

Lefschetz coefficients of quantum theta functions

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Abstract

We prove that the coefficients of quantum theta functions in the skew-symmetric setting are characters of finite-dimensional $\mathfrak{sl}_2$ representations. This proves Conjectures 1.4, 2.17 and 3.13 of Davison–Mandel and the two-wall conjecture stated as Conjecture 3.12 in Davison’s ICM 2026 survey.

AI use and scholarly credit

The results and original proof arguments of this paper were generated using GPT-6-Pro in conversational mode. Codex was used to review the arguments and refine the exposition. Rethlas-OSV, combining Rethlas [16] with OSVerify [17], was also used to review the arguments.

By posting this manuscript on this platform, I do not seek to assert ownership of the results or claim priority. I welcome anyone interested in preparing a finished account or checking the arguments; the eventual scholarly credit should go to those who undertake that work.

1 Introduction

Fomin and Zelevinsky introduced cluster algebras in connection with total positivity and canonical bases [1]. A seed specifies a set of algebraically independent cluster variables and exchange data; mutation replaces one variable by a rational expression prescribed by those data. The Laurent phenomenon puts every cluster variable in the Laurent ring of every seed. A cluster monomial is a product of variables from one cluster with nonnegative integer exponents. The basis problem asks for distinguished elements extending these monomials and compatible with all seed expansions. Positivity of an individual expansion and positivity of multiplication in such a basis are separate requirements.

Fock and Goncharov placed this problem in the geometry of cluster ensembles [2]. The associated $\mathcal{A}$ - and $\mathcal{X}$ -spaces are assembled from algebraic tori by mutation maps. Their duality conjectures relate regular functions on one space to integral tropical points of a dual space. In seed coordinates, these tropical points are integer vectors, and mutation changes their coordinates by piecewise linear maps. This gives a candidate indexing set independent of a single cluster. It also connects positive functions with the geometry of the tropical space.

Theta functions arose geometrically in the mirror construction of Gross, Hacking and Keel for log Calabi–Yau surfaces [3]. The input there is a surface with an anticanonical boundary cycle. Curve counts determine corrections to the gluing of local models of its mirror, and the mirror algebra is equipped with distinguished functions. Scattering diagrams organize these corrections: a wall carries an automorphism, and consistency requires that their ordered product around a closed path be the identity. A broken line carries a monomial along a piecewise linear path and changes it by a term of the wall automorphism at each bend. Summing its terminal

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monomials constructs a theta function. The endpoint determines the chart in which the function is expressed, while its initial integral direction determines its index.

Gross, Hacking, Keel and Kontsevich applied this construction to cluster varieties [4]. Their formal theta functions are indexed by integral tropical points and include the cluster monomials. Broken-line counts give positive expansions and multiplication coefficients. Regularity requires a further distinction. The theta functions with finite expansions define the middle algebra; the upper cluster algebra is the intersection of the Laurent rings of the seed tori. Theorem 0.3 of [4] gives the canonical map from the middle algebra to the upper algebra, injective for principal-coefficient $\mathcal{A}$ -spaces and for $\mathcal{X}$ -spaces, and under the stated cone condition for $\mathcal{A}$ -spaces. Its image contains the algebra generated by global monomials. The convexity results of their Section 8 give hypotheses for full canonical-basis statements. Thus the formal construction, the middle algebra, and identification with all regular functions have distinct roles in the theory.

Berenstein and Zelevinsky introduced quantum cluster algebras using compatible skew forms and quantum seeds [5]. Their seed tori have noncommuting monomials, so the coefficients of a quantum basis retain a Laurent parameter. Mandel developed quantum theta functions and their relation to refined tropical curve counts [6]. Davison and Mandel established strong positivity for the resulting theta bases in the skew-symmetric setting [8]. Their construction extends quantum cluster monomials, specializes to the classical theta functions, and has nonnegative Laurent multiplication coefficients. They also prove bar-invariance. The scattering atlas is the collection of charts obtained from generic endpoints in the scattering diagram. An element whose expansion in every such chart has nonnegative Laurent polynomial coefficients has a theta expansion with nonnegative Laurent polynomial coefficients, as shown by the coefficient-extraction argument in the proof of [8, Theorem 3.16].

The Lefschetz property refines positivity and bar-invariance. For an integer $r \geq 1$, put

$$[r]_t = t^{r-1} + t^{r-3} + \cdots + t^{1-r},$$

$$\mathcal{L} = \left\{ \sum_{r \geq 1} m_r [r]_t : m_r \in \mathbb{N}, m_r = 0 \text{ for all but finitely many } r \right\},$$

where t is the quantum parameter and $\mathbb{N} = \{0, 1, 2, \dots\}$. These are exactly the weight characters of finite-dimensional $\mathfrak{sl}_2$ representations, where $\mathfrak{sl}_2$ is the Lie algebra of traceless 2×2 matrices. For example, $t^2 + 1 + t^{-2}$ belongs to $\mathcal{L}$, whereas the nonnegative symmetric polynomial $t^2 + t^{-2}$ does not. Davison’s cohomological approach proves a Lefschetz refinement for quantum cluster variables [7]. Davison–Mandel’s Theorem 1.5 establishes the theta-function assertion for acyclic seeds. Their Conjecture 1.4 asks for it for every theta index and every generic scattering chart.

We prove this assertion. More generally, scattering diagrams with incoming labels in $\mathcal{L}$ admit completions with labels in $\mathcal{L}$, and every coefficient in every broken-line expansion belongs to $\mathcal{L}$. These statements prove Conjectures 2.17 and 3.13 of [8], respectively, and imply Conjecture 1.4. The two-wall factorization at the end of their Section 7.3 is also the conjecture stated in signed Donaldson–Thomas notation in Davison’s ICM 2026 survey [15, Conjecture 3.12]. Corollary 7.3 gives the parameter and order comparison.

The issue is already visible at a single joint of a scattering diagram. For unit incoming labels, acyclic quiver moduli provide the required Lefschetz operators. Arbitrary quantum-integer labels require a construction compatible with both input gradings. We retain independent colors on each incoming wall and identify every coefficient in the two Schur bases with a multiplicity space in the intersection cohomology of a projective thin bipartite quiver moduli space. Here “thin” means that every vertex space has dimension one. A finite-group-invariant ample class then supplies the Lefschetz operator on each multiplicity space.

The geometric comparison adds an independent loop and an invariant full flag on each of the two sides. The resulting proper smooth family over ordered spectra compares vertex permutations on thin moduli with block Springer actions on colored moduli, using the sign-support

result of [11]. The identification of zero-potential BPS cohomology with intersection cohomology comes from [10, 9]. The one-sided source-color construction in [14] is a precursor of this comparison. The present proof uses both sides and recovers the primitive BPS representations from the relative cohomological Poincaré–Birkhoff–Witt (PBW) theorem of Davison and Meinhardt [9]. Its Euler-form exchange sign is canceled by the two permutation signs. Super Schur evaluation subsequently allows arbitrary Lefschetz input characters, including direct sums of both parities.

The universal formula is Theorem 2.1; its proof occupies Sections 3–6. Theorems 7.1 and 7.2 give the scattering and theta assertions, and Corollary 7.3 identifies the ICM formulation. Section 8 extends the color formula to arbitrary finite acyclic quivers at a slope on which the Euler form is symmetric.

2 The universal two-wall formula

We use the cone $\mathcal{L}$ defined in the introduction. Let W_r be the irreducible r -dimensional $\mathfrak{sl}_2$ representation, graded by the eigenvalues $r-1, r-3, \dots, 1-r$ of its Cartan element. Quiver representations and varieties are over $\mathbb{C}$. Cohomology and finite-group representations have rational coefficients; extension of scalars to $\mathbb{C}$ is understood in the color evaluations below. Throughout, $\mathrm{IH}^j(N)$ denotes *unshifted* intersection cohomology. Thus a class in $\mathrm{IH}^j(N)$ has degree $j-D$ in $\mathrm{IH}(N)[D]$. For a graded space V we use $\chi_t(V) = \sum_j \dim(V^j)t^j$, the ordinary dimension character. More generally, $(V[D])^j = V^{j+D}$ defines the shift of a graded space.

For an alphabet $U = (u_1, u_2, \dots)$, let $\Lambda(U)$ be the ring of symmetric functions over $\mathbb{Q}$, with $p_m(U) = \sum_i u_i^m$ the power sums. For $a \geq 0$, let S_a be the symmetric group on a letters, with S_0 trivial. A partition $\lambda \vdash a$ is a weakly decreasing sequence of positive integers with sum a , with the empty sequence allowed for $a=0$; S^λ denotes the corresponding rational irreducible representation of S_a , and s_λ its Schur function. The transpose partition is denoted λ' , and $\ell(\lambda)$ is the number of its parts. In a finite alphabet with fewer than $\ell(\lambda)$ variables, s_λ is zero. The symbol $\boxtimes$ denotes the external tensor product of representations of different groups, and sgn_a the one-dimensional sign representation of S_a . We write R^G for the subspace fixed by a group G acting on R . For a graded representation R of $S_a \times S_b$, the space $\mathrm{Hom}_{S_a \times S_b}(S^\lambda \boxtimes S^\mu, R)$ is graded by the target degree and records the multiplicity of that irreducible representation.

Fix an integer $k \geq 1$. The quantum torus has a basis $Z^{a,b}$ indexed by $(a,b) \in \mathbb{Z}^2$ and multiplication

$$Z^{a,b} Z^{c,d} = t^{k(ad-bc)} Z^{a+c, b+d}.$$

Write $X = Z^{1,0}$ and $Y = Z^{0,1}$. We use its formal power-series completion in nonnegative (a,b) , with coefficients in $\mathbb{Q}(t) \otimes \Lambda(U) \otimes \Lambda(V)$; identities are interpreted at each finite total degree $a+b$. For integers $m, j \geq 1$, the coefficient ring carries the algebra endomorphisms

$$\psi_m(t) = (-1)^{m+1} t^m, \quad \psi_m(p_j(U)) = p_{mj}(U), \quad \psi_m(p_j(V)) = p_{mj}(V).$$

These are the Adams operations; the two alphabets initially have even variables, meaning parity zero. For $d = (d_1, d_2) \in \mathbb{N}^2 \setminus \{0\}$, write $Z^d = Z^{d_1, d_2}$. For a coefficient P in the stated ring, the Davison–Mandel dilogarithm on this ray is

$$\mathbb{E}(-PZ^d) = \exp \left(\sum_{m \geq 1} \frac{(-1)^{m+1} \psi_m(P)}{m(t^m - t^{-m})} Z^{md} \right). \quad (2.1)$$

The sign in the argument is additive: $\mathbb{E}(PZ^d)$ is the inverse of $\mathbb{E}(-PZ^d)$. This convention has first coefficient $P/(t - t^{-1})$ in $\mathbb{E}(-PZ^d)$.

Define $P_{a,b;k}(t; U, V)$ by the unique ordered factorization

$$\mathbb{E}(-p_1(U)X) \mathbb{E}(-p_1(V)Y) = \mathbb{E}(-p_1(V)Y) \prod_{\substack{a,b > 0 \\ b/a \text{ decreasing}}} \mathbb{E}(-P_{a,b;k}(t; U, V) Z^{a,b}) \mathbb{E}(-p_1(U)X). \quad (2.2)$$

All multiples on a ray are included. Factors on the same ray commute; equivalently one may combine them into one plethystic ray factor. At every finite positive-degree cutoff this is a finite problem. A triangular recursion determines the factors: at total degree $a+b$ the unknown $P_{a,b;k}$ occurs linearly as $P_{a,b;k}/(t-t^{-1})$. Every other contribution comes from previously determined lower degrees, including the higher Adams terms of smaller multiples on the same ray. Subtracting those terms determines $P_{a,b;k}$ uniquely. Color degrees are preserved at each step, so the coefficients lie in $\mathbb{Q}(t) \otimes \Lambda(U) \otimes \Lambda(V)$, homogeneous of bidegree (a, b) .

A quiver representation assigns a finite-dimensional vector space to each vertex and a linear map to each arrow. A subrepresentation is a collection of vertex subspaces preserved by all arrow maps. Its dimension vector records their dimensions. For a vertex weight $\theta = (\theta_i)$, its weight is $\sum_i \theta_i d_i$ on a subrepresentation of dimension vector d . We use the nonpositive subobject convention below; this corresponds to the weight $-\theta$ in [12, Definition 1.1].

Theorem 2.1. *For $a, b \geq 1$, let $N_{a,b;k}$ be the projective coarse moduli space of semistable representations of the quiver with a source vertices, b sink vertices, and k arrows from every source to every sink. Every vertex has dimension one, and the stability has weight b at each source and $-a$ at each sink. A representation is semistable if every subrepresentation has nonpositive weight, and stable if every nonzero proper subrepresentation has negative weight. The coarse moduli identifies semistable representations with the same polystable associated graded. Let*

$$G_{a,b} = S_a \times S_b, \quad D_{a,b} = kab - a - b + 1,$$

and use ordinary geometric vertex permutation on intersection cohomology. Then

$$P_{a,b;k}(t; U, V) = \sum_{\lambda \vdash a, \mu \vdash b} C_{\lambda,\mu;k}(t) s_\lambda(U) s_\mu(V), \quad (2.3)$$

where

$$C_{\lambda,\mu;k}(t) = \chi_t \operatorname{Hom}_{G_{a,b}} \left(S^\lambda \boxtimes S^\mu, (\operatorname{sgn}_a \boxtimes \operatorname{sgn}_b) \otimes \operatorname{IH}(N_{a,b;k})[D_{a,b}] \right). \quad (2.4)$$

In particular, every $C_{\lambda,\mu;k}$ belongs to $\mathcal{L}$.

The stable locus in this moduli space is nonempty. Indeed, choose every arrow nonzero. A subrepresentation containing a source must contain all sinks; its weight is strictly negative unless it is the full representation. A nonzero subrepresentation containing no source also has negative weight. Irreducibility of the representation space implies that the stable open subset is dense. The representation space has dimension kab . On the stable locus its effective gauge torus has dimension $a+b-1$, giving dimension $D_{a,b}$. For an acyclic quiver the affine invariant ring consists of constants, so relative GIT projectivity gives a projective coarse moduli space. The locus of nonzero arrows is a torus modulo the effective gauge torus; its image is dense, giving the toric structure.

For $a = b = 1$, the moduli is $\mathbb{P}^{k-1}$, so the formula gives $P_{1,1;k} = [k]_t p_1(U) p_1(V)$. The proof of the general formula will be completed in Section 6.

3 Finite colors and BPS cohomology

For integers $r, s \geq 1$, choose finite even alphabets $U = (u_1, \dots, u_r)$ and $V = (v_1, \dots, v_s)$. Let $Q_{r,s;k}$ be the corresponding complete bipartite quiver. For dimension compositions $\alpha \in \mathbb{N}^r$, $\beta \in \mathbb{N}^s$, set

$$P = |\alpha| = \sum_i \alpha_i, \quad Q = |\beta| = \sum_j \beta_j, \quad \delta_{\alpha,\beta} = kPQ - \sum_i \alpha_i^2 - \sum_j \beta_j^2. \quad (3.1)$$

For fixed totals $P, Q > 0$, put weight Q on each source and $-P$ on each sink. This is the stability of projected slope Q/P , with the same subrepresentation convention as above. Write

$u^\alpha = \prod_i u_i^{\alpha_i}$ and $v^\beta = \prod_j v_j^{\beta_j}$, with an empty product equal to one. For dimension vectors d, e of a quiver the Euler form is $\chi(d, e) = \sum_v d_v e_v - \sum_{v \rightarrow w} d_v e_w$, counting every arrow. A slope is called μ -generic when this form is symmetric on all dimension vectors of that slope. For projected dimensions (P, Q) and (P', Q') , the antisymmetric Euler form is $k(P'Q - PQ')$. It vanishes when $Q/P = Q'/P'$, proving μ -genericity on every positive finite slope.

For a variety X with an algebraic group G acting on it, $[X/G]$ denotes the quotient stack, and $H^*([X/G], \mathbb{Q})$ denotes equivariant cohomology. The group of changes of bases at the colored vertices is $\prod_i \mathrm{GL}_{\alpha_i} \times \prod_j \mathrm{GL}_{\beta_j}$, with GL_0 the trivial group. Its quotient of the semistable open subset is denoted by $\mathfrak{S}_{\alpha, \beta}$. Put

$$H_{\alpha, \beta} = H^*(\mathfrak{S}_{\alpha, \beta}, \mathbb{Q})[\delta_{\alpha, \beta}]. \quad (3.2)$$

This is the ordinary-cohomology presentation of the dual compactly supported virtual character used in DT integration. In particular, for one isolated one-dimensional vertex it is

$$T = H^*(BC^*)[-1], \quad \chi_t(T) = \frac{t}{1 - t^2}; \quad (3.3)$$

Here BC^* is the classifying stack of scalar automorphisms; $H^*(BC^*) = \mathbb{Q}[c]$ with c in degree two. In T , the vector c^m has degree $2m + 1$ for $m \geq 0$, so T is odd in every degree in which it is nonzero. For zero potential the BPS object is normalized intersection cohomology of the stable closure, or zero if the stable locus is empty [10, Theorem 4.6], [9, Theorem 4.10 and Corollary 4.11]. When the stable locus is nonempty, the normalization is a shift by the coarse dimension $\delta_{\alpha, \beta} + 1$. Write its ordinary character as $b_{\alpha, \beta}(t)$. We use “BPS cohomology” for this primitive object; the cohomological integrality theorem reconstructs semistable stack cohomology from its graded symmetric powers tensored with T .

The Harder–Narasimhan and integrality identities imply that the finite alphabet evaluation of (2.2) has

$$P_{P, Q; k}(t; U, V) = \sum_{\substack{|\alpha| = P \\ |\beta| = Q}} b_{\alpha, \beta}(t) u^\alpha v^\beta. \quad (3.4)$$

Here the collapse of degree lattices sends each source degree to X , each sink degree to Y , retaining its color monomial. The antisymmetric Euler form depends only on (P, Q) , so this map preserves the quantum torus multiplication. It also respects the Adams operations on each ray. Consequently the ordered product identity descends in every degree, including all multiples on each output ray.

The Hall identity with positive arguments is

$$\mathbb{E}(p_1(V)Y)\mathbb{E}(p_1(U)X) = \mathbb{E}(p_1(U)X) \prod_{b/a \text{ increasing}} \mathbb{E}(P_{a, b; k} Z^{a, b}) \mathbb{E}(p_1(V)Y). \quad (3.5)$$

The full representation stack has character

$$\frac{t^{\sum_i \alpha_i^2 + \sum_j \beta_j^2 - kPQ}}{\prod_i \prod_{h=1}^{\alpha_i} (1 - t^{2h}) \prod_j \prod_{h=1}^{\beta_j} (1 - t^{2h})}. \quad (3.6)$$

Indeed, for one vertex the dilogarithm expansion is

$$\mathbb{E}(uX) = \sum_{n \geq 0} \frac{t^{n^2}}{\prod_{h=1}^n (1 - t^{2h})} u^n X^n,$$

with the empty denominator equal to one. For a commuting even variable z , formula (2.1) gives

$$\log \mathbb{E}(z) = \sum_{m \geq 1} \frac{(-1)^{m+1} t^m z^m}{m(1 - t^{2m})}, \quad \mathbb{E}(z) = (1 + tz) \mathbb{E}(t^2 z).$$

Writing $\mathbb{E}(z) = \sum_{n \geq 0} a_n z^n$, comparison of coefficients gives $a_0 = 1$ and $(1 - t^{2n})a_n = t^{2n-1}a_{n-1}$ for $n \geq 1$. Iteration yields the displayed coefficient. Multiplying it over all source and sink colors gives the two sums of squares in the numerator, and the sink-before-source Weyl multiplication contributes t^{-kPQ} . This gives the normalization in (3.5). Inverting the whole identity gives (2.2), as in [8, §7.3].

For zero potential the monodromic vanishing-cycle normalization agrees with the centered ordinary object; see [9, Example 2.11]. The purity and Tate assertions in [8, §6.3.4, Propositions 6.21–6.22] identify the DT weight character with this χ_t . Consequently (3.4) is expressed in the original DM parameter, with the indicated cohomological shifts.

It remains to identify (3.4) with the Frobenius characteristic in (2.3). The next two sections supply that identification.

4 Double-loop Springer polarization of stacks

We extend the source-color construction of [14] to both sides. The sheaf-theoretic input is the standard Grothendieck–Springer sign-support statement, in the formulation of [11, §3]. We use its Betti version: the Springer correspondences and sign idempotents act on rational constructible complexes, and the zero nilpotent fiber is the full complex flag variety.

4.1 A proper smooth family over ordered spectra

Fix $P, Q > 0$, and an integer $f \geq 1$. On vector spaces $U_0 = \mathbb{C}^P$, $W_0 = \mathbb{C}^Q$, consider two independent endomorphisms L_U, L_W , k arbitrary maps $U_0 \rightarrow W_0$, and f framing vectors in U_0 . Add a one-dimensional framing vertex, with f arrows to U_0 . Choose rational $0 < \epsilon < 1$ and stability

$$\theta_U = Q - \epsilon/P, \quad \theta_W = -P, \quad \theta_\infty = \epsilon. \quad (4.1)$$

The weight of a subrepresentation of dimensions (A, B, η) , $\eta \in \{0, 1\}$, is

$$QA - PB + \epsilon(\eta - A/P). \quad (4.2)$$

The integer $QA - PB$ must vanish if this expression is zero, since $|\epsilon(\eta - A/P)| < 1$. Then $\eta = A/P$. If $\eta = 0$ this forces $A = B = 0$; if $\eta = 1$ it forces $A = P$ and $B = Q$. Thus semistability equals stability.

Let M^f be the resulting stable GIT quotient. Fixing the framing basis identifies the effective gauge group with $\mathrm{GL}_P \times \mathrm{GL}_Q$, and its action is free: a stable object's endomorphisms are scalars, and fixing the framing basis forces that scalar to be one. Let F^f additionally record full flags in U_0, W_0 preserved by the corresponding loops. The flag map $r : F^f \rightarrow M^f$ is projective. Concretely, construct F^f as the closed loop-invariant incidence scheme in the two relative full flag bundles over the free reductive quotient M^f . The Borel presentation below describes this scheme.

Put

$$\mathcal{T} = \mathbb{A}^P \times \mathbb{A}^Q, \quad \mathcal{S} = \mathrm{Sym}^P \mathbb{A}^1 \times \mathrm{Sym}^Q \mathbb{A}^1,$$

and let $\pi : \mathcal{T} \rightarrow \mathcal{S}$ be the finite quotient. Here $\mathrm{Sym}^d \mathbb{A}^1 = (\mathbb{A}^1)^d / S_d$ parametrizes unordered multisets of d complex numbers, including repeated values. Reading the induced endomorphism on each one-dimensional flag quotient defines $h : F^f \rightarrow \mathcal{T}$.

For a morphism f , Rf_* denotes derived direct image of rational constructible complexes, and $R^j f_*$ its degree- j cohomology sheaf; pullbacks in this argument are derived pullbacks.

Lemma 4.1. *The map h is proper and smooth, of relative dimension*

$$fP + kPQ - P - Q. \quad (4.3)$$

Hence $E_f^j = R^j h_* \mathbb{Q}$ is a constant local system on $\mathcal{T}$.

Proof. The affine invariant quotient of all loop, arrow and framing data is $\mathcal{S}$. One first uses the common scalar action on U_0, W_0 to exclude framing variables from invariants, and then a relative scalar action to exclude cross-arrow variables. The remaining invariants are the two characteristic polynomials. Relative GIT projectivity [12, §2], together with stable equals semistable, gives a projective map $M^f \rightarrow \mathcal{S}$. Composing with r gives a proper map. Its lift h through the separated finite morphism π is proper: its graph is closed in $F^f \times_{\mathcal{S}} \mathcal{T}$ and the second projection is a base change of a proper map.

Let $B_d \subset \mathrm{GL}_d$ be the upper triangular subgroup and $\mathfrak{b}_d$ its Lie algebra. A superscript s on the representation space denotes its stable open subset. Fixing standard full flags gives

$$F^f \simeq \left[\left(\mathfrak{b}_P \oplus \mathfrak{b}_Q \oplus \mathrm{Hom}(U_0, W_0)^k \oplus U_0^f \right)^s / (B_P \times B_Q) \right]. \quad (4.4)$$

Projection from the stable open set to the two diagonals is smooth. The Borel action is free; smoothness descends along this smooth torsor. Subtracting the Borel dimensions cancels the dimensions of the two Borel Lie algebras, leaving total dimension $kPQ + fP$. This gives (4.3). Proper smooth base change makes E_f^j a local system, which is constant because the complex affine space $\mathcal{T}$ is simply connected. $\square$

4.2 The geometric permutation action

Over the open set $\mathcal{T}_{\mathrm{reg}}$ of distinct eigenvalues on each side, put $\mathcal{S}_{\mathrm{reg}} = \pi(\mathcal{T}_{\mathrm{reg}})$. For distinct source eigenvalues $\lambda_1, \dots, \lambda_P$, the projector onto the i th eigenline is $\prod_{j \neq i} (L_U - \lambda_j) / (\lambda_i - \lambda_j)$; the analogous formula applies to L_W . These projectors decompose U_0, W_0 into ordered eigenlines. Loop-invariant subspaces are precisely sums of these eigenlines. Consequently

$$F^f|_{\mathcal{T}_{\mathrm{reg}}} \simeq \mathcal{T}_{\mathrm{reg}} \times M_{\mathrm{thin}}^f, \quad (4.5)$$

where M_{thin}^f is the stable moduli of the thin bipartite quiver with P source and Q sink vertices, a common one-dimensional framing vertex, and f arrows from it to each source. Each source has weight $Q - \epsilon/P$, each sink weight $-P$, and the framing vertex weight ϵ . Reordering the spectra corresponds exactly to relabeling its source and sink vertices.

The product description over $\mathcal{T}_{\mathrm{reg}}$ gives its geometric $S_P \times S_Q$ -action. We also write E_f^j for this constant system viewed as its fiber when taking a group multiplicity. Together with the coordinate action on $\mathcal{T}$, this defines an equivariant structure on the entire constant system.

We write $\mathfrak{gl}_d = \mathrm{End}(\mathbb{C}^d)$. The notation $\mathcal{IC}_X$ means the intersection complex normalized to restrict to $\mathbb{Q}[\dim X]$ on the smooth locus of a pure-dimensional variety X . Intermediate extension means the perverse extension having no subobject or quotient supported on the complement of that locus. The other construction of an action uses the Cartesian pullback of the product of two Grothendieck resolutions along

$$M^f \longrightarrow [\mathfrak{gl}_P / \mathrm{GL}_P] \times [\mathfrak{gl}_Q / \mathrm{GL}_Q].$$

It gives the Springer action on $Rr_*\mathbb{Q}$. These two constructions agree. Indeed, if $c : M^f \rightarrow \mathcal{S}$ is the characteristic-polynomial map,

$$R^j(cr)_*\mathbb{Q} = \pi_* E_f^j. \quad (4.6)$$

The finite map π is small, since all its fibers have dimension zero. Since E_f^j is constant on the smooth space $\mathcal{T}$, $\pi_* E_f^j[P + Q]$ is the intermediate extension of its restriction to $\mathcal{S}_{\mathrm{reg}}$. Endomorphisms of this extension are determined on that open set. There both actions are the same reordering of flags and eigenlines, so they agree everywhere. At a collision point, the action of its Young stabilizer is therefore the corresponding block Springer action. This identifies the action with geometric vertex permutations.

4.3 Block signs and the exact shift

For compositions α, β choose a spectrum having the indicated multiplicities within each side. Its Young stabilizer is

$$G_{\alpha, \beta} = \prod_i S_{\alpha_i} \times \prod_j S_{\beta_j}.$$

For $d \geq 1$, let $\mathcal{N}_d$ be the nilpotent cone in $\mathfrak{gl}_d$, and let ρ_d be the map from pairs consisting of an endomorphism and a preserved full flag to the endomorphism. With the Grothendieck action for which the top cohomology of the zero nilpotent flag fiber is sign, the Springer sign summand on the nilpotent cone is

$$\mathrm{Hom}_{S_d}(\mathrm{sgn}_d, R\rho_{d,*}\mathbb{Q})|_{\mathcal{N}_d} = \mathbb{Q}_{\{0\}}[-d(d-1)]. \quad (4.7)$$

The block action is the product action; these facts are precisely [11, Corollary 3.7 and proof of Proposition 3.8]. Applying (4.7) separately to every block on both sides selects the locus where each block loop is scalar. That locus is the stable moduli $M_{\alpha, \beta}^f$ of the colored quiver $Q_{r, s; k}$ with dimensions α, β , a common one-dimensional framing vertex, and f arrows to each source color. The weights are $Q - \epsilon/P$ on source colors, $-P$ on sink colors, and ϵ at the framing vertex. Source and sink spectra vary independently.

Set

$$2g_{\alpha, \beta} = \sum_i \alpha_i(\alpha_i - 1) + \sum_j \beta_j(\beta_j - 1).$$

In the following projector, sgn is the restriction of $\mathrm{sgn}_P \boxtimes \mathrm{sgn}_Q$ to $G_{\alpha, \beta}$. Taking a finite-group sign summand is exact over $\mathbb{Q}$, hence commutes with pullback. The Cartesian Grothendieck resolution and proper base change therefore pull (4.7) back to the scalar-block locus. The product of its top flag lines gives

$$\mathrm{Hom}_{G_{\alpha, \beta}}(\mathrm{sgn}, E_f^j) \cong H^{j-2g_{\alpha, \beta}}(M_{\alpha, \beta}^f, \mathbb{Q}). \quad (4.8)$$

On this fixed ordered-spectrum locus with distinguished blocks, the top flag cohomology line has trivial monodromy: the structure groups are connected general linear groups and preserve the fundamental class. The global permutation action is retained separately. Put $\delta_{\mathrm{thin}} = kPQ - P - Q$, and let $H_{P, Q}^{\mathrm{thin}}$ be the cohomology of the thin semistable stack shifted by δ_{thin} . The shift satisfies

$$\delta_{\mathrm{thin}} - \delta_{\alpha, \beta} = 2g_{\alpha, \beta}. \quad (4.9)$$

4.4 Framing and stabilization

The map from $M_{\alpha, \beta}^f$ to $\mathfrak{S}_{\alpha, \beta}$ is the stable open subset of the rank- fP framing bundle. Its complement has codimension tending to infinity with f . If the underlying representation is not semistable, a positive-weight subrepresentation, taken with framing dimension zero, still has positive perturbed weight: its original positive integer weight decreases by less than one. For a semistable underlying representation, the only possible destabilizing framed subobject has framing dimension one and original weight zero. Thus bad framing lands in the source subspace of a proper equal-slope subrepresentation. That source dimension A satisfies $A < P$. For fixed subspaces the codimension is $f(P - A) \geq f$; allowing the subspaces to vary subtracts at most $\sum_i A_i(\alpha_i - A_i) + \sum_j B_j(\beta_j - B_j)$, the dimension of their product of Grassmannians. Here A_i, B_j are the subdimensions at the colored vertices. Thus ordinary H^j stabilizes for every fixed j . More explicitly, allowing all vertex subspaces loses at most

$$C_{\alpha, \beta} = \sum_i \lfloor \alpha_i^2/4 \rfloor + \sum_j \lfloor \beta_j^2/4 \rfloor$$

dimensions. The bad locus has codimension at least $f - C_{\alpha, \beta}$, so the comparison is an isomorphism in ordinary degrees $j < 2(f - C_{\alpha, \beta}) - 1$. Indeed, on smooth finite-dimensional equivariant

approximations, local cohomology supported on a closed subset of complex codimension c vanishes below degree $2c$. The localization exact sequence gives this range, which persists as the approximation dimension increases.

For each desired centered degree choose f large enough for both ordinary degrees in (4.8) to lie in their stable ranges. The thin framed-to-stack comparison is geometrically $S_P \times S_Q$ -equivariant. Thus we first stabilize ordinary cohomology and only then shift by δ_{thin} or $\delta_{\alpha,\beta}$. The equality $j - \delta_{\text{thin}} = (j - 2g_{\alpha,\beta}) - \delta_{\alpha,\beta}$ from (4.9) shows that the centered degrees match. We obtain

$$\chi_t H_{\alpha,\beta} = \chi_t \left[\left((\text{sgn}_P \boxtimes \text{sgn}_Q) \otimes H_{P,Q}^{\text{thin}} \right)^{G_{\alpha,\beta}} \right] \quad (4.10)$$

with geometric source/sink permutations on the right. The identity concerns each fixed ordered color composition. It is this dimension identity that enters the primitive comparison below.

5 Permutation actions on BPS cohomology

The stack comparison must now be passed to primitive BPS cohomology while retaining the geometric permutation action.

Fix coprime integers $p, q > 0$. For $n \geq 1$, consider label sets of sizes $(P, Q) = (np, nq)$. A PBW decomposition of a thin dimension vector partitions the source labels and sink labels into disjoint supports. Every nonzero factor on this slope has projected dimensions $(P_i, Q_i) = n_i(p, q)$. There are no source-only or sink-only factors on a positive finite slope.

Let d_i be the thin dimension vector of support i . For two disjoint supports, writing $\chi_{ij} = \chi(d_i, d_j)$, the Euler forms are

$$\chi_{12} = -kP_1Q_2, \quad \chi_{ii} = P_i + Q_i - kP_iQ_i.$$

The extra symmetry in the relative cohomological PBW theorem is

$$\tau(d_1, d_2) = \chi_{12} + \chi_{11}\chi_{22} \pmod{2} \quad (5.1)$$

in addition to ordinary cohomological Koszul symmetry [9, Theorem C, formula (15)]. In the field $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$, substituting $P_i = n_i p$, $Q_i = n_i q$ gives $\chi_{ii} = n_i(p + q + kpq)$, and hence

$$\tau(d_1, d_2) = n_1 n_2 (kpq + (p + q + kpq)^2) = n_1 n_2 (p + q).$$

Consequently

$$\tau(d_1, d_2) = n_1 n_2 (p + q) = P_1 P_2 + Q_1 Q_2 \pmod{2}. \quad (5.2)$$

The block exchange in the restriction of $\text{sgn}_P \boxtimes \text{sgn}_Q$ has exactly the last sign. Twisting the whole thin symmetric sequence by this representation cancels the extra τ , while leaving the ordinary cohomological Koszul symmetry intact.

5.1 Naturality and the relative PBW theorem

Lemma 5.1. *For the thin complete bipartite quivers at a fixed positive slope, the relative PBW morphism is natural under all source and sink relabelings. In projected degree (P, Q) , its primitive factor has the geometric action on $\text{IH}(N_{P,Q;k}[D_{P,Q}])$ and the trivial action on T .*

Proof. All quivers on different finite label sets are related by relabeling; restriction to a support gives the corresponding smaller complete bipartite quiver. The moduli constructions, direct-sum morphisms, perverse filtrations and Hall correspondences respect these maps. The correspondence stacks and their complex algebraic push/pull maps are preserved by relabeling, as are the Euler normalization and the zero-potential constant/intersection complexes. The relative product uses the τ symmetry of (5.1).

For zero potential and a thin positive support, the stable open subset is nonempty. The primitive term is the canonical strict-full-support intersection-complex summand in the direct image of the stack cohomology. On the stable open its multiplicity is T , the cohomology of the scalar automorphism group. Vertex relabeling fixes the scalar automorphism λid , so acts trivially on T . The canonical full-support construction and its intermediate extension therefore identify the action on the primitive with the geometric action on $\text{IH}(N_{P,Q;k})[D_{P,Q}]$.

For a thin support dimension vector d , let $\mathfrak{S}_d$ be its semistable stack and N_d its coarse moduli. The map $p : \mathfrak{S}_d \rightarrow N_d$ sends a representation to the direct sum of its stable Jordan–Hölder factors of the fixed slope. Over the stable open, each $R^{2m}p_*\mathbb{Q}$ has its canonical rank-one fiber generated by c^m in $H^*(B\mathbb{C}^*)$; a band-preserving relabeling fixes this generator. On the thin stack an invariant representative of the degree-two fiber generator is

$$\frac{1}{|\text{supp } d|} \sum_{v \in \text{supp } d} c_1(\mathcal{L}_v),$$

where $\text{supp } d$ is the set of vertices of dimension one, $\mathcal{L}_v$ is the tautological line at vertex v , and c_1 denotes the first Chern class. Each summand restricts to the generator c on the scalar fiber. Base change for relabeling gives coherent actions on $Rp_*\mathcal{IC}$ and its perverse cohomology. The strict-full-support part consists of the simple perverse summands whose support is the entire coarse moduli space. It is intrinsic, and its restriction to the stable open determines it by intermediate extension, giving the equivariant primitive inclusion.

Hall covariance has a direct description on thin supports. For a thin output dimension vector, the block supports are disjoint. For $0 \rightarrow E_1 \rightarrow E \rightarrow E_2 \rightarrow 0$, arrows from source support 1 to sink support 2 are zero; reverse cross arrows are arbitrary extension coordinates. Thus the extension stack is an affine bundle of rank kP_2Q_1 over the two block stacks and maps to the total stack by a coordinate-zero embedding of codimension kP_1Q_2 . Relabeling preserves both maps. Smooth pullback, Gysin/trace and direct image commute with these isomorphisms; permuting complex coordinates preserves their complex orientation. On the common slope $P_1Q_2 = P_2Q_1$, and the shift for the union minus the two block shifts is

$$\begin{aligned} & k(P_1 + P_2)(Q_1 + Q_2) - (P_1 + P_2) - (Q_1 + Q_2) \\ & - (kP_1Q_1 - P_1 - Q_1) - (kP_2Q_2 - P_2 - Q_2) = 2kP_1Q_2. \end{aligned}$$

This equals the Gysin degree. Hence the normalized multiplication is relabeling-equivariant. An exchange of the ordered factors is a separate operation and uses the prescribed τ braiding, already accounted for in (5.2).

Apply the relative canonical PBW theorem and push to dimension vectors. The natural identification with the perverse associated graded of absolute cohomology [9, Proposition 5.7, Corollary 4.11, Theorem C] suffices for the resulting graded group character. The finite group preserves the filtration, hence a representation and its associated graded have the same character in every cohomological degree. More explicitly, the morphism in [9, Theorem 6.9] is the composition of the canonical inclusion of Corollary 4.11, the τ -symmetric projector, and iterated relative Hall multiplication. Each of these three maps is preserved by relabeling. The known nonequivariant isomorphism is therefore an equivariant isomorphism. $\square$

Let

$$R_H(P, Q) = (\text{sgn}_P \boxtimes \text{sgn}_Q) \otimes H_{P,Q}^{\text{thin}}, \quad R_B(P, Q) = (\text{sgn}_P \boxtimes \text{sgn}_Q) \otimes \text{IH}(N_{P,Q;k})[D_{P,Q}].$$

At $(0, 0)$ set $R_H(0, 0) = \mathbb{Q}$ and $R_B(0, 0) = 0$; away from the chosen positive slope set both sequences to zero. A bivariate symmetric sequence assigns to each pair of finite label sets a graded vector space, functorially under bijections; equivalently it assigns an $S_P \times S_Q$ representation to each (P, Q) . Its induction product is

$$(R \star R')(P, Q) = \bigoplus_{\substack{P_1 + P_2 = P \\ Q_1 + Q_2 = Q}} \text{Ind}_{(S_{P_1} \times S_{Q_1}) \times (S_{P_2} \times S_{Q_2})}^{S_P \times S_Q} (R(P_1, Q_1) \otimes R'(P_2, Q_2)).$$

The groups in the subscript act on disjoint label blocks. We define $\text{Sym}_{\text{seq}}(R) = \bigoplus_{m \geq 0} (R^{\star m})^{S_m}$, where S_m permutes factors with sign $(-1)^{ij}$ when degrees i, j are exchanged. The $m = 0$ term is the unit at $(0, 0)$. Brackets below record degree-wise characters, so extensions and their associated graded have the same class. Lemma 5.1 and the sign cancellation give

$$[R_H] = [\text{Sym}_{\text{seq}}(T \otimes R_B)]. \quad (5.3)$$

5.2 Shifts and tensor permutations

There are three distinct sources of signs. A geometric automorphism of one variety N acts on its globally shifted $\text{IH}(N)[\dim N]$ without an additional sign character. Exchanging two already shifted tensor factors, however, uses their cohomological Koszul sign. Finally, the normalized Hall symmetry has the extra term τ in (5.1). The degree-zero representation $\text{sgn}_P \boxtimes \text{sgn}_Q$ cancels only this last term, leaving the Koszul symmetry and the odd grading of T .

This distinction is the issue discussed in [10, Proposition 3.5 and Remark 3.6]: a naive geometric permutation of several factors and the sheaf-theoretic permutation of their shifted intersection complexes can differ by a dimension sign. In [9, §6.1], moving the half-Tate normalization factors explicitly produces the term $\chi_{11}\chi_{22}$ in τ .

With sink dimension one, a colored source of dimension α_i is stable precisely when its k outgoing linear forms define an injection $\mathbb{C}^{\alpha_i} \rightarrow \mathbb{C}^k$. The coarse moduli is therefore $\prod_i \text{Gr}(\alpha_i, k)$, where $\text{Gr}(m, k)$ parametrizes m -dimensional subspaces of $\mathbb{C}^k$, empty when some $\alpha_i > k$. The centered Grassmannian character is $\chi_t \Lambda^{\alpha_i} W_k$. To see the grading, the Schubert cells of $\text{Gr}(m, k)$ are indexed by $1 \leq i_1 < \dots < i_m \leq k$ and have complex dimensions $\sum_j (i_j - j)$. After shifting by $m(k - m)$ their cohomological degrees are $2 \sum_j i_j - m(k + 1)$. These are the sums of the corresponding weights $2i_j - k - 1$ of W_k , which proves the character identity. For a vector space W define the ordinary Schur functor by $S_\lambda(W) = \text{Hom}_{S_a}(S^\lambda, W^{\otimes a})$ for $\lambda \vdash a$, with ordinary tensor permutations. The ordinary dual Cauchy identity gives, for all a ,

$$P_{a,1;k}(t; U, V) = p_1(V) \sum_{\lambda \vdash a} s_\lambda(U) \chi_t S_{\lambda'}(W_k).$$

Geometrically $N_{a,1;k} = (\mathbb{P}^{k-1})^a$ and its external action uses geometric factor permutations; the global sign gives exactly the transpose λ' .

For example, $N_{2,1;2} = \mathbb{P}^1 \times \mathbb{P}^1$. Its geometric S_2 factor swap gives

$$\text{IH}(N_{2,1;2})[2] \cong \mathbf{1} \otimes W_3 \oplus \text{sgn}_2 \otimes W_1,$$

where $\mathbf{1}$ is the trivial S_2 representation. After the source-sign twist this yields $C_{(1,1),(1);2} = [3]_t$ and $C_{(2),(1);2} = 1$. The two-arrow Kronecker moduli of dimension $(2, 1)$ is a point, which also gives the latter coefficient.

5.3 Evaluation on colors and induction

For even vector spaces $A = \mathbb{C}^r, B = \mathbb{C}^s$, equipped with their diagonal color tori of weights u_i, v_j , define the evaluation of a bivariate symmetric sequence R by

$$\text{Ev}_{A,B}(R) = \bigoplus_{P, Q \geq 0} \left(R(P, Q) \otimes A^{\otimes P} \otimes B^{\otimes Q} \right)^{S_P \times S_Q}. \quad (5.4)$$

In characteristic zero this is symmetric monoidal for the induction product on sequences and ordinary graded tensor product. For the monoidality assertion, apply Frobenius reciprocity to each induced summand in $R \star R'$ and restrict $A^{\otimes P} \otimes B^{\otimes Q}$ to its two label blocks. The resulting invariants are the tensor product of the evaluations of R and R' . Block exchange gives the same graded permutation on both sides. For a fixed color weight (α, β) , the tensor basis

is the permutation module $\text{Ind}_{G_{\alpha,\beta}}^{S_P \times S_Q} \mathbf{1}$. Frobenius reciprocity identifies its contribution with $R(P, Q)^{G_{\alpha,\beta}}$.

Applying (5.4) to (5.3), stack polarization (4.10) identifies every color-weight graded dimension on the left with that of the actual colored semistable stack. Let H_{colored} be the sum of colored semistable stack cohomologies on the fixed slope, with their virtual shifts and dimension gradings, including the degree-zero unit. Let B_{colored} be the sum of their positive-degree BPS objects. For a graded space V , write $\text{Sym}(V) = \bigoplus_{m \geq 0} (V^{\otimes m})^{S_m}$, with Koszul permutations and degree-zero term $\mathbb{Q}$. Ordinary character integrality gives

$$[H_{\text{colored}}] = [\text{Sym}(T \otimes B_{\text{colored}})]. \quad (5.5)$$

The equality of underlying symmetric-object characters follows from [9, Proposition 6.2]: the extra Euler-form symmetry can be untwisted, and $\tau(d, d) = 0$ preserves the symmetric or exterior power assigned to repeated identical dimension labels.

Compare (5.3)–(5.5) inductively in n . At total projected degree (np, nq) , the only unknown primitive term is T times the degree- n BPS character; every product with two or more factors uses smaller positive n_i and is already identified. Subtract these lower terms and cancel $t/(1-t^2)$ in $\mathbb{Q}((t))$, the field of Laurent series bounded below in t , separately at each dimension. Its inverse is $(1-t^2)/t$. The induction starts at $n = 1$, where there are no lower products. This proves

$$b_{\alpha,\beta}(t) = \chi_t R_B(P, Q)^{G_{\alpha,\beta}}. \quad (5.6)$$

Schur–Weyl decomposition gives

$$\text{Ev}_{A,B}(R_B)_{P,Q} \cong \bigoplus_{\lambda \vdash P, \mu \vdash Q} \text{Hom}_{S_P \times S_Q}(S^\lambda \boxtimes S^\mu, R_B(P, Q)) \otimes S_\lambda(A) \otimes S_\mu(B).$$

Taking the color-torus character, the last two factors contribute $s_\lambda(U)s_\mu(V)$. Thus (5.6) gives (2.3)–(2.4) for every number of finite colors. Compatibility on adding unused colors makes this an identity of universal symmetric functions.

6 Lefschetz operators and input specialization

6.1 The multiplicity spaces

Completion of the proof of Theorem 2.1. The space $N_{a,b;k}$ is projective. Its intersection cohomology satisfies Hard Lefschetz [13]. Choose an ample line bundle A . The first Chern class of $\bigotimes_{g \in G_{a,b}} g^*A$ is ample and $G_{a,b}$ -invariant, where g^*A denotes pullback along the vertex-permutation automorphism g . Let L be cup product with this class. Hard Lefschetz gives isomorphisms $L^m : \text{IH}^{D_{a,b}-m}(N_{a,b;k}) \rightarrow \text{IH}^{D_{a,b}+m}(N_{a,b;k})$ for $m \geq 0$. These maps commute with vertex permutations. Applying the exact functor $\text{Hom}_{G_{a,b}}(S^\lambda \boxtimes S^\mu, -)$ preserves the isomorphisms, and the primitive decomposition gives the corresponding finite-dimensional $\mathfrak{sl}_2$ module. The external sign twist does not change grading or this operator. Equation (2.4) thus proves $C_{\lambda,\mu;k} \in \mathcal{L}$. $\square$

6.2 Super Schur specialization

Lemma 6.1. *For $R \in \mathcal{L}$ and a partition λ , the specialization of s_λ defined by $p_m \mapsto \psi_m(R)$ belongs to $\mathcal{L}$.*

Proof. Put $a = |\lambda|$, the sum of the parts of λ . Let $R(t) \in \mathcal{L}$, choose an $\mathfrak{sl}_2$ representation V_R with character R , and give its weight- j subspace parity $j \bmod 2$. The raising and lowering operators change the weight by two, so the $\mathfrak{sl}_2$ action is even. Graded permutations of $V_R^{\otimes a}$ therefore commute with $\mathfrak{sl}_2$.

Choose a homogeneous weight basis. A tensor basis vector contributes to the trace of an m -cycle only when all its factors are the same basis vector. For a degree- j vector the cyclic permutation has sign $(-1)^{(m-1)j}$ and total degree mj . Thus an m -cycle has ordinary graded trace

$$\sum_j (-1)^{(m-1)j} \dim V_R^j t^{mj} = \psi_m(R(t)). \quad (6.1)$$

For $\sigma \in S_a$, the graded trace on $V_R^{\otimes a}$ is the product of (6.1) over its disjoint cycles. Here the graded trace means the sum of the ordinary traces on degree- j subspaces, weighted by t^j . Taking S_a -invariants in $(S^\lambda)^* \otimes V_R^{\otimes a}$ uses the averaging projector $a!^{-1} \sum_{\sigma \in S_a} \sigma$; its graded trace is

$$\frac{1}{a!} \sum_{\sigma \in S_a} \text{Tr}(\sigma^{-1} | S^\lambda) \prod_{C \text{ a cycle of } \sigma} \psi_{|C|}(R).$$

The Frobenius character formula identifies this expression with the specialization of s_λ . Denote that specialization by $s_\lambda^\psi(R)$. Thus

$$s_\lambda^\psi(R) = \chi_t \text{Hom}_{S_a} (S^\lambda, V_R^{\otimes a}), \quad (6.2)$$

where S_a acts by graded permutations. This is an actual finite-dimensional $\mathfrak{sl}_2$ representation, so $s_\lambda^\psi(R) \in \mathcal{L}$.

For $R = [r]_t$ with r odd, this is ordinary $S_\lambda(W_r)$. For r even, all weights are odd and the sign twist changes $S_\lambda(W_r)$ to $S_{\lambda'}(W_r)$. Formula (6.2) also covers direct sums of both parities. $\square$

Corollary 6.2. *For every $k \geq 1$ and $R, S \in \mathcal{L}$, replace $p_1(U), p_1(V)$ in (2.2) by R, S . Every resulting coefficient $P_{a,b;k}(t; R, S)$ belongs to $\mathcal{L}$. All positive a, b , including nonprimitive pairs, are allowed.*

Proof. For $R, S \in \mathcal{L}$, the ring map

$$p_m(U) \mapsto \psi_m(R), \quad p_m(V) \mapsto \psi_m(S) \quad (6.3)$$

fixes t and commutes with all ray Adams operations: indeed $\psi_n \psi_m(t) = (-1)^{m+1+m(n+1)} t^{mn} = (-1)^{mn+1} t^{mn} = \psi_{mn}(t)$. Apply it to the whole universal factorization. Theorem 2.1 gives

$$P_{a,b;k}(t; R, S) = \sum_{\lambda, \mu} C_{\lambda, \mu; k}(t) s_\lambda^\psi(R) s_\mu^\psi(S) \in \mathcal{L} \quad (6.4)$$

because sums and tensor products preserve finite-dimensional $\mathfrak{sl}_2$ characters. $\square$

For pairing zero the two factors commute. A negative pairing is reduced to a positive one by exchanging the inputs and the orientation.

7 Scattering diagrams and theta functions

7.1 The scattering statement

Let L be a finite-rank free abelian group, put $L_{\mathbb{R}} = L \otimes_{\mathbb{Z}} \mathbb{R}$, and let $\omega : L \times L \rightarrow \mathbb{Q}$ be a skew-symmetric bilinear form, extended to $L_{\mathbb{R}}$. Let $L_0 \subset L$ be a sublattice satisfying $\omega(L_0, L) \subset \mathbb{Z}$. Choose a strongly convex rational polyhedral cone $\sigma \subset L_{\mathbb{R}}$; write $L^+ = (L \cap \sigma) \setminus \{0\}$ and $L_0^+ = L_0 \cap L^+$. Choose an integer $D > 0$ with $D\omega(L, L) \subset \mathbb{Z}$. The Weyl monomials are the basis elements z^p , $p \in L$, with $z^p z^{p'} = t^{\omega(p, p')} z^{p+p'}$ over $\mathbb{Q}(t^{1/D})$. Choose an integral linear functional strictly positive on $\sigma \setminus \{0\}$. Complete by its degree, allowing supports in translates of $\sigma \cap L$. Each bounded-degree slice of this monoid is finite, so coefficient identities are checked in finite degree quotients. Different such functionals define the same completion.

For $v \in L_0^+$ not in the radical of ω , put $v^{\omega^\perp} = \{x \in L_{\mathbb{R}} : \omega(v, x) = 0\}$. A wall consists of a rational polyhedral codimension-one cone $\mathfrak{d} \subset v^{\omega^\perp}$ and an element supported in positive multiples of v , such as $\mathbb{E}(-R(t)z^v)$. The dilogarithm is defined by (2.1), with Z^{md} replaced by z^{mv} . A wall is incoming if $x + \mathbb{R}_{\geq 0}v \subset \mathfrak{d}$ for every $x \in \mathfrak{d}$; otherwise it is outgoing. A diagram has finitely many nontrivial walls at each degree cutoff. At a transverse crossing by a path γ , its wall element acts by conjugation, raised to the power $\text{sign } \omega(v, -\gamma')$. The path-ordered product is the composition of these actions in traversal order. A diagram is consistent if every admissible closed path has identity product; equivalence means equality of path-ordered products. Admissible paths avoid wall boundaries and intersections of nonparallel walls. The completion theorem [8, Theorem 2.13] supplies the consistent completion of a finite full-hyperplane incoming diagram by outgoing walls.

Theorem 7.1. *For finitely many $v_i \in L_0^+$ with $\omega(v_i, \cdot) \neq 0$ and $R_i \in \mathcal{L}$, the consistent completion of*

$$\{(v_i^{\omega^\perp}, \mathbb{E}(-R_i(t)z^{v_i})) : 1 \leq i \leq s\}$$

has a representative all of whose walls have the form $(\mathfrak{d}, \mathbb{E}(-R_{\mathfrak{d}}(t)z^{v_{\mathfrak{d}}}))$ with $R_{\mathfrak{d}} \in \mathcal{L}$ and $v_{\mathfrak{d}} \in L_0^+$.

Proof. Work at a fixed degree cutoff. The change-of-monoid and generic perturbation construction of [8, Section 5] separates incoming factors and reduces new joints to two-wall problems. Split $R_i = \sum_r m_{i,r}[r]_t$ into irreducible characters; additivity of the dilogarithm writes each incoming factor as a product of the corresponding coincident factors. Each joint has integral pairing because its degrees lie in L_0 . Corollary 6.2 supplies its outgoing factors with labels in $\mathcal{L}$. This is exactly the sufficient two-wall criterion at the end of [8, Section 7.3].

The lattice projection in that construction adds the degree labels and preserves the skew pairing and t . Coincident factors may be retained separately or combined by adding their labels. Both operations preserve $\mathcal{L}$. The construction retains all degrees on a ray, so projection also retains nonprimitive factors. The compatible degreewise completions give the asserted diagram by inverse limit. This proves Conjecture 2.17 of [8]. $\square$

7.2 Broken-line coefficients

A broken line ending at a generic point $\mathcal{Q} \in L_{\mathbb{R}}$ is a piecewise affine path $\gamma : (-\infty, 0] \rightarrow L_{\mathbb{R}}$, with $\gamma(0) = \mathcal{Q}$, together with a monomial $c(t)z^m$ on each linear segment. Its velocity is $-m$. It begins with z^p for a prescribed $p \in L$ and changes monomial at a wall by selecting a term in the corresponding conjugation action. It avoids the joints; generic endpoints are chosen outside the exceptional loci for these paths. The theta expansion $\vartheta_{p,\mathcal{Q}}$ is the sum of terminal monomials. The construction of [8, Section 3] makes this sum well-defined degreewise. Write

$$\vartheta_{p,\mathcal{Q}} = z^p + \sum_{v \in L^+} c_{p,v,\mathcal{Q}}(t)z^{p+v},$$

where an absent monomial has coefficient zero; for $p = 0$ the theta function is 1.

Theorem 7.2. *For the incoming data of Theorem 7.1, every $c_{p,v,\mathcal{Q}}(t)$ belongs to $\mathcal{L}$, for all $p \in L$ and all generic $\mathcal{Q}$. Consequently Conjectures 3.13 and 1.4 of Davison–Mandel hold.*

Proof. Use the representative from Theorem 7.1. We compute the crossing in the Weyl basis. It suffices first to take one label $[r]_t$ and an incoming monomial z^p . Put $b = \omega(v, p) \in \mathbb{Z}$. For $b \neq 0$, the crossing sign is $\varepsilon = \text{sign } b$, since the velocity is $-p$. Along the ray extend ψ_m by $\psi_m(z^v) = z^{mv}$, and define $\text{Exp}_{\psi}(F) = \exp(\sum_{m \geq 1} \psi_m(F)/m)$. Equation (2.1) reads $\mathbb{E}(-Pz^v) = \text{Exp}_{\psi}(Pz^v/(t - t^{-1}))$. Since $z^v z^p = t^{2b} z^p z^v$, conjugation gives

$$\mathbb{E}(-\varepsilon[r]_t z^v) z^p \mathbb{E}(-\varepsilon[r]_t z^v)^{-1} = z^p \text{Exp}_{\psi} \left(\frac{\varepsilon[r]_t (t^{2b} - 1)}{t - t^{-1}} z^v \right) = z^p \text{Exp}_{\psi} ([r]_t [|b|]_t t^b z^v).$$

The weights of $W_r \otimes W_{|b|}$ shifted by b all have parity $r + |b| - 2 + b \equiv r \pmod{2}$. The coefficient of z^{nv} in the last exponential is therefore t^{nb} times the character of $\Lambda^n(W_r \otimes W_{|b|})$ if r is odd, and of $\text{Sym}^n(W_r \otimes W_{|b|})$ if r is even. Left multiplication by z^p contributes $t^{n\omega(p,v)} = t^{-nb}$, canceling the shift. Thus each resulting Weyl coefficient belongs to $\mathcal{L}$. For $b = 0$ the action is the identity. This also proves directly the special case of [8, Lemma 3.5] used here.

For a sum of irreducible labels, compose the commuting same-ray actions. An existing coefficient in $\mathcal{L}$ multiplies these characters; tensor products and direct sums of $\mathfrak{sl}_2$ modules preserve $\mathcal{L}$.

Starting with coefficient 1, induction over bends proves the assertion for each broken line. At a fixed output degree the defining sum is finite, so its coefficient is in $\mathcal{L}$. This proves Conjecture 3.13. Here $\mathcal{A}_{\text{prin}}$ is the cluster $\mathcal{A}$ -space with principal coefficients. The cluster seed diagrams for $\mathcal{A}$, $\mathcal{A}_{\text{prin}}$ and $\mathcal{X}$ enter this framework by [8, Lemma 4.3]; the quantum $\mathcal{A}$ case has its standing compatible-pair assumption. In these diagrams the incoming labels are 1, and $\vartheta_{p,\mathcal{Q}}$ is the expansion of the theta function with index p in the scattering chart indexed by $\mathcal{Q}$. Applying the result to each of these seed lattices proves Conjecture 1.4 for every index and generic chart. $\square$

7.3 The ICM 2026 formulation

Write $x = q^{1/2}$ for the parameter in [15]. Its signed quantum integer is

$$[\pm r]_q = (-1)^{r+1} \frac{x^r - x^{-r}}{x - x^{-1}} = [r]_{-x}.$$

Thus evaluation $t = -x$ sends $\mathcal{L}$ to nonnegative sums of these signed integers. This is a translation of parameter conventions.

Corollary 7.3. *Conjecture 3.12 of [15] holds: for every $n_1, n_2, k \geq 1$, all the ray labels in its ordered factorization are nonnegative sums of $[\pm r]_q$, $r \geq 1$.*

Proof. Use the positive-argument Hall identity (3.5), specialized to $R = [n_2]_t$ on the source and $S = [n_1]_t$ on the sink. This is the sink-before-source order of the ICM statement. With $t = -q^{1/2}$, its twisted multiplication is the same Euler-form quantum torus multiplication. Indeed, for $d = (a, b)$ and $e = (c, d_2)$, the Euler form is $\chi(d, e) = ac + bd_2 - kad_2$, so $-(\chi(d, e) - \chi(e, d)) = k(ad_2 - bc)$, the exponent in our torus. This is the convention of [15, Section 3.3]. The plethystic operation also agrees. Under $t = -x$ the Adams operation becomes ordinary substitution $x \mapsto x^m$, and

$$\log \mathbb{E}(P(t)Z^d) \Big|_{t=-x} = \sum_{m \geq 1} \frac{P(-x^m)}{m(x^m - x^{-m})} Z^{md}.$$

This is $\log \text{Exp}(P(-x)Z^d/(x - x^{-1}))$, where $\text{Exp}(F) = \exp(\sum_{m \geq 1} \Psi_m(F)/m)$, with $\Psi_m(x) = x^m$ and $\Psi_m(Z^d) = Z^{md}$, is ordinary plethystic exponentiation on this ray. Since $1/(x - x^{-1}) = -q^{1/2}/(1 - q)$, it is the dilogarithm in [15, Example 3.2]. Consequently the ordered factors in the two identities agree by uniqueness, including every multiple on each ray. Corollary 6.2 and $[r]_{-x} = [\pm r]_q$ give the result. More explicitly, for positive source and sink degrees a, b , its coefficient is $P_{a,b;k}(-x; [n_2]_{-x}, [n_1]_{-x})$; the axis coefficients are the two prescribed inputs. $\square$

8 Colored BPS invariants of acyclic quivers

8.1 The multialphabet formula

The two-sided construction extends to one alphabet at each vertex. It gives Schur–Lefschetz positivity in all alphabets simultaneously.

Let Q be a finite acyclic quiver with vertex set Q_0 and arrow set Q_1 , allowing multiple arrows. For $d, e \in \mathbb{N}^{Q_0}$ put

$$A(d, e) = \sum_{a: i \rightarrow j} d_i e_j, \quad \chi_Q(d, e) = \sum_i d_i e_i - A(d, e).$$

For a real weight $\zeta = (\zeta_i)$ define $|d| = \sum_i d_i$ and $\mu(d) = \zeta \cdot d/|d|$ for $d \neq 0$. A nonzero representation is semistable when every nonzero subrepresentation has slope at most its slope, and stable when the inequality is strict for every proper nonzero subrepresentation. Fix a value μ for which the stability is μ -generic: $\chi_Q(d, e) = \chi_Q(e, d)$ for dimensions on that slope. Replace vertex i by $r_i \geq 1$ colors; replace every arrow by all pairs between its colored endpoints. A colored dimension vector is an array $\alpha = (\alpha_{i,c})$, with $i \in Q_0$, $1 \leq c \leq r_i$ and $\alpha_{i,c} \in \mathbb{N}$. Write $\pi(\alpha)_i = \sum_c \alpha_{i,c}$ and let $b_\alpha(t)$ be the normalized zero-potential BPS character for the inherited slope stability. Explicitly, this is the character of the intersection cohomology of the stable closure shifted by $A(d, d) - \sum_{i,c} \alpha_{i,c}^2 + 1$, for $d = \pi(\alpha)$, and zero when the stable locus is empty. Each vertex has the even alphabet $U_i = (u_{i,1}, \dots, u_{i,r_i})$; $\mathbf{u}^\alpha = \prod_{i,c} u_{i,c}^{\alpha_{i,c}}$. Put $G_d = \prod_i S_{d_i}$. For a graded G_d representation R , define its multialphabet Frobenius characteristic by

$$\text{ch}_{G_d}(R) = \sum_{\lambda_i \vdash d_i} \chi_t \text{Hom}_{G_d}(\boxtimes_i S^{\lambda_i}, R) \prod_i s_{\lambda_i}(U_i).$$

For $d_i = 0$, the group and its representation are trivial and the empty partition contributes 1.

For nonzero d on the slope let Q^d have d_i one-dimensional clones at vertex i , with all arrow pairs. Put $\epsilon_d = \boxtimes_i \text{sgn}_{d_i}$. Give the clones at i the weight inherited from ζ_i . Let N_d denote its semistable coarse moduli and N_d^{st} its stable open locus. Define

$$I_d = \begin{cases} \text{IH}(N_d)[A(d, d) - |d| + 1], & N_d^{\text{st}} \neq \emptyset, \\ 0, & N_d^{\text{st}} = \emptyset, \end{cases}$$

where the shift in the first branch is the dimension of N_d .

Theorem 8.1. *For every μ -generic slope and every nonzero d on it,*

$$\sum_{\pi(\alpha)=d} b_\alpha(t) \mathbf{u}^\alpha = \text{ch}_{G_d}(\epsilon_d \otimes I_d). \quad (8.1)$$

The right-hand side is evaluated in the same finite alphabets as the left. Equivalently, its universal coefficient of $\prod_i s_{\lambda_i}(U_i)$ is

$$\chi_t \text{Hom}_{G_d}(\boxtimes_i S^{\lambda_i}, \epsilon_d \otimes I_d) \in \mathcal{L}. \quad (8.2)$$

The group G_d permutes clones within each fixed original vertex.

Proof. We extend the constructions of Sections 4–6. All colored and thin quivers inherit μ -genericity: the antisymmetrized Euler form depends only on projected dimensions. At fixed d , only finitely many subdimensions matter. One can choose an integral King weight θ preserving their slope comparisons and equalities. Indeed, the conditions for equality are homogeneous linear equations with integer coefficients in the weight vector. Their solution space has a rational basis. The finitely many strict inequalities cut out an open subset of this space containing $(\zeta_i - \mu)_i$. Choose a rational point in it and clear positive denominators. This also gives $\theta(d) = 0$.

Give every original vertex i an auxiliary endomorphism, an invariant full flag and $f \geq 1$ framing vectors from a common one-dimensional vertex. Choose rational $0 < \epsilon < 1$. Use $\theta_i - \epsilon/|d|$ and $\theta_\infty = \epsilon$. For a subobject of original dimension $e \leq d$ and framing dimension $\eta \in \{0, 1\}$, its weight is $\sum_i \theta_i e_i + \epsilon(\eta - |e|/|d|)$. Its integer part and perturbation vanish simultaneously only when $e = 0, \eta = 0$ or $e = d, \eta = 1$. Thus stable equals semistable. The effective gauge action is free.

Choose positive topological heights $w_j > w_i$ along every arrow. The central one-parameter subgroup with weights w_i gives all original arrows and framing variables nonzero weights of the same sign, and all auxiliary loops weight zero. The invariant ring therefore contains only the independent loop characteristic polynomials. This proves that the affine GIT base is $\prod_i \text{Sym}^{d_i} \mathbb{A}^1$. The all-flag ordered-spectrum map to $\prod_i \mathbb{A}^{d_i}$ is proper by relative GIT and the closed graph argument, and smooth by its free Borel presentation. Its relative dimension is

$$A(d, d) + (f - 1)|d|.$$

Regular eigenline decomposition and finite-pushforward IC uniqueness identify its Springer action with the geometric G_d action exactly as before.

For the colored composition α , the block sign projector has shift $\sum_{i,c} \alpha_{i,c}(\alpha_{i,c} - 1)$. The centers are

$$\delta_{\text{thin}} = A(d, d) - |d|, \quad \delta_\alpha = A(d, d) - \sum_{i,c} \alpha_{i,c}^2,$$

whose difference is that shift. Framing at every vertex ensures that a bad frame for a proper equal-slope subobject e has codimension at least $f(|d| - |e|) - \sum_{i,c} \lfloor \alpha_{i,c}^2/4 \rfloor$. Here H_α is the cohomology of the colored semistable stack shifted by δ_α , and $H_{\text{thin},d}$ is the thin semistable stack cohomology shifted by δ_{thin} . Ordinary-degree stabilization therefore proves

$$[H_\alpha] = [(\epsilon_d \otimes H_{\text{thin},d})^{G_\alpha}], \quad G_\alpha = \prod_{i,c} S_{\alpha_{i,c}}. \quad (8.3)$$

If the thin stable locus is empty, all colored stable loci are empty: otherwise choose bases in a stable colored representation and expand its matrices into thin arrows. Every thin subobject would be a colored coordinate-subspace subobject, so the thin object is stable. In this branch both sides of (8.1) are zero by the definition of BPS. When the stable locus is nonempty it is dense, since the representation space is irreducible; the canonical primitive action is then the geometric action on I_d , as in the main proof.

For two disjoint thin blocks of projected sizes d, e , their Euler pairing is $-A(d, e)$. The self-pairing of the first block is $|d| - A(d, d)$, which is congruent to $\chi_Q(d, d)$ modulo two because $d_i^2 \equiv d_i$. Substituting these expressions into (5.1) and adding the clone-sign exponent $\sum_i d_i e_i$ gives the residual exchange form

$$\rho(d, e) = \chi_Q(d, e) + \chi_Q(d, d)\chi_Q(e, e) \pmod{2}. \quad (8.4)$$

For this calculation put $q_Q(d) = \chi_Q(d, d) \pmod{2}$. On the mod-two span of the same-slope lattice, μ -genericity gives $q_Q(d + e) = q_Q(d) + q_Q(e) + \chi_Q(d, e) + \chi_Q(e, d) = q_Q(d) + q_Q(e)$. Hence ρ is symmetric bilinear, and $\rho(d, d) = q_Q(d) + q_Q(d)^2 = 0$. Choose a basis of this $\mathbb{F}_2$ vector space. Define ψ to equal ρ on ordered pairs of basis vectors above the diagonal, and zero on and below the diagonal, extending bilinearly. Then $\psi + \psi^\top = \rho$. On an ordered support partition with projected block sizes $d_1, \dots, d_r$, multiply by

$$(-1)^{\sum_{a < b} \psi(d_a, d_b)}. \quad (8.5)$$

Under an adjacent block exchange its scalar changes by $(-1)^{\rho(d_a, d_{a+1})}$. Thus this scalar conjugation takes the residual twisted factor permutations to ordinary graded permutations. Every external clone permutation preserves each ordered block's projected size, so the conjugation commutes with G_d . It therefore also commutes with Young invariants and color evaluation. In tensor length one the scalar is 1; it introduces no sign on the primitive or its geometric action. This is the explicit equivariant underlying-object version of [9, Proposition 6.2], with the choice made on original dimensions rather than on clone labels. On a single ray, writing $d = nu$ and $e = mu$ gives $\rho(d, e) = nm(q_Q(u) + q_Q(u)^2) = 0$.

The relative Hall maps are also compatible with clone permutations: for disjoint supports d, e , the extension bundle has rank $A(e, d)$ and the coordinate-zero embedding has codimension $A(d, e)$. Genericity gives $A(d, e) = A(e, d)$, and the shift difference is their sum, exactly the Gysin degree. The primitive inclusion is natural as in Lemma 5.1. Relative PBW followed by (8.5), ordinary color evaluation, and (8.3) now gives the same character series as colored integrality. Induction on $|d|$, subtraction of lower products, and cancellation of $t/(1 - t^2)$ identify each primitive, proving (8.1). The thin moduli is projective toric because Q is acyclic. A G_d -invariant ample class gives Hard Lefschetz on every multiplicity space, proving (8.2). $\square$

Remark 8.2. For one isolated original vertex and $d = 2$, all colored degree-two BPS vanish, while the thin semistable coarse moduli is a point. Using the entire semistable IH without the empty-stable rule would incorrectly yield $s_{(1,1)}(U)$.

For example, take the chain $1 \rightarrow 2 \rightarrow 3$ with weights $(1, 0, -1)$. The slope-zero vectors are (x, y, x) , and their Euler pairing is $2xu + yv - xv - yu$ on $(x, y, x), (u, v, u)$, so this slope is μ -generic. For $d = (1, 2, 1)$, the thin quiver has two length-two paths between its endpoints. Write the arrows on the two paths as a_1, b_1 and a_2, b_2 . The weight-zero condition at each middle clone forces a semi-invariant monomial to contain equal exponents of a_i and b_i . Its degree is the sum of these two exponents. Consequently the graded semi-invariant ring is $\mathbb{C}[a_1b_1, a_2b_2]$, with both generators in degree one. Its projective spectrum is $N_d \cong \mathbb{P}^1$. The middle S_2 exchanges these coordinates and acts trivially on cohomology. Hence

$$\sum_{\pi(\alpha)=(1,2,1)} b_\alpha(t) \mathbf{u}^\alpha = [2]_t s_{(1)}(U_1) s_{(1,1)}(U_2) s_{(1)}(U_3).$$

In particular, distinct middle colors contribute a nonzero Lefschetz character, whereas one middle color gives zero. The latter also follows directly: a map from a two-dimensional middle space to the last vertex has a nonzero kernel, which, supported only at the middle vertex, is a slope-zero proper subrepresentation.

8.2 A cyclic zero-potential example

Remark 8.3. For the two-cycle $1 \xrightarrow{a} 2 \xrightarrow{b} 1$, take dimension $(1, 1)$, zero potential and King weight $(1, -1)$. Stability is equivalent to $a \neq 0$: when $a = 0$ the first vertex is a positive-weight subobject, whereas with $a \neq 0$ the only possible proper subobject is supported at the second vertex. Gauge fixing $a = 1$ leaves $b \in \mathbb{A}^1$ free. The stable moduli is therefore $\mathbb{A}^1$, whose centered ordinary cohomology has character t^{-1} , outside $\mathcal{L}$. This nonprojective moduli space therefore fails the Lefschetz conclusion of Theorem 8.1. Its degree is central, so its adjoint action on the coefficient-free quantum torus is the identity.

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