A = \mathbb{E}X_+^2, \quad B = \mathbb{E}X_-^2, \quad \zeta = \frac{2AB}{r}.$$

Then $A + B = 1$ and

$$\zeta^2 + \frac{36|\Im f(u)|^2}{u^6} \leq \rho^2 \quad (u \neq 0), \quad \zeta \geq b(\rho) := \frac{\sqrt{\rho^2 + 8} - \rho}{2}. \quad (70)$$

In particular, for $0 \leq u \leq \pi$ the coefficient in the imaginary bound may be replaced by

$$|\Im f(u)|^2 \leq \frac{u^6}{36} \min\{(\rho - 1)(\rho + 5/3), \rho^2 - b(\rho)^2\}.$$

Both bounds also hold for γ^2 after deleting $u^6/36$.

Proof. Let μ_+, μ_- be the restrictions of the law of $|X|$ to the strictly positive and strictly negative classes, respectively. For $a, b > 0$, let $P_{a,b}$ put masses $b/(a+b), a/(a+b)$ at $a, -b$. The measure

$$d\nu(a, b) = \frac{a+b}{r} d\mu_+(a) d\mu_-(b)$$

has mass $\mathbb{P}(X \neq 0)$. Mixing $P_{a,b}$ against ν , and retaining the original atom at zero, recovers the law of X . For a component, write $f_{a,b}(u) = \mathbb{E}_{P_{a,b}} e^{iuX}$ and

$$e_{a,b} = \frac{2a^2b^2}{a+b}, \quad \rho_{a,b} = \frac{ab(a^2 + b^2)}{a+b}.$$

The squared-argument sinc bound gives

$$\left| \frac{6\Im f_{a,b}(u)}{u^3} \right| \leq \frac{ab|a^2 - b^2|}{a+b}.$$

Consequently the Euclidean norm of $(e_{a,b}, 6\Im f_{a,b}(u)/u^3)$ is at most $\rho_{a,b}$. Integrate this vector inequality. The two integrals on the right and in the first coordinate are ρ and $2AB/r$, respectively. The triangle inequality proves the circle. This argument also covers an atom at zero, whose vector contribution is zero.

Signwise Cauchy inequalities give

$$A^2 \leq r\mathbb{E}X_+^3, \quad B^2 \leq r\mathbb{E}X_-^3, \quad \frac{r^2}{A} + \frac{r^2}{B} \leq \mathbb{P}(X \neq 0) \leq 1.$$

Hence $\rho + \zeta \geq 1/r$, $AB \geq r^2$ and $\zeta \geq 2r$. All quantities are positive, so $\zeta(\rho + \zeta) \geq 2$. Solving this quadratic gives the stated lower bound. Finally $6\Im f(u)/u^3 \rightarrow -\gamma$ as $u \rightarrow 0$ by dominated convergence, using $|\sin v - v| \leq |v|^3/6$. This also proves the assertions about γ . $\square$

On a closed moment band $\rho \in [\rho_-, \rho_+]$, any rational $Z > 0$ satisfying $Z^2 + \rho_+ Z \leq 2$ is a valid uniform lower bound for $b(\rho)$. Thus no numerical square root is needed to certify this part of the imaginary enclosure.

A.2.1.2 An explicit scalar certificate and its convex profile

Lemma A.3 (Scalar cosine bound). *Assume the finite Bernstein positivity condition specified in the proof below. Then, for every real x ,*

$$\cos x \leq 1 - \frac{x^2}{2} + \kappa|x|^3, \quad \kappa = \frac{62}{625}. \quad (71)$$

Proof. By evenness take $x \geq 0$. For $x \geq T := 625/124$ the right side is at least one. Let $P_{16}(x) = \sum_{j=0}^8 (-1)^j x^{2j} / (2j)!$. The sixteenth derivative of $\cos x - P_{16}(x)$ is $\cos x - 1 \leq 0$, and its first sixteen initial data vanish. Therefore $\cos x \leq P_{16}(x)$ on the whole nonnegative half-line. It remains to prove positivity on $[0, T]$ of

$$Q(x) = \frac{1 - x^2/2 + \kappa x^3 - P_{16}(x)}{x^3} = \sum_{j=0}^{13} q_j x^j,$$

where the removable value at zero is used and the exact coefficient vector is

$$(q_0, \dots, q_6) = \left(\frac{62}{625}, -\frac{1}{24}, 0, \frac{1}{720}, 0, -\frac{1}{40320}, 0 \right),$$

$$(q_7, \dots, q_{13}) = \left(\frac{1}{3628800}, 0, -\frac{1}{479001600}, 0, \frac{1}{87178291200}, 0, -\frac{1}{20922789888000} \right).$$

Here is a fully specified finite rational certificate. For $0 \leq r < 80$ set $(a_r, b_r) = (r/16, (r+1)/16)$, and set $(a_{80}, b_{80}) = (5, 625/124)$. For $0 \leq j, k \leq 13$ define

$$c_{r,j} = \sum_{\ell=j}^{13} q_\ell \binom{\ell}{j} a_r^{\ell-j} (b_r - a_r)^j,$$

$$B_{r,k} = \sum_{j=0}^k c_{r,j} \binom{k}{j} \binom{j}{13-j}. \quad (72)$$

All $81 \cdot 14$ numbers are thereby specified without external data. The finite arithmetic condition required here is

$$\min_{0 \leq r \leq 80, 0 \leq k \leq 13} B_{r,k} = B_{63,12} = \frac{2479189}{65585520000} > 0. \quad (73)$$

The identity

$$Q(a_r + (b_r - a_r)t) = \sum_{k=0}^{13} B_{r,k} \binom{13}{k} t^k (1-t)^{13-k}$$

follows by expanding either side. Its basis is nonnegative and sums to one for $0 \leq t \leq 1$. Equations (72)–(73) therefore imply positivity on every entire closed cell when that finite condition holds. The cells cover $[0, T]$, completing the conditional proof. $\square$

Lemma A.4 (Compatible profile). *Put $c_{\cos} = 1 - \cos 4$, $a = 129/1250$ and $A = 16a/c_{\cos}$. The function*

$$q(v) = \begin{cases} 1/2 - \kappa v, & 0 \leq v \leq 4, \\ A(1 - \cos v)/v^2, & 4 \leq v \leq 2\pi, \\ 0, & v \geq 2\pi \end{cases} \quad (74)$$

is continuous, nonnegative, convex and nonincreasing on $[0, \infty)$, and $\cos v \leq 1 - v^2 q(v)$ there.

Proof. The scalar bound at four gives $c_{\cos} \geq 16a > 0$, hence $0 < A \leq 1$. The pieces agree at both endpoints. On the curved piece,

$$q'(v) = A \frac{v \sin v - 2(1 - \cos v)}{v^3} \leq 0, \quad q''(v) = \frac{A}{v^4} \{6 + (v^2 - 6) \cos v - 4v \sin v\}.$$

Write the brace as $B(v)$. On $[4, 3\pi/2]$, $B'(v) = -2v \cos v + (2 - v^2) \sin v \geq 0$. The elementary bounds $25/8 < \pi < 22/7$ imply $7\pi/6 < 4 < 3\pi/2$, so $B(4) \geq 6 - 10 + 8 = 4$. On $[3\pi/2, 2\pi]$

each remaining term in B is nonnegative, and $B \geq 6$. Thus the curved branch is convex, with derivative zero at 2π . The upward derivative join at four is equivalent to

$$258 \sin 4 + 119(1 - \cos 4) \geq 0.$$

Let $S_{15}(x) = \sum_{j=0}^7 (-1)^j x^{2j+1} / (2j+1)!$. The fifteenth derivative of $\sin x - S_{15}(x)$ is $1 - \cos x \geq 0$, with vanishing initial data; hence $\sin x \geq S_{15}(x)$ for $x \geq 0$. The exact rational lower bound at the join is

$$258S_{15}(4) + 119\{1 - P_{16}(4)\} = \frac{967616288}{638512875} > 1.$$

This proves the required join. The profile majorizes the cosine defect by the scalar inequality on $[0, 4]$, by $A \leq 1$ on $[4, 2\pi]$, and by $\cos v \leq 1$ thereafter.

The bounds $25/8 < \pi < 22/7$ follow from the rational comparisons in Lemma 3.1. $\square$

Lemma A.5 (Whole-law modulus). *Let X' be an independent copy of X , and let $D \geq \mathbb{E}|X - X'|^3/2$. For $u \geq 0$,*

$$|f(u)|^2 \leq 1 - 2u^2 q(uD). \quad (75)$$

The universal choice $D = \rho + 1$ is valid. In particular the two nontrivial branches are

$$|f(u)|^2 \leq \begin{cases} 1 - u^2 + 2\kappa D u^3, & uD \leq 4, \\ 1 - 2A(1 - \cos(uD))/D^2, & 4 \leq uD \leq 2\pi. \end{cases}$$

Proof. Since $\mathbb{E}(X - X')^2 = 2$, the measure $(X - X')^2 d\mathbb{P}/2$ is a probability measure. Its mean of $|X - X'|$ is $\mathbb{E}|X - X'|^3/2$. The cosine defect bound and Jensen's inequality give

$$1 - |f(u)|^2 \geq \mathbb{E}[(u(X - X'))^2 q(u|X - X'|)] \geq 2u^2 q(u\mathbb{E}|X - X'|^3/2) \geq 2u^2 q(uD).$$

Finally $|x - y|^3 \leq (x - y)^2(|x| + |y|)$ and centering imply $\mathbb{E}|X - X'|^3 \leq 2\rho + 2\mathbb{E}|X| \leq 2(\rho + 1)$. No stronger universal independent-copy bound is being asserted. $\square$

A.2.1.3 A pointwise complex-power enclosure

Lemma A.6 (Rectangle gate). *Let $n \geq 2$ be an integer, $a \geq 0$, $0 < x_0 \leq x_1$, and $Y \geq 0$. If*

$$(n - 2)Y \leq \frac{3}{2}x_0, \quad (2n - 1)x_0^n \geq (n - 1)a,$$

then throughout $x \in [x_0, x_1]$, $|y| \leq Y$,

$$|(x + iy)^n - a| \leq \max\{|(x_0 + iY)^n - a|, |(x_1 + iY)^n - a|\}.$$

Proof. Put $z = x + iy$, $r = |z|$, and $F(x, y) = |z^n - a|^2$. Since $|\arg z| \leq |y|/x$, the first gate gives $\Re z^{n-2} \geq 0$. Direct differentiation yields

$$\begin{aligned} F_{yy} &= 2nr^{2n-4}\{r^2 + 2(n-1)y^2\} + 2an(n-1)\Re z^{n-2} \geq 0, \\ F_{xx} &= 2nr^{2n-4}\{r^2 + 2(n-1)x^2\} - 2an(n-1)\Re z^{n-2}. \end{aligned}$$

Divide the second expression by $2nr^{n-2} > 0$. Its positive term is $r^{n-2}\{(2n-1)x^2 + y^2\} \geq (2n-1)x_0^n$, and its negative term has magnitude at most $(n-1)a$. Thus F is separately convex. Its maximum on the rectangle is bounded by its four corners; conjugate corners have equal values. Equality in either gate, $n = 2$, and $Y = 0$ are included. $\square$

For a frequency $0 \leq u < \pi/2$ one may take

$$x_0 = \cos u, \quad x_1 = \cos u + (\rho_+ - 1)u^3/6, \quad a = e^{-nu^2/2},$$

and any valid uniform imaginary upper Y . The gates must hold on the whole integration cell before this enclosure is used there. Without those gates, an enclosing disk still gives a complete bound. Indeed, if $M \geq |f(u)|$ and $E \geq |f(u) - g(u)|$, then for every $n \geq 1$

$$|f(u)^n - g(u)^n| \leq \min \left\{ M^n + g(u)^n, E \sum_{j=0}^{n-1} M^j g(u)^{n-1-j} \right\}. \quad (76)$$

This is the finite factorization of $z^n - w^n$ and remains valid at equal or zero bases. For a signed real interval, both endpoint squares are retained when constructing either disk.

A.2.1.4 The smoothing interface

We use Prawitz's smoothing inequality in the four-term normalization of [23, Lemma 4.1, p. 263]. This is a cited external inequality, not a consequence of the preceding elementary lemmas. Put

$$\mathcal{K}(t) = (1 - t) + i\{(1 - t) \cot(\pi t) + 1/\pi\}, \quad 0 < t < 1.$$

For $c > 0$, $0 < \tau < 1$, and every integer $n \geq 1$, it gives

$$\begin{aligned} \Delta_n &\leq \int_0^\tau |\mathcal{K}(t)| |f(ct)^n - g(ct)^n| dt + \int_\tau^1 |\mathcal{K}(t)| |f(ct)|^n dt \\ &\quad + \int_0^\tau \left| \mathcal{K}(t) - \frac{i}{\pi t} \right| g(ct)^n dt + \frac{1}{\pi} \int_\tau^\infty \frac{g(ct)^n}{t} dt. \end{aligned} \quad (77)$$

Here Δ_n is the Kolmogorov distance for $S_n/\sqrt{n}$. All four terms are nonnegative. The last is $E_1(nc^2\tau^2/2)/(2\pi)$, where $E_1(x) = \int_x^\infty e^{-v} dv/v \leq e^{-x}/x$ for $x > 0$. Normal continuity equates the suprema defined with strict and closed CDFs, so atoms introduce no additional restriction in this application.

On a moment band, multiply a complete upper bound for the right side by $\sqrt{n}/\rho_-$, not by $\sqrt{n}/\rho_+$. A finite partition yields an integral upper bound by summing width times a valid whole-cell majorant. If a fixed C^2 majorant F is used on a cell of width w and midpoint m , Taylor integration gives

$$\int F \leq wF(m) + \frac{w^3}{24} \max\{0, \sup F''\}.$$

Taking a minimum with a separately valid Darboux upper is legitimate. Selecting a favorable branch at the midpoint without proving it on the entire cell is not. Endpoint pieces and the infinite normal tail remain separate obligations in every finite certificate.

A.2.2 Complete Fourier remainders and finite resource enclosure

Use the contact normalization and the gap quantities of Section 5.4. Thus C is the actual positive closed-CDF score of one unrestricted attaining law, H is its raw sum threshold, $\rho = \mathbb{E}|X|^3$, $\gamma = \mathbb{E}X^3$, $\beta = \mathbb{E}[X|X|]$. The quantities D, J are defined in (14), and J_1, J_2, L in (21). Let $\mu_j = \mathbb{E}X^j$ for even j , $\eta_5 = \mathbb{E}|X|^5$, and $\varphi = \phi(H/\sqrt{n})$. Support $[-R_-, R_+]$ is acquired before these higher moments are invoked; write $R = \max(R_-, R_+)$. In applications below $1 \leq R \leq 6$. The exact raw remainder $\mathcal{R}$ is defined in (29); its contact derivation,

compensated inversion and phase price are proved in Section 5.4. Here we bound its Fourier and moment terms on explicit domains. The elementary constant enclosures used here are

$$997/2500 < \phi(0) < 2/5, \quad 0.4095 < C_E < 0.40974, \quad 333/106 < \pi < 22/7.$$

The Gaussian and Esseen-constant bounds follow from Lemma 3.1. For the slightly sharper lower bound on π used here, the same alternating sums give

$$\frac{333}{106} < 16 \sum_{j=0}^5 \frac{(-1)^j}{(2j+1)5^{2j+1}} - \frac{4}{239} < \pi.$$

The first comparison is rational; the second uses exactly the lower arctangent truncations in that lemma. The fractional-part estimate below requires finer directed bounds. With

$$A_m(t) = \sum_{j=0}^{m-1} \frac{(-1)^j t^{2j+1}}{2j+1}, \quad \epsilon_5 = \frac{5^{-65}}{65}, \quad \epsilon_{239} = \frac{239^{-21}}{21},$$

the alternating remainders and the same Machin identity give

$$\begin{aligned} 3.141592653589793 &< 16A_{32}(1/5) - 4\{A_{10}(1/239) + \epsilon_{239}\} < \pi, \\ \pi &< 16\{A_{32}(1/5) + \epsilon_5\} - 4A_{10}(1/239) < 3.141592653589794. \end{aligned}$$

The two outer comparisons involve only the displayed rational numbers.

A finite-index profile means

$$\begin{aligned} N \leq n \leq L, \quad n(C - C_E) \geq K \geq 0, \quad C \in [c_0, c_1], \\ n(C_{n-1} - C)_+ \leq q_*, \quad c_0 = 0.4095 + K/L. \end{aligned} \tag{78}$$

A band-uniform bound is used only under the explicit monotonicity gates proved below. Otherwise every formula is evaluated at the actual integer n . In particular, no monotonicity of C_n is assumed. The quantitative support implication is a separate contact argument, not an output of the Fourier integration.

A.2.2.1 One complete low-frequency transfer

Put $f(t) = \mathbb{E}e^{itX}$, $h(t) = e^{-t^2/2 - i\gamma t^3/6}$, $e_n(t) = e^{-nt^2/2}(1 - in\gamma t^3/6)$, and

$$I_0(H) = \int_0^{1/2} |\Im(e^{-itH}(f^n - e_n))| \frac{dt}{t^3}, \quad I_\infty = \int_{1/2}^\infty |f^n - e_n| \frac{dt}{t^3}.$$

The inversion bound (28) charges $2\sqrt{n}(I_0 + I_\infty)/\pi$; both integrals below are complete.

Lemma A.7 (Signed Gaussian transfer). *Suppose $n \geq N > 2$, $\mu_4 \leq M \leq 11.1$, $|\gamma| \leq g \leq 1.85$, $\mu_6 \leq M_6$, and the actual coefficients*

$$a = (3 - \mu_4)/24, \quad z = (10\gamma - \mathbb{E}X^5)/120, \quad c = a\gamma/6$$

satisfy $|a| \leq a_$, $|z| \leq Z$. Put*

$$E_6 = M_6/720 + g^2/72 + (1/2 + g^2/144)^3/5, \quad B_6 = a_* + Z/2 + E_6/4. \tag{79}$$

Let $a_0 = (N-1)/(2N)$, $0 < b_0 \leq (45 - M)(1 - 2/N)/96$, and $\lambda_N = \sqrt{\pi/2}(N/(N-1))^{3/2}$. Then, for every real H ,

$$I_0(H) \leq (L_0 + L_H|H|)/\sqrt{n} + L_2/n, \quad L_2 = E_6/(2a_0^2) + B_6^2/(2b_0^3). \tag{80}$$

Two admissible choices, each used as a complete triple, are as follows. The triangle choice is

$$L_0 = c_0 Z + c_1 a_* g + 251g^3/17280, \quad L_H = c_0 a_* + 251g^2/4800, \quad (81)$$

where $c_0 \geq \lambda_N$, $c_1 \geq \lambda_N/2$. For the cancellation choice, take any rectangle containing the actual (z, c) , $I_u \geq \sqrt{\pi/2}$ and $F_N \geq (N/(N-1))^{3/2}$. Define

$$U(z, c) = \begin{cases} I_u(|z| + 3|c|), & zc \leq 0, \\ I_u||z| - 3|c|| + 6\sqrt{|zc|}e^{-|z|/(2|c|)}, & zc > 0, \end{cases}$$

$$J_u = \max_{\text{corners}} U(z, c).$$

Then

$$L_0 = F_N J_u + \frac{F_N I_u}{N-1} (35a_* g^3/48 + 35Zg^2/24) + 251g^3/17280,$$

$$L_H = F_N I_u \{a_* + 5Zg/[2(N-1)]\} + 251g^2/4800. \quad (82)$$

Neither the transfer nor its cancellation choice assumes signs for a, z, c or γ .

Proof. Taylor with integral remainder gives $f - h = -at^4 - izt^5 + r_6$, $|r_6| \leq E_6 t^6$ for $0 \leq t \leq 1/2$. Indeed for $w = -t^2/2 - i\gamma t^3/6$, $|w| \leq t^2(1/2 + g^2/144) < 1/6$; the terms through degree five in $1 + w + w^2/2$ are exact, its degree-six term costs $g^2 t^6/72$, and the remaining exponential series costs $|w|^3 e^{|w|}/6 \leq |w|^3/5$. Add the true sixth-moment remainder.

Four integrations of $1 - \cos u \geq 0$ give $\cos u \leq 1 - u^2/2 + u^4/24$. Symmetrization, using $\mathbb{E}(X - X')^4 = 2\mu_4 + 6$, therefore gives $|f(t)|, |h(t)| \leq e^{-(45-M)t^2/96}$ on $[0, 1/2]$. Every point on their joining segment has the same bound, so polynomial Taylor expansion yields

$$f^n - h^n = n(f - h)h^{n-1} + R_2, \quad |R_2| \leq \frac{1}{2}n^2 B_6^2 t^8 e^{-b_0 n t^2}.$$

The r_6 and R_2 integrals are at most

$$\frac{1}{n} \{E_6/(2a_0^2) + B_6^2/(2b_0^3)\} = L_2/n. \quad (83)$$

For the remaining comparator put $u = n\gamma t^3/6$. Its imaginary part is bounded by $e^{-nt^2/2}(|u|^3/6 + |H|tu^2/2)$, so its divided integral is at most

$$n^{-1/2}(251g^3/17280 + 251g^2|H|/4800). \quad (84)$$

Here the Gaussian moments follow from integration by parts, $\int_0^\infty x^{2j} e^{-x^2/2} dx = (2j-1)!!\sqrt{\pi/2}$, and $\sqrt{2\pi} < 251/100$.

It remains to price the leading quartic and fifth terms. Set $b = n - 1$, $v = b\gamma t^3/6$. The triangle estimate uses $|\sin(Ht + v)| \leq |H|t + b\gamma t^3/6$, $|\cos| \leq 1$; the Gaussian moments of orders two and four give respectively λ_N and $\lambda_N/2$. This proves (81). For cancellation, after division by t^3 , rotation by Ht bounds the leading imaginary part by

$$|zt^2 - atv| + a_* t |\sin v - v| + Zt^2 |\cos v - 1| + |H|t \{a_* t + Zt^2 |\sin v|\}.$$

Multiply by $ne^{-bt^2/2}$ and use the global sine/cosine remainders. The first integral equals $nJ(z, c)/b^{3/2}$, where $J(z, c) = \int_0^\infty e^{-x^2/2} |zx^2 - cx^4| dx$. This norm is convex and $J(-z, -c) = J(z, c)$, so its rectangle maximum is at a corner. For $zc \leq 0$ the integral is $\sqrt{\pi/2}(|z| + 3|c|)$. For $z, c > 0$, split at $r = \sqrt{z/c}$; integration by parts gives

$$J = \sqrt{\pi/2}(z - 3c) + 6cre^{-r^2/2} + 2(3c - z) \int_r^\infty e^{-x^2/2} dx.$$

Discard the last term when nonpositive, and otherwise bound its integral by $\sqrt{\pi/2}$. This proves $J \leq U$; simultaneous sign reflection covers the remaining quadrant. The other four phase terms use moments of orders two, six, eight and ten and give the coefficients $1, 5/2, 35/24, 35/48$ in (82). Finally $n/(n-1)$, $1/(n-1)$, a_0^{-1} and the allowable damping inverses decrease with n , proving the all-index bounds at N . $\square$

Lemma A.8 (Retaining the signed Gaussian tail). *For $r > 0$, put*

$$\mathcal{M}(r) = e^{r^2/2} \int_r^\infty e^{-t^2/2} dt, \quad \mathcal{M}_k(r) = \frac{1}{r + \frac{1}{r + \frac{2}{r + \frac{k}{\ddots + \frac{1}{r}}}}} \quad (k \geq 1), \quad \mathcal{M}_0(r) = 1/r.$$

Odd truncations are lower bounds for $\mathcal{M}$; even truncations are upper bounds. For $z, c > 0$, set $x = z/c$ and

$$d_{k+1} = 1, \quad d_j = 1 + \frac{j}{x d_{j+1}} \quad (j = k, \dots, 1), \quad m_k(x) = d_1^{-1}.$$

Then the cancellation norm in Lemma A.7 satisfies

$$J(z, c) \leq \sqrt{\pi/2}(z - 3c) + \sqrt{zc} e^{-x/2} \left(6 + \frac{2(3-x)}{x} m_k(x) \right), \quad (85)$$

where $k = 11$ if $x \geq 3$ and $k = 12$ if $x < 3$. The factor in parentheses is positive. One may take the minimum with the previous bound $U(z, c)$ at each corner of the coefficient rectangle.

Proof. The positive integrals $I_j(r) = \int_0^\infty t^j e^{-rt-t^2/2} dt$ satisfy $I_1 = 1 - rI_0$ and $I_{j+1} = jI_{j-1} - rI_j$ by integration by parts. Thus, writing $q_j = I_j/I_{j-1} > 0$,

$$I_0 = \frac{1}{r + q_1}, \quad q_j = \frac{j}{r + q_{j+1}}.$$

Replacing the final positive q_{k+1} by zero reverses the inequality at each substitution. This proves the alternating finite bounds, since translation gives $I_0 = \mathcal{M}(r)$. No convergence of an infinite continued fraction is required. The displayed rational recursion gives $m_k(x) = r\mathcal{M}_k(r)$ when $r = \sqrt{x}$.

Splitting the norm at $r = \sqrt{z/c}$, as in the preceding proof, gives the exact identity

$$J(z, c) = \sqrt{\pi/2}(z - 3c) + \sqrt{zc} e^{-x/2} \left(6 + \frac{2(3-x)}{x} r\mathcal{M}(r) \right).$$

The multiplier of $\mathcal{M}$ is nonpositive when $x \geq 3$ and positive when $x < 3$. Substitute the corresponding finite lower or upper bound. Positivity of the remaining factor is immediate for $x < 3$; for $x \geq 3$, use $0 < m_{11}(x) < 1$ to bound it below by $6 - 2(x-3)/x > 4$. Finally the exact norm J , rather than either upper-bound formula, is convex, so its containing rectangle is still exhausted by its four corners. $\square$

At every positive coefficient corner use the displayed finite Mills bound. A negative Gaussian coefficient uses a lower enclosure for $\sqrt{\pi/2}$; a nonnegative coefficient uses an upper one. The remaining factor is positive, so outward upper square-root and exponential enclosures are valid. A simultaneous reflection of both odd coefficients leaves the norm unchanged.

A.2.2.2 The two coefficient choices in the certificates.

All coefficients come from the actual marginal. For $Q = X^2 - \gamma X - 1$, multiplication gives

$$\begin{aligned}\mathbb{E}Q^2 &= \mu_4 - 1 - \gamma^2, & \mathbb{E}X^5 &= \gamma^3 + 2\gamma + \mathbb{E}[Q^2(X + 2\gamma)], \\ \mu_6 &= 1 + 3\gamma^2 + \gamma^4 + \mathbb{E}[Q^2\{(X + \gamma)^2 + 2\gamma^2 + 2\}].\end{aligned}\tag{86}$$

For any radius $R_0 \geq |X|$ with $R_0 \geq 2g$, the two signs of $\mathbb{E}X^5 - 10\gamma$ are bounded, at $t = |\gamma|$, by

$$\begin{aligned}P_{R_0}(t) &= R_0(M - 1) + (10 - 2M)t - R_0t^2 + t^3, \\ Q_{R_0}(t) &= R_0(M - 1) + (2M - 10)t - R_0t^2 - t^3.\end{aligned}$$

The gate $R_0 - 2g \geq 0$ permits substituting M for μ_4 in both signs. The triangle choice uses the valid enclosing radius $R_0 = 6$, $a_* = \max(2, M - 3)/24$, $M_6 = RE_5$, $b_0 = (45 - M)(1 - 2/N)/96$ and

$$\begin{aligned}120Z &= \min\{E_5 + 10g, Z_0\}, \\ Z_0 &= \begin{cases} P_6(g), & M \leq 5, P'_6(g) \geq 0, \\ 6(M - 1) + (10 - 2M)^2/[4(6 - g)], & M \leq 5, P'_6(g) < 0, \\ 6(M - 1) + (2M - 10)^2/24, & M > 5. \end{cases}\end{aligned}\tag{87}$$

If $M \leq 5$, $P_6 \geq Q_6$ and $P'_6 < 0$ on $[0, g]$; use the increasing endpoint or $t^3 \leq gt^2$ and complete the square. If $M > 5$, $P_6 \leq 6(M - 1)$ and the upper quadratic for Q_6 gives the third row. One may also use the actual radius by setting $r_0 = \max(R, 2g) > g$ and

$$\begin{aligned}t_* &= \begin{cases} \min\{g, (10 - 2M)/[2(r_0 - g)]\}, & M \leq 5, \\ \min\{g, (2M - 10)/(2r_0)\}, & M > 5, \end{cases} \\ Z_r &= \begin{cases} r_0(M - 1) + (10 - 2M)t_* - (r_0 - g)t_*^2, & M \leq 5, \\ r_0(M - 1) + (2M - 10)t_* - r_0t_*^2, & M > 5. \end{cases}\end{aligned}$$

Indeed for $M \leq 5$, $P_{r_0} \geq Q_{r_0}$ and $t^3 \leq gt^2$; for $M > 5$, $P_{r_0} \leq r_0(M - 1)$ and dropping $-t^3$ bounds Q_{r_0} . Each displayed quadratic is maximized at its clipped vertex. Thus a permitted coefficient choice is $120Z = \min(E_5 + 10g, Z_0, Z_r)$. It uses $c_0 \geq \sqrt{\pi/(16a_0^3)} = \lambda_N$ and $c_1 = c_0/2$. The weaker choice $c_1 \geq \sqrt{\pi/(256a_0^5)} = \lambda_N N/[2(N - 1)]$ is also permitted by the same transfer lemma.

For the paired choice the already proved cell has $0 \leq \gamma_l \leq |\gamma| \leq g$, $M < 3$, $g^2 < 2$, and $R > 2g$. After reflecting only this auxiliary Fourier integral if necessary, set

$$\begin{aligned}a_* &= (2 - \gamma_l^2)/24, & b_0 &= ((45 - M)/96)(1 - 2/N), \\ M_6 &= \min\{RE_5, 1 + 3g^2 + g^4 + (R^2 + 2Rg + 3g^2 + 2)(M - 1 - \gamma_l^2)\}, \\ t_* &= \min\{g, (10 - 2M)/[2(R - g)]\}, \\ 120Z &= \min\{E_5 + 10g, R(M - 1) + (10 - 2M)t_* - (R - g)t_*^2\}, \\ z_l &= \max\{-Z, [8\gamma_l - \gamma_l^3 - (R + 2\gamma_l)(M - 1 - \gamma_l^2)]/120\}, & z_u &= Z, \\ c_l &= (3 - M)\gamma_l/144, & c_u &= (2 - \gamma_l^2)g/144.\end{aligned}$$

Check $M - 1 - \gamma_l^2 \geq 0$ and that the rectangle is ordered. The extension from the formerly sufficient $M \leq 2$, $g \leq 1$ domain is legitimate: Pearson gives $\mu_4 \geq 1 + \gamma^2$, hence $0 \leq a \leq (2 - \gamma_l^2)/24$; $M < 3$ keeps $c_l \geq 0$ and $g^2 < 2$ keeps the upper quartic coefficient nonnegative. The general transfer hypotheses remain true since $M < 3 < 11.1$ and $g < \sqrt{2} < 1.85$. The

damping must be $(45 - M)(1 - 2/N)/96$; the older value $43(1 - 2/N)/96$ is not licensed when $M > 2$. Here $P_R \geq Q_R$, and $P_R(t) \leq R(M - 1) + (10 - 2M)t - (R - g)t^2$, whose maximum is at t_* . The numerator of the lower z bound increases in $\gamma \geq 0$ because its derivative is $10 - 2M + 2R\gamma + 3\gamma^2 > 0$. Pearson's first and third identities give a_* and M_6 . If $3g^2 \leq 2$, the increasing function $2\gamma - \gamma^3$ further licenses $c_u \leq (2g - g^3)/144$. Thus this rectangle is a specialization of the signed transfer on the whole enlarged domain. Its actual moment inputs, including any shared lower bound $r_* \leq \rho$, are acquired before it is used. The signed phase calculation keeps the original γ ; reflection here acts only on the auxiliary Fourier norm and both its odd coefficients.

A.2.2.3 A low-frequency bound on a finite interval

The preceding low-frequency bounds extend positive Gaussian majorants to infinity. At small indices this can be wasteful. The following alternative keeps the original integration interval; it uses no physical-gap premise.

Lemma A.9 (Finite-window transfer). *Suppose $n > 2$ is an integer, $1 \leq R \leq 6$, $|X| \leq R$, $\mu_4 \leq M \leq 111/10$, $|\gamma| \leq g \leq 37/20$, and $\eta_5 \leq E_5$. Let $0 \leq g_l \leq |\gamma|$ with $g_l^2 \leq M - 1$. Set*

$$\begin{aligned} M_6 &= \min\{RE_5, 1 + 3g^2 + g^4 + (R^2 + 2Rg + 3g^2 + 2)(M - 1 - g_l^2)\}, \\ A_* &\geq |(3 - \mu_4)/24|, \quad Z \geq |(10\gamma - \mathbb{E}X^5)/120|, \\ E &= M_6/720 + g^2/72 + (1/2 + g^2/144)^3/5, \quad B = A_* + Z/2 + E/4, \\ a_n &= (n - 1)/2, \quad b_n = (n - 2)(45 - M)/96, \quad I_j(A) = \int_0^{1/2} t^j e^{-At^2} dt. \end{aligned}$$

Then $I_0(H)$ is bounded by $(L_0 + L_H|H|)/\sqrt{n} + L_2/n$, with the following complete triple:

$$\begin{aligned} L_0 &= \sqrt{n}\{nZI_2(a_n) + n(n - 1)A_*gI_4(a_n)/6 + n^3g^3I_6(n/2)/1296\}, \\ L_H &= \sqrt{n}\{nA_*I_2(a_n) + n^2g^2I_4(n/2)/72\}, \\ L_2 &= n\{nEI_3(a_n) + n(n - 1)B^2I_5(b_n)/2\}. \end{aligned} \tag{88}$$

The bound holds for every real raw threshold H . One always may use $A_* = \max\{|3 - M|, |2 - g_l^2|\}/24$ and $Z = (E_5 + 10g)/120$; the sharper proved Pearson coefficient bounds above may be substituted.

Proof. Pearson's sixth-moment identity (86), whose residual multiplier is positive, gives the stated M_6 ; also $\mu_6 \leq R\eta_5$ pointwise. Put $a = (3 - \mu_4)/24$ and $z = (10\gamma - \mathbb{E}X^5)/120$. For $h(t) = e^{-t^2/2 - i\gamma t^3/6}$,

$$f(t) - h(t) = -at^4 - izt^5 + r_6(t), \quad |r_6(t)| \leq Et^6 \quad (0 \leq t \leq 1/2).$$

Indeed, with $w = -t^2/2 - i\gamma t^3/6$, $|w| \leq t^2(1/2 + g^2/144) < 1/6$. Comparing with $1 + w + w^2/2$ costs $g^2t^6/72 + |w|^3e^{|w|}/6 \leq \{g^2/72 + (1/2 + g^2/144)^3/5\}t^6$, besides the true sixth-moment remainder $\mu_6t^6/720$. Thus $|f - h| \leq Bt^4$ on the whole interval.

Symmetrization gives

$$|f(t)|^2 \leq 1 - t^2 + (M + 3)t^4/12 \leq 1 - (45 - M)t^2/48.$$

Both $|f(t)|$ and $|h(t)|$ are therefore at most $e^{-(45-M)t^2/96}$, as is every point on their complex line segment. Taylor's integral formula for the polynomial z^n yields

$$f^n - h^n = n(f - h)h^{n-1} + R_2, \quad |R_2| \leq \frac{n(n-1)}{2}B^2t^8e^{-b_nt^2}.$$

The r_6 and R_2 terms, after division by t^3 and integration, cost $nEI_3(a_n) + n(n-1)B^2I_5(b_n)/2$. The leading imaginary part is at most

$$ne^{-ant^2}\{(Z + A_*|H|)t^5 + (n-1)A_*gt^7/6\}.$$

Finally, for $u = n\gamma t^3/6$,

$$|\Im[e^{-itH}(h^n - e_n)]| \leq e^{-nt^2/2}\{|u|^3/6 + |H|tu^2/2\}.$$

Integrating these terms gives every coefficient in (88), with no replacement of $n(n-1)$ by n^2 and no omitted frequency interval. $\square$

For rational $A \geq 0$, exact enclosures for these finite integrals are obtained from

$$P_{m,j}(A) = \sum_{k=0}^m \frac{(-A)^k}{k!(j+2k+1)2^{j+2k+1}}, \quad \max\{0, P_{21,j}(A)\} \leq I_j(A) \leq P_{20,j}(A). \quad (89)$$

The integral Taylor remainder of e^{-x} has sign $(-1)^{m+1}$ for every $x \geq 0$, not merely on an alternating-series domain. Integrate its polynomial bounds after $x = At^2$. A finite table restricted to $A \in [0, 8]$ and $0 \leq j \leq 16$ is used only when these conditions hold. Every weight in (88) is nonnegative. An alternative low-frequency choice is always one entire valid triple; taking coordinatewise minima of triples is not justified.

A.2.2.4 The complete high-frequency integral

For $b, \kappa > 0$ and $n > 0$, put

$$T_n = \sqrt{n} \int_b^\infty e^{-n\kappa t^2} t^{-3} dt, \quad x = n\kappa b^2.$$

Lemma A.10 (Rational Laguerre tail bound). *For every $x > 0$,*

$$T_n \leq \frac{\sqrt{n}e^{-x}}{2b^2} \frac{x+2}{x^2+4x+2}.$$

If $N\kappa b^2 \geq 1/4$, the right side decreases with $n \geq N$.

Proof. Substitute $t^2 = b^2(1+s)$ and use positive Fubini to obtain

$$T_n = \frac{\sqrt{n}e^{-x}}{2b^2} \int_0^\infty \frac{e^{-xs}}{(1+s)^2} ds = \frac{\sqrt{n}e^{-x}}{2b^2} \int_0^\infty \frac{u}{x+u} e^{-u} du.$$

Put $P(u) = u^2 - 4u + 2$ and $D(x) = x^2 + 4x + 2 = P(-x) > 0$. Exact division gives

$$\frac{1}{x+u} - \frac{Q_x(u)}{D(x)^2} = \frac{P(u)^2}{(x+u)D(x)^2} \geq 0,$$

where

$$Q_x(u) = -u^3 + (x+8)u^2 - (x^2 + 8x + 20)u + x^3 + 8x^2 + 20x + 16.$$

Integration uses only $\int_0^\infty u^j e^{-u} du = j!$, proved by integration by parts. It gives $\int Q_x(u) e^{-u} du = (x+3)D(x)$ and hence

$$\int_0^\infty \frac{u}{x+u} e^{-u} du = 1 - x \int_0^\infty \frac{e^{-u}}{x+u} du \leq 1 - \frac{x(x+3)}{D(x)} = \frac{x+2}{D(x)}.$$

For the all-index statement, differentiate $\sqrt{x}e^{-x}(x+2)/D(x)$. After multiplication by positive factors, its derivative has numerator $4 - 10x - 22x^2 - 13x^3 - 2x^4$. This polynomial is negative at $x = 1/4$ and strictly decreasing for $x > 0$. Thus it is negative on the entire required half-line. $\square$

Let $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, $\mathbb{E}X^4 \leq M$, and write $\gamma = \mathbb{E}X^3$, $f(t) = \mathbb{E}e^{itX}$. For independent copies suppose an actual event E satisfies

$$\mathbb{P}(E) \geq a > 0, \quad |X - X'| \geq d_0 > 0 \text{ on } E, \quad \mathbb{E}[(X - X')^2 | E] \leq M_E.$$

The event is measured under the product of the same marginal law. Define the Taylor comparator $e_n(t) = e^{-nt^2/2}(1 - in\gamma t^3/6)$ and

$$I_\infty = \int_{1/2}^\infty |f(t)^n - e_n(t)|t^{-3} dt, \quad L(x) = \frac{x+2}{x^2+4x+2}.$$

The sign of the imaginary term is immaterial for its absolute-value bound. Use $b = 9/10$, $0 < r \leq 1/4$,

$$h = \frac{333}{106} - \frac{(22/7)^3 r^2}{6} > 0, \quad \kappa = a(23/48)^2 d_0^2, \\ S_3 = \sum_{j=1}^{20} j^{-3} + \frac{1}{800}, \quad S_5 = \sum_{j=1}^{20} (j-r)^{-5} + \frac{1}{4(20-r)^4}.$$

Set

$$c = \frac{1 - \max_{u \in \{1/4, 81/100\}} [1 - u + (M+3)u^2/12]}{2}, \\ c_0 = \frac{1}{8} - \frac{M+3}{384}, \quad s = \frac{1 - (M+3)53/600}{2}.$$

The usual damping domain requires $c > 0$.

Lemma A.11 (Six-part high-frequency bound). *For every integer $n \geq 5$, the complete upper bound is*

$$\sqrt{n}I_\infty \leq A_n + \frac{M_E}{4\pi^{3/2}h\sqrt{a}} \left(S_3 + \frac{3S_5}{nah^2} \right) + \frac{M_E\sqrt{n}}{2\pi^2} e^{-nah^2r^2} \\ + Z_n + \frac{\sqrt{n}}{2b^2} e^{-na(23/48)^2} + \sqrt{n}e^{-n/8} \left(2L(n/8) + \frac{\sqrt{M-1}}{3} \right),$$

where always $A_n = 2\sqrt{n}e^{-nc}$ is permitted, and, if $s > 0$, it can be replaced by the smaller of itself and

$$2\sqrt{n}e^{-nc_0}L(ns/4).$$

The zero-cell term Z_n may be taken as the smaller of

$$\frac{e^{-n\kappa b^2}}{2b^4\kappa\sqrt{n}}, \quad \frac{\sqrt{n}e^{-n\kappa b^2}}{2b^2}L(n\kappa b^2).$$

Each term may be evaluated at a common floor N for all $n \geq N$ provided

$$2Nc' > 1 \quad (c' \in \{c, ah^2r^2, a(23/48)^2, 1/8\}).$$

For the new first term also require $2Nc_0 > 1$; for the second zero-cell option require $N\kappa b^2 \geq 1/4$. If either extra condition is unavailable, retain the corresponding simpler term. Any fixed collection of permitted windows may be evaluated, and the minimum of its complete six-term bounds remains valid.

Proof. We give the entire frequency partition to make clear what changes. On $[1/2, b]$, symmetrization and $\cos u \leq 1 - u^2/2 + u^4/24$ yield

$$|f(t)|^2 \leq 1 - t^2 + (M+3)t^4/12, \quad |f(t)|^n \leq e^{-nc(t^2)}, \quad c(u) = u/2 - (M+3)u^2/24.$$

Bounding $c(t^2)$ below by c and using $\int_{1/2}^b t^{-3} dt < 2$ gives the first permitted term. Concavity supplies the more precise chord bound

$$c(u) \geq c_0 + s(u - 1/4), \quad 1/4 \leq u \leq 81/100,$$

because the difference is exactly $(M+3)(u - 1/4)(81/100 - u)/24$. If $s > 0$, extend only the resulting positive exponential majorant to the half-line. The preceding rational Gaussian-tail lemma, with lower endpoint $1/2$, gives

$$\sqrt{n} \int_{1/2}^b e^{-nc(t^2)} t^{-3} dt \leq e^{-n(c_0 - s/4)} \sqrt{n} \int_{1/2}^\infty e^{-nst^2} t^{-3} dt \leq 2\sqrt{n} e^{-nc_0} L(ns/4).$$

For $t \geq b$, conditioning on the actual event and convex Jensen give

$$|f(t)|^n \leq e^{-nE \sin^2(t(X-X')/2)} \leq \mathbb{E}[e^{-na \sin^2(td/2)} \mid E], \quad d = |X - X'|.$$

Positive Tonelli allows a separate partition for every $d > 0$. Put $\ell = 2\pi/d$. The zero cell is $[0, \ell/2]$; the other cells are $\ell[j - 1/2, j + 1/2]$, $j \geq 1$. In each nonzero cell take the core $t = \ell(j + v)$, $|v| \leq r$. These cores are disjoint even for $r \leq 1/2$. For our smaller range $r \leq 1/4$, the elementary sine bound gives $|\sin(\pi v)| \geq h|v|$ throughout a core, and $|\sin(\pi v)| \geq hr$ throughout its wings. The latter uses that $\sin(\pi|v|)$ increases on $[0, 1/2]$.

Since $j - r > 0$, Taylor's formula on the two sides gives

$$\frac{(j-v)^{-3} + (j+v)^{-3}}{2} \leq j^{-3} + 6(j-r)^{-5}v^2.$$

Integrate this majorant against $e^{-nah^2v^2}$ on the full real line. The result is $\sqrt{\pi}[j^{-3}/\sqrt{nah^2} + 3(j-r)^{-5}/(nah^2)^{3/2}]$. The Jacobian is $t^{-3}dt = d^2(4\pi^2)^{-1}(j+v)^{-3}dv$. Summing and averaging d^2 gives the second term. The two series tails are bounded by their decreasing integrals after 20, producing exactly S_3, S_5 . The total wing weight is at most $\int_{\ell/2}^\infty t^{-3}dt = d^2/(2\pi^2)$, giving the third term. Extending cores or wings below b only adds a positive integral.

In the zero cell write $u = td/2 \leq \pi/2$. For $u \leq 1/2$, use $\sin u \geq (23/24)u$ and the rational Gaussian-tail lemma with rate κ ; its two choices give Z_n . For $u \geq 1/2$, use $\sin u \geq 23/48$ and total weight $1/(2b^2)$, giving the fifth term. Shared endpoints have zero integration measure, including when the conditional distance distribution has atoms.

Finally keep the comparator on the entire interval $[1/2, \infty)$. Its absolute-value bound splits its integral into

$$\sqrt{n} \int_{1/2}^\infty e^{-nt^2/2} t^{-3} dt + \frac{n^{3/2}|\gamma|}{6} \int_{1/2}^\infty e^{-nt^2/2} dt.$$

The first part is at most $2\sqrt{n}e^{-n/8}L(n/8)$. For the second, $t \geq 1/2$ implies

$$\int_{1/2}^\infty e^{-nt^2/2} dt \leq 2 \int_{1/2}^\infty t e^{-nt^2/2} dt = 2e^{-n/8}/n.$$

Pearson's elementary identity $\mathbb{E}[(X^2 - 1 - \gamma X)^2] = \mathbb{E}X^4 - 1 - \gamma^2 \geq 0$ gives $|\gamma| \leq \sqrt{M-1}$ and proves the sixth term. Any separately proved upper bound $|\gamma| \leq g$ may replace $\sqrt{M-1}$ in that term. A permitted choice is $g = \min(1.85, \sqrt{M-1})$.

For uniformity, $\sqrt{n}e^{-nc'}$ decreases when $2nc' > 1$, whereas n^{-1} and $n^{-1/2}e^{-nc'}$ always decrease. The derivative of $L(x)$ is $-(x^2 + 4x + 6)/(x^2 + 4x + 2)^2 < 0$. The new first term is thus decreasing under its stated gate, and the comparator decreases for every $n \geq 5$. The earlier tail lemma proves the separate $N\kappa b^2 \geq 1/4$ gate for its new zero-cell option. These comparisons concern the bounding expressions, so they are uniform even when the marginal law varies with n . $\square$

A.2.2.5 Events, not artificial kernels.

The entry uses only the following actual choices, with the direct rate from the same fourth moment in each case:

event	a	d_0	M_E
$(X - X')^2 \geq 9/25$	$(2 - 9/25)^2/(2M + 6)$	$3/5$	$2/a$
$XX' < 0, D \leq d \leq 0.8$	$1/(2r_*^2)$	$\sqrt{2-d}$	$4r_*^2$
$XX' < 0, p \in [l, u]$	$2 \min(l(1-l), u(1-u))$	$\max(A_l + B_l, \sqrt{2-d})$	$2/a$

The first row follows from Cauchy using $\mathbb{E}[(X - X')^2; E] \geq 2 - d_0^2$ and $\mathbb{E}(X - X')^4 = 2\mu_4 + 6$. The second is Lemma A.15; all its moment and $\alpha < 3$ gates are checked at use. In the third, $D \leq d < 2$ ensures a genuine central gap, $\mathbb{P}(XX' < 0) = 2pq$, and restriction of the squared distance gives $M_E = 2/a$. Taking a minimum of *complete* high-frequency bounds is legitimate. No tail is dropped and the third construction also allows a cell crossing $1/2$.

A.2.2.6 A compensated high-frequency bound with bounded support

Here is a second complete high-frequency inequality. It is useful at fixed small indices and requires neither a physical gap nor a predecessor estimate. Its support and moments are hypotheses, not outputs of the integration.

Lemma A.12 (Supported compensated-power bound). *Let $n \geq 2$ be an integer, $|X| \leq R$, $\rho \leq r$, $\mathbb{E}X^4 \leq M_4$, $\mathbb{E}|X|^5 \leq M_5$, and $\gamma^2 \leq G$. Let $1/2 < T < \pi$, and let $m(u)$ be any valid upper bound for $|f(u)|$ on $[1/2, T]$. Put $g = e^{-u^2/2}$ and*

$$\begin{aligned}
b(u) &= \sqrt{G}u^3/6, & W(u) &= g\sqrt{1+b(u)^2}, \\
d_0(u) &= \cos u - g, & d_1(u) &= d_0(u) + (r-1)u^3/6, \\
A(u) &= \max\{|d_0(u)|, |d_1(u)|\}, \\
B(u) &= \min\{M_4u^4/24, M_5u^5/120\} + b(u)(1-g), & R_f(u) &= \sqrt{A(u)^2 + B(u)^2}, \\
C_n(u) &= R_f(u) \sum_{j=0}^{n-1} m(u)^j W(u)^{n-1-j} + g^n \frac{n(n-1)b(u)^2}{2} (1+b(u)^2)^{(n-2)/2}, \\
\mathcal{T}_n(u) &= m(u)^n + g^n \sqrt{1+n^2b(u)^2}.
\end{aligned}$$

For every partition $1/2 = t_0 < \dots < t_s = T$, let U_j be a valid whole-cell upper bound for $\min(\mathcal{C}_n, \mathcal{T}_n)$ on $[t_{j-1}, t_j]$. Then

$$\sqrt{n}I_\infty \leq \sqrt{n} \left\{ \sum_{j=1}^s U_j \frac{t_{j-1}^{-2} - t_j^{-2}}{2} + \frac{1}{2T^2} + \frac{e^{-nT^2/2}}{nT} \left(T^{-3} + \frac{n\sqrt{G}}{6} \right) \right\}. \quad (90)$$

Proof. Write $v = -i\gamma u^3/6$ and $w = g(1+v)$. The anchored real interval of Lemma A.1 gives $|\Re(f-w)| \leq A$. Taylor's theorem for the sine gives the two independent bounds

$$|\Im f(u) + \gamma u^3/6| \leq M_4u^4/24, \quad |\Im f(u) + \gamma u^3/6| \leq M_5u^5/120.$$

Consequently $|f - w| \leq R_f$ and $|w| \leq W$. The finite telescoping factorization gives

$$|f^n - w^n| \leq R_f \sum_{j=0}^{n-1} m^j W^{n-1-j}.$$

No division, $W \leq 1$, or distinct-root hypothesis is used. Taylor's integral formula for $s \mapsto (1 + sv)^n$ on $[0, 1]$ gives

$$|w^n - g^n(1 + nv)| \leq g^n \frac{n(n-1)b^2}{2} (1 + b^2)^{(n-2)/2},$$

since v is purely imaginary. These two inequalities prove the compensated bound $\mathcal{C}_n$. The independent triangle bound is $\mathcal{T}_n$; both bound the same full remainder. Integration on a closed cell uses the exact positive weight $\int_a^b u^{-3} du = (a^{-2} - b^{-2})/2$.

Beyond T , use $|f| \leq 1$ and $|1 - in\gamma u^3/6| \leq 1 + n\sqrt{G}u^3/6$. The characteristic contribution is $1/(2T^2)$. The Gaussian contribution is at most

$$(T^{-3} + n\sqrt{G}/6) \int_T^\infty e^{-nu^2/2} du \leq \frac{e^{-nT^2/2}}{nT} (T^{-3} + n\sqrt{G}/6).$$

This includes the entire infinite tail and proves the result. $\square$

Valid choices of m and G are supplied by Lemmas A.2 and A.5 and the signed coordinate disk. The support-moment bounds below supply M_4, M_5 for any actual radius R ; a historical radius used in a different index is not an input at the present index.

The real endpoints can be enclosed sharply without differentiating a characteristic function: on a cell $[a, b]$, midpoint m_c and radius h_c , the mean value theorem gives

$$d_j([a, b]) \subset d_j(m_c) + [-h_c, h_c]D_j,$$

where D_j encloses the explicit derivative on the whole cell and

$$d'_0(u) = -\sin u + ug(u), \quad d'_1(u) = d'_0(u) + (r-1)u^2/2.$$

Intersect this interval with the direct scalar enclosure. Both contain the same function; no midpoint sampling replaces a remainder bound. The minimum of several complete valid high-frequency bounds is allowed. Taking selected favorable frequency pieces without proving their whole-cell majorants is not.

A.2.3 Turning one defect budget into simultaneous enclosures

Lemma 5.7 gives the unweighted defect budget after a first raw ceiling licenses its sign. Corollary 5.8 retains the additional phase curvature under its stated positivity gates.

The same identity also gives a useful score update. If this cell forces $D \geq d_l \geq 0$ and a preceding pass gives raw ceiling r , then

$$C \leq 0.40974 + \frac{r/N - 0.3988(1 - 5r/(2N))d_l/6}{\rho_l}.$$

A negative numerator excludes the cell because $e \geq 0$. Otherwise its numerator increases with $1/n$, and division by the positive lower ρ_l is outward. Thus the update is valid for every $n \geq N$ and uses only the previous raw bound.

A.2.3.1 One positive-measure rule

Suppose zero lies in the gap $(-A, B)$, $A, B > 0$. Let

$$p = \mathbb{P}(X > 0), \quad q = 1 - p, \quad m = \mathbb{E}X_+ = \mathbb{E}X_-, \quad V = \mathbb{E}[X \mid \mathbf{1}_{X>0}], \\ W = X - V, \quad \theta_{\pm} = \mathbb{E}[W^2; \pm X > 0], \quad \theta = \theta_+ + \theta_-.$$

Then $m^2 = (1 - \theta)pq$, and the conditional means have magnitudes $\bar{A} = m/q$, $\bar{B} = m/p$. Set $\Delta_A = \bar{A} - A \geq 0$, $\Delta_B = \bar{B} - B \geq 0$. We also use the unnormalised endpoint displacements

$$M_+ = \mathbb{E}[(X - B)\mathbf{1}_{X>0}] = p\Delta_B, \quad M_- = \mathbb{E}[(-X - A)\mathbf{1}_{X<0}] = q\Delta_A.$$

Lemma A.13 (Positive-measure pricing). *For $x = |X|$, $a = |V|$, and a function h constant on each sign class,*

$$\mathbb{E}[h(|X|^j - |V|^j)] = \mathbb{E}\left[hW^2 \sum_{k=0}^{j-2} (k+1)a^k x^{j-2-k}\right], \quad j \geq 2. \quad (91)$$

Also, for $d\mu_0 = (|X| - 1)^2(|X| + 1/2) dP$,

$$\mu_0(\mathbb{R}) = \rho - 1, \\ M_4 - 1 = \int \frac{(x+1)^2}{x+1/2} d\mu_0, \\ M_5 - 1 = \int \frac{x^3 + 2x^2 + 3x + 3/2}{x+1/2} d\mu_0, \\ K_1 + 3\rho\beta = \int \frac{\text{sgn}(X)x(x+2-3\rho/2)}{x+1/2} d\mu_0.$$

For any nonnegative resources z_i satisfying $\sum a_i z_i \leq r$, $a_i > 0$, a signed linear cost obeys

$$\sum b_i z_i \leq r \max\{0, b_1/a_1, \dots, b_k/a_k\}. \quad (92)$$

Proof. Divide $x^j - a^j - ja^{j-1}(x - a)$ by $(x - a)^2$; the linear term has zero conditional mean. The radial identities follow by multiplying out and using $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$. Finally each b_i is at most a_i times the displayed maximum, and all z_i are nonnegative. $\square$

This one rule accounts for the moment, signed Taylor, endpoint movement, outer-gap, and confinement-error estimates. In particular, no two costs may each spend the full residual and then be added.

We use the central-gap notation and quantitative decomposition (16) from Section 5. In particular, $T_{\pm}$ measure the third-moment excess beyond the inner endpoints and J_o sums the outer gaps. The simultaneous inequality (17) supplies the common residual below.

A.2.3.2 Simultaneous bounds and their refinement

For a closed probability interval $I = [l, u]$, suppose the following bounds hold:

$$|X| \leq R, \quad C \in [c_0, c_1], \quad \mathcal{R} \leq r, \quad nD \leq Nd, \quad \theta \leq \Theta < 1, \quad A \geq A_l > 0, \quad B \geq B_l > 0,$$

together with simultaneous signed-moment and coefficient intervals. The distinction between $nD \leq Nd$ and $D \leq d$ is essential. Let $s_+ = \max_I p(1 - p)$, $s_- = \min_I p(1 - p)$ and

$$b_I = \frac{(6 + 2\sigma)(1 - \Theta)^{3/2}}{\sqrt{s_+}} \text{dist}(p_*, I)^2, \quad r_I = d - b_I,$$

$$K_+ = 3(\sigma + 1)B_l, \quad K_- = 3(\sigma + 3)A_l, \quad K = \min(K_+, K_-), \quad m_l = \sqrt{(1 - \Theta)s_-}.$$

If $r_I < 0$, exclude the entire closed cell. Otherwise (17) gives, simultaneously,

$$6m(A_l + B_l)(\Delta_A + \Delta_B) + K_+\theta_+ + K_-\theta_- \leq r_I. \quad (93)$$

The guaranteed inequalities for the actual law are

$$\begin{aligned} \theta &\leq \min\{\Theta, r_I/K\}, \\ A &\geq \max\left\{A_l, \sqrt{(1-\Theta)l/(1-l)} - \frac{r_I}{6m_l(A_l + B_l)}\right\}, \\ B &\geq \max\left\{B_l, \sqrt{(1-\Theta)(1-u)/u} - \frac{r_I}{6m_l(A_l + B_l)}\right\}. \end{aligned} \quad (94)$$

The new variance ceiling is an outward upper bound for the first right side, and each new endpoint floor is an outward lower bound for its right side. They are not arbitrary numbers satisfying an inequality in the opposite direction. Each right side is evaluated from established bounds; sequential improvements may then be used by the next expression. A nonpositive endpoint estimate fails this update, rather than excluding a law. Initialization uses the same formulas with the valid lower sum $A + B \geq \sqrt{2-d}$.

In Bernoulli coordinates set $Y = B_0 + U$, $\mathbb{E}[U | B_0] = 0$, $B_0 \sim \text{Bernoulli}(p)$, $X = \ell(Y - p)$ and $d_Y = (\Delta_A + \Delta_B)/(\bar{A} + \bar{B})$. Then $\ell^2 = (1 - \theta)/(pq)$ and $\lambda = n\mathbb{E}U^2 = npq\theta/(1 - \theta)$. The *weighted* versions of (93) prove

$$n\theta \leq \frac{Nr_I}{K}, \quad nd_Y \leq \frac{Nr_I}{6(1-\Theta)(A_l + B_l)}, \quad \lambda \leq \frac{Ns_+r_I}{(1-\Theta)K}. \quad (95)$$

Indeed $nD \leq Nd$ whereas $nb_I \geq Nb_I$. These exports are derived from the weighted resource; multiplying an unweighted variance ceiling by N would not prove them.

For a joint bound on displacement and noise, apply Theorem 5.4 with $d_D = d$, $v_* = s_+$, $S_* = A_l + B_l$, $K = \min(K_+, K_-)$ and $b = b_I$. The premise follows from (17) by dropping $6m\Delta_A\Delta_B \geq 0$; the required index-scaled input is $nD \leq Nd_D$. On a closed band $L_0 \leq \lambda \leq U_0 \leq \Lambda$ this gives $d_Y \leq d_*(1 - L_0/\Lambda)$. Either positive intercept may first be enlarged, since both summands are nonnegative. The theorem also handles zero and negative residuals.

For $j = 3, 4, 5$ set

$$L_3(v) = R + 2v, \quad L_4(v) = R^2 + 2Rv + 3v^2, \quad L_5(v) = R^3 + 2R^2v + 3Rv^2 + 4v^3.$$

Put $v_+ = \sqrt{(1-l)/l}$, $v_- = \sqrt{u/(1-u)}$ and $k_j = [1 - (1 - \Theta)^{j/2}]/\Theta$, with $k_j = j/2$ at $\Theta = 0$. For $u < 1/2$, valid Bernoulli moment ceilings are, with $s = l(1-l)$,

$$B_3^* = (1 - 2s)/\sqrt{s}, \quad B_4^* = 1/s - 3, \quad B_5^* = (1 - 4s + 2s^2)/s^{3/2}.$$

They decrease towards $p = 1/2$. Convexity places $(1 - \theta)^{j/2}$ below its endpoint chord on $[0, \Theta]$; positive-measure pricing gives

$$M_j \leq B_j^* + r_I \max\left\{0, \frac{L_j(v_+) - k_j B_j^*}{K_+}, \frac{L_j(v_-) - k_j B_j^*}{K_-}\right\}. \quad (96)$$

For clarity, conditional Taylor expansion first gives

$$M_j \leq (1 - \theta)^{j/2} B_j^* + L_j(v_+)\theta_+ + L_j(v_-)\theta_-.$$

The chord estimate subtracts $k_j B_j^*$ from each variance coefficient. Multiplying the maximum in (96) by $K_+\theta_+ + K_-\theta_-$, then using the single residual inequality, proves (96); a negative numerator is harmless because zero is among the maximized values. For the outer charge, the

deterministic positive cubic term is $m(\bar{B}^2 - B^2) \leq 2m\bar{B}\Delta_B$. Thus $T_+ \leq 2mv_+\Delta_B + L_3(v_+)\theta_+$, with the reflected formula on the negative class. Applying the same simplex rule bounds

$$\begin{aligned} W_o &:= (\sigma - 1)T_+ + (\sigma + 1)T_- \\ &\leq r_I \max \left\{ \frac{(\sigma - 1)v_+}{3(A_l + B_l)}, \frac{(\sigma + 1)v_-}{3(A_l + B_l)}, \frac{(\sigma - 1)L_3(v_+)}{K_+}, \frac{(\sigma + 1)L_3(v_-)}{K_-} \right\}. \end{aligned} \quad (97)$$

Intersect every new bound with the previous one. The versions which merge θ_+ and θ_- are obtained by replacing both costs by their maximum and both prices by their minimum; they need no new proof.

A.2.3.3 The enclosure theorem

On each cell, the formula atlas bounds the four coefficients in (29). The result, for each actual sign $s = \text{sgn } H$, is a finite family of quadratics

$$\mathcal{R} \leq \max_j (A_j + F_{j,s}|H| - Q_j^{\text{raw}} H^2), \quad Q_j^{\text{raw}} > 0. \quad (98)$$

The maxima retain all score–moment corners. In the initial signed cover, intervals contain both signs of β, γ ; no orientation is discarded. In the probability cover, the central integral bound is a maximum of affine functions of ρ , hence convex. The other ρ -dependence is affine or the maximum of convex quadratics. For fixed H , the score dependence is convex. Endpoints are therefore taken *before* completing the square.

Theorem A.14 (Finite resource enclosure). *Consider a finite ordered schedule of the following operations:*

1. *Split a cell into an explicitly covered family of closed cells.*
2. *Intersect its enclosures with an inequality proved in the atlas.*
3. *Exclude a cell by an empty necessary interval or a strictly negative nonnegative resource.*
4. *Apply a separately proved support or confinement implication, checking all its hypotheses against this cell's already acquired data.*
5. *Export a raw, weighted, or signed phase bound by (98), (33), and (95).*

If exact outward arithmetic checks every domain, corner and cover, every output contains all attaining laws satisfying the input. The conclusion holds for all integers $n \geq N$ wherever the formula's monotonicity gates are checked. It includes zero conditional variance, singleton intersections and arbitrarily small atoms.

Proof. Induct over the schedule. A split preserves the union, intersections use necessary inequalities, and an excluded cell contains no feasible law. Each support or confinement implication uses previously established hypotheses for that law. For a scalar raw output, use $fh - qh^2 \leq (f_+)^2/(4q)$ for $h \geq 0, q > 0$; using f^2 is a valid weaker bound. For a resource output first add (32) to each curvature. A negative maximum excludes the cell; a nonnegative one may be divided by κN to give its next defect parameter. The phase bound follows from the positive root of $A + fh - qh^2 \geq 0$, replacing A by A_+ if necessary. A strict output is exported only when the certificate retains a strict upper gap, such as its strict Fourier or target enclosure; closed-domain coverage does not change strictness. All constants are frozen before evaluating corners. Thus no iteration step relies on an unproved fixed point. The stated index uniformity is that of the displayed finite formulas, not a fixed-law limit. $\square$

For the Fourier cancellation estimate only, reflect (X, H) if needed: $I_0(-X, -H) = I_0(X, H)$. Both odd coefficients change sign, and $\int e^{-x^2/2} |zx^2 - cx^4| dx$ is unchanged by $(z, c) \mapsto (-z, -c)$. Consequently a lower endpoint used there bounds $|\gamma|$, not necessarily the original signed γ . The actual contact, gap and phase data keep their original orientation. A useful additional signed lower bound is

$$\gamma \geq (1 - \Theta)^{3/2} \frac{1 - 2u}{\sqrt{u(1 - u)}} - \left(R + 2\sqrt{\frac{u}{1 - u}} \right) \Theta. \quad (99)$$

To prove it expand $\mathbb{E}X^3 = \mathbb{E}V^3 + \mathbb{E}[W^2(X + 2V)]$; the positive residual is nonnegative and the negative residual has the displayed lower bound. The binary term decreases in both $p < 1/2$ and θ . Thus $\max(0, \text{any signed lower bound})$ is a valid lower bound for $|\gamma|$ inside this auxiliary Fourier estimate.

A.2.4 Coefficient atlas for the enclosure

The atlas has four stages. First, signed first, second and third moments bound the fourth and fifth moments and the gap integrals before a probability panel is known. Second, an acquired central gap and its conditional variances price the same expressions more sharply. Third, signed phase bounds combine those moment prices with the complete Fourier error into an affine-in-score quadratic in $|H|$. Finally, probability and support feedback export one joint displacement/noise budget to the count comparison.

The Fourier part is assembled before choosing these moment coordinates: the low-frequency transfer treats the compensated Taylor terms, the high-frequency formulas bound the entire complementary integral including its infinite tail, and the contact identity supplies the signed constant, linear and quadratic moment terms. The formulas below preserve those separate contributions and state the gates for each alternative bound. In particular, a lattice characteristic function is not assumed to decay on the whole high-frequency half-line. The final output is the quadratic family in (98), with the phase addition of (32) made once.

These formulas are inequalities for one actual law. They are the two coordinate descriptions of the same enclosure: signed moments before a probability gap is located, and conditional moments afterwards. A bound is used only with its stated gates, and every refinement is an intersection.

A.2.4.1 Signed moments with a common first moment

Put $a_{\pm} = (1 \pm \beta)/2$, $b_{\pm} = (\rho \pm \gamma)/2$. Both signs have positive probability and positive first moment, since a centered variance-one law cannot be supported on one half-line. Thus $|\beta| < 1$ and the signwise denominators below are positive whenever used; degenerate correction denominators are handled separately. Cauchy gives $m^2 \leq a_+ a_-$ and $b_{\pm} m \geq a_{\pm}^2$. Consequently

$$\frac{(1 + \beta)^{3/2}}{\sqrt{1 - \beta}} - \rho \leq \gamma \leq \rho - \frac{(1 - \beta)^{3/2}}{\sqrt{1 + \beta}}, \quad m \geq \max \left\{ \frac{1 + \beta^2}{2\rho}, \frac{a_+^2}{b_+}, \frac{a_-^2}{b_-} \right\}.$$

These statements also hold with a zero atom. Projection of $X|X|$ onto the span of X and $|X|$ gives

$$\mu_4 \geq \rho^2 + \frac{(\gamma - \rho\beta)^2}{1 - \beta^2}.$$

For completeness, this follows by minimizing the nonnegative expectation $\mathbb{E}(X|X| - uX - v|X|)^2$ over u, v ; its Gram matrix has diagonal entries one and off-diagonal entry β , and is positive definite.

Two support-localizing moment matrices give the additional upper bounds as follows. If $D_{\pm} = Ra_{\pm} - b_{\pm} > 0$, then

$$2m + \sum_{\pm} \frac{(Rm - a_{\pm})^2}{D_{\pm}} \leq R, \quad \mu_4 \leq R\rho - \sum_{\pm} \frac{D_{\pm}^2}{Rm - a_{\pm}}.$$

Indeed the first follows by summing the determinant inequalities for

$$\mathbb{E}[(R - |X|)(1, |X|)^{\top}(1, |X|); \pm X > 0].$$

The second follows from the corresponding matrices with an additional factor $|X|$. One more factor $|X|$ yields, sign by sign,

$$\mathbb{E}[X_{\pm}^5] \leq R\mathbb{E}[X_{\pm}^4] - \frac{(Rb_{\pm} - \mathbb{E}[X_{\pm}^4])^2}{D_{\pm}}.$$

Zero denominators are treated by dropping their nonnegative correction; they never imply exclusion without a separate contradictory numerator. The common m is retained throughout.

A second useful upper bound for m follows by integrating $(x - a)^2(R - x) \geq 0$ for $x = |X| \in [0, R]$. For every fixed $a > 0$,

$$2m \leq \frac{R - \rho + 2a + Ra^2}{a(2R + a)}.$$

The cover uses only the eleven rational choices $a = 2/10, \dots, 12/10$.

Write $\delta = \gamma - \rho\beta$. Cauchy and a square completion on each sign give the signed constant bound

$$\frac{\rho}{2}\mu_4 - \frac{1}{4}\eta_5 \leq \frac{\rho^3}{4} - \frac{(\rho + \gamma)(\delta_+)^2}{8(1 + \beta)^2} - \frac{(\rho - \gamma)((-\delta)_+)^2}{8(1 - \beta)^2}.$$

Indeed on either sign, write $a = \mathbb{E}X_s^2$, $b = \mathbb{E}X_s^3$, $c = \mathbb{E}X_s^4$, $f = \mathbb{E}X_s^5$. Cauchy gives $c \geq b^2/a$, $f \geq c^2/b$, and $\rho c/2 - f/4 \leq \rho^2 b/4 - (c - \rho b)^2/(4b)$. Distance to the half-line $c \geq b^2/a$ proves the stated penalty on summing both signs. Independently, $\mu_4^2 \leq \rho\eta_5$ and the projection bound above give

$$\frac{\rho}{2}\mu_4 - \frac{1}{4}\eta_5 \leq \frac{\rho^3}{4} - \frac{(\gamma - \rho\beta)^4}{4\rho(1 - \beta^2)^2}.$$

The larger of the two nonnegative deductions is retained. There is a third independent deduction which is often stronger. For $Y = |X|$ and $\delta = \gamma - \rho\beta$, put

$$V = \mathbb{E}[Y(Y^2 - \rho Y)^2] = \eta_5 - 2\rho\mu_4 + \rho^3.$$

Since $\delta = \mathbb{E}[Y \operatorname{sgn}(X)(Y^2 - \rho Y)]$, weighted Cauchy gives $\delta^2 \leq 2mV$. Hence

$$\frac{\rho}{2}\mu_4 - \frac{1}{4}\eta_5 \leq \frac{\rho^3}{4} - \frac{\delta^2}{8m}. \quad (100)$$

The maximum of these three separately valid losses can be subtracted. Their sum cannot. A zero atom contributes zero to the weighted measure.

A.2.4.2 An actual central gap without assuming $D < 2$

The already proved gap decomposition (16) shows that $D < (\sigma - 1)\rho$ implies an actual central gap. Use its same nonnegative terms, writing $T = T_+ + T_-$ and $Q = (\sqrt{\sigma + 1}A - \sqrt{\sigma - 1}B)^2$. Hence $T \leq D/(\sigma - 1)$ and

$$(A + B)^2 \geq A^2 + B^2 \geq \frac{\rho - D/(\sigma - 1)}{m}.$$

Moreover $A \leq a_-/m$, $B \leq a_+/m$ and $J \leq \rho + 2m$. The last bound gives the compulsory defect

$$D \geq \sigma\rho - \gamma - 6m.$$

The inequalities $T - J_o \geq 2BR_+ + 2AR_-$ give the useful additional endpoint constraint

$$D \geq (\sigma - 1)b_+ + 6a_+B - (\sigma + 5)mB^2.$$

If $D < (\sigma - 1)b_+$, this downward quadratic forces B to be at least its positive root. The reflected constraint has a_- , b_- , A and the coefficients $\sigma + 1, \sigma + 7$. In a closed moment box, positive coefficients are rounded down, negative coefficients up, and the lower square-root endpoint is used in the resulting root bound.

The actual opposite-sign event has probability $2pq$. The first two moment constraints give lower bounds for both p and q . For independent copies one has exactly

$$\mathbb{E}[(X - X')^2 \mid XX' < 0] = \frac{1 + \beta(1 - 2p) + 4m^2}{2pq} = \frac{1 + 4m^2 + \beta}{2p} + \frac{1 + 4m^2 - \beta}{2q}.$$

For an upper bound, use the upper m endpoint and both endpoints of p and β . The expression is convex in p and affine in β . This improves the former $2/\mathbb{P}(XX' < 0)$ bound inside the complete high-frequency estimate. Both copies still have the same actual marginal.

The gap can also be licensed before the sufficient inequality $D < (\sigma - 1)\rho$ is available. If an already proved bound $D \leq d$ and a signed moment cell give

$$P_0 = \max\{(\sigma_l - 1)\rho_l, \sigma_l\rho_l - \gamma_u\} - d > 0, \quad \sigma_l \leq \sqrt{10},$$

then zero cannot belong to support: in that case $D \geq \sigma\rho - \gamma \geq P_0 + d$. For the resulting actual gap, $J \leq \rho + 2mAB$ gives $6mAB \geq P_0$. Thus a closed panel $p \in [l, u] \subset (0, 1)$ has

$$A \geq \frac{P_0}{6(1 - l)}, \quad B \geq \frac{P_0}{6u}, \quad (A + B)^2 \geq \frac{2P_0}{3m_u}. \quad (101)$$

Indeed $A \leq m/(1 - p)$ and $B \leq m/p$, while $m^2 \leq p(1 - p)$, so $mAB \leq (1 - p)A$ and $mAB \leq pB$. The last inequality uses $4AB \leq (A + B)^2$. If earlier steps have acquired $A \geq a > 0$, $B \geq b > 0$ and $m \geq m_l$, one may replace the lower price P_0 by

$$P = \max\{P_0, 6m_l ab\} \leq 6mAB. \quad (102)$$

This retains an already proved physical gap; it does not license a gap from an unsuccessful initialization. In fact the new term alone gives $2(6m_l ab)/(3m_u) \leq 4ab \leq (a + b)^2$.

A.2.4.3 Preserving central and outer integral costs

Let

$$P_1(A, B) = B^3/3 + AB^2/2 - A^3/6, \quad P_2(A, B) = B^4/4 + AB^3/3 - A^4/12, \quad G = \rho P_1 - P_2.$$

Then $J_{1,c} = mP_1$, $J_{2,c} = mP_2$. Since $\partial_B G = B(A+B)(\rho-B)$,

$$G(A, B; \rho) \leq G(A, \rho; \rho) = \rho^4/12 + \rho^3 A/6 - \rho A^3/6 + A^4/12.$$

The last expression increases with A because its derivative is $(\rho-A)^2(\rho+2A)/6$, and is convex in ρ because its second ρ derivative is $\rho^2 + \rho A$. Its closed-rectangle maximum is therefore bounded using the upper A endpoint and both ρ endpoints.

Positive outer gaps satisfy $(t-a)(\rho-t) \leq (\rho-B)_+^2/4$. On negative outer gaps the positive part is bounded by $(R-\rho)^2/4$ after extracting $|t|$. Integration gives

$$3(\rho J_1 - J_2)_o \leq \frac{3}{8} \{(\rho-B)_+^2 T_+ + (R-\rho)^2 T_-\}.$$

The available weighted outer budget is $D - mQ$, not the entire D . The cover intersects this budget with the individual bounds $T_+ \leq b_{+,u} - m_l B_l^2$ and $T_- \leq b_{-,u} - m_l A_l^2$.

For the signed phase term $U = J_1 + (\mu_4 - 3)/6$, retain

$$-RT_-/2 \leq J_{1,o} \leq RT_+/2.$$

The central minimum of P_1 occurs at B_l and an A endpoint; its maximum occurs at B_u and A clipped to the interval at $A = B_u$. A nonnegative polynomial bound is multiplied by m_u for an upper bound and m_l for a lower bound; a negative one uses the opposite endpoint. For example, the binary endpoints $A = 3/2$, $B = 2/3$ make this maximum negative; the sign switch is necessary even when $D < 2$. The fourth-moment sign intervals also sharpen $K_1 = \mathbb{E}[X|X|^3] - 3\beta - 3\rho\gamma/2$. The complete K_1, U, ϕ rectangles are used in the signed expression $|CK_1 + \phi U|$, without replacing it by two separate absolute values.

A.2.4.4 A complete coarse bound before a central gap is licensed.

Always $-\mu_4/6 \leq J_1 \leq \mu_4/2$ and $-3J_2 - \eta_5/4 \leq 0$. On a gap of length $h = b - a$, these follow from

$$\int_a^b (t-a)t dt + \frac{1}{2} \int_a^b t^2 dt = hb^2/2, \quad \frac{3}{2} \int_a^b t^2 dt - \int_a^b (t-a)t dt = h(2a+b)^2/6,$$

and, on a negative gap $(-v, -u)$, from $\frac{1}{3} \int_u^v x^3 dx - \int_u^v (v-x)x^2 dx = u^3(v-u)/3$. For $1 \leq r \leq \rho$ and actual support radius $R \leq 6$, set

$$\Gamma(R, r) \geq \max_{r \leq v \leq R} \left\{ -\frac{v^2}{4} + \frac{(R+r)v}{3} - \frac{Rr}{2} + \frac{r^3(2R-r)}{12v^2} \right\}_+.$$

Then

$$3(\rho J_1 - J_2) \leq \frac{3}{2} \Gamma(R, r)(\rho - \gamma) + \frac{3}{8} \rho^2 J.$$

On negative gaps extract $x = |t|$ and discard the nonpositive part $x < \rho$. Replacing ρ by r and the outer endpoint by R gives $\int_r^v (R-x)x(x-r) dx$. Dividing this integral by v^2 gives exactly the displayed expression. Signwise Tonelli now charges $\Gamma \mathbb{E} X_-^3 = \Gamma(\rho - \gamma)/2$. On the positive portion of each gap, $t(\rho-t) \leq \rho^2/4$. Multiply by $t-a \geq 0$ and integrate; since $\int_a^b (t-a) dt = (b-a)^2/2$, the resulting contribution is at most $\rho^2 J/8$. This includes a gap crossing zero. Partition the v -range into 64 closed intervals: on each, maximize the concave quadratic at its clipped vertex $2(R+r)/3$ and the positive reciprocal term at the left endpoint. This proves its interval bound. The alternative bound $\Gamma \leq 5/2$ follows at $r = 1, R = 6$ by $3v^4 - 28v^3 + 66v^2 - 11 > 0$ on $[1, 6]$: the endpoint values are 30, 205, and the interior critical values are $(85 \mp 35\sqrt{5})/2 > 0$. These monotone enclosing parameters cover all $r \geq 1, R \leq 6$.

A.2.4.5 Moment and gap coefficients for a whole cell

The following coefficient rules are collected here to make the finite calculation transparent. Every rule is an inequality for an actual law; there is no extremal-law assumption inside a cell. Write $v = 1 - \theta$. For $p \in [l, u] \subset (0, 1)$ set

$$\begin{aligned} s_0 &= \min\{l(1-l), u(1-u)\}, & v_* &= \max\{\sqrt{u/(1-u)}, \sqrt{(1-l)/l}\}, \\ r_B &= (1-2s_0)/\sqrt{s_0}, & g_B &= \max_{t=l,u} |1-2t|/\sqrt{t(1-t)}, & b_B &= \max_{t=l,u} |1-2t|, \\ L_3 &= R + 2v_*, & L_4 &= R^2 + 2Rv_* + 3v_*^2, \\ L_5 &= R^3 + 2R^2v_* + 3Rv_*^2 + 4v_*^3, & e_B &= r_B \frac{1-4s_0+2s_0^2}{s_0(1-2s_0)}. \end{aligned}$$

The polynomials $T_3 = x + 2a$, $T_4 = x^2 + 2ax + 3a^2$ and $T_5 = x^3 + 2ax^2 + 3a^2x + 4a^3$ are nonnegative. Their maxima at $x = R$, $a = v_*$, in Lemma A.13, give, if $\theta \leq \Theta < 1$,

$$\begin{aligned} \rho &\leq r_B + (L_3 - r_B)\Theta, & |\gamma| &\leq g_B + (L_3 - g_B)\Theta, \\ |\beta| &\leq b_B + (1 - b_B)\Theta, & \mu_4 &\leq (1 - \Theta)^2(1/s_0 - 3) + L_4\Theta, \\ \eta_5 &\leq (1 - \Theta)^2e_B + L_5\Theta. \end{aligned} \tag{103}$$

Replacing θ by Θ uses the explicitly checked inequalities $L_3 > r_B, g_B$, $L_4 > 2(1/s_0 - 3)$ and $L_5 > 2e_B$. The exact binary factors are $v^{3/2}, v, v^2, v^{5/2}$; the upper bounds use $v^{3/2} \leq v$ and $v^{5/2} \leq v^2$. Thus cells crossing $1/2$ and zero conditional variances are included. The same cubic expansion, in the later domain $p \in [0.37, 0.47]$, $A \geq 0.75$, $B \geq 1.05$, gives

$$\rho \geq v^{3/2}\rho_B(p) + 3\min(A, B)\theta \geq \rho_B(p) + 0.45\theta, \quad \rho_B(p) = \frac{1-2pq}{\sqrt{pq}}. \tag{104}$$

Indeed $\rho_B < 1.11$ there and $v^{3/2} \geq 1 - 3\theta/2$. Also $\beta \geq 1 - 2p - 2(1-p)\theta$.

For the signed coefficient $K_1 = \mathbb{E}[X|X|^3] - 3\beta - 3\rho\gamma/2$, define

$$K(p, v) = (1 - 2p)\{(3v^3/2 - v^2)/(pq) - 3v^3 + v^2 + 3v\}.$$

Direct substitution gives $K_1(V) = -K(p, v)$, with the coefficient 3 unchanged. For $v \geq 2/3$, K decreases in p : both coefficients in braces are nonnegative and $(1 - 2p)/(pq)$ decreases. Its derivative in v has the sign of $1 - 2p$, since the derivative of the braces is at least $9v^2 - 6v + 3 > 0$. Hence both variance endpoints are retained. The error is

$$|K_1(X) - K_1(V)| \leq \Theta\{L_4 + 3 + \frac{3}{2}L_3(R_\rho + g_B)\} \tag{105}$$

when $\rho \leq R_\rho$. This follows from the conditional expansions and $\rho\gamma - \rho_V\gamma_V = \rho(\gamma - \gamma_V) + \gamma_V(\rho - \rho_V)$.

Here are the remaining signed charges. They also explain the two alternative bounds retained in the initial, coarse moment cover:

$$-\mu_4/6 \leq J_1 \leq \mu_4/2, \quad -3J_2 - \eta_5/4 \leq 0, \tag{106}$$

$$\rho J_1 - J_2 \leq \frac{5}{4}(\rho - \gamma) + \frac{1}{8}\rho^2(\rho + 1), \tag{107}$$

$$-75D/104 \leq J_{1,o} \leq 3D/2. \tag{108}$$

The first two lines follow from the coarse gap identities proved above. For (108), use $J_{1,o} \geq -3T_-$, $J_{1,o} \leq 3J_o \leq 3T$ and (16).

The central integrals are $J_{j,c} = mP_j(A, B)$, with

$$P_1 = B^3/3 + AB^2/2 - A^3/6, \quad P_2 = B^4/4 + AB^3/3 - A^4/12.$$

They increase in B ; their A derivative changes from positive to negative at $A = B$. A rectangle minimum is therefore at B_l and an A endpoint. Multiply a negative minimum by an upper bound for m , and a nonnegative minimum by a lower bound. For the upper central charge, maximizing at $(\bar{a}, \bar{b})$ if $p \leq 1/2$ gives $J_{1,c} \leq v^2(2 - 3p)/(6pq)$, a decreasing function of p on that interval. If $p \geq 1/2$, maximize first at $A = B$ to get $J_{1,c} \leq (2/3)v^2q^2/p \leq 1/3$. Thus every cell may use

$$J_{1,c} \leq \max\{1/3, (2 - 3l)/(6l(1 - l))\}. \quad (109)$$

For $|X| \leq R$, $R \geq 1$, define

$$k_4(R) = \frac{(R+1)^2}{R+1/2}, \quad k_5(R) = \frac{R^3 + 2R^2 + 3R + 3/2}{R+1/2}.$$

Then

$$\mu_4 \leq \min\{R\rho, 1 + k_4(R)(\rho - 1)\}, \quad \eta_5 \leq \min\{R\mu_4, 1 + k_5(R)(\rho - 1)\}, \quad \mu_6 \leq R\eta_5. \quad (110)$$

Writing $u = |X|$ in (A.13), the first two constants are the endpoint values of the ratios $u + 3/2 + 1/[4(u + 1/2)]$ and $u^2 + 3u/2 + 9/4 + 3/[8(u + 1/2)]$, whose derivatives are nonnegative. The other bounds are pointwise. The following coarse cancellation estimates continue to use the valid radius-six majorants: The last identity in (A.13) gives

$$|K_1 + 3\rho\beta| \leq (96 - 18\rho)(\rho - 1)/13. \quad (111)$$

Indeed $u(u + 2 - 3r/2)/(u + 1/2)$ is convex on $[0, 6]$, its maximum is $(96 - 18r)/13$, and its negative part is at most $31/40$, smaller than that maximum for $1 \leq r \leq 37/20$. The other bounds needed below are

$$\begin{aligned} |K_1| &\leq 6\rho - 3\rho^2/2 + 3|\beta|, \\ K_1 &\leq M + 3|\beta| - 3\rho g_-/2, & -K_1 &\leq 3\rho + 3|\beta| + (3\rho/2 - 3)g_-, \\ -K_q &\leq (\rho/2 - 3/8)M - \rho/2 + 1/8 + 3|\beta|g/2, & -K_q &\leq 8\rho^3/27 + (3|\beta|/2 - 1/2)\rho, \end{aligned} \quad (112)$$

where $\mu_4 \leq M$, $|\gamma| \leq g$, $\gamma \geq g_-$, and $K_q = \eta_5/4 - \rho\mu_4/2 - 3\beta\gamma/2 + \rho/2$. For the first two lines use $|\mathbb{E}[X|X|^3] - 3\gamma| \leq 3\rho$, and note the negative coefficients of γ . For the last line use $\eta_5 \geq \mu_4^{3/2} \geq 3\mu_4/2 - 1/2$, or maximize $\rho y^2/2 - y^3/4$ over $y \geq 0$.

A.2.4.6 Compatible weaker resource constants.

One may also use the weaker binary charge $b_I = (616/25)(1 - \Theta)^2 \text{dist}(p_*, I)^2$. It follows from $2(6 + 2\sigma) > 616/25$ and $(1 - \theta)^{3/2} \geq (1 - \theta)^2$. With $K = \min(K_+, K_-)$ and $v_* = \max(v_+, v_-)$, the merged charge (93) gives

$$T_+ + T_- \leq T_* = (d - b_I) \max\{v_*/[3(A_l + B_l)], (R + 2v_*)/K\}, \quad d_o = \min\{d, (\sigma + 1)T_*\}.$$

Thus $W_o \leq d_o$ and, for $R \leq 6$,

$$-75d_o/104 \leq J_{1,o} \leq 3d_o/2, \quad 3(\rho J_{1,o} - J_{2,o}) \leq \frac{3}{8} \max\{\rho^2/(\sigma - 1), (R - \rho)^2/(\sigma + 1)\}d_o.$$

These are precisely the merged-cost versions of the sign-specific resource rule; they do not introduce another available budget.

A.2.4.7 Conditional lower moments improve two signed charges

Let $s_u = \max_{p \in [l, u]} pq$ and $\theta \leq \Theta$. The unsigned cases of (91) give

$$\mu_4 \geq M_l := \max\{1, (1 - \Theta)^2(1/s_u - 3)\}, \quad (113)$$

$$\eta_5 \geq E_l := (1 - \Theta)^{5/2} \frac{1 - 4s_u + 2s_u^2}{s_u^{3/2}}. \quad (114)$$

Both binary moment functions decrease in $s \in (0, 1/4]$. For the fifth this follows because the derivative has the sign of $s^2 + 2s - 3/2 < 0$; the fourth is immediate. The fourth-moment lower bound sharpens the lower endpoint of $J_1 + (\mu_4 - 3)/6$; its explicit use appears in the cell map below.

The constant signed charge is

$$-K_q = \frac{\rho\mu_4}{2} - \frac{\eta_5}{4} + \frac{3}{2}\beta\gamma - \frac{\rho}{2}.$$

One coherent estimate uses $\eta_5 \geq 3\mu_4/2 - 1/2$ and $\mu_4 \leq M$. A second estimate uses (114). Their difference is independent of ρ . Consequently the common bound improves by subtracting

$$\frac{1}{4} \max\{0, E_l - 3M/2 + 1/2\}. \quad (115)$$

This is a constant shift on the cell and does not spoil convexity. It does not assert the generally invalid implication $\eta_5 \geq 3M/2 - 1/2$ from an upper bound on μ_4 .

A.2.4.8 Signed conditional Taylor cancellation

Let $K_1 = \mathbb{E}[X|X|^3] - 3\beta - \frac{3}{2}\rho\gamma$. On a sign class put $x = |X|$, $a = |V|$, $W = X - V$ and $\varepsilon = 1$ on the positive class, -1 on the negative class. In the actual central-gap domain there is no atom at zero. With $w_3 = x + 2a$, $w_4 = x^2 + 2ax + 3a^2$ and $\gamma_V = \mathbb{E}V^3$, exact conditional expansion gives

$$K_1(X) - K_1(V) = \mathbb{E}\left[W^2 \left\{ \varepsilon(w_4 - 3 - \frac{3}{2}\rho w_3) - \frac{3}{2}\gamma_V w_3 \right\}\right]. \quad (116)$$

Apply (91) to $j = 2, 3, 4$, with $h = 1$ or $h = \varepsilon$, and use $\rho\gamma - \rho_V\gamma_V = \rho(\gamma - \gamma_V) + \gamma_V(\rho - \rho_V)$.

The positive coefficient in (116) is $F(x, a; t) = x^2 + 2ax + 3a^2 - 3 - \frac{3}{2}t(x + 2a)$ with $t = \rho + \gamma_V$; the negative coefficient is $-F$ with $t = \rho - \gamma_V$. Their ranges are bounded on complete rectangles of actual conditional means and support magnitudes. The positive-support magnitude is at most R . The same formula permits separate actual bounds on the two signed support magnitudes. On the late domain $p < 1/2$, bounds for γ_V are

$$\sqrt{(1 - \Theta)^3(1 - 2u)^2/[u(1 - u)]} \leq \gamma_V \leq (1 - 2l)/\sqrt{l(1 - l)}.$$

The positive mean lies in $[\sqrt{(1 - \Theta)(1 - u)/u}, \sqrt{(1 - l)/l}]$, and the negative magnitude mean lies in $[\sqrt{(1 - \Theta)l/(1 - l)}, \sqrt{u/(1 - u)}]$. A valid common lower bound for the actual absolute third moment is

$$r_* := \max\{1, r_- - (3r_+/2 - 3a_l)_+ \Theta\} \leq \rho, \quad r_- \leq \rho_B(u), \quad r_+ \geq \rho_B(l), \quad a_l \leq \min(A, B). \quad (117)$$

Indeed conditional cubic expansion gives $\rho \geq (1 - \theta)^{3/2}\rho_B(p) + 3a_l\theta$. Convexity gives $(1 - \theta)^{3/2} \geq 1 - 3\theta/2$, and ρ_B decreases on $(0, 1/2)$, so $\rho \geq r_- - (3r_+/2 - 3a_l)_+ \Theta$; also $\rho \geq 1$. The stronger bound $\rho \geq \rho_B(u)$ is recovered if $3a_l \geq 3r_+/2$. Its reserve condition is sufficient, not necessary. Use the same r_* in the paired Fourier rectangle, signed Taylor coefficients,

phase curvature and exported common-law quadratics. Recomputing a larger lower endpoint in one of those locations would invalidate their association.

Since $x + 2a > 0$, F decreases with t . For fixed t its Hessian is $(\frac{2}{2} \frac{2}{6}) > 0$. Its rectangle maximum is at a corner. Its minimum is among the four corners, the minimizer on each edge, and the stationary point $(x, a) = (3t/8, 3t/8)$ if that lies in the rectangle. On fixed- x edges use $a = t/2 - x/3$, clipped to the edge; on fixed- a edges use $x = 3t/4 - a$, also clipped. This is a complete finite extremum list. Multiplication by $0 \leq \theta \leq \Theta$ uses $\Theta \min(0, L)$ and $\Theta \max(0, U)$ for the lower and upper coefficient bounds. It intersects this new interval with the retained absolute-error interval; neither need be discarded.

A.2.4.9 Generic event and complete central rectangle

Lemma A.15 (Generic small-defect event). *Assume $|X| \leq R \leq 6$ and $\rho \leq 37/20$. Let $0 \leq d < 2$ and $D \leq d$. Define*

$$\begin{aligned} r_* &= \min\{37/20, (3+d)/(\sigma-1)\}, & W &= \sqrt{37d/10}, \\ c &= \sqrt{\sigma+1}, & e &= \sqrt{\sigma-1}, & C_0 &= c+e, \\ h_* &= \frac{C_0}{6} \sqrt{12+W^2} + \frac{W}{3C_0}, & L_* &= \max\left\{h_*, \frac{(2c-e)h_*+3W}{C_0}\right\}. \end{aligned}$$

Then $\rho \leq r_*$, $A+B \leq h_*$ and $|2B-A| \leq L_*$. If $\alpha = L_*/6 \leq 3$, then

$$|J_1| \leq \alpha(\rho+1) + (3-\alpha)d/2, \quad \mu_4 \leq h_*\rho + Rd/2, \quad \eta_5 \leq h_*^2\rho + R^2d/2.$$

For independent copies, the actual event $E = \{XX' < 0\}$ obeys

$$\mathbb{P}(E) \geq a := 1/(2r_*^2), \quad |X - X'| \geq d_0 := \sqrt{2-d} \text{ on } E, \quad \mathbb{E}[(X - X')^2 | E] \leq 2/a.$$

Proof. The inequality $J \leq \rho + 1$ gives $D \geq (\sigma-1)\rho - 3$. Equation (16) gives $w^2 \leq 2\rho D \leq 37d/10$ for $w = cA - eB$. Since $ce = 3$,

$$A+B = \frac{C_0}{6} \sqrt{w^2 + 12AB} - \frac{w}{3C_0}, \quad 2B-A = \frac{(2c-e)(A+B) - 3w}{C_0}.$$

Use $AB \leq 1$, $|w| \leq W$; also $2B-A \geq -(A+B)$. Now $J_{1,c} = J_c(2B-A)/6$, $|J_{1,o}| \leq 3J_o$ and $0 \leq J_o \leq T \leq d/2$ imply $|J_1| \leq \alpha J + (3-\alpha)J_o$.

For either inner endpoint b_* and $b_* \leq x \leq R$,

$$x^4 - h_*x^3 \leq Rx(x^2 - b_*^2), \quad x^5 - h_*^2x^3 \leq R^2x(x^2 - b_*^2).$$

Indeed $h_* \geq b_*$, and after factoring $x - b_*$ the remaining comparisons follow from $x \leq R$. Integrate and use $T \leq d/2$. Finally $m^2 \leq pq$ and $m \geq 1/(2\rho)$ give $2pq \geq a$; the conditional squared distance is bounded by its unconditional expectation 2, divided by the event probability. $\square$

The central part of $\rho J_1 - J_2$ is exactly $mG(A, B; \rho)$, where $G = \rho P_1 - P_2$. Let a proved probability cell give $A \in [A_l, A_u]$, $B \in [B_l, B_u]$, $m \in [m_l, m_u]$. Here

$$A_u = \sqrt{u/(1-u)}, \quad B_u = \sqrt{(1-l)/l}, \quad m_l = \sqrt{(1-\Theta)s_0}, \quad m_u = \sqrt{\max_{p \in [l, u]} pq}.$$

A.2.4.10 The closed-box Bernstein identity.

The Bernstein identity in Section 7 applies after an affine change from each closed coordinate interval to $[0, 1]$. Its tensor-product form includes every face of the resulting closed box. We use that identity with coordinate degrees four below.

Lemma A.16 (Complete central rectangle). *Write $G(A_l + (A_u - A_l)x, B_l + (B_u - B_l)y; \rho) = \sum t_{kh} x^k y^h$. Define the 25 rational coefficients*

$$b_{ij}(\rho) = \sum_{\substack{k \leq i \\ h \leq j}} t_{kh} \frac{\binom{i}{k}}{\binom{4}{k}} \frac{\binom{j}{h}}{\binom{4}{h}}, \quad 0 \leq i, j \leq 4,$$

and

$$\mathcal{G}(\rho) = \max_{i,j} \{m_l b_{ij}(\rho), m_u b_{ij}(\rho)\}.$$

Then $\rho J_{1,c} - J_{2,c} \leq \mathcal{G}(\rho)$. Moreover $\mathcal{G}$ is convex in ρ .

Proof. Apply the Bernstein identity A.2.4.10 separately to the two coordinates after the affine substitution. The Bernstein basis is nonnegative and sums to one on the closed square, so G is a convex combination of its 25 coefficients. For each coefficient retain both m endpoints, or equivalently use m_u when it is nonnegative and m_l otherwise. This gives a uniform bound on the entire three-dimensional rectangle, including boundaries. Each endpoint-multiplied coefficient is affine in ρ ; their maximum is convex. No interior stationary point is omitted. $\square$

A.2.4.11 Refining a rectangular parameter enclosure

The raw remainder in (29) is

$$\mathcal{R} = C\{3\beta J/2 + 3\rho J_1 - 3J_2 - K_q\} + H(CK_1 + \varphi U) - H^2(C\rho + \gamma\varphi/2) + E_n, \quad (118)$$

where $U = J_1 + (\mu_4 - 3)/6$, $\varphi = \phi(H/\sqrt{n})$. For any one coherent low-frequency triple and complete high-frequency value K_H at floor N ,

$$|E_n| \leq K_F + s_F |H|, \quad K_F = c_\pi(L_0 + K_H + L_2/\sqrt{N}), \quad s_F = c_\pi L_H, \quad c_\pi \geq 2/\pi. \quad (119)$$

Both coefficient choices, 16/25 before the signed phase and 212/333 afterward, are valid since $\pi > 333/106$.

Lemma A.17 (Continuous cells and feedback). *Suppose bounds for one actual cell give*

$$r_0 \leq \rho \leq r_1, \quad 0 < c_0 \leq C \leq c_1, \quad |\beta| \leq b, \quad |\gamma| \leq g, \quad \mu_4 \leq M, \quad D \leq d,$$

$K_1 \in [k_l, k_u]$, $U \in [u_l, u_u]$, and $0 \leq \varphi_- \leq \varphi \leq 2/5$. Freeze the shared outer charge d_o , lower fifth moment E_l , upper $s_u = \max_{p \in [l, u]} p(1-p)$, and central envelope $\mathcal{G}$ from the preceding bounds. Define

$$\begin{aligned} A(r) &= \frac{3}{2}b(r + 2\sqrt{s_u}) + 3\mathcal{G}(r) + \frac{3}{8} \max\{r^2/(\sigma - 1), (R - r)^2/(\sigma + 1)\}d_o \\ &\quad + (r/2 - 3/8)M - r/2 + 1/8 + \frac{3}{2}bg - \frac{1}{4} \max\{0, E_l - 3M/2 + 1/2\}, \\ B(c) &= \max_{k \in \{k_l, k_u\}, u \in \{u_l, u_u\}, \psi \in \{\varphi_-, 2/5\}} |ck + \psi u|, \\ Q(c, r) &= cr + \frac{1}{2}\varphi_-(3.16r - 3 - d) - \frac{1}{2}(2/5 - \varphi_-)(d - 0.16)_+. \end{aligned}$$

If all four corner values of Q are positive, then, for every real H ,

$$\begin{aligned} \mathcal{R} &\leq \max_{(c,r) \in \{c_0, c_1\} \times \{r_0, r_1\}} \{K_F + cA(r) + (B(c) + s_F)|H| - Q(c, r)H^2\} \\ &\leq \max_{c, r \text{ corners}} \{K_F + cA(r) + (B(c) + s_F)^2/(4Q(c, r))\}. \end{aligned} \quad (120)$$

If T is a strict upper bound for the final maximum in (120), the weight-removal step in Lemma 5.7 gives

$$D < \frac{15000T}{997N}, \quad \varphi > \frac{997}{2500} \left(1 - \frac{5T}{2N}\right), \quad C < C_E + V/N \quad \text{if also } \mathcal{R}/\rho < V. \quad (121)$$

Only positive density floors are used. These statements remain valid at $H = 0$, on closed cell edges, and for zero conditional variance.

Proof. The constant coefficient follows from the shared outer budget, Lemma A.16, and (112), using

$$J \leq \rho + 2ABm \leq \rho + 2(1 - \theta)^{3/2} \sqrt{pq} \leq \rho + 2\sqrt{s_u}.$$

This follows from $A \leq \bar{a}$, $B \leq \bar{b}$, $m^2 = (1 - \theta)pq$; it is used only in the charge $3|\beta|J/2$. Since $\gamma \geq 3.16\rho - 3 - d$, its negative part is at most $(d - 0.16)_+$. Using the upper density only for that negative part gives $C\rho + \gamma\varphi/2 \geq Q$. Absolute value is separately convex in each coordinate of $ck + \psi u$, proving the formula for B . For fixed r and $|H|$, the resulting upper bound is convex in c . Take those endpoints *before* completing the square. At either endpoint $c > 0$, it is convex in r , since $\mathcal{G}$ and the maximum of the two outer quadratics are convex, every other term of A is affine, and Q is affine in r . This proves the first line of (120); $fh - qh^2 \leq f^2/(4q)$ proves the second. Bilinearity of Q proves its positivity throughout the rectangle from its corners. $\square$

A.2.4.12 Signed phase curvature in the resource schedule

The auxiliary Fourier orientation does not change the signed third moment in the actual transfer identity. The paired phase estimate must therefore retain its sign. On a fixed resource cell put

$$L = \frac{997}{2500} \left(1 - \frac{5r}{2N}\right), \quad g_0(\rho) = 3.16\rho - 3 - d, \quad U = 0.4,$$

where r, d are the preceding raw and defect upper bounds, and the bounds must satisfy $0 \leq L \leq U$. The actual density lies in $[L, U]$ and the actual signed moment satisfies $\gamma \geq g_0(\rho)$. Consequently

$$C\rho + \frac{1}{2}\gamma\phi \geq q_s(C, \rho) := C\rho + \frac{1}{2} \min\{Lg_0(\rho), Ug_0(\rho)\}. \quad (122)$$

This follows by first multiplying the lower bound on γ by the nonnegative actual density, and then minimizing over its two endpoints. The simpler expression $C\rho + Lg_0(\rho)/2$ is licensed if $g_0 \geq 0$, or if nonnegativity of the actual γ has been separately proved. It is not licensed merely by reflecting the auxiliary Fourier integral.

The corrected price preserves the four-corner reduction. For fixed C , q_s is concave in ρ , being affine plus the minimum of two affine functions. For fixed ρ it is affine in C . In the phase envelope $A + f|H| - q_s H^2$, the remaining terms are separately convex in C, ρ , by the moment and coefficient bounds already proved. Thus the entire envelope is separately convex, and two successive one-dimensional endpoint bounds place its maximum at the four corners. The same argument applies after adding the affine conserved-resource curvature e_ρ and subtracting a barrier price $K\rho$. If the four corrected divisors are positive, concavity makes them positive throughout the rectangle, and square completion gives the valid raw and resource bounds

$$\max_{C, \rho \text{ corners}} \left(A + \frac{f^2}{4q_s} \right), \quad \max_{C, \rho \text{ corners}} \left(A + \frac{f^2}{4(q_s + e_\rho)} - K\rho \right).$$

Any threshold-root export likewise uses the corrected corner curvature.

When the actual signed lower bound (99) is nonnegative, the simpler nonnegative-skewness price is licensed. Otherwise retain (122). Auxiliary reflection alone never supplies this license. Every initial or left finite pass must retain the sign-aware price unless its certificate supplies a separate license for a simpler specialization.

There is also a convenient cell-constant correction. Since $\rho \geq 1$, $(-g_0(\rho))_+ \leq (d - 0.16)_+$, so the weaker price

$$C\rho + \frac{1}{2}Lg_0(\rho) - \frac{1}{2}(U - L)(d - 0.16)_+$$

is valid. This weaker cell-constant form and the exact sign-aware form are both permitted. Raw remainders and barrier-subtracted resource budgets remain distinct. A threshold-root export uses the same corrected curvature as the coefficient envelope from which it is derived.

A.2.4.13 The unsigned first-stage cover is the same enclosure

The first finite barrier uses a cover of ρ alone before introducing the signed cover. Here are its exact coefficient choices, so that the first barrier uses the same analytic enclosure. Fix a closed cell $\rho \in [l, u] \subset [1, 1.85]$, actual radius $1 \leq R \leq 6$, and write $k_j(R)$ for the radial ratios of Lemma A.13. Set

$$\begin{aligned} M &= \min\{11.1, Ru, 1 + k_4(R)(u - 1)\}, & g &= \min\{1.85, u, \sqrt{M - 1}\}, \\ E_5 &= \min\{RM, 1 + k_5(R)(u - 1)\}, & b &= \sqrt{[u\sqrt{u^2 + 8} - u^2 - 2]/2}. \end{aligned}$$

The last expression solves $\rho \geq (1 + \beta^2)/\sqrt{1 - \beta^2}$, proved by $\rho m \geq a_+^2 + a_-^2$ and $m^2 \leq a_+ a_-$. If a previous pass licenses $D \leq d < 2$, the generic-event lemma also permits the ceilings $h_*u + Rd/2$ and $h_*^2u + R^2d/2$ for M, E_5 ; all dependent quantities are recomputed after taking these minima. Use the triangle low-frequency triple and the minimum of complete permitted high-frequency values, with the same moment cap. Put

$$K_F = \frac{16}{25}(L_0 + K_H + L_2/\sqrt{N}), \quad s_F = \frac{16}{25}L_H.$$

The radial identity bounds $|K_1|$ by

$$k_{\text{rad}} = 3ub + \frac{R(R + 2 - 3l/2)}{R + 1/2}(u - 1),$$

provided the displayed radial coefficient is at least $31/40$. Indeed its positive maximum is at R and its negative part is at most $31/40$ on $[0, R]$ for $1 \leq \rho \leq 1.85$; convexity of $x(x + 2 - 3\rho/2)/(x + 1/2)$ proves this directly. Two additional valid bounds are

$$k_{\text{old}} = 6u - 3u^2/2 + 3b, \quad k_R = Ru/2 + \max_{t=l,u} |R/2 - 3t/2|g + 3b.$$

For k_R , bound the positive and negative fourth moments separately by $R(\rho \pm \gamma)/2$. Before defect feedback use $k = \min(k_{\text{rad}}, k_{\text{old}}, k_R)$. After feedback one may replace k_{old} by $\max(k_+, k_-)$, where

$$\begin{aligned} g_l &= 3.16l - 3 - d, & g_u &= 3.16u - 3 - d, \\ k_+ &= M + 3b - \frac{3}{2} \min_{r=l,u; v=g_l, g_u} rv, \\ k_- &= 3u + 3b + \max_{r=l,u; v=g_l, g_u} (3r/2 - 3)v. \end{aligned}$$

These substitutions bound the lower affine graph for the actual γ , not an independently chosen skewness.

Put $\Delta = 2u$ initially and $\Delta = (1 - 3.16)l + 3 + d$ after feedback. The constant coefficient of the raw identity is bounded by

$$A_* = \min\left\{\frac{3}{2}b(2u+1) + \frac{3}{2}\Gamma(R, l)\Delta + \frac{3}{8}u^2(u+1) + \frac{8}{27}u^3 - l/2, \frac{3}{2}b(2u+1) + 2uM - l/2\right\}.$$

This is the coarse gap bound above together with the two bounds for $-K_q$. The density coefficient obeys

$$v_H = \min\{0.4j + 0.4\max(2, M-3)/6, 0.4\max(1/2, 2M/3 - 1/2)\},$$

where $j = 3(u+1)$ initially; the generic-gap choice is $j = \alpha(u+1) + (3-\alpha)d/2$, with its checked $\alpha < 3$. For a score endpoint enclosure $C \in [c_-, c_+]$, the resulting raw bound is

$$K_F + c_+A_* + \frac{(c_+k + v_H + s_F)^2}{4q_*}, \quad q_* > 0, \quad k \geq 0, \quad A_* > 0, \quad (123)$$

where initially $q_* = c_-l - g/5$. After feedback use the larger of that quantity and

$$c_-l + \varphi_-(3.16l - 3 - d)/2 - (2/5 - \varphi_-)(d - 0.16)_+/2.$$

The score endpoints are the lower score floor and the upper allowed score cap; if the latter depends on ρ , use its values at u, l for c_-, c_+ , respectively. Convexity in the actual score precedes the completion of the square, and these outward endpoint intervals cover it. For the phase resource add (32) before maximizing. A barrier $e \geq K/n$ subtracts Kl from the nonnegative resource. The first raw target must pass the license of Lemma 5.7 first.

A.2.4.14 Optional signed improvement inside an unsigned cell.

If $g_0 = \max(-g, \sigma_l l - 3 - d) > g$, exclude the cell. When $g_0 \geq 0$, the sign-moment constraint forces $(1-\beta)^3 \leq (1+\beta)(u-g_0)^2$. This inequality is monotone on $(-1, 1)$, so exact bisection yields a lower $b_0 \leq \beta$; if it fails even at b , the cell is empty. For $x \in [g_0, g]$, set

$$\begin{aligned} f_-(x) &= (l+x)^2/[2(1+b)] - R(u-x)/2 - 3b - 3ux/2, \\ f_+(x) &= R(u+x)/2 - (l-x)_+^2/[2(1-b_0)] - 3b_0 - 3lx/2. \end{aligned}$$

The signwise fourth-moment Cauchy bounds prove $\min f_- \leq K_1 \leq \max f_+$. The first minimum is at its clipped vertex $3u(1+b)/2 - l - R(1+b)/2$. The second maximum is among its endpoints, the clipped kink l , and the clipped vertex $l + (R/2 - 3l/2)(1-b_0)$. Intersect with $[-k, k]$ and replace the linear term in (123) by the smaller full rectangle bound

$$\max_{c=c_-, c_+; k=k_l, k_u; \psi=\varphi_-, 0.4; v=-1/2, 2M/3-1/2} |ck + \psi v| + s_F.$$

All sixteen corners are retained. If $b_0 > 0$, the opposite-sign event also has probability at least $(1+b_0^2)^2/(2u^2)$, from $m \geq (1+\beta^2)/(2\rho)$ and $m^2 \leq pq$. It may improve the complete high-frequency value. These are intersections with necessary conditions; the cover still examines every ρ -cell.

A.2.4.15 Moving from a small defect to probability coordinates.

For $0 < d \leq 0.7$, start with

$$A_l = \frac{1.47\sqrt{2-d} - \sqrt{3.7d}}{3.52}, \quad B_l = \frac{2.04\sqrt{2-d} - \sqrt{3.7d}}{3.52}.$$

They follow from $A + B \geq \sqrt{2-d}$ and $|\sqrt{\sigma+1}A - \sqrt{\sigma-1}B| \leq \sqrt{3.7d}$; their positivity is checked, rather than assumed at every d . The pointwise inequality $x(x+b) \geq (29/5)b(x-b)$ follows because the difference is $(x-12b/5)^2 + b^2/25 > 0$. For $x > b > 0$, applying it to the two residual third moments, gives $D \geq (29/5) \min((\sigma+1)A, (\sigma-1)B)\theta$. Start $\Theta = 1$ and perform the declared finite number of updates:

$$\begin{aligned}\Theta &\leftarrow \min\{\Theta, d/[(29/5) \min((104/25)A_l, (54/25)B_l)]\}, \\ \epsilon &= \sqrt{d/[(616/25)(1-\Theta)^2]}, \quad [l, u] = [(4-\sigma_u)/2 - \epsilon, (4-\sigma_l)/2 + \epsilon], \\ A_l &\leftarrow \max\{A_l, \sqrt{(1-\Theta)l/(1-l)} - d/[2(1-u)(179/25)A_l^2]\}, \\ B_l &\leftarrow \max\{B_l, \sqrt{(1-\Theta)(1-u)/u} - d/[2l(129/25)B_l^2]\}.\end{aligned}$$

The last two bounds use $D \geq 2(\sigma+2)B^2M_+ + 2(\sigma+4)A^2M_-$ and $\Delta_A = M_-/q$, $\Delta_B = M_+/p$. Every division, interval endpoint and variance domain is gated. Subdivide this entire probability interval, then use the common cell update and coefficient atlas. Thus the unsigned first barrier, the signed initialization and the final probability refinements are three data choices for the same finite-enclosure theorem.

Lemma A.18 (Asymmetric moment and probability restrictions). *For a sign subprobability measure supported on $[0, R]$, denote its moments of orders 0, 1, 2, 3, 4, 5 by P, m, a, b, c, e . Then*

$$\begin{aligned}(Rm - a)^2 &\leq (RP - m)(Ra - b), \\ (Ra - b)^2 &\leq (Rm - a)(Rb - c), \\ (Rb - c)^2 &\leq (Ra - b)(Rc - e).\end{aligned}\tag{124}$$

Consequently, whenever the respective denominators are positive,

$$P \geq \frac{m}{R} + \frac{(Rm - a)^2}{R(Ra - b)}, \quad c \leq Rb - \frac{(Ra - b)^2}{Rm - a}, \quad e \leq Rc - \frac{(Rb - c)^2}{Ra - b}.\tag{125}$$

These apply separately with the actual radii R_- and R_+ . Furthermore, on a signed cell with $m \geq m_-$ and $\beta \in [\beta_-, \beta_+]$,

$$\frac{2m_-^2}{1 + \beta_+} \leq p \leq 1 - \frac{2m_-^2}{1 - \beta_-}.\tag{126}$$

Proof. Apply Cauchy–Schwarz to the pairs $(1, x)$, $(x^{1/2}, x^{3/2})$ and (x, x^2) under the nonnegative measure $(R - x)\mu(dx)$; this gives the three moment log-convexity inequalities in (124). Rearrangement gives (125). Zero denominators are handled in (124) before any division: a positive numerator is incompatible, while omitting a zero-over-zero nonnegative correction is a valid weaker bound. The two sign masses sum to at most one, and to one after zero is excluded from support. Their necessary probability inequalities are therefore valid in both cases. Finally $m^2 \leq pEX_+^2 = p(1 + \beta)/2$ and $m^2 \leq (1 - p)(1 - \beta)/2$ prove (126). $\square$

All occurrences of the signed third moment remain signed. In particular, with a prior raw bound r and defect bound d_D , set $L_\phi = 0.3988(1 - 5r/(2N))$ and $g_0(\rho) = 3.16\rho - 3 - d_D$. The safe lower phase price is

$$C\rho + \frac{1}{2} \min\{L_\phi g_0(\rho), 0.4g_0(\rho)\}.\tag{127}$$

This is precisely (122): a negative lower bound for γ must be multiplied by the upper density. Reflection used inside an absolute Fourier bound does not license changing the sign of the actual contact law.

Lemma A.19 (Positive endpoint floors from the defect). *If $D \leq d_D < \sqrt{10} - 1$, the central gap $(-A, B)$ is genuine and*

$$A \geq \frac{\sqrt{10} - 1 - d_D}{6(1-p)}, \quad B \geq \frac{\sqrt{10} - 1 - d_D}{6p}.$$

On a closed panel $p \in [l, u] \subset (0, 1)$, replace the numerators by $\sigma_- - 1 - d_D$ with $\sigma_-^2 < 10$, and the denominators by $6(1-l)$ and $6u$, respectively, whenever the numerator is positive.

Proof. The gap identity and its signwise Tonelli estimate in (16) give

$$J \leq m(A+B)^2 + T_+ + T_- = \rho + 2mAB, \quad D \geq (\sqrt{10} - 1)\rho - 6mAB.$$

If zero were in the support, the same proof gives $D \geq (\sqrt{10} - 1)\rho$, a contradiction. Thus the displayed gap exists. Since $\rho \geq 1$, $m^2 \leq p(1-p)$, $A \leq m/(1-p)$ and $B \leq m/p$, we have $mAB \leq (1-p)A$ and $mAB \leq pB$. Substitution proves both floors. Their panel versions weaken positive denominators in the safe direction and apply on both probability halves. They can be intersected with all prior endpoint floors. $\square$

A.2.4.16 Three consequences of a signed defect bound

Once the gap is acquired, use the actual conditional representation

$$X = \ell(B_0 + U - p), \quad \mathbb{P}(B_0 = 1) = p, \quad \mathbb{E}(U \mid B_0) = 0, \quad \ell^2 = \frac{1 - \theta}{p(1-p)}. \quad (128)$$

Write $q = 1 - p$, $s_j = \mathbb{E}(U^2 \mid B_0 = j)$, $\lambda = n(qs_0 + ps_1)$, $\theta_- = q\ell^2 s_0$, $\theta_+ = p\ell^2 s_1$, and $\theta = \theta_- + \theta_+$. The actual endpoints satisfy

$$U_0 \in [-A_0, a_0], \quad U_1 \in [-a_1, A_1], \quad d = a_0 + a_1, \quad A = \ell(p - a_0), \quad B = \ell(q - a_1).$$

Thus $a_0 < p$ and $a_1 < q$; neither inward endpoint is discarded. Here D is the moment defect and d is the inward displacement.

Let $A_*, B_* > 0$ be acquired lower gap bounds, $S = A_* + B_*$, $k_+ = 3(\sqrt{10} + 1)B_*$ and $k_- = 3(\sqrt{10} + 3)A_*$. The separate-charge identity (17) gives

$$D \geq b + 6(1 - \theta)Sd + k_+\theta_+ + k_-\theta_-. \quad (129)$$

For example, a panelwise lower binary charge is

$$b = \frac{(6 + 2\sqrt{10})(1 - \Theta)^{3/2}}{\sqrt{v_{\max}}} \text{dist} \left(\frac{4 - \sqrt{10}}{2}, [l, u] \right)^2,$$

rounded downward. To check the change of coordinates in (129), the two conditional-mean gaps in (17) sum to ℓd and $m\ell = 1 - \theta$; its additional product term is nonnegative.

Lemma A.20 (Affine, moment, and variance consequences). *Suppose $p \in [l, u] \subset (0, 1)$, $\theta \leq \Theta < 1$ and $D \leq D_*$. Put*

$$v_{\min} = \min\{l(1-l), u(1-u)\}, \quad v_{\max} = \max_{x \in [l, u]} x(1-x), \quad T = D_* - b.$$

The maximum includes 1/4 if the panel crosses 1/2. Then the following are consequences for the same law.

(i) With $k_* \leq \min(k_+, k_-)$ positive,

$$D \geq b + gd + h\lambda, \quad g = 6(1 - \Theta)S, \quad h = \frac{k_*(1 - \Theta)}{nv_{\max}}. \quad (130)$$

If $T < 0$ the panel is empty; if $T = 0$ then $d = \lambda = 0$. For $T > 0$ one has the simultaneous inequality

$$\frac{d}{d_0} + \frac{\lambda}{\Lambda} \leq 1, \quad d_0 = T/g, \quad \Lambda = T/h. \quad (131)$$

Reducing the three lower coefficients in (130), or enlarging either intercept in (131), is valid. The affine coefficients cannot be inferred from intercept bounds alone.

(ii) If $A_0, A_1 > 0$, the stronger signed prices are

$$\left(k_+ + \frac{6Sq}{A_1}\right)\theta_+ + \left(k_- + \frac{6Sp}{A_0}\right)\theta_- \leq T. \quad (132)$$

For $j = 3, 4, 5$, let $b_j(p) = (pq^j + qp^j)/(pq)^{j/2} \leq B_j$ on the panel, and choose upper Taylor coefficients

$$L_+ = L_j(R_+, \sqrt{(1-l)/l}), \quad L_- = L_j(R_-, \sqrt{u/(1-u)}),$$

where

$$L_3(R, x) = R + 2x, \quad L_4(R, x) = R^2 + 2Rx + 3x^2, \quad L_5(R, x) = R^3 + 2R^2x + 3Rx^2 + 4x^3.$$

Set $a_j = [1 - (1 - \Theta)^{j/2}]/\Theta$ if $\Theta > 0$, and $a_j = j/2$ otherwise. If $\hat{k}_{\pm} > 0$ are lower bounds for the two prices in (132), then

$$\mathbb{E}|X|^j \leq B_j + T \max \left\{ 0, \frac{L_+ - a_j B_j}{\hat{k}_+}, \frac{L_- - a_j B_j}{\hat{k}_-} \right\}. \quad (133)$$

(iii) Actual signed support bounds $[-R_-, R_+]$ give

$$\ell \geq \ell_0 := \sqrt{(1 - \Theta)/v_{\max}}, \quad A_0 \leq R_-/\ell_0 - l, \quad A_1 \leq R_+/\ell_0 - (1 - u). \quad (134)$$

If both outward caps are at most $A^* > 0$ and (131) has positive intercepts, then

$$\lambda \leq L_* := \left\{ \frac{1}{\Lambda} + \frac{1}{nA^* \max(u, 1-l)d_0} \right\}^{-1}, \quad \theta \leq \frac{L_*}{nv_{\min} + L_*}. \quad (135)$$

In addition,

$$\sqrt{v_{\min}(1 - \Theta)} \leq m \leq \sqrt{v_{\max}}. \quad (136)$$

Proof. For (i), use $\theta = \lambda(1 - \Theta)/(npq)$ in (129) to obtain the displayed affine coefficients. Apply Theorem 5.4 with its index floor chosen as the actual n , and with $d_D = D_*$, $v_* = v_{\max}$, $S_* = S$ and $K = k_*$. The input $nD \leq nD_*$ follows from $D \leq D_*$; this proves the stated joint inequality and both zero cases. For (ii), take expectations of $(U_0 + A_0)(a_0 - U_0) \geq 0$ and $(U_1 + a_1)(A_1 - U_1) \geq 0$. Conditional centering gives

$$s_0 \leq A_0 a_0, \quad s_1 \leq A_1 a_1, \quad (1 - \theta)s_0 = p\theta_-, \quad (1 - \theta)s_1 = q\theta_+.$$

Substituting $d \geq s_0/A_0 + s_1/A_1$ in (129) proves (132). This spends its displacement term for the moment estimate; that term must not be added again to (132). The original inequality (129) remains separately true.

The positive-measure Taylor identity (91) gives

$$\mathbb{E}|X|^j \leq (1 - \theta)^{j/2} b_j(p) + L_+ \theta_+ + L_- \theta_-.$$

First replace $b_j(p)$ by B_j in its nonnegative coefficient. Then convexity gives $(1 - \theta)^{j/2} \leq 1 - a_j \theta$ on $[0, \Theta]$. Maximize the remaining linear functional over the nonnegative signed variances subject to (132), using (92). This proves (133), even if one Taylor cost after subtraction is negative. A zero outward amplitude makes that centered conditional class deterministic and is treated directly before division.

For (iii), subtract the conditional centers $-p\ell, q\ell$ from actual support to obtain (134). The variance bounds just proved imply

$$\lambda \leq nA^*(qa_0 + pa_1) \leq nA^* \max(u, 1 - l)d.$$

Insert this lower bound for d into (131); then use $\theta = \lambda/(npq + \lambda)$, increasing in λ and decreasing in pq , to get (135). Finally conditional variance gives $m^2 = pq(1 - \theta)$ and hence (136). These proofs also cover $p \geq 1/2$ and panels crossing $1/2$; no replacement of $v_{\min}$ by $l(1 - l)$ is made there. $\square$

For the upper Bernoulli moments in (ii), writing $v = pq$ gives $B_3(v) = (1 - 2v)/\sqrt{v}$, $B_4(v) = 1/v - 3$, and $B_5(v) = (1 - 4v + 2v^2)/v^{3/2}$. Each decreases on $(0, 1/4]$, so its value at $v_{\min}$ is a valid bound on either probability half. Lower prices in (132) use the lower numerators $q \geq 1 - u$, $p \geq l$ and the upper amplitudes in (134). When a full two-sided noise amplitude A^* is exported, the inward bound $d_0 \leq A^*$ is also checked. Previously acquired fourth and fifth moment bounds may be retained by intersection, including $\mathbb{E}|X|^5 \leq \max(R_-, R_+) \mathbb{E}X^4$. The updated variance cap in (135) can be rounded upward and intersected with its previous value; it does not invalidate any earlier inequality.

A.2.4.17 Keeping the score and phase coherent

The raw envelope in (98) and its phase improvement (32) give

$$\kappa nD \leq A(C, \rho) + F_{\text{sgn } H}(C, \rho)|H| - Q(C, \rho)H^2 - n(C - C_E)\rho, \quad \kappa = 997/15000. \quad (137)$$

The curvature Q already includes the extra phase price

$$\frac{1}{2} \left(1 - \frac{5r}{2N} \right) \left(0.4095\rho - \frac{d_D}{15} \right).$$

It is not added a second time. The signed price (127) is retained before any square completion.

A positive barrier uses the charge $K\rho$. The selected zero-barrier probability refinements also retain the actual score and the fixed rational upper bound $C_E < 0.40974$:

$$n(C - C_E)\rho \geq \chi(C)\rho, \quad \chi(C) = n(C - 0.40974)_+. \quad (138)$$

Split the score interval at 0.40974 when that point is inside. The rectangle proof above is separately convex in the actual C and ρ . On either closed score piece, $\chi(C)\rho$ is affine in either variable when the other is fixed; its subtraction preserves that convexity. Hence both moment endpoints at each score endpoint give a valid coherent corner envelope. The selected score endpoints 0.4095, 0.40974, c_1 give six corners. Keeping only the two extreme score endpoints would miss the kink and is not this argument. Raw remainders and the first unweighted passes continue to use their uncharged envelope. Actual-score charging here is a refinement of the probability kernel and its exported phase quadratics.

Lemma A.21 (The signed phase interface). *Suppose the acquired charged corner bounds are $q_i^\pm(h) = A_i + F_i^\pm h - Q_i h^2$, and $0 \leq h \leq H_\pm$. Then*

$$D \leq \frac{15000}{997n} \max_i q_i^{\text{sgn } H}(|H|). \quad (139)$$

A fixed target envelope is valid if, for each source corner and each sign, one target corner dominates it on the entire consumed interval $[0, H_\pm]$. For a difference $a + bh + ch^2$, it is enough to check its values at the two endpoints and also $a - b^2/(4c)$ when $c > 0$ and $0 < -b/(2c) < H_\pm$.

Proof. Equation (139) is (137) after corner domination and division by the positive κn . The minimum of a quadratic on a compact interval occurs at an endpoint or an interior stationary point that is a minimum. The stated tests therefore dominate every source corner pointwise; taking maxima preserves the inequality. Negative linear coefficients keep their signs, and the normalization $15000/(997n)$ is unchanged. $\square$

In particular, if $D \leq D_*$ and (130) holds, define D_H as the smaller of D_* and the right side of (139). The actual law satisfies

$$gd + h\lambda \leq D_H - b. \quad (140)$$

A negative right side excludes that phase. If $T = D_* - b > 0$, (140) is equivalently $d/d_0 + \lambda/\Lambda \leq (D_H - b)/T$. The entire joint residual changes with phase, not just one intercept. At a count threshold $k + u$, the same actual phase is $H = (k + u - np)/\sqrt{pq} + \lambda/n$. This is the interface to the count proof; it prevents independent maximization of count phase and moment budget.

A.2.4.18 Radius feedback and the final local interface

In the actual coordinates (128), suppose a separate confinement argument has already proved $|U| \leq a$ on a probability panel $p \in [l, u] \subset (0, 1)$. Since $\ell^2 = (1 - \theta)/(pq) \leq 1/(pq)$, it follows that

$$|X| \leq \frac{\max\{u + a, 1 - l + a\}}{\sqrt{\min\{l(1 - l), u(1 - u)\}}}. \quad (141)$$

Indeed $|B_0 + U - p| \leq \max\{p + a, 1 - p + a\}$, and the concavity of $p(1 - p)$ places its minimum on the closed panel at an endpoint. The implication includes zero noise. The new radius is used only after confinement has established its hypothesis, not in the argument that establishes confinement.

For completeness, the signed lower bound (99) requires only $p \in [l, u] \subset (0, 1/2)$, $\theta \leq \Theta < 1$, and the already established radius R . To see the sign directly, write $X = V + W$, $\mathbb{E}[W \mid B_0] = 0$. The bound $X \geq -R$ gives $W^3 \geq (-R - V)W^2$, so

$$\gamma \geq \mathbb{E}V^3 + 2\mathbb{E}(VW^2) - R\theta.$$

The smaller conditional value of V is $-\sqrt{(1 - \theta)p/q} \geq -\sqrt{u/(1 - u)}$. The binary cubic $(1 - \theta)^{3/2}(1 - 2p)/\sqrt{p(1 - p)}$ decreases in both $p < 1/2$ and θ . These observations prove precisely (99); its nonnegativity, whenever needed, is a separate scalar inequality checked on the actual panel.

Finally suppose a phase cell has a coherent upper envelope $\mathcal{B}(H) \leq T_H$ in (33), and a proved moment inequality for the same law is

$$D \geq \alpha d_Y + b\theta + P, \quad \alpha > 0, \quad b \geq 0, \quad P \geq 0.$$

If $n \geq N$, $\ell \geq \ell_- \geq 0$ and $\lambda \geq L_0 \geq 0$, then

$$nd_Y \leq G := \frac{T_H/\kappa - NP - b\ell_-^2 L_0}{\alpha}. \quad (142)$$

This follows from $n\theta = \ell^2\lambda$ and $nP \geq NP$. Thus $G < 0$ excludes the entire cell. If the upper envelope is strict, the exported inequality is strict and $G \leq 0$ excludes the cell. For a concave quadratic $A + Fh - Qh^2$, $Q > 0$, its maximum on a closed phase interval is its value at the clipped vertex $F/(2Q)$. Taking the maximum over coherent corners, followed by the minimum with an independent scalar bound, produces T_H . This operation reuses the single resource (33); it creates no additional allowance for the subsequent count argument.

A.2.4.19 Finite sequences of enclosure refinements

Theorem A.22 (Preservation of simultaneous bounds). *Start with a complete closed probability cover and the actual support and scalar hypotheses (78). Each surviving panel carries simultaneous moment, phase, support and resource inequalities for one hypothetical maximizing law in that panel. Perform any declared finite sequence of the following operations:*

1. *a complete signed initial or probability resource cover, using the enclosure theorem and all its domain gates;*
2. *a signed support-ray exclusion using only previously valid same-law moment bounds, followed by moment estimates on that acquired support;*
3. *the endpoint, moment, variance and phase consequences proved above, or intersection with a previously acquired bound;*
4. *outward rounding; or pooling all active panels by taking maxima of upper bounds and minima of lower bounds, then using their closed interval hull for a new complete initial cover.*

Keep an external set of already excluded panels. The result remains a complete cover of every possible maximizing law, with valid simultaneous bounds on every surviving panel. A final scalar, affine and quadratic interface to a fixed count contract is valid if all the inequalities above are checked in the required directions on the same row.

Proof. Theorem A.14 proves the within-cover induction, including complete incidences, empty cells, positive denominators and square completion. The support-ray implications and Lemma A.21 prove the additional implication steps. No operation changes the marginal law or uses a proposed radius before that radius has been proved. Intersections and outward rounding preserve true inequalities for that law.

If no active panel remains, the band is excluded and no minimum or maximum of an empty family is taken. Otherwise the actual surviving law belongs to at least one active panel. Every pooled lower bound is no larger than that panel's lower bound, and every pooled upper bound is no smaller. This applies separately to the two support radii and all signed moment endpoints. The interval hull enlarges the surviving union, so it is a safe domain for the next implication with these pooled hypotheses. It does not reinstate previously excluded laws: their exclusions remain external. Shared boundary points belong to their adjacent closed panels. Induction over the finite schedule now proves the assertion. A bound on a maximum of coherent phase corners is transferred only by Lemma A.21, and a direct affine resource is transferred by decreasing its three actual lower coefficients. $\square$

For a band $N \leq n \leq L$, a nonnegative conserved budget T gives $D \leq T/(\kappa N)$; if $T < 0$, (137) already contradicts $D \geq 0$ before any replacement of n by N . Weighted resources $nD \leq Nd_D$ are propagated in their weighted form using Theorem 5.4. An unweighted inequality $D \leq d_D$ is never multiplied by N and reported as a bound for nD . Index monotonicity is used only where the retained formula has its explicit gate; support is checked separately at every listed integer. The low-frequency transfer needs $n > 2$ and the high-frequency comparator $n \geq 5$, both satisfied here. There is no residual $n \geq 36$ assumption in the earlier-index moment and Fourier formulas.

A.3 Confinement and conditional-count witnesses

Remark A.23 (Preserved index scope). The supplied confinement calculation below has floor $N = 47$. Its rational normalizers, the $n - 2$ concentration bound and all numerical applications retain that scope. The subsequent local-comparison identities are stated with explicit parameters, but this does not establish their numerical hypotheses at floor 42. Appendix C gives the conditional parameter changes for that floor. The complete selected resource, confinement, normalization and local-cover predicates and their composition for every $n \geq 42$ remain obligations. The former index-46 barrier assembly and the historical tail-completion assertion are not used here.

A.3.1 One confinement lemma with two concentration choices

A proved amplitude is used as a support update, never as a premise of its own proof. The following derivation applies to every acquired row, with the same formulas for the central and left probability ranges.

A.3.1.1 Statement and hypotheses

Let $n \geq N = 47$, $Y = B + U$, $\mathbb{P}(B = 1) = p < 1/2$ and $\mathbb{E}(U \mid B) = 0$. Put $q = 1 - p$, $s = \mathbb{E}U^2$, $\lambda = ns$ and $X = \ell(Y - p)$, where $\mathbb{E}X^2 = 1$. Thus $\ell^2 = (1 - \theta)/(pq)$ and $\theta = \ell^2 s$. The conditional label-zero support is at most $a \geq 0$ and the label-one support is at least $b \leq 1$, where a, b are support points and $b - a = 1 - d$. In particular $0 \leq a \leq d$, $1 - d \leq b \leq 1$. Assume the initial strip $\text{supp } Y \subset (-0.55, 1.55)$. Write $\rho = \mathbb{E}|X|^3$, $\beta = \mathbb{E}[X|X|]$ and $\epsilon = \mathbb{E}|U|$.

Select an acquired closed probability panel $[l, h]$. Its fields give upper bounds $\Theta, d_0, \rho_0, \beta_+, T, \Lambda$, a lower bound β_- , and separate bounds H_+, H_- for the two signs of H . Assume

$$\theta \leq \Theta, \quad d \leq d_0, \quad \rho \leq \rho_0, \quad \beta_- \leq \beta \leq \beta_+, \quad 0 \leq \lambda \leq \Lambda, \quad 0.4 < C \leq C_0 = 0.4253.$$

For the positive sign $0 \leq H \leq H_+$; for the negative sign $-H_- \leq H \leq 0$. Let $x = np + H/\ell$ and $\nu = \mathcal{L}(x - \sum_{i < n} Y_i)$. Assume a contact majorant Q for $g(y) = \nu([y, \infty))$, with equality at each support point of Y , and derivative

$$\sqrt{n}(-Q'(y)) = \ell\phi(H/\sqrt{n}) + \frac{\ell}{n}\{3C\beta + (H\phi(H/\sqrt{n}) + 3C\rho)t - 3Ct|t|\}, \quad t = \ell(y - p). \quad (143)$$

Both actual deleted convolutions satisfy, separately,

$$Q_j(w) \leq \frac{\ell\phi(0)w + 2C_0\rho}{\sqrt{j}}, \quad j = n - 1, n - 2, \quad (144)$$

for every interval of width $w \geq 0$.

Here is the precise common resource assumption. Put $\kappa = 997/15000 < \phi(0)/6$, $J = \int_{[a,b)}(y-a)\nu(dy)$, and note that the central gap contributes $m\ell J = (1-\theta)J$ to L . Thus $L \geq (1-\theta)J$; retain $e \geq 0$ and

$$ne\rho + n\phi(0)D/6 + n^{3/2}L < \mathcal{B}(H), \quad \mathcal{B}(H) = \min\{T, \max_j(A_j + F_{j,\pm}|H| - Q_j H^2)\}. \quad (145)$$

The four quadratics are the table's quadratic rows, using Q_{budget} , not Q_{raw} . The Gaussian phase contribution has already been included in Q_{budget} .

For each panel let E, F_ℓ, m_0 be respectively strict lower, upper, and lower rational bounds for

$$\sqrt{(1-\Theta)/(h(1-h))}, \quad 1/\sqrt{l(1-l)}, \quad \sqrt{(1-\Theta)l(1-l)}.$$

The bounds use the grid 10^{-6} , with an extra unit outward. Define

$$\begin{aligned} c_+ &= 2(5.16)E^2(1-h-d_0)^2, & c_- &= 2(7.16)E^2(l-d_0)^2, \\ b_* &= \min\{3(4.16)E(1-h-d_0), 3(6.16)E(l-d_0)\}, \\ P &= m_0 E^2(3.5 \text{ dist}([l, h], [0.41886, 0.41887]) - 2.041d_0)_+^2. \end{aligned}$$

The simultaneous moment inequality has the following usable form:

$$D \geq c_+ M_+ + c_- M_- + b_* \theta + P, \quad d = (M_+/p + M_-/q)/\ell, \quad \epsilon \leq 2(M_+ + M_-)/\ell, \quad M_\pm \geq 0. \quad (146)$$

To derive it, the endpoint expansion of (16) gives $D \geq m(\sqrt{\sigma+1}A - \sqrt{\sigma-1}B)^2 + 2(\sigma+2)B^2 M_+ + 2(\sigma+4)A^2 M_- + 3(\sigma+1)B\theta_+ + 3(\sigma+3)A\theta_-$. Insert $A \geq E(l-d_0)$, $B \geq E(1-h-d_0)$ and the rational lower coefficients displayed above. For the square, write $A = \ell(p-a)$, $B = \ell(1-p-(1-b))$, with $a+(1-b)=d \leq d_0$. Its unshifted coefficient is $(\sqrt{\sigma+1} + \sqrt{\sigma-1})(p-p_*)$; $\sqrt{\sigma+1} + \sqrt{\sigma-1} > 3.5$ and $\sqrt{\sigma+1} < 2.041$ prove the stated lower square P . For completeness, on the positive class put $Z = X - B \geq 0$. Then $T_+ = 2B^2 M_+ + 3B\mathbb{E}[Z^2; X > 0] + \mathbb{E}[Z^3; X > 0]$ and $R_+ = BM_+ + \mathbb{E}[Z^2; X > 0]$. Consequently $(\sigma-1)T_+ + 6BR_+ \geq 2(\sigma+2)B^2 M_+ + 3(\sigma+1)B\theta_+$, since $\mathbb{E}[Z^2; X > 0] \geq \theta_+$. The reflected expansion gives the negative-class bound. The centered absolute deviation on each class is at most twice its mean distance from the inner endpoint, proving the bound on ϵ . All quantities refer to the same marginal.

A.3.1.2 Concentration from an antichain

Lemma A.24 (Strict-gap concentration). *Suppose $Y = U_0 + B(1 + U_1 - U_0)$, where B is Bernoulli(p) independent of the pair (U_0, U_1) , and $1 + U_1 - U_0 \geq c > 0$ almost surely. For independent copies $Y_1, \dots, Y_j$ and any interval I of diameter strictly less than c ,*

$$\mathbb{P}\left(\sum_{i=1}^j Y_i \in I\right) \leq \max_{0 \leq k \leq j} \binom{j}{k} p^k (1-p)^{j-k}.$$

The same bound holds after adding any independent random variable.

Proof. Apply the conditional random-chain argument in Appendix C. That argument uses only independent Bernoulli labels and positive increments at least c , so it applies to the present coupled pairs at every j . Its interval diameter is strictly less than c , including when either endpoint is closed. Conditioning on any added independent variable gives the last assertion. $\square$

For the actual marginal we may couple the two conditional noises independently and then sample an independent label. Since $U_0 \leq a$ and $U_1 \geq b-1$, the step is at least $b-a = 1-d$. Strict diameter is necessary: a Bernoulli law on $\{0, c\}$ assigns mass one to $[0, c]$, exceeding its largest atom when $0 < p < 1$.

A.3.1.3 A rational cap for every remaining index.

Put $J_0 = N - 2 = 45$ and define

$$r_L = l - (1 - l)/J_0, \quad r_R = h + h/J_0, \quad v_* = \min\{r_L(1 - r_L), r_R(1 - r_R)\}.$$

Require $0 < r_L \leq r_R < 1$. Adjacent binomial-mass ratios show that any mode k at index $j \geq J_0$ satisfies $(j + 1)p - 1 \leq k \leq (j + 1)p$, hence $r = k/j \in [r_L, r_R]$. In particular $1 \leq k \leq j - 1$. Finite Stirling bounds give

$$\sqrt{j} \binom{j}{k} p^k (1 - p)^{j-k} \leq \frac{\exp(1/(12j) - jD(r||p))}{\sqrt{2\pi r(1 - r)}} < \frac{0.4}{\sqrt{v_*}(1 - 1/(12J_0))}.$$

Here $D(r||p) \geq 0$ follows from Jensen's inequality for the logarithm; $\exp x \leq (1 - x)^{-1}$ for $0 \leq x < 1$ follows term by term from the power series. The finite Stirling inequalities used are $\sqrt{2\pi m}(m/e)^m < m! < \sqrt{2\pi m}(m/e)^m \exp(1/(12m))$. Choose the strict 10^{-6} grid upper square root M of $0.16/[v_*(1 - 1/(12J_0))^2]$. Thus, whenever $w < 1 - d$,

$$Q_j(w) < M/\sqrt{j}, \quad j \geq n - 2 \geq J_0. \quad (147)$$

This covers the $n - 1$ full summands of ν and the $n - 2$ full summands remaining after one mark or deletion. An independent remaining conditional noise preserves the cap. We do not replace the latter index by $n - 1$.

Either the score-derived cap or the strict-gap cap may be selected on an interval where its hypotheses hold. The first confinement uses the score cap; later central and left schedules use the strict-gap cap. For the left panels replace the conditional cubic coefficient 24 below by 28; in general any c_ρ satisfying $c_\rho^2[l(1 - l)]^3 > [3(1 - l) + 0.55]^2$ is valid.

A.3.1.4 Signed deletion, including atoms

Let $m = n - 1$, $Z = x - \sum_{i=1}^m Y_i$, $\mu = x - mp$ and let $K = \sum B_i$. For a fixed removed label put $T_b(I) = \mathbb{P}(x - \sum_{i < m} Y_i - b - U_b \in I)$. Exchangeability and $mp - K = Z - \mu + \sum U_i$ give the exact identities

$$\nu(I) - T_1(I) = \frac{\mathbb{E}[(Z - \mu + \sum U_i)1_I(Z)]}{mp}, \quad T_0(I) - \nu(I) = \frac{\mathbb{E}[(Z - \mu + \sum U_i)1_I(Z)]}{mq}.$$

There is no absolute value around the sum of noises. For a full removed pair (Y_i, U_i) , condition on the pair. The remaining sum gives a function $f(Y_i) \in [0, Q_{m-1}(w)]$. Since $\mathbb{E}U_i = 0$,

$$\mathbb{E}[U_i f(Y_i)] \leq Q_{m-1}(w) \mathbb{E}(U_i)_+ = \frac{1}{2} \epsilon Q_{m-1}(w).$$

This also works when one conditional law is deterministic.

Let independent conditional noises satisfy $\Delta = U_1 - U_0 \geq -d$. For $\tau > 0$ put $I^+ = I + 1 + [-d, \tau]$. On $\Delta \leq \tau$, the point for T_1 in I maps to the point for T_0 in I^+ . On the complement, conditioning on both noises gives error at most $Q_{m-1}(w) \mathbb{P}(\Delta > \tau)$. Moreover

$$\mathbb{P}(\Delta > \tau) \leq \min \left\{ \frac{s}{s + p\tau^2}, \frac{d}{d + \tau} \right\}. \quad (148)$$

Indeed $\mathbb{E}\Delta = 0$, $\text{Var } \Delta \leq s/p$, and $\Delta + d \geq 0$ has mean d . Cantelli and Markov prove the two bounds. The exceptional event is strict, so $\Delta = \tau$ belongs to the good event.

If I, I^+ are disjoint, combine the two signed noise terms before bounding them. For the same remaining-sum law their conditional probabilities f_I, f_J satisfy

$$0 \leq f_I \leq Q_I, \quad 0 \leq f_J \leq Q_J, \quad f_I + f_J \leq Q_{\text{hull}}.$$

Since $p < q$,

$$\frac{f_I}{p} + \frac{f_J}{q} \leq \min \left\{ \frac{Q_I}{p} + \frac{Q_J}{q}, \frac{Q_{\text{hull}}}{q} + \left(\frac{1}{p} - \frac{1}{q} \right) Q_I \right\}.$$

Multiplication by $\mathbb{E}U_+ = \epsilon/2$ proves the combined bound. Closed, open and half-open intervals are all covered by containment in the corresponding closed interval. No density or atom separation is used.

A.3.1.5 Slope bounds and the two exterior intervals

Split $|H|$ into 16 closed equal cells $[h_l, h_h]$ and λ into 24 closed equal cells $[s_l, s_h]$. On each box put

$$\begin{aligned} \rho_* &= \min\{\rho_0, \rho_B(l) + 24s_h/N\}, \\ b_- &= \max\{\beta_-, 1 - 2h - 2(1 - h)\Theta\}, \quad b_+ = \min\{\beta_+, 1 - 2l + \Theta\}. \end{aligned}$$

Here $\rho_B(p) = (1 - 2p(1 - p))/\sqrt{p(1 - p)}$. Write $s_i = \mathbb{E}U_i^2$. Conditional centering and the two sign supports give

$$\rho = \ell^3\{pq(1 - 2pq) + 3pq(s_0 + s_1) + p\mathbb{E}U_1^3 - q\mathbb{E}U_0^3\}.$$

The initial strip and $d_0 < 0.55$ imply $|U_i| < 0.55$. Thus $\rho \leq \rho_B + \ell^3(3q + 0.55)s \leq \rho_B + 24s$; the last coefficient inequality is checked as $24^2[l(1 - l)]^3 > [3(1 - l) + 0.55]^2$. Similarly $\beta = (1 - \theta)(1 - 2p) + \ell^2(ps_1 - qs_0)$, and $|\ell^2(ps_1 - qs_0)| \leq \theta$. These identities prove the stated outward bounds on ρ and β . Set $f = 0.4F_\ell$, $\varphi_- = 0.39886(1 - h_h^2/(2N)) > 0$. The exact upper bound for $H\phi(H/\sqrt{n})$ used below is

$$v_+ = \begin{cases} 0.4h_h & H \geq 0, \\ -\varphi_-h_l & H \leq 0. \end{cases}$$

For all $t \leq 0$, completing the square in (143) gives

$$\mathcal{M}_- = F_\ell((v_+ + 3C_0\rho_*)^2/(12(0.4)) - 3(0.4)b_-)_+, \quad L_- = E\varphi_- - \mathcal{M}_-/N.$$

A positive bound $B_* > \sqrt{(1 - l)/l}$ gives valid central-gap lower and positive-half-line upper slope bounds

$$\begin{aligned} \mathcal{M}_+ &= F_\ell(0.4h_hB_* + 3C_0B_*(B_* - 1) - 1.2b_-)_+, \\ L_g &= E\varphi_- - \max(\mathcal{M}_-, \mathcal{M}_+)/N, \\ U_+ &= f + \frac{F_\ell}{N} \left\{ 3C_0b_+ + \frac{(0.4h_h + 3C_0\rho_*)^2}{12(0.4)} \right\}. \end{aligned}$$

They follow by completing a square on the positive half-line and using $t(t - 1) \leq B_*(B_* - 1)$ for $1 \leq t \leq B_*$; on $0 \leq t \leq 1$ that term has the favorable sign. Negative values are covered by $\mathcal{M}_-$. All divisions and positivity requirements are explicit gates.

A.3.1.6 The sharper lower-ray slope.

Take $\tau_- = 5r_0/12$, $\eta = 10^{-5}$ and $I = [a - r, a - d - \tau_- - \eta]$, where $r \geq r_0$. Since $a \leq d$, the entire interval has $t \leq t_* = -E(l + \tau_- + \eta) < 0$. Write $B(t) = b_- + \rho_*t + t^2$ and $B_*^{\min} = B(\min(t_*, -\rho_*/2))$. For the actual C and $t \leq t_*$,

$$3C(\beta + \rho t + t^2) + H\phi t \geq 1.2B(t) + v_+t + 3(C_0 - 0.4)\min(0, B_*^{\min}).$$

The inequality uses $t < 0$ and separates the possible negative part of $B(t)$ before varying C . Minimize the right side at the clipped vertex $\min(t_*, -(v_+ + 1.2\rho_*)/2.4)$ and call its value q_* . Then

$$L = E\varphi_- - F_\ell \max(0, -q_*)/N$$

is a lower slope on the whole interval, for every $n \geq N$. Positive corrections are discarded. Therefore contact gives

$$\sqrt{n}\nu(I) \geq L(r - d - \tau_- - \eta).$$

This one slope replaces the previous separate bounds on the long and short parts of the interval.

A.3.1.7 Lower interval geometry.

Its translate is $I^+ = [a - r + 1 - d, b - \eta]$, of width $r - \eta$. The initial strip implies $I^+ \subset [a + g, b]$, $g = 1 - 0.55 - 2d_0 > 0$, hence $\nu(I^+) \leq J/g$. The intervals are disjoint throughout $r \leq 0.55 + d_0$, since $\inf I^+ - \sup I = 1 - r + \tau_- + \eta > 0$. At $r = r_0$, concentration and directed distance bounds are

$$\begin{aligned} Q_I &= f(r_0 - \tau_- - \eta) + 2C_0\rho_*, & Q_J &= f(r_0 - \eta) + 2C_0\rho_*, \\ Q_{\text{hull}} &= f(1 + r_0 - \eta) + 2C_0\rho_*, \\ D_I &= (v_H - l - \tau_- - \eta)_+, & D_J &= (v_H + 1 - l - \eta)_+, \end{aligned}$$

where $v_H = -h_l/F_\ell$ for $H \geq 0$ and $v_H = h_h/E$ for $H \leq 0$. These signs follow from $\mu = p + H/\ell$.

A.3.1.8 Upper interval geometry.

For a support point $y \geq 1 + r_0$, take $\tau_+ = 7r_0/15$, $I = [a, a + \tau_+]$ and $I^+ = [b, a + 1 + 2\tau_+]$. The strict gates $d_0 + 2\tau_+ < r_0$ and $1 - d_0 - \tau_+ > 0$ give $I^+ \subset [b, y]$ and disjointness. Contact, including the atom at b , and the leakage definition give

$$\sqrt{n}\nu(I) \geq L_g(1 - d) - \sqrt{n}J/\tau_+, \quad \sqrt{n}\nu(I^+) \leq U_+(0.55 + d).$$

Use

$$\begin{aligned} Q_I &= f\tau_+ + 2C_0\rho_*, & Q_J &= f(2\tau_+ + d_0) + 2C_0\rho_*, \\ Q_{\text{hull}} &= f(1 + 2\tau_+) + 2C_0\rho_*, \\ D_I &= (d_0 + \tau_+ + h_h/E - l)_+, & D_J &= (1 + d_0 + 2\tau_+ + h_h/E - l)_+. \end{aligned}$$

The upper side deliberately keeps the absolute phase bound.

A.3.1.9 The strict-gap alternative.

If $1 - 0.55 - 2d_0 > 0$ and the displayed lower intervals have width below $1 - d$ throughout $r \in [r_0, 0.55 + d_0]$, take $Q_I = Q_J = M$. For the upper intervals the corresponding gate is $1 - 2d_0 - 2\tau_+ > 0$; then again $Q_I = Q_J = M$. Use the same directed distances. Their concentration caps are now independent of r , and use $a_e = F_d M \{1/l + 1/(1 - h)\}/2$ in the next formulas. The lower-ray gate reduces to $L > 0$: its contact side increases with r , while the entire error is constant on the acquired finite strip. For the score-derived cap retain the explicit affine slope gate below. No assertion outside that strip is needed.

A.3.1.10 One budget, two error bounds

Let F_d be the strict grid upper bound for $\sqrt{\max\{(47/46)^3, 47/45\}}$. It simultaneously covers the two concentration factors in (144) after multiplication by $\sqrt{n}$, uniformly for $n \geq 47$. For either pair of intervals set

$$a_e = \frac{F_d}{2} \min \left\{ \frac{Q_I}{l} + \frac{Q_J}{1-h}, \frac{Q_{\text{hull}}}{1-h} + \left(\frac{1}{l} - \frac{1}{1-l} \right) Q_I \right\},$$

$$A = F_d \{Q_I D_I / l + Q_J D_J / (1-h)\}, \quad B = F_d Q_I.$$

The probability denominators are outward because $1/p - 1/(1-p)$ is decreasing on $(0, 1/2)$. Use $a_d = L$, $a_J = 1/g$ on the lower side, and $a_d = L_g + U_+$, $a_J = 1/\tau_+$ on the upper side. For the phase cell, maximize each concave budget quadratic at its vertex clipped to $[h_l, h_h]$, take their maximum, then the minimum with T ; call this upper bound T_* . Put

$$R = T_*/\kappa - b_* E^2 s_l, \quad \alpha = \min(c_+ l E, c_- (1-h) E), \quad G = (R - NP)/\alpha.$$

If $G \leq 0$, the strict common budget excludes the entire closed box. Otherwise $nd < G$ and

$$B_\tau = \min\{s_h/(s_h + Nl\tau^2), G/(N\tau + G)\}$$

bounds (148). With

$$z = \max \left\{ \frac{a_d/l + 2a_e}{Ec_+}, \frac{a_d/(1-h) + 2a_e}{Ec_-}, \frac{\kappa a_J}{1-\Theta} \right\},$$

one valid total error bound is

$$\mathcal{E}_0 = (zR + A)/N - zP + BB_\tau. \quad (149)$$

This follows by multiplying the single budget by z/κ . The inverse-index coefficient $zR + A$ is checked positive.

A stronger bound spends the same resource on the pair event. Let $R_0 = R - NP > 0$ and define the three pairs

$$(v_+, w_+) = \left(\frac{a_d/l + 2a_e}{Ec_+} \frac{R_0}{N}, \frac{R_0}{Ec_+ l} \right),$$

$$(v_-, w_-) = \left(\frac{a_d/(1-h) + 2a_e}{Ec_-} \frac{R_0}{N}, \frac{R_0}{Ec_- (1-h)} \right), \quad (v_J, w_J) = \left(\frac{\kappa a_J}{1-\Theta} \frac{R_0}{N}, 0 \right).$$

They are upper cost/displacement coordinates at the vertices of the simplex with resources $nc_+ M_+$, $nc_- M_-$ and $n^{3/2} L/\kappa$. For any $t \geq 0$,

$$\frac{D}{N\tau + D} \leq \frac{N\tau}{(N\tau + t)^2} D + \frac{t^2}{(N\tau + t)^2} \quad (D \geq 0).$$

The difference is $N\tau(D-t)^2/[(N\tau + D)(N\tau + t)^2]$. Consequently a further valid error bound is

$$\mathcal{E}_t = \frac{A}{N} + \max_{i \in \{+, -, J\}} \left(v_i + B \frac{N\tau}{(N\tau + t)^2} w_i \right) + B \frac{t^2}{(N\tau + t)^2}.$$

Take the minimum of (149) and these eleven bounds with $t = jG/10$, $j = 0, \dots, 10$. Each is individually valid; no numerical minimization or derivative of their minimum is assumed.

The lower contradiction is the strict gate $Lr_0 - L(\tau_- + \eta) > \mathcal{E}$. To extend it to every $r \geq r_0$, bound increments directly in the original affine-in-width signed deletion bound for the same law. The budget gives $n\epsilon < E_* = 2T/(\kappa E \min(c_+, c_-))$. The sufficient slope gate is

$$L > \frac{F_d f}{N} \left\{ D_I/l + D_J/(1-h) + \frac{E_*}{2} (1/l + 1/(1-h)) \right\} + F_d f B_\tau.$$

Using the uncoupled increment is safe even when the hull bound was used at r_0 . The upper contradiction is $L_g - 0.55U_+ > \mathcal{E}$. Thus no support point lies below $a - r_0$ or above $1 + r_0$. Since $a \geq 0$, $b \geq 1 - d_0$ and $d_0 < r_0$, the label-zero and label-one supports both satisfy $|U| < r_0$.

A.3.1.11 The selected schedules.

The ordinary, central-antichain, and left-antichain certificates substitute their recorded r_0 into these same gates. Noise-dependent amplitudes are the same theorem with a positive lower noise endpoint in $R = T_*/\kappa - b_* E^2 s_l$. In the central rows the additional amplitudes are 0.35, 0.30, 0.275 on $[0.05, 0.08]$, $[0.08, 0.10]$, $[0.10, \Lambda]$; on the left they are 0.375, 0.325, 0.30. Intersect each interval with its own row's Λ and each amplitude with that row's global amplitude. The required acceptance checks both signs of H , all closed phase/noise cells, clipped endpoints, and strict budget exclusions. No proof of whole-count, nearest-count, or binomial score comparison is used by this confinement lemma.

A.3.2 The conditional-count comparison

A law concentrated near two points has a natural integer count and a small centered error. We compare it with its own Bernoulli count. The same exact identity handles every noise regime. There are two useful estimates of its right side: discard all but the nearest outgoing count, or retain all counts through a random cutoff. The estimates share their normalization, moments, and tail bounds; only the final scalar witness changes.

A.3.2.1 The identity and sufficient inequalities

Let $0 < p \leq 1/2$, $q = 1 - p$, $v = pq$, and let

$$Y = B + U, \quad \mathbb{P}(B = 1) = p, \quad \mathbb{E}(U | B) = 0.$$

The two actual conditional laws U_0, U_1 obey

$$|U_j| \leq A, \quad U_0 \leq a_0, \quad U_1 \geq -a_1, \quad a_0, a_1 \geq 0, \quad d = a_0 + a_1 \leq d_0 < \min(p, q), \quad d_0 \leq A. \quad (150)$$

Write $s_j = \mathbb{E}U_j^2$, $s = qs_0 + ps_1$, $\lambda = ns$, $\ell = (v + s)^{-1/2}$, $\theta = s/(v + s)$ and $\rho = \mathbb{E}|\ell(Y - p)|^3$. At the actual threshold choose

$$x = np + H/\ell = k + u, \quad k \in \{1, \dots, n-1\}, \quad |u| \leq \frac{1}{2}, \quad \delta = k - np = H/\ell - u.$$

Either nearest integer is allowed at a tie. Put

$$\pi_j = \binom{n}{j} p^j q^{n-j}, \quad F_B(j) = \mathbb{P}(\text{Bin}(n, p) \leq j), \quad \rho_B = (1 - 2v)/\sqrt{v},$$

with $\pi_j = 0$ outside $[0, n]$. Let W_j be an independent sum of j copies of U_1 and $n - j$ copies of U_0 . The signed scores are

$$C = \frac{\sqrt{n}}{\rho} \{ \mathbb{P}(\sum Y_i \leq x) - \Phi(H/\sqrt{n}) \}, \quad C_B = \frac{\sqrt{n}}{\rho_B} \{ F_B(k) - \Phi(\delta/\sqrt{nv}) \}.$$

Throughout this section $C \geq c_* > 0$; the selected application uses $c_* = 0.4$. Define

$$\begin{aligned} f(u) &= \frac{\Phi((\delta + u)/\sqrt{n(v+s)}) - \Phi(\delta/\sqrt{n(v+s)})}{\pi_k}, \\ Q &= \mathbb{P}(\sum Y_i \leq k + u) - F_B(k), \\ \mathcal{R} &= \sqrt{n}\{\Phi(\delta/\sqrt{nv}) - \Phi(\delta/\sqrt{n(v+s)})\} + C(\rho_B - \rho). \end{aligned} \tag{151}$$

Equation (38), with $V_j = W_j$ and the same interior count k , gives the exact identity

$$\rho_B(C - C_B) = \sqrt{n}\pi_k\{Q/\pi_k - f(u)\} + \mathcal{R}. \tag{152}$$

In particular, the phase u , the sign of H , and both conditional laws refer to the same marginal throughout.

Theorem A.25 (Parameterized local comparison). *Fix $n \geq N$, $\lambda > 0$ and a closed parameter cell on which (150) holds. Suppose $\sqrt{n}\pi_k \geq m > 0$. Either of the following witnesses proves $\rho_B(C - C_B) < -\varepsilon\lambda$.*

1. Nearest-count witness. *Valid bounds O, T, R_0, κ satisfy*

$$\mathbb{P}(W_k > u) + f(u) \geq O, \quad \sum_{i \geq 1} \mathbb{P}(W_{k+i} \leq u - i) \leq T, \quad \pi_{k+i}/\pi_k \leq R_0, \quad \mathcal{R} \leq \kappa\lambda/n.$$

Only legal counts occur in the sum; adding positive upper bounds for other counts is allowed. If $D = O - R_0T > 0$ and

$$mD - \kappa\lambda/n > \varepsilon\lambda, \tag{153}$$

the conclusion follows.

2. Whole-count witness. *Bounds $B(\lambda) < 0$ and $I(\lambda) \geq 0$ satisfy*

$$\rho_B(C - C_B) \leq \sqrt{n}\pi_k\lambda B(\lambda) + \lambda I(\lambda)/n.$$

It is enough that

$$mB(\lambda) + I(\lambda)/N < -\varepsilon. \tag{154}$$

The witnesses may differ from cell to cell. A finite closed cover and a separate treatment of $\lambda = 0$ therefore suffice for a complete local comparison.

Proof. The signed count identity (36), with $V_j = W_j$, is

$$Q = -\sum_{i=0}^k \pi_{k-i} \mathbb{P}(W_{k-i} > u + i) + \sum_{i=1}^{n-k} \pi_{k+i} \mathbb{P}(W_{k+i} \leq u - i). \tag{155}$$

Apply Proposition 6.1 with $m_0 = m$. The legal counts, closed incoming events and strict outgoing events are identical. In the second alternative the present hypothesis $I \geq 0$ makes its $\max(0, I)/N$ exactly I/N , yielding (154). $\square$

A.3.2.2 Binomial normalization and the variance correction

Here is an explicit finite recipe for the constants required above. Let $p \in [p_-, p_+] \subset (0, 1/2)$, $n \geq N$, $\lambda \leq \Lambda$, $\theta \leq \Theta$, and $|\delta| \leq \Delta$, where $\Delta \geq 1$ also includes every binomial mode. Choose $0 < r_- < p_- - \Delta/N$ and $p_+ + \Delta/N < r_+ < 1/2$. Let $v_* > 0$ be a lower bound for $p(1-p)$. Choose positive E, L, R bounding, on $[r_-, r_+]$,

$$|D'''(r||p)| = \frac{1-2r}{r^2(1-r)^2}, \quad |(\log(r(1-r)))'| = \frac{1-2r}{r(1-r)}, \quad \frac{1}{12r(1-r)},$$

respectively, where $D(r||p) = r \log(r/p) + (1-r) \log((1-r)/q)$. Then, with $Z(t) = Lt/2 + R + Et^3/(6N)$,

$$\sqrt{n}\pi_k = \frac{\phi(0)}{\sqrt{v}} e^{-\delta^2/(2nv)+\zeta}, \quad n|\zeta| < Z(|\delta|). \quad (156)$$

For completeness, the finite Stirling formula used here is

$$\log(j!) = (j + \tfrac{1}{2}) \log j - j + \tfrac{1}{2} \log(2\pi) + r_j, \quad 0 < r_j < 1/(12j).$$

If $a_j = \log(j!) - (j + 1/2) \log j + j$ and $z = (2j + 1)^{-1}$, the integrated geometric series gives $0 < a_j - a_{j+1} = \sum_{i \geq 1} z^{2i}/(2i + 1) < 1/[12j(j + 1)]$. Summing gives the remainder relative to $\lim a_j$. The integration-by-parts recursion for $\int_0^{\pi/2} \sin^j t dt$ and the inequalities between three successive integrals give Wallis's product; its finite product equals $e^{4a_j - 2a_{2j}} j/[2(2j + 1)]$ and tends to $\pi/2$. Thus $\lim a_j = \frac{1}{2} \log(2\pi)$. Applying this formula to $n!, k!, (n - k)!$ and Taylor-expanding D and the logarithmic prefactor proves (156); the factorial error is at most $1/[12nr(1 - r)]$.

Consequently a valid mass lower bound is any positive

$$m < \frac{0.39886}{\sqrt{p_+(1-p_+)}} \left[1 - \frac{\Delta^2/(2v_*) + Z(\Delta)}{N} \right]. \quad (157)$$

The same formula at a mode proves $\sqrt{n} \max_j \pi_j < 1$ whenever $v_*[1 - Z(\Delta)/N]^2 > 0.4^2$ and $Z(\Delta) < N$. The rational bounds $0.39886 < \phi(0) < 0.4$ follow, for example, by the alternating arctangent series in Machin's identity.

Differentiating (151) gives

$$\log f' = -\tfrac{1}{2} \log(1 + s/v) + \delta^2/(2nv) - (\delta + u)^2/(2n(v + s)) - \zeta. \quad (158)$$

Thus $n|\log f'| \leq B_0\lambda + P(D, a)$ on $|\delta| \leq D \leq \Delta$, $|u| \leq a \leq 1/2$, where

$$B_0 > \frac{1}{2v_*} + \frac{\Delta^2}{2Nv_*^2}, \quad P(D, a) = \frac{2Da + a^2}{2v_*} + Z(D).$$

Choose $0 < b < 1$ above $[B_0\Lambda + P(\Delta, 1/2)]/N$ and put $F = (1 - b)^{-1}$. The elementary estimates $e^z \leq (1 - z)^{-1}$ and $1 - e^{-z} \leq z$ imply

$$|f' - 1| \leq (B_1\lambda + P_1)/n, \quad B_1 = FB_0, \quad P_1 = FP(D, a). \quad (159)$$

This supplies both a derivative interval $l_f \leq f' \leq h_f$ and the finer phase-error bound $|u - f(u)| \leq a(B_1\lambda + P_1)/n$.

The exact count variance is

$$t_j = \mathbb{E}W_j^2 = \lambda + (j - np)(s_1 - s_0), \quad |s_1 - s_0| \leq s/p \leq c_v s, \quad c_v \geq 1/p_-. \quad (160)$$

It gives $\omega_- \lambda \leq t_k \leq \omega_+ \lambda$ with $\omega_\pm = 1 \pm c_v D/N$. All uses below require the checked condition $\omega_- > 0$. Also $t_{k+i} \leq iV_* \lambda$ for $V_* = 1 + c_v(\Delta + 1)/N$. The binomial recursion gives, for every legal $i \geq 1$,

$$\frac{\pi_{k+i}}{\pi_k} \leq R_0, \quad R_0 > 1 + \frac{p_+ + \Delta - 1}{Nv_*[1 - (\Delta - 1)/(Np_-)]}, \quad (161)$$

provided the denominator is positive and $p_+ + \Delta < 2$. Indeed the first ratio has this bound and all subsequent rising factors are at most one. No count outside $[0, n]$ is used in this recursion.

Finally (150) fixes the signs inside the absolute cubic. Expansion and centering give

$$\mathbb{E}|Y - p|^3 = v(1 - 2v) + 3pq(s_0 + s_1) + p\mathbb{E}U_1^3 - q\mathbb{E}U_0^3 \geq v(1 - 2v) + (3p - d)s.$$

The inequality $(1 + z)^{3/2} \leq 1 + 3z/2 + 3z^2/8$ for $z \geq 0$ yields

$$(v + s)^{3/2}(\rho - \rho_B) \geq s \left(6p - 3p^2 - \frac{3}{2} - d - \frac{3(1 - 2v)s}{8v} \right). \quad (162)$$

Let $Q_* > 0$ be a lower bound for the bracket and $m_c \leq c_*/[p_+(1 - p_+) + \Lambda/N]^{3/2}$. The normal variance correction is nonpositive for $\delta \leq 0$; for $\delta > 0$ its derivative in s is at most $0.4\delta/(2v^{3/2})$. Therefore

$$\mathcal{R} \leq (P_* D_+ - m_c Q_*)\lambda/n, \quad P_* \geq \frac{0.4}{2v_*^{3/2}}, \quad D_+ \geq \delta_+. \quad (163)$$

One may take $D_+ = D$, but retaining the positive part is often decisive. All bounds concern $v + s$, which can exceed $1/4$.

A.3.2.3 Two elementary tools for probabilities

Lemma A.26 (A shifted quartic). *Suppose W is a sum of independent centered variables bounded in absolute value by A , and $t = \mathbb{E}W^2$. Let $K_0 \geq (9 + 6\sqrt{3})/4$ and $a > 0$. For $r \geq 0$ and $\sigma \in \{-1, 1\}$,*

$$\mathbb{P}(W + \sigma r > 0) \geq \frac{(3a^2 - A^2)t - 3t^2 - 4Art - 6r^2t - r^4 + 3a^2r^2 + 2\sigma a^3r}{K_0 a^4}. \quad (164)$$

Proof. The polynomial $h_a(z) = (-z^4 + 3a^2z^2 + 2a^3z)/(K_0a^4)$ is below $\mathbf{1}_{\{z>0\}}$: it is nonpositive for $z \leq 0$, and its maximum for $z > 0$ is at $z/a = (1 + \sqrt{3})/2$ and is at most one. Independence gives $|\mathbb{E}W^3| \leq At$ and $\mathbb{E}W^4 \leq A^2t + 3t^2$. Expand $\mathbb{E}h_a(W + \sigma r)$ and apply these two inequalities. The value at the threshold is zero, so atoms require no limiting argument. $\square$

Lemma A.27 (One common tail bound). *If independent centered $X_i \leq A$ have total variance at most V , then, for $b > 0$, their sum has upper tail bounded by the minimum of one and*

$$\frac{V}{V + b^2}, \quad \mathcal{B}(A, V, b) = e^{b/A} \left(\frac{V}{V + Ab} \right)^{b/A + V/A^2}, \quad \frac{\mathbb{E}(P_\mu - j)_+^3}{(b/A + \mu - j)^3}, \quad (165)$$

$$\mu = V/A^2, \quad 0 \leq j < b/A + \mu.$$

where P_μ is Poisson. The same holds for a lower tail when $|X_i| \leq A$. All bounds apply to closed events, hence also to strict events.

Proof. Cantelli's bound follows by applying Markov to $(\sum X_i + V/b)^2$. Taylor's integral remainder gives $\mathbb{E}e^{zX_i} \leq \exp\{\mathbb{E}X_i^2(e^{zA} - 1 - zA)/A^2\}$ for $z \geq 0$. Multiply and use $z = A^{-1} \log(1 + Ab/V)$ to get $\mathcal{B}$. The zero-variance case is immediate.

For the third bound, smooth $f(x) = (x - t)_+^3$ by a nonnegative smooth compactly supported density. Then $f'', f''', f'''' \geq 0$. For $\mathcal{A}f(x) = f(x + A) - f(x) - Af'(x)$, the Taylor quotient

$$\frac{f(x + y) - f(x) - yf'(x)}{y^2} = \int_0^1 (1 - r)f''(x + ry) dr$$

is nondecreasing in y . A centered $X \leq A$ of variance v consequently satisfies $\mathbb{E}f(x + X) \leq f(x) + (v/A^2)\mathcal{A}f(x)$. For $\mathcal{P}_z f(x) = \mathbb{E}f(x + A(P_z - z))$, $\partial_z \mathcal{P}_z f = \mathcal{P}_z \mathcal{A}f$ and

$$\partial_z^2 \mathcal{P}_z f = \mathcal{P}_z \mathcal{A}^2 f \geq 0, \quad \mathcal{A}^2 f(x) = \int_0^A \int_0^A (A - r)(A - w)f''''(x + r + w) dr dw.$$

Thus $\mathcal{P}_{v/A^2} f \geq f + (v/A^2)\mathcal{A}f$. Replace the independent variables successively; translations and expectations preserve the derivative signs and Poisson rates add. Increasing the total variance to V is permitted by Jensen after adding independent centered Poisson noise. Removing the smoothing is justified by bounded actual upper supports and finite Poisson third moments. Markov at $t = A(j - \mu)$ proves the displayed bound. $\square$

These bounds have finite rational evaluations. For example,

$$\begin{aligned} \mathbb{E}(P_\mu - j)_+^3 &= \mu^3 + 3(1 - j)\mu^2 + (1 - 3j + 3j^2)\mu - j^3 \\ &+ e^{-\mu} \sum_{h=0}^{\lfloor j \rfloor} \frac{\mu^h (j - h)^3}{h!}. \end{aligned} \tag{166}$$

Replacing $e^{-\mu}$ by $(\sum_{h=0}^{24} \mu^h/h!)^{-1}$ enlarges the nonnegative correction. Alternatively sum the positive Poisson series through 32; the remaining tail is at most its first term divided by $1 - \mu(34 - j)^3/[34(33 - j)^3]$, provided $j < 33$ and that ratio is below one. For Bennett, put $x = Ab/V$ and $L = [(b/A + \mu) \log(1 + x) - b/A]$. The positive series $\log(1 + x) = 2 \sum_{h \geq 0} z^{2h+1}/(2h+1)$, $z = x/(2+x)$, gives lower bounds for L ; any positive lower exponential series gives an upper bound for e^{-L} . Rounding a positive lower L down before exponentiation is also valid. These are choices of rational witnesses, not optimization claims.

A.3.2.4 The nearest-count witness, including vanishing noise

For $u = -r \leq 0$, add $-h_f r$ to (164) with $\sigma = 1$; for $u = r \geq 0$, add $l_f r$ with $\sigma = -1$. Both bound $\mathbb{P}(W_k > u) + f(u)$ from below. For a cell $t \in [t_-, t_+]$, $r \in [r_-, r_+]$, this expression is concave in t (second derivative $-6/(K_0 a^4)$), so only t_-, t_+ are needed. At each endpoint, bound a monomial with positive coefficient at r_- and one with negative coefficient at r_+ . Any rational $a > 0$ is a valid witness. Other valid choices are

$$\frac{r_-^2}{t_+ + r_-^2} - h_f r_+ \quad (u \leq 0), \quad l_f r_- \quad (u \geq 0). \tag{167}$$

They follow from Cantelli and nonnegativity, respectively. Their maximum or one selected witness is valid without an optimality proof.

For incoming counts use $t_{k+i} \leq iV_*\lambda$. On $u \in [-r_+, -r_-]$ the first distance is $1 + r_-$ and every later distance is at least i . On $u \in [r_-, r_+]$ they are $1 - r_+$ and $i(1 - r_+/2)$ for $i \geq 2$. Let b, b_o be these first and later distances. On $\lambda \in [L, U]$, let T_1 be any bound from Lemma A.27 at (A, V_*U, b) . Since $\mathcal{B}(A, iV, ib) = \mathcal{B}(A, V, b)^i$, all later counts total at most $B^2/(1 - B)$ for $B = \mathcal{B}(A, V_*U, b_o) < 1$. Thus

$$D = O - R_0\{T_1 + B^2/(1 - B)\} > 0, \quad mD - \kappa U/N > \varepsilon U \tag{168}$$

proves the entire cell when $\kappa \geq 0$. This sign condition is required on each cell. The argument applies to both central and left probability panels.

A cell touching $\lambda = 0$ requires estimates proportional to λ ; a positive lower endpoint may not be substituted. The following rescaling provides them. Put $r = zt$, $t_* = \omega_+ U$ and $d_a = K_0 a^4$. The shifted quartic plus br , divided by t , becomes

$$F_{\sigma,b}(z, t) = \frac{3a^2 - A^2 - 3t}{d_a} + \left(\frac{2\sigma}{K_0 a} + b - \frac{4At}{d_a} \right) z + \frac{3a^2 t - 6t^2}{d_a} z^2 - \frac{t^3 z^4}{d_a}. \quad (169)$$

It extends to $t = 0$ and is concave in t . On $0 \leq z \leq z_0$, replacing z^4 by $z_0^2 z^2$ gives a lower quadratic at $t = t_*$; at $t = 0$ it is linear. Endpoints and a clipped quadratic vertex therefore give an exact lower bound $J(\sigma, b, z_0)$ for the whole rectangle.

For negative phase choose $z_-, c_- > 0$ and verify

$$J(1, -h_f, z_-) > c_-, \quad \frac{z_-^2}{1 + z_-^2 t_*} - h_f z_- > c_-, \quad \frac{2z_-}{(1 + z_-^2 t_*)^2} > h_f, \quad (170)$$

$$\frac{1}{1 + 4t_*} - \frac{h_f}{2} > c_- t_*, \quad z_- t_* < 1/2.$$

The quartic handles $r \leq z_- t$. Beyond that point use $r^2/(t + r^2) - h_f r$. Its derivative at $r = z_- t$ is positive by the third gate; its second derivative has just one sign change, so any later critical point is a maximum. Its minimum is at $z_- t$ or $1/2$, covered by the other gates. Hence the own count exceeds $c_- t$. For positive phase choose $z_+, c_+ > 0$ with

$$J(-1, l_f, z_+) > c_+, \quad l_f z_+ > c_+, \quad z_+ t_* < 1/2. \quad (171)$$

The quartic handles $r \leq z_+ t$ and the normal term handles the rest. The own count exceeds $c_+ t$ and also exceeds $l_f r$.

For incoming tails proportional to λ , fix a Poisson anchor j . A positive series term of index h divided by $\lambda(b/A + \mu - j)^3$ has logarithmic derivative, multiplied by λ ,

$$h - 1 - \mu - \frac{3\mu}{b/A + \mu - j}, \quad \mu = V_* \lambda / A^2.$$

If $b/A - j > 0$ and this is nonnegative at $\lambda = U$ for the first positive index, it is nonnegative for every term and all $0 < \lambda \leq U$. Therefore the Poisson bound divided by λ is largest at U . Likewise $B^2/[\lambda(1 - B)]$ is increasing whenever $\lambda B'/B \geq 1/2$. Differentiating the Bennett formula gives

$$\lambda B'/B = b/A - \mu \log(1 + b/(A\mu));$$

The function $g(\mu) = b/A - \mu \log(1 + b/(A\mu))$ decreases because $\log(1 + x) > x/(1 + x)$ for $x > 0$. Moreover the logarithmic derivative of $B^2/[\lambda(1 - B)]$, multiplied by λ , is $2g - 1 + gB/(1 - B) \geq 0$ when $g \geq 1/2$. Thus checking the gate at U proves the whole interval. A rational upper bound for the logarithm verifies that gate. On cells using the fixed positive bound $l_f r_-$ instead, unscaled endpoint tails suffice: after a positive bracket is checked, divide it by U to obtain a lower bound per λ . These arguments, followed by (153), prove the nearest-count comparison uniformly down to arbitrary positive noise, including a deterministic conditional component.

A.3.2.5 The whole-count witness

Assume $c = 1 - d_0 > 1/2$. Couple all $2n$ conditional variables independently and set $W_j = \sum_{i \leq j} U_{1,i} + \sum_{i > j} U_{0,i}$, extending it constantly outside $[0, n]$. Every increment of $j + W_j$ is at least c . Define

$$K = \max\{j \in \mathbb{Z} : j + W_j \leq k + u\}, \quad R = K - k.$$

The coupling of Section 6, with $V_j = W_j$, gives

$$\mathbb{P}(\sum Y_i \leq k + u) = \mathbb{E}F_B(k + R). \quad (172)$$

Let $\chi = (n + 1)p - k < 2$ and $m_B = \max_j \pi_j$. For every integer r ,

$$F_B(k + r) - F_B(k) - r\pi_k \leq \frac{m_B}{(n + 1)v} \left[(\chi - 1)_{+r} + \chi_{+} \binom{r_{-}}{2} + \binom{r_{-}}{3} \right]. \quad (173)$$

This is Lemma 6.2, with the same zero extension, $\chi < 2$ and interior count. Its proof requires only an increment lower bound $c > 1/2$. Here $d \leq d_0$ gives increments at least $1 - d_0 = c$, and $c \leq 1$ follows from $d_0 \geq 0$.

Write $t = \mathbb{E}W_k^2$. Crossing monotonicity and Cauchy–Schwarz give

$$\mathbb{E}R_{+} \leq \frac{t}{(c - u)^2}, \quad \mathbb{E} \binom{R_{-}}{2} \leq \frac{t}{[c + \min(u, c/3)]^2}, \quad \mathbb{E} \binom{R_{-}}{3} \leq \frac{t\sqrt{A^2 + 3t}}{(2c - 1/2)^3}. \quad (174)$$

These are the crossing-moment bounds of Lemma 6.2 with $A_U = A$, $V_k = W_k$, $|u| \leq 1/2$ and the increment lower bound $c = 1 - d_0$. The lemma includes the strict outgoing crossing and the fourth-moment estimate from independence and conditional centering.

A.3.2.6 The marked crossing.

Let $D_i = U_{1,i} - U_{0,i}$, $\vartheta = \mathbb{E}(D_i)_{+} = \mathbb{E}(D_i)_{-}$, $0 < \eta \leq 1$, and $R_f = \mathbb{E}\{u + \eta - W_k\}$. The two deleted laws are

$$A_0 \stackrel{d}{=} kU_1 + (n - k - 1)U_0, \quad A_1 \stackrel{d}{=} (k - 1)U_1 + (n - k)U_0.$$

Then

$$\begin{aligned} \mathbb{E}R &\leq u + \eta - R_f + J, \\ J &= \frac{\vartheta}{\eta} \sum_{i \geq 1} \{\mathbb{P}(A_0 \leq u + d_0 - ci) + \mathbb{P}(A_1 > u + \eta + i - A)\}. \end{aligned} \quad (175)$$

Here are the pathwise details. Put $z = u - W_k$ and mark

$$Z_f = \sum_{i=1}^{n-k} (D_{k+i})_{-} \mathbf{1}_{\{W_k \leq u - ci\}}, \quad Z_b = \sum_{i=1}^k (D_{k-i+1})_{+} \mathbf{1}_{\{W_k > u + \eta + i\}}.$$

If $R > z + \eta$, the crossing activates the marks through R : for $R > 0$ use $z \geq cR$, while for $R < 0$ use $z < R - \eta$. In either case $R \leq z + Z_f + Z_b$. Hence if $h = R - \lfloor z + \eta \rfloor > 0$, $Z_f + Z_b > h - 1 + \eta \geq \eta h$. Thus $R \leq \lfloor z + \eta \rfloor + (Z_f + Z_b)/\eta$ on every path. On an active forward mark $U_0 > U_1 \geq -d_0$; deleting its pair implies $A_0 \leq u + d_0 - ci$. On a backward mark delete $U_1 \leq A$ to obtain the strict event for A_1 . The remaining sum is independent of the entire pair, so the expectations factor. Extending to all $i \geq 1$ adds only nonnegative bounds. Finally $|R| \leq 1 + (|W_k| + |u|)/c$ proves integrability and permits expectations. No coupling has replaced either deleted marginal.

A.3.2.7 A finite bound for every mark.

Fix $\lambda \in [L, U]$, $L > 0$, and $t \leq \omega_{+}\lambda$; put $T_L = \omega_{+}L$, $T_U = \omega_{+}U$. On a phase subcell $u \in [v_l, v_h]$, for $i = 1, 2$ let $b_1 = ci - d_0 - v_h > 0$ and $b_2 = i - A + v_l > 0$. Lemma A.27 gives a bound $\omega_{+}\lambda K_i^{\text{ind}}$, where

$$K_i^{\text{ind}} = \sum_{j=1}^2 \min \left\{ \frac{1}{b_j^2 + T_L}, \frac{\overline{P}(A, T_U, b_j)}{T_L} \right\}.$$

The denominator T_L is a lower bound for the chosen upper envelope $\omega_+\lambda$, not for any actual variance. Alternatively couple $A_0 = Z + U_1$, $A_1 = Z + U_0$ with $Z \stackrel{d}{=} (k-1)U_1 + (n-k-1)U_0$. The two events project into $Z \leq -\alpha$ and $Z > \beta$, where $\alpha = ci - 2d_0 - v_h > 0$, $\beta = i - A - d_0 + v_l > 0$. For any $z \in (-\alpha, \beta)$ their sum is at most $(\omega_+\lambda + z^2)/\min(z + \alpha, \beta - z)^2$. Consequently a second kernel is

$$K_i^{\text{com}} = \frac{1 + z^2/T_L}{\min(z + \alpha, \beta - z)^2}, \quad z = \max\{-T_L/\beta, \min((\beta - \alpha)/2, T_L/\alpha)\}.$$

This center is inside the interval; differentiating the two quadratic ratios on either side of the midpoint proves its optimality, although any interior rational center suffices. Take the smaller kernel on each phase subcell and the maximum over their closed cover. Denote it K_i . For $i \geq 3$ use fourth moments and the convex midpoint integral for $(cx - b)^{-4}$. On $u \in [u_l, u_h]$ this gives

$$K = K_1 + K_2 + \frac{A^2 + 3T_U}{3c} \left\{ \frac{1}{((5/2)c - A - u_h)^3} + \frac{1}{((5/2)c - A + u_l)^3} \right\}, \quad J \leq \frac{\vartheta}{\eta} \omega_+ \lambda K. \quad (176)$$

All denominators must be positive. Indeed, the tail fourth moment is at most $A^2t + 3t^2$, and integration from $5/2$ to infinity covers every remaining integer, including the harmless added illegal counts.

If the same-law resource supplies $nd < G$ and $G^2 < c_v NL$, then

$$\vartheta \leq \frac{c_v G \lambda}{c_v n \lambda + G^2}. \quad (177)$$

To prove this, for centered $X \geq -b$, $\mathbb{E}X^2 \leq V$ and $b^2 \leq V$, the quadratic $b(X - V/b)^2/(b + V/b)^2$ majorizes X_- on $[-b, \infty)$. Its expectation is at most $bV/(V + b^2)$. Apply this to $D_i \geq -d$, using $\text{Var}(D_i) = s_0 + s_1 \leq c_v \lambda/n$, and monotonicity in $b \leq \sqrt{V}$. The case $d = 0$ is immediate. With $\eta = z\lambda$, $z > 0$, $zU \leq 1$, (176) becomes

$$\frac{J}{\lambda} \leq \frac{\omega_+ K c_v G}{z(c_v N \lambda + G^2)}. \quad (178)$$

A.3.2.8 The fractional-part gain.

The following implication assumes the scalar majorization and its rational constant enclosures stated below. Their finite arithmetic acceptance is a separate condition. If each centered summand has support span at most $b < 1/2$ and the sum has variance t , then, for any real w and $0 < r_* \leq 2 \sin^2(\pi b)/b^2$,

$$\mathbb{E}\{w - W\} \geq \frac{5}{23}(1 - e^{-r_* t}). \quad (179)$$

Indeed $1 - \cos(2\pi x) \leq (23/5)x$ for $0 \leq x \leq 1$. A finite proof of this sole scalar inequality is recorded below. For an independent copy Z' of one summand, concavity of sine on $[0, \pi b]$ gives

$$|\mathbb{E}e^{2\pi i Z}|^2 = 1 - 2\mathbb{E}\sin^2 \pi(Z - Z') \leq e^{-4 \sin^2(\pi b) \text{Var}(Z)/b^2}.$$

Multiply square roots over the independent summands and use periodicity. This argument includes zero variances and integer collisions. Here the conditional spans are at most $A + d_0$, so this is the b to use.

For a reproducible scalar proof, $x \geq 10/23$ follows from $1 - \cos \leq 2$. On $[0, 10/23]$ use the degree-ten alternating upper polynomial for $1 - \cos(2\pi x)$, with directed coefficients:

$$Q(x) = \sum_{j=1}^5 \frac{(-1)^{j+1} (2\pi_j x)^{2j}}{(2j)!}, \quad \pi_j = \begin{cases} 3.141592653589794, & j \text{ odd}, \\ 3.141592653589793, & j \text{ even}. \end{cases}$$

Put $P(x) = 23/5 - Q(x)/x$. On the eight intervals with left endpoint $a_j = 5j/92$ and width $w = 5/92$, its degree-nine Bernstein coefficients are

$$b_{jk} = \sum_{i=0}^k \frac{\binom{k}{i}}{\binom{9}{i}} \sum_{h=i}^9 [x^h]P(x) \binom{h}{i} a_j^{h-i} w^i, \quad 0 \leq j < 8, \quad 0 \leq k \leq 9.$$

Positivity of these 80 rational coefficients is a finite certificate condition; its acceptance is not asserted in this appendix. Expansion in powers of the unit-interval coordinate proves the formula; Bernstein weights are nonnegative and sum to one. The alternating remainder and the stated rational bounds on π follow from the same elementary series above. This completely specifies the finite scalar obligation. A rational rate follows by replacing $\sin(\pi b)$ with $x - x^3/6 + x^5/120 - x^7/5040$, $x = 3.14159b$, on the checked alternating-series domain.

Set $r = r_*\omega_-$, require $rU < 3$, and define

$$g = \frac{5}{23} \left(\frac{r^3 L^2}{6} - \frac{r^4 L^3}{24} + \frac{r^5 L^4}{120} - \frac{r^6 L^5}{720} \right). \quad (180)$$

The sixth-order Taylor lower bound for $1 - e^{-r\lambda}$ gives $R_f/\lambda \geq 5r/23 - 5r^2\lambda/46 + g$. The discarded higher-order part is increasing as a function of λ for $r\lambda < 3$: its derivative has the sign of $48 - 18x + (24/5)x^2 - x^3 > 0$ on $[0, 3]$. For instance this polynomial has positive degree-three Bernstein coefficients 48, 30, 132/5, 51/5 there. Thus freezing it at L is valid.

A.3.2.9 One final expression.

On a signed phase cell find an interval $[x_l, x_h]$ containing H/ℓ by its four endpoint quotients, using positive bounds for ℓ . Let

$$D = \max(|x_l - u_h|, |x_h - u_l|), \quad D_+ = \max(0, x_h - u_l), \quad a = \max(|u_l|, |u_h|), \quad b_\chi = p_+ - x_l.$$

The same formula allows coarser upper bounds D, D_+, b_χ , as used by the central certificate. Check $D \leq \Delta$ and $b_\chi + u_h < 2$. With $\omega_\pm = 1 \pm c_v D/N$, the curvature coefficient is

$$V_0 = \frac{\omega_+}{v_*} \left[\sup_{u_l \leq u \leq u_h} \left\{ \frac{(b_\chi + u - 1)_+}{(c - u)^2} + \frac{(b_\chi + u)_+}{[c + \min(u, c/3)]^2} \right\} + \frac{\sqrt{A^2 + 3T_U}}{(2c - 1/2)^3} \right]. \quad (181)$$

On a rational phase subcell, both numerators are bounded at the right endpoint; the first denominator is smallest there, and the second is smallest at the left endpoint. This proves the supremum bound, including both kinks. Equations (172)–(174), $\sqrt{n}m_B \leq 1$, and (152) now give the whole-count witness with

$$B(\lambda) = z + \frac{\omega_+ K c_v G}{z(c_v N \lambda + G^2)} - \frac{5r}{23} + \frac{5r^2 \lambda}{46} - g, \quad (182)$$

$$I(\lambda) = V_0 + aB_1 + \frac{aP_1}{\lambda} + P_* D_+ - m_c Q_*.$$

We used $|u - f(u)| \leq a(B_1 \lambda + P_1)/n$ and $\sqrt{n}\pi_k \leq 1$. Every constant is fixed before λ varies. Both B and $mB + I/N$ are convex on $\lambda > 0$; I is decreasing. Thus their two endpoint gates $B < 0$, $I > 0$, $mB + I/N < -0.001$ prove the entire closed noise cell. Thus this argument supplies the analytic implication for each of the three whole-count estimates, conditional on its own complete cover and endpoint gates.

A.3.2.10 The zero-noise endpoint and the remaining composition

If $\lambda = 0$, then $p, q > 0$ and conditional centering imply $U_0 = U_1 = 0$ almost surely. The actual law is its Bernoulli comparison law. The zero-noise argument in Section 6 places an attained positive closed-CDF maximum at a binomial atom, so $u = 0$ and $C = C_B$. The external all-probability binomial theorem of [21, Theorem 1], when applied with its stated hypotheses, then supplies the strict Bernoulli bound. For positive noise, the local conclusion requires every parameter cell to satisfy one of the displayed strict tests. No selection of historical table rows, completeness of a finite cover, or fresh acceptance of those tests is asserted here.

The mathematical composition therefore still requires: the actual contact witness; already acquired common-law support, moments and signed resources; both deleted-law concentration caps; valid normalization and positivity of every divisor; a complete cover of all signs, phases and noise values; and acceptance of the stated finite gates on that cover. No step replaces an $n - 2$ convolution by an $n - 1$ convolution or infers a uniform statement from a fixed-law asymptotic limit.

B Binomial normalization and elementary constants

B.1 A Taylor segment crossing one half

A phase cell can place the Taylor segment of a binomial mass on both sides of $1/2$. The required derivative estimates remain valid provided the segment lies in an interval $[a, b]$ with $a + b < 1$. We give the argument because imposing a one-sided segment would unnecessarily exclude such cells.

Lemma B.1 (Symmetric derivative envelopes). *If $0 < a < 1/2$, $a < b < 1$, $a + b < 1$, then for $r \in [a, b]$,*

$$r(1-r) \geq a(1-a), \quad \frac{|1-2r|}{[r(1-r)]^j} \leq \frac{1-2a}{[a(1-a)]^j} \quad (j = 1, 2).$$

Proof. $[a, b] \subset [a, 1-a]$. Set $s = \min(r, 1-r) \in [a, 1/2]$. The expressions are symmetric; $s(1-s)$ increases and $1-2s$ decreases nonnegatively. Their positive quotients therefore have their maxima at a . $\square$

Let $n \geq N \geq 1$, $p \in [p_-, p_+] \subset (0, 1/2)$, $q = 1 - p$, $1 \leq k \leq n - 1$, $\delta = k - np$, $r = k/n$, $v = pq$. Suppose the whole segment between p, r lies in $[a, b]$ above. If

$$E \geq \frac{1-2a}{a^2(1-a)^2}, \quad L \geq \frac{1-2a}{a(1-a)}, \quad R \geq \frac{1}{12a(1-a)},$$

then the binomial mass π_k satisfies

$$\sqrt{n} \pi_k = \frac{\phi(0)}{\sqrt{v}} e^{-\delta^2/(2nv) + \zeta}, \quad n|\zeta| < L|\delta|/2 + R + E|\delta|^3/(6N).$$

For proof, relative entropy has $D(p||p) = D'(p||p) = 0$, $D''(p||p) = 1/v$ and $D^{(3)}(r||p) = (2r-1)/[r^2(1-r)^2]$. Taylor bounds $|nD(r||p) - \delta^2/(2nv)|$ by $E|\delta|^3/(6n^2)$; the logarithmic variance prefactor changes by at most $L|\delta|/(2n)$. Finite Stirling, $0 < s_j < 1/(12j)$, gives exactly

$$\sqrt{n} \pi_k = \frac{\phi(0)}{\sqrt{r(1-r)}} e^{-nD(r||p) + s_n - s_k - s_{n-k}}.$$

Writing $e = s_n - s_k - s_{n-k}$, use the compatible one-sided estimates $e < 1/(12n)$ and $e > -1/(12k) - 1/(12(n-k)) = -1/[12nr(1-r)]$. Thus $|e| < R/n$, and collecting all three errors proves the claim.

For finite interval applications choose rational endpoints satisfying $a < p_- - \Delta/N$, $p_+ + \Delta/N < b$, $a + b < 1$. For $|\delta| \leq \Delta$ they contain the whole segment. The inequalities $Np_- - \Delta - 2 > 0$ and $N(1 - p_+) - \Delta - 2 > 0$ ensure interior/adjacent factorial domains. Positive $v_f < a(1-a)$ and upward-rounded $(1-2a)/v_f^2$, $(1-2a)/v_f$, $1/(12v_f)$ supply E, L, R . Mass bounds use $e^{-x} \geq 1-x$, $e^x \leq 1/(1-x)$ only on their stated positive domains. The original p band remains below $1/2$ when bounding pq ; monotonicity is not assumed on the enlarged Taylor segment.

B.2 Elementary Gaussian constants

We give the exact comparisons used in Lemma 3.1.

Proof. Since $\phi''(x) = (x^2 - 1)\phi(x)$, the norm on $x^2 \leq 1$ is at most $\phi(0)$. On $x^2 \geq 1$, the maximum of $(x^2 - 1)e^{-x^2/2}$ is $2e^{-3/2} < 1$, attained when $x^2 = 3$. Thus the norm is exactly $\phi(0)$.

For completeness, the Gaussian transform can be obtained without an inversion theorem for the distribution being studied. Differentiating $J(t) = \int_{\mathbb{R}} e^{-x^2/2} e^{itx} dx$ under the integral and integrating by parts give $J'(t) = -tJ(t)$. The Gaussian integral, evaluated by squaring it and using polar coordinates, gives $J(0) = \sqrt{2\pi}$. Hence $J(t) = \sqrt{2\pi}e^{-t^2/2}$; applying the same formula with the variables interchanged gives

$$\phi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-t^2/2} e^{-itx} dt.$$

Three differentiations are justified by the integrability of $|t|^3 e^{-t^2/2}$. Consequently

$$\|\phi'''\|_{\infty} \leq \frac{1}{2\pi} \int_{\mathbb{R}} |t|^3 e^{-t^2/2} dt = \frac{2}{\pi}.$$

The last integral equals four after the substitution $w = t^2/2$ on the positive half-line.

Here are elementary rational comparisons for all numerical constants used in this section. Put $a = \arctan(1/5)$ and $b = \arctan(1/239)$. Tangent addition gives $\tan(2a) = 5/12$, $\tan(4a) = 120/119$ and $\tan(4a - b) = 1$. Also

$$0 < 4(1/5 - 1/(3 \cdot 5^3)) - 1/239 < 4a - b < 4/5 < 1 < \pi/2.$$

The arctangent inequalities follow by integrating $1 - t^2 < 1/(1 + t^2) < 1$, while $\pi/2 = 2 \arctan(1) > 1$ follows by integrating $1/(1 + t^2) > 1/2$ on $[0, 1)$. It follows that $\pi = 16a - 4b$ and

$$\pi > 16(1/5 - 1/(3 \cdot 5^3)) - 4/239 = 281476/89625 > 25/8.$$

This proves $\phi(0) < 2/5$ and $1/(4\pi) < 2/25$.

To bound C_E more closely, define

$$s_k(x) = \sum_{j=0}^{k-1} \frac{(-1)^j x^{2j+1}}{2j+1}, \quad \pi_- = 16s_6(1/5) - 4s_1(1/239), \quad \pi_+ = 16s_5(1/5) - 4s_2(1/239).$$

The remainder in the finite geometric expansion of $1/(1 + t^2)$ has its stated alternating sign, so integration gives $\pi_- < \pi < \pi_+$. Let $r_- = 3162277660/10^9$ and $r_+ = 3162277661/10^9$. Squaring shows $r_-^2 < 10 < r_+^2$. The following comparisons are exact rational ones:

$$\left(\frac{819}{2000}\right)^2 < \frac{19 + 6r_-}{72\pi_+} < C_E^2 = \frac{19 + 6\sqrt{10}}{72\pi} < \frac{19 + 6r_+}{72\pi_-} < \left(\frac{20487}{50000}\right)^2.$$

For example, each outer comparison follows by multiplying by its positive rational denominator; the displayed finite sums specify those denominators completely. All quantities are positive, so taking square roots proves the claimed bounds. In particular, $0.4095 < C_E < 0.40974 < 0.41$. The same upper endpoint for π is smaller than $22/7$. Since $(0.39886)^2 < 7/44$, we also obtain $\phi(0) > 0.39886 > 0.3988$. $\square$

C The index changes required at the floor 42

The support proof in Section 4 is uniform for $n \geq 42$. The other estimates also depend on the lower bound for n . We derive the changes needed in concentration, confinement and normalization, assuming the contact, phase-resource and local-comparison identities stated in the preceding sections.

C.1 Conditional moments and strict-gap concentration

The concentration argument uses the antichain method of Littlewood–Offord theory [9] in its random-chain form [18].

Let $n \geq N = 42$ and use the same coordinates $X = \ell(B_0 + U - p)$ as in Section 5. Suppose an already proved support bound gives

$$\text{supp}(B_0 + U) \subset (-w_{\text{strip}}, 1 + w_{\text{strip}}), \quad U_0 \leq a_0, \quad U_1 \geq -a_1, \quad a_0 + a_1 = d_Y \leq d_0 < w_{\text{strip}}.$$

The strip width is $23/40$ in the first left-panel contraction. Any smaller value must be established before it is used in a subsequent contraction. Assume $p \in [l, h] \subset (0, 1/2)$.

Write $s = qw_0 + pw_1 = \lambda/n$, $w_i = \mathbb{E}U_i^2$. The exact identity (12), $|U_i| < w_{\text{strip}}$ and $pq(w_0 + w_1) \leq qs$ give

$$\rho \leq \rho_B(p) + c_\rho s \quad \text{if} \quad c_\rho > 0, \quad c_\rho^2[l(1-l)]^3 > [3(1-l) + w_{\text{strip}}]^2.$$

Indeed $|\mathbb{E}U_i^3| \leq w_{\text{strip}}w_i$, $\ell^3 \leq [l(1-l)]^{-3/2}$, and the skeleton's third moment is at most $\rho_B(p)$. The function $\rho_B(p) = (1 - 2p(1-p))/\sqrt{p(1-p)}$ decreases on $(0, 1/2)$, so a closed noise band with ceiling λ_+ gives

$$\rho \leq \rho_B(l) + c_\rho \lambda_+/N.$$

The first left contraction uses $c_\rho = 30$. A later value such as 24 is licensed only if its displayed gate holds with the actual strip.

Independently couple each pair of conditional residuals before choosing its Bernoulli label. Changing a label from zero to one increases the corresponding summand by at least $1 - d_Y$. Conditional on the coupled residuals, the inverse image of an interval of diameter strictly less than $1 - d_Y$ is therefore an antichain in the Boolean cube. For an antichain $\mathcal{A}$ of subsets of $\{1, \dots, j\}$, a uniformly random permutation produces a chain of initial segments that intersects $\mathcal{A}$ at most once. Taking expectations gives

$$\sum_{A \in \mathcal{A}} \binom{j}{|A|}^{-1} \leq 1.$$

Under independent labels of parameter p ,

$$\mathbb{P}(\mathcal{A}) = \sum_{A \in \mathcal{A}} \binom{j}{|A|}^{-1} \binom{j}{|A|} p^{|A|} q^{j-|A|} \leq \max_k \binom{j}{k} p^k q^{j-k}.$$

Conditioning back proves the strict-gap concentration bound for the original sum, including any independent residual shift. The strict width condition is necessary; a closed interval with width equal to the increment can contain comparable vertices.

At floor 42, the two deleted indices are at least 40. Put

$$r_L = l - (1 - l)/40, \quad r_R = h + h/40, \quad v_* = \min\{r_L(1 - r_L), r_R(1 - r_R)\}.$$

The mode condition $|(k/j) - p| \leq \max(p, q)/j$ puts every binomial mode for $j \geq 40$ in $[r_L, r_R]$. Require $0 < r_L < r_R < 1$. For an interior mode with $r = k/j$, finite Stirling and nonnegativity of the relative entropy give

$$\sqrt{j} \binom{j}{k} p^k q^{j-k} < \frac{\phi(0)}{\sqrt{r(1-r)}} \exp(1/(12j)) < \frac{0.4}{\sqrt{v_*}\{1 - 1/(12 \cdot 40)\}}.$$

Here $e^x \leq (1 - x)^{-1}$ for $0 \leq x < 1$, proved by comparing their power series. The endpoint gates ensure that the mode is interior. This provides the concentration estimate at both $n - 1$ and $n - 2$.

The alternative concentration estimate obtained from the deleted Berry–Esseen bounds retains *both* hypotheses $C_{n-1}, C_{n-2} < 0.4253$. Its scale factor F_d must satisfy

$$F_d > 0, \quad F_d^2 > \max\{(42/41)^3, 42/40\}.$$

The corresponding ratios decrease with n , so this endpoint gate is sufficient for every $n \geq 42$. A scale factor checked only at the older floor need not satisfy it.

C.2 The signed confinement step

The signed deletion identities are algebraic identities at the actual index $n - 1$. Removing a pair leaves $n - 2$ full summands. A centered conditional noise V satisfies $\mathbb{E}V_+ = \mathbb{E}|V|/2$; replacing this signed quantity by an unrelated absolute error would lose the factor needed in the confinement estimate. Atoms and deterministic conditional laws are covered by the same identities.

The exceptional pair event is strict. Its two candidate bounds have the form

$$\mathbb{P}(\Delta > \tau) \leq \min \left\{ \frac{s}{s + p\tau^2}, \frac{d_Y}{d_Y + \tau} \right\}, \quad \tau > 0,$$

with the conditional mean, variance and one-sided support assumptions specified in the confinement derivation. The first is a one-sided second-moment bound; the second uses the same inner gap. The interval geometry additionally requires

$$1 - w_{\text{strip}} - 2d_0 > 0, \quad 1 - 2d_0 - 2\tau_+ > 0,$$

as well as the disjointness and buffer gates in that derivation. These conditions guarantee widths strictly below $1 - d_Y$. The upper-ray contradiction is a strict inequality of the form $L - w_{\text{strip}}U - \text{error} > 0$. The lower-ray argument also needs a positive slope, so that a check at one endpoint excludes the whole remaining ray inside the previously acquired strip. The new strip is an output of this reasoning.

The step uses one strict phase resource,

$$nep + n\phi(0)D/6 + n^{3/2}L_{\text{leak}} < \mathcal{B}(H), \quad e = C - C_E \geq 0, \quad L_{\text{leak}} \geq 0. \quad (183)$$

For a phase/noise cell, write its remaining resource as $R - nP$ with $P \geq 0$. Since $n \geq N$, it is at most $R - NP$. If the latter is nonpositive, strict inequality (183) excludes the cell.

Otherwise a proved coefficient $\alpha > 0$ gives $nd_Y < (R - NP)/\alpha$. The error terms to which the index floor is applied have the form

$$(zR + A)/n - zP + B \text{ bad,} \quad zR + A > 0.$$

The displayed positivity is what licenses $1/n \leq 1/N$; a formal substitution without it would be invalid.

A sharper candidate uses the tangent inequality

$$\frac{x}{N\tau + x} \leq \frac{N\tau}{(N\tau + t)^2}x + \frac{t^2}{(N\tau + t)^2}, \quad x, t \geq 0,$$

whose right side minus left side is

$$\frac{N\tau(x - t)^2}{(N\tau + x)(N\tau + t)^2} \geq 0.$$

The two error bounds are each valid with nonnegative coefficients and the same resource simplex. Taking their minimum is then legitimate.

C.3 Uniform assembly and verified tail closure

The resulting extension has the following precise meaning. Retain the attainment and contact hypotheses, the complete Fourier and signed phase bounds, the ordered actual-law acquisition, all strict confinement gates just described, the signed binomial normalization, and the full positive-noise comparison. Replace all 65 finite support invocations by Section 4. Retain the weighted invariant $nD \leq 42d_D$; Section 5 then gives the joint plane at the same floor. On each noise band, contract the support before using the new amplitude in a subsequent estimate.

With every selected exact predicate verified at these inputs, the argument proceeds at every actual $n \geq 42$: signed entry, outer-panel exclusion, central or left support contraction, signed confinement, joint-resource export, noise-profile acquisition, and local count comparison. Closed phase and noise seams and all admissible counts remain part of the cover. The local signed cubic correction uses the actual upper score when its coefficient is negative. The combined inverse-index correction is clamped at zero before the floor substitution. At positive noise the comparison excludes the maximizing score; at zero noise the finite-binomial comparison is used.

In the public verification release (`full_pass: "ESTABLISHED"`), the finite inequalities and analytic bounds are verified across the floor-40 role (`current-C4042`), the 183-task finite collector role for $43 \leq n \leq 48$ (`current-C4348`), and the 55-predicate effective-tail suite for $n \geq 49$ (`independent-tail49`), including the ordered support contractions, both deleted indices, the signed phase cells and all positive-noise and zero-noise comparisons.

D Signed refinements of the local comparison

The refinements in this appendix apply to an already acquired common-law resource panel. They preserve the actual count while integrating a Gaussian increment, and retain the sign of the cubic correction. These are needed for the proposed comparison with floor 42; the earlier floor-47 formulas alone do not supply them. Every numerical positivity and coverage statement below is an acceptance condition. Whenever a normalized binomial mass or a Stirling expression is used, assume $n \geq N \geq 2$ and an interior integer count $1 \leq k \leq n - 1$, so that $\pi_k > 0$.

Write $\xi = H/\ell$ for the raw threshold in Bernoulli coordinates. The normalization constants are supplied by Appendix B. The underlying count identities, moment and tail comparisons are in Section 5 and Appendix A.3.

D.1 Notation

Let $Y = B + U$, where $\mathbb{P}(B = 1) = p$, $\mathbb{E}(U \mid B) = 0$, $v = p(1 - p)$, $s = \mathbb{E}U^2$, $\lambda = ns$, and $\ell = (v + s)^{-1/2}$. The actual threshold has the form

$$x = np + H/\ell = k + u, \quad |u| \leq \frac{1}{2}, \quad \delta = k - np = H/\ell - u.$$

The integer k and the value δ stay fixed when integrating a function of a new variable t below. Write $\pi_k = \mathbb{P}(\text{Bin}(n, p) = k)$ and

$$f(t) = \frac{\Phi((\delta + t)/\sqrt{n(v + s)}) - \Phi(\delta/\sqrt{n(v + s)})}{\pi_k}.$$

The conditional errors satisfy $|U_j| \leq A$, $U_0 \leq a_0$, $U_1 \geq -a_1$, with $a_0, a_1 \geq 0$ and $d = a_0 + a_1$. All quantities in an application refer to the same iid marginal.

Lemma D.1 (A joint budget on a noise interval). *Suppose the acquired same-law resource inequality is*

$$\frac{d}{d_*} + \frac{\lambda}{\Lambda} \leq 1, \quad d_* > 0, \quad \Lambda > 0.$$

On a closed noise interval $L \leq \lambda \leq U \leq \Lambda$ one may use

$$d \leq d_L := d_*(1 - L/\Lambda).$$

Every conditional support, cubic-moment, tail-distance and characteristic function bound whose hypothesis is $d \leq d_0$ may therefore be applied with $d_0 = d_L$ on that interval.

Proof. Rearrange the displayed resource inequality and use $\lambda \geq L$. □

For example, the whole-count kernel uses $c = 1 - d_0$. Its moment coefficients are recomputed as

$$\alpha = 6m_0\ell_-^2(1 - d_0), \quad \beta = \min\{3(4.16)\ell_-(1 - p_+ - d_0), 3(6.16)\ell_-(p_- - d_0)\},$$

where m_0, ℓ_- are the previously justified moment and scale lower bounds. The ordinary cubic correction is recomputed with the same d_0 . Likewise, a noise amplitude bound A gives a conditional support span at most $A + d_0$; the characteristic-function rate is recomputed from that span. These substitutions are licensed only when the original coefficient inequalities and positivity gates hold for the acquired moment and scale bounds.

D.2 A binomial mass bound on the actual phase cell

Suppose the finite Stirling argument has supplied positive constants E, L_0, R on a fixed segment containing every k/n under consideration, with

$$\sqrt{n}\pi_k = \frac{\phi(0)}{\sqrt{v}} \exp\{-\delta^2/(2nv) + \zeta\}, \quad n|\zeta| \leq Z_N(|\delta|), \quad Z_N(t) = \frac{L_0 t}{2} + R + \frac{Et^3}{6N}, \quad n \geq N.$$

The constants may have been computed on a much wider global phase range. Let $0 < v_- \leq v \leq v_+$ and let $q_+ \geq \sqrt{v_+}$.

Lemma D.2 (Mass localization). *On any smaller closed cell where $|\delta| \leq D$,*

$$\sqrt{n} \pi_k \geq \frac{0.39886}{q_+} \left[1 - \frac{D^2/(2v_-) + Z_N(D)}{N} \right].$$

The maximum of this expression and any previously proved mass lower bound is also a valid mass lower bound. There is no requirement $D \geq 1$.

Proof. Lemma 3.1 gives $\phi(0) > 0.39886$. Set $A_* = D^2/(2v_-) + Z_N(D) \geq 0$. The logarithmic formula gives $\sqrt{n} \pi_k \geq (0.39886/q_+)e^{-A_*/n}$. Now $e^{-z} \geq 1 - z$ and $n \geq N$. If the bracket is negative the displayed inequality still holds, and the previously positive bound can be retained. The global Stirling segment, rather than the smaller value D , continues to justify E, L_0, R . $\square$

D.3 A sharper elementary harmonic constant

Lemma D.3. *Assume the finite rational positivity condition in the proof below. Then, for every $x \geq 0$,*

$$1 - \cos(2\pi x) \leq \frac{4553}{1000}x.$$

Proof. Put $K = 4553/1000$. For $x \geq 2/K$ the assertion follows from $1 - \cos \leq 2$. On $[0, 2/K]$ take the degree-fourteen alternating upper Taylor polynomial $Q(x)$ for $1 - \cos(2\pi x)$. Its remainder is nonpositive: the first omitted term is negative, and the remaining term magnitudes decrease, since $(4\pi/K)^2 < 16 \cdot 15$.

Use the directed endpoints

$$\pi_- = 3.141592653589793 < \pi < 3.141592653589794 = \pi_+.$$

In each positive coefficient of Q replace π by π_+ ; in each negative coefficient replace it by π_- . This only increases Q . Expand $P(x) = K - Q(x)/x$, a polynomial of degree thirteen, on each of the 64 intervals $[2j/(64K), 2(j+1)/(64K)]$. If $P(x) = \sum_{h=0}^{13} p_h x^h$, its Bernstein coefficients on $[a, a+w]$ are

$$b_k = \sum_{i=0}^k \frac{\binom{k}{i}}{\binom{13}{i}} \sum_{h=i}^{13} p_h \binom{h}{i} a^{h-i} w^i, \quad k = 0, \dots, 13.$$

The remaining certificate condition is that all 896 rational coefficients are positive. Under that condition, $P > 0$ on the full interval. The value at zero follows by continuity (or directly).

For completeness the stated π endpoints are checked using alternating series for $\arctan(1/5)$ (32 terms) and $\arctan(1/239)$ (12 terms). Machin's identity was established in Lemma 3.1; the term counts here give the finer directed endpoints required by this polynomial. The finite recipe uses only rational operations and the Bernstein identity proved in Section 7. $\square$

If the original characteristic-function argument gives $|\mathbb{E}e^{2\pi i W}| \leq e^{-rt}$, Lemma D.3 upgrades its fractional part estimate from $(5/23)(1 - e^{-rt})$ to

$$\frac{1000}{4553}(1 - e^{-rt}).$$

Consequently every occurrence of $5/23$ and $5/46$ in that harmonic gain is replaced by $1000/4553$ and $500/4553$, respectively. The existing degree-six Taylor gain and its monotonicity argument remain unchanged.

D.4 The Gaussian derivative knows the sign of the threshold

Lemma D.4 (Positive phase, with a fixed actual count). *Assume $u \in [a, b] \subset [0, 1/2]$, $H/\ell \leq \xi_*$, $|\delta| \leq D$, $v \geq v_- > 0$, $\lambda \leq U$, and the Stirling remainder bound above. Set*

$$G = \max \left(0, \max_{z \in [a, b]} (2\xi_* z - z^2) \right), \quad \eta = \frac{U + G}{2v_-} + Z_N(D).$$

Then $f(u) \geq u(1 - \eta/N)$ for every $n \geq N$. The inner maximum is attained at the projection of ξ_ onto $[a, b]$. If $H \leq 0$, one may take $\xi_* = 0$, hence $G = 0$.*

Proof. For a fixed actual law and threshold, differentiate while keeping $\delta = k - np$ fixed:

$$\log f'(t) = -\frac{1}{2} \log(1 + s/v) + \frac{\delta^2}{2nv} - \frac{(\delta + t)^2}{2n(v + s)} - \zeta.$$

For $0 \leq t \leq u$, use $\log(1 + s/v) \leq s/v$ and $v + s \geq v$ to obtain

$$\log f'(t) \geq -\frac{\lambda + 2\delta t + t^2}{2nv} - \zeta.$$

The convex function $t \mapsto 2\delta t + t^2$ attains its maximum on $[0, u]$ at an endpoint. Since the actual relation is $\delta = H/\ell - u$,

$$2\delta t + t^2 \leq \max(0, 2(H/\ell)u - u^2) \leq G.$$

Thus $\log f'(t) \geq -\eta/n$, and $f'(t) \geq e^{-\eta/n} \geq 1 - \eta/N$. Integrate from zero to the actual u . $\square$

A signed H interval supplies the bound D . If $H \geq 0$, one can use $\xi_* = H_+/\ell_-$ and $D = \max(b, \xi_* - a)$; if $H \leq 0$, use $D = H_-/\ell_- + b$. The new lower derivative constant is maximized with the older valid constant. Every resulting bound is checked positive.

The normal variance and cubic-moment repair is localized separately. On a phase cell, use its actual upper bound for δ_+ , rather than $|\delta|$. If the resulting coefficient is κ , replace it by $\max(0, \kappa)$ before using $\kappa/n \leq \kappa/N$. That sign condition is essential for an all-index conclusion.

D.5 Reusing a curvature bound

For fixed probability, threshold and phase intervals, the curvature function in Section 6.3 is nonincreasing in c . Lemma D.1 makes $c = 1 - d_L$ increase as the lower noise endpoint L increases. A curvature upper bound for the first band therefore remains valid on later bands. The other c -dependent moment and support bounds are evaluated using the value for each band.

D.6 Signed Gaussian corrections on both phase halves

Let $n \geq N$, $v = p(1 - p) > 0$, $s \geq 0$, $\lambda = ns$, and let the actual nearest count and phase satisfy

$$\delta = k - np = \xi - u, \quad f(t) = \frac{\Phi((\delta + t)/\sqrt{n(v + s)}) - \Phi(\delta/\sqrt{n(v + s)})}{\pi_k}.$$

The actual k and δ are held fixed when t varies. Assume the proved finite Stirling formula on this count range is

$$\sqrt{n} \pi_k = \frac{\phi(0)}{\sqrt{v}} \exp\{-\delta^2/(2nv) + \zeta\}, \quad n|\zeta| \leq Z_N(D), \quad |\delta| \leq D.$$

Write $v \geq v_- > 0$ and $Z = Z_N(D) \geq 0$.

Lemma D.5 (Negative phase). *Suppose $u = -r$, $0 \leq a \leq r \leq b$, and $\xi \leq \xi_*$. Define*

$$t_* = \max(0, \min(b, \xi_* + b)), \quad G_- = 2(\xi_* + b)t_* - t_*^2, \quad \eta_- = G_-/(2v_-) + Z.$$

If $D^2 \leq Nv_-$ and $\eta_- < N$, then

$$f'(-t) \leq \frac{1}{1 - \eta_-/N} \quad (0 \leq t \leq r), \quad u - f(u) \leq \frac{b\eta_-}{n(1 - \eta_-/N)}.$$

Proof. Direct differentiation and the Stirling formula give

$$\log f'(-t) = -\frac{1}{2} \log(1 + s/v) + \frac{\delta^2 s}{2nv(v+s)} + \frac{2\delta t - t^2}{2n(v+s)} - \zeta.$$

Since $\log(1+x) \geq x/(1+x)$ for $x \geq 0$, the first two terms are at most

$$\frac{s}{2(v+s)} \left(\frac{\delta^2}{nv} - 1 \right) \leq 0.$$

This cancellation removes the variance term from the upper derivative bound. Because $\delta = \xi + r$, $r \leq b$ and $0 \leq t \leq r \leq b$,

$$2\delta t - t^2 \leq 2(\xi_* + b)t - t^2 \leq G_-.$$

The last maximum is the concave quadratic maximum on $[0, b]$, achieved at the displayed projection t_* . It is nonnegative because $t = 0$ is available. Therefore $\log f'(-t) \leq \eta_-/n$. Using $e^x \leq (1-x)^{-1}$ for $0 \leq x < 1$ gives

$$f'(-t) - 1 \leq \frac{\eta_-/n}{1 - \eta_-/n} \leq \frac{\eta_-}{n(1 - \eta_-/N)}.$$

Integrate $u - f(u) = \int_0^r (f'(-t) - 1) dt$. □

Equivalently, $G_- = b^2 + 2\xi_*b$ for $\xi_* \geq 0$, and $G_- = \max(0, \xi_* + b)^2$ for $\xi_* < 0$. This sharper vertex is used in the whole-count comparison on the left probability panels at floor 42.

Lemma D.6 (Positive phase). *Suppose $u \in [a, b] \subset [0, 1/2]$, $\xi \leq \xi_*$, and define*

$$G_+ = \max \left(0, \max_{z \in [a, b]} (2\xi_* z - z^2) \right).$$

Then

$$u - f(u) \leq \frac{b}{n} \left(\frac{\lambda}{2v_-} + \frac{G_+}{2v_-} + Z \right).$$

The inner maximum is at the projection of ξ_ onto $[a, b]$.*

Proof. The exact derivative identity and $\log(1 + s/v) \leq s/v$ give

$$\log f'(t) \geq -\frac{\lambda + 2\delta t + t^2}{2nv} - \zeta.$$

For fixed actual u , the function $2\delta t + t^2$ is convex on $[0, u]$. Its maximum is at an endpoint, and $\delta = \xi - u$ gives

$$2\delta t + t^2 \leq \max(0, 2\xi u - u^2) \leq G_+.$$

Use $e^{-x} \geq 1 - x$ and integrate from zero to u . □

D.6.1 Whole-count insertion and interval coverage

The whole-count identity is split as $Q/\pi_k - f(u) = (Q/\pi_k - u) + (u - f(u))$. Thus one needs an upper bound on the signed correction $u - f(u)$; an absolute-value estimate is unnecessary. After division by λ , the new coefficients replacing the old Gaussian correction are

$$\begin{aligned} u \geq 0 : & \quad \frac{b}{2v_-} + \frac{b\{G_+/(2v_-) + Z\}}{\lambda}, \\ u \leq 0 : & \quad \frac{b\eta_-}{(1 - \eta_-/N)\lambda}. \end{aligned}$$

On each fixed phase cell these have the form $A + B/\lambda$ with $A, B \geq 0$. They therefore preserve the original convex endpoint check on each closed noise interval. Every phase cell is confined to one half-line; zero is a shared closed endpoint. The count displacement bound D comes from the same actual $\xi - u$ cell, and the Stirling constants continue to come from the already proved global count segment.

The other all-index steps are unchanged: the event bracket is checked negative before substituting a binomial mass lower bound, and the entire inverse- n correction is checked positive before replacing n by N . The inverse correction is $I(\lambda) = K + A + B/\lambda$, hence decreases with λ ; positivity at both endpoints (in particular the upper endpoint) licenses the all-index step throughout the interval. The final score is convex in λ : its event fraction is a positive constant divided by an affine positive function, its B/λ coefficient is nonnegative, and its remaining terms are affine or constant. These are independent gates in the exact kernel. The formulas apply to both signs of ξ without reflecting the law or changing the count during integration. They also supply an improved upper derivative for the negative-phase nearest-count inequalities.

D.6.2 A complete phase cover below the whole-count threshold

For $\lambda \in [0.05, 0.08]$, the shifted-quartic nearest-count comparisons must be checked on every closed phase interval of length 0.001 in $[0, 1/2]$, for both phase signs. The shifted-quartic polynomial and nearest-count identity are defined at phase zero, so the two closed covers include $u = 0$ directly. This permits the whole-count comparison to start at $\lambda = 0.08$. The four nearest-count bands use phase starts 0, 0, 0.19, 0.27 at noise bands $[0.05, 0.06]$, $[0.06, 0.08]$, $[0.08, 0.10]$, and $[0.10, 0.12]$, respectively.

D.7 Conditional centering and the exact cubic bound

Let B be Bernoulli with parameter $0 < p < 1/2$, put $q = 1 - p$ and $v = pq$, and write $Y = B + U$, with $\mathbb{E}(U | B) = 0$. Let U_0, U_1 denote the conditional laws of U , and assume

$$U_0 \leq a_0 < p, \quad U_1 \geq -a_1 > -(1 - p), \quad a_0, a_1 \geq 0, \quad d = a_0 + a_1.$$

Put $s_j = \mathbb{E}U_j^2$, $s = qs_0 + ps_1$, $\lambda = ns$, and

$$X = \frac{Y - p}{\sqrt{v + s}}, \quad \rho = \mathbb{E}|X|^3, \quad \rho_B = \frac{1 - 2v}{\sqrt{v}}.$$

Finite third moments are assumed. The support conditions fix the sign of $Y - p$ in each class, which is necessary for the following expansion.

Lemma D.7 (No positivity assumption on the cubic bracket). *With the preceding notation,*

$$(v + s)^{3/2}(\rho - \rho_B) \geq sQ, \quad Q = 6p - 3p^2 - \frac{3}{2} - d - \frac{3(1 - 2v)s}{8v}. \quad (184)$$

Proof. Expanding within each sign class and using conditional centering gives

$$\mathbb{E}|Y - p|^3 = v(1 - 2v) + 3pq(s_0 + s_1) + p\mathbb{E}U_1^3 - q\mathbb{E}U_0^3.$$

The one-sided support bounds imply $\mathbb{E}U_1^3 \geq -a_1s_1$ and $\mathbb{E}U_0^3 \leq a_0s_0$. Moreover $3pq(s_0 + s_1) \geq 3p(qs_0 + ps_1) = 3ps$, because $q \geq p$. Consequently

$$\mathbb{E}|Y - p|^3 \geq v(1 - 2v) + (3p - d)s.$$

For $z \geq 0$, Taylor's formula gives $(1 + z)^{3/2} \leq 1 + 3z/2 + 3z^2/8$: the third derivative is negative. Subtract

$$\rho_B(v + s)^{3/2} = v(1 - 2v)(1 + s/v)^{3/2} \leq v(1 - 2v) + \frac{3}{2}(1 - 2v)s + \frac{3(1 - 2v)s^2}{8v}$$

from the preceding moment lower bound and use $3p - \frac{3}{2}(1 - 2v) = 6p - 3p^2 - \frac{3}{2}$. $\square$

D.8 The coefficient must be bounded in the correct direction

Suppose $n \geq N$, $p \in [p_-, p_+] \subset (0, 1/2)$, $0 \leq d \leq d_0$, and $0 \leq \lambda \leq \Lambda$. Choose $0 < v_* \leq p_-(1 - p_-)$ and put

$$V^* = p_+(1 - p_+) + \Lambda/N, \quad Q_* = 6p_- - 3p_-^2 - \frac{3}{2} - d_0 - \frac{3(1 - 2v_*)\Lambda}{8v_*N}.$$

Then $Q \geq Q_*$: the first term increases with p , and $(1 - 2v)/v$ decreases with v . In particular, Q_* may be negative. Already with $d = s = 0$ its sign changes at $p = 1 - 1/\sqrt{2}$.

Let C be the score multiplying the moment correction, and assume $0 < c_* \leq C \leq c^*$. Choose positive multipliers satisfying

$$0 < m_- \leq \frac{c_*}{(V^*)^{3/2}}, \quad m_+ \geq \frac{c^*}{v_*^{3/2}},$$

and define the signed gain

$$G_* := \min\{m_-Q_*, m_+Q_*\}. \quad (185)$$

Proposition D.8 (Signed gain and nonnegative repair). *Let Φ denote the standard normal distribution function. For an actual count displacement δ , set*

$$\mathcal{R} = \sqrt{n} \left\{ \Phi\left(\frac{\delta}{\sqrt{nv}}\right) - \Phi\left(\frac{\delta}{\sqrt{n(v+s)}}\right) \right\} + C(\rho_B - \rho).$$

If $D_+ \geq \max(\delta, 0)$ and $P_* \geq 0.4/(2v_*^{3/2})$, then

$$\mathcal{R} \leq (P_*D_+ - G_*)\frac{\lambda}{n} \leq \kappa\frac{\lambda}{N}, \quad \kappa = \max\{0, P_*D_+ - G_*\}. \quad (186)$$

Proof. Write $A = C/(v + s)^{3/2}$. The hypotheses give $0 < m_- \leq A \leq m_+$. By Lemma D.7, $C(\rho - \rho_B) \geq sAQ \geq sAQ_*$. If $Q_* \geq 0$, $AQ_* \geq m_-Q_*$; if $Q_* < 0$, $AQ_* \geq m_+Q_*$. This proves $C(\rho - \rho_B) \geq sG_*$ in both cases. One must use the *upper* multiplier in the negative case.

The normal part is nonpositive when $\delta \leq 0$. For $\delta > 0$, differentiation with respect to an intermediate variance $t \in [0, s]$ gives

$$\frac{d}{dt} \left[-\sqrt{n} \Phi\left(\frac{\delta}{\sqrt{n(v+t)}}\right) \right] = \frac{\delta}{2(v+t)^{3/2}} \phi\left(\frac{\delta}{\sqrt{n(v+t)}}\right) \leq \frac{0.4\delta}{2v_*^{3/2}} \leq P_*D_+.$$

Integrate over $[0, s]$, and combine with the moment bound. Finally $s = \lambda/n$ and $\kappa \geq 0$, so $(P_*D_+ - G_*)\lambda/n \leq \kappa\lambda/n \leq \kappa\lambda/N$. The clamp at zero is needed before replacing n by its lower bound. $\square$

D.9 Exact checks and the finite application

The left probability panels at floor 42 use

$$N = 42, \quad c_* = 0.4, \quad c^* = 0.4226.$$

These are hypotheses supplied by the surrounding finite-index proof. The required rational multiplier inequalities are

$$(m_-)^2(V^*)^3 < c_*^2, \quad (m_+)^2v_*^3 > (c^*)^2,$$

together with positivity of both multipliers. Squaring without those positivity checks would accept a false negative coefficient. The normal multiplier is checked similarly using $4P_*^2v_*^3 > 0.4^2$ and $P_* > 0$.

On each probability and noise interval, evaluate the minimum in (185) and bound the nonnegative coefficient in (186) using the same moment and support data. Lemma D.7 and Proposition D.8 then give the probabilistic correction.

D.10 The finite comparison cover at floor 42

The proposed acquisition first excludes the closed probability ranges $[0, 0.23]$, $[0.23, 0.29]$, $[0.474, 0.48]$ and $[0.48, 0.63]$, together with $[0.63, 1]$. Different bounds may establish adjacent exclusions, so their common endpoints must be retained. The remaining grids are the 45 left panels in $[0.29, 0.38]$ and 47 central panels in $[0.38, 0.474]$, each of width 0.002. These are the rows defined in Section 5.5. The impossible endpoints $p = 0, 1$ are excluded by centering and unit variance. Each exclusion and each resulting moment interval is a certificate obligation.

On a surviving row, intersect every noise interval with its acquired ceiling Λ . The following table specifies the original comparison cover. A whole-count radius r means $|u| \leq r$; the nearest-count comparison covers the complementary closed phase intervals up to $|u| = 1/2$. Both signs of H are checked separately. At $\lambda = 0$, the actual law is Bernoulli and the separately cited binomial theorem applies.

Noise interval	Central panels	Left panels
$(0, 0.05]$	Small-noise bound, all phases	Small-noise bound, all phases
$[0.05, 0.06]$	Whole count: $ u \leq 0.09$; nearest count outside	Nearest count, all phases
$[0.06, 0.08]$	Whole count: $ u \leq 0.14$; nearest count outside	Nearest count, all phases
$[0.08, 0.10]$	Whole count: $ u \leq 0.19$; nearest count outside	Whole count: $ u \leq 0.19$; nearest count outside
$[0.10, 0.12]$	Whole count: $ u \leq 0.29$; nearest count outside	Whole count: $ u \leq 0.27$; nearest count outside
$[0.12, \Lambda]$	Whole count, all phases	Whole count, all phases

All seams are closed and shared. The comparison's amplitude caps must also be acquired: on the four bands ending at 0.08, 0.10, 0.12, Λ , the central-panel caps are 0.375, 0.325, 0.275, 0.25 and the left-panel caps are 0.40, 0.35, 0.30, 0.275, respectively, intersected with the row's global cap. A narrower cap is an output of confinement before it can be used here. The joint gap cap is recomputed from the same weighted resource in every band. The additional exclusions of Section 5.5 reduce this cover once their required moment and resource bounds have been established.

E Support and strip arguments with predecessor loss

The finite-index proofs use support information before they enter a Fourier or conditional-count calculation. Some of those implications allow a positive predecessor loss, a positive score excess, or an index below 42. They are not consequences of the zero-loss theorem in Section 4. This appendix supplies their analytic arguments with their literal index domains. The numerical polynomial inequalities, moment rows and threshold enclosures remain separately stated certificate premises.

We use the attained positive closed pair and contact identity of Lemma 2.2. Thus the law is fixed throughout every deletion, moment estimate and support contraction; H is its raw sum threshold. The structural bounds (3), the sign-moment inequality of Lemma 2.4, and, when used to acquire a signed moment row, Theorem 2.3, retain their stated hypotheses. Write $C = C_n$ for the attained score, $\rho = \mathbb{E}|X|^3$ and $\beta = \mathbb{E}[X|X|]$. Every certificate below must contain the same law's parameters and the whole closed domain on which it is applied.

E.1 The scale correction

Write $\phi_0 = \phi(0)$. We use the rational comparisons of Lemma 3.1. For $n > 1$ put $s_n = \sqrt{n/(n-1)}$ and $e_n = n(s_n - 1) - 1/2$.

Lemma E.1. *For every integer $n \geq 36$, one has $0 < e_n < 0.384/n$.*

Proof. Put $x = 1/n$ and $c = 0.384 = 48/125$. Direct expansion gives

$$(1-x)(1+x/2+cx^2)^2 - 1 = x^2\{(2c-3/4) - (c+1/4)x + (c^2-c)x^2 - c^2x^3\}.$$

The bracket is decreasing for $x \geq 0$, since $0 < c < 1$. At $x = 1/36$ its exact rational value is $823/4050000 > 0$. Squaring positive quantities gives $s_n < 1 + x/2 + cx^2$ for $0 < x \leq 1/36$. Similarly $(1-x)(1+x/2)^2 - 1 = -3x^2/4 - x^3/4 < 0$, which gives the lower bound. $\square$

E.2 A signed Gaussian envelope

Define, for every real H, t and $n > 1$,

$$\mathcal{H}_n(H, t) = n^{3/2}\{\Phi((H-t)/\sqrt{n-1}) - \Phi(H/\sqrt{n})\} + nt\phi(H/\sqrt{n}) + \frac{H}{2}(t^2 - 1)\phi(H/\sqrt{n}).$$

Lemma E.2 (Direct support at every index). *Let $n \geq 2$, and let (P, H) be an unrestricted attained positive closed maximizing pair, with $C = C_n > 0$. Put $s = \sqrt{n/(n-1)}$, $e = n(s-1) - 1/2$, $A = \sqrt{n-1} \Delta_{n-1}(P)/\rho$, and $d_P = n(A-C)$. Suppose $q \geq 0$, $S > 0$, $E \geq 0$, $r > 0$, $b \geq 0$, $(d_P)_+ \leq q$, $s \leq S$, $e \leq E$, $\rho \leq r$, $|\beta| \leq b$, and $0 < c \leq C$. Define*

$$Q(v) = -v^3 + \frac{3}{2}rv^2 + 3bv + rE, \quad L(v) = \frac{2}{5}(v^2/2 + 1/2 + E) + cQ'(v).$$

If $R > 0$ satisfies

$$Q(R), Q'(R), Q''(R), L(R), L'(R) < 0, \quad 1/5 - 3c < 0,$$

and, for both $\eta \in \{-1, 1\}$,

$$\sup_{H \in \mathbb{R}} \mathcal{H}_n(H, \eta R) + cQ(R) + rSq < 0,$$

then $\text{supp } P \subset (-R, R)$.

Proof. The contact identity (4), the active discrepancy $C\rho/\sqrt{n}$, and the predecessor estimate for this same law imply, at every support point t ,

$$0 \leq \mathcal{H}_n(H, t) + C\{-|t|^3 + \frac{3}{2}\rho t^2 + 3\beta t + \rho e\} + \rho s d_P.$$

Here $n(sA - C) = s d_P + n(s - 1)C$; the scale correction has no omitted term. Also $A \leq C_{n-1}$, hence $(d_P)_+ \leq n(C_{n-1} - C)_+$. Put $z = H/\sqrt{n}$, $y = (H - t)/\sqrt{n}$. Exact differentiation gives

$$\partial_t \mathcal{H}_n = n\phi(z) + Ht\phi(z) - ns\phi(sy).$$

Taylor's theorem and $\|\phi''\|_\infty = \phi(0)$ give $|n\phi(y) - n\phi(z) - Ht\phi(z)| \leq \phi(0)t^2/2$. Moreover $\partial_s[s\phi(sy)] = -\phi''(sy)$, so $|s\phi(sy) - \phi(y)| \leq \phi(0)(s - 1)$. Therefore

$$|\partial_t \mathcal{H}_n(H, t)| \leq \phi(0)(t^2/2 + 1/2 + e) \quad (n \geq 2, H, t \in \mathbb{R}).$$

The norm identity follows by maximizing $(x^2 - 1)e^{-x^2/2}$; its interior extrema occur at $0, \pm\sqrt{3}$. Since $Q''' = -6$, its first three gates imply $Q(v) < 0$ for every $v \geq R$. At $t = \eta v$, the bracket multiplying C is at most $Q(v)$, so its multiplier may be replaced by c . The signed predecessor term is at most rSq . The contact expression is consequently bounded by $\mathcal{H}_n(H, \eta v) + cQ(v) + rSq$, whose derivative in v is at most $L(v)$. Since $L'' = 2/5 - 6c < 0$ and $L(R), L'(R) < 0$, this envelope strictly decreases on the entire closed ray. Its strict endpoint upper bound contradicts contact on both rays, including endpoints, exterior atoms and exterior accumulation points. $\square$

The whole threshold line.

No sampled threshold maximum is needed in this lemma. For $R > 0$, Gaussian differentiation gives

$$|\partial_H^2 \mathcal{H}_n(H, R)| \leq M, \quad M = \frac{\sqrt{n}}{4} \left(\frac{n}{n-1} + 1 \right) + \frac{2}{5}R + \frac{2|R^2 - 1|}{5\sqrt{n}}.$$

This follows from $\|\phi'\|_\infty < 1/4$, $\|\phi''\|_\infty < 2/5$, and $\|\phi'''\|_\infty \leq 2/\pi < 4/5$. For a closed threshold cell of midpoint m and halfwidth w , Taylor's theorem bounds both signs by

$$\pm \mathcal{H}_n(H, R) \leq \pm \mathcal{H}_n(m, R) + |\partial_H \mathcal{H}_n(m, R)|w + Mw^2/2.$$

If $H_0 \geq 5\sqrt{n}$, $H_0 - R \geq 5\sqrt{n-1}$, and $\tau \geq \Phi(-5)$, Mills' bound and monotonicity of $H\phi(H/\sqrt{n})$ give, on both threshold tails,

$$|\mathcal{H}_n(H, R)| \leq n\sqrt{n}\tau + (nR + H_0|R^2 - 1|/2)\phi(H_0/\sqrt{n}), \quad |H| \geq H_0.$$

Finally $\mathcal{H}_n(H, -R) = -\mathcal{H}_n(-H, R)$. Finite rational CDF, density and square-root enclosures on all closed cells, together with these two tails, therefore certify the displayed whole-line endpoint bounds.

In particular the positive-budget support certificate at index 27 uses

$$\begin{aligned} R &= 13/5, \quad q = 813/2000, \quad c = 8171/18000, \quad r = 6413121/5478500, \\ b &= 326352124353/10^{12}, \quad E = 14331929737/10^{12}, \quad S = 1019049330731/10^{12}. \end{aligned}$$

The contrary hypothesis $C_{27} \geq C_E + 2/45$, the finite cap and the signed-moment inequality supply these bounds. Its retained numerical premise checks every polynomial gate and both strict endpoint inequalities on 580 adjacent closed threshold cells covering $[-29, 29]$, and on the two tails. Conditional on those complete checks and the stated moment bounds, this support implication applies at 27 without extending an estimate proved only for $n \geq 36$.

Write $v = n - 1$, $s = \sqrt{n/v}$, $u = |t|$, and

$$A = \frac{u^3 - 3u}{6}, \quad B = \frac{u^4 - 6u^2 + 3}{24\sqrt{v}}, \quad D = \frac{u^5 - 15u}{120v}.$$

Lemma E.3 (Fifth-order identity with a uniform finite remainder). *If $u > \sqrt{3}$, then, with $z_v = H/\sqrt{v}$,*

$$\frac{\mathcal{H}_n(H, \pm u)}{s^3} = \pm A(1 - z_v^2)\phi(z_v) + B\phi^{(3)}(z_v) \mp D\phi^{(4)}(z_v) + R,$$

where the sign of the remainder need not be fixed and

$$|R| \leq E_v^{(5)}(u) := \frac{51u^6}{14400v^{3/2}} + \frac{51(3u^2 - 2)}{960v^{3/2}} + \frac{u}{8v^2}.$$

This statement is uniform over every real H .

Proof. Let $F_w(x) = \Phi(x/\sqrt{w})$, with primes denoting x derivatives. The heat identity $\partial_w F_w = F_w''/2$ and ordinary Taylor's theorem give, with unlabelled derivatives at (v, H) ,

$$\begin{aligned} F_{v+1} &= F_v + F_v''/2 + F_v^{(4)}/8 + R_0, & |R_0| &\leq \sup_{w \in [v, v+1]} \|F_w^{(6)}\|_\infty/48, \\ F_{v+1}' &= F_v' + F_v'''/2 + F_v^{(5)}/8 + R_1, & |R_1| &\leq \sup_{w \in [v, v+1]} \|F_w^{(7)}\|_\infty/48, \\ F_{v+1}'' &= F_v'' + F_v^{(4)}/2 + R_2, & |R_2| &\leq \sup_{w \in [v, v+1]} \|F_w^{(6)}\|_\infty/8. \end{aligned}$$

Expand $F_v(H - t)$ through degree five, with remainder at most $u^6 \|F_v^{(6)}\|_\infty/720$. In

$$\frac{\mathcal{H}_n}{n^{3/2}} = F_v(H - t) - F_{v+1}(H) + tF_{v+1}'(H) - \frac{t^2 - 1}{2}F_{v+1}''(H),$$

the first three derivative coefficients cancel. The next three are

$$\left(\frac{t}{2} - \frac{t^3}{6}\right)F_v''' + \frac{t^4 - 6t^2 + 3}{24}F_v^{(4)} + \left(\frac{t}{8} - \frac{t^5}{120}\right)F_v^{(5)}.$$

The remaining absolute value is bounded by

$$\left\{ \frac{u^6}{720} + \frac{1}{48} + \frac{u^2 - 1}{16} \right\} \sup_{w \in [v, v+1]} \|F_w^{(6)}\|_\infty + \frac{u}{48} \sup_{w \in [v, v+1]} \|F_w^{(7)}\|_\infty.$$

Fourier inversion of the Gaussian, differentiated under its absolutely integrable Fourier transform, gives

$$\|\phi^{(k)}\|_\infty \leq \frac{1}{\pi} \int_0^\infty y^k e^{-y^2/2} dy.$$

Integration by parts yields $\|\phi^{(5)}\|_\infty \leq 8/\pi < 2.55$ and $\|\phi^{(6)}\|_\infty \leq 15\phi_0 < 6$. Since $F_w^{(j)} = w^{-j/2}\phi^{(j-1)}$, the displayed remainder becomes $s^3 E_v^{(5)}(u)$ after multiplication by $n^{3/2}$. The three retained derivative coefficients give the identity. $\square$

Lemma E.4 (A lower tangent bound for the fourth Gaussian derivative). *For every real z ,*

$$\phi^{(4)}(z) \geq \phi_0(3 - 15z^2/2).$$

Proof. The same Gaussian Fourier integral gives $\phi^{(4)}(z) = \pi^{-1} \int_0^\infty y^4 e^{-y^2/2} \cos(yz) dy$. Use the globally valid inequality $\cos w \geq 1 - w^2/2$ inside this integral. Its fourth and sixth Gaussian moments are $3\phi_0$ and $15\phi_0$, respectively. $\square$

Lemma E.5 (Uniform signed fifth-order envelope). *Suppose $u > \sqrt{3}$ and $A, B, D \geq 0$. If $W = A - 15D/2 > 0$, put*

$$T = A - 3D + \frac{9B^2}{4W}.$$

If $T \geq A/2 + 2B + 3D$, then, for every real H and either sign,

$$\mathcal{H}_n(H, \pm u) \leq s^3 \{\phi_0 T + E_v^{(5)}(u)\}.$$

Proof. Put $z_v = H/\sqrt{v}$. For positive $t = u$ and $|z_v| \leq 1$, divide the retained terms by ϕ_0 and use the preceding tangent bound to obtain

$$A - 3D - (A - 15D/2)z_v^2 + 3B|z_v| \leq A - 3D + 9B^2/(4W).$$

Here $(1 - z_v^2)\phi(z_v) \leq (1 - z_v^2)\phi_0$ and $|\phi^{(3)}(z_v)| \leq 3|z_v|\phi_0$ on this interval. For $|z_v| \geq 1$ the first term is nonpositive, and the Fourier bounds $\|\phi^{(3)}\|_\infty < 2\phi_0$ and $\|\phi^{(4)}\|_\infty \leq 3\phi_0$ give $2B + 3D$. For negative $t = -u$, the first term has global upper bound $2e^{-3/2}A\phi_0 < A\phi_0/2$, and the other terms give $(2B + 3D)\phi_0$. The hypothesis on T covers both exterior cases. The inequality $2e^{-3/2} < 1/2$ follows from four positive terms of the exponential series at $3/2$. Finally add the uniform remainder. $\square$

E.3 A direct Gaussian bound on both complete rays

Lemma E.6 (A global cubic Gaussian bound). *For every integer $n \geq 2$ and all real H, t ,*

$$|\mathcal{H}_n(H, t)| < \frac{|t|^3}{15} + \frac{|t|}{5} + \frac{1}{2}.$$

Proof. Put $z = H/\sqrt{n}$, $u = z - t/\sqrt{n}$, and $f(\lambda) = \Phi(u/\sqrt{1-\lambda})$. Taylor expansion in the spatial variable through degree two gives

$$n^{3/2}\{\Phi(u) - \Phi(z)\} = -nt\phi(z) - \frac{1}{2}Ht^2\phi(z) + R_{\text{tr}}, \quad |R_{\text{tr}}| \leq \phi(0)|t|^3/6,$$

using $\|\phi''\|_\infty = \phi(0)$. Expansion of f through degree one at zero gives

$$n^{3/2}\{f(1/n) - f(0)\} = \frac{1}{2}\sqrt{n}u\phi(u) + R_{\text{sc}}.$$

For $0 \leq \lambda \leq 1/2$, writing $r = u/\sqrt{1-\lambda}$, differentiation gives

$$f''(\lambda) = \frac{r(3-r^2)\phi(r)}{4(1-\lambda)^2}.$$

The exact maxima of $|r|\phi(r)$ and $|r|^3\phi(r)$ are $\phi(0)e^{-1/2} < 1/4$ and $3\sqrt{3}\phi(0)e^{-3/2} < 1/2$ respectively. These strict rational bounds follow from $\phi(0) < 2/5$, $e > 8/3 > 64/25$, $e^{3/2} > 21/5$, and $3\sqrt{3} < 21/4$; the exponential inequalities follow from finite positive terms of their series. Therefore $\|f''\|_{[0,1/2]} < 5/4$, and

$$|R_{\text{sc}}| < \frac{5}{8\sqrt{n}} \leq \frac{5}{8\sqrt{2}} < \frac{1}{2}.$$

Substitution in the definition of $\mathcal{H}_n$ cancels the terms in nt and Ht^2 , leaving

$$\mathcal{H}_n(H, t) = R_{\text{tr}} + R_{\text{sc}} + \frac{1}{2}\sqrt{n}\{u\phi(u) - z\phi(z)\}.$$

Since $(x\phi(x))' = -\phi''(x)$, the last term has absolute value at most $\phi(0)|t|/2 < |t|/5$ for $t \neq 0$, and is zero at $t = 0$. Together with $\phi(0)/6 < 1/15$ and the strict scale-remainder bound, this proves the assertion for both signs and every threshold. In particular there is no restriction to sampled thresholds or to a bounded part of a support ray. $\square$

E.4 Finite bands and the exact support polynomial

For integers $36 \leq N \leq L$ and tested radii $u \geq 5/2$, use the simultaneous endpoint bounds

$$B_+ = \frac{u^4 - 6u^2 + 3}{24\sqrt{N-1}}, \quad D_+ = \frac{u^5 - 15u}{120(N-1)}, \quad D_- = \frac{u^5 - 15u}{120(L-1)}.$$

For an infinite index range, set $D_- = 0$ instead. Put

$$W_* = A - 15D_+/2, \quad T_* = A - 3D_- + \frac{9B_+^2}{4W_*}.$$

The proof of Lemma E.5 applies uniformly on the whole band if $W_* > 0$ and $T_* \geq A/2 + 2B_+ + 3D_+$. The restriction $u \geq 5/2$ ensures that all the displayed A, B_+, D_+, D_- are nonnegative. In particular the favorable term is frozen at the upper index endpoint, and the adverse terms at the lower one. No monotonicity of a signed coefficient is assumed.

Let (P, H) be an attained positive closed maximizing pair, standardized to mean zero and variance one. Write $C = C_n \geq C_E$, $\rho = \mathbb{E}|X|^3$, $\beta = \mathbb{E}[X|X|]$, and $d = n(C_{n-1} - C_n)_+$. The finite contact/predecessor bound is

$$\rho I(t) \leq \mathcal{H}_n(H, t) + C\{-|t|^3 + 3\rho t^2/2 + 3\beta t + \rho e_n\} + \rho s_n d,$$

with $I = 0$ on the entire closed support. Suppose also, for a finite band, that $C_n - C_E \geq K/n$, where $K > 0$. The published finite cap gives

$$c = 0.4095 + K/L < C_n, \quad \rho < r = \min \left\{ \frac{0.3031(0.646)}{c - 0.3031}, \frac{0.3322(0.429)}{c - 0.3322} \right\}.$$

For the unrestricted nonnegative case put $K = 0$, $c = 0.4095$. The sign-moment inequality $\rho \geq (1 + \beta^2)/\sqrt{1 - \beta^2}$ supplies $|\beta| < b$ whenever $(1 + b^2)^2 > r^2(1 - b^2)$ and $0 < b < 1$. All these statements concern the same current law. The lower endpoint cannot replace L in K/L .

When a preceding resource enclosure supplies sharper actual bounds $\rho \leq r$ and $|\beta| \leq b$, these may instead be used in the same polynomials. Their validity is the only extra premise.

Use positive rational ℓ, S with $\ell^2 \leq N - 1$ and $S^2 > N/(N - 1)$. For $u = |t|$ define

$$Q(u) = -u^3 + 3ru^2/2 + 3bu + 0.384r/N.$$

First check $Q < 0$, before replacing C by its lower bound c . Replace $1/\sqrt{N-1}$ by $1/\ell$ in the adverse remainder and use $\widehat{B}_+ = (u^4 - 6u^2 + 3)/(24\ell) \geq B_+$ in the linear term of the domination gate. Explicitly, a sufficient rational remainder is

$$E_*(u) = \frac{51u^6}{14400(N-1)\ell} + \frac{51(3u^2 - 2)}{960(N-1)\ell} + \frac{u}{8(N-1)^2}.$$

The finite-band formulas are used only where their denominators and listed sign gates have been proved. Strict negative contact for every n in the band follows from the polynomial inequality

$$\frac{2}{5}S^3 \left\{ (A - 3D_-)W_* + \frac{(u^4 - 6u^2 + 3)^2}{256(N-1)} \right\} + S^3 E_* W_* + \{cQ + rSq\}W_* < 0.$$

The second polynomial gate is $W_* > 0$. The third is

$$\{A/2 - 3D_- - 3D_+ - 2\widehat{B}_+\}W_* + \frac{(u^4 - 6u^2 + 3)^2}{256(N-1)} > 0.$$

This is a rational sufficient version of the cleared-denominator domination hypothesis; its quadratic B_+^2 term still uses the exact factor $1/(N-1)$. The bound on T_* is positive, so the replacement $\phi_0 < 0.4$ and $s_n < S$ both have their correct direction.

The retained exact acceptance condition tests these polynomials on all adjacent closed intervals of length 0.05 from R to 4.5. Positive Bernstein coefficients prove their signs on every point of each cell, including endpoints. On the rays $u \geq 4.5$ use Lemma E.6, $|\mathcal{H}_n(H, t)| < u^3/15 + u/5 + 1/2$. The ray certificate must prove that the resulting cubic and its first three derivatives are strictly negative at 4.5, and that the coefficient Q and its derivative gates permit replacing C by c on the whole ray. This excludes the whole rays. Thus, if $d \leq q$, actual support is in $(-R, R)$, including exclusion of arbitrarily small exterior atoms and exterior accumulation points.

E.5 Retained support rows and their application domains

The older local route specified $K = 0$, $R = 3.84$, $q = 0.19$ on

$$[45, 49], [50, 59], [60, 89], [90, 134], [135, 199], [200, \infty).$$

That route used only indices at least 47; this paragraph records its literal support premises and makes no global exclusion claim. The last row uses $D_- = 0$; there is no upper index restriction. A complete acceptance must reconstruct all closed u cells and both exterior rays. Missing an index band or the infinite row is an acceptance failure.

The older finite-index route specified five further support rows:

$[N, L]$	K	R	q
$[40, 40]$	1.5	3.3	0.88
$[41, 49]$	1.5	2.9	1.5/40
$[45, 45]$	0.7	3.5	0.8341
$[46, 98]$	0.7	3.5	0.7/45
$[46, 46]$	0.54	3.4	79/450

The second and fourth are used only on $41 \leq n \leq 45$ and $46 \leq n \leq 50$, respectively. Retaining their originally certified upper endpoints makes their inequalities weaker, so restricting the application introduces no new arithmetic assertion. These are support implications, not yet exclusions: the separately proved and accepted resource calculation would supply the latter. They are not an independent proof of Assumption 7.1.

E.6 One bounded-threshold series

Put $c_k = (-1)^k/[2^k k!(2k+1)]$, and let $(x)_j$ denote the rising factorial. Define the following rational polynomials:

$$P_j(H, t) = \sum_{k=0}^{j+1} \frac{c_k (k+1/2)_{j+1-k}}{(j+1-k)!} (H-t)^{2k+1} - c_{j+1} H^{2j+3} \\ + \frac{(-1)^{j+1} t H^{2j+2}}{2^{j+1} (j+1)!} + \frac{(-1)^j H^{2j+1} (t^2 - 1)}{2^{j+1} j!}.$$

In particular,

$$P_0 = (t^3 - 3t)/6, \quad P_1 = \frac{3t - t^3}{4} H^2 + \frac{t^4 - 6t^2 + 3}{8} H + \frac{-t^5 + 10t^3 - 15t}{40}.$$

Fix $h > 0$, $E(x) = (193/71)^{\lceil x \rceil}$, and

$$B_r = \frac{7(h+r)}{6} E((h+r)^2/6) + \left(h+r + \frac{h(r^2+1)}{8} \right) E(h^2/8).$$

Lemma E.7. For every $n \geq 4$, $|H| \leq h$ and $|t| \leq r$,

$$\left| \mathcal{H}_n / \phi(0) - \sum_{j=0}^5 P_j / n^j \right| \leq 4^7 B_r / n^6.$$

Proof. Put $z = 1/n$. Integration of the Gaussian exponential series gives $\mathcal{H}_n / \phi(0) = S(z)/z$, where

$$\begin{aligned} S(z) &= \sum_{k \geq 0} c_k \{ (H - t)^{2k+1} (1 - z)^{-k-1/2} - H^{2k+1} \} z^k \\ &\quad + t e^{-H^2 z/2} + \frac{1}{2} z H (t^2 - 1) e^{-H^2 z/2}. \end{aligned}$$

Its constant coefficient vanishes, and expansion of the binomial factor gives $[z^{j+1}]S = P_j$. At $z_0 = 1/4$, the sum obtained by replacing all coefficients by their absolute-value majorants is at most B_r : use $(1 - z_0)^{-1/2} < 7/6$, $1/(2k+1) \leq 1$, and $e < 193/71$. The latter bound follows, for example, by summing the exponential series through degree six and bounding the rest by a geometric series with ratio $1/8$. All products and rearrangements are therefore absolutely convergent. The discarded terms have degrees at least seven in S , so their sum at $0 \leq z \leq z_0$ is at most $(z/z_0)^7 B_r$. Divide by z to get the result. This proves a finite uniform bound, including every threshold in the stated interval and every $n \geq 4$. $\square$

E.7 A common contact test for radius and strip refinement

Suppose $n \geq 47$, $0 \leq C - C_E \leq 0.7/n$, $d = n(C_{n-1} - C)_+ \leq 0.19$ and $|X| \leq R_0$. A previously acquired row gives

$$\rho \leq r, \quad 0 \leq b_l \leq \beta \leq b_u < 1, \quad |H| \leq h.$$

Define $Q(t) = -|t|^3 + 3\rho t^2/2 + 3\beta t$ and $L_P(t) = C_E Q(t) + \phi(0)(t^3 - 3t)/6$. For $t > 0$, upper polynomials for $L_P(t), Q(t)$ are

$$(1/15 - 0.4095)t^3 + 1.5(0.40974)rt^2 + (3(0.40974)b_u - 0.3988/2)t, \quad -t^3 + 1.5rt^2 + 3b_u t.$$

For $t < 0$, write $u = |t|$ and use

$$(-0.3988/6 - 0.4095)u^3 + 1.5(0.40974)ru^2 + (0.2 - 3(0.4095)b_l)u, \quad -u^3 + 1.5ru^2 - 3b_l u.$$

These coefficient choices are valid because $b_l \geq 0$ is checked first. On any closed cell of one sign, let L_J be an upper bound for the first polynomial and let $Q_J \geq 0$ bound the positive part of the second. For $j = 1, 2, 3$, take m_j to be the positive part of the largest tensor Bernstein coefficient of P_j on $[-h, h] \times J$. For $j = 4, 5$, the absolute monomial bound $\sum_{a,b} |[H^a t^b]P_j| h^a R_J^b$ suffices, where $R_J = \max_{t \in J} |t|$. The support contact is then bounded above by

$$L_J + \frac{0.7Q_J}{47} + 0.4 \left(\sum_{j=1}^5 \frac{m_j}{47^j} + \frac{4^7 B_{R_J}}{47^6} \right) + r \left(0.19S + \frac{0.384(0.425)}{47} \right), \quad S > \sqrt{47/46}. \quad (187)$$

Indeed substitute the finite series in contact, split $C = C_E + (C - C_E)$, and use $C < 0.40974 + 0.7/47 < 0.425$. Every error retained in (187) is nonnegative and decreases with n . A strictly negative rational result therefore excludes the entire cell for every integer $n \geq 47$, including atoms and accumulation points.

For a coordinate strip, conditional centering gives

$$X = \ell(Y - p), \quad \ell = \sqrt{(1 - \theta)/(p(1 - p))}, \quad \beta = (1 - \theta)(1 - 2p) + \kappa, \quad |\kappa| \leq \theta.$$

If $p \in [p_l, p_u] \subset (0, 1/2)$ and $\theta \leq \Theta$, put $\ell_l \leq \sqrt{(1 - \Theta)/(p_u(1 - p_u))}$. The forbidden strip $Y \notin (-0.55, 1.55)$ maps into

$$X \leq -\ell_l(p_l + 0.55), \quad X \geq \ell_l(1.55 - p_u).$$

Apply (187) between these thresholds and $\pm R_0$. An interval beyond the known radius is empty; an equal endpoint is a singleton that is still checked. If only an upper β bound is given, use $b_l = (1 - \Theta)(1 - 2p_u) - \Theta$, after checking its nonnegativity.

The retained schedule lists the following conditional consequences; each requires the complete numerical tests just described:

input rows	prior radius	conclusion	closed cells
10 central preparation rows	3.84	$ X < 2.5$	1080
45 central rows	2.5	$-0.55 < Y < 1.55$	1431
38 left rows	2.7	$-0.55 < Y < 1.55$	3073

The corresponding upper contact bounds must be strictly negative on every listed cell. Each input moment interval must first be established by the preceding resource estimate. The left radii are applications of the all-threshold support polynomial, with $(N, L, K, q) = (47, \infty, 0, 0.19)$ and the actual moment caps

$$(r, b, R) = (1.390, 0.430, 3), \quad (1.2563, 0.3926, 2.7).$$

Conditional on the corresponding polynomial checks, the same heat proof supplies the initial radius, both left contractions, and the five finite-base rows. The bounded-threshold series supplies the central contraction and both coordinate strips.

E.8 Relation to the zero-loss bound

The 65 contractions of Section 4 use zero predecessor loss and moment bounds established in the stated order. The index-27 condition, the five positive-excess finite-band cases and their support strips have different hypotheses. Their proofs require the separate polynomial inequalities and closed-cell bounds given above.

F Finite-index reductions and certificate conditions

The finite-index arguments use one standardized iid marginal at each actual index, with the notation $S_n = X_1 + \dots + X_n$ and $R_n(P) = \sqrt{n} \Delta_n(P)/\rho(P)$. The notation $C_n(P)$ below also denotes $R_n(P)$; C_n is its supremum over standardized laws. The numerical conclusions are established by the verified certificate conditions in the public release (`full_pass: "ESTABLISHED"`).

The first part develops the conditional-count bounds shared by several finite-index comparisons. The small-index reductions then proceed by a closed cubic dual at $n = 1$ (Section F.2), a rational tensor majorant at $n = 2$ (Section F.3), independence-null polynomials at $n = 3$ (Section F.4), a canonical reference law and physical Prawitz comparison at $n = 4$ (Section F.5), and a moment-domain cover with a gap-free count comparison at $n = 5$ (Section F.6). The later sections supply signed moment and gap refinements, an exact phase helper for $n \in \{6, 7, 8, 9\}$, a B1 sine interface for $n = 26, \dots, 30$, and conditional resource transfers at floors 25, 31 and 40. Together with the complete thirteen-role partition in Section 7 (N1, G2, G3, G4/R00T4, C5-one-free, projected-C69, projected-C1019, current-C2024, current-C25, current-C2639, current-C4042, current-C4348, and independent-tail49), these reductions and verified certificates cover all finite indices and the unbounded tail.

In count calculations $m = \mathbb{E}X_+$ and h is the reciprocal span. In the later index-transfer section m denotes the actual integer index, and the first positive mean is written μ . The symmetrization section states its separate first-absolute-moment convention for h .

Assumption F.1 (Analytic inputs and their scope). Each application requires the following inputs at its actual index, for the same marginal, signed branch, raw threshold and moments.

1. Global contrary-witness attainment, positive closed-edge orientation, contamination contact, and the stated global caps, for the unrestricted maximizing law.
2. The actual-index contact exclusion on both complete support rays and the whole threshold line, with every Taylor remainder and tail included and favourable coefficients evaluated on their stated index domains. The separate two- and three-index support proofs are given below.
3. The sharp moment-domain inequality $\rho^2(\gamma^2 + 4) \geq (\gamma^2 + 2)^2$, together with its stated moment hypotheses. Its use only enlarges the realizable moment domain.
4. The closed-CDF Prawitz majorant with the exact normalization used below, and the complete four-term Fourier bound, including the infinite normal tail.
5. The gap functional with its half-open endpoint convention and conserved contact identity for the same law, including all nonnegative slack terms that are discarded in the displayed inequalities.
6. The resource enclosure and finite propagation implications on their original index domains, with a genuine acquired central gap, complete signed resource fields, positive divisors and the same physical defect throughout.
7. Directed rational enclosures of elementary constants, Gaussian functions and trigonometric functions, with analytic remainder bounds and their domain gates.

The corresponding analytic arguments are located as follows. The attainment and contact proofs are in Section 2; the external moment caps are in (3); and the sharp moment domain is proved in Section 2.3. The conserved identity is in Section 5.4. The Prawitz construction, Fourier bounds and resource enclosure are in Sections A.1 and A.2.1, including Section A.14. Every application also retains the numerical conditions and index restrictions of the result it uses.

F.1 The conditional count comparison

We develop the outgoing and incoming probability bounds used after support and resource acquisition, keeping the same marginal throughout.

F.1.1 One marginal, all conditional counts

Fix $n \geq 2$ and a standardized marginal with $\rho < \infty$. Use the sign coordinates of (11): $p = \mathbb{P}(X > 0)$, $q = 1 - p$, $v = pq$, $m = \mathbb{E}X_+ = \mathbb{E}X_- > 0$, with zero atoms assigned to the nonpositive class. In the reciprocal-span notation,

$$h = \ell^{-1} = v/m = \sqrt{v + s}, \quad I = B_0.$$

Thus the main representation becomes

$$hX = I - p + U, \quad \mathbb{E}(U \mid I) = 0. \quad (188)$$

Retain the same conditional laws U_0, U_1 and moments $w_i = \mathbb{E}U_i^2$, $\mu_i = \mathbb{E}U_i^3$. Their total noise is $s = qw_0 + pw_1 = h^2 - v$ and $\lambda = ns$. If $s > 0$, write $\alpha = pw_1/s \in [0, 1]$, so that $w_0 = s(1 - \alpha)/q$ and $w_1 = s\alpha/p$. At $s = 0$ both conditional laws vanish and any $\alpha \in [0, 1]$ is allowed, without division by s . Higher moments below require the stated bounded-support hypotheses.

Use the conditional sums W_j from (35), and write $b_j(p) = \binom{n}{j}p^j q^{n-j}$. Conditional independence gives

$$V_j := \mathbb{E}W_j^2 = (n - j)w_0 + jw_1 = \lambda \left(1 + \frac{(j - np)(\alpha - p)}{npq} \right). \quad (189)$$

Lemma F.2 (Closed count comparison). *For $z = np + hH = k + u$, with $0 \leq k \leq n$,*

$$\mathbb{P}(S_n \leq H) \leq F_{\text{Bin}(n,p)}(k) - b_k(p)\mathbb{P}(W_k > u) + \sum_{j>k} b_j(p)\mathbb{P}(W_j \leq u - (j - k)). \quad (190)$$

The outgoing event is strict and every incoming event is closed.

Proof. In the exact signed identity (36), discard the nonpositive outgoing terms with count $j < k$ and retain the $j = k$ term. The strict/closed conventions in that identity include every event-plane atom. $\square$

All ensuing formulas hold on either probability half. On a closed interval $[l, r] \subset (0, 1)$, the exact range of $v = p(1 - p)$ is

$$v_- = \min\{l(1 - l), r(1 - r)\}, \quad v_+ = \begin{cases} 1/4, & l \leq 1/2 \leq r, \\ \max\{l(1 - l), r(1 - r)\}, & \text{otherwise.} \end{cases}$$

The maximum of b_j is attained at j/n if that point belongs to the interval, and otherwise at an endpoint; its minimum is at an endpoint. Also

$$\frac{d}{dp} F_{\text{Bin}(n,p)}(k) = -\frac{(n - k)b_k(p)}{1 - p} \leq 0.$$

Thus no reflection of p , the count, or the signed moments is needed when a parameter interval meets or lies to the right of $1/2$.

F.1.2 Moment identities and necessary contractions

Suppose the acquired conditional supports are

$$U_0 \in [-O_0, a_0], \quad U_1 \in [-a_1, O_1], \quad O_i, a_i \geq 0. \quad (191)$$

For a centered variable $T \in [-L, R]$, of variance w , integration of nonnegative polynomials gives

$$w \leq LR, \quad w^2/L - Lw \leq \mathbb{E}T^3 \leq Rw - w^2/R, \quad \mathbb{E}T^4 \leq (R - L)\mathbb{E}T^3 + LRw. \quad (192)$$

For the cubic bounds use respectively $(T + L)(T - w/L)^2 \geq 0$ and $(R - T)(T + w/R)^2 \geq 0$; for the fourth bound multiply $(T + L)(R - T) \geq 0$ by T^2 . If either endpoint is zero, centering forces $T = 0$; all corresponding moment terms vanish.

For later substitutions, the identities (12) read as follows in reciprocal-span units:

$$\rho h^3 = v(1 - 2v) + 3v(w_0 + w_1) - q\mu_0 + p\mu_1, \quad (193)$$

$$\gamma h^3 = v(1 - 2p) + 3v(w_1 - w_0) + q\mu_0 + p\mu_1, \quad (194)$$

$$\beta = \frac{v(1 - 2p) + s(2\alpha - 1)}{v + s}, \quad m = \frac{v}{\sqrt{v + s}}, \quad (195)$$

Here $\gamma = \mathbb{E}X^3$ and $\beta = \mathbb{E}[X|X|]$; the differing signs of μ_0 follow from the two actual sign classes. The expressions for β and m use $pw_1 - qw_0 = s(2\alpha - 1)$ and $h^2 = v + s$.

These identities give necessary closed-domain contractions. For example, if $m_- \leq m \leq m_+$, then

$$\lambda = n(v^2/m^2 - v), \quad v \geq m_-^2.$$

On the resulting closed variance interval a lower bound is obtained by minimizing $n(v^2/m_+^2 - v)$, and, when $m_- > 0$, an upper bound by maximizing $n(v^2/m_-^2 - v)$. These are convex quadratics: the minimum occurs at an endpoint or its stationary point, and the maximum at an endpoint. If θ is the within-sign variance of X , then

$$\theta = \frac{s}{v + s}, \quad \lambda = \frac{nv\theta}{1 - \theta}.$$

Thus $\theta \leq \Theta < 1$ implies $\lambda \leq nv_+\Theta/(1 - \Theta)$.

If a simultaneous acquired resource gives

$$\frac{a_0 + a_1}{d_0} + \frac{\lambda}{\Lambda} \leq 1, \quad d_0, \Lambda > 0, \quad (196)$$

then (192) also implies

$$\lambda \leq \left\{ \frac{1}{\Lambda} + \frac{1}{nd_0} \min_{p, \alpha} \left[\frac{1 - \alpha}{(1 - p)O_0} + \frac{\alpha}{pO_1} \right] \right\}^{-1}.$$

Here the minimum is over the incoming probability/allocation rectangle. First minimize at an α endpoint. For each such endpoint the remaining function is $A/(1 - p) + B/p$, with $A, B \geq 0$; its minimum is at an endpoint or the stationary value $A + B + 2\sqrt{AB}$. A smaller rational lower bound for this minimum remains valid. These independent caps intersect the working λ interval; they do not replace the intercept Λ in (196).

Signed β bounds in (195) similarly restrict α , or give a necessary positive λ floor. For example, if $\beta \leq b < 1 - 2p_+$, then

$$s(b + 1 - 2\alpha) \geq v(1 - 2p - b) \geq v_-(1 - 2p_+ - b) > 0.$$

A nonpositive upper bound for the left coefficient excludes the law; otherwise division gives a valid lower bound for s . Every exclusion uses a strict contradiction. Equal interval endpoints and zero variance remain included.

There is also a rho-derived noise floor. The rough cubic bounds $-\mu_0 \leq O_0w_0$, $\mu_1 \leq O_1w_1$ in (193) imply

$$r(v + s)^{3/2} \leq v(1 - 2v) + Ms, \quad M = \max_{p, \alpha \text{ corners}} \{(3p + O_0)(1 - \alpha) + (3q + O_1)\alpha\},$$

when $\rho \geq r$. The maximum is at rectangle corners because the expression is affine in each variable. Convexity gives

$$A(v) := rv^{3/2} - v + 2v^2 \leq (M - \frac{3}{2}r\sqrt{v})s.$$

Let $t \leq \sqrt{v_-}$ be a nonnegative rational lower enclosure and $A_- = rv_-t - v_- + 2v_-^2 > 0$. Then $rt > 1 - 2v_-$, so $A'(v_-) \geq 3rt/2 - 1 + 4v_- > 1/2 + v_- > 0$. Since $A'' > 0$, $A(v) \geq A_-$ on the whole variance interval. Thus $M - 3rt/2 \leq 0$ is impossible; otherwise $\lambda \geq nA_-/(M - 3rt/2)$. If the positive A_- gate fails, this optional floor is simply omitted.

F.1.3 The physical-defect halfplane

Assume compact support and a genuine central gap $(-A, B)$, with $A, B > 0$. Use τ, J, D from (14), $\sigma = \sqrt{10}$, and the nonnegative residuals $T_{\pm}, R_{\pm}$ of Lemma 5.1. Its outer-gap estimates and the identity (16) give

$$D \geq \sigma\rho - \gamma - 6mAB + 6BR_+ + 6AR_- . \quad (197)$$

We now express this inequality in the common conditional cubic moments.

Now $A = (p - a_0)/h$, $B = (q - a_1)/h$. Substitution in (197), followed by (193)–(194), yields

$$\begin{aligned} Dh^3 &\geq B_{\sigma}(p) + 6vf(a_0, a_1) \\ &\quad + qw_0[3p(\sigma + 3) - 6a_0] + pw_1[3q(\sigma + 1) - 6a_1] \\ &\quad - (\sigma + 1)q\mu_0 + (\sigma - 1)p\mu_1, \\ f(a, b) &= a + b - a^2 - b^2 - ab, \quad B_{\sigma}(p) = v[(\sigma - 1) + 2p - (2\sigma + 6)v]. \end{aligned} \quad (198)$$

This supplies another necessary halfplane in the two common cubic moments. Inserting the one-sided cubic bounds from (192) gives the stronger lower bound

$$\begin{aligned} Dh^3 &\geq B_{\sigma} + 6vf(a_0, a_1) \\ &\quad + q \left\{ [3p(\sigma + 3) - (\sigma + 7)a_0]w_0 + (\sigma + 1)w_0^2/a_0 \right\} \\ &\quad + p \left\{ [3q(\sigma + 1) - (\sigma + 5)a_1]w_1 + (\sigma - 1)w_1^2/a_1 \right\}. \end{aligned} \quad (199)$$

At $a_i = 0$, its conditional variance is zero and its contribution is defined as zero. Lower substitution for a variance in a linear term requires a nonnegative coefficient; otherwise a signed interval bound is used. The concave polynomial f has its minimum on a clipped rectangle at a vertex. On $a + b \leq d < 1/2$ it is also increasing in each coordinate. Any stronger monotone inversion of (199) requires its own derivative-sign bound on the whole interval; failure of that bound gives no exclusion.

F.1.4 Strict outgoing bounds

Lemma F.3 (A family of polynomial minorants). *For $a, c > 0$, put*

$$Q_{a,c}(x) = x(ca - x)(x + a)^2, \quad K_c = \max_{0 \leq t \leq c} t(c - t)(t + 1)^2.$$

Then $K_c > 0$ and

$$Q_{a,c}(x) \leq a^4 K_c \mathbf{1}_{\{x > 0\}} \quad (x \in \mathbb{R}).$$

If W is centered, of variance V , third moment M_3 and fourth moment M_4 , then, writing $B_1 = (c - 2)a$, $B_2 = (2c - 1)a^2$, $B_3 = ca^3$,

$$\begin{aligned} \mathbb{P}(W > u) &\geq \frac{1}{a^4 K_c} \{ -M_4 + (B_1 + 4u)M_3 + (B_2 - 3B_1u - 6u^2)V \\ &\quad - u^4 - B_1u^3 + B_2u^2 - B_3u \}. \end{aligned} \quad (200)$$

Proof. The polynomial is nonpositive on $x \leq 0$ and $x \geq ca$. On the remaining interval scale by a . Expand $Q_{a,c}(W - u)$ and use $\mathbb{E}W = 0$. $\square$

The positive maximizer defining K_c is

$$t_c = \frac{3c - 2 + \sqrt{9c^2 + 4c + 4}}{8}.$$

For $c = 2$, $K_2 = 9/4 + 3\sqrt{3}/2$. There are also exact rational normalizations. For $c = 3/2$,

$$2 - t(3/2 - t)(t + 1)^2 = (t - 1)^2((t + 5/4)^2 + 7/16).$$

More generally, if $t_* > 0$,

$$c = \frac{2t_*(2t_* + 1)}{3t_* + 1}, \quad K = \frac{t_*^2(t_* + 1)^3}{3t_* + 1},$$

then

$$K - t(c - t)(t + 1)^2 = (t - t_*)^2 \left[\left(t + \frac{t_*^2 + 3t_* + 1}{3t_* + 1} \right)^2 + \frac{t_*^2(2t_*^2 + 4t_* + 1)}{(3t_* + 1)^2} \right].$$

These identities certify the pointwise inequality globally, rather than at selected values of the random variable.

For $W = W_k$, independence gives

$$M_3 = (n - k)\mu_0 + k\mu_1, \quad M_4 = (n - k)\mathbb{E}U_0^4 + k\mathbb{E}U_1^4 + 3\{V_k^2 - (n - k)w_0^2 - kw_1^2\}.$$

Using (192) makes the outgoing lower bound affine in the *same* pair (μ_0, μ_1) used by the incoming bounds. An alternative independent bound is

$$\mathbb{E}(U_i^4 - 4uU_i^3) \leq w_i \max\{L_i^2 + 4uL_i, R_i^2 - 4uR_i\}, \quad U_i \in [-L_i, R_i].$$

It follows by maximizing the convex quadratic $x^2 - 4ux$ on the support. A phase interval crossing an endpoint switch retains both cases.

For $u < 0$, Cantelli also gives $\mathbb{P}(W_k > u) \geq u^2/(V_k + u^2)$. A further elementary bound is useful when a positive-class residual is forced upwards. For $T \in [-a, O]$, $\mathbb{E}T = 0$, $\mathbb{E}T^2 = w$, and $-a \leq b < O$,

$$\mathbb{P}(T > b) \geq \max\left\{0, \frac{w - ab}{(O + a)(O - b)}\right\}.$$

Indeed $(T + a)(T - b)$ is nonpositive on $T \leq b$, and is at most $(O + a)(O - b)$ otherwise. For the own count, choose a buffer $B \geq 0$ and set $b = u + B + (k - 1)a_1$. Any one of the k class-one variables exceeding b makes their sum exceed $u + B$. Independence gives the lower probability $1 - (1 - r)^k$. The other group is at least $-B$ with probability at least

$$\max\left\{\frac{B^2}{(n - k)w_0 + B^2}, \frac{B}{B + (n - k)a_0}\right\}.$$

The two groups are independent, so their lower probabilities may be multiplied. At $B = 0$ use one only if the other-group variance is zero, and otherwise zero. Thresholds outside $[-a, O]$ use the corresponding certain or empty strict event. No atomic CDF is differentiated.

F.1.5 Incoming exponential bounds

For a centered $Y \leq B$, $B > 0$, of variance at most w ,

$$\mathbb{E}e^{tY} \leq 1 + \frac{w}{B^2}(e^{tB} - 1 - tB) \quad (t \geq 0). \quad (201)$$

The pointwise inequality follows because $(e^{tx} - 1 - tx)/x^2 = t^2 \int_0^1 (1-r)e^{trx} dr$ is increasing in x . Taking expectations proves (201).

Lemma F.4 (Sharp one-sided MGF and its tangent). *For the same variables,*

$$\mathbb{E}e^{tY} \leq M(w; B, t) := \frac{B^2 e^{-tw/B} + we^{tB}}{B^2 + w}.$$

For fixed B, t , the function M is increasing and concave in $w \geq 0$. Hence every tangent $A + Dw$ bounds it above on that whole half-line, with $A \geq 1$, $D \geq 0$.

Proof. For $w > 0$, put $a = -w/B$. The quadratic matching e^{tx} and its derivative at a , and matching its value at B , majorizes e^{tx} for every $x \leq B$: its Hermite remainder has sign $(x-a)^2(x-B) \leq 0$. Its quadratic coefficient is nonnegative. The two-point law on $\{a, B\}$ with masses $B^2/(B^2+w)$, $w/(B^2+w)$ has mean zero and variance w , and therefore evaluates this majorant's expectation exactly. Variance upper bounds remain valid. Differentiating gives

$$M'(w) = \frac{B^2}{(B^2+w)^2} \left[e^{tB} - e^{-tw/B} \left(1 + \frac{t(B^2+w)}{B} \right) \right] \geq 0.$$

For $x = t(B^2+w)/B$,

$$M''(w) = \frac{B^2 e^{-tw/B}}{(B^2+w)^3} [x^2 + 2x + 2 - 2e^x] \leq 0.$$

At zero variance the law is concentrated at zero. Concavity proves the tangent assertion, including its value at zero. $\square$

Apply these bounds to $-U_0$ and $-U_1$, whose upper supports are O_0, a_1 . Conditional independence bounds $\mathbb{E}e^{-tW_j}$ by the product of the two conditional MGF bounds, with multiplicities $n-j, j$. A zero upper support forces that conditional variable to vanish and contributes the factor one. The original common allocation may be retained. For example, tangents to the sharp bounds give a product

$$(A_0 + D_0 c_0 (1-\alpha))^{n-j} (A_1 + D_1 c_1 \alpha)^j, \quad c_0 = \frac{\lambda_+}{n(1-p_+)}, \quad c_1 = \frac{\lambda_+}{np_-}.$$

Its logarithm is concave. Its maximum on a closed allocation interval is at an endpoint or its sole stationary point; zero-slope cases are monotone or constant. The same α is used in both factors.

If $M \geq \mathbb{E}e^{tY}$, $\mathbb{E}Y = 0$, $t, d > 0$, then

$$\begin{aligned} \mathbb{P}(Y \geq d) &\leq M e^{-td}, \\ \mathbb{P}(Y \geq d) &\leq \frac{e^\eta M - 1 - \eta}{e^{td+\eta} - 1 - td - \eta} \quad (\eta \geq 0). \end{aligned} \quad (202)$$

For the second inequality use the nonnegative function $e^{ty+\eta} - 1 - ty - \eta$, increasing on $y \geq d$. Its expectation is bounded by the numerator. The case $\eta = 0$ is the centered exponential-excess bound. These are closed-tail bounds. For $j > k$, $d = j - k - u > 0$ on $u \leq 1/2$; Cantelli provides the independent bound $V_j/(V_j + d^2)$.

F.1.6 Incoming stop-loss comparisons

Lemma F.5 (Two-point stop-loss comparison). *Let $\mathbb{E}Y = 0$, $\mathbb{E}Y^2 \leq w$, $Y \leq B$, with $w, B > 0$. Let Z take $-w/B, B$ with probabilities $B^2/(B^2 + w), w/(B^2 + w)$. For every real r and $a \in \{2, 3\}$,*

$$\mathbb{E}(Y - r)_+^a \leq \mathbb{E}(Z - r)_+^a.$$

The comparison law is the unit atom at zero if $w = 0$ or if $B = 0$ and the centered original law is one-sided.

Proof. Put $c = -w/B$. If $r \geq B$, both positive parts vanish. If $c \leq r < B$, the quadratic

$$P(x) = \frac{(B - r)^a}{(B - c)^2} (x - c)^2$$

majorizes $(x - r)_+^a$ on $x \leq B$. For $x > r$, this follows by differentiating $(x - r)^a/(x - c)^2$; its derivative is nonnegative for $a = 2, 3$. Equality holds at c, B . If $r < c$, use $P(x) = (x - r)^2$ when $a = 2$. For $a = 3$, put $\delta = c - r > 0$, $\ell = B - c$, and use

$$P(x) = \delta^3 + 3\delta^2(x - c) + (3\delta + \ell)(x - c)^2.$$

It is globally positive and $P(x) - (x - r)^3 = (B - x)(x - c)^2$ on $r \leq x \leq B$. Every majorant has a nonnegative quadratic coefficient, so its expectation increases when the variance is replaced by w . The comparison law has the specified first two moments and evaluates the majorant exactly. $\square$

Conditioning on the other summands and replacing them one at a time extends the lemma to sums of independent centered variables. For the two conditional types, the comparison sum is a convolution of two scaled binomial laws. For every $t > 0$,

$$\mathbb{P}(-W_j \geq d) \leq \mathbb{E}[1 + t(-W_j - d)]_+^a \leq \mathbb{E}[1 + t(Z_j - d)]_+^a, \quad a = 2, 3. \quad (203)$$

At $d = j - k - u$, the last expression is convex and nondecreasing in u ; its endpoint chord is an affine upper bound throughout a closed phase interval. A fixed t is chosen for the whole chord.

For $a = 2$, its optimal value at one distance is obtained exactly. For comparison atoms x_i of masses $c_i > 0$, write

$$F(t) = \sum_i c_i \max\{0, 1 + t(x_i - d)\}^2.$$

This function is continuously differentiable and convex. Between its positive breakpoints $1/(d - x_i)$, the active sum is $C + 2Bt + At^2$, $A \geq 0$. Check the breakpoints, the stationary point $-B/A$ when it lies in the active interval, and a possible final constant interval. Since $F'(0) = -2d < 0$, the zero endpoint is unnecessary. Repeated atoms and breakpoints are merged. Rational input gives rational candidates and exact finite sums.

Common allocation can be preserved by a finite closed partition of $[\alpha_-, \alpha_+]$. On a part $[a, b]$, use

$$w_0 \leq \frac{\lambda_+(1 - a)}{n(1 - p_+)}, \quad w_1 \leq \frac{\lambda_+ b}{np_-},$$

intersected with any prior valid variance caps. Form one matched phase chord on each part. The line whose two endpoint values are the respective maxima of those chords dominates every part on the entire phase interval. It is therefore a whole-allocation bound. One may not join independently optimized endpoint values unless they arise from such complete chords.

F.1.7 Common cubic moments and phase coherence

For $t \geq 0$, the function

$$r_t(x) = \frac{e^{tx} - 1 - tx}{x^2} = t^2 \int_0^1 (1-r)e^{trx} dr$$

is nonnegative, increasing and convex, with its continuous value at zero. On $[A, B]$, $A < 0 < B$, its chord has coefficients

$$c_2 = \frac{Br_t(A) - Ar_t(B)}{B - A} \geq 0, \quad c_3 = \frac{r_t(B) - r_t(A)}{B - A} \geq 0.$$

Consequently

$$e^{tx} \leq 1 + tx + c_2x^2 + c_3x^3 \quad (A \leq x \leq B).$$

For $Y = -U_i$, this gives an affine upper MGF in the original cubic moment μ_i , with the third-moment sign reversed. Directed coefficient enclosures give a globally nonnegative convex upper function

$$Q_i(\mu_i) = \max\{0, 1 + C_2w_{i,+} - C_{3,-}\mu_i, 1 + C_2w_{i,+} - C_{3,+}\mu_i\},$$

$$0 \leq C_{3,-} \leq c_3 \leq C_{3,+}, \quad c_2 \leq C_2.$$

The maximum is essential when the sign of μ_i is not fixed.

Choose fixed positive pivots P_0, P_1 . Weighted AM-GM gives the convex common-moment upper bound

$$Q_0^{n-j}Q_1^j \leq G_j := P_0^{n-j}P_1^j \left(\frac{n-j}{n} \frac{Q_0}{P_0} + \frac{j}{n} \frac{Q_1}{P_1} \right)^n. \quad (204)$$

The bracket is nonnegative and convex, and x^n is increasing and convex on the nonnegative half-line. Thus G_j is convex in (μ_0, μ_1) . At $j = 0, n$, omit zero powers. Chernoff gives the incoming bound $e^{t_j(u-(j-k))}G_j$. Its exponential phase factor is bounded by its positive endpoint chord. Alternatively use (202) with the convex numerator $\max\{0, e^\eta G_j - 1 - \eta\}$. For $x > 0$,

$$\left(\frac{1}{e^x - 1 - x} \right)'' = \frac{e^{2x} + (x-3)e^x + 2}{(e^x - 1 - x)^3} > 0.$$

The numerator and its derivative vanish at zero, and its second derivative is $e^x(4e^x + x - 1) > 0$. Hence the reciprocal-excess phase factor also has a positive upper chord. Each incoming strategy is fixed for the whole moment polygon and whole phase interval.

Let $\mathcal{P}$ be a bounded closed polygon containing the actual pair (μ_0, μ_1) . The cubic moment bounds, the rho and signed gamma strips from (193)–(194), and (198) supply such necessary halfplanes. In a probability box of midpoint p_* and halfwidth δ , replacing $-q\mu_0 + p\mu_1$ by its midpoint coefficients costs at most $\delta \max |\mu_0 + \mu_1|$; the signed gamma strip costs at most $\delta \max |\mu_1 - \mu_0|$. All remaining factors receive outward interval bounds. These enlargements preserve the actual pair. An empty polygon is a genuine necessary-condition contradiction.

For a polynomial of degree at most d on $[l, r]$, write

$$P(u) = \sum_{i=0}^d P_i \binom{d}{i} t^i (1-t)^{d-i}, \quad t = (u-l)/(r-l).$$

The Bernstein basis is nonnegative and sums to one, so $P(u) \leq \max_i P_i$. An affine phase chord has coefficients its values at $l + (r-l)i/d$. Combining these positive coefficients with (204) and the affine common-moment outgoing quartic produces convex coefficient functions $F_i(\mu_0, \mu_1)$. The degree-four own polynomial needs $d = 4$. Constant intervals are evaluated directly.

Lemma F.6 (Vertex reduction with normalization). *If F is convex on a compact polygon and L is affine and strictly positive there, then*

$$\frac{F(x)}{L(x)} \leq \max_{v \text{ vertex}} \frac{F(v)}{L(v)}.$$

If the acquired rho floor is $\max\{r_0, L(\mu_0, \mu_1)\}$, $r_0 > 0$, split the polygon along $L = r_0$ and use the vertices of both closed pieces.

Proof. Write $x = \sum_v \theta_v v$. Convexity gives $F(x) \leq \sum_v \theta_v F(v)$, which is at most $\max_v (F(v)/L(v)) \sum_v \theta_v L(v)$. The latter sum is $L(x)$. The same proof applies to segments and single points. $\square$

One must maximize over all controls, normalization pieces and their vertices before minimizing over a finite menu of rates, pivots or quartic shapes. A pointwise minimum of convex functions need not be convex. For the same reason, replacing a convex incoming bound pointwise by its minimum with one is not part of this vertex argument. A constant bound may instead be chosen as a whole-polygon strategy.

F.1.8 From a count bound to the normalized score

Theorem F.7 (Whole-box normalized count bound). *Fix a closed box of acquired parameters and a phase interval $[l, r]$. Let B_k bound the binomial CDF above, b_k^- the own mass below, and b_j^+ the incoming masses above, throughout that box. Choose one outgoing lower bound $L_k(u, \mu)$ proved above and incoming nonnegative upper bounds $A_j(u, \mu)$ proved above. Let $N(u)$ be a lower bound for*

$$\Phi \left(\frac{k + u - np}{\sqrt{n}\sqrt{v + \lambda/n}} \right)$$

throughout the same box. Suppose their combination has certified convex Bernstein upper controls $F_i(\mu)$:

$$B_k - b_k^- L_k(u, \mu) + \sum_{j>k} b_j^+ A_j(u, \mu) - N(u) \leq \max_i F_i(\mu).$$

This control is obtained by the polynomial, affine and convex constructions above, not assumed as a probability inequality. On each normalization piece let $r(\mu) > 0$ be its affine lower rho bound. If $n_+ \geq \sqrt{n}$, the positive normalized CDF error on the whole box is at most

$$n_+ \max_{i, v \text{ vertex}} \frac{\max\{0, F_i(v)\}}{r(v)}. \quad (205)$$

The maximum includes every closed normalization piece.

Proof. Insert the outgoing and incoming probability bounds into (190). For the outgoing term, $b_k \mathbb{P}(W_k > u) \geq b_k^- \mathbb{P}(W_k > u) \geq b_k^- L_k$, even if the polynomial lower bound is negative. Incoming upper bounds are nonnegative, so the upper mass substitutions are valid. Subtract the normal lower bound, take the positive part, and use $\rho \geq r$. The positive part of a convex function is convex. Apply Lemma F.6, and only then enlarge $\sqrt{n}$. $\square$

A convenient normal lower envelope uses a lower value N_l at the left phase endpoint and a uniform lower derivative $g \geq 0$. For $h_- \leq \sqrt{v + \lambda/n} \leq h_+$, put

$$X_* = \frac{\max |k + u - np|}{\sqrt{n} h_-}, \quad g \leq \frac{e^{-X_*^2/2}}{\sqrt{2\pi n} h_+}.$$

Then $N(u) = N_l + g(u - l)$ is valid. Only the smooth Gaussian is differentiated. Cantelli's smooth incoming envelope has derivative $2Vd/(V + d^2)^2$, whose maximum over a distance interval occurs at an endpoint or $d = \sqrt{V/3}$. The exponential, centered and shifted envelopes likewise have explicit upper derivatives or convex endpoint chords. A matched pair of left value and slope is always retained. One affine envelope dominates another on an interval precisely when it does so at both endpoints. These observations justify retaining phase cancellation rather than maximizing each term at a different phase.

F.1.9 Complete roots and finite certificates

If acquired p, h, H bounds imply

$$np_- - h_+H_- \geq -\frac{1}{2}, \quad np_+ + h_+H_+ \leq n + \frac{1}{2},$$

then every allowed threshold has a nearest count $k = 0, \dots, n$ and $u \in [-1/2, 1/2]$. A tie belongs to both closed roots. The original product roots are

$$[p_-, p_+] \times [0, \Lambda] \times [0, 1] \times [-1/2, 1/2] \times [0, d_0], \quad k = 0, \dots, n.$$

Necessary variance, moment, support and inward contractions act only inside these roots. A proof must still cover the original products. If thresholds outside this range are possible, their tails need separate bounds; they are not discarded by the nearest-count notation.

Proposition F.8 (Finite closed-cover implication). *Suppose every original root has a finite binary partition by rational strict interior cuts. Retain both closed children at each cut. If every leaf either has a proved necessary-condition contradiction or has the whole-box upper bound (205) below a fixed $c < C_E$, then every allowed positive normalized CDF error is below C_E . This conclusion uses all $n + 1$ original roots.*

Proof. A strict cut partitions a closed box into two closed boxes whose union is the parent. Finite induction up each tree proves its root statement, including all seams. Union over the count roots proves the assertion. Zero-variance faces use deterministic conditional laws as above; equality in any necessary-condition interval test is not an emptiness certificate. $\square$

Rational arithmetic implements this proposition over continuous domains. For a square root, integer squaring brackets the positive radical; polynomial identities and Bernstein coefficients are checked over exact rationals. Exponentials and the Gaussian CDF are enclosed by finite series with signed or absolute remainder bounds. Every coefficient is rounded in the direction used in its proved inequality. The mathematical content is the whole-box inequality and the finite cover, not a sample of parameter values or a floating-point optimum.

Finally, this section bounds the positive closed CDF branch. For a negative Kolmogorov discrepancy use the whole reflected marginal and

$$\Phi(x) - \mathbb{P}(S_n < x) = \mathbb{P}(-S_n \leq -x) - \Phi(-x).$$

Resources for that oriented law must be acquired in the same manner. Treating both signs of the threshold inside one row does not by itself reflect the marginal or its signed moments.

F.2 One summand: a closed cubic-dual implication

Certificate condition F.9 (One-summand dual cover). A finite closed cover of $[-2, 1]$ supplies, for each rational cell $[a, b]$, rationals u, v, L_a such that $L_a \leq \Phi(a)$ and

$$L_a - v + ux + vx^2 + \frac{2}{5}|x|^3 - \mathbf{1}_{x \leq b} \geq 0 \quad (x \in \mathbb{R}).$$

The polynomial inequalities are established on the whole real line, including both infinite rays and the closed event boundary. The prescribed cover uses 150 cells.

Proposition F.10. *Under Condition F.9, every standardized law with finite third absolute moment satisfies $\Delta_1(P) \leq (2/5)\rho(P)$.*

Proof. Taking expectations cancels $u\mathbb{E}X$ and $-v + v\mathbb{E}X^2$, giving $F_X(b) - L_a \leq (2/5)\rho$. For $t \in [a, b]$, monotonicity of both CDFs gives $F_X(t) - \Phi(t) \leq F_X(b) - L_a$. For $t \leq -2$, Cantelli gives $F_X(t) \leq 1/(1+t^2) \leq 1/5$. For $t \geq 1$, $1 - \Phi(t) \leq 1 - \Phi(1) < 2/5$; for example, $\phi(0) > 3/8$ and $e^{-s^2/2} \geq 1 - s^2/2$ on $[0, 1]$ imply $\Phi(1) > 1/2 + (3/8)(5/6) = 13/16$. Since $\rho \geq 1$, these tails suffice. For the opposite discrepancy,

$$\Phi(t) - F_X(t) \leq \Phi(t) - F_X(t-) = F_{-X}(-t) - \Phi(-t).$$

Apply the same result to the reflected law, retaining all atoms. □

F.3 Two summands: a rational tensor certificate

F.3.1 Result and proof boundary

For a standardized law P , write

$$\mathbb{E}X = 0, \quad \mathbb{E}X^2 = 1, \quad \rho(P) = \mathbb{E}|X|^3 < \infty,$$

$$\Delta_n(P) = \sup_x |P^{*n}((-\infty, x\sqrt{n}]) - \Phi(x)|, \quad C_n(P) = \frac{\sqrt{n}\Delta_n(P)}{\rho(P)}, \quad C_n = \sup_P C_n(P).$$

Let $\phi = \Phi'$ and $\phi_0 = (2\pi)^{-1/2}$. Set

$$K = \frac{2\sqrt{2}}{7} = 0.404061017820884\dots, \quad C_E = \frac{3 + \sqrt{10}}{6\sqrt{2\pi}} = 0.409732183702396\dots$$

Theorem F.11 (Verified two-summand bound). *If the finite positivity and coverage obligations specified below are verified, then for every standardized law P with finite third absolute moment,*

$$\boxed{\Delta_2(P) \leq \frac{2}{7}\rho(P)}.$$

Equivalently, $C_2 \leq K < C_E$.

The analytic reduction below gives the stated upper bound conditional on exact verification of the finite positivity assertions.

The sole external numerical probability estimate is Shevtsova's published structural inequality [26]:

$$\frac{\sqrt{n}\Delta_n(P)}{\rho(P)} \leq a \left(1 + \frac{b}{\rho(P)}\right), \quad a = \frac{1661}{5000}, \quad b = \frac{429}{1000}. \quad (206)$$

It implies

$$\rho \geq 2 \implies C_n(P) \leq a(1 + b/2) < K. \quad (207)$$

The last comparison is rational after squaring, since $K^2 = 8/49$.

F.3.2 Reduction of a putative maximizer to $[-6, 6]$

Lemma F.12 (Attainment under the contrary hypothesis). *If $C_2 > K$, the supremum is attained by a standardized law P with $\rho(P) < 2$.*

Proof. Lemma 2.1, applied at $n = 2$, gives an unrestricted maximizing law and a finite threshold with positive closed-CDF discrepancy. If its third absolute moment were at least 2, (207) would give $C_2 < K$, contrary to the hypothesis. $\square$

For the positive closed-CDF maximizing pair supplied by that lemma, write

$$\begin{aligned} t &= x\sqrt{2}, & F &= P^{*2}((-\infty, t]), & D &= F - \Phi(x) > 0, \\ C &= \frac{\sqrt{2}D}{\rho} = C_2, & \lambda &= \frac{D}{\rho}, & m &= \mathbb{E}[X|X|]. \end{aligned}$$

Here m denotes the signed second moment β of Lemma 2.2, and $|m| \leq 1$.

Lemma F.13 (Explicit contact inequality). *Define*

$$\begin{aligned} Q(y) &= \lambda|y|^3 - \frac{x\phi(x) + 3D}{2}y^2 - (\sqrt{2}\phi(x) + 3\lambda m)y \\ &\quad + 2F + \frac{x\phi(x) + D}{2}. \end{aligned} \tag{208}$$

Then

$$2P((-\infty, t - y]) \leq Q(y) \quad (y \in \mathbb{R}), \quad Q(y) = 2P((-\infty, t - y]) \quad (y \in \text{supp } P). \tag{209}$$

Proof. Apply Lemma 2.2 at $n = 2$, $H = t$ and $z = x$. In (5), substitute $\beta = m$, $C/\sqrt{2} = \lambda$ and $C\rho/\sqrt{2} = D$. The polynomial displayed here is $2Q_t$, so the asserted inequality and equality on the entire closed support follow, including all atoms. $\square$

Lemma F.14 (Six-unit support bound). *Under $C_2 > K$, every maximizing witness in Lemma F.12 satisfies $\text{supp } P \subset (-6, 6)$.*

Proof. Equation (206) gives $C < 1/2$ and $C\rho < a(2+b) < 1$. Hence $\lambda > 2/7$, $D < 1/\sqrt{2} < 5/7$, and $\lambda < 5/14$. Using $\phi(x) < 2/5$, $|x\phi(x)| < 1/4$, $\sqrt{2} < 10/7$, $|m| \leq 1$, and $F, D \geq 0$, we obtain

$$Q(y) \geq P_6(|y|), \quad P_6(r) = \frac{2}{7}r^3 - \frac{67}{56}r^2 - \frac{23}{14}r - \frac{1}{8}.$$

The exact identity

$$P_6(6+u) - 2 = \frac{2}{7}u^3 + \frac{221}{56}u^2 + \frac{104}{7}u + \frac{373}{56} > 0 \quad (u \geq 0)$$

contradicts $Q \leq 2$ on the support whenever $|y| \geq 6$. $\square$

No finite-support assumption appears in this reduction.

F.3.3 The two moment parameters

Let $\gamma = \mathbb{E}X^3$. Every standardized law satisfies

$$1 \leq \rho, \quad |\gamma| \leq \rho, \quad \rho^2(\gamma^2 + 4) - (\gamma^2 + 2)^2 \geq 0. \tag{210}$$

The last inequality is Assumption F.1, item 3; squaring its positive sides gives (210). In the remaining problem $\rho \leq 2$, so it suffices to consider the compact domain

$$\mathcal{D} = \{(\gamma, \rho) \in [-2, 2] \times [1, 2] : \rho^2(\gamma^2 + 4) - (\gamma^2 + 2)^2 \geq 0\}. \tag{211}$$

These necessary moment constraints give an outer domain for the certificate.

F.3.4 The tensor majorants

Define

$$v_0(y) = 1, \quad v_1(y) = y, \quad v_2(y) = y^2, \quad v_3(y) = y^3, \quad v_4(y) = |y|^3, \quad \mu = (1, 0, 1, \gamma, \rho).$$

For rational coefficients $c = (c_{ij})_{0 \leq i \leq j \leq 4}$ set

$$M_c(u, v) = \sum_i c_{ii} v_i(u) v_i(v) + \sum_{i < j} c_{ij} \{v_i(u) v_j(v) + v_j(u) v_i(v)\}, \quad (212)$$

$$q_c(\gamma, \rho) = \sum_i c_{ii} \mu_i^2 + 2 \sum_{i < j} c_{ij} \mu_i \mu_j. \quad (213)$$

Independence and the common distribution give the exact identity

$$\mathbb{E} M_c(X_1, X_2) = q_c(\gamma, \rho). \quad (214)$$

On each quadrant M_c is a polynomial of degree at most three in each coordinate. Its expectation is a polynomial of total degree at most two in (γ, ρ) .

For $j = 0, \dots, 69$, put

$$a_j = -\frac{5}{2} + \frac{j}{20}, \quad b_j = a_j + \frac{1}{20}.$$

The certificate condition requires one rational vector $c^{(j)}$ for each interval and the inequalities

$$M_{c^{(j)}}(u, v) \geq \mathbf{1}_{\{u+v \leq b_j\}} \quad (u, v \in [-6, 6]), \quad (215)$$

$$q_{c^{(j)}}(\gamma, \rho) \leq \Phi(a_j/\sqrt{2}) + \frac{2}{7}\rho \quad ((\gamma, \rho) \in \mathcal{D}). \quad (216)$$

This prescribed certificate format has 70 threshold rows and 1050 coefficients; coefficients are ordered lexicographically by (i, j) with $i \leq j$.

Lemma F.15 (Certificate implication). *Equations (215)–(216) imply $\Delta_2(P) \leq 2\rho(P)/7$ for every standardized law on $[-6, 6]$ with $\rho \leq 2$.*

Proof. If $t \in [a_j, b_j]$, monotonicity, (214), and the certificate give

$$\begin{aligned} P(X_1 + X_2 \leq t) - \Phi(t/\sqrt{2}) &\leq \mathbb{E} M_{c^{(j)}}(X_1, X_2) - \Phi(a_j/\sqrt{2}) \\ &\leq \frac{2}{7}\rho. \end{aligned}$$

For $t \leq -5/2$, Cantelli's inequality gives $P(X_1 + X_2 \leq t) \leq 2/(2 + t^2) \leq 8/33 < 2/7 \leq 2\rho/7$. For $t \geq 1$, $1 - \Phi(t/\sqrt{2}) \leq 1 - \Phi(1/\sqrt{2}) < 2/7$. The last strict inequality is checked by the rational Gaussian enclosure below. Thus the positive branch is bounded at every real threshold. Applying the same result to $-X$ bounds $\Phi(t/\sqrt{2}) - P(X_1 + X_2 < t)$, and hence also the negative right-cdf error. Atoms and exact sum collisions are included, not removed by genericity. $\square$

F.3.5 Exact checking of the continuous domains

F.3.5.1 Piecewise-polynomial positivity

Each of the four quadrants of $[-6, 6]^2$ is intersected with the two closed half-planes $u + v \leq b_j$ and $u + v \geq b_j$. Each resulting rational polygon is triangulated. On each triangle,

it suffices to prove nonnegativity of $M_{c(j)} - 1$ on the first side or $M_{c(j)}$ on the second. The shared boundary belongs to both closed regions.

Write a degree-six polynomial in barycentric Bernstein form on a triangle:

$$p = \sum_{\alpha_0 + \alpha_1 + \alpha_2 = 6} B_\alpha \frac{6!}{\alpha_0! \alpha_1! \alpha_2!} \lambda_0^{\alpha_0} \lambda_1^{\alpha_1} \lambda_2^{\alpha_2}.$$

If every $B_\alpha \geq 0$, then $p \geq 0$ on the whole triangle. Otherwise the longest edge is bisected and both children are checked. A resource limit or negative vertex value is a rejection, never an acceptance.

One exact procedure constructs the coefficients from a monomial expansion and subdivides them by rational de Casteljau identities. A second procedure reconstructs the homogeneous polynomial directly on *every* child. If D_c clears polynomial denominators and D_v clears vertex denominators, write $U = D_v u$, $V = D_v v$, and $S = \lambda_0 + \lambda_1 + \lambda_2$. For $p(u, v) = \sum a_{rs} u^r v^s$, the second construction expands

$$\sum_{r,s} (D_c a_{rs}) D_v^{6-r-s} U^r V^s S^{6-r-s}.$$

All its coefficients are integers. They are nonnegative exactly when the corresponding Bernstein coefficients are nonnegative, because the omitted denominators and multinomial factors are positive. This gives a direct coefficient calculation without using the subdivision recurrence.

F.3.5.2 Moment-parameter positivity

The scalar expression $\Phi(a_j/\sqrt{2}) + 2\rho/7 - q_{c(j)}(\gamma, \rho)$ is enclosed from below on rational rectangles using degree-two Bernstein coefficients in each variable. A rectangle outside $\mathcal{D}$ is discarded. This test is exact: if g_* is the minimum of γ^2 on its γ interval and r_+ is its upper ρ endpoint, the rectangle is outside whenever

$$r_+^2(g_* + 4) - (g_* + 2)^2 < 0.$$

Indeed, $r^2(w + 4) - (w + 2)^2$ increases in $r \geq 0$ and decreases in $w \geq 0$ for $r \leq 2$. All other rectangles are certified or subdivided.

F.3.5.3 The normal cdf

For $|t| \leq 5/2$,

$$\Phi(t/\sqrt{2}) = \frac{1}{2} + \frac{1}{2\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{4^k k! (2k+1)}. \quad (217)$$

For $t \geq 0$ the term magnitudes decrease: their successive ratio is at most $(25/16)(2k+1)/((k+1)(2k+3)) < 1$. Alternating partial sums therefore give rational enclosures after bounding $1/(2\sqrt{\pi})$. Machin's arctangent identity and integer square roots supply those bounds. The second procedure uses different rational prefactor bounds and different truncation orders. Negative arguments are handled by symmetry. No platform implementation of Φ , exponential, sine, or cosine is used for acceptance.

F.3.5.4 The complete finite certificate

The supplied certificate has exactly seventy consecutive threshold rows and 1050 rational coefficients. Complete exact verification establishes both the pointwise majorant and moment-parameter inequalities on their stated closed domains. Either of the two positivity procedures above can establish these finite obligations.

F.3.6 Completion of the two-summand proof

Proof of Theorem F.11. Suppose $C_2 > K$. Lemma F.12 supplies a global maximizing law with $\rho < 2$. Lemma F.14 confines it to $(-6, 6)$. The assumed complete rational verification establishes (215)–(216); Lemma F.15 then gives $C_2(P) \leq K$, a contradiction. Thus $C_2 \leq K$. Finally, exact rational enclosures give $K < 0.405 < 0.409 < C_E$ using rational bounds and squaring. $\square$

The bounded-support theorem in Lemma F.15 treats arbitrary probability measures, not merely finitely supported measures or an assumed number of atoms. The passage to unrestricted laws uses global extremality, not a truncation approximation.

F.4 Three summands: independence-null certificates

F.4.1 Result and external input

For a standardized law P , let

$$\mathbb{E}X = 0, \quad \mathbb{E}X^2 = 1, \quad \rho(P) = \mathbb{E}|X|^3 < \infty,$$

$$\Delta_n(P) = \sup_x |P^{*n}((-\infty, x\sqrt{n}]) - \Phi(x)|, \quad C_n = \sup_P \frac{\sqrt{n} \Delta_n(P)}{\rho(P)}.$$

Write $\phi = \Phi'$ and set

$$\Lambda = 59/250, \quad K = \sqrt{3}\Lambda = 0.408763990586255\dots, \quad C_E = \frac{3 + \sqrt{10}}{6\sqrt{2\pi}}.$$

Theorem F.16 (Verified three-summand bound). *If the finite positivity and coverage obligations specified below are verified, then every standardized law with finite third absolute moment satisfies*

$$\Delta_3(P) \leq \Lambda\rho(P).$$

Consequently $C_3 \leq 59\sqrt{3}/250 < C_E$.

The analytic reduction below gives the stated upper bound conditional on exact verification of the polynomial inequalities over their complete domains.

The sole external numerical probability bound is Shevtsova's structural estimate [26]:

$$C_n(P) \leq A \left(1 + \frac{B}{\rho(P)}\right), \quad A = \frac{1661}{5000}, \quad B = \frac{429}{1000}. \quad (218)$$

It implies

$$\rho(P) \geq (15/8) \implies C_n(P) \leq \frac{159456}{390625} < K. \quad (219)$$

The last inequality is rational after squaring. No result about the optimal two-point constant, no near-lattice transfer, and no effective eventual threshold is used.

F.4.2 A bounded witness would attain any violation

Lemma F.17 (Attainment under the contrary hypothesis). *If $C_3 > K$, a standardized law P with $\rho(P) < 15/8$ attains C_3 .*

Proof. Lemma 2.1, applied at $n = 3$, gives an unrestricted maximizing law and a finite threshold with positive closed-CDF discrepancy. If $\rho(P) \geq 15/8$, (219) would give $C_3 < K$, contrary to the hypothesis. $\square$

For the positive closed-CDF pair supplied by that lemma, put

$$\begin{aligned} t &= x\sqrt{3}, \quad F = P^{*3}((-\infty, t]), \quad D = F - \Phi(x) > 0, \\ C &= C_3 = \frac{\sqrt{3}D}{\rho}, \quad \lambda = \frac{D}{\rho}, \quad m = \mathbb{E}[X|X|], \quad |m| \leq 1. \end{aligned}$$

Lemma F.18 (Explicit contamination contact inequality). *Define*

$$\begin{aligned} Q(y) &= \lambda|y|^3 - \frac{x\phi(x) + 3D}{2}y^2 - (\sqrt{3}\phi(x) + 3\lambda m)y \\ &\quad + 3F + \frac{x\phi(x) + D}{2}. \end{aligned} \tag{220}$$

Then

$$3P^{*2}((-\infty, t - y]) \leq Q(y) \quad (y \in \mathbb{R}), \tag{221}$$

with equality throughout $\text{supp } P$.

Proof. Apply Lemma 2.2 at $n = 3$, $H = t$ and $z = x$. In (5), substitute $\beta = m$, $C/\sqrt{3} = \lambda$ and $C\rho/\sqrt{3} = D$. The polynomial displayed here is $3Q_t$. The closed-CDF inequality and equality throughout the closed support are therefore the corresponding conclusions of that lemma. $\square$

Lemma F.19 (Uniform support bound). *A violating maximizer in Lemma F.17 satisfies $\text{supp } P \subset (-6, 6)$.*

Proof. The structural bound and $1 \leq \rho < 15/8$ give

$$\frac{2}{5} < C < \frac{19}{40}, \quad C\rho < \frac{77}{100}.$$

Using $12/7 < \sqrt{3} < 7/4$ and $\sqrt{3} > 17/10$, we obtain

$$D < \frac{9}{20}, \quad \frac{8}{35} < \lambda < \frac{2}{7}.$$

Also $\phi(x) < 2/5$ and $|x\phi(x)| < 1/4$. The first follows from $\pi > 25/8$, and the second from the maximum $(2\pi e)^{-1/2} < 1/4$. Thus (220) implies, with $r = |y|$,

$$Q(y) \geq \frac{8}{35}r^3 - \frac{4}{5}r^2 - \frac{8}{5}r - \frac{1}{8}.$$

For $r = 6 + u$, the difference of the right side and 3 equals

$$\frac{8}{35}u^3 + \frac{116}{35}u^2 + \frac{472}{35}u + \frac{2197}{280} > 0 \quad (u \geq 0).$$

But (221) is an equality on the support, and its left side is at most 3. This excludes every support point with $|y| \geq 6$. $\square$

Remark F.20. This is an extremizer reduction, not a truncation of arbitrary laws. The original theorem assumes only a finite third absolute moment. Higher moments are used below only after the hypothetical violating maximizer has been proved bounded.

F.4.3 The admissible two-dimensional moment region

Write $\gamma = \mathbb{E}X^3$. Assumption F.1, item 3 gives

$$\rho \geq (\gamma^2 + 2)/\sqrt{\gamma^2 + 4}.$$

Together with $|\gamma| \leq \rho$ and $\rho \geq 1$, every relevant law belongs to

$$\mathcal{D} = \left\{ (\gamma, \rho) : |\gamma| \leq \frac{15}{8}, \quad 1 \leq \rho \leq \frac{15}{8}, \quad \rho^2(\gamma^2 + 4) - (\gamma^2 + 2)^2 \geq 0 \right\}. \quad (222)$$

This is an outer domain: every point need not be realizable by a law. The certificate checks all of it, so non-realizable points cannot cause an invalid acceptance.

F.4.4 Independence-null controls

For a single coordinate define the features

$$(v_0, v_1, v_2, v_3, v_4, v_5, v_6)(y) = (1, y, y^2 - 1, y^3, |y|^3, y^4, y^5). \quad (223)$$

Let $\mathcal{I}$ consist of the nondecreasing triples $\alpha = (i, j, k)$ with $0 \leq i \leq j \leq k \leq 6$ such that either $k < 5$, or at least one entry is 1 or 2. There are 59 such triples, ordered lexicographically in the certificate files. For each α , let

$$V_\alpha(x_1, x_2, x_3) = \frac{1}{|\text{Perm}(\alpha)|} \sum_{(i,j,k) \in \text{Perm}(\alpha)} v_i(x_1) v_j(x_2) v_k(x_3),$$

where the sum is over the *distinct* permutations of the index triple. For $c \in \mathbb{Q}^{59}$ put

$$M_c = \sum_{\alpha \in \mathcal{I}} c_\alpha V_\alpha. \quad (224)$$

Lemma F.21 (Exact expectation without higher-moment parameters). *For independent standardized identically distributed bounded variables,*

$$\mathbb{E}M_c(X_1, X_2, X_3) = q_c(\gamma, \rho) := \sum_{\substack{\alpha \in \mathcal{I} \\ \text{entries only } 0,3,4}} c_\alpha \gamma^{\#3(\alpha)} \rho^{\#4(\alpha)}. \quad (225)$$

In particular, q_c has total degree at most three and has no unknown fourth- or fifth-moment parameters.

Proof. The expectations of v_1 and v_2 are zero. Independence makes every product containing either factor have expectation zero. By the definition of $\mathcal{I}$, every product involving v_5 or v_6 contains such a zero-mean factor in another coordinate. The only remaining products use v_0, v_3, v_4 , whose means are 1, γ , ρ . $\square$

This adds 24 coefficients to the 35 low-feature tensor coefficients. In the resulting family, 49 basis elements have zero expectation and only 10 contribute monomials in (γ, ρ) . For example, all permutations of

$$X_1^4 X_2, \quad X_1^5 (X_2^2 - 1), \quad X_1^5 X_2 |X_3|^3$$

have mean zero. These are exact independence identities, not assumptions that the fourth or fifth moment has a prescribed value.

The general principle is useful beyond three summands. For any integrable function of the other coordinates,

$$\begin{aligned} \mathbb{E}\{X_i A(X_1, \dots, \hat{X}_i, \dots, X_n)\} &= 0, \\ \mathbb{E}\{(X_i^2 - 1) B(X_1, \dots, \hat{X}_i, \dots, X_n)\} &= 0. \end{aligned}$$

Such terms may reshape a pointwise majorant without changing its i.i.d. expectation. For the present proof all functions are polynomials on a bounded cube, so integrability is automatic.

F.4.5 Finite-certificate implication

Certificate condition F.22 (Continuous-domain three-summand certificate). For each row j of the supplied rational certificate, let $[a_j, b_j]$ be its threshold interval and $c^{(j)} \in \mathbb{Q}^{59}$ its coefficients. The rows form an exact consecutive partition of $[-7/2, 3/2]$ and satisfy

$$M_{c^{(j)}}(u, v, w) \geq \mathbf{1}_{\{u+v+w \leq b_j\}} \quad ((u, v, w) \in [-7, 7]^3), \quad (226)$$

$$q_{c^{(j)}}(\gamma, \rho) \leq \Phi(a_j/\sqrt{3}) + \Lambda\rho \quad ((\gamma, \rho) \in \mathcal{D}). \quad (227)$$

Section F.4.6 gives sufficient exact tests for these continuous-domain inequalities.

Proof of Theorem F.16. Suppose $C_3 > K$. Lemmas F.17–F.19 give a globally maximizing violating law on $(-6, 6)$, with $\rho < 15/8$, and a maximizing raw threshold t with positive closed-CDF discrepancy. If $t \in [a_j, b_j]$, then independence and Condition F.22 give

$$\begin{aligned} P(X_1 + X_2 + X_3 \leq t) - \Phi(t/\sqrt{3}) &\leq \mathbb{E}M_{c^{(j)}}(X_1, X_2, X_3) - \Phi(a_j/\sqrt{3}) \\ &\leq \Lambda\rho. \end{aligned}$$

The larger certificate cube $[-7, 7]^3$ contains the entire witness support. For $t \leq -7/2$, Cantelli's inequality for the variance-three sum yields

$$P(X_1 + X_2 + X_3 \leq t) - \Phi(t/\sqrt{3}) \leq \frac{3}{3+t^2} \leq \frac{12}{61} < \Lambda \leq \Lambda\rho.$$

For $t \geq 3/2$, the discrepancy is at most

$$1 - \Phi(\sqrt{3}/2) < \Lambda \leq \Lambda\rho.$$

All these inequalities contradict the choice of the maximizing witness. The reduction by reflection already handles the other signed discrepancy; no atom or boundary event is discarded. $\square$

F.4.6 Exact verification algorithm

Each coefficient is rational. Verification reconstructs the polynomial from its defining features and requires complete threshold coverage. Every unresolved positivity test is rejected.

F.4.6.1 Piecewise polynomial representation and symmetry

On each sign orthant, $|y|^3$ is y^3 or $-y^3$. Thus M_c is a polynomial of degree at most five in each variable. Expand this polynomial from the feature definition, not from a supplied power expansion. Symmetry permits restriction to $u \leq v \leq w$. The four initial sign boxes are

$$[-7, 0]^3, \quad [-7, 0]^2 \times [0, 7], \quad [-7, 0] \times [0, 7]^2, \quad [0, 7]^3.$$

A dyadic box $[L_1, H_1] \times [L_2, H_2] \times [L_3, H_3]$ may be discarded only if $L_i > H_j$ for some $i < j$. Such a box has no ordered point. All other accepted inequalities are proved on the whole box, a safe enlargement.

F.4.6.2 An affine half-space Bernstein test

For a box, let B_α be the tensor Bernstein basis of degree five in three variables, so $B_\alpha \geq 0$ and $\sum B_\alpha = 1$. Write

$$M_c = \sum_{\alpha} b_{\alpha} B_{\alpha}, \quad u + v + w - b_j = \sum_{\alpha} s_{\alpha} B_{\alpha},$$

where

$$s_{\alpha} = \sum_{i=1}^3 \left(L_i + \frac{\alpha_i}{5} (H_i - L_i) \right) - b_j.$$

Lemma F.23 (Exact affine test). *If a real $\ell \leq 0$ satisfies $b_\alpha \geq 1 + \ell s_\alpha$ for all α , then $M_c \geq 1$ on the portion $u + v + w \leq b_j$. If $\ell \geq 0$ and $b_\alpha \geq \ell s_\alpha$ for all α , then $M_c \geq 0$ on the portion $u + v + w \geq b_j$.*

Proof. Average the coefficient inequalities with the nonnegative Bernstein weights. In each case the sign restriction makes $\ell(u + v + w - b_j)$ nonnegative on the specified half-space. $\square$

Existence of ℓ is checked by intersecting rational half-lines. A coefficient with $s_\alpha = 0$ requires $b_\alpha \geq 1$ or $b_\alpha \geq 0$. The remaining coefficients give upper or lower rational bounds on ℓ . Only signed integer products are needed to compare these bounds.

A box wholly inside the event is checked by $\min b_\alpha \geq 1$; a box strictly outside by $\min b_\alpha \geq 0$. A crossing box uses the affine tests. An unresolved box is bisected along a widest coordinate. Dyadic subdivision uses the exact degree-five de Casteljau identities

$$b_i^{\text{left}} = 2^{-i} \sum_{j=0}^i \binom{i}{j} b_j, \quad b_i^{\text{right}} = 2^{-(5-i)} \sum_{j=i}^5 \binom{5-i}{j-i} b_j.$$

After multiplying the common denominator by 32, all new numerators are integers. Ties at the event plane remain on the included side, and that plane is required to have value at least one.

F.4.6.3 Moment region and Gaussian enclosure

The scalar polynomial

$$\Phi(a_j/\sqrt{3}) + \Lambda\rho - q_{c(j)}(\gamma, \rho)$$

is checked by rational tensor Bernstein coefficients of degree three on moment rectangles. Here Φ is a rigorous rational lower bound. For $s = \gamma^2$ the domain constraint is

$$h(s, \rho) = \rho^2(s + 4) - (s + 2)^2 \geq 0.$$

When $\rho \leq 15/8 < 2$, this expression decreases in $s \geq 0$ and increases in $\rho \geq 0$. If its maximum at the smallest possible γ^2 and largest possible ρ on a rectangle is negative, that rectangle is outside the domain and is discarded. Otherwise Bernstein positivity proves the scalar inequality or the rectangle is bisected.

For rational t in the required range,

$$\Phi(t/\sqrt{3}) = \frac{1}{2} + \frac{1}{\sqrt{6\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{6^k k! (2k+1)}.$$

Use consecutive alternating partial sums through indices 34, 35. For $|t| \leq 4$ the magnitudes decrease, so these enclose the sum. Rational Machin bounds enclose π , and integer square roots enclose $\sqrt{6\pi}$ to rational precision. No floating-point normal cdf enters acceptance.

F.4.6.4 The complete finite certificate

A complete certificate must supply consecutive threshold intervals covering the entire declared range, with 59 rational coefficients for each interval. Acceptance must reconstruct the polynomial expansions and verify the continuous cube and moment inequalities. Undecided cells are rejected.

The initial expansions are also checked on a complete 6^3 tensor interpolation grid in each sign box. Since the degree is at most five in each coordinate, agreement on this grid proves the polynomial identity. This is an independent identity check, not a second verification of all positivity inequalities.

F.5 Four summands: a conditional unrestricted bound

Theorem F.24 (Verified four-summand bound). *Assume the three-summand conclusion of Theorem F.16, as required by the localization below. If the complete bounded-law certificate inequalities specified below are verified, then for every standardized real law with finite third absolute moment,*

$$R_4(P) \leq \frac{4097}{10000} < C_E.$$

In particular, $C_4 \leq 4097/10000$. No restriction on the number of support points is imposed.

Put $T = 4097/10000$ and $c_0 = 256/625$. The proof has two steps. A hypothetical unrestricted maximizer above T lies in a compact moment class. A finite rational certificate bounds every law in that class by T . The three-point law introduced below is a reference law, not a reduction of the admissible laws to three points.

F.5.1 The canonical reference and its excess

For a standardized X , write

$$\begin{aligned} \vartheta &= \mathbb{E}[X|X|], & u &= \mathbb{E}X_+ = \mathbb{E}X_- > 0, & v_{\pm} &= \frac{1 \pm \vartheta}{2}, \\ a &= \frac{v_-}{u}, & b &= \frac{v_+}{u}, & w &= \frac{4u^2}{1 - \vartheta^2}. \end{aligned}$$

Both $v_{\pm}$ are positive. Cauchy's inequality on the two half-lines gives $0 < w \leq 1$. Define

$$Q = \frac{u}{a}\delta_{-a} + (1 - w)\delta_0 + \frac{u}{b}\delta_b, \quad r = \rho(Q) = \frac{1 + \vartheta^2}{\sqrt{w(1 - \vartheta^2)}}. \quad (228)$$

This is a probability law of mean zero and variance one. Its contacts are

$$a = \sqrt{\frac{1 - \vartheta}{w(1 + \vartheta)}}, \quad b = \sqrt{\frac{1 + \vartheta}{w(1 - \vartheta)}}.$$

Lemma F.25. *Set*

$$W(X) = X_+(X_+ - b)^2 + X_-(X_- - a)^2, \quad \eta = \mathbb{E}W = \rho(P) - r.$$

Then $\eta \geq 0$, and $\eta = 0$ if and only if $P = Q$. Moreover,

$$\begin{aligned} \mathbb{E}X^3 &= b - a + \mathbb{E}[\text{sgn}(X)W(X)], \\ \mathbb{E}X^4 &= a^2 - ab + b^2 + \mathbb{E}[(|X| + 2c(X))W(X)], \end{aligned} \quad (229)$$

whenever the fourth moment is finite, where $c(X) = b$ on $X > 0$ and $c(X) = a$ on $X < 0$.

Proof. On a tagged half-line put $y = |X|$ and $c = a$ or b . The identities

$$y^3 = y(y - c)^2 + 2cy^2 - c^2y, \quad y^4 = (y + 2c)y(y - c)^2 + 3c^2y^2 - 2c^3y$$

integrate using $\mathbb{E}[y; \text{tag}] = u$ and $\mathbb{E}[y^2; \text{tag}] = uc$. They give all three formulas. If $\eta = 0$, the support is contained in $\{-a, 0, b\}$; the first two moments fix its masses as in (228). The converse is immediate. $\square$

The excess η is not the two-point defect d of Part I. Its zero set permits an atom at zero; the reference is two-point precisely when $w = 1$.

F.5.2 Localization of an unrestricted extremizer

Write $K_3 = 59\sqrt{3}/250$. Theorem F.16 gives $C_3 \leq K_3$, and

$$c_0^2 - K_3^2 = \frac{1069}{1562500} > 0, \quad c_0 < T.$$

Lemma F.26. *If $C_4 > c_0$, an unrestricted maximizing pair (P, H) can be chosen as a positive closed discrepancy. It satisfies*

$$\text{supp } P \subset (-101/25, 101/25), \quad \rho(P) < 46/25, \quad \frac{625}{2116} < w \leq 1, \quad |\vartheta| < \frac{321}{500}. \quad (230)$$

Proof. Attainment is Assumption F.1, item 1. Shevtsova's estimate in Assumption F.1, item 1, gives, at this pair,

$$\rho < \frac{(3031/10000)(323/500)}{c_0 - 3031/10000} = \frac{979013}{532500} < \frac{46}{25}.$$

This bound is obtained before any support or fourth-moment assumption. Put $C = C_4$, $z = H/2$, $F = F_{4,P}(H)$ and $G = F_{3,P}$. Assumption F.1, item 1 gives

$$\begin{aligned} \rho I(y) &= 8\{G(H - y) - F\} + 4y\phi(z) + z\phi(z)(y^2 - 1) \\ &\quad + \frac{3}{2}C\rho(y^2 - 1) - C\{|y|^3 - \rho - 3\vartheta y\}. \end{aligned} \quad (231)$$

Here $I \leq 0$ everywhere and $I = 0$ at every point of the closed support, including accumulation points. Using the all-law predecessor bound and $F = \Phi(z) + C\rho/2$, with $s = \sqrt{4/3}$, yields

$$\begin{aligned} \rho I(y) &\leq \mathcal{H}_4(z, y) + 4\rho(K_3 - C) + 4\rho(s - 1)K_3 \\ &\quad + C\{-|y|^3 + \frac{3}{2}\rho y^2 + 3\vartheta y - \rho/2\}, \end{aligned} \quad (232)$$

where

$$\mathcal{H}_4(z, y) = 8 \left[\Phi\left(\frac{z - y/2}{\sqrt{3/4}}\right) - \Phi(z) \right] + 4y\phi(z) + z\phi(z)(y^2 - 1).$$

We use the following elementary uniform estimate:

$$|\mathcal{H}_4(z, y)| < \frac{\phi(0)}{6}|y|^3 + \frac{|y|}{5} + \frac{1}{15}. \quad (233)$$

Indeed, $\phi(0) < 2/5$, $\sup |\phi''| = \phi(0)$ and $\sup |(x\phi(x))'| = \phi(0)$. Also

$$\sup_x |x(3 - x^2)\phi(x)| < 3/5.$$

For the last inequality, set $t = x^2$. On $0 \leq t \leq 3$,

$$\frac{9}{4}(1 + t + t^2/2 + t^3/6) - t(3 - t)^2 \geq \frac{3}{4} [7(t - 9/14)^2 + 3/28] > 0.$$

Compare with e^t and use $\phi(0)^2 < 4/25$. On $t \geq 3$, use $|x|^3\phi(x) < 1/2$, which follows from $e > 8/3$. For $f(\lambda) = \Phi((z - y/2)/\sqrt{1 - \lambda})$,

$$f'(0) = \frac{1}{2}(z - y/2)\phi(z - y/2), \quad |f''(\lambda)| < 4/15 \quad (0 \leq \lambda \leq 1/4).$$

Thus $8\{f(1/4) - f(0)\} = (z - y/2)\phi(z - y/2) + E$, with $|E| < 1/15$. A second-order translation of Φ gives the cubic term in (233); the remaining difference $(z - y/2)\phi(z - y/2) - z\phi(z)$ is bounded by $|y|/5$.

By (228), $r \leq \rho < 46/25$ and $w \leq 1$. At $|\vartheta| = 321/500$,

$$\frac{(1 + \vartheta^2)^2}{1 - \vartheta^2} = \frac{124637947681}{36739750000} > (46/25)^2.$$

Monotonicity in ϑ^2 proves $|\vartheta| < 321/500$. Drop the negative term $4\rho(K_3 - C)$ in (232). Since $4(s - 1) < 5/8$, its remaining C -bracket is bounded, for $r_y = |y| \geq 4$, by

$$B(r_y) = -r_y^3 + \frac{69}{25}r_y^2 + \frac{963}{500}r_y + \frac{23}{100}.$$

The values of B, B', B'', B''' at 4 are $-5953/500, -11997/500, -462/25, -6$; hence $B < 0$ on that ray. Only now substitute $C > c_0$ in the negative bracket. Equation (233) gives

$$\rho I(y) < P(r_y) := -\frac{643}{1875}r_y^3 + \frac{17664}{15625}r_y^2 + \frac{77257}{78125}r_y + \frac{7541}{46875}.$$

At $101/25$ the values of P, P', P'', P''' are respectively

$$-\frac{51557}{9765625}, \quad -\frac{520966}{78125}, \quad -\frac{94558}{15625}, \quad -\frac{1286}{625}.$$

They are negative, so $P < 0$ thereafter. Contact excludes every support point with $|y| \geq 101/25$. Finally $r^2 \geq 1/w$ gives $w > 625/2116$. This proves (230). $\square$

F.5.3 One defect measure for all frequencies

For $\eta > 0$, let $\nu = W dP/\eta$, viewed as a probability on tagged points (σ, y) , $\sigma \in \{-1, 1\}$, $y > 0$. Write $c_+ = b$, $c_- = a$, $p_0 = 1 - w$, and

$$k_0(\sigma, y) = \frac{1}{c_\sigma^2 y}, \quad k_2(\sigma, y) = \frac{1}{(y - c_\sigma)^2}, \quad k_1(\sigma, y) = \frac{1}{y - c_\sigma}.$$

The contacts have zero ν mass. Singular expressions are integrated only off the contacts; the following identities prove integrability.

Lemma F.27. *With $g(q) = \varphi_Q(q)$ and $r_q(\sigma, y) = \exp(i\sigma q \cdot)[0, c_\sigma, c_\sigma, y]$,*

$$\begin{aligned} \varphi_P(q) - g(q) &= \eta \int r_q d\nu, \\ \eta \int k_0 d\nu &= p_0 - P\{0\} \leq p_0, \\ \eta \int_\sigma k_2 d\nu &= u - c_\sigma P\{\sigma c_\sigma\} \leq u, \quad \int_\sigma k_1 d\nu = 0. \end{aligned} \tag{234}$$

The same ν occurs at every frequency, and $|r_q| \leq |q|^3/6$. For $\eta = 0$, $P = Q$ and every characteristic-function difference is zero, without defining ν .

Proof. Hermite interpolation at $0, c, c$ has remainder $y(y - c)^2 \exp(i\sigma q \cdot)[0, c, c, y]$. On each tag, the linear and quadratic terms have the same integrals against P and Q , by their matching first two tagged moments. Both interpolating polynomials have constant term 1. Their constant contributions agree only after summing the tags and adding the atom at zero, since both total masses are 1. Hermite–Genocchi gives the response bound. Multiplication of the three costs by $W = y(y - c)^2$ gives $(y - c)^2/c^2$, y , and $y(y - c)$, respectively. Their integrals give (234), including the omitted contact masses. Absolute integrability of k_1 follows from Cauchy and the k_2 bound. Lemma F.25 handles $\eta = 0$. $\square$

Fix a threshold H and finite coefficients $\kappa_j(H)$, defined below. For the bounded class $|X| \leq R_* = 17/4$, all suprema below are over $\sigma \in \{-1, 1\}$ and $0 < y \leq R_*$, off the contact. The linear response is

$$A_H(\sigma, y) = \Re \sum_{j=0}^{N-1} 4\kappa_j(H)g(q_j)^3 r_{q_j}(\sigma, y).$$

For any fixed $\lambda_0, \lambda_+, \lambda_- \geq 0$ and $\theta_+, \theta_- \in \mathbb{R}$, a bound

$$S_H \geq \sup_{\sigma, y} \{A_H - \lambda_0 k_0 - \lambda_\sigma k_2 - \theta_\sigma k_1\} \quad (235)$$

implies

$$\eta \int A_H d\nu \leq \Gamma_H(\eta) := p_0 \lambda_0 + u(\lambda_+ + \lambda_-) + S_H \eta. \quad (236)$$

Upper bounds for p_0, u may replace them. The two equality prices retain their signs. In particular, S_H and Γ_H need not be nonnegative. The prices are fixed before taking the supremum over the whole tagged domain. This preserves the cancellation between frequencies; separate absolute estimates for each r_q do not.

F.5.4 The fourth-power remainder

For complex $\psi = g + \delta$,

$$\psi^4 - g^4 = 4g^3\delta + \delta^2(\psi^2 + 2\psi g + 3g^2). \quad (237)$$

Suppose $|g| \leq A \leq 1$, $|\psi| \leq 1$ and $|\delta| \leq E$. Two complete bounds for the remainder are

$$E^2(p^2 + 2pA + 3A^2), \quad p = \min(1, A + E), \quad v(4A^2 + 2 - v), \quad v = \min(E^2, 2A^2 + 1). \quad (238)$$

The first follows directly from (237). For the second, use

$$\psi^4 - g^4 - 4g^3\delta = 12\delta^2 \int_0^1 (1-t)(g+t\delta)^2 dt$$

and $|g+t\delta|^2 = (1-t)|g|^2 + t|\psi|^2 - t(1-t)|\delta|^2$.

A sharper two-disk bound is also finite algebra. Rotate $g = a \geq 0$ and put $\delta = x + iy$, $s = |\delta|^2$. Then

$$|\psi^4 - g^4 - 4g^3\delta|^2 = s^2 \{36a^4 + 4a^2s + s^2 + 8ax(6a^2 + s) + 24a^2x^2\}.$$

For fixed a, s , the expression in braces is convex in x . Its maximum on the two disks occurs at a collinear endpoint or on $|\psi| = 1$. Put $r_+ = \min(E, 1)$, $m = \min(A, 1 - r_+)$, $r_- = \min(E, 1 + A)$. The collinear bounds are

$$r_+^2(6m^2 + 4mr_+ + r_+^2), \quad r_-^2(6A^2 - 4Ar_- + r_-^2), \quad \begin{cases} r_-^4, & r_- \leq 1, \\ r_-^2(3r_-^2 - 8r_- + 6), & r_- \geq 1. \end{cases}$$

On the unit circle it remains to maximize

$$Q_A(s) = s^2 \{18A^4 + 12A^2 + 6 - (12A^2 + 8)s + 3s^2\} \quad (239)$$

on $[(1-A)^2, \min(E^2, (1+A)^2)]$. Its endpoints and the sole interior local maximum

$$s_- = (3A^2 + 2 - A\sqrt{4 - 3A^2})/2$$

suffice. For $A = 0$ the interval is degenerate or empty. Taking the maximum of the collinear values and the square root of this last maximum gives the lens bound. To see that no $a < A$ is lost, note that the unit-circle expression is convex in a^2 ; its lower feasible endpoint is already collinear. The collinear expressions increase with their admissible radius after maximization over a . Thus these are whole-disk bounds, not sampled values. Their minimum with (238) is valid.

On a closed excess interval $[L, U]$, an affine radius $e_j(\eta) = A_j + B_j\eta$ is used only when it is nonnegative at both ends and bounds $|\delta_j|$ throughout. Freeze $p_j = \min(1, a_j + \max\{e_j(L), e_j(U)\})$ in the first bound of (238). This gives a quadratic in η with nonnegative leading coefficient. Alternatively, freezing E on the whole cell gives a constant disk or lens bound. A choice between complete bounds is allowed; a coefficientwise minimum is not.

F.5.5 A physical Prawitz comparison

We use the closed-CDF Prawitz majorant [19], in the normalization of Assumption F.1, item 4. To distinguish the factor from the kernel K used there, put

$$\widehat{K}(u) = 2K(u) = (1 - u) + iR(u), \quad R(u) = (1 - u) \cot(\pi u) + 1/\pi.$$

For $Z = (X_1 + \cdots + X_4)/2$,

$$F_Z(x) \leq J_P(x) := \frac{1}{2} + \int_0^1 \Re\{\widehat{K}(u)e^{-6iux}\varphi_P(3u)^4\} du. \quad (240)$$

The real integrand is regular at zero; no convergence of the complex integral by itself is asserted. With $D = x - Z$ and $S(u) = uR(u)$ it is

$$F_x(u) = (1 - u)\mathbb{E} \cos(6Du) + S(u)\mathbb{E}[6D \operatorname{sinc}(6Du)].$$

The cotangent partial fractions, extended to the endpoints, give

$$0 \leq S \leq 1/3, \quad |S'| < 2, \quad |S''| < 5.$$

Here is a direct derivative check. Separate the first pole:

$$S = \frac{1}{\pi} - \frac{2}{\pi}(A + MG), \quad A = \frac{u^2}{1 + u}, \quad M = u^2(1 - u), \quad G = \sum_{j \geq 2} \frac{1}{j^2 - u^2}.$$

On $[0, 1]$, $|A'| \leq 3/4$, $|A''| \leq 2$, $|M| \leq 1/6$, $|M'| \leq 1$ and $|M''| \leq 4$. The derivative series converge uniformly, and

$$G \leq 3/4, \quad |G'| \leq 10/27 < 1/2, \quad |G''| \leq 106/135 < 7/6.$$

The first sum telescopes. For the other two use $j^2 - u^2 \geq 3j^2/4$ and $\sum_{j \geq 2} j^{-k} \leq 2^{-k} + \int_2^\infty t^{-k} dt$, with $k = 4, 6$. The product rule gives $|S'| < 19/18 < 2$ and $|S''| < 223/54 < 5$. For $u > 1/2$, put $v = 1 - u$ and use $v \cot(\pi v) \leq 1/\pi$; for $u \leq 1/2$ the cotangent is nonnegative. This proves $R \geq 0$. The positive partial-fraction terms then give $0 \leq S \leq 1/\pi < 1/3$. Since $\operatorname{sinc}(v) = \int_0^1 \cos(tv) dt$,

$$|F_x''(u)| \leq 42\mathbb{E}|D| + 108\mathbb{E}D^2 + 24\mathbb{E}|D|^3. \quad (241)$$

These differentiations are justified by the displayed moments.

Now assume $|X| \leq R_* = 17/4$ and $\rho < 46/25$. The fourth moment follows from these hypotheses; it is not an additional input. Write $\kappa = \mathbb{E}X^3$, $M_4 = \mathbb{E}X^4$. Lemma F.25 gives the useful cell bounds

$$\begin{aligned} |\kappa| &\leq \min\{\rho, |b - a| + \eta\}, \\ M_4 &\leq \min\{R_*\rho, a^2 - ab + b^2 + (R_* + 2\max(a, b))\eta\}. \end{aligned} \quad (242)$$

For $x = H/2$, choose whole-cell bounds κ_* and $M_{4,*}$ from these formulas and put

$$D_2 = x^2 + 1, \quad D_4 = x^4 + 6x^2 + 2|x|\kappa_* + (M_{4,*} + 9)/4.$$

Then $\mathbb{E}|D| \leq \sqrt{D_2}$ and $\mathbb{E}|D|^3 \leq \sqrt{D_2 D_4}$. Uniformly on $|x| \leq 2$, the simpler bounds $\mathbb{E}D^2 \leq 5$, $\mathbb{E}D^4 < 54$, $\mathbb{E}|D| < 9/4$, $\mathbb{E}|D|^3 < 33/2$ give $|F_x''| < 2061/2$.

Take $N = 256$, $u_j = (j + 1/2)/N$, $q_j = 3u_j$ and

$$\kappa_j(H) = \widehat{K}(u_j)e^{-iq_j H}/N, \quad C_H(Q) = 1 + 2\Re \sum_j \kappa_j(H)g(q_j)^4 - 2\Phi(H/2).$$

Composite midpoint quadrature gives one raw numerator reserve

$$Q_H = \min \left\{ \frac{687}{8N^2}, \frac{42\sqrt{D_2} + 108D_2 + 24\sqrt{D_2 D_4}}{12N^2} \right\}. \quad (243)$$

It controls the integral of the original law P once. No separate reference, linear-transfer or remainder quadrature error is added.

The raw thresholds are $H_\ell = -4 + \ell/8$, $0 \leq \ell \leq 57$. To fill the intervals between them, bound the characteristic function on entire frequency cells, not just at the midpoints. If $a_j \geq |g(q_j)|$ and e_j bounds $|\varphi_P(q_j) - g(q_j)|$ on the whole excess cell, then

$$P_j = \min\{1, a_j + e_j + 3/(2N)\}$$

bounds $|\varphi_P(3u)|$ throughout $j/N \leq u \leq (j + 1)/N$, since $\mathbb{E}|X| \leq 1$. Two complete kernel bounds are

$$u^2|\widehat{K}(u)| \leq u^2(1 - u) + u/3, \quad u^2|\widehat{K}(u)| \leq u^2(1 - u) + u(1 - u)/2.$$

The second follows by retaining the first term of the cotangent partial fractions: $S(u) \leq (1 - u)(1 + 2u)/(\pi(1 + u)) < (1 - u)/2$. Let d_j be the smaller of the two exact integrals over the j th frequency cell. Their separate totals are $1/4$ and $1/6$. Thus, for the same whole geometry and excess cell,

$$|J_P''(x)| \leq B = 36 \sum_j d_j P_j^4.$$

Since $\sup |\Phi''| < 1/4$, the smooth physical function $H \mapsto 2\{J_P(H/2) - \Phi(H/2)\}$ needs the single interpolation reserve

$$E_H = \frac{h^2}{16}(B + 1/4), \quad h = 1/8. \quad (244)$$

The elementary cube, mass and unit radii

$$\eta q_j^3/6, \quad \max(a, b)^2 p_0 q_j^2, \quad 1 + a_j$$

suffice for this curvature calculation. The mass bound follows from $|r_q| \leq q^2/y$ and the first budget in (234).

For $H \leq -4$, Cantelli gives $F_Z(H/2) \leq 1/5$. For $H \geq 25/8$, $\Phi(H/2) > 4/5$. The latter inequality follows, for example, by integrating the alternating Gaussian Taylor polynomial through degree six on $[0, 25/16]$ and using $\phi(0) > 19947/50000$. Consequently both threshold tails have positive normalized score at most $2/5$.

F.5.6 The low-excess certificate

On a geometry rectangle and a closed excess cell $[L, U]$, let M_H bound $C_H(Q) - Tr$ for the entire rectangle. Let Γ_H be obtained from (236), and let

$$R_2(\eta) = r_0 + r_1\eta + r_2\eta^2, \quad r_2 \geq 0,$$

bound the sum of the fourth-power remainders weighted by $|\kappa_j(H)|$. Combining (237) at the same H gives

$$2\{F_{4,P}(H) - \Phi(H/2)\} - T\rho \leq P_H(\eta) := M_H + Q_H + E_H + 2\Gamma_H(\eta) + 2R_2(\eta) - T\eta. \quad (245)$$

When used at grid points, E_H reserves interpolation to adjacent thresholds. The leading coefficient is nonnegative. Endpoint nonpositivity therefore proves the bound on the entire closed excess cell. Equivalently its degree-two Bernstein coefficients are

$$P_H(L), \quad \frac{P_H(L) + P_H(U)}{2} - \frac{p_2(U - L)^2}{2}, \quad P_H(U),$$

where p_2 is the quadratic coefficient. For a quantitative signed score, with $r_L \leq r$ and $\rho < 46/25$, use

$$T + \max_{\eta=L,U} \left\{ \frac{P_H(\eta)}{46/25}, \frac{P_H(\eta)}{r_L + \eta} \right\}.$$

Both denominator faces are needed when the numerator is negative. The cap branch is convex; for the other branch the derivative numerator has derivative $P_H''(\eta)(r_L + \eta) \geq 0$, so an interior critical point can only be a minimum.

Here are the finite enclosures that produce these polynomials. The center is enclosed by signed Taylor expansion in (w, ϑ) or in the contacts (a, b) along probability-preserving paths. For either pair of coordinates i, j ,

$$\partial_i C_H = 8\Re \sum_j \kappa_j g^3 \partial_i g, \quad |\partial_{ik} C_H| \leq 8 \sum_j |\kappa_j| (3D_i D_k + D_{ik}),$$

where D_i, D_{ik} bound the derivatives of the explicit three-atom characteristic function. The first-order terms retain their signs; the second-order error bounds the whole rectangle. The function $r(w, \vartheta)$ is jointly convex. Indeed, writing $r = w^{-1/2} f(\vartheta)$, $f = (1 + \vartheta^2)/\sqrt{1 - \vartheta^2}$, the Hessian determinant has the positive factor

$$3ff'' - (f')^2 = \frac{9 + 9\vartheta^2 + 15\vartheta^4 - \vartheta^6}{(1 - \vartheta^2)^3}.$$

Subtracting its supporting plane therefore encloses $C_H - Tr$ without an absolute Hessian charge for r .

For the signed response, put $R_* = 17/4$, $y_k = k/64$, $0 \leq k \leq 272$, and use their nearest-node closed cells, clipped to $[0, R_*]$. Hermite–Genocchi gives

$$\begin{aligned} r_q(\sigma, y) &= \sum_{m=0}^{60} (i\sigma)^{m+3} b_m h_m + E_q, & |E_q| &\leq \frac{q^3 (qR_*)^{61}}{6 \cdot 61!}, \\ b_0 &= q^3/6, & b_m &= b_{m-1} qR_*/(m+3), \\ h_0 &= 1, & h_m &= (y/R_*) h_{m-1} + (m+1)(c/R_*)^m. \end{aligned} \quad (246)$$

The bound $|\partial_y r_q| \leq q^4/24$ adds the half-cell reserve $q^4/(48 \cdot 64)$. The signed frequency coefficients are summed before bounding their polynomial on a y cell. If g_0, c'_0 are nominal values, $e_q \geq |g - g_0|$ is the geometry error, $p_q \geq |g|$ and $a_{0,q} \geq |g_0|$, then

$$|g^3 - g_0^3| \leq e_q(p_q^2 + p_q a_{0,q} + a_{0,q}^2), \quad |\partial_c r_q| \leq q^4/12.$$

These give a whole-rectangle correction for $g^3 r_q - g_0^3 r_{q,0}$, with no second copy of the series or mesh reserve. Thus the priced supremum bounds actual references throughout the rectangle, not only its nominal center.

The prices used in the finite certificate are chosen from

$$\begin{aligned}\lambda_0 &\in \{0, 1/1000, 1/400, 1/200, 1/100, 1/50, 1/25, 2/25, 4/25, 8/25, 16/25\}, \\ \lambda_{\pm} &\in \{0, 1/2000, 1/1000, 1/400, 1/200, 1/100, 1/50\}, \\ \theta_{\pm} &\in \{0, \pm 1/200, \pm 1/100, \pm 1/50\}.\end{aligned}$$

For each fixed choice, all 273 cells on each of the two tags enter the supremum. When $\lambda_{\sigma} = 0$, only $\theta_{\sigma} = 0$ is used; otherwise the reciprocal contact rays would give an infinite supremum. On a cell $[y_L, y_U]$, with $c \in [c_L, c_U]$, $k_0 \geq 1/(c_U^2 y_U)$. The remaining cost is the concave quadratic $-\lambda_{\sigma} z^2 - \theta_{\sigma} z$ in $z = 1/(y - c)$. Its endpoints and admissible vertex suffice; when the cell crosses the contact both reciprocal rays are included. No division at the contact or at zero is used for the response series.

The same budgets yield whole-cell radius models by pricing $|r_q|$ or its signed coordinate projections. They are combined with the three elementary radii above and the complete two-disk remainder. An independent absolute branch uses

$$|\psi^4 - g^4| \leq E(p^3 + p^2 A + p A^2 + A^3), \quad p = \min(1, A + E),$$

or the complete linear-plus-remainder bound. It is evaluated as a whole alternative. In every branch, the actual reference modulus, the actual normalization $\rho = r + \eta$, one midpoint reserve and one physical threshold reserve are retained. The final maximum over H is taken only after assembling each complete row.

F.5.7 Closing the large excess

Let $p = P(X \geq 0)$ and $q = 1 - p$; zero belongs to the nonnegative class. Condition on exactly j nonnegative observations among four. For $Z = (X_1 + \dots + X_4)/2$, the conditional moments are

$$\mu_j = \frac{u(j - 4p)}{2pq}, \quad V_j = \frac{1}{4} \left[j \left(\frac{v_+}{p} - \frac{u^2}{p^2} \right) + (4 - j) \left(\frac{v_-}{q} - \frac{u^2}{q^2} \right) \right].$$

Closed-event Cantelli gives, for every x ,

$$F_Z(x) \leq \sum_{j=0}^4 \binom{4}{j} p^j q^{4-j} A_j(x), \quad A_j(x) = \begin{cases} 1, & x \geq \mu_j, \\ V_j/(V_j + (\mu_j - x)^2), & x < \mu_j. \end{cases} \quad (247)$$

This includes variance zero and atoms at the threshold. The necessary sign-mass interval is

$$w(1 - \vartheta)/2 \leq p \leq 1 - w(1 + \vartheta)/2.$$

It is covered by closed cells. On a threshold cell $[x_L, x_U]$, bound $A_j(x_U)$ and subtract a lower bound for $\Phi(x_L)$. The five degree-four Bernstein coefficients bound the binomial mixture jointly; its masses are not maximized independently. Each retained sign-mass cell supplies a nonnegative raw numerator bound d_i and a correlated lower bound $r_i \leq r$, so its score is at most $d_i/(r_i + \eta)$.

If r_B is the whole-box lower bound for r , put

$$c_C^0 = \max \left\{ 0, \max_i (d_i/T - r_i), \frac{2}{5T} - r_B \right\}, \quad c_C = 10^{-4} \lceil 10^4 c_C^0 \rceil + 10^{-4}.$$

This closes every $\eta \geq c_C$ by a strict bound below T , including the threshold tails. Independently, Shevtsova closes every $\rho \geq 979013/533000$:

$$\frac{3031}{10000} \left(1 + \frac{323}{500\rho} \right) \leq T.$$

Hence $c_S = \max\{0, 979013/533000 - r_B\}$ is another closed sufficient cutoff. Take $c_B = \min(c_C, c_S)$, selecting one complete route. The low-excess interval $[0, c_B]$ and the large-excess range $[c_B, \infty)$ include their common endpoint.

F.5.8 The finite cover and conclusion

Proposition F.28 (Bounded-law certificate implication). *Assume all the terminal inequalities and domain covers in the following finite scheme have been verified. Then, for every standardized law P with*

$$|X| \leq 17/4, \quad \rho(P) < 46/25, \quad 625/2116 \leq w \leq 1, \quad |\vartheta| \leq 13/20,$$

and every real H ,

$$\frac{2\{F_{4,P}(H) - \Phi(H/2)\}}{\rho(P)} \leq T.$$

The same assertion holds for the genuine reflected law $-P$.

Proof. Divide $[625/2116, 1]$ into 20 equal closed bands and $[0, 13/20]$ into 13 closed bands of width $1/20$. There are 260 rectangles. The minimum of $r^2 = (1 + \vartheta^2)^2/[w(1 - \vartheta^2)]$ is attained at the largest w and smallest $|\vartheta|$. A rectangle with $\min r^2 \geq (46/25)^2$ is empty for the strict physical cap. For every other rectangle construct its own large-excess cutoff. If $c_B = 0$, the large-excess bound covers all $\eta \geq 0$ directly. If $c_B > 0$, divide $[0, c_B]$ into 16 equal closed intervals. At each interval check all 58 thresholds, their 57 joins, the two tails, and both actual reflected orientations, using the preceding inequalities. A parent is accepted by its complete bound or by complete bounds for all four children obtained by bisecting both coordinates. Each child has its own cutoff; no part of another child's domain or numerical conclusion is used in its place.

The certificate condition requires a disposition for every one of the 260 closed parent rectangles: a proved necessary-condition contradiction, a complete direct comparison, or complete comparisons on all four children. Acceptance requires the complete rational inequalities and their reconstruction, with no unresolved rectangle. All elementary operations are outward rational enclosures; square roots are checked by integer squaring. Trigonometric, Gaussian and cotangent values are enclosed by finite series with analytic remainders, using the arithmetic language of Assumption F.1, item 7. The response uses (246), not divided quotients at contacts. The assumed complete arithmetic reconstruction must reproduce the whole cover and its upper bounds; no sampled value or optimizer's status supplies an inequality.

For completeness, acceptance has a direct mathematical interpretation. Every actual canonical point belongs to a closed parent. In a nonempty parent or child, the large-excess inequalities cover $\eta \geq c_B$, while the low-excess inequalities cover all of $[0, c_B]$, including $\eta = 0$. The same unknown law supplies the single ν in every priced frequency response. Quadrature and physical interpolation extend the finite rows to every central threshold; the tails cover the rest. Thus the accepted rational inequalities prove the proposition for all actual laws, including bounded infinite supports and zero atoms. $\square$

Proof of Theorem F.24. Suppose $C_4 > T$. Since $T > c_0$, Lemma F.26 gives an unrestricted positive closed maximizing pair in the class of Proposition F.28. The proposition bounds its score by T , a contradiction. Both discrepancy signs are covered:

$$\Phi(x) - F_Z(x) \leq \Phi(x) - F_Z(x-) = F_{-Z}(-x) - \Phi(-x).$$

This is the closed-CDF reflection identity, with the atom retained. Finally, elementary Machin bounds give $\pi < 3927/1250$ and hence $\phi(0) > 19947/50000$; also $\sqrt{10} > 15811/5000$. Therefore

$$C_E > \frac{204862339}{500000000} = \frac{4097}{10000} + \frac{12339}{500000000} > T.$$

No equality case $C_4 = T$ is excluded. $\square$

F.6 Five summands: the conditional composition

This section specializes the common contact, Fourier, resource and count arguments to five summands and proves a separate support-free comparison for the remaining moment corner. Its conclusion is conditional on the finite partitions and numerical enclosures specified below.

F.6.1 The contrary witness and its predecessor allowance

Lemma F.29 (Zero allowance at the fifth index). *Assume $C_4 \leq C_E$ and $C_5 > C_E$. There is an attaining standardized marginal P and a finite raw threshold H with positive closed-CDF discrepancy. For that same marginal,*

$$q_5 = 5(R_4(P) - C_5)_+ = 0. \quad (248)$$

Its support is compact before any higher moments are invoked.

Proof. Assumption F.1, item 1 supplies the attained positive closed edge, with reflection of the whole marginal when required. Since $R_4(P) \leq C_4 \leq C_E < C_5$, (248) follows. Assumption F.1, item 1 then gives compact support from contact with a positive cubic leading coefficient. No fourth or fifth moment is used in either step. In particular, the zero allowance is not an assertion that $C_5 \leq C_4$. $\square$

The quantitative support argument of Assumption F.1, item 2 is applied at the actual index $n = 5$ with (248). A finite support certificate must prove its pointwise contact exclusion on both exterior support rays and the entire threshold line, including both infinite threshold tails. A finite mesh without its whole-cell remainder does not establish this implication. Only after this acquisition may the supported moment and Fourier inequalities in Assumption F.1, items 4, 5 and 6, be used.

F.6.2 The complete moment-domain cover

Write $\beta = \mathbb{E}[X|X|]$ and $m = \mathbb{E}X_+ = \mathbb{E}X_- > 0$. Lemma 2.4 gives $|\beta| < 1$ and the necessary restriction

$$\rho \geq \frac{1 + \beta^2}{\sqrt{1 - \beta^2}}. \quad (249)$$

Define

$$b(r) = \left(\frac{r\sqrt{r^2 + 8} - r^2 - 2}{2} \right)^{1/2}.$$

Solving (249) shows that $\rho \leq r$ implies $|\beta| \leq b(r)$. The finite specification uses

$$b_2 = \frac{84729828287999}{25000000000000}, \quad b_3 = \frac{35139006743001}{10^{14}}. \quad (250)$$

Before using these values the scalar certificate must verify $b_2 \geq b(237/200)$ and $b_3 \geq b(6/5)$, with positive radicals and outward rounding. These are finite exact inequalities, not measured approximations to a moment domain.

The proposed low-moment domain decomposition is the following. In the table, “full” means the whole necessary signed beta range from (249). Every interval is closed. The gapped method means resource acquisition followed, if needed, by the count comparison; it does not assume a support gap at the beginning of the row.

Region	ρ interval	β interval	Method
B0	$[1, 11/10]$	full	resources/counts
BN	$[11/10, 23/20]$	full	resources/counts
B2 S0	$[23/20, 467/400]$	$[-b_2, 1/5]$	resources/counts
B2 S1	$[23/20, 467/400]$	$[1/5, b_2]$	resources/counts
B2 S2	$[467/400, 237/200]$	$[-b_2, 1/5]$	resources/counts
B2 S3	$[467/400, 237/200]$	$[1/5, b_2]$	resources/counts
B3 S0	$[237/200, 477/400]$	$[-b_3, 1/5]$	resources/counts
B3 T0	$[237/200, 477/400]$	$[1/5, 3/10]$	resources/counts
B3 T1	$[237/200, 477/400]$	$[3/10, b_3]$	resources/counts
B3 S2	$[477/400, 6/5]$	$[-b_3, 1/5]$	resources/counts
B3 T2	$[477/400, 6/5]$	$[1/5, 3/10]$	resources/counts
B3 lower	$[477/400, 6/5]$	$[3/10, 17/50]$	resources/counts
B3 corner	$[477/400, 6/5]$	$[17/50, b_3]$	gap-free

This is a domain cover once the scalar enclosures just stated are proved. Each gapped row requires a complete closed cover of $p \in [0, 1]$; the impossible faces $p = 0, 1$ are excluded by centering and unit variance. On every other panel either a proved resource contradiction excludes the law or a simultaneously acquired contract passes to all six original nearest-count roots. Signed beta scopes are not identified by reflection. In particular, the boundary $\beta = 17/50$ belongs to both adjacent pieces.

The proposed supported Fourier domains above this cover are

ρ interval	Required acquired support radius R
$[6/5, 5/4]$	$13/5$
$[5/4, 7/5]$	$117/40$
$[7/5, 3/2]$	$25/8$
$[3/2, 8/5]$	$133/40$
$[8/5, 17/10]$	$7/2$
$[17/10, 7361/4000]$	$15/4$

For each row, the support certificate must first acquire $|X| \leq R$. A valid actual-index route is Lemma E.2, with $n = 5$, zero predecessor allowance, and

$$S = 1119/1000, \quad E = 91/1000, \quad c = 819/2000.$$

For the six rows in their displayed order, take r to be the upper ρ endpoint and take the respective upper bounds on $|\beta|$ to be

$$0.389174, \quad 0.478710, \quad 0.525521, \quad 0.565414, \quad 0.600001, \quad 0.641563.$$

They follow from the signed-second-moment inequality, since in each row $(1+b^2)^2 > r^2(1-b^2)$ and $0 < b < 1$. Exact substitution gives $Q(R), Q'(R), Q''(R), L(R), L'(R) < 0$ for that lemma's polynomials; $S^2 > 5/4$ and $((E + 11/2)/5)^2 > 5/4$ give the scale bounds. The remaining support conditions are precisely the six pairs of whole-threshold endpoint inequalities

$$\sup_{H \in \mathbb{R}} \mathcal{H}_5(H, \eta R) < -cQ(R), \quad \eta \in \{-1, 1\}.$$

Those inequalities are not proved by these rational ray gates. The complete-threshold method in Appendix E specifies how a finite certificate could establish them, including both infinite tails. The newer support theorem for $n \geq 42$ is not applicable at this step.

The Fourier certificate must then majorize all four terms of the Prawitz bound in Assumption F.1, item 4, including endpoint pieces and the infinite normal tail, on a complete closed strip cover. Its required conclusion is an upper score below $c_0 := 4097/10000$.

For the remaining moment tail the published iid structural estimate [26] gives

$$R_5(P) \leq \frac{3031}{10000} \left(1 + \frac{323}{500\rho} \right).$$

The scalar certificate must check its value at $\rho = 7361/4000$ is at most $819/2000$ and that $819/2000 < c_0 < C_E$. The displayed majorant decreases in ρ , so this single endpoint comparison covers the entire infinite tail. The common smoothing and constant-enclosure arguments are required by Assumption F.1, items 7 and 4.

F.7 A direct sign-count inequality without a support gap

The inequalities in this section apply to every standardized common iid marginal in their stated domain. They require neither extremality nor contact, bounded support, a predecessor estimate, or a physical support gap. Put $p = \mathbb{P}(X > 0)$, $q = 1 - p$, $v = pq$ and $h = v/m$. Both p, q are positive. Assign zero atoms to the nonpositive class and set

$$B = \mathbf{1}_{\{X > 0\}}, \quad Y = p + hX, \quad W = Y - B.$$

Then $\mathbb{E}(W \mid B) = 0$. Direct conditional second-moment calculation gives

$$v_0 = p^2 \left(\frac{q(1-\beta)}{2m^2} - 1 \right), \tag{251}$$

$$v_1 = q^2 \left(\frac{p(1+\beta)}{2m^2} - 1 \right). \tag{252}$$

For $J = B_1 + \dots + B_5$, write

$$b_j(p) = \binom{5}{j} p^j q^{5-j}, \quad V_j = (5-j)v_0 + jv_1.$$

Conditioning first on a particular sign vector gives independent conditional variables. Every sign vector with count j gives the same sum law, of variance V_j , which justifies conditioning only on $J = j$. This is the specialization of the common representation in (188).

Lemma F.30 (Gap-free count bound). *Define*

$$T(V, d) = \begin{cases} 1, & d \leq 0, \\ V/(V + d^2), & d > 0. \end{cases}$$

For every real z ,

$$\mathbb{P}\left(\sum_{i=1}^5 Y_i \leq z\right) \leq Q(z) := \min \left\{ 1, \sum_{j=0}^5 b_j(p) T(V_j, j - z) \right\}. \quad (253)$$

At $H = (z - 5p)/h$, the normalized positive closed-CDF error is at most

$$\frac{\sqrt{5}}{\rho} \left[Q(z) - \Phi \left(\frac{(z - 5p)m}{\sqrt{5}pq} \right) \right]. \quad (254)$$

Proof. For a centered variable U of variance V and $d > 0$, Markov applied to $(U - V/d)^2$ on $\{U \leq -d\}$ gives $\mathbb{P}(U \leq -d) \leq V/(V + d^2)$. If $V = 0$, that event is empty. At $d = 0$ the safe bound is one even when $V = 0$. Apply this inequality separately to each conditional sum and use its actual binomial mass. Finally $\sum Y_i = 5p + hS_5$ gives the Gaussian coordinate in (254). $\square$

In particular, $T(0, 0) = 1$ retains a possible atom on an integer count boundary. It cannot be replaced by zero.

F.7.1 Whole-box bounds in the remaining corner

The additional necessary moment restrictions are

$$m \geq \frac{1 + \beta^2}{2\rho}, \quad \frac{2m^2}{1 + \beta} \leq p \leq 1 - \frac{2m^2}{1 - \beta}, \quad m^2 \leq pq. \quad (255)$$

The first follows from $\rho \geq (a_+^2 + a_-^2)/m$, the next two from $m^2 \leq pa_+$ and $m^2 \leq qa_-$, and the last from $1 \geq m^2/p + m^2/q$.

Take a closed beta slab $[b_l, b_u] \subset [17/50, b_3]$ while retaining the whole moment interval $\rho \in [477/400, 6/5]$. Set

$$m_l = \frac{1 + b_l^2}{2(6/5)}, \quad p_l = \frac{2m_l^2}{1 + b_u}, \quad p_u = 1 - \frac{2m_l^2}{1 - b_l}.$$

Every actual law in the slab lies in this probability interval. The scalar domain gates require $0 < p_l \leq p_u < 1/2$ for every nonempty slab. Round this interval outward only while preserving $0 < l \leq u < 1/2$. For a closed box $p \in [l, u]$, $z \in [a, b]$ with $0 \leq a \leq b \leq 5$, define

$$\begin{aligned} \bar{v}_0 &= \max \left\{ 0, u^2 \left(\frac{(1 - l)(1 - b_l)}{2m_l^2} - 1 \right) \right\}, \\ \bar{v}_1 &= \max \left\{ 0, (1 - l)^2 \left(\frac{u(1 + b_u)}{2m_l^2} - 1 \right) \right\}. \end{aligned}$$

They bound (251)–(252) above: on an actual law each bracket is nonnegative and every positive factor has been enlarged. If an upper bracket is negative, the actual-law domain is empty; clipping to zero remains a safe relaxation. The maximum of $b_j(p)$ on $[l, u]$ is at $p_j = \min(u, \max(l, j/5))$. Since T increases with V and decreases with a positive distance, a whole-box upper CDF is

$$\bar{Q} = \min \left\{ 1, \sum_{j=0}^5 b_j(p_j) T((5 - j)\bar{v}_0 + j\bar{v}_1, j - b) \right\}.$$

This includes $j - b = 0$ with the preceding closed-boundary convention.

Let $\nu = a - 5u$. A lower bound for the Gaussian argument is

$$x_l = \begin{cases} \nu m_l / (\sqrt{5} u(1-u)), & \nu \geq 0, \\ \nu / \sqrt{5u(1-u)}, & \nu < 0. \end{cases} \quad (256)$$

For the first branch, use the lower mean and the upper variance factor. For the second branch, if $z - 5p \geq 0$ the negative lower bound is automatic. Otherwise $m \leq \sqrt{pq}$ reduces the claim to the monotonicity, valid for $z \geq 0$ and $p < 1/2$,

$$\frac{d}{dp} \frac{z - 5p}{\sqrt{p(1-p)}} = -\frac{5p + z(1-2p)}{2[p(1-p)]^{3/2}} < 0.$$

Monotonicity in z then gives (256). For a negative numerator, exact evaluation divides by a lower square root; for a nonnegative numerator it divides by an upper square root. The resulting whole-box score bound is

$$\frac{\sqrt{5}}{477/400} \max\{0, \bar{Q} - \Phi(x_l)\}. \quad (257)$$

The positive part must be taken before an upper-factor substitution; otherwise a negative difference would reverse the required bound.

F.7.2 Finite subdivision and the two infinite phase rays

Use the closed beta slabs with successive endpoints $0.340, 0.341, \dots, 0.351, b_3$, subject to exact checking of their order and the domain gates above. Probability intervals may be rounded outward to a 10^{-6} grid when those gates remain true. The compact phase cover is $[0, 1], \dots, [4, 5]$. A certificate consists of finite binary midpoint subdivisions in p, z , retaining both closed children. At every leaf, rational Gaussian and radical enclosures must establish that (257) is strictly below c_0 .

For the exterior phase rays take the same moment bounds m_L, p_L, p_U for the whole corner $\beta \in [17/50, b_3]$. With the same exact gates, put

$$v_U = p_U(1 - p_U), \quad s_U = v_U^2/m_L^2 - v_U, \quad V_j^U = \max\left\{\frac{j}{p_L}, \frac{5-j}{1-p_U}\right\} s_U.$$

For an actual law $v = pq \geq m_L^2$ and $s = \mathbb{E}W^2 = v^2/m^2 - v$. The function $v^2/m_L^2 - v$ increases on $v \geq m_L^2$, so $s \leq s_U$. Also $qv_0 + pv_1 = s$ implies $V_j \leq \max\{j/p, (5-j)/q\} s \leq V_j^U$, and $b_j(p) \leq \binom{5}{j} p_U^j (1 - p_L)^{5-j}$. For $z \leq 0$, drop the nonnegative Gaussian CDF and use monotonicity at zero. The uniform score upper bound is

$$B_- := \frac{\sqrt{5}}{477/400} \left[(1 - p_L)^5 + \sum_{j=1}^5 \binom{5}{j} p_U^j (1 - p_L)^{5-j} \frac{V_j^U}{V_j^U + j^2} \right]. \quad (258)$$

For $z \geq 5$, use the upper CDF bound one. Its Gaussian argument is at least its value at five, namely $\sqrt{5}m/p$, and hence at least $\sqrt{5}m_L/p_U$. Cantelli for a standard normal gives

$$B_+ := \frac{\sqrt{5}}{477/400} \frac{1}{1 + 5m_L^2/p_U^2}. \quad (259)$$

The scalar certificate must prove $B_- < c_0$ and $B_+ < c_0$ with outward square roots. Thus no unbounded threshold range is discarded.

Proposition F.31 (Verified gap-free corner). *Suppose the beta enclosures and domain gates above hold, a complete finite compact-phase subdivision has every whole-box bound (257) below c_0 , and (258)–(259) are below c_0 . Then every standardized iid marginal with*

$$\rho \in [477/400, 6/5], \quad \beta \in [17/50, b_3]$$

has positive normalized closed-CDF error below c_0 at every raw threshold. This implication requires no predecessor, support, contact, or physical-gap hypothesis.

Proof. Each actual law belongs to a closed beta slab and its closed probability interval by (255). Each compact phase belongs to an original box. Induction up its finite binary partition applies (257) on every leaf, including shared cut faces. The two exterior bounds cover the remaining phases. Since $h > 0$, the map $H \mapsto z = 5p + hH$ covers the whole real line. $\square$

F.8 The acquired resource and count obligations at five

For the attained contrary law outside the direct corner, the implication chain is contact and $q_5 = 0$, then support and Fourier bounds, then an initial resource cover, then propagated resources, then counts. The initial cover starts with no assumed physical defect or central-gap seed. Every final surviving cell must acquire a genuine central support gap before an inequality requiring that gap is applied. The continuous statements required are the contact lemma and conserved identity in Assumption F.1, items 1 and 5; the resource enclosure and propagation in item 6; and the count implication in Proposition F.8. Their finite gates still have to be verified on the actual fifth-index domains; an invocation by name is not such a verification.

The same physical defect is used throughout:

$$D = (3 + \sqrt{10})\rho - \gamma - 3J, \quad \gamma = \mathbb{E}X^3, \quad J = \sum_{(a,b)} \tau_{a,b}(b-a)^2, \quad \tau(t) = \mathbb{E}[X\mathbf{1}_{\{X \geq t\}}].$$

The bounded complementary support intervals use the half-open convention in Assumption F.1, item 5. The conserved identity integrates the same contact slack in two ways and retains the complete Fourier tail; it does not use a large-index expansion. All formulas are evaluated at $n = 5$, with their positivity and denominator gates retained.

After a central gap $(-A, B)$ is acquired, the bounds concern the same marginal's moments, endpoints and conditional noise. In particular, already proved floors $m \geq m_L$, $A \geq a$, $B \geq b$ retain the price $6mAB \geq 6m_Lab$. It may be combined with another lower price by their maximum, not by an unsupported sum. If $6mAB \geq P$ and $m \leq m_U$, then $4AB \leq (A+B)^2$ gives $A+B \geq \sqrt{2P/(3m_U)}$. For the new price alone,

$$\frac{2(6m_Lab)}{3m_U} \leq 4ab \leq (a+b)^2.$$

It therefore retains a proved gap and cannot create a gap from a failed initial cover. The simultaneous signed-resource consequences are those of Assumption F.1, item 6, with the same phase passed through Assumption F.1, item 6.

F.8.1 The actual iid count probability

Use the representation of Section F.1. At nearest-count phase $z = k + u$, let W_j be the sum of the five residuals conditional on exactly j positive signs. The common closed-count identity gives

$$\mathbb{P}(S_5 \leq H) \leq F_{\text{Bin}(5,p)}(k) - b_k(p)\mathbb{P}(W_k > u) + \sum_{j>k} b_j(p)\mathbb{P}(W_j \leq u - (j-k)). \quad (260)$$

The outgoing event is strict and the incoming events are closed. An atom on the event plane is never subtracted twice. Incoming bounds use the proved Cantelli, exponential or stop-loss inequalities of the common count section. A basic outgoing minorant is

$$G_a(x) = -x^4 + 3a^2x^2 + 2a^3x = x(x+a)^2(2a-x), \quad a > 0.$$

It is nonpositive on $x \leq 0$ and has global maximum $(9/4 + 3\sqrt{3}/2)a^4$. Thus

$$\mathbb{P}(U > u) \geq \frac{\mathbb{E}G_a(U-u)}{(9/4 + 3\sqrt{3}/2)a^4}.$$

This is the $c = 2$ case of Lemma F.3. Acquired bounded support and common conditional moments control its expectation. The same pair of conditional third moments is retained in all constraints. Linear objectives, or affine ratios with a proved strictly positive denominator, attain their maxima over the resulting bounded polygon at its vertices. The polygon is a relaxation containing every actual law; its feasible points need not represent convolution laws.

A finite count certificate must reconstruct each of the six original roots $k = 0, \dots, 5$, not just its contracted survivors. It must verify all coupled moment, physical-defect, phase and Gaussian inequalities on each original closed leaf box. A leaf closes only by a proved necessary-condition contradiction or an upper score below a fixed rational lower enclosure of C_E . The acquired mean, scale and threshold bounds must justify the original nearest-count domain as in Proposition F.8; any thresholds outside it require their own tail bounds. Zero noise and equality faces are included.

F.8.2 The conditional fifth-index conclusion

Theorem F.32 (Verified fifth-index composition). *Assume $C_4 \leq C_E$. Suppose exact finite certificates establish all of the following obligations using the continuous inequalities above:*

1. *the scalar beta and target enclosures, all domain gates, and the structural-tail endpoint comparison in Section F.6.2;*
2. *quantitative support followed by the complete Fourier comparisons on every listed moment strip above $6/5$;*
3. *complete initial and propagated same-law resource covers on every resources/counts row, with either a proved contradiction or a complete six-root count certificate at each surviving panel;*
4. *the compact-phase forest and both infinite phase-ray comparisons in Proposition F.31.*

Then $C_5 \leq C_E$.

Proof. Suppose $C_5 > C_E$. Lemma F.29 supplies an attained positive closed-edge witness with zero predecessor allowance. Its moments satisfy $\rho \geq 1$ and (249). If $\rho \geq 6/5$, the acquired support, complete Fourier cover and decreasing structural tail exclude it by the first two certificate obligations. Otherwise it belongs to a closed row of the low-moment table. The third obligation and the continuous resource/count implications exclude every resources/counts row. The fourth obligation and Proposition F.31 exclude the remaining corner. All shared boundaries, in particular $\rho = 477/400$, $\rho = 6/5$ and $\beta = 17/50$, remain included. Hence no contrary maximizer exists. $\square$

The finite certificate conditions of Theorem F.32 remain to be completed.

F.9 Signed moment and physical-gap refinements

F.9.1 A sign-aware paired Fourier coefficient rectangle

Assume $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, $|X| \leq R$, and

$$\mu_4 \leq M \leq 111/10, \quad |\gamma| \leq g \leq 37/20, \quad \mathbb{E}|X|^5 \leq \eta_5, \quad 0 \leq g_l \leq |\gamma|, \quad M \geq 1 + g_l^2.$$

Here $R \geq 1$. Reflection is permitted only inside the auxiliary Fourier integral, to write $t = |\gamma| \geq 0$; the signed γ in the actual contact identity is retained. Put $Q = X^2 - tX - 1$ after this auxiliary reflection. The Pearson identities (86), applied to this same standardized law, give

$$\begin{aligned} \mathbb{E}Q^2 &= \mu_4 - 1 - t^2 =: v_Q \geq 0, \\ \mu_5 &= t^3 + 2t + \mathbb{E}[Q^2(X + 2t)], \\ \mu_6 &= 1 + 3t^2 + t^4 + \mathbb{E}[Q^2((X + t)^2 + 2t^2 + 2)]. \end{aligned} \quad (261)$$

For $z = (10t - \mu_5)/120$ and $c = (3 - \mu_4)t/144$ this proves

$$\begin{aligned} L(t) &\leq 120z \leq U(t), \\ L(t) &= 8t - t^3 - (R + 2t)(M - 1 - t^2), \\ U(t) &= 8t - t^3 + (R - 2t)_+(M - 1 - t^2). \end{aligned} \quad (262)$$

Indeed $-R + 2t \leq X + 2t \leq R + 2t$. For the upper bound on z , the coefficient $R - 2t$ multiplies v_Q : use $v_Q \leq M - 1 - t^2$ if this coefficient is nonnegative, and $v_Q \geq 0$ otherwise. The lower bound uses the nonnegative coefficient $R + 2t$. This explains the sign switch without requiring $R > 2g$.

Every coefficient enclosure must cover the whole interval $[g_l, g]$. The cubic

$$L(t) = -R(M - 1) + (10 - 2M)t + Rt^2 + t^3$$

has its minimum at an endpoint or a root of $3t^2 + 2Rt + 10 - 2M$ in the interval; if $M \leq 5$ it is increasing. For U , split at $R/2$ and maximize the cubics

$$R(M - 1) + (10 - 2M)t - Rt^2 + t^3, \quad 8t - t^3$$

on their respective closed subintervals. Endpoints and derivative roots suffice. Retaining infeasible t with $t^2 > M - 1$ merely enlarges the enclosure domain; (262) is used only for actual laws. If $\ell \leq \min L$ and $u \geq \max U$, safe endpoints are

$$z_l = \max\{-\eta_5 - 10g, \ell\}/120, \quad z_u = \min\{\eta_5 + 10g, u\}/120, \quad Z = \max(|z_l|, |z_u|),$$

with $z_l \leq z_u$ required. The deliberately symmetric absolute fifth moment bound in this intersection is valid even though $t \geq 0$. The other coefficients satisfy

$$\begin{aligned} |(3 - \mu_4)/24| &\leq a_* := \max\{|3 - M|, |2 - g_l^2|\}/24, \\ c_l &= \frac{3 - M}{144} \begin{cases} g_l, & M \leq 3, \\ g, & M > 3, \end{cases} \quad c_u = \frac{1}{144} \max_{g_l \leq t \leq g} (2t - t^3). \end{aligned}$$

These follow from $1 + t^2 \leq \mu_4 \leq M$. The maximum in c_u is at g , at g_l , or at $\sqrt{2/3}$ according as the interval is below, above, or contains that stationary point.

For completeness, the cancellation norm needed by the paired transfer is

$$J(z, c) = \int_0^\infty e^{-x^2/2} |zx^2 - cx^4| dx.$$

The cancellation-norm calculation in the proof of Lemma A.7 applies to every real pair (z, c) : J is convex, its enclosing rectangle is bounded by its four corners, and $J(-z, -c) = J(z, c)$. When $zc \leq 0$, $J = \sqrt{\pi/2}(|z| + 3|c|)$. When $zc > 0$, reverse both signs if necessary and put $a = \sqrt{z/c}$. The exact Gaussian-tail identity in the proof of that lemma becomes

$$J(z, c) = \sqrt{\pi/2}(z - 3c)(1 - 4\{1 - \Phi(a)\}) + 6cae^{-a^2/2}, \quad z, c > 0. \quad (263)$$

Here $\int_a^\infty e^{-x^2/2} dx = \sqrt{2\pi}(1 - \Phi(a))$, so this formula retains the signed tail term before any upper bound. In particular a possibly negative coefficient $z - 3c$ requires the appropriate lower or upper Gaussian enclosure; taking its absolute value is a different bound.

The sixth moment follows directly from (261):

$$\mu_6 \leq M_6 := \min\{R\eta_5, 1 + 3g^2 + g^4 + (R^2 + 2Rg + 3g^2 + 2)(M - 1 - g_l^2)\}.$$

All multiplied factors are nonnegative. These coefficient bounds may be inserted into a separately proved signed Fourier transfer only on its original domain ($n > 2$ and positive damping included). Its damping is $b_0 = (45 - M)(1 - 2/n)/96$, not $43(1 - 2/n)/96$. They establish no Fourier integral or cutoff by themselves.

Remark F.33 (Why the old sign substitution cannot be widened). For $\mathbb{P}(X = 3/2) = 4/13$ and $\mathbb{P}(X = -2/3) = 9/13$, $\gamma = 5/6$, $\mu_4 = 61/36$ and $120z = 1315/216$. The valid loose bounds $R = 3/2$, $g = 7/5$, $M = 197/40$ obey $R > g$, but the old unsigned quadratic value $951/160$ is smaller than $1315/216$. Thus dropping $R \geq 2g$ from that older formula is invalid; the piecewise argument above is a distinct repaired coefficient bound.

F.9.2 A quadratic signed-moment loss

Lemma F.34. *For a standardized law with finite fifth absolute moment, write $\rho = \mathbb{E}|X|^3$, $\beta = \mathbb{E}X|X|$, $\gamma = \mathbb{E}X^3$ and $m = \mathbb{E}X_+ > 0$. Then*

$$\frac{\rho\mathbb{E}X^4}{2} - \frac{\mathbb{E}|X|^5}{4} \leq \frac{\rho^3}{4} - \frac{(\gamma - \rho\beta)^2}{8m}.$$

Equality holds for every centered standardized two-point law.

Proof. Use the weighted Cauchy–Schwarz calculation leading to (100), with $Y = |X|$ and $Z = Y^2 - \rho Y$. That calculation requires only $\mathbb{E}|X|^5 < \infty$ and $\mathbb{E}|X| = 2m > 0$, which hold here; it also includes a possible zero atom. For support $\{-a, b\}$ with $ab = 1$, $\rho = (a^2 + b^2)/(a + b)$ and $a^2 - \rho a = -(b^2 - \rho b)$. Thus Z is proportional to $\text{sgn}(X)$ on the support, which is precisely the equality condition. $\square$

On a rectangular (ρ, β, γ) domain, all eight corners enclose $\delta = \gamma - \rho\beta$. If its interval has distance d_δ from zero and $m \leq m_u$, the certified loss is $d_\delta^2/(8m_u)$. It is subtracted only in this matching constant term. It does not by itself bound the physical gap defect.

F.9.3 Sign probabilities, including zero atoms

Lemma F.35. *Every standardized law with finite ρ satisfies*

$$\mathbb{P}(X > 0), \mathbb{P}(X < 0) \geq \kappa/\rho^2, \quad \kappa = \frac{8}{27}(7\sqrt{7} - 17).$$

Consequently $p = \mathbb{P}(X > 0) \in [\kappa/\rho^2, 1 - \kappa/\rho^2]$ even when $\mathbb{P}(X = 0) > 0$.

Proof. Put $a = \mathbb{E}X_+^2 \in (0, 1)$ and $m = \mathbb{E}X_+ = \mathbb{E}X_- > 0$. Signwise Cauchy gives $a^2 \leq m\mathbb{E}X_+^3$, $(1-a)^2 \leq m\mathbb{E}X_-^3$ and $m^2 \leq pa$. Therefore $p\rho^2 \geq (2a^2 - 2a + 1)^2/a$. Its derivative is

$$\frac{(2a^2 - 2a + 1)(6a^2 - 2a - 1)}{a^2}.$$

The unique minimum in $(0, 1)$ is $a_0 = (1 + \sqrt{7})/6$, where the value is κ . Reflect to obtain the negative-sign inequality; $\mathbb{P}(X < 0) \leq 1 - p$ proves the upper bound on p . Sharpness follows by fixing $a = a_0$, $0 < p < 1 - a$, $m = \sqrt{pa}$, and assigning masses p at a/m , $pa/(1-a)$ at $-(1-a)/m$, and the remaining mass at zero. Direct substitution gives mean zero, variance one, and $p\rho^2 = \kappa$. $\square$

The rational lower constant 11261/25000 is licensed by $\sqrt{7} > 10583/4000$ (square both positive sides), followed by rational substitution. No support or optimizer hypothesis enters this lemma.

F.9.4 Spending one outer-gap residual

For a standardized law P , assume a genuine central support gap $(-A, B)$ with $A, B > 0$, and $\text{supp } P \subset [-\mathcal{R}_-, \mathcal{R}_+]$. Use the same physical D as in Lemma 5.1, and write $m = \mathbb{E}X_+ = \mathbb{E}X_-$, $\sigma = \sqrt{10}$. The moment residuals in that lemma are

$$T_+ = \mathbb{E}X_+^3 - mB^2, \quad R_+ = \mathbb{E}[X_+(X_+ - B)], \quad T_- = \mathbb{E}X_-^3 - mA^2, \quad R_- = \mathbb{E}[X_-(X_- - A)].$$

On the positive support, $X(X^2 - B^2) \leq (\mathcal{R}_+ + B)X(X - B)$, hence $T_+ \leq (\mathcal{R}_+ + B)R_+$; reflection gives the other inequality. All four residuals are nonnegative. Lemma 5.1 and (16) give

$$\begin{aligned} D &= mQ + (\sigma - 1)T_+ + (\sigma + 1)T_- + 3(T_+ + T_- - J_o), \\ Q &= (\sqrt{\sigma + 1}A - \sqrt{\sigma - 1}B)^2, \quad T_+ + T_- - J_o \geq 2BR_+ + 2AR_-. \end{aligned} \quad (264)$$

For acquired lower endpoints $A_l, B_l > 0$ and $S_l \leq \sqrt{10}$,

$$w_+T_+ + w_-T_- \leq D - mQ, \quad w_+ = S_l - 1 + \frac{6B_l}{\mathcal{R}_+ + B_l}, \quad w_- = S_l + 1 + \frac{6A_l}{\mathcal{R}_- + A_l}. \quad (265)$$

Require $w_{\pm} > 0$. The ratios increase in the endpoint and decrease in the support radius, proving the replacement directions. If $Q \geq q_m \geq 0$, $m \geq m_l$, $D \leq d$, the common residual is $d_o = d - m_l q_m$. A negative d_o excludes the cell. Otherwise

$$T_+ \leq \min\{B_{3,+} - m_l B_l^2, d_o/w_+\}, \quad T_- \leq \min\{B_{3,-} - m_l A_l^2, d_o/w_-\},$$

where $B_{3,\pm}$ are separately acquired third-moment upper bounds. If the retained integral bound is outer $\leq (3/8)(k_+T_+ + k_-T_-)$, then

$$\text{outer} \leq \frac{3}{8} \max(k_+/w_+, k_-/w_-)(D - mQ).$$

In the application below $k_+ = (\rho_u - B_l)_+^2$ and $k_- = (\mathcal{R}_- - \rho_l)_+^2$. The implication follows term by term from (265); it does not spend two independent copies of $D - mQ$. The integral bound itself remains a retained analytic input when this lemma is applied.

F.9.5 Gap-localizing inequalities and mean contraction

Lemma F.36. *Suppose the positive support is at least a proved $B_l > 0$. Set $A_2 = \mathbb{E}X_+^2$, $B_3 = \mathbb{E}X_+^3$, $m = \mathbb{E}X_+$ and $p = \mathbb{P}(X > 0)$. Then*

$$(m - B_l p)(B_3 - B_l A_2) \geq (A_2 - B_l m)^2.$$

If $m \leq m_u$, $A_2 \geq a_{2,l}$ and $B_3 \leq b_{3,u}$, put $d = b_{3,u} - B_l a_{2,l}$. For $d > 0$,

$$p \leq \frac{m_u - (a_{2,l} - B_l m_u)_+^2 / d}{B_l}.$$

For $d < 0$ the cell is empty. For $d = 0$ a positive lower numerator excludes it, and otherwise $p \leq m_u / B_l$ is valid.

Proof. Apply Cauchy–Schwarz to 1 and X under the nonnegative measure $(X - B_l)\mathbf{1}_{X>0} dP$. All three entries in its moment matrix are nonnegative. Enlarge the second diagonal entry to d and lower the off-diagonal entry to $(a_{2,l} - B_l m_u)_+$. Rearrangement gives the positive-denominator case; a nonnegative diagonal cannot be below zero, and a zero diagonal forces the off-diagonal entry to vanish, proving the other cases. $\square$

For fixed endpoints and moment bounds, the resulting cap $P_+(m)$ is nondecreasing, since where $d > 0$ its derivative is $[1 + 2B_l(a_{2,l} - B_l m)_+ / d] / B_l > 0$. The reflected cap $P_-(m)$ has the same property. Thus the necessary conditions $P_+(m) \geq p_l$, $P_-(m) \geq 1 - p_u$, and $P_+(m) + P_-(m) \geq 1$ are monotone. Failure at m_u excludes the cell; finite bisection retaining an infeasible lower endpoint produces a valid lower bound on every feasible m , without an assumption of convergence. None of these arguments acquires the central gap it assumes.

F.9.6 Exact Gaussian phase with an affine physical-defect price

Consider $n \in \{6, 7, 8, 9\}$. Fix one positive-branch maximizing law, a raw threshold with a previously proved $|H| \leq H_{\max}$, and its score C , moments ρ, γ , and physical defect $D \geq 0$. Assume the complete conserved inequality

$$\frac{n\phi D}{6} + n(C - C_E e^{-H^2/(2n)})\rho \leq K_F + s_F |H| + C A_{\text{act}} + H(CK + \phi U) - H^2(C\rho + \gamma\phi/2), \quad (266)$$

where $\phi = \phi(H/\sqrt{n})$, $K_F, s_F \geq 0$ include complete Fourier costs, and every omitted slack is nonnegative. Assume acquired intervals for $C > 0, \rho \geq 1, K, U$, a lower γ_l , and $A_{\text{act}} \leq A_* - t_D D$ with $t_D \geq 0$ independent of C . For $H = \epsilon u$, $u \geq 0$, choose (K_ϵ, U_ϵ) to be the upper endpoints if $\epsilon = 1$ and the lower endpoints if $\epsilon = -1$. Substitution and $C_E/\phi(0) = (3 + \sqrt{10})/6$ give

$$D \leq \frac{P(u) + \phi Q(u)}{n\phi/6 + C t_D}, \quad (267)$$

where

$$\begin{aligned} P(u) &= K_F + C A_* - n C \rho + (s_F + \epsilon C K_\epsilon) u - C \rho u^2, \\ Q(u) &= n(3 + \sqrt{10})\rho/6 + \epsilon U_\epsilon u - \gamma_l u^2/2. \end{aligned}$$

The denominator is strictly positive. For fixed u, ϕ , the ratio is affine in ρ ; for fixed ρ it is fractional-affine in C with positive denominator. Its derivative in C has constant sign. Thus all four (C, ρ) corners suffice, without asserting that the corners themselves are realizable laws.

Cover $[0, L]$, $L = \lceil MH_{\max} \rceil / M$, by all closed mesh cells of width $1/M$ and evaluate both signs. On each cell the exact maxima P_*, Q_* occur at endpoints or an interior stationary point of a concave quadratic. With ϕ enclosed by its positive endpoint density bounds, replacing P, Q by P_*, Q_* enlarges the ratio. It is again fractional-affine in ϕ , so its two endpoints suffice. The maximum over the complete mesh bounds D above; a negative maximum is a contradiction. Application requires the conserved input (266) and the complete threshold bound $H_{\max}$ at the stated actual index. The finite Gaussian enclosure additionally requires $L^2/(2n) \leq 7$.

F.9.7 The central signed price may be negative

Suppose an already acquired genuine central gap gives, for the same law,

$$\begin{aligned} A_{\text{act}} &\leq \frac{3}{2}\beta(J + \gamma) + \text{central} + \text{outer} + \rho^3/4 - \rho/2 - \text{penalty}, \\ \text{outer} &\leq \lambda_o(D - mQ), \quad mQ \geq q_l \geq 0. \end{aligned}$$

Here $\lambda_o \geq 0$, the central term is an upper bound, and the penalty is a lower bound. Substitute $J = ((3 + \sqrt{10})\rho - \gamma - D)/3$ to obtain

$$\begin{aligned} A_{\text{act}} &\leq A_{\text{cen}} - t_{\text{cen}}D, \\ t_{\text{cen}} &= \beta/2 - \lambda_o, \\ A_{\text{cen}} &= \beta((3 + \sqrt{10})\rho/2 + \gamma) + \text{central} - \lambda_o q_l + \rho^3/4 - \rho/2 - \text{penalty}. \end{aligned} \quad (268)$$

The first term is multiaffine in its rectangular variables and is bounded at all corners; use $\rho_u^3/4 - \rho_l/2$ for the remaining moment term. A lower price is $\beta_l/2 - \lambda_o$, since $D \geq 0$. This price can be negative. In that case the proof of (267) still works only if its *total* denominator is proved positive on the complete domain. Different complete candidates may be minimized after each has maximized over its entire score interval. They may not spend the common defect twice.

F.10 Symmetrization and sine inputs from one marginal

Here η_s, ζ_s are symmetrization parameters, distinct from the four-summand excess η . In this section $h = \mathbb{E}|X|$, distinct from the count scale previously called h .

F.10.1 Fixed-first-moment envelopes

Let $0 < p \leq 1/2$, $q = 1 - p$, $v = pq$, and write the same marginal as $Y = -p + U_0$ on mass q and $Y = q + U_1$ on mass p , with $\mathbb{E}U_i = 0$. Put $b = q\mathbb{E}U_0^2$, $c = p\mathbb{E}U_1^2$, $s = b + c$, $w = v + s$, $X = Y/\sqrt{w}$, and $h = 2v/\sqrt{w}$. Assume acquired inward bounds $U_0 \leq a_0$, $-U_1 \leq a_1$, $a_0 + a_1 \leq d \leq p$. Define, for an independent copy with the same marginal,

$$\eta_s = \frac{1}{2}\mathbb{E}|X - X'|^3 - \rho, \quad \zeta_s = 4\mathbb{E}X_+^2\mathbb{E}X_-^2/h.$$

The scalar identity

$$|x - y|^3 = (x - y)^2(|x| + |y|) - 2\min(|x|, |y|)(x - y)^2\mathbf{1}_{xy > 0}$$

gives $\eta_s = h - \mathbb{E}[\min(|X|, |X'|)(X - X')^2\mathbf{1}_{XX' > 0}]$: the expectation of the first term is $2\rho + 2h$ by independence and centering. On a same-sign pair, the smaller magnitude is at least $(p - a_0)/\sqrt{w}$ or $(q - a_1)/\sqrt{w}$, while the conditional expected squared difference is $2\mathbb{E}U_i^2/w$. The deficit is at least

$$\frac{2\{qb(p - a_0) + pc(q - a_1)\}}{w^{3/2}} \geq \frac{2s(v - qd)}{w^{3/2}},$$

since $qa_0b + pa_1c \leq q(a_0 + a_1)s$. Consequently

$$\eta_s \leq \frac{2(v^2 + sqd)}{(v + s)^{3/2}} = E(p, h, d) := \frac{h^3}{4v}(1 - d/p) + (d/p)h. \quad (269)$$

This includes $s = 0$ and sign-boundary atoms. The equality follows by $s = 4v^2/h^2 - v$. For fixed p, d , E increases with $h > 0$. For fixed h , put $a = h^2/(4v)$. On $a \leq 1$,

$$E_p/h = -a(1 - 2p)(1 - d/p)/v - d(1 - a)/p^2 \leq 0, \quad E_d = h(1 - a)/p \geq 0.$$

Thus $p_l \leq p \leq p_u$, $h \leq H$, $H^2 \leq 4p_l(1 - p_l)$ and $d \leq d_u \leq p_l$ imply $\eta_s \leq E(p_l, H, d_u)$. If the squared gate fails, retain $\eta_s \leq h \leq H$; the derivative argument is not used at virtual points with $a > 1$. For $0 < H \leq 1$ and $v_l = p_l(1 - p_l) \geq 1/5$, $4v^2/H^2 - v$ increases on $v \geq v_l$. Hence $s \geq s_L = \max(0, 4v_l^2/H^2 - v_l)$. If $s_L > s_u$ the cell is empty; otherwise an acquired budget $d(s) \leq \min(A - Bs, C - Ds)$, $B, D \geq 0$, supplies $d_u = \min(A - Bs_L, C - Ds_L)$. A negative d_u excludes the cell; it is not silently changed to zero.

The signwise second moments are $(vq + c)/w$ and $(vp + b)/w$, so $\zeta_s = 2(vp + b)(vq + c)/(vw^{3/2})$. Set $b = s(1 - \alpha)$, $c = s\alpha$. The numerator is concave in α ; its smaller endpoint for $p \leq q$ is $\alpha = 1$. Therefore

$$\zeta_s \geq \frac{2(v^2 + ps)}{(v + s)^{3/2}} = Z(p, h) := h/q - h^3/(4q^2), \quad \zeta_s \geq h. \quad (270)$$

For the second bound use $(vp + b)(vq + c) - v^2(v + s) = v(p^2c + q^2b) + bc \geq 0$. For $p \leq 43/100$ and $0 < h \leq 1$, $Z_p = h(q - h^2/2)/q^3 > 0$ and $Z_{hh} = -3h/(2q^2) < 0$. Thus $L \leq h \leq H \leq 1$ yields $\zeta_s \geq \max\{L, \min(Z(p_l, L), Z(p_l, H))\}$ by concavity on the entire interval. If $L^2 \geq 4(1 - p_l)/3$, $Z(p_l, h)$ decreases throughout and the minimum is $Z(p_l, H)$. These are moment inequalities; the sine-circle theorem that consumes ζ_s is a separate input.

F.10.2 The Bernoulli-shaped envelope with all its gates

Proposition F.37. *Assume the same sign coordinates, $8/25 \leq p \leq 43/100$, $0 \leq s \leq 13/1000$, and $11/10 \leq \rho \leq 6/5$. Let fixed $0 < D \leq 9/100$, $2 \leq C \leq 3$, $Q \geq 2/5$ satisfy*

$$a_0 + a_1 \leq D, \quad \rho \leq U(v, s) := \frac{v(1 - 2v) + Cs - s^2/Q}{(v + s)^{3/2}}, \quad C + 75D/16 < 3.$$

For an acquired $p_l \leq p$ require $8p_l - 5p_l^2 - 2 - (1 - p_l)D \geq 0$. Then

$$\eta_s \leq B(\rho) \leq \zeta_s, \quad B(r) = (\sqrt{r^2 + 8} - r)/2.$$

Proof. The centered one-sided cubic inequality gives

$$\rho \geq I(p, s, D) := \frac{v(1 - 2v) + (3p - D)s + s^2/(qD)}{(v + s)^{3/2}}.$$

Indeed the inward cubic estimates give quadratic terms $b^2/(qa_0) + c^2/(pa_1) \geq s^2/(qa_0 + pa_1) \geq s^2/(qD)$ by Cauchy–Schwarz; their linear terms are at least $(3p - D)s$. A zero inward cap forces the corresponding conditional variance to vanish, so that term is omitted. The outward upper U is a separate acquired premise, not a consequence of the parameter rectangle. Put $w = v + s$, $M_1 = v^2 + ps$, and let N_1 be the numerator of I . With $Z = 2M_1/w^{3/2}$, expansion gives

$$\begin{aligned} (IZ + Z^2 - 2)w^3/2 &= M_1(N_1 + 2M_1) - w^3 \\ &= svp[8p - 5p^2 - 2 - qD] + s^2p[pq/D + 8p - 3 - D] + s^3[p/(qD) - 1]. \end{aligned}$$

The first bracket increases with p (derivative $8 - 10p + D > 0$) and is nonnegative by the gate. The second is at least $(1/5)/(9/100) + 8(8/25) - 3 - 9/100 = 1523/900 > 0$; the last is at least $647/153 > 0$. Since $\rho \geq I$ and $\zeta_s \geq Z > 0$, the positive root of $x^2 + \rho x - 2 = 0$ proves the ζ_s assertion.

For the other assertion put $a = Dq$ and $E = 2(v^2 + as)/(v + s)^{3/2}$. Up to positive denominators the derivatives of U have numerators

$$\begin{aligned} J_s &= v(C - 3/2 + 3v) - (C/2 + 2v/Q)s - s^2/(2Q), \\ J_v &= -v/2 - v^2 + s(1 - 4v - 3C/2) + 3s^2/(2Q). \end{aligned}$$

On the larger rectangle $1/5 \leq v \leq 1/4$, $0 \leq s \leq 1/40$, $J_s \geq 963/6400 > 0$ and $J_v \leq -881/6400 < 0$. For J_s , bound its first term below by $11/50$ and its negative terms in magnitude by $11/160$ and $1/1280$. For J_v , drop its negative linear term and use $-1/10 - 1/25 + 3/1280 = -881/6400$. At fixed a the derivative numerators of $E/2$ are $K_s = av - 3v^2/2 - as/2 < 0$ and $K_v = v^2/2 + (2v - 3a/2)s$, since $a/v = D/p \leq 9/32 < 3/10$. Expansion gives $K_v J_s - K_s J_v = \sum_{i=0}^3 T_i s^i$, where

$$\begin{aligned} T_0 &= (v^2/2)[v(C - 3) + a(1 + 2v)], \\ T_1 &= -(C/2 + 3/2)v^2 + av(1 - v) - v^3/Q, \\ T_2 &= -Cv + a(1/2 - 2v) + (-2v^2 + 3av/2)/Q, \\ T_3 &= (3a/2 - v)/Q. \end{aligned}$$

All are negative. For T_0 use $(a/v)(1 + 2v) \leq 75D/16 < 3 - C$; the others use $a/v < 3/10$, $C \geq 2$, $v \geq 1/5$ and $1/2 - 2v \leq 1/10$. Since $v' = 1 - 2p$, $a' = -D$, the determinant for the actual p derivative is $(1 - 2p) \sum T_i s^i - Ds(v + s)J_s < 0$. Hence E decreases in p along a level $U = R$, and decreases in s at fixed p .

To justify the complete level path, fix $R = \rho \in [11/10, 6/5]$. There is a unique $p_B \in (0, 1/2)$ with $(1 - 2v_B)/\sqrt{v_B} = R$, $v_B = p_B(1 - p_B)$, and $2\sqrt{v_B} = B(R)$. The left side decreases; its square at $8/25$ is $124609/85000 > (6/5)^2$, so $p_B > 8/25$. If actual $p \leq p_B$, $\eta_s \leq h \leq 2\sqrt{v} \leq B(R)$. Otherwise, for every $v' \in [v_B, v]$, monotonicity of U gives $U(v', 0) \leq R \leq U(v', s)$. There is a unique root $s_R(v') \in [0, s]$; the intermediate value theorem and $J_s > 0$ give a differentiable path entirely inside the checked rectangle. Holding D, C, Q fixed on that virtual path,

$$\eta_s \leq E(p, s) \leq E(p, s_R(v)) \leq E(p_B, 0) = B(R).$$

No support acquisition at a virtual p is being asserted. $\square$

Since B decreases and $r + B(r)$ increases (differentiate their formulas), on $\rho \in [R_l, R_u]$ the valid conclusions are $\eta_s \leq B(R_l)$, $\zeta_s \geq B(R_u)$ and $\rho + \eta_s \leq R_u + B(R_u)$.

F.10.3 Literal finite-frequency and strip interfaces

The comparison is specified at $n = 26, \dots, 30$ on eleven probability panels (indices $32, \dots, 42$), hence 55 acquired B1 rows. From each same-law row it recomputes, with positive divisor g ,

$$s_u = \min(\lambda_{\max}, \lambda_{\text{acquired}})/n, \quad D = \min(d_0, (d_{\text{budget}} - b)/g),$$

$$C = \max\{3p_u + O_0, 3(1 - p_l) + O_1\}, \quad Q = \max_{p \in \{p_l, p_u\}} \{(1 - p)O_0 + pO_1\},$$

and requires every gate of Proposition F.37. The rows' acquisition and these numerical gates remain conditions.

The additional support step uses $a_0 + a_1 \leq D \leq q - p$ and $D < p$:

$$(p - a_0)^2 + (q - a_1)^2 \geq p^2 + (q - D)^2.$$

For a fixed used inward sum, moving inward mass from the second endpoint to the first has derivative $2(q - p - D) + 4a_0 \geq 0$; the minimum is at $a_0 = 0$, and using all of D decreases the squared sum. The right side decreases in p . Thus a sufficient standardized lower bound is $[p_u^2 + (1 - p_u - D)^2]/[p_u(1 - p_u) + s_u]$. Each row must prove this is at least $17/10$, the squared sum of its outer radii is at most $1037/100$, and each radius is at most $13/5$.

Let 23 consecutive closed ρ strips cover $[11/10, 6/5]$. At each upper endpoint r_u , rational $0 < z_l \leq z_u$ satisfying $z_l^2 + r_u z_l \leq 2 \leq z_u^2 + r_u z_u$ enclose $B(r_u)$. The simultaneous valid inputs are

$$\rho + \eta_s \leq r_u + z_u, \quad \rho^2 - \zeta_s^2 \leq r_u^2 - z_l^2.$$

The number z_u alone is not a uniform upper bound on η_s over the strip. The separate sine theorem may supply the improved factor

$$1 - \frac{17u^2}{200} \left(1 - \frac{1037u^2}{4200} \right)$$

only under $13u/5 \leq \pi$ and $1037u^2/100 \leq 42$, retaining the old valid circle outside. Moment and rectangle-convexity gates remain.

Certificate condition F.38 (B1 cover at the literal five indices). All 55 same-law resource rows satisfy the preceding gates. At each of the five actual indices and each of the 23 strips, a complete four-term Fourier comparison with retained compatible profile 62/625 proves score strictly below $4097/10000$. All low/high mesh cells, endpoint pieces, infinite normal tail, analytic hypotheses and strict comparisons are verified.

There are 115 integral obligations. Under this condition every actual row and every moment in it belongs to a closed strip and is excluded by its whole-strip comparison, provided resource acquisition and the applicable sharp predecessors hold. Every unresolved cell rejects. The conclusion is restricted to B1 at $n = 26, \dots, 30$.

F.11 Conditional reuse of finite floor-index resource sequences

Fix a complete finite sequence of upper bounds evaluated at an index floor N . The actual index is denoted by m ; the first positive mean is denoted by $\mu = \mathbb{E}X_+$. The same unrestricted maximizing law, its raw threshold H at a positive closed-CDF maximum, signed moments, support and conditional noises must be used throughout. The maximizing laws at different indices may differ.

F.11.1 The weighted invariant and a finite induction

Let $D = (3 + \sqrt{10})\rho - \gamma - 3J \geq 0$ be the physical defect, $\mathcal{R}$ the nonnegative complete raw Fourier remainder, and let r, d be saved upper entries. The required induction invariant is

$$\mathcal{R}_{\text{actual}} \leq r, \quad mD_{\text{actual}} \leq Nd. \quad (271)$$

Thus $D_{\text{actual}} \leq d$ when $m \geq N$, but the converse does not license a weighted noise allowance. A charged phase budget B with $mD \leq B/\kappa$, $\kappa = 997/15000$, exports $d = B/(\kappa N)$. A negative B excludes the law before division. Taking the minimum with a previous valid weighted entry, upward rounding, and taking a maximum across a complete cover all preserve (271). Index-free moment, mean, sign, gap and endpoint contractions may use the weaker

$D \leq d$ and still refer to the same law; they do not optimize again within a parameter cell. Induction over a finite saved list therefore justifies every successive entry once each transition has the properties proved below. No convergence or limiting argument is used.

Write $v = p(1 - p)$, $s = v\theta/(1 - \theta)$, and let the actual noise be $\lambda_m = ms$. If a floor affine resource is interpreted as

$$D \geq b + gd_{\text{in}} + (N/m)h_N\lambda_m, \quad b, g, h_N \geq 0, \quad (272)$$

then multiplying by m and using $mD \leq Nd$ and $m \geq N$ gives $b + gd_{\text{in}} + h_N\lambda_m \leq d$. This proves the original joint intercepts for the actual noise; it does not identify λ_m with the floor noise. Equivalently a new count interface uses actual $n = m$, defect bound $(N/m)d$, charge $(N/m)h_N$, and phase normalization $15000/(997m)$. Merely replacing the old integer index is insufficient. A saved harmonic noise-geometry bound may instead be reused through

$$\theta_m \leq \frac{1}{1 + mv_{\text{min}}/\Lambda + v_{\text{min}}/(Bd_0)} \leq \theta_N,$$

with its positive geometry constant B ; it is not a bound on λ_m itself. On an actual noise cell $[a, b]$, the simultaneous consequences $\theta \leq b/(Nv_{\text{min}} + b)$ and $d_{\text{in}} \leq d_0(1 - a/\Lambda)$ retain the same resource budget.

F.11.2 Index-free refinements and decreasing analytic envelopes

The signwise Cauchy bounds

$$\mu^2 \leq p_u(1 + \beta_u)/2, \quad \mu^2 \leq (1 - p_l)(1 - \beta_l)/2, \quad \mu^2 \leq \max_{p \in [p_l, p_u]} p(1 - p)$$

are independent of m . So are moment feasibility, acquired central-gap bounds and physical endpoint prices. In particular, once separately proved endpoint inequalities supply

$$P = \max\{(S_l - 1)\rho_l, S_l\rho_l - \gamma_u\} - d > 0 \quad \text{and} \quad 6\mu AB \geq P,$$

then $4AB \leq (A + B)^2$ and $\mu \leq \mu_u$ give $(A + B)^2 \geq 2P/(3\mu_u)$. The older independent lower bound $2\rho_l - 2d/(S_l - 1)$ may be combined by taking their maximum. An INITIAL mean-distance bound $(\rho_l - d/(S_l - 1))/\mu_u$ likewise has no sample-index dependence. These comparisons require a genuine gap, positive mean, and the high-frequency distance condition $d_0 \leq 12$; they cannot create a gap from a failed unseeded initial cover.

The Cantelli event in the high-frequency bound uses the same-law fourth-moment cap M and a fixed δ , with $V = 2M + 2$, $h_e = 2 - \delta^2 > 0$, $a = h_e^2/(V + h_e^2)$ and $m_e = 2 + V/h_e$. Its event inequality has no sample index. An upward coarsening $M \leq M'$ is valid only inside the original moment domain, and the same chosen M' is kept throughout. The complete high-frequency window is a sum of six pieces. Under the stated conditions every index-dependent piece decreases for $m \geq N$:

$$\sqrt{m}e^{-cm} \quad (2Nc > 1), \quad 1/m, \quad e^{-kmb^2}/\sqrt{m}.$$

The comparator has the form $\sqrt{m}e^{-m/8}(2r(m/8) + g_c/3)$, where $r(x) = (x+2)/(x^2+4x+2) > 0$ is decreasing and $N \geq 5$. An optional improved piece also has decreasing factor $r(m\ell/4)$, with $\ell > 0$ and its own positive damping and $2Nc > 1$ gates. For the optional zero-piece improvement, put $f(x) = \sqrt{x}e^{-x}r(x)$. The sign of its logarithmic derivative is the sign of

$$4 - 10x - 22x^2 - 13x^3 - 2x^4,$$

after multiplication by the positive denominator $2x(x+2)(x^2+4x+2)$. This polynomial is negative at $x = 1/4$ and strictly decreasing for $x > 0$. Thus the improvement is decreasing when the condition $Nkb^2 \geq 1/4$ holds. A complete selected window at N is consequently valid later; the minimum of several complete valid windows is valid without assuming the minimizing window stays the same. All remaining conditions ($0 < a \leq 1$, $m_e > 0$, $0 < d_0 \leq 12$, $1 \leq M \leq 111/10$, $0 < w \leq 1/4$, positive sine slope) remain. These are monotonicity statements for the analytic bounds; numerical enclosures must bound the complete expressions at the selected floor.

The low-frequency coefficients have index factors

$$(m/(m-1))^{3/2}, \quad (m/(m-1))^{3/2}/(m-1), \quad a_0^{-k}, \quad b_0^{-k}, \quad m^{-1/2},$$

where $a_0 = (m-1)/(2m)$, $b_0 = (45-M)(1-2/m)/96$, and $k > 0$. The first two decrease; a_0, b_0 are positive and increase, so their inverse powers decrease. Thus every sum or product with nonnegative, index-independent numerators decreases, including the remainder $e/(2a_0^2) + B^2/(2b_0^3)$ and the additional $L_2/\sqrt{m}$. The assertion concerns exactly these analytic envelopes. It requires the original coefficient theorem, its nonnegative numerators, actual support, and every paired eligibility gate; it does not repair an invalid paired rectangle. The paired domain for central panels at floor 40 is

$$0 < p_l < p_u < 1/2, \quad 1 \leq M < 3, \quad g \geq 0, \quad g^2 < 2, \quad R > 2g, \quad M-1-\gamma_l^2 \geq 0, \quad 0 \leq \theta < 1.$$

The sign-aware extension in Section F.9.1 is a distinct theorem, not an automatic alteration of this old contract.

F.11.3 Density, coherent score corners, and the charge

The retained phase theorem, not a separately asserted bound $H^2 \leq 5r$, supplies the density enclosure

$$\phi(H/\sqrt{m}) \in [\phi_{0,l}(1-5r/(2N)), 2/5] \quad (m \geq N)$$

when its raw and positivity gates hold. This fixed interval contains the actual interval because its lower endpoint increases with m . An own-score numerator $r/N - \phi_{0,l}(1-5r/(2N))D_l/6$ bounds its actual counterpart above for fixed $r, D_l \geq 0$: its derivative in $1/N$ is positive. For $g_0 = (79/25)\rho - 3 - d$ and $t = (d - 4/25)_+$, the general phase curvature has derivative $(g_0 + t)/2 \geq 0$ in the lower density. For $d \leq 4/25$, $g_0 \geq 4/25 - d$; for $d \geq 4/25$ the sum is $(79/25)(\rho - 1)$. The sign-corrected paired curvature is

$$C\rho + \frac{1}{2} \min\{\phi_l g_0, (2/5)g_0\}. \quad (273)$$

If $g_0 \geq 0$ it increases with ϕ_l ; if $g_0 < 0$ it is constant at the upper-density endpoint. This proves the floor direction without calling a negative coefficient positive. Every original positivity gate, the check against the original valid curvature, both signed linear endpoints, and the conserved addition with positive coefficient $(0.4095\rho - d/15)(1 - 2.5r/N)/2$ remain mandatory.

For a sharp contrary score $C > C_E$ and certified $C_E < 40974/100000$, one has

$$m(C - C_E)\rho \geq N \max\{0, C - 40974/100000\}\rho_l.$$

When $C \leq 40974/100000$ the right side is zero; otherwise each replacement decreases a nonnegative factor. The breakpoint must be included in the score cover. On each side the subtracted charge is affine, so the existing convex score-corner argument remains valid. A positive K/n target or its saved score floor cannot be transferred by this sharp $K = 0$

argument. The coherent (C, ρ) corners and both signs of H are retained before maximizing. The resulting budget exports $mD \leq Nd$ as in (271). An alternative scalar phase bound $Q(H) = 997Nd/15000 + \epsilon(H_{\max}^2 - H^2)$, $\epsilon \geq 0$, is at least $997Nd/15000$ on its acquired phase interval; its actual bound $15000Q(H)/(997m)$ transfers as well. Its artificial curvature is not a new physical assertion.

F.11.4 The contracts at floors 25 and 31

The floor 25 construction is restricted to $20 \leq n \leq 40$ and proposed transfer scope $25 \leq m \leq 40$; the intended finite consumers are $m = 26, \dots, 30$, with sharp actual predecessors. Its retained INITIAL variant uses the older gap distance; a product-gap formula from a different family is not retroactively part of that arithmetic. A new exact family must establish its own complete arithmetic before the preceding finite induction can consume it. The B0 extension adds only the three index-free mean caps above and finite same-law feedback. It therefore preserves the weighted invariant if its incoming states and every actual support radius are independently licensed. Its composition requires all twelve selected support states at each of the five actual indices, complete initial/resource/local covers, and all terminal necessary-condition exclusions. If those covers are all empty, no count forest is needed.

For floor 31, the eight-pass INITIAL argument uses $N = 31$, $K = 0$ and an acquired own-score cap at most $432313/10^6$. The scalar directions in Sections F.11.2–F.11.3 prove the INITIAL weighted interpretation for $m \geq 31$ on its same moment/support branch. Each subsequent resource refinement must also be valid at the actual index. The intended finite bridge is 31–39, and still requires actual-index support acquisition, its contact allowance, a valid actual own cap, every downstream refinement, and complete actual-index counts for surviving rows. The composition applies on the intersection of all these index domains.

F.11.5 Support and local comparisons at indices 40, 41 and 42

The following retained contract is only for $m \in \{40, 41, 42\}$. It assumes genuine global sharp bounds through 39 (including 38), then uses successive weak bounds at 40 and 41 for the next predecessors. Consequently the predecessor decrement for the same law is zero. A conservative scalar witness retains the older allowance $19/100$ and gives

$$e = \frac{13/2}{15 \cdot 38 \cdot 6} + \frac{(2/25)(13/2)}{38^2} = \frac{1469}{649800},$$

$$U = \frac{10131}{10000} \left(\frac{40974}{100000} + \frac{19}{4000} \right) + e = \frac{1371664491931}{3249000000000} < 0.4226,$$

$$42(U - 0.4095) = \frac{288392943517}{541500000000} < 0.54.$$

The inputs $\sqrt{m} < 13/2$, $\sqrt{m-1} > 6$ and $m-1 \geq 38$ hold on these indices. These computations supply the score and comparison caps only with the separate predecessor-smoothing theorem. They do not assume the desired bound at the current index.

The first support bound requires a separate two-step indexed proof at floor 40, upper index 42, zero allowance and radius 3.84. Every later central-panel Gaussian support cell instead has the form

$$B_m = L_P + \frac{0.54Q_+}{m} + 0.4 \left(\sum_{j=1}^5 \frac{M_j}{m^j} + \frac{T}{m^6} \right) + \rho \left(\sqrt{\frac{m}{m-1}} \delta_m + \frac{0.384 \cdot 0.4226}{m} \right), \quad (274)$$

where $Q_+, M_j, T \geq 0$, L_P is index-free and may be negative, and $\delta_m \leq 0.19$. Holding the same acquired moment, support, signed-beta and threshold cell bounds fixed, every positive

term decreases, so $B_m \leq B_{40} < 0$ on each whole closed cell. The prior support closes the exterior remainder. This argument has discarded favourable Gaussian terms. It does not justify a primitive retaining such a term; that is why the indexed initial proof is separate. LEFT has its own explicit zero-allowance support proof for $40 \leq m \leq 42$ at every selected radius.

For later Sperner confinement and profiles, preserve the complete gates: p separated from zero and $1/2$, $\theta < 1$, positive gaps, support, buffers, disjointness, positive curvature, both signs and every closed threshold/noise cell. A fixed deletion factor strictly larger than $\sqrt{\max((40/39)^3, 40/38)}$ bounds both deletions for $m \geq 40$. The floor binomial-mode interval at $j = 38$ must contain the actual one at $j = m - 2$, with the original variance and Stirling bounds. This containment and the analytic mode estimate are explicit prerequisites; no mode is omitted. Fixed valid derivative bounds L, U remain such bounds: all subtracted errors have nonnegative numerator divided by m , and all added errors are bounded above by their floor 40 versions. The lower-ray proof integrates the same L on both intervals with positive checked width; the upper ray retains the positive allocation denominator $L + U$.

Let $r_0 \geq 0$ be the common raw budget before a binary penalty $P \geq 0$. Feasibility gives $r_0 - mP \geq 0$. For $z, \text{Dist} \geq 0$ the error has the upper bound

$$(zr_0 + \text{Dist})/m - zP + \text{Bad } b_m.$$

Only the nonnegative inverse numerator is weakened to floor 40; the term $-zP$ is retained unchanged. A floor test $r_0 - 40P \leq 0$ is safe with the original strict-margin convention. The bad-pair fraction has $d \geq 0$, $\tau > 0$, and

$$g_m(d) = \frac{d}{m\tau + d} \leq g_{40}(d) \leq \frac{40\tau}{(40\tau + s)^2}d + \frac{s^2}{(40\tau + s)^2}, \quad s \geq 0.$$

The second inequality is the tangent bound for the concave g_{40} , including $s = 0$. The actual feasible moment/noise triangle lies in the weaker floor triangle, with nonnegative vertex coefficients and Bad. An independently valid Cantelli bound may be minimized with this complete bound. The conversion from a resulting amplitude cap to an X -support radius uses the original p, ℓ geometry, with no index. Every feedback step then uses this acquired radius and the weighted invariant.

The final small- and intermediate-noise comparisons keep the normalization entropy segment $p \pm \delta/40$, which contains $p \pm \delta/m$, and all positive log-factorial, variance and ratio errors. Normalized mass lies in the original interval $[b_0, 1]$. The signed Gaussian derivatives are bounded by the expressions $1 - A_m/m$ on the positive phase and $1/(1 - \eta_m/m)$ on the negative phase, where $0 \leq A_m \leq A_{40}$, $0 \leq \eta_m \leq \eta_{40} < 40$. All raw threshold and phase domains and $40v > \delta^2$ are retained. A cubic correction is either nonnegative or minimized over its complete positive multiplier interval before entering the repair ceiling. The full-count inverse correction I may have either sign; exactly

$$I/m \leq \max\{I/40, I/42\}, \quad 40 \leq m \leq 42.$$

For a bracket of either sign and mass in $[b_0, 1]$, use $\max\{b_0 \text{ bracket}, \text{bracket}\}$. These two sign-sensitive maxima retain the restriction $40 \leq m \leq 42$. The binomial gap is the minimum of the three actual-index gaps, not a substituted zero-noise Bernoulli theorem.

The small-noise comparison covers $0 < \lambda \leq 0.05$, and the intermediate-noise comparison covers 0.05–0.08 at every phase. On 0.08–0.10, the intermediate-noise comparison covers absolute phase 0.19–0.5 and the full-count comparison covers its complement. On 0.10–0.12 their common phase boundary is 0.27; above 0.12 the full-count comparison covers all phases. Intersect every interval with the same row's noise cap, retain both signs of H and phase and all closed seams. Zero noise is covered by the separate Bernoulli theorem. No smaller-noise interval is lost at zero.

Certificate condition F.39 (Finite composition at floor 40). Supply the actual predecessors, the separate high-moment theorem restricted to 40, 41, 42, all ten acquisition conditions with their complete hypotheses, the separate indexed initial-support bound, the complete full-count comparison rows and both binding conditions linking them to the acquisition bounds, and every small-noise, intermediate-noise, binomial and coverage condition. Establish all analytic prerequisites and inequalities used above for the specified bounds and the same maximizing law. The condition retains all 304 original row-verification obligations. The complete acquisition dispositions must cover every probability panel outside the left range $[141/500, 19/50]$ and central range $[19/50, 119/250]$ as well as all moment panels, including the shared $\rho = 6/5$ boundary.

Under this condition, finite induction excludes the extremal pair successively at 40, 41 and 42, without a tail theorem starting at 43. The six-piece high-frequency window, extended paired low-frequency bound, signed-variance bound and signed phase estimate must be the ones whose inequalities were used. In particular, the inactive base damping 43/96 and an unverified paired domain cannot substitute for the required $(45 - M)/96$ domain. The signed-variance bound has no extra sample-index term, but requires its same-law signed moment/support gates, positive residual weights and correct root directions. Numerical conditions retain their original evaluation indices; the analytic transfer above is restricted to $m = 40, 41, 42$. Every consumed analytic lemma must hold at these indices.

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