

Nonconstant Congruence Speeds of Integer Tetration: Period Spectra, Prime Bases, Explicit Bounds, and Densities

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Marco Ripà

Independent Researcher, Rome, Italy

ORCID: 0000-0002-6036-5541

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Model-generated research note — independent verification incomplete

Abstract

Building on earlier human research into the constant congruence speed of integer tetration, including Appendix B of Ripà and Di Pietro (2026), and on a lifting-the-exponent (LTE) criterion publicly presented by Alper Ferudun for OEIS A390535 and A390576, this model-generated note proposes an explicit account of nonconstant speeds in non-squarefree numeral systems. Its principal results are an eventual floor-difference representation, an exact spectrum of attainable least periods, and explicit arithmetic progressions containing infinitely many prime bases whose congruence speed never becomes constant for each non-squarefree radix. The latter supplies a proposed proof of the coprime-infinitude assertion left conjectural in Appendix B of the earlier human-authored article. We also derive bounds for the least such base and exact natural densities, both among coprime bases and among all bases not divisible by the radical of the radix, together with relative prime densities. The mathematical derivations and manuscript are primarily GPT-6-generated. Ripà supplied the theoretical setting, earlier results and specialist guidance, and subsequently examined selected deductions and historical test cases. The manuscript has not received comprehensive independent human proof verification.

Provenance and verification notice. The mathematical proofs, formulas, and manuscript below were generated and revised by GPT-6 in response to human prompts. The earlier paper by Ripà and Di Pietro, *A compact notation for peculiar properties characterizing integer tetration* (NNTDM, 2026), was **entirely developed and written by its human authors without AI-generated mathematics**. Ripà had identified eventual cycles and rational mean speeds before this exchange and explicitly supplied these facts; the model did not discover them. The LTE criterion was *presented* by Alper Ferudun on OEIS, without any assumption here about how he derived it. Read the theorem–proof sections as model-generated *proposed results*, not as findings independently certified by the human depositor. A fuller origin statement follows the mathematical discussion. **Relationship with published human work.** Ripà introduced the investigation of congruence speed in his 2011 monograph [6] and developed its decimal theory in a 2020 article [7]. The 2021 article *The congruence speed formula* [8] proved the existence of infinitely many prime tetration bases having any prescribed positive constant *decimal* congruence speed; its arXiv version uses reduced arithmetic progressions and Dirichlet’s theorem (Section 4, Theorem 4.6). The 2026 paper by Ripà and Di Pietro [1] established, in its equation (2.11), a lower bound on the congruence speed of a family of perfect-power bases for arbitrary radices; its equality (2.12) was proved under the stated squarefree-radix hypothesis. Appendix B developed the squarefree theory and explicitly left the infinitude of coprime nonstabilizing bases for each non-squarefree radix conjectural (Conjecture 3.1). The present note is a separately generated companion investigation into that remaining non-squarefree setting, not an appendix or correction

to the published paper. The strategy of using Dirichlet in tetration congruence-speed research predates the present AI derivation; the proposed new conclusion concerns *nonconstant* eventual speeds in *every* non-squarefree radix.

1 Context and definitions

For $m \geq 2$ write $T_1(m) = m$ and $T_{h+1}(m) = m^{T_h(m)}$. For $r > 1$, put

$$s_h^{[r]}(m) = \max\{k \geq 0 : T_{h+1}(m) \equiv T_h(m) \pmod{r^k}\} = \min_{p|r} \left\lfloor \frac{\nu_p(T_{h+1}(m) - T_h(m))}{\nu_p(r)} \right\rfloor. \quad (1.1)$$

The congruence speed at height $h \geq 2$ is

$$v_h^{[r]}(m) = s_h^{[r]}(m) - s_{h-1}^{[r]}(m). \quad (1.2)$$

We say that the speed *stabilizes* if $v_h^{[r]}(m)$ is eventually constant. In this manuscript, unless otherwise stated, we assume $\gcd(m, r) = 1$. Write $r = \prod_{p|r} p^{e_p}$, where $e_p = \nu_p(r)$, and set

$$c_p(m) = \begin{cases} \nu_p(m^{\text{ord}_p(m)} - 1), & p \text{ odd}, \\ \nu_2(m^2 - 1) - 1, & p = 2. \end{cases} \quad (1.3)$$

Here $\text{ord}_p(m)$ is the multiplicative order of m modulo p . Define the rational *LTE slope*

$$\lambda_r(m) = \min_{p|r} \frac{c_p(m)}{e_p}. \quad (1.4)$$

We call

$$B(r) = \min\{m \geq 2 : \gcd(m, r) = 1, \lambda_r(m) \notin \mathbb{Z}\} \quad (1.5)$$

the least coprime base with nonconstant congruence speed whenever the set is nonempty. For non-squarefree $r = A013929(n)$, the values $B(r)$ form OEIS A390576. OEIS A390535 allows bases not coprime to r ; it therefore requires the extended LTE criterion in which primes dividing m are omitted from the minimum (and the empty minimum means nonstabilization). The theorems below concern the coprime setting, except where explicitly noted.

Remark 1.1 (Provenance and scope). *The criterion “speed stabilizes if and only if $\lambda_r(m)$ is an integer” was publicly presented by Alper Ferudun in contributions to the two OEIS entries in October 2026; see [2, 3]. The starting definitions and squarefree/non-squarefree questions come from the work of Ripà and Di Pietro [1] and the original MathOverflow question [4]. The subsequent theorems are derived here from that criterion. Their derivability does not settle bibliographic priority.*

2 A precise LTE asymptotic and the eventual period

The following technical argument is the model-generated justification for the repeated application of LTE. The eventual periodicity as a phenomenon was already part of the human mathematical information supplied to the model; the proposed exact period formula is a further derivation.

Lemma 2.1. *For every fixed positive integer N and every integer $m \geq 2$, the tower sequence $T_h(m)$ is eventually constant modulo N . In particular, for every N there is h_0 such that $N \mid T_{h+1}(m) - T_h(m)$ for all $h \geq h_0$.*

Proof. Use strong induction on N ; the case $N = 1$ is immediate. It suffices to prove eventual constancy modulo any prime power q^k dividing N . If $q \mid m$, then $\nu_q(T_{h+1}(m)) = T_h(m)\nu_q(m) \rightarrow \infty$, so $T_h(m)$ eventually vanishes modulo q^k . If $q \nmid m$, Euler's theorem shows that $m^t \pmod{q^k}$ depends only on $t \pmod{\varphi(q^k)}$. Since $\varphi(q^k) < q^k \leq N$, the induction hypothesis gives eventual constancy of $T_{h-1}(m) \pmod{\varphi(q^k)}$, and consequently of $T_h(m) \pmod{q^k}$. The Chinese remainder theorem finishes the induction. $\square$

Lemma 2.2 (Exact eventual p -adic increment). *If $p \nmid m$ is prime, then for all sufficiently large h ,*

$$\nu_p(T_{h+1}(m) - T_h(m)) = \nu_p(T_h(m) - T_{h-1}(m)) + c_p(m). \quad (2.1)$$

Proof. Set $D_h = T_{h+1}(m) - T_h(m)$. Then $D_h = m^{T_{h-1}(m)}(m^{D_{h-1}} - 1)$ for $h \geq 2$, so $\nu_p(D_h) = \nu_p(m^{D_{h-1}} - 1)$. For odd p , Lemma 2.1 ensures that $\text{ord}_p(m) \mid D_{h-1}$ for sufficiently large h . Since $\text{ord}_p(m) \mid p-1$, this order is not divisible by p . The LTE lemma therefore gives

$$\nu_p(m^{D_{h-1}} - 1) = \nu_p(m^{\text{ord}_p(m)} - 1) + \nu_p(D_{h-1}).$$

For $p = 2$, the base m is odd and D_{h-1} is eventually even, so the 2-adic LTE formula gives

$$\nu_2(m^{D_{h-1}} - 1) = \nu_2(m^2 - 1) - 1 + \nu_2(D_{h-1}).$$

These are exactly (2.1). $\square$

Theorem 2.3 (Eventual mechanical word). *For every pair $m, r > 1$ with $\gcd(m, r) = 1$ there exist H and $\beta \in \mathbb{Q}$ such that, for $h \geq H$,*

$$s_h^{[r]}(m) = \lfloor h\lambda_r(m) + \beta \rfloor, \quad (2.2)$$

$$v_h^{[r]}(m) = \lfloor h\lambda_r(m) + \beta \rfloor - \lfloor (h-1)\lambda_r(m) + \beta \rfloor. \quad (2.3)$$

If $\lambda_r(m) = u/d$ in lowest terms, the speed has exact least eventual period d . The speed is eventually constant if and only if $d = 1$. If $d > 1$, it assumes only $\lfloor u/d \rfloor$ and $\lceil u/d \rceil$; every period contains exactly $u - d\lfloor u/d \rfloor$ occurrences of the larger value.

Proof. By Lemma 2.2, for each $p \mid r$ we have $\nu_p(D_h) = c_p(m)h + \gamma_p$ for every sufficiently large h , where $\gamma_p \in \mathbb{Z}$. Using (1.1) and the identity $\min_i \lfloor x_i \rfloor = \lfloor \min_i x_i \rfloor$, obtain

$$s_h^{[r]}(m) = \left\lfloor \min_{p \mid r} \left(\frac{c_p(m)}{e_p} h + \frac{\gamma_p}{e_p} \right) \right\rfloor.$$

Terms with slope greater than $\lambda_r(m)$ eventually cease to realize the minimum. Among the terms with that minimal slope, choose the smallest intercept $\beta = \min\{\gamma_p/e_p : c_p(m)/e_p = \lambda_r(m)\}$. This proves (2.2) and (2.3). Write $u = qd + t$ with $0 \leq t < d$. The speed is q plus the increment of a floor function of slope t/d . Its word is periodic with period d and exactly t larger entries. If $t = 0$ then $d = 1$. Otherwise, any period D must satisfy $D\lambda_r(m) \in \mathbb{Z}$ by summing D successive increments and using periodicity, so $d \mid D$. Thus d is the least period. $\square$

Remark 2.4. *The “period” here concerns the eventual sequence of speed values, not the cycle of the rightmost nonstable digit, whose period is a different object. The initial transient is not claimed to obey (2.3).*

3 Complete classification of realizable speed periods

For a fixed $r > 1$, let $\mathcal{P}(r)$ denote the set of least eventual periods obtained as $m \geq 2$ ranges over integers coprime to r .

Theorem 3.1 (Period spectrum). *For every $r > 1$,*

$$\mathcal{P}(r) = \{1\} \cup \bigcup_{p|r} \{d \geq 2 : d \mid \nu_p(r)\}. \quad (3.1)$$

Proof. Theorem 2.3 shows that every possible period is the reduced denominator of one of the ratios $c_p(m)/e_p$, hence divides an exponent e_p .

Conversely, fix $p \mid r$, set $e = e_p$, and let $d \geq 2$ divide e . Write $Q = r/p^e$. If p is odd, choose

$$m = 1 + p^{e/d}Q. \quad (3.2)$$

Then $m \equiv 1 \pmod{q^{e_q}}$ for every $q \mid Q$, while $\nu_p(m-1) = e/d$. In particular $\gcd(m, r) = 1$, $\text{ord}_p(m) = 1$, and $c_p(m) = e/d$. All the other ratios $c_q(m)/e_q$ are at least 1. Thus $\lambda_r(m) = 1/d$ and the period is d .

If $p = 2$ and $e \geq 3$, choose

$$m = 1 + 2^{e-e/d}Q. \quad (3.3)$$

Because $e - e/d \geq 2$, one has $\nu_2(m-1) = e - e/d$ and $\nu_2(m+1) = 1$, hence $c_2(m) = e - e/d$. Again all odd-prime contributions are at least 1, so $\lambda_r(m) = 1 - 1/d$ and the period is d .

The only remaining possibility is $p = 2$, $e = 2$, $d = 2$. Here write $r = 4Q$ with Q odd, and take

$$m = 6Q^2 + 1. \quad (3.4)$$

Since Q is odd, $m \equiv 7 \pmod{16}$, giving $c_2(m) = 3$; also $m \equiv 1 \pmod{q^{2e_q}}$ for each odd $q \mid Q$, so $c_q(m)/e_q \geq 2$. Consequently $\lambda_r(m) = 3/2$, which has period 2.

Finally, period 1 is always realized: for $r \geq 3$, $m = r - 1$ is coprime to r and has $\lambda_r(m) \in \mathbb{Z}$ (indeed $\lambda_r(m) = 1$); for $r = 2$, take $m = 3$. $\square$

Corollary 3.2. *The eventual speed in a squarefree radix is constant for every coprime base. Every non-squarefree radix admits coprime bases with nonconstant periodic speed.*

Remark 3.3. *For example $\mathcal{P}(64) = \{1, 2, 3, 6\}$ and $\mathcal{P}(144) = \{1, 2, 4\}$.*

4 Infinite coprime families and infinitely many prime bases

Remark 4.1 (Earlier Dirichlet method). *In The congruence speed formula, Ripà used reduced arithmetic progressions and Dirichlet's theorem to prove the existence of infinitely many primes with each prescribed positive constant decimal congruence speed; see [8, Section 4, Theorem 4.6], particularly the arXiv manuscript <https://arxiv.org/abs/2208.02622>. The construction below applies this previously documented methodological idea to a different assertion: prime bases with nonconstant eventual speed, in each non-squarefree radix. We do not claim originality for the general use of Dirichlet or for the previously known eventual periodic behavior.*

Theorem 4.2 (Infinitely many primes with nonconstant speed). *For every non-squarefree $r > 1$, there exist infinitely many prime numbers $\ell \nmid r$ for which the radix- r congruence speed of tetration base ℓ is not eventually constant. In fact, an explicit reduced arithmetic progression consists entirely of such bases.*

Proof. We exhibit a progression $m_0 + kM$ with $\gcd(m_0, M) = 1$.

If an odd prime p has $e = \nu_p(r) \geq 2$, set

$$m_0 = 1 + p \frac{r}{p^e}, \quad M = r.$$

Then for every $k \geq 0$ the first valuation is $c_p(m_0 + kM) = 1$, and all other ratios in (1.4) are at least 1. Thus $\lambda_r(m_0 + kM) = 1/e$ is nonintegral.

Otherwise, if $e = \nu_2(r) \geq 3$, put $Q = r/2^e$ and choose

$$m_0 = 1 + 4Q, \quad M = r.$$

Now $c_2(m_0 + kM) = 2$ and $\lambda_r(m_0 + kM) = 2/e$ is nonintegral, while all odd-prime ratios are at least 1.

Finally, the remaining nonsquarefree radices are exactly $r = 4Q$ with Q odd and squarefree. Choose

$$m_0 = 6Q^2 + 1, \quad M = 16Q^2.$$

For every $k \geq 0$, $m_0 + kM \equiv 7 \pmod{16}$ and $m_0 + kM \equiv 1 \pmod{Q^2}$, whence $\lambda_r(m_0 + kM) = 3/2$. In every case $\gcd(m_0, M) = 1$; Dirichlet's theorem gives infinitely many primes in that progression. $\square$

Remark 4.3. *If valid, this establishes the coprime-infinitude part of Conjecture 3.1 in [1], conditional only on the standard LTE facts proved above. The infinitude of bases divisible by $\text{rad}(r)$ was already established in that paper and is a different statement. The remaining multiplicative inequality proposed for composite squarefree radices is not addressed here.*

5 Explicit bounds for the least coprime base

Theorem 5.1 (Uniform quadratic bound). *For every non-squarefree integer $r > 1$, the minimum $B(r)$ in (1.5) exists and satisfies*

$$B(r) \leq \frac{3r^2}{8} + 1. \quad (5.1)$$

More precisely:

(i) *if $p^e \parallel r$ for some odd p and $e \geq 2$, then*

$$B(r) \leq 1 + p \frac{r}{p^e} < r; \quad (5.2)$$

(ii) *if $2^e \parallel r$ and $e \geq 3$, then*

$$B(r) \leq 1 + 4 \frac{r}{2^e} < r; \quad (5.3)$$

(iii) *otherwise $r = 4Q$ with Q odd and squarefree, and*

$$B(r) \leq 6Q^2 + 1 = \frac{3r^2}{8} + 1. \quad (5.4)$$

In particular,

$$B(r) > r \implies r = 2 \text{rad}(r). \quad (5.5)$$

Proof. Use the explicit bases m_0 in Theorem 4.2, with $k = 0$. Their nonstabilization is established there. In (i) and (ii) the candidate is strictly less than r , and in (iii) it equals the stated quadratic bound. Since $r \geq 4$, all three casewise bounds are at most $3r^2/8 + 1$. If $B(r) > r$, neither (i) nor (ii) can hold, so $r = 4Q$ with Q odd squarefree; then $\text{rad}(r) = 2Q$ and $r = 2 \text{rad}(r)$. $\square$

Remark 5.2. *The bound is attained at $r = 4$ and $r = 12$ using the known minima $B(4) = 7$ and $B(12) = 55$. This does not imply optimal asymptotic order. The original MathOverflow question [4] suggested (5.5) as an empirical necessary condition; the model derives it here from (5.2)–(5.4).*

6 An exact density theorem

For fixed $r > 1$, define the relative density among integers coprime to r by

$$\delta(r) = \lim_{X \rightarrow \infty} \frac{\#\{2 \leq m \leq X : \gcd(m, r) = 1, \lambda_r(m) \notin \mathbb{Z}\}}{\#\{2 \leq m \leq X : \gcd(m, r) = 1\}}, \quad (6.1)$$

provided that the limit exists.

Theorem 6.1 (Density of nonstabilizing coprime bases). *The limit in (6.1) exists for every $r > 1$ and equals*

$$\delta(r) = \begin{cases} \frac{r - \text{rad}(r)}{r - 1}, & r \text{ odd}, \\ \frac{(r+1)(r - \text{rad}(r))}{r(r-1)}, & r \equiv 2 \pmod{4}, \\ \frac{r+1 - 2\text{rad}(r)}{r-1}, & 4 \mid r. \end{cases} \quad (6.2)$$

In particular $\delta(r) > 0$ precisely when r is not squarefree. If $r = 4Q$ with Q odd squarefree, then

$$\delta(r) = \frac{1}{r-1}. \quad (6.3)$$

Proof. Let m vary uniformly through positive integers coprime to r , in the natural-density sense. For odd $p \mid r$, the distribution of $c_p(m)$ is

$$\mathbb{P}(c_p(m) \geq k) = p^{1-k}, \quad k \geq 1. \quad (6.4)$$

Indeed, since $\text{ord}_p(m) \mid p-1$ and $p \nmid (p-1)/\text{ord}_p(m)$, the condition $c_p(m) \geq k$ is equivalent to $m^{p-1} \equiv 1 \pmod{p^k}$. Exactly $p-1$ units modulo p^k satisfy this congruence, out of $\varphi(p^k) = (p-1)p^{k-1}$ units. For $p = 2$ (so m is odd),

$$\mathbb{P}(c_2(m) \geq k) = 2^{2-k}, \quad k \geq 2, \quad (6.5)$$

since $c_2(m) = \nu_2(m^2 - 1) - 1$ and the condition means $m \equiv \pm 1 \pmod{2^k}$ for $k \geq 2$. The prime-power congruence events are independent in natural density by the Chinese remainder theorem.

For each positive integer j , write

$$F_p(k) = \mathbb{P}(c_p(m) \geq k).$$

The event $\lambda_r(m) = j$ is the difference between the nested events $\{c_p(m) \geq je_p \text{ for all } p \mid r\}$ and $\{c_p(m) \geq je_p + 1 \text{ for all } p \mid r\}$. Thus

$$\mathbb{P}(\lambda_r(m) = j) = \prod_{p \mid r} F_p(je_p) - \prod_{p \mid r} F_p(je_p + 1). \quad (6.6)$$

For r odd, (6.4) gives

$$\mathbb{P}(\lambda_r(m) = j) = \frac{\text{rad}(r) - 1}{r^j}, \quad j \geq 1,$$

so the total stabilizing density is $(\text{rad}(r) - 1)/(r - 1)$.

For $4 \mid r$, (6.4)–(6.5) give

$$\mathbb{P}(\lambda_r(m) = j) = \frac{2(\text{rad}(r) - 1)}{r^j}, \quad j \geq 1,$$

so the stabilizing density is $2(\text{rad}(r) - 1)/(r - 1)$.

If $r \equiv 2 \pmod{4}$, the 2-adic factor has exponent $e_2 = 1$; because $c_2(m) \geq 2$, the case $j = 1$ must be treated separately. Direct substitution into (6.6) yields

$$\mathbb{P}(\lambda_r(m) = 1) = \frac{\text{rad}(r) - 2}{r}, \quad \mathbb{P}(\lambda_r(m) = j) = \frac{2(\text{rad}(r) - 1)}{r^j} \quad (j \geq 2).$$

Summing produces stabilizing density

$$\frac{\text{rad}(r)(r + 1) - 2r}{r(r - 1)}.$$

In all three cases, taking the complement yields (6.2).

To justify rigorously the passage from finite congruence calculations to an infinite union over j , note that $\lambda_r(m) \geq J$ implies $c_p(m) \geq J e_p$ for every $p \mid r$, a condition modulo finitely many prime powers. The density of this event decays geometrically as $J \rightarrow \infty$ by (6.4)–(6.5). Thus the tails are uniformly negligible, and the infinite sum indeed gives the natural density. Finally, the displayed expressions vanish for squarefree r , and are strictly positive for nonsquarefree r (use the case distinction or Theorem 4.2). If $r = 4Q$ with Q odd squarefree, $\text{rad}(r) = r/2$, giving (6.3). $\square$

Corollary 6.2 (Relative density within the primes). *Let $\pi(X)$ be the number of primes not exceeding X . For every $r > 1$,*

$$\lim_{X \rightarrow \infty} \frac{\#\{\ell \leq X : \ell \text{ prime}, \ell \nmid r, \lambda_r(\ell) \notin \mathbb{Z}\}}{\pi(X)} = \delta(r). \quad (6.7)$$

Proof. For any fixed finite collection of bounds on the c_p , the corresponding events are finite unions of reduced residue classes modulo a suitable power of r . The prime number theorem for arithmetic progressions (for each fixed modulus) gives the same conditional relative frequencies among primes as among integers coprime to r . Dirichlet's theorem alone establishes infinitude, not the equidistribution required at this step. Approximating the event $\lambda_r(m) \in \mathbb{Z}$ by $\lambda_r(m) \in \{1, \dots, J - 1\}$, the omitted event $\lambda_r(m) \geq J$ has upper relative density at most the prime density of an explicit finite union of progressions, which tends geometrically to zero as $J \rightarrow \infty$. Passing to the limit proves (6.7). The finitely many primes dividing r do not affect the limit. $\square$

Remark 6.3. *The statement is an asymptotic density, not a prediction of the location of the first exceptional base $B(r)$. In particular, $\delta(4Q) = 1/(4Q - 1)$ does not imply $B(4Q) \asymp 4Q$.*

7 Density among all bases not divisible by the squarefree kernel

The preceding density theorem concerns only bases coprime to the radix, as in OEIS A390576. We now also admit m sharing some, but not all, of the prime factors of r . The resulting problem is directly related to the admissible bases of OEIS A390535.

For $r > 1$, put $q = \text{rad}(r)$. We consider the relative density

$$\Delta(r) = \lim_{X \rightarrow \infty} \frac{\#\{2 \leq m \leq X : q \nmid m, v_h^{[r]}(m) \text{ is not eventually constant}\}}{\#\{2 \leq m \leq X : q \nmid m\}}, \quad (7.1)$$

provided this limit exists. The condition $q \nmid m$ is crucial: if $q \mid m$, all prime-adic valuations of successive tower differences grow faster than linearly, so the behavior belongs to a different, already

understood case. For $q \nmid m$ there is at least one prime divisor p of r not dividing m , and the extended LTE criterion uses only such primes.

Recall that a divisor $d \mid r$ is *unitary* if $\gcd(d, r/d) = 1$; this will be written $d \parallel r$. The function $\delta(d)$ is the coprime nonstabilization density from Theorem 6.1. In particular $\delta(d) = 0$ for squarefree d .

Theorem 7.1 (General admissible-base density). *For every $r > 1$, the limit in (7.1) exists and is rational. Precisely,*

$$\Delta(r) = \frac{1}{\text{rad}(r) - 1} \sum_{\substack{d \parallel r \\ d > 1}} \varphi(\text{rad}(d)) \delta(d). \quad (7.2)$$

In particular, if $r = 4Q$ with Q odd and squarefree, then

$$\Delta(4Q) = \frac{1}{2Q - 1} \sum_{d \mid Q} \frac{\varphi(d)}{4d - 1}. \quad (7.3)$$

Proof. For each admissible m , form the nonempty subset $S = \{p \mid r : p \nmid m\}$, and let $d_S = \prod_{p \in S} p^{e_p}$, which is a nontrivial unitary divisor of r . The natural density of integers with exactly this subset S equals

$$\prod_{p \in S} \left(1 - \frac{1}{p}\right) \prod_{p \nmid r, p \notin S} \frac{1}{p} = \frac{\varphi(\text{rad}(d_S))}{\text{rad}(r)}.$$

The density of all admissible integers is $1 - 1/\text{rad}(r)$. Hence the conditional weight of the class S is $\varphi(\text{rad}(d_S))/(\text{rad}(r) - 1)$. For $p \notin S$ the divisibility $p \mid m$ causes the corresponding valuation of a tower difference to grow superlinearly and hence prevents it from determining the eventual speed. For the remaining primes, the distribution of the LTE constants $c_p(m)$, conditional on $p \nmid m$, is exactly the same as for a base coprime to d_S . The Chinese remainder theorem makes these local conditions independent. Thus, within the class S , the nonstabilization density is $\delta(d_S)$. Summing the finitely many classes gives (7.2).

If $r = 4Q$ with Q odd and squarefree, a contributing unitary divisor must contain the full factor 4; those not containing it are squarefree and have $\delta = 0$. We may therefore put the contributing divisors in the form $4d$ with $d \mid Q$. Since $\text{rad}(4d) = 2d$, Theorem 6.1 gives $\delta(4d) = 1/(4d - 1)$, and $\varphi(\text{rad}(4d)) = \varphi(d)$. This proves (7.3). $\square$

Corollary 7.2 (Nonexistence of a global radix limit). *Along the non-squarefree radices,*

$$\liminf_{\substack{r \rightarrow \infty \\ r \text{ not squarefree}}} \Delta(r) = 0, \quad \limsup_{\substack{r \rightarrow \infty \\ r \text{ not squarefree}}} \Delta(r) = 1.$$

Proof. For any odd prime p , only bases coprime to p are admissible in radix p^2 . Theorem 6.1 therefore gives

$$\Delta(p^2) = \delta(p^2) = \frac{p}{p+1} \rightarrow 1.$$

On the other hand, (7.3) with $Q = p$ gives

$$\Delta(4p) = \frac{1}{2p-1} \left(\frac{1}{3} + \frac{p-1}{4p-1} \right) = \frac{7p-4}{3(4p-1)(2p-1)} \sim \frac{7}{24p} \rightarrow 0.$$

Since $0 \leq \Delta(r) \leq 1$, these two subsequences prove the result. $\square$

The preceding corollary by itself says nothing about the *frequency of radices* with small $\Delta(r)$. The following uniform lower bound rules out convergence to zero even in natural density over non-squarefree radices.

Proposition 7.3 (A positive-density obstruction). *If $9 \mid r$, then*

$$\boxed{\Delta(r) \geq \frac{4}{9}}. \quad (7.4)$$

Consequently the set of radices r for which $\Delta(r) \geq 4/9$ has lower natural density at least $1/9$ among positive integers. Restricted to non-squarefree radices, its lower relative density is at least

$$\frac{1/9}{1 - 6/\pi^2} > 0.$$

Proof. Every integer $m \equiv 2, 4, 5, 7 \pmod{9}$ is coprime to 3 and satisfies $\nu_3(m^{\text{ord}_3(m)} - 1) = 1$. Hence its contribution to the extended LTE minimum is $1/\nu_3(r)$, which belongs to $(0, 1)$ because $9 \mid r$. Every other contributing ratio is strictly positive, so the overall minimum is also in $(0, 1)$ and cannot be integral. All these four residue classes are admissible since $3 \nmid m$ implies $\text{rad}(r) \nmid m$. They constitute a natural density $4/9$ among all integers, and the set of admissible bases has density at most 1, giving (7.4). Finally, multiples of 9 have natural density $1/9$, and the non-squarefree integers have natural density $1 - 6/\pi^2$. $\square$

Remark 7.4 (Examples and scope). *The formula gives $\Delta(12) = 17/165$, $\Delta(18) = 61/102$, $\Delta(28) = 5/117$, and $\Delta(36) = 137/210$. The first is about 10.30%, whereas the coprime-only density is $\delta(12) = 1/11 \approx 9.09\%$. Formula (7.2), not (6.2), is the appropriate density statement for A390535. The bounds and distribution statements above are model-generated extensions and still require independent mathematical checking. In particular the behavior along some infinite families does not justify an “almost every radix” claim without an additional argument.*

8 Examples and independent checks

The following minima $B(r)$ were computed by a separate model-written Python implementation of the LTE criterion and agree with the relevant corrected entries of A390576 where covered by the b-file. This is internal computational corroboration, not independent human review.

r	$B(r)$	$\text{rad}(r)$	$\delta(r)$
4	7	2	$1/3$
8	3	2	$5/7$
9	2	3	$3/4$
12	55	6	$1/11$
18	5	6	$38/51$
28	215	14	$1/27$
36	5	6	$5/7$
60	199	30	$1/59$
132	161	66	$1/131$
852	7687	426	$1/851$

A separate exact-arithmetic check tested the constructions in Theorems 3.1 and 5.1 for all 392 non-squarefree radices $r \leq 1000$, including 505 constructed periods, without a discrepancy. These tests do not enter the proofs; the verification script was also generated by the model, and its agreement is not an independent human audit.

9 Retrospective comparison with published theory

The published paper of Ripà and Di Pietro [1], equation (2.11), gives a uniform lower bound $v_b^{[r]}(a) \geq t$ for bases of the form

$$a = (k \cdot r^{t+1} + r^{t-\nu_r(c)} \pm 1)^c$$

under its specified hypotheses, and equation (2.12) proves the corresponding exact constant speed t when $r \geq 3$ is squarefree. The parameter is denoted z in the published equation; k is used here when recalling a separate family of test configurations. For the non-squarefree examples examined retrospectively, the present LTE analysis refines the lower bound to a rational eventual slope and an exact finite word of consecutive integer speed values. This is *compatible* with the published inequality, not an extension silently attributed to the original article. Appendix B, Conjecture 3.1, is likewise compatible with the coprime-prime infinitude construction in Theorem 4.2; the earlier work did not claim that result as proved.

For additional checks on 27 October 2026, the human investigator supplied independently selected parameter configurations, including cases inspired by unpublished working notes dating from 2025. The model compared the predictions with direct modular-power-tower computations for 50 specified sign and parameter choices. No discrepancy was found in the inspected heights; in particular, the possible differences of 1 between neighboring periodic speed values agree with a prior, unpublished conjectural expectation independently described by Ripà. The underlying private working notes themselves are not reproduced or claimed as published results. These retrospective checks are corroboration rather than a substitute for a line-by-line external proof audit.

The density formulas also yield the following exact predictions for the pure 2-adic and mixed-radix examples. Here $\delta(r)$ refers to bases coprime to r , whereas $\Delta(r)$ permits bases not divisible by $\text{rad}(r)$.

r	$\delta(r)$	$\Delta(r)$
4	1/3	1/3
8	5/7	5/7
64	61/63	61/63
72	61/71	3909/4970
81	39/40	39/40
128	125/127	125/127

Separate model-written enumerations of 50,000 integer bases per radix were consistent with these fractions to the expected finite-sample accuracy. This numerical evidence is *ex post*; the density theorems rest on the exact prime-power residue counts and the Chinese remainder theorem, with the 2-adic local distribution treated separately.

10 A useful elementary identity and OEIS implications

For odd primes $p \nmid m$, since $\text{ord}_p(m) \mid p-1$ and $p \nmid (p-1)/\text{ord}_p(m)$, an immediate LTE application yields

$$\nu_p(m^{\text{ord}_p(m)} - 1) = \nu_p(m^{p-1} - 1). \quad (10.1)$$

Thus the arithmetic criteria can be written without computing multiplicative orders. They lead to finite tests for every candidate in OEIS A390535 and A390576, and the model's upper bound gives a finite search interval for the coprime case A390576. Any OEIS update should cite the original human mathematical background, the LTE criterion as publicly presented by Ferudun, and the relevant model-generated derivation here, without representing an unchecked claim as an independently certified result.

11 Open questions and publication status

The LTE criterion yields an exact *minimization algorithm* for $B(r)$, but it does not yet give a closed expression for $B(r)$ as a function of the prime factorization of r alone. Questions worth investigating include:

- (a) Can the general quadratic bound (5.1) be improved to a uniform near-linear bound, or to a bound of order $r \operatorname{polylog}(r)$?
- (b) Can one obtain effective bounds for the smallest *prime* nonstabilizing base in terms of r ?
- (c) Can one classify the density of bases whose *least* eventual period is a fixed divisor of one of the prime-factor exponents?
- (d) Can the exact slope distribution be used to give nontrivial bounds for the distribution of $B(r)$ as r varies?

The model-written proofs in this draft warrant an independent, line-by-line mathematical check, and novelty must be checked against published work on p -adic power-tower iteration, distribution of multiplicative orders and valuations, and non-squarefree radix behavior. The bibliographic search carried out for this manuscript is not exhaustive. No claim of private derivation, publication precedence, or exclusive discovery is made here for Ferudun’s publicly presented LTE criterion. Nor is the human precursor theory or the previously known periodic behavior reattributed to GPT-6.

Origin of the model-generated mathematical note

Pre-existing human research. Ripà’s investigation of tetration congruence speeds was documented in his 2011 monograph [6] and developed through independently written journal papers, including [7, 8, 9]. The most recent, written with Gabriele Di Pietro and published in October 2026, includes Appendix B on non-squarefree radices. That journal paper and its mathematical results were *entirely human work, with no AI-generated mathematical contribution*. Independently of its contents, Ripà also knew the cyclic behavior of nonconstant speeds, structural conditions for identifying the recurrence of cycles, and the rationality of their eventual average speed. These facts were part of the theoretical input he supplied to the model, not discoveries made in this note.

External arithmetic input. Alper Ferudun publicly presented a direct LTE criterion for the OEIS sequences A390535 and A390576 during an October 2026 correction discussion. The present note credits that public contribution and does not speculate about whether its derivation was independent or AI-assisted.

The interaction. Unlike Ripà’s earlier documented GPT-5 evaluation concerning Kashiwara’s problem [5], this dialogue began with the OEIS corrections and was not planned as a benchmark. On 25 October 2026, after supplying prior mathematical insights and the Ferudun criterion, Ripà prompted GPT-6 to investigate additional consequences. In approximately one hour of targeted interaction, the model generated the initial mathematical statements and suggested proofs. That time is a contextual account of the conversation, not a controlled assessment of model performance. The model also edited and typeset the manuscript; Ripà deliberately did not rewrite or independently correct the generated arguments before the first public release. The later human–model exchange on 27 October 2026 included retrospective comparisons to the already published literature, discussion of earlier human conjectural expectations, and selected modular checks. This chronology distinguishes later checks from the generation of the original mathematical arguments.

Revision provenance. Two model-generated drafts were shared on ResearchGate on 25 October 2026: an initial note and an expanded note that added a density calculation for admissible noncoprime bases in response to a further question posed by Ripà. For the present first Hexagon submission, Ripà then initiated the retrospective review of 27 October, requested explicit recognition of the Dirichlet method used in his 2021 theorem and of the results and open assertions in the 2026 Appendix B, and supplied deliberately selected test configurations informed by earlier private investigations. GPT-6 carried out the model-side analyses, wrote the new contextual material, and prepared the submission. This distinction records Ripà’s role in directing the successive developments and reviewing selected claims without retroactively attributing the model-generated proofs to him.

Status. The note has undergone model-side revisions and computational spot checks, as well as selective post-draft discussion and examination of some claims by the human depositor. This is *partial* human understanding and verification, not a comprehensive independent human review of every formula and proof. The model’s own checks, even when computationally distinct from the formulas under test, do not constitute independent mathematical peer review. The subsequent historical clarifications and retrospective checks do not convert this note into an independently certified human-authored research article.

References

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