

ON SOME POSITIVITY PROPERTIES OF THE RIEMANN Ξ FUNCTION

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ABSTRACT. We prove strict log-concavity of the positive functions Θ and Λ with one-sided and bilateral Laplace transforms $1/\xi(\frac{1}{2} + \sqrt{s})$ and $1/\Xi$, respectively, where $\Xi(z) = \xi(\frac{1}{2} + iz)$. For either function, log-concavity is the Pólya frequency property of order two (PF₂). Without assuming RH, but using known zero verification through height $3 \cdot 10^{12}$, we further prove that Λ is a PF function of order ten (PF₁₀). We also prove local PF positivity for Θ : for each fixed order, the ordered minors of $\Theta(x - y)$ are positive when all differences $x_i - y_j$ are sufficiently small and positive. Additionally, we prove the generalized Laguerre inequalities: if RH holds up to height $T > 1/2$, then $L_n[\Xi](x) \geq 0$ for every $n \geq 0$ and $|x| \leq T - 1/2$; the case $n = 1$ is due to Csordas, Ruttan and Varga (1991).

1. INTRODUCTION

For a connoisseur of positivity, the Riemann Ξ function offers a tantalizing array of questions. While the hardest of these are equivalent to the Riemann hypothesis (RH) itself, we find that even weak necessary conditions for RH already repay study. The ideas permeating our study are classical: Fourier–Laplace transforms, the Laguerre–Pólya class, Pólya frequency functions, among others. Our goal is a richer understanding of these positivity properties without assuming RH, and ideally with as little information about the zeta zeros as possible.

We use the usual normalization (Titchmarsh, 1986, Section 2.1) for Ξ :

$$(1.1) \quad \xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \quad c_0 = \xi(\frac{1}{2}) > 0, \quad \Xi(z) = \xi(\frac{1}{2} + iz),$$

and define the following functions

$$(1.2) \quad F(z) := \Xi(z)/c_0, \quad h(z) := \xi(\frac{1}{2} + \sqrt{z})/c_0, \quad B(s) := \log h(s) \quad (s \geq 0).$$

Let $\mathcal{Z}_+$ denote the multiset of nontrivial zeta zeros $\rho = \beta + i\gamma$ with $\gamma > 0$, and define

$$H(t) := \sum_{\rho \in \mathcal{Z}_+} e^{-t(\gamma - i(\beta - \frac{1}{2}))^2} \quad (t > 0).$$

Under RH three positivity properties hold: (a) H is *completely monotone* (i.e., $(-1)^n H^{(n)} \geq 0$ for all $n \geq 0$); (b) by Schoenberg’s theorem (Schoenberg, 1951; Gröchenig, 2020), $1/(c_0 h)$ and $1/\Xi$ are the one-sided and bilateral Laplace transforms of Pólya frequency (PF) functions of every order; and (c) Ξ satisfies every generalized Laguerre inequality on $\mathbb{R}$. Conversely, each of these properties implies RH (Proposition A.1; Csordas and Varga, 1990, Theorem 2.9). Without RH, an off-line zero ρ and its reflection $1 - \bar{\rho}$ both lie in $\mathcal{Z}_+$ and contribute complex-conjugate terms to H . So H stays real, but these terms oscillate in t and can in principle make it negative. We ask what parts of (a)–(c) follow from the available bounds on those oscillations.

Our main results are the following. Items marked * overlap in part with earlier or parallel work; the precise relation is described in Section 5 and in the discussion of the corresponding result.

Key words and phrases. Riemann Xi; Pólya frequency; log-concavity; generalized Laguerre; complete monotonicity.

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First public release: 20 October 2026, as v1 in the author’s preprint repository, <https://github.com/suvrit/preprints/tree/main/preprints/riem-xi>; every version is kept there with its timestamp.

- (1)* Let Θ on $(0, \infty)$ and Λ on $\mathbb{R}$ be the positive densities with one-sided and bilateral Laplace transforms $1/(c_0 h)$ and $1/\Xi$, respectively. They exist without RH. More generally, for every $\alpha > 0$, $h^{-\alpha}$ is the Laplace transform of a positive self-decomposable law, and $[c_0/\xi(\frac{1}{2} + \|u\|)]^\alpha$ is strictly positive definite on $\mathbb{R}^d$ for every d (Theorem 1.1). The existence question of Roynette and Yor (2005) was answered by Polson (2026b, v24, Theorem A); the proof here uses zeros only through height 100 and adds the regularity of the densities. This result answers a positive-definiteness question raised by Gröchenig (2020).
- (2) The densities Θ and Λ are log-concave, so $\Theta, \Lambda \in \text{PF}_2$ (Theorem 1.2); the proof uses zeros only through height 100. Using zeros through height $3 \cdot 10^{12}$ gives $\Lambda \in \text{PF}_{10}$ (Theorem 1.2). A minor result shows PF_k for all k , when restricted to small positive arguments (Proposition C.8).
- (3)* If RH holds up to height $T > \frac{1}{2}$, then Ξ satisfies the generalized Laguerre inequalities $L_n[\Xi](x) \geq 0$ of every order n for $|x| \leq T - \frac{1}{2}$, where $L_n[\Xi](x)$ is the coefficient of y^{2n} in $|\Xi(x + iy)|^2$ (Theorem 1.3). The case $n = 1$ goes back to Csordas et al. (1991).

The inputs to our results are the critical strip, explicit zero counting, and published zero verification. Our conclusions do not extend the verified height, and they are merely weak necessary conditions. The global Laguerre inequality $L_1[\Xi] \geq 0$ (Csordas, 1998, Problem 2) remains open, and in our view, it is a much stronger necessary condition than those studied in this paper. Below, in §§1.1–1.3 we present the formal statements of our main results.

1.1. Construction of the reciprocal laws Θ and Λ . Assuming RH, the densities Θ and Λ exist and are PF_∞ by Schoenberg’s theorem (Schoenberg, 1951, Theorems 1 and 2), specialized to Ξ in Gröchenig (2020, Theorems 3–5). But without RH, at least their existence follows easily. Using the notation from (1.1) and (1.2), from Hadamard’s product for h over the zeros (Lemma 2.1) we obtain $B'(s) = \int_0^\infty e^{-st} H(t) dt$. So $H > 0$ makes B a *Bernstein function*, nonnegative with completely monotone derivative, and $h^{-\alpha} = e^{-\alpha B}$ the Laplace transform of a probability law.

Theorem 1.1 (Reciprocal xi and probability laws). *For $t > 0$, $H(t) > 0$ and $H'(t) < 0$. The function B is Bernstein, with the representation*

$$(1.3) \quad B(s) = \int_0^\infty (1 - e^{-st}) \frac{H(t)}{t} dt.$$

Moreover, for every $\alpha > 0$ there is a positive self-decomposable random variable S_α with $\mathbb{E}e^{-sS_\alpha} = h(s)^{-\alpha}$. It has a C^∞ density positive on $(0, \infty)$. For every $d \geq 1$,

$$(1.4) \quad h(\|u\|^2)^{-\alpha} = c_0^\alpha \xi(\tfrac{1}{2} + \|u\|)^{-\alpha}$$

is strictly positive definite on $\mathbb{R}^d$ and is the characteristic function of an infinitely divisible law. Its Gaussian-mixture density is positive on $\mathbb{R}^d$ and extends to an entire function on $\mathbb{C}^d$.

1.2. Log-concavity and finite PF orders. As noted, RH implies Θ and Λ are PF_∞ functions; without RH, we can still obtain lower PF orders. We say a function is PF_m when $\det[f(x_i - y_j)]_{i,j=1}^m \geq 0$ for $x_1 < \dots < x_m$ and $y_1 < \dots < y_m$. On an interval log-concavity is equivalent to PF_2 .

Theorem 1.2 (Log-concavity and PF_k). *The densities Θ and Λ are PF_2 : $(\log \Theta)''(x) < 0$ for $x > 0$, while $(\log \Lambda)''(x) < 0$ for $x \in \mathbb{R}$. In fact, even more is true: $\Lambda \in \text{PF}_{10}$.*

1.3. Generalized Laguerre inequalities for Ξ . Under RH, Ξ satisfies the generalized Laguerre inequalities on $\mathbb{R}$; we prove that it satisfies all of them on an interval, which is once again merely a weak necessary check. For a real entire function f , define the (generalized) Laguerre transform

$$(1.5) \quad L_n[f](x) := \frac{1}{(2n)!} \sum_{j=0}^{2n} (-1)^{j+n} \binom{2n}{j} f^{(j)}(x) f^{(2n-j)}(x), \quad n \in \mathbb{N}_0.$$

These are the coefficients of y^{2n} in $|f(x + iy)|^2$; in particular, $L_0[f] = f^2$ and $L_1[f] = (f')^2 - ff''$. Satisfying the generalized Laguerre inequality $L_n[f] \geq 0$ for each $n \geq 0$ implies that f is in the Laguerre–Pólya class, so for instance the truth of $L_n[\Xi] \geq 0$ would imply RH. [Csordas et al. \(1991\)](#) proved $L_1[\Xi] \geq 0$ on an interval using known zero verification ([Csordas, 2015](#), Remark 4.8); we extend that result to show that $L_n[\Xi] \geq 0$ on that interval.

Theorem 1.3 (Generalized Laguerre on an interval). *Let $T > 1/2$, and suppose all nontrivial zeta zeros $\beta + i\gamma$ with height $0 < \gamma \leq T$ satisfy $\beta = 1/2$. Then,*

$$L_n[\Xi](x) \geq 0 \quad (n = 0, 1, 2, \dots, \quad |x| \leq T - \tfrac{1}{2}).$$

Moreover, $L_n[\Xi](x) > 0$ unless x is a zero of Ξ of multiplicity exceeding n .

By [Platt and Trudgian \(2021, Theorem 1\)](#) the critical line hypothesis holds for $T = T_* := 3 \cdot 10^{12}$, so every $L_n[\Xi] \geq 0$ on $|x| \leq T_* - \frac{1}{2}$. For $n = 1$, this use of verified zeros goes back to [Csordas et al. \(1991, Corollary to Lemma 4\)](#), who assume that the zeros are simple; see also [Csordas et al. \(1992, p. 398\)](#), [Csordas \(1998, remark after Problem 2\)](#) and [Csordas \(2015, Remark 4.8\)](#).

2. PROOF OF THEOREM 1.1: THE RECIPROCAL LAWS

This section proves Theorem 1.1. The construction needs two analytic facts: the canonical product must yield a convergent Laplace integral, and H must stay positive and decreasing. We establish the convergence first, then compare the possible negative tail with one retained real zero. Sections 2.3 and 2.4 then construct the laws and the densities Θ and Λ . Endpoint asymptotics, used for the regularity of the laws, are proved in Appendix B.

2.1. Analytic and numerical inputs. We use the classical analytic continuation, functional equation, zero symmetries, and critical strip of ζ ([Titchmarsh, 1986](#), Chapter II). In particular $\delta_\rho := \beta - \frac{1}{2}$ satisfies $|\delta_\rho| < 1/2$, ξ has order one, and the nontrivial zeros are its only zeros. An elementary theta representation, recalled in Appendix B, also gives $c_0 > 0$.

Let $N(Y)$ count upper-half-plane zeros of ordinate at most Y , with multiplicity, and write

$$M(Y) = \frac{Y}{2\pi} \left(\log \frac{Y}{2\pi} - 1 \right), \quad M(0) = 0.$$

We use the following consequence of [Bellotti and Wong \(2025, Theorem 1.1\)](#):

$$(2.1) \quad |N(Y) - M(Y)| \leq \tfrac{1}{8} \log(Y + 2) + 4 \quad (Y \geq 0).$$

For all $U \geq 100$, (2.1) implies

$$(2.2) \quad N(Y) \leq Y \log Y \quad (Y \geq U), \quad \sum_{\gamma > U} \gamma^{-2} \leq \frac{2(\log U + 1)}{U} < 1.$$

The sum estimate follows by Stieltjes integration by parts, dropping the nonpositive boundary term.

The proved finite zero inputs are

$$(2.3) \quad \gamma \leq T_* \implies \delta_\rho = 0, \quad 14 < \gamma_1 < 15,$$

where γ_1 is the first positive zero ordinate. The first assertion follows from [Platt and Trudgian \(2021, Theorem 1\)](#), which verifies height 3 000 175 332 800. The first-zero enclosure is contained in rigorously isolated zero data ([Platt, 2017](#); [The LMFDB Collaboration, 2026](#)). Only the portion through height 100 is needed for Theorem 1.1 and the log-concavity in Theorem 1.2. Table 1 lists the numerical inputs of each result; the certificates are in Appendix C.

For $\rho \in \mathcal{Z}_+$ put $\lambda_\rho := (\gamma - i\delta_\rho)^2$, so that $H(t) = \sum_{\rho \in \mathcal{Z}_+} e^{-t\lambda_\rho}$.

Lemma 2.1 (Product and convergence). *The function h is entire of order $1/2$, and*

$$(2.4) \quad h(z) = \prod_{\rho \in \mathcal{Z}_+} (1 + z/\lambda_\rho).$$

The product converges locally uniformly. If $a_\rho = \operatorname{Re} \lambda_\rho$, then $a_\rho > 0$, $\sum a_\rho^{-1} < \infty$, and $h(s) > 0$ for $s \geq 0$. The series for H and its derivatives converge locally absolutely on $\operatorname{Re} t > 0$.

Proof. Evenness turns the order-one Xi function into an order- $1/2$ entire function under squaring. The pair $\rho, 1 - \rho$ yields one squared zero; exactly one member is in $\mathcal{Z}_+$. Hadamard factorization at order less than one therefore yields (2.4), with constant one. For large ordinates $a_\rho = \gamma^2 - \delta_\rho^2 \geq \gamma^2 - 1/4$, and zero counting gives reciprocal-square summability. At low ordinates (2.3) makes $a_\rho = \gamma^2 > 0$. Conjugate factors in (2.4) combine into positive squared moduli on the positive axis; real factors are positive. On a compact subset of $\operatorname{Re} t > 0$, the Gaussian decay $e^{-c\gamma^2 + C\gamma}$ absorbs every fixed power of γ , proving the last claim. $\square$

2.2. Positivity of H and $-H'$. A high zero can oscillate only after its exponential weight has become small. The estimate below makes this delay quantitative for both H and $-H'$. Keeping a single low real zero gives a common positive term against which all negative contributions of the tail can be measured. The same comparison is the large-time step in Polson (2026b, v24, Proposition 6.5) and Polson (2026a, v9, proof of Theorem 22); the estimate below makes it valid for every $t > 0$.

Put $q = \gamma_1^2 \in (196, 225)$ and

$$a_\rho = \gamma^2 - \delta_\rho^2, \quad b_\rho = 2\gamma\delta_\rho, \quad \lambda_\rho = a_\rho - ib_\rho.$$

Then $|b_\rho| \leq \gamma$ and $|\lambda_\rho| = \gamma^2 + \delta_\rho^2$.

Lemma 2.2 (One retained real zero). *Suppose $100 \leq U \leq T_*$, and define*

$$H_{>U}(t) = \sum_{\gamma > U} e^{-t\lambda_\rho}, \quad K_U(t) = e^{-qt} + H_{>U}(t), \quad \varepsilon_U = U^4 e^{-U}.$$

For every $t > 0$,

$$K_U(t) \geq (1 - \varepsilon_U)e^{-qt} > 0, \quad -K'_U(t) \geq (1 - \varepsilon_U)qe^{-qt} > 0.$$

In particular $H > 0$ and $-H' > 0$.

Proof. The real summands of H and $-H'$ are respectively

$$e^{-a_\rho t} \cos(b_\rho t), \quad e^{-a_\rho t} [a_\rho \cos(b_\rho t) + b_\rho \sin(b_\rho t)].$$

Both are nonnegative when $|b_\rho|t \leq \pi/2$. A negative contribution therefore requires $t > \pi/(2\gamma)$. For $\gamma \geq 100$, $a_\rho - q \geq \gamma^2 - 1/4 - 225 \geq 3\gamma^2/4$. At every time when the corresponding contribution is negative,

$$e^{-(a_\rho - q)t} \leq e^{-3\pi\gamma/8} \leq e^{-\gamma}.$$

Its negative part is consequently at most $e^{-qt}e^{-\gamma}$ in the first sum, and at most $qe^{-qt}\gamma^2 e^{-\gamma}$ in the second, using $|\lambda_\rho| \leq q\gamma^2$. By (2.2) and monotonicity of $y^2 e^{-y}$ and $y^4 e^{-y}$ for $y \geq 100$,

$$\sum_{\gamma > U} e^{-\gamma} < U^2 e^{-U}, \quad \sum_{\gamma > U} \gamma^2 e^{-\gamma} < U^4 e^{-U}.$$

Subtract from the retained terms e^{-qt} and qe^{-qt} . Finally $H - K_U$ is a finite sum of positive real exponentials, proving the assertions about H . $\square$

2.3. The Bernstein representation and the laws. We first derive the Lévy representation of B from the canonical product. Termwise it is Frullani's integral $\log(1 + s/\lambda) = \int_0^\infty (1 - e^{-st})e^{-\lambda t} dt/t$; for a positive real rate λ this is the Lévy representation of a gamma law. When every $\lambda_\rho > 0$, summation gives Thorin's representation of a generalized gamma convolution (Bondesson, 1992, Chapter 3). Without RH the exponents λ_ρ may be complex, and the positivity of the resulting Lévy density $H(t)/t$ is the content of Lemma 2.2. For the xi function the identity is written in Polson (2026b, v24, Theorem 5.3). Lemma 2.4 supplies smooth positive densities and negative moments. Gaussian subordination gives the radial characteristic functions, and the growth of B gives their entire inverse Fourier transforms.

The integral characterization of Bernstein functions and the complete Bernstein subclass are described in Schilling et al. (2012, Chapters 3 and 6); here a Poisson random measure constructs the laws directly.

Proposition 2.3 (Bernstein representation). *Equation (1.3) holds, and for $s \geq 0$, $n \geq 0$,*

$$(-1)^n B^{(n+1)}(s) = \int_0^\infty t^n e^{-st} H(t) dt > 0.$$

Moreover $\int_0^\infty |H(t)| dt < \infty$.

Proof. Logarithmic differentiation of (2.4) yields $B'(s) = \sum_\rho (s + \lambda_\rho)^{-1}$. Interchanging with the elementary Laplace integral is justified by

$$\sum_\rho \int_0^\infty |e^{-(s+\lambda_\rho)t}| dt = \sum_\rho (s + a_\rho)^{-1} < \infty.$$

Thus $B'(s) = \int e^{-st} H(t) dt$ and $\int |H| \leq \sum a_\rho^{-1} < \infty$. Integrate in s , using $B(0) = 0$, to obtain (1.3). The positivity follows from Lemma 2.2; differentiation is justified by integrability at zero and exponential decay at infinity. The Lévy condition follows from $\int (1 \wedge t) H(t) dt/t < \infty$. $\square$

For $\alpha > 0$ take a Poisson random measure on $(0, \infty)$ of intensity $\alpha H(t) dt/t$, and let S_α be the sum of its points. It is finite almost surely because its expectation is $\alpha \int H(t) dt < \infty$. The Laplace functional proves

$$(2.5) \quad \mathbb{E} e^{-sS_\alpha} = e^{-\alpha B(s)} = h(s)^{-\alpha}.$$

Independent sums add the parameter α , so these laws form a convolution semigroup. For $0 < c < 1$,

$$(2.6) \quad B(s) - B(cs) = \int_0^\infty (1 - e^{-st}) \frac{H(t) - H(t/c)}{t} dt.$$

Since H is decreasing, the Lévy density $(H(t) - H(t/c))/t$ in (2.6) is nonnegative. Equation (2.6) therefore constructs an independent $R_c \geq 0$ with $S_\alpha \stackrel{d}{=} cS'_\alpha + R_c$, proving self-decomposability. This is the classical criterion: a law on $[0, \infty)$ whose Lévy density has the form $k(t)/t$ with k nonincreasing is self-decomposable (Sato, 1999, Corollary 15.11). For the xi function see also Polson (2026b, v24, Theorem 5.6(ii)).

Lemma 2.4 (Density, support, and moments). *For every $\alpha > 0$, S_α has a $C^\infty(\mathbb{R})$ density θ_α , zero on $(-\infty, 0]$ and positive on $(0, \infty)$. Its Laplace transform is holomorphic on $\operatorname{Re} s > -q$. It has exponential moments of all orders $r < q$, no exponential moment of order $r \geq q$, and finite negative moments of all orders.*

The proof extracts gamma convolution factors of arbitrarily high order from the Lévy measure. This gives smoothness and full positive support; the remaining moment estimates are in Appendix B.4. The exponential-moment threshold is also in Polson (2026b, v24, Corollary 6.8(iii)) and Polson (2026a, v9, Theorem 22(iii)).

Let G_d be standard Gaussian and independent of S_α . The vector $X_{\alpha,d} = \sqrt{2S_\alpha}G_d$ has characteristic function (1.4) and density

$$p_{\alpha,d}(x) = \int_0^\infty (4\pi u)^{-d/2} e^{-\|x\|^2/(4u)} \theta_\alpha(u) du > 0.$$

Its infinite divisibility follows by adding the parameter α , and its self-decomposability follows from (2.6) with c^2 instead of c . This is the usual Gaussian-subordination mechanism (Schoenberg, 1938); see also Roynette and Yor (2005, Proposition I.8.1). For the xi function with $\alpha = d = 1$ it is Polson (2026b, v24, Theorem A). The Gaussian Gram matrix on distinct points is positive definite for each $u > 0$: its quadratic form is the integral of the squared modulus of a nonzero exponential polynomial against a positive Gaussian density. Integration against θ_α proves strictness for (1.4). By (B.5), its characteristic function is integrable against $e^{A\|t\|}$ for every $A > 0$. The inverse Fourier integral therefore defines an entire extension of $p_{\alpha,d}$ to $\mathbb{C}^d$. This proves Theorem 1.1.

Theorem 1.1 also removes the hypothesis from Roynette and Yor (2005, eq. (V.36)), which they prove under RH: for every $\alpha > \frac{1}{2}$ there is a random variable $\tilde{H}_\alpha \geq 0$ with

$$(2.7) \quad \mathbb{E}e^{-\lambda^2 \tilde{H}_\alpha/2} = \frac{\xi(\alpha)}{\xi(\alpha + \lambda)} \quad (\lambda \geq 0).$$

Indeed, $\tilde{H} = 2S_1$ satisfies $\mathbb{E}e^{-\beta^2 \tilde{H}/2} = c_0/\xi(\frac{1}{2} + \beta)$ for $\beta \geq 0$, and their lemma (Roynette and Yor, 2005, Lemme V.3.4), applied to $\tilde{H}$ with shift $\alpha - \frac{1}{2} > 0$, gives (2.7). For $\alpha \geq 1$, $\tilde{H}_\alpha/2$ can be taken to be a generalized gamma convolution (Polson, 2026a, v9, Theorem 18). The question (V.35) of Roynette and Yor, which asks for the additional factor $\exp(\lambda\xi'(\alpha)/\xi(\alpha))$ on the right of (2.7), is not settled by this argument; see Polson (2026a, v9, Remark 20).

2.4. The densities Θ and Λ . In the notation of Gröchenig (2020, Theorem 5), $\Xi(z) = \Xi_1(-z^2)$ with $\Xi_1 = c_0 h$. Set

$$(2.8) \quad \Theta(u) = \frac{\theta_1(u)}{c_0}, \quad \Lambda(x) = \frac{p_{1,1}(x)}{c_0} = \frac{1}{c_0} \int_0^\infty \frac{e^{-x^2/(4u)}}{\sqrt{4\pi u}} \theta_1(u) du.$$

Both integrate to $1/c_0$ on their respective domains.

Corollary 2.5 (Positive versions of the two Laplace representations). *The density Θ is smooth, supported in $[0, \infty)$, and strictly positive on $(0, \infty)$. The density Λ is even, strictly positive on $\mathbb{R}$, and extends to an entire function. They satisfy*

$$(2.9) \quad \begin{aligned} \frac{1}{c_0 h(s)} &= \int_0^\infty e^{-su} \Theta(u) du, & \operatorname{Re} s &> -\gamma_1^2, \\ \frac{1}{\Xi(z)} &= \int_{\mathbb{R}} e^{-zx} \Lambda(x) dx, & |\operatorname{Re} z| &< \gamma_1. \end{aligned}$$

In particular, for arbitrary $c_j \in \mathbb{C}$ and $t_j \in \mathbb{R}$,

$$(2.10) \quad \sum_{j,k} \frac{c_j \overline{c_k}}{\xi(\frac{1}{2} + t_j - t_k)} = \int_{\mathbb{R}} \left| \sum_j c_j e^{it_j x} \right|^2 \Lambda(x) dx \geq 0.$$

The inequality is strict for distinct t_j and a nonzero coefficient vector.

Proof. The first transform follows from (2.5) and Lemma 2.4; equality on the full half-plane follows by holomorphic continuation. For real $|z| < \gamma_1$, conditioning the Gaussian mixture gives

$$\int_{\mathbb{R}} e^{-zx} \Lambda(x) dx = c_0^{-1} \mathbb{E}e^{z^2 S_1} = \frac{1}{c_0 h(-z^2)}.$$

The exponential moments of the resulting symmetric variable give holomorphy of the bilateral transform throughout the strip, so the identity theorem proves (2.9) there. Take $z = it$ and use the

functional equation and evenness to obtain the Fourier transform $1/\xi(\frac{1}{2} + t)$. This proves (2.10). Strictness follows because $\Lambda > 0$ and a nonzero exponential polynomial with distinct frequencies cannot vanish identically. $\square$

Consequently the exact inverse Fourier transform specified in Gröchenig (2020, Theorem 4) is

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{-itx}}{\xi(\frac{1}{2} + t)} dt = \Lambda(x) > 0.$$

The Gaussian mixture in (2.8) gives a positive representation of this inverse transform and proves the positive-definiteness property discussed after Theorem 5 of Gröchenig (2020). Konstantopoulos et al. (2024, Section 4.1) ask whether F belongs to their subclass $\mathcal{D}_P$ of van Dantzig functions. By Theorem 1.1, F has the properties that Konstantopoulos et al. (2024, Theorem 4.1) derive for members of $\mathcal{D}_P$: $\log F(it) = B(t^2)$, and the Lévy density ℓ of B satisfies $(t\ell(t))' = H'(t) < 0$, so B lies in their class $\mathcal{B}_J$. These properties therefore cannot exclude F from $\mathcal{D}_P$. The same observation is made in Polson (2026b, v24, Section 1.2), and Polson (2026b, v24, Section 13) further restricts the possible representations.

3. PROOF OF THEOREM 1.2: LOG-CONCAVITY AND ORDER TEN

This section proves Theorem 1.2, which has two parts with different inputs. Log-concavity needs only the low zeros (Section 3.1). Order ten for Λ needs the verified height and a determinant certificate (Section 3.2, with the estimates in Appendix C).

3.1. Log-concavity from the low verified zeros. The one-sided density has a boundary at zero, where a small additive error need not preserve logarithmic curvature. Exponential tilting avoids that difficulty. A second low real zero keeps the tilted residual self-decomposable; its unimodality and its shape before the mode then control the required primitive on the whole positive axis. The same tilting argument also gives bilateral log-concavity. Only the zeros through height 100 are used in this subsection.

Log-concavity is additional information: the self-decomposable law $\text{Gamma}(\alpha, b)$, with $0 < \alpha < 1$ and $b > 0$, has density $g(x) = Cx^{\alpha-1}e^{-bx}$ and $(\log g)''(x) = (1 - \alpha)/x^2 > 0$. Here the work is to verify that the required exponential tilts of the laws of S_1 and $X_{1,1}$ remain self-decomposable, so that classical shape theorems apply. These reciprocal densities also differ from the Riemann Fourier kernel Φ , whose PF_2 property is recorded in Csordas (1998, Corollary 2.11), and from the xi Taylor coefficient sequence studied by Katkova (2007).

We use two facts from the theory of self-decomposable laws. Every such law on $\mathbb{R}$ is unimodal (Yamazato, 1978). A drift-free positive self-decomposable law with a smooth positive density has Yamazato's property: its density is log-concave before its mode and nonincreasing after it. This is Yamazato's result, as recalled by Watanabe (1992, pp. 50–51), whose Theorem 1.2 extends it to unimodal one-sided Lévy processes without drift. For a variable V with a finite exponential moment at r , its *exponential tilt* is the probability law $e^{rx} \Pr(V \in dx) / \mathbb{E}e^{rV}$. Tilting a positive infinitely divisible law with Lévy density $k(t)/t$ replaces $k(t)$ by $e^{rt}k(t)$.

The next elementary lemma is a localized strict version of the classical passage from a log-concave density to its distribution function; compare Bagnoli and Bergstrom (2005, Theorem 1 and Corollary 1). They credit Prékopa for the integration principle and Dierker for the elementary proof. We give the precise version needed here.

Lemma 3.1 (Log-concavity of a primitive). *Let $g > 0$ be C^1 on $(0, \infty)$ and locally integrable at zero. Suppose that, whenever $g'(x) > 0$, $\log g$ is concave on $(0, x)$. Then $G(x) = \int_0^x g(u) du$ satisfies $(\log G)''(x) < 0$ for every $x > 0$. In particular this holds for a positive smooth density with Yamazato's property, or for a positive log-concave function.*

Proof. If $g'(x) \leq 0$, then $G(x)g'(x) - g(x)^2 < 0$. Otherwise put $r = g'(x)/g(x) > 0$. The tangent inequality for $\log g$ gives

$$G(x) \leq g(x) \int_0^x e^{r(u-x)} du = g(x) \frac{1 - e^{-rx}}{r} < \frac{g(x)}{r}.$$

Again $Gg' - g^2 < 0$, which is exactly the asserted curvature sign. $\square$

Proposition 3.2 (The one-sided reciprocal density). *The one-sided density satisfies*

$$(\log \Theta)''(x) < 0 \quad (x > 0).$$

Its extension by zero to $(-\infty, 0]$ belongs to PF_2 .

Proof. Put $q = \gamma_1^2 \in (196, 225)$ and let m_1 be the multiplicity of the first ordinate. Choose a real ordinate $\eta \in (21, 22)$; its existence is certified in Appendix C.6. We claim that

$$(3.1) \quad J(t) = e^{qt}(H(t) - m_1 e^{-qt}) > 0, \quad J'(t) < 0 \quad (t > 0).$$

To see this, retain the term at η and compare it with $e^{qt}H_{>100}(t)$. Put $\tau = \eta^2 - q > 216$. The shifted tail exponents are $\lambda_\rho - q$. Both their real contributions and negative derivatives are nonnegative when $|b_\rho|t \leq \pi/2$. A negative contribution therefore requires $t > \pi/(2\gamma)$, while

$$(a_\rho - q) - \tau = a_\rho - \eta^2 \geq 3\gamma^2/4, \quad |\lambda_\rho - q| \leq 2\gamma^2 \leq \tau\gamma^2.$$

The proof of Lemma 2.2 now applies verbatim, giving

$$e^{-\tau t} + e^{qt}H_{>100}(t) \geq (1 - \varepsilon_{100})e^{-\tau t}, \quad -\frac{d}{dt}(e^{-\tau t} + e^{qt}H_{>100}(t)) \geq (1 - \varepsilon_{100})\tau e^{-\tau t}.$$

The other terms in J are positive real exponentials with positive rates, which proves (3.1).

Subtracting all m_1 first-zero factors gives $S_1 \stackrel{d}{=} \text{Gamma}(m_1, q) + V$, with independence and with V having Lévy density $(H(t) - m_1 e^{-qt})/t$. All remaining rates have real part greater than q , so $M := \mathbb{E}e^{qV} < \infty$. The tilted law $V^{(q)}$ has zero drift and Lévy density $J(t)/t$, hence is self-decomposable (a tilt in the opposite direction appears in Polson, 2026a, v9, Theorem 22(ii)). The gamma-extraction argument of Appendix B.4 gives it a smooth positive density: removing finitely many factors and tilting preserve the small-time divergence of J and its positive exponential lower bound.

Let F_q be its distribution function. Yamazato's property and Lemma 3.1 give $(\log F_q)'' < 0$. The density of $\text{Exp}(q) + V$ is

$$f(x) = qMe^{-qx}F_q(x), \quad x > 0,$$

so it has the same strict logarithmic curvature. If $m_1 > 1$, convolve with the other $m_1 - 1$ exponential factors. Each step replaces f by $qe^{-qx} \int_0^x e^{qu} f(u) du$; since $e^{qu} f(u)$ is log-concave, Lemma 3.1 preserves strict curvature. This proves the claim for $\theta_1 = c_0\Theta$, without assuming simplicity. A log-concave function on its interval support, extended by zero, is PF_2 . The support can force some ordered minors to vanish. $\square$

Proposition 3.3 (The bilateral reciprocal density). *The density Λ is strictly log-concave on $\mathbb{R}$. Every PF_2 minor is strictly positive.*

Proof. Gaussian subordination (Sato, 1999, Theorem 30.1) gives the Lévy density of $X_{1,1}$ as $A(|x|)/|x|$ (for the Xi function, cf. Polson, 2026b, v24, Section 2 and Proposition 12.1), where

$$A(x) = \sum_{\rho \in \mathbb{Z}_+} e^{-\gamma x} \cos(\delta_\rho x), \quad x > 0.$$

Indeed, the Gaussian integral applied to each term of H gives $e^{-(\gamma - i\delta_\rho)|x|}/|x|$. Absolute convergence for $x \neq 0$ justifies the interchange: the integrals of the absolute values sum to $|x|^{-1} \sum_\rho e^{-\sqrt{a_\rho}|x|} < \infty$. Conjugation gives the displayed real sum. Put $a = \gamma_1 < 15$. For every $|s| < a$, both $e^{sx}A(x)$ and

$e^{-sx}A(x)$ are positive and decreasing. Retain the same ordinate $\eta \in (21, 22)$. For either sign, the tail summands and their negative derivatives are nonnegative for $x \leq \pi$, because $|\delta_\rho| \leq 1/2$. A negative term therefore has $x > \pi$ and

$$e^{-(\gamma-\eta)x} \leq e^{-\pi(\gamma-22)}, \quad |\gamma \mp s - i\delta_\rho| \leq 2\gamma \leq (\eta \mp s)\gamma^2.$$

The reciprocal-square tail bound makes the negative tail contributions in either estimate, relative to the retained term, at most

$$\varepsilon = 100^4 e^{-78\pi} < 1.$$

All remaining verified terms have positive coefficients and rates $\gamma \mp s > 0$. The tilt exists because $\mathbb{E}e^{sX_{1,1}} = \mathbb{E}e^{s^2S_1} < \infty$ for $|s| < a$. Its Lévy density is $e^{sx}A(|x|)/|x|$, so the two monotonicity statements show that it is self-decomposable (Sato, 1999, Theorem 15.10), and hence unimodal.

Choose η as the retained ordinate in Theorem C.3. This extracts a $\text{Lap}(a)$ factor, with a remaining law having positive density on the whole line. If $p = c_0\Lambda$, convolution with that factor gives $|p'(x)| < ap(x)$ for every x : the positive contributions from each side of x make the triangle inequality strict. Were $(\log p)''(x_0) > 0$, the tilt with $s = -p'(x_0)/p(x_0) \in (-a, a)$ would have a strict local minimum at x_0 , contradicting unimodality. Therefore $\log p$ is concave. If it were affine on an interval, analyticity would force $p(x) = Ce^{bx}$ everywhere, which cannot be a probability density. Thus it is strictly concave, and all PF_2 minors are positive. $\square$

These arguments establish order two without the high verified height. They do not by themselves give order three. For example, if $0 < \alpha < 1$ and $b > q$, the q -tilt of a $\text{Gamma}(\alpha, b)$ residual is self-decomposable. Yet the density f of $\text{Exp}(q) + \text{Gamma}(\alpha, b)$ satisfies $f(x) = Cx^\alpha(1 + O(x))$ near zero, with a convergent power series in the last factor, and hence

$$-\det[f^{(i+j)}(x)]_{i,j=0}^2 \sim 2C^3\alpha^2(\alpha-1)x^{3\alpha-6} < 0.$$

Thus this density fails PF_3 . Higher orders need further information. For Λ , Section 3.2 supplies it from the verified height; Appendix C.5 treats each fixed order of Θ near the one-sided boundary.

3.2. Order ten. For order ten we retain a zero near the verified height, so that the residual law is very small, and compare Λ with a convolution of eleven verified Laplace densities.

Proof of Theorem 1.2. Propositions 3.2 and 3.3 prove the log-concavity assertions. For order ten, extract eleven verified Laplace factors and a symmetric residual Y_U at $U = T_*$. Lemma C.5 bounds $\mathbb{E}Y_U^2$ and $\mathbb{E}\cosh(10^5 Y_U) - 1$. Lemma C.7, using the interval bounds in Appendix C.7, makes the signed derivative determinants positive through size ten. Lemma C.6 converts these signs into strictly positive PF_{10} minors. Convoluting the remaining Laplace factors preserves strictness by the composition formula for totally positive kernels. The factorization and composition step are detailed in Appendix C.4. $\square$

The order-ten result also has an operator interpretation. Let $v(f)$ be the largest number of sign changes of f along increasing finite sequences, with zero values discarded. For every bounded real Borel function f , the finite-order variation-diminishing theorem (Karlin, 1968) gives

$$v(\Lambda * f) \leq v(f) \quad \text{if } v(f) \leq 9.$$

The convolution converges absolutely because Λ is integrable. This is the finite-order counterpart of Gröchenig (2020, Theorem 6).

4. PROOF OF THEOREM 1.3: LAGUERRE INEQUALITIES ON AN INTERVAL

This section proves Theorem 1.3 by expressing each L_n as a coefficient of a product with nonnegative coefficients.

4.1. Coefficients of the squared modulus. The definition (1.5) also applies to real C^{2n} functions. For example,

$$L_2[f] = \frac{1}{4}(f'')^2 - \frac{1}{3}f'f''' + \frac{1}{12}ff^{(4)}.$$

Lemma 4.1 (Generating and multiplication identities). *For a real entire f and real x, y ,*

$$|f(x + iy)|^2 = f(x + iy)f(x - iy) = \sum_{n=0}^{\infty} L_n[f](x)y^{2n}.$$

For real entire f, g ,

$$(4.1) \quad L_n[fg](x) = \sum_{k=0}^n L_k[f](x)L_{n-k}[g](x).$$

Local uniform convergence of real entire functions $f_j \rightarrow f$ implies $L_n[f_j] \rightarrow L_n[f]$ locally uniformly, for each fixed n .

Proof. Multiply the two Taylor series at x . Odd powers of y cancel, and the coefficient of y^{2n} is $\sum_{j=0}^{2n} i^j(-i)^{2n-j} f^{(j)}(x)f^{(2n-j)}(x)/(j!(2n-j)!)$. Since $i^j(-i)^{2n-j} = (-1)^{j+n}$, this is (1.5). Absolute convergence on compact sets justifies the multiplication. Multiplying the generating identities for f and g proves (4.1). Cauchy's integral formula implies convergence of each fixed derivative under local uniform convergence; the last statement follows from the finite sum in (1.5). $\square$

4.2. The zero-exclusion principle. The Xi function satisfies the hypothesis of the next lemma by Lemma 2.1.

Lemma 4.2 (All-order zero-exclusion principle). *Let $f \not\equiv 0$ be a real entire function with a representation*

$$(4.2) \quad f(z) = Cz^d e^{-a_0 z^2 + b_0 z} \prod_{\zeta \in \mathcal{Z}_f^*} E_1(z/\zeta), \quad E_1(w) = (1-w)e^w,$$

where $C \in \mathbb{R} \setminus \{0\}$, $d \in \mathbb{N}_0$, $a_0 \geq 0$, $b_0 \in \mathbb{R}$, and $\mathcal{Z}_f^*$ is the multiset of nonzero zeros, invariant under complex conjugation, with $\sum_{\zeta \in \mathcal{Z}_f^*} |\zeta|^{-2} < \infty$. The product is grouped into real zeros and conjugate pairs. Let $I \subset \mathbb{R}$ be an interval such that

$$|x - \operatorname{Re} \zeta| \geq |\operatorname{Im} \zeta| \quad (x \in I, \zeta \in \mathcal{Z}_f^* \setminus \mathbb{R}).$$

Then $L_n[f](x) \geq 0$ for every $n \in \mathbb{N}_0$ and every $x \in I$.

In particular, if every nonreal zero $\zeta = a + ib$ satisfies $|a| > T$ and $|b| \leq \Delta$, with $T > \Delta \geq 0$, then

$$L_n[f](x) \geq 0 \quad (n \in \mathbb{N}_0, |x| \leq T - \Delta).$$

Likewise, if $|\operatorname{Im} \zeta| \leq \Delta$ and no nonreal zero has real part in $[x_-, x_+]$, where $x_+ - x_- > 2\Delta$, the conclusion holds on $[x_- + \Delta, x_+ - \Delta]$.

Proof. Fix $x \in I$ and regard $Y = y^2$ as a formal variable. For a real zero $r \neq 0$,

$$E_1((x + iy)/r)E_1((x - iy)/r) = \frac{e^{2x/r}}{r^2}((x - r)^2 + Y),$$

which has nonnegative coefficients in Y . The factor z^d similarly contributes $(x^2 + Y)^d$.

For a conjugate pair $\zeta = a + ib, \bar{\zeta} = a - ib$, $b \neq 0$, set $Q_{a,b}(z) = (z - a)^2 + b^2$. Their canonical-product factors multiply to

$$E_1(z/\zeta)E_1(z/\bar{\zeta}) = \frac{Q_{a,b}(z)}{a^2 + b^2} \exp\left(\frac{2az}{a^2 + b^2}\right).$$

With $u = x - a$, direct multiplication gives

$$Q_{a,b}(x + iy)Q_{a,b}(x - iy) = (u^2 + b^2)^2 + 2(u^2 - b^2)Y + Y^2.$$

Its coefficients are nonnegative by $|u| \geq |b|$. The exponential and normalizing factors contribute the positive, Y -independent factor $e^{4ax/(a^2+b^2)}/(a^2+b^2)^2$.

Finally, the leading factor in (4.2) contributes

$$C^2 e^{-2a_0 x^2 + 2b_0 x} e^{2a_0 Y},$$

whose series in Y has nonnegative coefficients because $a_0 \geq 0$. Every finite grouped product therefore has nonnegative coefficients in Y , by ordinary series multiplication. The canonical products converge locally uniformly, and hence so do their products at $x + iy$ and $x - iy$ on compact complex y -disks. Cauchy's coefficient formula shows that every coefficient converges to the corresponding coefficient of $f(x + iy)f(x - iy)$. Those limiting coefficients are $L_n[f](x)$. This proves the first assertion without division by $f(x)$ and so also at real zeros, including multiple zeros.

For $|x| \leq T - \Delta$, $|x - a| \geq |a| - |x| > \Delta \geq |b|$, proving the symmetric-interval assertion. For the final assertion, $a < x_-$ or $a > x_+$, so every $x \in [x_- + \Delta, x_+ - \Delta]$ satisfies $|x - a| > \Delta \geq |b|$. $\square$

Proof of Theorem 1.3. A zeta zero $\rho = \beta + i\gamma$ corresponds to the Xi zero $z_\rho = \gamma - i(\beta - \frac{1}{2})$. Thus the critical strip gives $|\operatorname{Im} z_\rho| < 1/2$, and the hypothesis gives $|\operatorname{Re} z_\rho| > T$ for every nonreal Xi zero. Apply Lemma 4.2 with this T and $\Delta = 1/2$. For the Xi function one can equivalently use the polynomial truncations $c_0 \prod_{\gamma \leq Y} (1 - z^2/\lambda_\rho)$ directly: real zeros occur as pairs and nonreal zeros as conjugation-symmetric quartets.

To obtain strictness, fix x in the stated interval and write $\Xi(z) = (z - x)^{m_x} g(z)$, where $g(x) \neq 0$. All nonreal zero pairs satisfy the strict inequality $|x - \operatorname{Re} \zeta| > |\operatorname{Im} \zeta|$. Thus each remaining real-zero factor has two positive coefficients in Y , and each nonreal pair has three. There are infinitely many remaining factors, by zero counting. For any $k \geq 0$, choose a finite product P whose polynomial in Y has degree at least k ; its k th coefficient is positive. The remaining factor g/P satisfies Lemma 4.2 and is nonzero at x , so $L_k[g](x) \geq L_k[P](x)(g/P)(x)^2 > 0$. Finally $|\Xi(x + iy)|^2 = y^{2m_x} |g(x + iy)|^2$ gives exactly the stated equality cases, and positive normalization preserves them for F . $\square$

5. RELATED WORK

5.1. The reciprocal law. The reciprocal law was asked for, and constructed under RH, by [Roynette and Yor \(2005, eq. \(0.29\) and Théorème V.3.2\)](#). Polson's work connects reciprocal entire functions, Hadamard products, and Thorin representations ([Polson, 2018, Section 2.1](#)). In [Polson \(2026b, v24, Theorems A and 5.3\)](#), whose T is our S_1 , the Lévy density $H(t)/t$, self-decomposability, and Gaussian-mixture construction coincide with ours. [Polson \(2026a, v9, Theorem 22\)](#) also gives the self-decomposable law. The two proofs use the explicit formula near zero and, at later times, verification through heights 1000 and 200, respectively. Our proof extends their large-time comparison with one low zero to every $t > 0$ (Section 2.2), so it needs neither the explicit formula nor verification beyond height 100. Positive powers and arbitrary dimension follow directly from infinite divisibility and Gaussian subordination; the smoothness and positivity of the densities, the negative moments of S_α , and the entire extension are not stated there.¹ Both papers also deduce unimodality of S_1 from Yamazato's theorem ([Polson, 2026b, v24, Corollary 6.8\(iv\)](#); [Polson, 2026a, v9, Theorem 22\(ii\)](#)), which Theorem 1.2 strengthens to log-concavity.

5.2. Pólya frequency orders. Gröchenig's Xi specialization of Schoenberg's transform theorem ([Schoenberg, 1951](#); [Gröchenig, 2020](#)) identifies $\Theta, \Lambda \in \operatorname{PF}_\infty$ with RH. For Λ this equivalence is already in [Roynette and Yor \(2005, Théorème V.3.2\)](#); see also [Konstantopoulos et al. \(2024, Corollary 2.4\)](#). Theorem 1.2 establishes finite PF orders for these densities. Its log-concavity argument combines Yamazato's one-sided shape property ([Yamazato, 1978](#); [Watanabe, 1992](#)) with

¹Theorem 1.1, the equivalence of RH with (ii)–(vi) in Proposition A.1, and the observation on [Konstantopoulos et al. \(2024, Theorem 4.1\)](#) in Section 2.4, were obtained independently of version 24 of [Polson \(2026b\)](#). They are in the first version of this manuscript, recorded in commit `ea8e7e0154b` of the author's repository on 11 September 2026 at 13:45 UTC; version 24 was submitted to arXiv at 16:43 UTC that day.

estimates for exponential tilts; the higher-order argument applies the classical Wronskian criterion and composition formula (Karlin and Studden, 1966; Karlin, 1968) to a finite convolution and its residual law.

5.3. Laguerre inequalities. Csordas et al. (1991, Lemma 4) prove the Laguerre inequality on intervals away from the nonreal zeros. Let $f(z) = Ce^{-az^2+bz} \prod_{\zeta} (1-z/\zeta)e^{z/\zeta}$ be a real entire function with $C \neq 0$, $a \geq 0$, and every nonreal zero in the strip $|\operatorname{Im} z| < 1$. If all zeros with real part in an interval $[x_-, x_+]$, where $x_+ - x_- > 2$, are real and simple, then $L_1[f] > 0$ on $[x_- + 1, x_+ - 1]$; see also Csordas (2015, Proposition 2.2). Their proof uses the second logarithmic derivative. Our coefficient argument is the classical product argument for Csordas and Varga (1990, Theorem 2.9), applied to conjugate zero pairs; it gives every L_n on the same interval (Lemma 4.2) and, for Ξ , determines equality from zero multiplicity.

5.4. Equivalent forms of RH. The equivalence of RH with complete monotonicity of H is a special case of Zhang (2023, v8, Lemma 3). Polson’s RH equivalences (Polson, 2026b, v24, Proposition 3.2 and Theorem 5.6; Polson, 2026a, v9, Theorem 26) identify complete monotonicity of H with the generalized gamma convolution property of S_1 ; Proposition A.1 also treats H/t^α for every finite $\alpha > 0$. Polson (2026b, v24, Theorem 7.1) shows moreover that Hankel positivity of every order at a single time, with both Hankel blocks of the Stieltjes moment problem, is equivalent to RH; Polson (2026c, v25, Section 8.3) discusses the unshifted test.

5.5. The xi function in probability. The xi function itself also occurs as a probability transform. Biane et al. (2001) survey the probability laws whose transforms are ξ and the Jacobi theta function. Nakamura (2015, Theorems 1.1–1.2) shows that $\xi(\sigma - it)/\xi(\sigma)$ is a characteristic function of t for every real σ , and formulates an RH equivalence through *pretended infinite divisibility* for every $1/2 < \sigma < 1$. Nakamura’s signed Lévy–Khinchine-type representations omit the usual integrability condition on the total variation of the signed measure. The quasi-infinitely divisible class studied by Lindner et al. (2018) retains that condition. Nakamura and Suzuki (2023, Theorems 1.1–1.2) construct from the explicit formula a function that is the characteristic function of an infinitely divisible law if and only if RH holds. Under RH its Lévy measure is $\sum_{\gamma} m_{\gamma} \gamma^{-2} \delta_{-\gamma}$, where γ ranges over distinct signed zero ordinates and m_{γ} is the multiplicity. For the reciprocal transform treated here, $1/h$ is infinitely divisible unconditionally, with the positive Lévy density $H(t)/t$; RH is instead the statement that this density is completely monotone (Proposition A.1). The normalized pair $(F(t), 1/F(it))$ is a van Dantzig pair (Konstantopoulos et al., 2024).

AI USAGE

The results in this paper were developed through extensive use of an AI assistant (OpenAI’s GPT Pro). None of the claims were obtained zero-shot: they emerged through sustained back-and-forth discussion, including failed attempts, counterexamples, audits, and repairs. The reciprocal-density revisions, verification programs, and attribution review also used OpenAI Codex. Anthropic’s Claude (Opus 5.5 and Fable 5.1) was used extensively to assist with the writing. Assistant checks and program reruns are not external peer review or formal verification.

ACKNOWLEDGEMENTS

The author (SS) thanks the Alexander von Humboldt (AvH) Foundation for their generous support via a Humboldt Professorship in AI.

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APPENDIX A. EQUIVALENT FORMS OF THE RIEMANN HYPOTHESIS

Extended to every order, the positivity properties of Sections 2.3 and 3 are each equivalent to RH. Schoenberg's characterization relates PF_∞ to real zeros of an entire reciprocal transform (Schoenberg, 1951). Its Xi specialization (Roynette and Yor, 2005, Théorème V.3.2; Gröchenig, 2020, Theorems 3–5) combines with the theory of complete Bernstein functions to give the following equivalences. A Bernstein function B is *complete*, written $B \in \text{CBF}$, when its Lévy density is completely monotone. The equivalence of (i) and (ii), and the continuation to the slit plane used in the proof, are also in Polson (2026b, v24, Propositions 3.2–3.3 and Theorem 5.6(iii)) and Polson (2026a, v9, Lemma 8 and Theorem 26); the continuation argument goes back to Polson (2017).

Proposition A.1 (Equivalent forms of the Riemann hypothesis). *The following assertions are equivalent:*

- (i) RH;
- (ii) H is completely monotone;
- (iii) $H(t)/t$ is completely monotone;
- (iv) for one (equivalently, every) finite $\alpha > 0$, $H(t)/t^\alpha$ is completely monotone;
- (v) $[H(u_i + u_j)]$ is positive semidefinite in every finite size for all $u_i > 0$;
- (vi) $B \in \text{CBF}$;
- (vii) $\Theta \in \text{PF}_\infty$;
- (viii) $\Lambda \in \text{PF}_\infty$.

Under these equivalent conditions the Hankel kernels of H and H/t^α are strictly totally positive in every order for every $\alpha > 0$.

The central implication is analytic. Complete monotonicity gives a positive Laplace representation, which extends h'/h or $\log h$ to the plane slit along the negative real axis. Such an extension excludes every zero of h off that axis, and hence every zeta zero off the critical line.

Proof. We prove the implications in the order

$$\begin{aligned} (i) \Rightarrow (ii) \Rightarrow \{(iii), (iv)\}, \quad (ii) \Leftrightarrow (v), \quad (iii) \Leftrightarrow (vi) \Rightarrow (i), \\ (iv) \Rightarrow (i), \quad (i) \Leftrightarrow (vii) \Leftrightarrow (viii). \end{aligned}$$

Under RH, $\lambda_\rho = \gamma^2 > 0$, so H is a sum of positive real exponentials and is completely monotone. Products of completely monotone functions are completely monotone, and $t^{-\alpha}$ is completely monotone for every $\alpha > 0$; hence (i) implies (ii)–(iv). Complete monotonicity immediately gives all-order Hankel positive semidefiniteness by the Gram representation. Conversely, the representation theorem for exponentially convex functions gives $H(t) = \int_{\mathbb{R}} e^{-tx} \nu(dx)$ with $\nu \geq 0$; see Widder (1934). Since $H(t) \rightarrow 0$ as $t \rightarrow \infty$, ν cannot charge $(-\infty, 0]$. This makes H completely monotone.

If H/t is completely monotone, the Lévy-density characterization of complete Bernstein functions (Schilling et al., 2012, Chapter 6) makes B complete Bernstein. Conversely a complete Bernstein function has a completely monotone Lévy density, so uniqueness of the Lévy representation gives complete monotonicity of H/t .

If B is complete Bernstein, its Pick extension is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$. The identity theorem extends $h = e^B$ from the positive axis to that domain, where the exponential cannot vanish. Every zero of the entire function h must therefore lie on the removed axis. Since $h(0) = 1$, its zeros are negative real. For $\rho = \beta + i\gamma \in \mathcal{Z}_+$ the corresponding zero is

$$(\rho - \tfrac{1}{2})^2 = \delta_\rho^2 - \gamma^2 + 2i\gamma\delta_\rho.$$

As $\gamma > 0$, this is negative real exactly when $\delta_\rho = 0$. Thus completeness forces RH by combining slit-plane holomorphy with the pre-existing entireness of h .

More generally, suppose $\ell_\alpha(t) = H(t)t^{-\alpha}$ is completely monotone for one finite $\alpha > 0$. Write $\ell_\alpha(t) = \int_0^\infty e^{-rt} \mu(dr)$ with $\mu \geq 0$. Then for $s > 0$, Tonelli gives

$$B'(s) = \int_0^\infty e^{-st} H(t) dt = \Gamma(\alpha + 1) \int_0^\infty (s + r)^{-\alpha-1} \mu(dr).$$

At $s = 1$ this gives $\int (1 + r)^{-\alpha-1} \mu(dr) < \infty$. For every compact $K \subset \mathbb{C} \setminus (-\infty, 0]$, $|z + r| \geq c_K(1 + r)$ for $z \in K$, $r \geq 0$, with $c_K > 0$. Thus the principal-branch integral defines a holomorphic function J on the slit plane, as in the proof of [Polson \(2026a, v9, Theorem 26\)](#). The identity theorem gives $h' = Jh$ there; a zero of h of multiplicity m would give orders $m - 1$ and at least m on the two sides. Hence all zeros of h are negative real, proving (iv) $\Rightarrow$ (i).

Under RH the genus-zero product [\(2.4\)](#) becomes

$$h(s) = \prod_{\gamma > 0} \left(1 + \frac{s}{\gamma^2}\right), \quad \Xi(z) = c_0 \prod_{\gamma > 0} \left(1 - \frac{z^2}{\gamma^2}\right),$$

where ordinates are counted with multiplicity and $\sum_{\gamma > 0} \gamma^{-2} < \infty$. These are respectively the restricted and even Laguerre–Pólya products. Schoenberg’s characterization applied to $1/(c_0 h)$ and $1/\Xi$ gives the PF densities; conversely, either PF representation forces the corresponding product and hence RH ([Roynette and Yor, 2005](#), Théorème V.3.2; [Gröchenig, 2020](#), Theorems 3–5). Uniqueness of the Laplace transforms identifies those densities with [\(2.8\)](#). Under RH the representing measures of H and of H/t^α have infinite support, so every nonzero polynomial square has strictly positive integral; Proposition [C.2](#) then gives strict total positivity in every finite order. $\square$

APPENDIX B. ESTIMATES FOR H AND THE RECIPROCAL LAWS

The reciprocal-law construction needs control of H at both ends of the time axis. Near zero, many high zeros contribute and zero counting determines the leading scale; at large times, the first zero determines the decay. These estimates justify the Lévy integrals and moment bounds used in the paper. We first check the counting constants and theta normalization, then derive the asymptotics of H at both ends, including their fixed-order derivative versions; the last subsection completes the regularity and moment statements of Lemma [2.4](#). Table [1](#) lists, for each result of the paper, the inputs beyond elementary calculus and linear algebra.

Table 1: Numerical inputs by result.

Assertion	Inputs beyond elementary calculus and linear algebra
Positive H and $-H'$; the Bernstein and self-decomposable laws	Critical strip, zero counting, the verified first zero, and all zeros through height 100
Global log-concavity of Θ and Λ	Reality through height 100, the real zero enclosed near 21.022 (Appendix C.6), the shifted-exponent estimates, and Yamazato’s shape theorems (Yamazato, 1978 ; Watanabe, 1992)
PF convolution factors below U	The estimate in Appendix C.2 and verified critical-line zeros below U
PF approximation in total variation	A retained zero in $(U/4, U/3]$, the reciprocal-square tail bound, the first exponential or Laplace factor, and the scalar check $225 b_{T_*} < 4.5 \cdot 10^{-9}$ in Corollary C.4
$\Lambda \in \text{PF}_{10}$ with strict minors	Verified height T_* , the residual exponential moment, eleven real-zero enclosures, the compact interval cover, and the analytic tail bound
All $L_n[\Xi] \geq 0$ on the interval determined by verified zeros	Canonical product, critical-strip width, and finite verification through T_*

B.1. The counting constant. For $Y \geq e$, [Bellotti and Wong \(2025, Theorem 1.1\)](#) give the upper bound

$$0.11200 \log Y + 0.12567 \log \log Y + 3.77417.$$

Put $u = \log Y \geq 1$. The excess over $(\log Y)/8$ is $0.12567 \log u - 0.013u + 3.77417$, whose maximum at $u = 0.12567/0.013$ is less than $3.934 < 4$. At $Y = e$ the same counting estimate gives $N(e) < 4$; thus $N(Y) \leq 3$ and $-1 < M(Y) \leq 0$ on $[0, e]$, which covers the remaining range. One-sided limits at zero ordinates give the same inequality for either endpoint convention. The counting theorem, including its analytic inputs and low-height checks, is used as published.

B.2. The theta normalization. The classical theta formula, in our normalization, is

$$(B.1) \quad \Xi(z) = \int_{\mathbb{R}} e^{izx} \Phi(x) dx, \quad \Phi(x) = \sum_{n \geq 1} (4\pi^2 n^4 e^{9x/2} - 6\pi n^2 e^{5x/2}) e^{-\pi n^2 e^{2x}}.$$

This normalization follows by differentiating the Jacobi theta identity and integrating by parts; see [Titchmarsh \(1986, Section 2.16\)](#) and [Konstantopoulos et al. \(2024, eq. \(2.7\)\)](#). The kernel is even by the theta transformation, and each summand is positive for $x \geq 0$, since $2\pi n^2 e^{2x} > 3$. It has faster-than-exponential decay at both ends. Thus F is the characteristic function of the positive density Φ/c_0 .

B.3. Asymptotics of H at both ends.

Proposition B.1 (Asymptotics). *As $t \downarrow 0$,*

$$(B.2) \quad H(t) = \frac{\log(1/t) - \gamma_E - 2 \log(4\pi)}{8\sqrt{\pi t}} + O(\log(1/t)).$$

For each fixed nonnegative integer j ,

$$(B.3) \quad (-1)^j H^{(j)}(t) \sim \frac{\Gamma(j + \frac{1}{2})}{8\pi} t^{-j-1/2} \log(1/t).$$

At infinity $H(t) = m_1 e^{-qt} + O(e^{-(q+\varepsilon)t})$ for some $\varepsilon > 0$, where m_1 is the multiplicity of the first zero. All fixed derivatives have analogous bounds.

The leading term in (B.2) agrees with [Polson \(2026b, v24, Corollary 5.5\)](#), who also obtains the constant term $7/8$ from the explicit formula.

Proof. First replace every λ_ρ by γ^2 . Stieltjes integration against (2.1) gives the main integral

$$\frac{1}{2\pi} \int_0^\infty e^{-ty^2} \log(y/(2\pi)) dy.$$

The error is bounded by $\int_0^\infty 2tye^{-ty^2} [\frac{1}{8} \log(y+2) + 4] dy = O(1 + \log(1/t))$. Set $x = \sqrt{t}y$ and use $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$ and $\int_0^\infty e^{-x^2} \log x dx = \sqrt{\pi}(-\gamma_E - 2 \log 2)/4$ to obtain the stated main term. For $0 < t \leq 1$,

$$|e^{t\delta^2} \cos(2t\gamma\delta) - 1| \leq C(t + t^2\gamma^2).$$

Zero counting then bounds the total replacement error by $O(\sqrt{t} [1 + \log(1/t)])$. For (B.3), repeat the argument with $y^{2j}e^{-ty^2}$. Its counting error is $O(t^{-j} \log(1/t))$; second-order Taylor expansion of $z^{2j}e^{-tz^2}$ on the short vertical segments from γ to $\gamma - i\delta$ gives an error $O(t^{1/2-j} \log(1/t))$. More explicitly, its real first-order term is zero, and the second derivative is bounded by a constant depending on j times $(y^{2j-2} + ty^{2j} + t^2y^{2j+2})e^{-ty^2/2}$, with the negative power omitted at $j = 0$. Integration of these bounds against N proves the claimed order. The Gaussian main integral has leading constant $\Gamma(j + 1/2)/(8\pi)$. For the large-time statement, low zeros are discrete and real, and every tail real part exceeds $100^2 - 1/4 > q$. There is therefore a positive gap above the minimum real part q . Local absolute convergence at $t = 1$ controls the remaining exponentially decaying sum, also after any fixed number of differentiations. $\square$

B.4. Regularity and moments of the reciprocal laws. The Bernstein representation gives a probability measure, but the later shape arguments need a smooth density that is positive in the interior of its support. We complete that step by extracting gamma convolution factors of arbitrarily high smoothness; the asymptotics of H then control the exponential moments.

Proof of Lemma 2.4. For each integer $N \geq 2$, (B.2) and the lower bound in Lemma 2.2 allow $b_N > 0$ such that $\alpha H(t) \geq N e^{-b_N t}$ for every $t > 0$. To see this explicitly, first choose $t_N > 0$ so that $\alpha H \geq N$ on $(0, t_N]$, and then choose $b_N \geq q$ large enough that $N e^{-(b_N - q)t_N} \leq \alpha(1 - \varepsilon_{100})$. Subtracting this Lévy density gives independent variables with

$$(B.4) \quad S_\alpha \stackrel{d}{=} \text{Gamma}(N, b_N) + R_N.$$

The gamma density, extended by zero to the negative half-line, has bounded continuous derivatives through order $N - 2$. Convolution with the law of R_N preserves them. As N is arbitrary, θ_α is smooth. Every R_N has zero drift: its Laplace exponent divided by s tends to zero, by dominated convergence with the finite first Lévy moment. For a nonnegative variable Y , the elementary Laplace bounds show $\lim_{s \rightarrow \infty} -s^{-1} \log \mathbb{E} e^{-sY} = \inf \text{supp } Y$. Hence $\inf \text{supp } R_N = 0$, and convolution in (B.4) makes $\theta_\alpha(x) > 0$ for every $x > 0$.

The large-time expansion of Proposition B.1 shows $\int_1^\infty (e^{rt} - 1)H(t) dt/t < \infty$ exactly for $r < q$. The Poisson exponential formula, first with truncations and then by monotone convergence, proves the exponential-moment assertions. Holomorphy follows by domination on compact subsets. Finally Stirling's formula in (1.1) gives

$$(B.5) \quad B(s) = \frac{1}{4}\sqrt{s} \log s + O(\sqrt{s}) \quad (s \rightarrow \infty).$$

In particular $h(s)^{-\alpha} \rightarrow 0$, so there is no atom at zero. For $p > 0$, Tonelli's theorem yields

$$\mathbb{E} S_\alpha^{-p} = \frac{1}{\Gamma(p)} \int_0^\infty s^{p-1} h(s)^{-\alpha} ds < \infty,$$

using (B.5). □

APPENDIX C. HIGHER PÓLYA FREQUENCY ORDERS

The order-ten part of Theorem 1.2 compares Λ with a finite convolution of Laplace densities coming from verified zeros. We first extract such factors and isolate the remaining law, a step also used in Section 3.1. We then bound the exponential moments of the residual and certify the derivative determinants. A separate argument treats each fixed order of Θ near zero. Every certified inequality was evaluated in directed interval arithmetic at 70 or more decimal digits from rational or exact decimal data; an inconclusive interval comparison never counts as a certificate. The published verified height and the first-zero isolation are external inputs and are not recomputed.

C.1. Wronskian transfers. The following is the classical Wronskian criterion for an extended complete Chebyshev system (Karlin and Studden, 1966). We include the short proof to fix the signs used in the PF arguments of this appendix and in the last assertion of Proposition A.1.

Lemma C.1. *Suppose real C^{k-1} functions $f_1, \dots, f_k$ on an interval have positive initial Wronskians, where $W(f_1, \dots, f_j) = \det[f_b^{(a-1)}]_{a,b=1}^j$: $W(f_1, \dots, f_j) > 0$ for $1 \leq j \leq k$. Then $\det[f_j(s_i)]_{i,j=1}^k > 0$ whenever $s_1 < \dots < s_k$.*

Proof. Induct on k . Since $f_1 > 0$, divide the evaluation row at s_i by $f_1(s_i)$. Set $g_j = (f_{j+1}/f_1)'$. Direct differentiation gives

$$W(g_1, \dots, g_j) = \frac{W(f_1, \dots, f_{j+1})}{f_1^{j+1}} > 0.$$

Subtract consecutive normalized rows, starting at the bottom, and expand in the first column. The remaining determinant is the integral, over $[s_1, s_2] \times \cdots \times [s_{k-1}, s_k]$, of the evaluation determinant of $g_1, \dots, g_{k-1}$. It is strictly positive in the interior by the induction hypothesis. $\square$

Positive differential Hankel determinants pass to distinct arguments by the same principle.

Proposition C.2. *If a smooth real f satisfies $[(-1)^{i+j} f^{(i+j)}(t)]_{i,j=0}^{k-1} \succ 0$ for all $t > 0$, then $f(x+y)$ is strictly totally positive through order k .*

Proof. Every initial differential Hankel determinant $\det[f^{(i+j)}(t)]_{i,j=0}^{r-1}$ is positive: the inserted row and column signs have product one. Fixing y , these are the initial Wronskians of $x \mapsto f(x+y), f'(x+y), \dots, f^{(r-1)}(x+y)$. Lemma C.1 first proves positivity of $\det[f^{(j-1)}(x_i+y)]_{i,j=1}^r$. Fixing the x_i , these determinants are the transposes of the initial Wronskians of the functions $y \mapsto f(x_i+y)$. A second application of the lemma proves the assertion. $\square$

C.2. Explicit PF convolution factors. For this factorization we use a variant of Lemma 2.2. Its bounds hold with q replaced by any real q_* satisfying $2 \leq q_* \leq U^2/4 - 1/4$, and with $K_U(t) = e^{-q_* t} + H_{>U}(t)$: the same proof applies because $a_\rho - q_* \geq 3\gamma^2/4$ and $|\lambda_\rho| \leq q_* \gamma^2$.

For $100 \leq U \leq T_*$, choose a verified ordinate γ_* with $2 \leq \gamma_*^2 \leq U^2/4 - 1/4$; the first ordinate is always a possible choice. Let $\mathcal{A}_U$ be the finite multiset of ordinates $\gamma \leq U$, with *one occurrence* of γ_* deleted, and define

$$P_U(s) = \prod_{\gamma \in \mathcal{A}_U} (1 + s/\gamma^2).$$

Repeated zeros are retained with their multiplicities.

Theorem C.3 (Extraction of verified-zero PF factors). *There is a positive self-decomposable random variable R_U such that*

$$(C.1) \quad S_1 \stackrel{d}{=} \sum_{\gamma \in \mathcal{A}_U} E_\gamma + R_U, \quad E_\gamma \sim \text{Exp}(\gamma^2),$$

$$(C.2) \quad X_{1,1} \stackrel{d}{=} \sum_{\gamma \in \mathcal{A}_U} L_\gamma + Y_U, \quad L_\gamma \sim \text{Lap}(\gamma),$$

where $\text{Lap}(\gamma)$ has density $\gamma e^{-\gamma|x|}/2$ and $Y_U \stackrel{d}{=} \sqrt{2R_U} G_1$ with G_1 independent of R_U . The summands on each right-hand side are independent. The density of either finite sum is a PF_∞ function. The residual R_U has a smooth positive density on $(0, \infty)$; Y_U is self-decomposable and has an entire positive density.

Under RH, $1/h(s) = \prod_\gamma (1 + s/\gamma^2)^{-1}$ is the Laplace transform of a convergent sum of independent exponential variables, a generalized gamma convolution in Thorin's sense (Bondesson, 1992, Chapter 3), and correspondingly $X_{1,1} = \sum_\gamma L_\gamma$; for xi these factorizations are written in Polson (2018, Section 2.1) and Polson (2026a, v9, Theorem 12(i)). The point here is the unconditional extraction of finitely many verified factors with a self-decomposable residual.

Proof. Set $C_U(s) = B(s) - \log P_U(s)$. By the product and integral formulas,

$$(C.3) \quad C_U(s) = \int_0^\infty (1 - e^{-st}) \frac{K_U(t)}{t} dt, \quad \mathbb{E} e^{-sR_U} = e^{-C_U(s)} = \frac{P_U(s)}{h(s)}.$$

Here $K_U(t) = e^{-\gamma_*^2 t} + H_{>U}(t)$, which is positive and decreasing by the variant of Lemma 2.2 stated above. Thus the Poisson and self-decomposability constructions from Section 2.3 apply. Multiplication of Laplace transforms proves (C.1). On the Fourier axis each factor becomes $(1 + t^2/\gamma^2)^{-1}$, the characteristic function of $\text{Lap}(\gamma)$, proving (C.2). The finite exponential and centered Laplace densities are PF_∞ by Schoenberg's product characterization (Schoenberg, 1951). Equivalently, their transform reciprocals have only real zeros of the required sign. Removing

finitely many exponentials leaves the small-time divergence of H unchanged and changes $B(s)$ by $O_U(\log(2+s))$. The density and entire-extension arguments in Section 2.3 therefore apply to both residuals. $\square$

The densities in (2.8) therefore have explicit finite PF_∞ convolution factors. A higher retained zero makes the residual small and yields quantitative approximations.

Corollary C.4 (Approximation by verified-zero PF laws). *For $1000 \leq U \leq T_*$, the retained ordinate can be chosen in $(U/4, U/3]$. Put*

$$b_U = \frac{16}{U^2} + \frac{2(\log U + 1)}{U}, \quad A_U = \sum_{\gamma \in \mathcal{A}_U} E_\gamma, \quad Z_U = \sum_{\gamma \in \mathcal{A}_U} L_\gamma.$$

Then $\mathbb{E}R_U \leq b_U$. If d_{TV} denotes total variation distance between probability laws, with values in $[0, 1]$, then

$$(C.4) \quad d_{\text{TV}}(S_1, A_U) \leq qb_U, \quad d_{\text{TV}}(X_{1,1}, Z_U) \leq qb_U, \quad q = \gamma_1^2 < 225.$$

Both bounds are less than $4.5 \cdot 10^{-9}$ at $U = T_*$. The approximating laws have PF_∞ densities.

Proof. The counting bound gives

$$N(U/3) - N(U/4) \geq \frac{U}{24\pi} \log \frac{U}{8\pi} - \frac{1}{4} \log(U+2) - 8 > 0 \quad (U \geq 1000).$$

Indeed, the first term exceeds $U/32$; the resulting lower bound is positive at 1000 and increasing thereafter. This supplies γ_* , and its square satisfies the hypothesis of Lemma 2.2. Differentiating (C.3) at zero gives

$$\mathbb{E}R_U = \gamma_*^{-2} + \sum_{\gamma > U} \text{Re} \frac{1}{\lambda_\rho} \leq \frac{16}{U^2} + \sum_{\gamma > U} \gamma^{-2} \leq b_U,$$

since $\text{Re}(\lambda_\rho^{-1}) = (\gamma^2 - \delta_\rho^2)/(\gamma^2 + \delta_\rho^2)^2 \leq \gamma^{-2}$.

The finite sums include the first-zero factors. Translation by $r \geq 0$ changes the law $\text{Exp}(q)$ in total variation by $1 - e^{-qr} \leq qr$. Convolution contracts total variation, so conditioning on R_U proves the first bound in (C.4). For the second, let $l(x) = \gamma_1 e^{-\gamma_1 |x|}/2$. Its second distributional derivative is $D^2 l = ql - q\delta_0$, of total variation $2q$. Consequently

$$\|l(\cdot - y) + l(\cdot + y) - 2l\|_1 \leq 2qy^2.$$

This follows by integrating translates of $D^2 l$ against the triangular weight $(|y| - |v|)_+$. The residual Y_U is symmetric and $\mathbb{E}Y_U^2 = 2\mathbb{E}R_U$. Averaging the displayed inequality and dividing the resulting L^1 bound by two gives $d_{\text{TV}}(L_{\gamma_1} + Y_U, L_{\gamma_1}) \leq q\mathbb{E}R_U$. Convolving the other Laplace factors proves the claim. The final numerical bound follows from $q < 225$ and $U = T_*$. $\square$

These estimates concern the normalized densities $c_0\Theta$ and $c_0\Lambda$.

C.3. Exponential moments of the residual. For order ten we use the much smaller residual obtained by retaining a zero near the verified height. The next bound controls the whole residual law and is the input to the finite determinant certificate.

Lemma C.5. *Take $U = T_*$ and $\gamma_* \in (U/4, U/3]$ in Corollary C.4, and put $b = b_U$ and $\Omega = 10^5$. For $Y_U = \sqrt{2R_U}G_1$,*

$$\mathbb{E}Y_U^2 \leq 2b, \quad \mathbb{E} \cosh(\Omega Y_U) - 1 \leq e^{2\Omega^2 b} - 1 < \frac{1}{2}.$$

Proof. The variance bound follows from $\mathbb{E}R_U \leq b$. For $0 \leq v \leq \Omega^2$,

$$\int_0^\infty e^{vt} K_U(t) dt \leq \frac{1}{\gamma_*^2 - v} + \sum_{\gamma > U} \frac{1}{\gamma^2 - 1/4 - v} \leq 2b.$$

Indeed, $\Omega^2 \leq U^2/32 < \gamma_*^2/2$ and $\Omega^2 + 1/4 \leq U^2/2$. The Poisson exponential-moment formula and Tonelli give

$$\log \mathbb{E} e^{\Omega^2 R_U} = \int_0^{\Omega^2} \int_0^\infty e^{vt} K_U(t) dt dv \leq 2\Omega^2 b.$$

Gaussian conditioning yields the cosh bound. Appendix C.7 certifies the final scalar inequality. $\square$

C.4. The order PF_{10} for Λ . Log-concavity controls only two-by-two minors. For higher orders we use signed differential determinants,

$$\mathcal{D}_k(p; x) = (-1)^{k(k-1)/2} \det[p^{(i+j)}(x)]_{i,j=0}^{k-1}.$$

Their positivity gives positivity of the PF_k minors by an iterated Wronskian criterion. Eleven Laplace factors supply two more derivatives than order ten needs. This permits a second-order estimate for every derivative error, using symmetry of the residual. A separate exponential moment makes the error decay faster than the reference determinants in the tails. These two choices leave a finite interval calculation with a positive margin.

Lemma C.6 (Differential criterion for PF minors). *If $p \in C^{2m-2}(\mathbb{R})$ and $\mathcal{D}_k(p; x) > 0$ for every $x \in \mathbb{R}$ and $1 \leq k \leq m$, then $p(x - y)$ has strictly positive ordered minors through order m .*

Proof. Fix y . The initial Wronskians of the functions $x \mapsto (-1)^j p^{(j)}(x - y)$, $0 \leq j < m$, are precisely $\mathcal{D}_k(p; x - y)$. Lemma C.1 makes their ordered evaluation determinants positive. For fixed increasing x_i , these determinants are the transposes of the initial Wronskians in y of $y \mapsto p(x_i - y)$. A second application of that lemma proves the claim. $\square$

Lemma C.7 (Eleven Laplace factors with a small symmetric residual). *Let $a_i \in I_i$, $1 \leq i \leq 11$, where the disjoint intervals I_i are specified in Appendix C.6. Their centers are approximately*

$$\begin{aligned} &14.13, 21.02, 32.94, 48.01, 72.07, 107.17, \\ &161.19, 241.05, 361.29, 544.32, 814.87. \end{aligned}$$

Let $L_{a_1}, \dots, L_{a_{11}}, Y$ be independent, with $L_{a_i} \sim \text{Lap}(a_i)$ and Y symmetric. Suppose

$$\mathbb{E} Y^2 \leq 2b, \quad \mathbb{E} \cosh(\Omega Y) - 1 \leq \kappa := \frac{1}{2}, \quad \Omega = 10^5, \quad b = b_{T_*}.$$

The density p of $Y + \sum_{i=1}^{11} L_{a_i}$ satisfies $\mathcal{D}_k(p; x) > 0$ for every $x \in \mathbb{R}$ and $1 \leq k \leq 10$.

Proof. Write f for the convolution of the eleven Laplace densities. Partial fractions give

$$f(x) = \sum_{i=1}^{11} c_i e^{-a_i |x|}, \quad c_i = \frac{\prod_{j=1}^{11} a_j^2}{2a_i \prod_{j \neq i} (a_j^2 - a_i^2)}.$$

The coefficients alternate in sign, starting positive. The function is C^{20} , with bounded derivatives through that order. Set $S = 100$, $r_i = a_i/S$, and $C_j = \sum_i |c_i| r_i^j$. Symmetry and Taylor's formula therefore give, for $0 \leq j \leq 18$,

$$(C.5) \quad p^{(j)}(x) = \mathbb{E} f^{(j)}(x - Y), \quad \left| \frac{p^{(j)}(x) - f^{(j)}(x)}{S^j} \right| \leq b S^2 C_{j+2}.$$

In particular there is no loss of derivative order at zero.

For $x > 0$ put $M_i = \mathbb{E} e^{a_i Y}$ and $g(x) = \sum_i c_i M_i e^{-a_i x}$. Since $a_i < \Omega$, comparison of the cosh power series gives

$$1 \leq M_i \leq 1 + \kappa(a_i/\Omega)^2 \leq \overline{M}, \quad \overline{M} = 1 + \kappa(a_{11}/\Omega)^2.$$

Only $Y > x$ separates p from g . On this event, $\cosh(\Omega Y) - 1 \geq \frac{1}{2} e^{\Omega Y} (1 - e^{-\Omega x})^2$, whence

$$(C.6) \quad \left| \frac{p^{(j)}(x) - g^{(j)}(x)}{S^j} \right| \leq C_j d(x), \quad d(x) = \frac{4\kappa e^{-\Omega x}}{(1 - e^{-\Omega x})^2}, \quad 0 \leq j \leq 18.$$

The matching derivatives of f at zero justify the derivative formulas; equivalently, one may first differentiate $p = f * \mathcal{L}(Y)$ using its bounded derivatives and then split the expectation at $Y = x$.

For $0 \leq u, v < k$ write $G_{uv} = g^{(u+v)}(x)/S^{u+v}$ and $E_{uv} = (p^{(u+v)} - g^{(u+v)})(x)/S^{u+v}$. Cauchy–Binet gives the reference determinant

$$(-1)^{k(k-1)/2} \det G = \sum_{|I|=k} w_I \left(\prod_{i \in I} M_i \right) e^{-S_I x}, \quad S_I = \sum_{i \in I} a_i, \quad w_I = (-1)^{k(k-1)/2} \prod_{i \in I} c_i \prod_{\substack{i < j \\ i, j \in I}} (r_j - r_i)^2.$$

For $I_0 = \{1, \dots, k\}$, write $w_0 = w_{I_0} > 0$ and $S_0 = S_{I_0}$. The other exponent sums are strictly larger. To bound the perturbation, use $C_j \leq C_0 r_{11}^j$ in (C.6) and $|G_{uv}| \leq C_0 r_{11}^{u+v} \bar{M} e^{-a_1 x}$. Expanding the determinant by permutations and by the number of error entries gives

$$(C.7) \quad \frac{|\det(G + E) - \det G|}{w_0 e^{-S_0 x}} \leq \mathcal{E}_k(x) := \frac{k! C_0^k r_{11}^{k(k-1)}}{w_0} \sum_{j=1}^k \binom{k}{j} \bar{M}^{k-j} (4\kappa)^j \frac{e^{-[j\Omega + (k-j)a_1 - S_0]x}}{(1 - e^{-\Omega x})^{2j}}.$$

All exponent differences on the right are positive for all $a_i \in I_i$. Thus each error term decreases on $(0, \infty)$.

Appendix C.7 certifies the remaining finite inequalities. On $[0, 1/250]$, it uses (C.5) and interval Gaussian elimination to obtain alternating pivot signs through size ten. On $[1/250, 1/10]$, it bounds the normalized reference determinant below by more than $\mathcal{E}_k(x)$, for $2 \leq k \leq 10$. For $x \geq 1/10$, discard all positive reference terms except the leading one. The remaining normalized lower bound exceeds 0.18 at $1/10$ and increases thereafter; $\mathcal{E}_k$ decreases and is smaller than this margin at $1/10$. This controls the entire tail. Positivity of p handles order one, and symmetry handles $x < 0$. $\square$

Proof of the order-ten assertion in Theorem 1.2. The endpoint sign certificates in Appendix C.6 supply a real zero in each I_i . These eleven ordinates occur among the extracted factors in Corollary C.4, with $U = T_*$ and $\gamma_* \in (U/4, U/3]$. Lemma C.5 supplies both residual bounds. Lemmas C.7 and C.6 give strictly positive ordered minors through order ten for these eleven factors convolved with Y_U .

Convolution with each remaining Laplace density preserves strictness. The classical composition formula for totally positive kernels (continuous Cauchy–Binet; Karlin, 1968) integrates the product of a strictly positive minor and a nonnegative Laplace minor over $u_1 < \dots < u_k$. The latter is positive near $u_i = y_i$: the matrix $[e^{-a|y_i - y_j|}]$ is positive definite for distinct y_i , as follows from its strictly positive Fourier density. Thus the integral is strictly positive. Iterating over the remaining finite set of factors proves the assertion for $c_0 \Lambda$, and hence for Λ . $\square$

C.5. Fixed-order positivity near the one-sided boundary. We use the exponential-tilting (saddlepoint) method of Daniels (1954) to obtain relative determinant estimates near the left endpoint of Θ . Under a sufficiently strong negative exponential tilt, the one-sided law becomes Gaussian after centering and scaling. To control determinants, weak convergence is insufficient: we prove convergence of every fixed number of density derivatives.

Proposition C.8 (Local PF positivity of Θ). *For each fixed integer $m \geq 1$, there is $\varepsilon_m > 0$ such that*

$$\det[\Theta(x_i - y_j)]_{i,j=1}^k > 0 \quad (1 \leq k \leq m)$$

whenever $x_1 < \dots < x_k$, $y_1 < \dots < y_k$, and $0 < x_i - y_j < \varepsilon_m$ for every i, j . More precisely, put $x_s = B'(s)$ and $v_s = -B''(s)$. As $s \rightarrow \infty$,

$$(C.8) \quad \mathcal{D}_k(\Theta; x_s) \sim \Theta(x_s)^k v_s^{-k(k-1)/2} \prod_{j=0}^{k-1} j! \quad \text{for every fixed } k.$$

Proof. We use the branch of $B(z) = \log h(z)$ that is real on the positive axis. Stirling's expansion (NIST, 2026, Section 5.11), together with $\log \zeta(w) = O(2^{-\operatorname{Re} w})$ for $\operatorname{Re} w \rightarrow \infty$, gives

$$(C.9) \quad B(z) = \frac{\sqrt{z}}{2} \left(\log \frac{\sqrt{z}}{2\pi} - 1 \right) + \frac{7}{8} \log z + C + O(|z|^{-1/2})$$

uniformly in the right half-plane as $|z| \rightarrow \infty$. Indeed, $|\arg \sqrt{z}| \leq \pi/4$ and $\operatorname{Re} \sqrt{z} \geq \sqrt{|z|}/2$ there, so the gamma expansion and Euler-product estimate are uniform. Cauchy's estimates permit termwise differentiation near the positive axis. In particular,

$$(C.10) \quad x_s \sim \frac{\log s}{8\sqrt{s}}, \quad v_s \sim \frac{\log s}{16s^{3/2}}, \quad B'''(z) = O\left(\frac{\log s}{s^{5/2}}\right) \quad (|z - s| \leq s/2).$$

For large s , $v_s > 0$ and x_s decreases continuously to zero.

The tilted density is $f_s(u) = e^{B(s)-su}\theta_1(u)$, with mean x_s and variance v_s . Write $\sigma_s = \sqrt{v_s}$ and let $g_s(z) = \sigma_s f_s(x_s + \sigma_s z)$ be its centered, standardized density. Its characteristic function is

$$\phi_s(t) = \exp \{B(s) - B(s - it/\sigma_s) - itx_s/\sigma_s\}.$$

Taylor expansion using (C.10) gives $\phi_s(t) \rightarrow e^{-t^2/2}$ for each fixed t .

To pass to density derivatives, we need an integrable bound for all frequencies. Write $\sqrt{1+iu} = a + ib$ with $a \geq 1$, and set $w = \sqrt{s}$, $L = \log(w/(2\pi)) - 1$. From (C.9),

$$\operatorname{Re} B(s + isu) - B(s) \geq \frac{wL}{2}(a - 1) - \frac{C_1}{w}.$$

Here $\operatorname{Re}[(a + ib) \log(a + ib)] \geq 0$: for $\eta = |b|/a < 1$, its quotient by a is $\operatorname{arctanh}(\eta^2) - \eta \arctan \eta$, whose derivative is $\eta/(1 - \eta^2) - \arctan \eta \geq 0$. The logarithmic term $\frac{7}{8} \log |1 + iu|$ is also nonnegative. Finally,

$$a - 1 = \frac{u^2}{2(a + 1)(\sqrt{1 + u^2} + 1)} \geq c_1 \min\{u^2, \sqrt{|u|}\}$$

for an absolute $c_1 > 0$. Put $A_s = s\sigma_s$, so $A_s^2 \sim wL/8$ and $A_s \rightarrow \infty$. Substitution of $u = t/A_s$ gives $|\phi_s(t)| \leq C_2 \exp\{-c \min(t^2, A_s^{3/2} \sqrt{|t|})\}$, with $c, C_2 > 0$ independent of large s . Since $A_s \geq 1$ eventually,

$$|\phi_s(t)| \leq C_2 \exp\{-c \min(t^2, \sqrt{|t|})\}.$$

The right side is integrable after multiplication by any fixed power of $|t|$. Dominated convergence in Fourier inversion therefore gives $g_s^{(r)}(0) \rightarrow \varphi^{(r)}(0)$ for every fixed r , where $\varphi(z) = (2\pi)^{-1/2} e^{-z^2/2}$.

The Gaussian moment determinant is

$$\mathcal{D}_k(\varphi; 0) = \varphi(0)^k \prod_{j=0}^{k-1} j! > 0.$$

For example, this follows by replacing monomials with monic Hermite polynomials in the Gaussian Gram matrix; their squared norms are $j!$. For $M_f = (f^{(i+j)}(x))_{i,j=0}^{k-1}$, Leibniz's rule gives $M_{e^{cx}} = e^{cx} T_c M_f T_c^\top$, where T_c is lower triangular with $(T_c)_{ij} = \binom{i}{j} c^{i-j}$ for $j \leq i$. Since $\det T_c = 1$, tilting multiplies the differential determinant by e^{kcx} . Scaling then gives

$$\frac{\mathcal{D}_k(\Theta; x_s)}{\Theta(x_s)^k} = \sigma_s^{-k(k-1)} \frac{\mathcal{D}_k(g_s; 0)}{g_s(0)^k},$$

which proves (C.8). All initial signed determinants through size m are therefore positive on $(0, \varepsilon_m)$ for some $\varepsilon_m > 0$. The proof of Lemma C.6 applies locally: both Wronskian integrations remain in this interval when all the cross differences do. This proves the stated PF_k minor inequalities. $\square$

C.6. The eleven real-zero enclosures. Define $I_i = [\bar{a}_i - 10^{-40}, \bar{a}_i + 10^{-40}]$ by the following exact decimal centers:

i	$\bar{a}_i$
1	14.13472514173469379045725198356247027078425711569924318
2	21.02203963877155499262847959389690277733434052490278175
3	32.93506158773918969066236896407490348881271560351703901
4	48.00515088116715972794247274942751604168684400114442512
5	72.06715767448190758252210796982616839048090662145669709
6	107.1686111842764075151233519630861912134767078814047653
7	161.1889641375960275194373441293695543649157903274754666
8	241.0491577962165864128379214103356705496456828447220938
9	361.2893616958046503902913112277152281908759416379958332
10	544.3238901010052629031213087557608362045030422788811585
11	814.8705396258725677840196869567854134266570546908545228

Interval evaluation gives opposite signs for Ξ at the two endpoints of every I_i , so continuity gives a real zero in each box. The boxes are positive, disjoint, and increasing. The numbering and simplicity of these zeros are not needed.

The evaluation uses Euler–Maclaurin with $N = 1000$, $p = 64$, and $s = 1/2 + it$:

$$\zeta(s) = \sum_{n=1}^{N-1} n^{-s} + \frac{N^{1-s}}{s-1} + \frac{1}{2}N^{-s} + \sum_{k=1}^p \frac{B_{2k}}{(2k)!} (s)_{2k-1} N^{-s-2k+1} + R_p(s),$$

$$|R_p(s)| \leq \frac{4|(s)_{2p}|}{(2\pi)^{2p}(\operatorname{Re} s + 2p - 1)N^{\operatorname{Re} s + 2p - 1}}.$$

Here $(s)_j = s(s+1)\cdots(s+j-1)$ and B_{2k} are exact rational Bernoulli numbers. The remainder follows by integration of the periodic Bernoulli remainder and $\|B_{2p}(\{\cdot\})\|_\infty/(2p)! \leq 4/(2\pi)^{2p}$. The resulting disc is enclosed by a complex rectangle and multiplied by $s(s-1)\pi^{-s/2}\Gamma(s/2)/2$, with the gamma factor also in interval arithmetic, to enclose $\Xi(t)$.

C.7. The order-ten certificate. With $U = T_*$, $b = b_U$ from Corollary C.4, and $\Omega = 10^5$, interval arithmetic certifies

$$\Omega^2 \leq U^2/32, \quad \Omega^2 + 1/4 \leq U^2/2, \quad e^{2\Omega^2 b} - 1 < \frac{1}{2}.$$

Together with the analytic mean bound $\mathbb{E}R_U \leq b$, these give the final inequality in Lemma C.5 and both hypotheses of Lemma C.7. In the notation of that lemma, the normalized reference determinant is

$$(C.11) \quad F_k(x) = \sum_{|I|=k} q_I e^{-d_I x}, \quad q_I = \frac{w_I}{w_0} \prod_{i \in I} M_i, \quad d_I = S_I - S_0.$$

The certificate checks $w_0 > 0$, all coefficient signs, and $d_I > 0$ for $I \neq I_0$; the leading exponent is identically zero rather than an interval subtraction of a quantity from itself. The moment factors range over the full boxes $1 \leq M_i \leq 1 + \frac{1}{2}(a_i/\Omega)^2$. Every exponent $j\Omega + (k-j)a_1 - S_0$ in (C.7) is certified positive over all rate boxes, so $\mathcal{E}_k$ decreases on $(0, \infty)$. The large value of Ω makes this permutation bound sufficient; no complementary-minor enumeration is needed.

The compact interval. Centered Taylor enclosures reduce cancellation in the finite sums. For a box $[l, u]$ with midpoint z and half-width h , set $V_j(x) = \sum_i c_i (-r_i)^j e^{-a_i x}$ with $r_i = a_i/S$ as in (C.5). Then

$$(C.12) \quad V_j([l, u]) \subseteq V_j(z) + S V_{j+1}(z)[-h, h] + [-A_j, A_j], \quad A_j = \frac{S^2 h^2}{2} \sum_i |c_i| r_i^{j+2} e^{-a_i l}.$$

Add the symmetric interval of radius bS^2C_{j+2} from (C.5), for $0 \leq j \leq 18$. On $[0, 1/250]$, these intervals form a ten-by-ten Hankel matrix. Interval Gaussian elimination accepts a box only if its successive pivot signs are $(-1)^j$, $j = 0, \dots, 9$, with zero excluded, and each division is made only after that pivot's sign is certified. The accepted signs give all ten initial signed determinants. On $[1/250, 1/10]$, use the corresponding Taylor enclosure of (C.11),

$$F_k([l, u]) \subseteq F_k(z) + F'_k(z)[-h, h] + [-A, A], \quad A = \frac{h^2}{2} \sum_{|I|=k} |q_I| d_I^2 e^{-d_I l},$$

and accept a box only if its lower bound exceeds the upper bound of $\mathcal{E}_k(l)$ for every $2 \leq k \leq 10$; monotonicity then controls the error throughout the box. Order one is positive analytically. The interval arithmetic includes every admissible rate and moment value, hence every residual law in the lemma. Each rejected interval is bisected at its exact rational midpoint, and an undecided comparison never counts as positive. The resulting covers are

interval	accepted subintervals	maximum depth
$[0, 1/250]$	388	11
$[1/250, 1/10]$	34	9

The closed boxes cover both endpoints without gaps, and each enclosure holds throughout its box, so the result is uniform rather than a grid test. Symmetry supplies the negative half-line.

The unbounded tails. For $x \geq 1/10$, retain the leading term of F_k , whose moment product is at least one, and discard all other positive terms. Bounding the negative terms at $1/10$ gives

$$F_k(x) \geq m_k := 1 - \sum_{w_I < 0} \frac{|w_I|}{w_0} \left(\prod_{i \in I} \overline{M}_i \right) e^{-(S_I - S_0)/10}, \quad \overline{M}_i = 1 + \frac{1}{2}(a_i/\Omega)^2.$$

For each $2 \leq k \leq 10$, the certificate gives $m_k > 0.18$ and $\mathcal{E}_k(1/10)/m_k < 10^{-3}$; the computed ratios are in fact below 10^{-1000} . Both adverse contributions decrease beyond $1/10$, so the determinant stays positive on the entire unbounded interval. The certificate establishes order ten for the bilateral density; it gives no order-uniform assertion.