

Stability and Convergence of a Regularized Immersed Boundary Method under Dynamic Cartesian Refinement. Part I: Fixed Geometry

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Abstract

We analyze a conforming variational immersed boundary discretization of the two-dimensional unsteady Stokes equations with a fixed interface. Exactly divergence-free velocities are obtained from Cartesian bicubic Hermite streamfunctions; regularized surface moments impose the boundary condition, and backward Euler time stepping uses constrained orthogonal transfer. Arbitrary local fine-scale enrichments may be added or removed at every step while the complete coarse space is retained. For homogeneous interface data, a common coarse comparison family yields an exact energy identity, a physical no-slip defect of order $\sqrt{\varepsilon} + \sqrt{\ell}$, and velocity convergence without a restriction coupling the time step to the number of mesh changes. We extend this analysis to a disk with prescribed time-dependent angular velocity. A coarse lifting reproduces rigid rotation exactly throughout every regularization support, independently of the refinement pattern and surface partition. Subtracting this lifting reduces the driven problem to the homogeneous constraint spaces with an explicitly controlled forcing perturbation. This gives driven stability, physical slip control, and a quantitative energy error estimate without assuming a uniform multiplier inf-sup condition. The method assumes exact integration and is not a standard staggered-grid AMR scheme. Manufactured unsteady stationary and rotating-disk calculations test the implementation and its quadrature sensitivity. General changes of interface geometry and fluid-driven rigid-body motion remain outside the analysis.

Keywords: immersed boundary method; dynamic mesh refinement; Stokes equations; prescribed rotation; constrained projection; regularization; convergence.

1 Introduction and precise scope

Regularized immersed boundary (IB) methods transfer information between a background fluid discretization and an independently represented interface. Adaptive mesh refinement introduces a second transfer: the fluid state must be represented in a different discrete space

after refinement or coarsening. Stability of either transfer in isolation does not settle the accuracy of their composition. In particular, a small bulk L^2 error may remain appreciable after averaging across a thin regularization layer.

The relevant ingredients have substantial prior theory. Peskin’s formulation supplies the regularized coupling structure [1]. Rigorous IB convergence theory includes Mori’s velocity estimates for periodic Stokes flow and a small-amplitude dynamic problem [2], the analysis of discrete-delta properties and local accuracy by Liu and Mori [3], and their stationary Stokes velocity and pressure estimates in L^p [4]. More recently, Li, Pan, and Ruiz-Álvarez analyzed stationary Stokes discretizations with prescribed outer boundary velocity [5]. In that setting the singular interface force is given; here the interface reaction is determined through regularized velocity constraints. These works establish essential IB accuracy results, but do not supply the changing-space argument used below.

Prescribed rigid motion in a background-domain multiplier formulation also has a long history [6]. Constraint formulations for rigid bodies and their mobility operators were studied by Kallemov et al. [7]; their reported implementation used uniform grids. Dynamic-mesh Stokes error analysis, including errors introduced by Helmholtz and Stokes projections, predates the present work [8]. General stable transfer operators for dynamic AMR were analyzed by Lundquist et al. [9], and projection jumps also appear in analyses of remeshed moving-domain problems [10]. Neither projection contractivity nor the observation that remeshing errors can accumulate is a new claim here.

The neighboring adaptive fictitious-domain literature is equally important. Berrone et al. [11] proved optimal adaptive convergence for an elliptic fictitious-domain formulation with an unknown boundary multiplier. Heltai and Lei developed regularization error estimates [12] and an adaptive analysis for regularized forcing data [13]. Their setting emphasizes that the regularization width and the bulk mesh size must be tracked separately. On the algorithmic side, adaptive IB methods go back at least to Roma, Peskin, and Berger [14]. Zeng, Bhalla, and Shen developed a DLM–IB framework on dynamically adaptive collocated grids, including subcycling and non-subcycling time advancement [15]; IB, multiple resolution levels, and lattice Green functions have also been combined [16]. The present analysis is for a different conforming realization and is not a convergence proof for those implementations. It does not claim that IB with AMR is new.

Computational IB studies also motivate distinguishing boundary enforcement from bulk flow accuracy. He, Yang, Sotiropoulos, and Shen [17] introduced an explicit immersed pressure boundary condition to mitigate spurious penetration through thin flexible structures in two-phase flow. For fixed geometries, Yang et al. [18] coupled large-eddy simulation with IB for a backward-facing step, while Chen et al. [19] used a level-set-based IB method to study airfoil flow transition and vortex evolution. He, Liu, and Shen [20] resolved flexible-stem canopy flows with IB, and Zheng et al. [21] studied a prescribed flapping plate near a free surface using the two-phase IB framework. These studies provide methodological and application context for accurate interface treatment; their turbulent, multiphase, or deforming-body configurations are not covered by the present Stokes analysis. In particular, their computational results are not used as analytical premises for our convergence proof.

Our result concerns a deliberately specified conforming method. The main construction is a family of coarse, divergence-free comparison functions that vanishes on a collar containing every regularization support used during the computation. These functions satisfy all discrete boundary moments exactly, even when the regularization width and surface partition change. Orthogonal transfer preserves their inner products. We use this fact to obtain estimates with no factor counting refinement or coarsening events. The approximation is constructed, rather than postulated through a mesh-uniform constraint inf–sup assumption.

There are three restrictions with practical consequences. First, the Cartesian velocity space is generated by C^1 streamfunctions and is not a MAC space. Second, boundary conditions are variational moments with exact integrals, not point-marker equations evaluated by a quadrature rule. Third, the interface geometry is fixed. The stationary-body analysis is extended to prescribed rotation of a circular disk, whose boundary as a geometric set does not move. The common collar construction does not provide the corresponding argument for translation or rotation of a noncircular body. These restrictions make it possible to separate a complete theorem from implementation-specific questions that remain open.

The prescribed-rotation extension is a fixed-geometry result, not a general moving-body theorem. We construct a common discrete lifting from a cutoff quadratic streamfunction, so that the nonzero regularized moments are feasible for every admissible mesh configuration. The transformed forcing retains both the angular-acceleration term and the endpoint viscous lifting term. The resulting convergence proof uses the homogeneous theory with a vanishing forcing perturbation. The background-domain extension includes prescribed interior forcing to represent rigid rotation; no conclusion about a physical torque is inferred from an uncorrected interface multiplier.

The analysis does not prescribe an error-based marking rule. It permits arbitrary choices of the admissible local enrichments, and therefore concerns stability and convergence under dynamic adaptation, not adaptive optimality. The spatial rate is expressed in the size of the retained coarse grid. Refining a small patch while holding the coarse grid fixed is not a convergence sequence covered by our limit theorem.

2 Continuous problem and regularized constraints

Let $\Omega = (0, 1)^2$. Let $B \Subset \Omega$ be a simply connected body with a connected C^5 boundary Γ , and put $F = \Omega \setminus \overline{B}$. The smoothness assumption is convenient for the quantitative collar construction; weaker assumptions suffice for several intermediate results. Define

$$U = \{v \in H_0^1(\Omega)^2 : \operatorname{div} v = 0\}, \quad \|v\|_U = \|\nabla v\|_{L^2(\Omega)}, \quad (1)$$

$$V = \{v \in U : \gamma_\Gamma v = 0\}, \quad H = \overline{V}^{L^2(\Omega)^2}, \quad (2)$$

$$a(v, w) = \nu (\nabla v, \nabla w)_{L^2(\Omega)}, \quad \nu > 0. \quad (3)$$

We write $\|\cdot\|_0$ for the bulk L^2 norm. By Poincaré's inequality, $\|\cdot\|_U$ is equivalent to the H^1 norm on U .

Given $u_0 \in H$ and $f \in L^2(0, T; U')$, the target problem is

$$u \in L^2(0, T; V) \cap C([0, T]; H), \quad \langle \partial_t u, v \rangle + a(u, v) = \langle f, v \rangle \quad (v \in V), \quad u(0) = u_0. \quad (4)$$

The derivative belongs to $L^2(0, T; V')$. Applying the coercive evolution-form theorem of J.-L. Lions to $V \subset H \subset V'$ gives a unique solution; see Brezis [22, Theorem 10.9]. The Hilbert-triple chain rule for variational evolution equations [23, Chapter 3] then gives the energy identity

$$\frac{1}{2} \|u(t)\|_0^2 + \int_0^t a(u, u) \, ds = \frac{1}{2} \|u_0\|_0^2 + \int_0^t \langle f, u \rangle \, ds. \quad (5)$$

This is a background-domain description. To represent a rigid solid rather than a second interior fluid, take $u_0 = 0$ in B and require f to vanish on tests supported in B . The interior subproblem then has zero data and its solution is zero. In particular, a bulk force supported in F , extended by zero into B , has the required property. The pressure is eliminated by testing in U ; the theory below does not establish a pressure estimate.

Fix a nonnegative, even $\rho \in C_c^\infty(\mathbb{R}^2)$ with $\int \rho = 1$ and support in the unit ball. Set $\rho_\varepsilon(x) = \varepsilon^{-2} \rho(x/\varepsilon)$. If Ev is the zero extension of $v \in H_0^1(\Omega)^2$, define

$$J_\varepsilon v = \gamma_\Gamma(\rho_\varepsilon * Ev), \quad b_\varepsilon(v, \mu) = \int_\Gamma \mu \cdot J_\varepsilon v \, ds. \quad (6)$$

The force operator is the adjoint of this variational interpolation, retaining the power-compatible coupling of the IB formulation [1]. Kernel moment conditions and their consequences for discrete IB accuracy are studied in Liu and Mori [3]; our operator is instead an exact continuum convolution. No sampling or quadrature operator is included in (6).

Let $\mathcal{P}$ partition Γ into arclength panels, and let ℓ be the largest panel length. Let $\Lambda_{\mathcal{P}}$ be the space of vector-valued panelwise constants and $\Pi_{\mathcal{P}}$ the $L^2(\Gamma)^2$ orthogonal projector onto it. The constraint is

$$b_\varepsilon(v_h, \mu) = 0 \quad \text{for all } \mu \in \Lambda_{\mathcal{P}}, \quad \text{equivalently } \Pi_{\mathcal{P}} J_\varepsilon v_h = 0. \quad (7)$$

The surface panels need not coincide with bulk cell intersections. Throughout a run the widths ε , the panels, and the local refinement patches may change. Their maximal widths over that run will be denoted by $\bar{\varepsilon}$ and $\bar{\ell}$.

3 A physical interface-defect estimate

Lemma 3.1 (Uniform regularized trace and moment defect). *For sufficiently small $\varepsilon > 0$ and $v \in U$,*

$$\|J_\varepsilon v\|_{H^{1/2}(\Gamma)} \leq C \|v\|_U, \quad (8)$$

$$\|\gamma_\Gamma v - J_\varepsilon v\|_{L^2(\Gamma)} \leq C \sqrt{\varepsilon} \|v\|_U. \quad (9)$$

If (7) holds, then

$$\|\gamma_\Gamma v\|_{L^2(\Gamma)} \leq C(\sqrt{\varepsilon} + \sqrt{\ell}) \|v\|_U. \quad (10)$$

The constants depend on the fixed geometry and kernel, not on the bulk mesh, panel positions, or the number of panels.

Proof. Zero extension is bounded from $H_0^1(\Omega)$ into $H^1(\mathbb{R}^2)$ [24, Theorems 3.18–3.19]. Young’s convolution inequality, also used in Heltai and Lei [12, Lemma 3], gives uniform H^1 boundedness of convolution. The trace theorem [24, Theorem 3.10(iii)], applied on a fixed neighborhood of Γ , proves (8). For $w = Ev - \rho_\varepsilon * Ev$, the first-order mollification bound [12, Lemma 4, equation (8), with $s = 1$] and convolution boundedness give

$$\|w\|_{L^2(\mathbb{R}^2)} \leq C\varepsilon \|v\|_U, \quad \|w\|_{H^1(\mathbb{R}^2)} \leq C \|v\|_U.$$

The multiplicative trace inequality yields the following bound on a fixed neighborhood of Γ . One obtains this version by transferring the face inequality of Ern and Guermond [24, Lemma 12.15], with $p = 2$, through fixed smooth charts and a finite partition of unity; chart derivatives are absorbed into the H^1 norm:

$$\|\gamma_\Gamma w\|_{L^2(\Gamma)}^2 \leq C \|w\|_{L^2(\mathbb{R}^2)} \|w\|_{H^1(\mathbb{R}^2)} \leq C\varepsilon \|v\|_U^2.$$

This proves (9).

The panel estimate is the fractional Poincaré inequality [24, Lemma 3.26]. For completeness, its direct proof here is short: on an arclength interval I of length ℓ_I , the panel average z_I satisfies

$$\int_I |z - z_I|^2 \, ds = \frac{1}{2\ell_I} \int_I \int_I |z(s) - z(t)|^2 \, ds \, dt \leq \frac{\ell_I}{2} \int_I \int_I \frac{|z(s) - z(t)|^2}{|s - t|^2} \, ds \, dt.$$

Summing over panels, with the equivalent arclength definition of the surface fractional norm, proves

$$\|(I - \Pi_{\mathcal{P}})z\|_{L^2(\Gamma)} \leq C\sqrt{\ell} \|z\|_{H^{1/2}(\Gamma)}.$$

Under (7), $J_\varepsilon v = (I - \Pi_{\mathcal{P}})J_\varepsilon v$. Combine this observation with (8), (9), and the triangle inequality. $\square$

Estimate (10) concerns the physical trace of the reconstructed conforming velocity, not a discrete residual that vanishes by definition. No multiplier inf-sup estimate is used. The estimate is useful in a time-integrated norm: viscous dissipation bounds $\int \|u_h\|_U^2$, but does not by itself bound the gradient at each isolated remeshing instant.

The distinction between a satisfied discrete boundary equation and physical impermeability is also relevant to the spurious-penetration issue addressed computationally by He et al. [17]. Their remedy acts through an immersed pressure boundary condition in a two-phase finite-difference solver. Here (10) controls the full velocity trace by regularized moments and viscous energy; it is a different mechanism and does not analyze that pressure-boundary algorithm.

4 Cartesian spaces and a common constrained approximation family

4.1 A concrete two-level conforming construction

Let $h = 1/M$, where M is a positive integer, and let $\mathcal{T}_h$ be the uniform partition of Ω into squares of side h . On a rectangle, write $\mathbb{Q}_3$ for polynomials of degree at most three in each variable. Let $S_h \subset H_0^2(\Omega)$ be the clamped Cartesian bicubic Hermite space. Its local degrees of freedom at each vertex are the value, the two first derivatives, and the mixed derivative ∂_{xy} . Matching these degrees of freedom gives a globally C^1 function. The boundary degrees of freedom are set to zero, so that both the function and its normal derivative vanish on $\partial\Omega$. This is the standard Cartesian Bogner–Fox–Schmit construction; see, for example, Burman et al. [25] for its tensor-product description and interpolation estimates.

For a coarse cell $K \in \mathcal{T}_h$, divide K into four squares of side $h/2$. Define $B_h(K)$ to consist of the C^1 piecewise bicubics on this subdivision whose value and gradient vanish on ∂K . Extending such a function by zero gives an element of $H_0^2(\Omega)$. Equivalently, all Hermite degrees of freedom at the eight boundary vertices are zero and the four degrees of freedom at the center vertex are free. Hence $\dim B_h(K) = 4$. For any set $\mathcal{A} \subset \mathcal{T}_h$, put

$$S_{h,\mathcal{A}} = S_h + \sum_{K \in \mathcal{A}} B_h(K), \quad X_{h,\mathcal{A}} = \text{curl } S_{h,\mathcal{A}}, \quad \text{curl } \psi = (\partial_y \psi, -\partial_x \psi). \quad (11)$$

Every $X_{h,\mathcal{A}}$ is a finite-dimensional subspace of U . Incompressibility holds exactly, since $\text{div curl } \psi = 0$ in distributions for $\psi \in H^2$. The zero boundary gradient of the streamfunction gives the outer homogeneous velocity condition.

This construction retains the full coarse space and adds local fine-grid degrees of freedom. All the spaces are contained in the uniform $h/2$ master space, but only cells in $\mathcal{A}$ carry the additional center degrees of freedom. Adding or deleting an arbitrary cell from $\mathcal{A}$ is a genuine local refinement or coarsening of this specified conforming space. The zero value and gradient of each local enrichment on the coarse-cell boundary prevent a conformity defect at a coarse–fine interface. We do not identify this space with the full space of every possible hanging-node implementation.

For a regularization width ε and a panel space $\Lambda_{\mathcal{P}}$, define

$$Z_{h,\mathcal{A},\varepsilon,\mathcal{P}} = \{v_h \in X_{h,\mathcal{A}} : b_\varepsilon(v_h, \mu) = 0 \text{ for every } \mu \in \Lambda_{\mathcal{P}}\}. \quad (12)$$

These are exact surface-moment constraints. Linear independence of their rows is neither assumed nor needed for the velocity approximation arguments below.

4.2 A family present in every configuration

Let $\bar{\varepsilon}$ be an upper bound for every regularization width in one run. Define the union of coarse cells

$$\begin{aligned} D_{h,\bar{\varepsilon}} &= \bigcup \{K \in \mathcal{T}_h : K \cap \{x : \text{dist}(x, \Gamma) \leq \bar{\varepsilon}\} \neq \emptyset\}, \\ K_{h,\bar{\varepsilon}} &= \{v_h \in \text{curl } S_h : v_h = 0 \text{ on } D_{h,\bar{\varepsilon}}\}. \end{aligned} \quad (13)$$

The equalities defining $K_{h,\bar{\varepsilon}}$ are polynomial identities on specified coarse cells, and therefore define a finite-dimensional linear space. They impose zero *velocity*; they do not impose zero streamfunction on the immersed collar. The latter would unnecessarily exclude circulation modes.

If $0 < \varepsilon \leq \bar{\varepsilon}$, every kernel support centered on Γ lies inside $D_{h,\bar{\varepsilon}}$. Thus

$$K_{h,\bar{\varepsilon}} \subset V, \quad K_{h,\bar{\varepsilon}} \subset Z_{h,\mathcal{A},\varepsilon,\mathcal{P}} \quad \text{for every } \mathcal{A}, \mathcal{P}, \quad 0 < \varepsilon \leq \bar{\varepsilon}. \quad (14)$$

In particular, an orthogonal transfer onto any of these constraint spaces fixes every element of $K_{h,\bar{\varepsilon}}$. The next result shows that this common family is large enough to approximate the physical solution space.

Lemma 4.1 (Common recovery family). *Let $h_m \rightarrow 0$ and $\bar{\varepsilon}_m \rightarrow 0$. For each $w \in V$ there exist $w_m \in K_{h_m,\bar{\varepsilon}_m}$ such that*

$$\|w_m - w\|_U \rightarrow 0. \quad (15)$$

The sequence w_m is independent of the refinement sets, the panel partitions, and the individual regularization widths used during run m , provided those widths do not exceed $\bar{\varepsilon}_m$. No relation between h_m and $\bar{\varepsilon}_m$ is required.

Proof. The restrictions of w to B and F belong to the zero-trace solenoidal H^1 spaces on these domains. The standard solenoidal density theorem on bounded Lipschitz domains, applied separately in the two domains, gives smooth divergence-free fields compactly supported in B and F that converge to these restrictions in H^1 ; see Galdi [26, Section III.4]. Their zero extensions and sums form a sequence $w^{(j)} \in C_c^\infty(\Omega)^2$ converging to w in U . For each j , $w^{(j)}$ vanishes on a neighborhood of Γ and is globally divergence free. This domainwise construction is valid in the multiply connected exterior region; it does not assert that the streamfunction has the same constant on its different boundary components.

For a fixed such field $w^{(j)}$, define explicitly

$$\psi^{(j)}(x, y) = \int_0^y w_1^{(j)}(x, s) ds.$$

Since $\text{div } w^{(j)} = 0$ and $w_2^{(j)}(x, 0) = 0$, one has $\partial_y \psi^{(j)} = w_1^{(j)}$ and $\partial_x \psi^{(j)} = -w_2^{(j)}$. The zero outer velocity implies $\psi^{(j)} = 0$ and $\nabla \psi^{(j)} = 0$ on the outer square. On the connected collar of Γ , $\psi^{(j)}$ is a constant, which need not be zero. Thus $\psi^{(j)}$ is a smooth clamped streamfunction.

Let $I_h \psi^{(j)} \in S_h$ be its bicubic Hermite interpolant. The Cartesian Hermite interpolation estimate, for example Burman et al. [25, equation (52), with $m = 2$], gives

$$\|I_h \psi^{(j)} - \psi^{(j)}\|_{H^2(\Omega)} \leq Ch^2 \|\psi^{(j)}\|_{H^4(\Omega)}. \quad (16)$$

If $w^{(j)}$ vanishes whenever $\text{dist}(x, \Gamma) < r_j$, and $\bar{\varepsilon}_m + \sqrt{2}h_m < r_j$, every coarse cell in $D_{h_m,\bar{\varepsilon}_m}$ lies in this constant-streamfunction collar. Hermite interpolation reproduces the constant

exactly on such a cell. Consequently, $\text{curl } I_{h_m} \psi^{(j)} \in K_{h_m, \bar{\varepsilon}_m}$ for all sufficiently large m , and it converges to $w^{(j)}$ in U .

Choose $j = j(m)$ tending to infinity sufficiently slowly that both this collar inclusion and the interpolation convergence hold, while $\|w^{(j(m))} - w\|_U \rightarrow 0$. The triangle inequality proves (15). $\square$

The collar is used to construct comparison functions. The actual discrete solution is constrained only by (12); it is not required to vanish on $D_{h, \bar{\varepsilon}}$.

4.3 A quantitative collar construction

For a velocity w that is smooth separately in the body and the fluid, write

$$\mathcal{M}(w) = \|w\|_{W^{3,\infty}(B)} + \|w\|_{W^{3,\infty}(F)}.$$

All restrictions and traces in the next lemma are understood in the Sobolev sense. A globally H^1 velocity with a jump in its first derivatives across Γ is allowed.

Lemma 4.2 (Quantitative common approximation). *There are constants $c_0, C > 0$, depending only on the fixed geometry, such that the following holds. If $h + \bar{\varepsilon} < c_0$, there is a linear operator $Q_{h, \bar{\varepsilon}}$ from $V \cap [W^{3,\infty}(B)^2 \times W^{3,\infty}(F)^2]$ into $K_{h, \bar{\varepsilon}}$ satisfying*

$$\|Q_{h, \bar{\varepsilon}} w - w\|_{H^1(\Omega)} \leq C \sqrt{h + \bar{\varepsilon}} \mathcal{M}(w). \quad (17)$$

For fixed $h, \bar{\varepsilon}$, this operator commutes with the weak time derivative whenever w belongs to the Bochner H^1 space in the stated piecewise regularity class, with $w(t), \partial_t w(t) \in V$ almost everywhere. Hence the same estimate applies almost everywhere to the time derivative of the approximation error, with $\mathcal{M}(\partial_t w)$ on the right.

Proof. The signed-distance tubular coordinates used below are standard for a smooth fixed interface; see also the geometric setup in Burman et al. [25]. The flattening and width-dependent estimates are derived here. As in the preceding proof, set $\psi(x, y) = \int_0^y w_1(x, s) ds$. Then $\text{curl } \psi = w$, $\psi \in H_0^2(\Omega)$, and ψ is separately $W^{4,\infty}$ on B and F . Its piecewise $W^{4,\infty}$ norm is bounded by $C\mathcal{M}(w)$. Since $\nabla \psi = 0$ on the connected interface, $\psi|_\Gamma = c$ for one scalar c . Taylor's formula along the interface normals gives, on either side and in a fixed tubular neighborhood,

$$|\psi - c| \leq Cd^2 \mathcal{M}(w), \quad |\nabla \psi| \leq C|d| \mathcal{M}(w), \quad (18)$$

where d is the signed distance to Γ .

Choose a fixed smooth cutoff χ with $\chi(s) = 1$ for $|s| \leq 1$ and $\chi(s) = 0$ for $|s| \geq 2$. Put

$$\delta = 4(\bar{\varepsilon} + \sqrt{2}h), \quad \psi_\delta = \psi - \chi(d/\delta)(\psi - c) \quad (19)$$

inside the tubular neighborhood, and set $\psi_\delta = \psi$ elsewhere. Choose c_0 sufficiently small that the 2δ collar is contained in that neighborhood and disjoint from $\partial\Omega$. This preserves the clamped outer condition. The modified streamfunction equals c on $|d| \leq \delta$, and thus removes the original interface derivative jumps completely: $\psi_\delta \in H^4(\Omega) \cap H_0^2(\Omega)$.

The difference $\psi_\delta - \psi$ is supported in a set of area $O(\delta)$. By (18), the zeroth, first, and second derivatives of this difference are bounded pointwise by $C\delta^2 \mathcal{M}(w)$, $C\delta \mathcal{M}(w)$, and $C\mathcal{M}(w)$, respectively. Consequently,

$$\|\psi_\delta - \psi\|_{H^2(\Omega)} \leq C\sqrt{\delta} \mathcal{M}(w). \quad (20)$$

On the transition region $\delta < |d| < 2\delta$, the fourth derivatives of the cutoff product are bounded by $C\delta^{-2}\mathcal{M}(w)$. Indeed, the terms with four and three cutoff derivatives are multiplied by $\psi - c = O(\delta^2)$ and $\nabla\psi = O(\delta)$, respectively; the terms with two cutoff derivatives multiply bounded second derivatives of ψ . The remaining terms are no larger for sufficiently small δ . Derivatives of the signed-distance function are bounded on the fixed tubular neighborhood. Since this transition region also has area $O(\delta)$, we obtain

$$\|\psi_\delta\|_{H^4(\Omega)} \leq C(1 + \delta^{-3/2})\mathcal{M}(w). \quad (21)$$

Define $Q_{h,\bar{\varepsilon}}w = \text{curl } I_h\psi_\delta$. A coarse cell meeting the $\bar{\varepsilon}$ collar is contained in the collar of width $\bar{\varepsilon} + \sqrt{2}h < \delta$, where $\psi_\delta = c$. Exact constant reproduction therefore proves $Q_{h,\bar{\varepsilon}}w \in K_{h,\bar{\varepsilon}}$. Combining (16), (20), and (21) gives

$$\|Q_{h,\bar{\varepsilon}}w - w\|_{H^1(\Omega)} \leq C(\sqrt{\delta} + h^2(1 + \delta^{-3/2}))\mathcal{M}(w).$$

Because $\delta \geq 4\sqrt{2}h$ and $\delta \leq C(h + \bar{\varepsilon})$, this is (17).

Finally, the construction $w \mapsto \psi$, extraction of its constant trace c , multiplication by the fixed cutoff, and Hermite interpolation are all linear. For fixed $h, \bar{\varepsilon}$, the resulting operator is bounded from the stated piecewise regularity class, equipped with the norm $\mathcal{M}$, into $H^1(\Omega)^2$; this follows also from (17) and the continuous inclusion of that class in $H^1(\Omega)^2$. The geometry, δ , and the coarse grid are fixed throughout the run. A bounded linear map commutes with Bochner weak differentiation, which proves the last assertion. $\square$

The error in (17) is expressed in the retained coarse mesh size. It proves a robust approximation property, rather than optimality of an adaptive marking strategy. It remains valid when the regularization support intersects coarse-fine boundaries because the comparison function is already zero on the entire support. No uniform estimate for the boundary multiplier is inferred from this construction.

5 Time stepping and exact energy balance

Choose a time partition $0 = t_0 < \dots < t_N = T$, with $\tau_n = t_n - t_{n-1}$. At time level n , choose any admissible set of refined coarse cells $\mathcal{A}^n$, any $\varepsilon_n > 0$, and any surface panel partition $\mathcal{P}^n$. Let

$$Z^n = \{v_h \in X_{h,\mathcal{A}^n} : \Pi_{\mathcal{P}^n} J_{\varepsilon_n} v_h = 0\}, \quad P^n : L^2(\Omega)^2 \longrightarrow Z^n \quad (22)$$

be the exact bulk L^2 orthogonal projector. All spaces use the same physical integrals and the same bilinear form a ; in particular the mass inner product is not changed by mass lumping.

Set $u^0 = P^0 u_0$ and $f^n = \tau_n^{-1} \int_{t_{n-1}}^{t_n} f(t) dt$. The scheme is

$$u^n \in Z^n, \quad \left((u^n - P^n u^{n-1}) / \tau_n, v_h \right) + a(u^n, v_h) = \langle f^n, v_h \rangle \quad (v_h \in Z^n). \quad (23)$$

All bulk inner products without subscripts are in $L^2(\Omega)^2$. Backward Euler with projection of the old velocity onto the new space is an established dynamic-mesh Stokes construction; compare Brenner et al. [8, Algorithm 2.2]. Here the projector also enforces the regularized interface moments. Because the mass form plus $\tau_n a$ is positive definite on Z^n , this problem has a unique velocity, whether or not the surface constraints are linearly independent.

Equivalently, one may solve on $X_{h,\mathcal{A}^n}$ with a multiplier in $\Lambda_{\mathcal{P}^n}$. In finite dimensions, the residual functional of the constrained minimizer vanishes on the kernel of the constraint map and therefore belongs to the range of its transpose. A multiplier exists, although it may not

be unique. This observation is not a claim of uniform conditioning or convergence of that multiplier.

Testing by the current velocity and using dual coupling to cancel constraint terms are established energy arguments; see, for example, Boffi et al. [27, Proposition 3] in a distributed-multiplier elastic FSI setting. We use the same elementary energy principle, not their elastic-body theorem. The identity below additionally keeps the loss from transfer between our changing constraint spaces; its contractive-transfer mechanism is consistent with Lundquist et al. [9, Section 2].

Proposition 5.1 (Energy identity). *For every step of (23),*

$$\|u^n\|_0^2 - \|u^{n-1}\|_0^2 + \|u^n - u^{n-1}\|_0^2 + 2\tau_n a(u^n, u^n) = 2\tau_n \langle f^n, u^n \rangle. \quad (24)$$

Moreover,

$$\|u^n - u^{n-1}\|_0^2 = \|u^n - P^n u^{n-1}\|_0^2 + \|(I - P^n)u^{n-1}\|_0^2. \quad (25)$$

Consequently,

$$\max_{0 \leq n \leq N} \|u^n\|_0^2 + \sum_{n=1}^N \tau_n \|u^n\|_U^2 + \sum_{n=1}^N \|u^n - u^{n-1}\|_0^2 \leq C_\nu \left(\|u_0\|_0^2 + \|f\|_{L^2(0,T;U')}^2 \right). \quad (26)$$

Proof. Orthogonality gives $(P^n u^{n-1}, u^n) = (u^{n-1}, u^n)$. Test (23) with u^n and apply

$$2(x - y, x) = \|x\|_0^2 - \|y\|_0^2 + \|x - y\|_0^2.$$

This is (24). The two terms $u^n - P^n u^{n-1} \in Z^n$ and $P^n u^{n-1} - u^{n-1} \perp Z^n$ are orthogonal, proving (25). Sum (24), use Young's inequality on the right-hand side, and use Jensen's inequality for the time averages of f . $\square$

Corollary 5.2 (Time-integrated physical no-slip defect). *Let $U_h(t) = u^n$ on $(t_{n-1}, t_n]$. Then*

$$\|\gamma_T U_h\|_{L^2(0,T;L^2(\Gamma))} \leq C_{\nu,\Gamma,\rho} (\sqrt{\bar{\varepsilon}} + \sqrt{\bar{\ell}}) \left(\|u_0\|_0 + \|f\|_{L^2(0,T;U')} \right). \quad (27)$$

Proof. Apply Lemma 3.1 on each time interval and then (26). $\square$

Lemma 5.3 (The common comparison cancels transfer). *If $w_h \in Z^n$ for every n , then*

$$(P^n u^{n-1}, w_h) = (u^{n-1}, w_h), \quad (u^n - u^{n-1}, w_h) = \tau_n \{ \langle f^n, w_h \rangle - a(u^n, w_h) \}. \quad (28)$$

If $z_h(t)$ takes values in a fixed subspace of every Z^n , then also $P^n z_h(t) = z_h(t)$ at all times. Thus no jump of the comparison function is introduced at a mesh change.

Proof. Both statements follow directly from the defining orthogonality of P^n . The collar construction in Section 4 supplies the comparison functions required by this lemma. $\square$

6 Convergence with arbitrarily frequent mesh changes

We index complete computations by m . The coarse mesh size is h_m ; the time partition, the selected refined cells, the regularization widths, and the surface partitions may all depend on m . Write

$$\bar{\varepsilon}_m = \max_{0 \leq n \leq N_m} \varepsilon_m^n, \quad \bar{\ell}_m = \max_{0 \leq n \leq N_m} \ell_m^n, \quad \tau_{\max,m} = \max_{1 \leq n \leq N_m} \tau_m^n.$$

The maxima include the choices used to define the initial projector. Let $U_m(t) = u_m^n$ on $(t_{n-1}, t_n]$ and $U_m(0) = u_m^0$. The following theorem is qualitative; it requires no regularity of an interface multiplier.

Theorem 6.1 (Strong velocity convergence). *Let $u_0 \in H$ and $f \in L^2(0, T; U')$. Suppose*

$$h_m + \bar{\varepsilon}_m + \bar{\ell}_m + \tau_{\max, m} \longrightarrow 0. \quad (29)$$

At every time level choose any admissible refinement patch and impose the exact moment constraints in (22). Let $u_m^0 = P_m^0 u_0$ and let u_m^n solve (23), with the time averages of f . Then, for the solution u of (4),

$$\|U_m - u\|_{L^2(0, T; U)} \longrightarrow 0, \quad \sup_{0 \leq t \leq T} \|U_m(t) - u(t)\|_0 \longrightarrow 0. \quad (30)$$

In addition, the total numerical loss tends to zero:

$$\sum_{n=1}^{N_m} \left\{ \|u_m^n - P_m^n u_m^{n-1}\|_0^2 + \|(I - P_m^n) u_m^{n-1}\|_0^2 \right\} \longrightarrow 0. \quad (31)$$

There is no restriction relating N_m or the number of patch changes to h_m , $\bar{\varepsilon}_m$, or $\bar{\ell}_m$, beyond (29).

Proof. Initial approximation and compactness. The common recovery property of Lemma 4.1 implies that, for every $w \in V$, there are w_m belonging to every Z_m^n with $w_m \rightarrow w$ in U . Since V is dense in H by definition, the best-approximation property of P_m^0 gives

$$\|u_m^0 - u_0\|_0 \longrightarrow 0. \quad (32)$$

Indeed, first approximate u_0 in L^2 by a fixed element of V , and then use its common recovery sequence as a competitor for P_m^0 .

The energy bound (26) gives, along a subsequence,

$$U_m \rightharpoonup \tilde{u} \quad \text{in } L^2(0, T; U), \quad U_m \overset{*}{\rightharpoonup} \tilde{u} \quad \text{in } L^\infty(0, T; L^2(\Omega)^2).$$

By Corollary 5.2, $\gamma_\Gamma U_m \rightarrow 0$ strongly in $L^2(0, T; L^2(\Gamma))$. The bounded trace operator therefore gives $\tilde{u} \in L^2(0, T; V)$.

The evolution equation from common test functions. Fix $w \in V$ and a common recovery sequence $w_m \rightarrow w$ in U . Define the continuous, piecewise affine scalar function g_m by its nodal values $g_m(t_n) = (u_m^n, w_m)$. Lemma 5.3 gives

$$g'_m(t) = \langle f_m(t), w_m \rangle - a(U_m(t), w_m) \quad \text{for a.e. } t, \quad (33)$$

where $f_m(t) = f_m^n$ on the n th time interval. The derivatives g'_m are uniformly bounded in $L^2(0, T)$, while $g_m(0) \rightarrow (u_0, w)$. Thus g_m is relatively compact in $C([0, T])$. Also,

$$\sup_{0 \leq t \leq T} |(U_m(t), w) - g_m(t)| \leq C \|w - w_m\|_0 + \sqrt{\tau_{\max, m}} \|g'_m\|_{L^2(0, T)} \longrightarrow 0. \quad (34)$$

Time averaging on partitions with maximum step tending to zero gives $f_m \rightarrow f$ strongly in $L^2(0, T; U')$. Passing to the limit in (33), first on a countable dense family in V and then by continuity, yields

$$\frac{d}{dt} (\tilde{u}(t), w) = \langle f(t), w \rangle - a(\tilde{u}(t), w), \quad (\tilde{u}(0), w) = (u_0, w) \quad (w \in V).$$

Consequently $\partial_t \tilde{u} \in L^2(0, T; V')$ and $\tilde{u}$ is the solution of (4). Uniqueness identifies $\tilde{u} = u$. It also implies, for every fixed $w \in V$,

$$\sup_{0 \leq t \leq T} |(U_m(t) - u(t), w)| \longrightarrow 0. \quad (35)$$

The same assertion holds for every $w \in H$, by density and the uniform bulk norm bound. The weak convergence and moment conclusions hold for the whole sequence, since every subsequence has the same possible limit.

Strong convergence in the dissipative norm. The terminal moment convergence implies

$$\|u(T)\|_0^2 \leq \liminf_m \|u_m^{N_m}\|_0^2. \quad (36)$$

This assertion does not require $u_m^{N_m}$ to lie in H : test against elements of the dense subset $V \subset H$ and take the supremum over unit bulk norm tests. Summing (24) gives

$$\frac{1}{2} \|u_m^{N_m}\|_0^2 + \int_0^T a(U_m, U_m) dt + \frac{1}{2} D_m = \frac{1}{2} \|u_m^0\|_0^2 + \int_0^T \langle f_m, U_m \rangle dt, \quad (37)$$

where $D_m = \sum_{n=1}^{N_m} \|u_m^n - u_m^{n-1}\|_0^2$. The right-hand side tends to $\frac{1}{2} \|u_0\|_0^2 + \int_0^T \langle f, u \rangle dt$: use (32), strong convergence of f_m , and weak convergence of U_m in $L^2(0, T; U)$. By (5), this limit is $\frac{1}{2} \|u(T)\|_0^2 + \int_0^T a(u, u) dt$. Weak lower semicontinuity for the dissipation, (36), and $D_m \geq 0$ therefore force

$$\int_0^T a(U_m, U_m) dt \longrightarrow \int_0^T a(u, u) dt, \quad \|u_m^{N_m}\|_0 \longrightarrow \|u(T)\|_0, \quad D_m \longrightarrow 0. \quad (38)$$

Since $a(v, v) = \nu \|v\|_U^2$, weak convergence together with norm convergence proves $U_m \rightarrow u$ in $L^2(0, T; U)$ by the Hilbert-space criterion in Brezis [22, Proposition 3.32]. Equation (25) then proves (31).

There is no hidden terminal component orthogonal to H . To see this explicitly, let Q_H be the bulk orthogonal projection onto H . Equation (35) gives $Q_H u_m^{N_m} \rightharpoonup u(T)$ in H . Hence

$$\|u(T)\|_0 \leq \liminf_m \|Q_H u_m^{N_m}\|_0 \leq \limsup_m \|Q_H u_m^{N_m}\|_0 \leq \lim_m \|u_m^{N_m}\|_0 = \|u(T)\|_0.$$

It follows that $Q_H u_m^{N_m} \rightarrow u(T)$ strongly and $\|(I - Q_H)u_m^{N_m}\|_0 \rightarrow 0$.

Uniform convergence in the bulk norm. Strong convergence in $L^2(0, T; U)$ implies

$$a(U_m, U_m) \longrightarrow a(u, u) \quad \text{in } L^1(0, T), \quad \langle f_m, U_m \rangle \longrightarrow \langle f, u \rangle \quad \text{in } L^1(0, T).$$

The corresponding cumulative integrals converge uniformly in their upper limits. The discrete energy identity at any node, after discarding the nonnegative accumulated loss, and the continuous energy identity therefore give

$$\limsup_m \max_n (\|u_m^n\|_0^2 - \|u(t_n)\|_0^2) \leq 0.$$

By continuity of $u : [0, T] \rightarrow H$ and $\tau_{\max, m} \rightarrow 0$, this upper bound also holds with arbitrary t and $U_m(t)$. The image $u([0, T])$ is a compact subset of H . Using a finite bulk-norm net for this image in (35) proves

$$\sup_{0 \leq t \leq T} |(U_m(t) - u(t), u(t))| \longrightarrow 0.$$

Finally, expand

$$\|U_m(t) - u(t)\|_0^2 = \|U_m(t)\|_0^2 - \|u(t)\|_0^2 - 2(U_m(t) - u(t), u(t)).$$

The preceding two bounds prove uniform bulk convergence. $\square$

Remark 6.2 (Why no remeshing count occurs). The cancellation uses a fixed family of comparison functions common to all spaces of one computation, not a bound for each mesh-transfer defect. This differs from the projection-difference terms in Brenner et al. [8, Theorems 3.1 and 3.4] and the remeshing contributions in Gawlik and Lew [10]. The dense family is supplied by the retained coarse grid and the collar construction. A collection of unrelated shifted grids need not have such a family. Thus Theorem 6.1 does not assert convergence for arbitrary stable interpolations or for arbitrary changing spaces.

7 A quantitative estimate for piecewise smooth solutions

We next give an a priori bound for a more regular solution. The estimate is in the retained coarse mesh size; it is not an adaptive optimality statement. The comparison-function and residual-energy strategy is standard; compare the dynamic-mesh Stokes error analysis in Brenner et al. [8, Section 3.1]. The specific additional ingredient here is one constructed comparison operator shared by all admissible spaces, so that its mesh-change residual vanishes exactly. Write

$$Y = W^{3,\infty}(B)^2 \times W^{3,\infty}(F)^2, \quad \|v\|_Y = \|v|_B\|_{W^{3,\infty}(B)} + \|v|_F\|_{W^{3,\infty}(F)}.$$

The notation $u \in H^1(0, T; Y)$ below means a Bochner absolutely continuous function whose value and time derivative have square-integrable Y norms. In particular u has a continuous representative with values in Y .

Theorem 7.1 (Coarse-grid energy error bound). *Assume the geometry and exact-integration discretization of the preceding sections. Suppose the target velocity satisfies*

$$u \in H^1(0, T; Y) \cap H^1(0, T; V).$$

Suppose also that there is $\lambda \in L^2(0, T; L^2(\Gamma)^2)$ such that, for almost every t ,

$$(\partial_t u(t), v) + a(u(t), v) + \int_{\Gamma} \lambda(t) \cdot \gamma_{\Gamma} v \, ds = \langle f(t), v \rangle \quad (v \in U). \quad (39)$$

This assumption specifies the regularity of the continuous reaction; it does not impose an inf-sup condition on the discrete multiplier. For sufficiently small $h_m + \bar{\varepsilon}_m$, the numerical velocity satisfies

$$\begin{aligned} & \max_{0 \leq n \leq N_m} \|u_m^n - u(t_n)\|_0 + \left(\sum_{n=1}^{N_m} \tau_m^n \|u_m^n - u(t_n)\|_U^2 \right)^{1/2} \\ & \leq C_{\nu, T, \Gamma, \rho} \left[\sqrt{h_m + \bar{\varepsilon}_m} (\|u\|_{L^\infty(0, T; Y)} + \|\partial_t u\|_{L^2(0, T; Y)}) \right. \\ & \quad \left. + (\sqrt{\bar{\varepsilon}_m} + \sqrt{\bar{\ell}_m}) \|\lambda\|_{L^2(0, T; L^2(\Gamma))} + \tau_{\max, m} \|\partial_t u\|_{L^2(0, T; U)} \right]. \end{aligned} \quad (40)$$

The constant is independent of the selected refined cells, their relative positions to Γ , the number of time steps, and the number of refinement or coarsening events. For fixed solution norms, the right-hand side is $O(\sqrt{h_m} + \sqrt{\bar{\varepsilon}_m} + \sqrt{\bar{\ell}_m} + \tau_{\max, m})$.

Proof. We suppress the computation index m . Let $L_h = Q_{h, \bar{\varepsilon}} : V \cap Y \rightarrow K_{h, \bar{\varepsilon}}$ be the fixed linear comparison operator constructed in Lemma 4.2, with $K_{h, \bar{\varepsilon}} \subset Z^n$ for every n . Put

$$z(t) = L_h u(t), \quad \eta(t) = u(t) - z(t), \quad e^n = u^n - z(t_n), \quad d = h + \bar{\varepsilon}.$$

The same, time-independent operator is used throughout this computation. The quantitative collar bound and its application to $\partial_t u$ give

$$\|\eta\|_{L^\infty(0, T; U)} \leq C\sqrt{d} \|u\|_{L^\infty(0, T; Y)}, \quad \|\partial_t \eta\|_{L^2(0, T; U)} \leq C\sqrt{d} \|\partial_t u\|_{L^2(0, T; Y)}. \quad (41)$$

Here differentiation commutes with L_h because it is a fixed linear map. This is precisely where using a different comparison function after each mesh change would introduce an additional term.

Let $\bar{u}^n$ and $\bar{\lambda}^n$ denote the exact time averages on (t_{n-1}, t_n) . Time averaging (39) and subtracting the resulting identity from (23), using $P^n z(t_{n-1}) = z(t_{n-1})$, gives

$$\left((e^n - P^n e^{n-1}) / \tau_n, v_h \right) + a(e^n, v_h) = \mathcal{R}_n(v_h) \quad (v_h \in Z^n), \quad (42)$$

where

$$\begin{aligned} \mathcal{R}_n(v_h) = & \left(\tau_n^{-1} \int_{t_{n-1}}^{t_n} \partial_t \eta(t) dt, v_h \right) + a(\eta(t_n), v_h) \\ & + a(\bar{u}^n - u(t_n), v_h) + \int_{\Gamma} \bar{\lambda}^n \cdot \gamma_{\Gamma} v_h ds. \end{aligned} \quad (43)$$

All four residuals have a specified origin: comparison differentiation, spatial comparison, time averaging, and the physical boundary reaction. There is no remeshing residual.

Use $\|v\|_a = \sqrt{a(v, v)} = \sqrt{\nu} \|v\|_U$, and let C_P be a Poincare constant for Ω . Lemma 3.1 gives $|\mathcal{R}_n(v_h)| \leq r_n \|v_h\|_a$, with

$$\begin{aligned} r_n = & \frac{C_P}{\sqrt{\nu}} \left\| \tau_n^{-1} \int_{t_{n-1}}^{t_n} \partial_t \eta dt \right\|_0 + \sqrt{\nu} \|\eta(t_n)\|_U \\ & + \sqrt{\nu} \|\bar{u}^n - u(t_n)\|_U + \frac{C_{\Gamma}}{\sqrt{\nu}} (\sqrt{\varepsilon_n} + \sqrt{\ell_n}) \|\bar{\lambda}^n\|_{L^2(\Gamma)}. \end{aligned} \quad (44)$$

Test (42) with e^n , use orthogonal-projection contractivity, and apply Young's inequality. This yields

$$\|e^n\|_0^2 - \|e^{n-1}\|_0^2 + \tau_n \|e^n\|_a^2 \leq \tau_n r_n^2.$$

Therefore

$$\max_n \|e^n\|_0^2 + \sum_n \tau_n \|e^n\|_a^2 \leq 2 \left(\|e^0\|_0^2 + \sum_n \tau_n r_n^2 \right). \quad (45)$$

The initialization is compatible with the common comparison: $e^0 = P^0 u_0 - z(0) = P^0 \eta(0)$, so $\|e^0\|_0 \leq \|\eta(0)\|_0$.

Jensen's inequality controls the averaged $\partial_t \eta$ and $\bar{\lambda}^n$ terms. For the time residual, the fundamental theorem of calculus and Cauchy–Schwarz give

$$\sum_n \tau_n \|\bar{u}^n - u(t_n)\|_U^2 \leq \tau_{\max}^2 \|\partial_t u\|_{L^2(0, T; U)}^2. \quad (46)$$

Indeed, on a single interval the norm of $u(t_n) - \bar{u}^n$ is at most $\sqrt{\tau_n} \|\partial_t u\|_{L^2(t_{n-1}, t_n; U)}$. Squaring the four terms of (44), using $(x_1 + \dots + x_4)^2 \leq 4 \sum_i x_i^2$, and then applying (41) proves

$$\begin{aligned} \sum_n \tau_n r_n^2 \leq & C_{\nu, \Gamma} [d\{T \|u\|_{L^\infty(0, T; Y)}^2 + \|\partial_t u\|_{L^2(0, T; Y)}^2\} + \tau_{\max}^2 \|\partial_t u\|_{L^2(0, T; U)}^2 \\ & + (\sqrt{\varepsilon} + \sqrt{\ell})^2 \|\lambda\|_{L^2(0, T; L^2(\Gamma))}^2]. \end{aligned}$$

Insert this into (45), take square roots, and use $u^n - u(t_n) = e^n - \eta(t_n)$ and Poincare's inequality. This proves (40). $\square$

Remark 7.2 (Nodal and continuous-time error norms). The same right-hand side as in (40), with a changed constant, controls $\|U_m - u\|_{L^2(0, T; U)}$, since

$$\sum_n \int_{t_{n-1}}^{t_n} \|u(t_n) - u(t)\|_U^2 dt \leq \tau_{\max, m}^2 \|\partial_t u\|_{L^2(0, T; U)}^2.$$

The first term in (40) is a maximum over time nodes. For the piecewise constant U_m , an $L^\infty(0, T; L^2)$ error estimate includes the additional bound $\sqrt{\tau_{\max, m}} \|\partial_t u\|_{L^2(0, T; L^2)}$. For a first-order bound on that interpolation term, one can assume $\partial_t u \in L^\infty(0, T; L^2)$. The qualitative uniform convergence in Theorem 6.1 does not require this stronger regularity.

8 Prescribed rotation with fixed immersed geometry

The position of an immersed interface need not change when the material on it moves. A disk rotating about its fixed center is a useful example: its Eulerian boundary remains fixed, but its no-slip velocity is nonzero and may vary in time. We prove an extension for this problem. The common homogeneous comparison family from Section 4 is retained, and a second, explicitly constructed coarse-grid function provides an exact lifting of every regularized boundary constraint. Prescribed rigid motion in fictitious-domain formulations predates this construction [6, 7]. The point here is the simultaneous feasibility of one coarse lift for every admitted refinement, regularization width and surface partition.

8.1 An exact common lifting

Let $B = \{x : |x - c| < R\} \Subset \Omega$ be a disk. Write

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad r(x) = J(x - c), \quad \omega \in H^1(0, T).$$

Thus the prescribed boundary velocity is $\omega(t)r$; positive ω means counterclockwise rotation. Choose $d_* > 0$ and $\chi \in C_c^5(\Omega)$ such that $\chi = 1$ on the open d_* -neighborhood of $\bar{B}$. Set

$$\psi(x) = -\frac{1}{2}\chi(x)|x - c|^2, \quad g = \text{curl } \psi, \quad g_h = \text{curl } I_h \psi. \quad (47)$$

The sign is consistent with $\text{curl } \psi = (\partial_y \psi, -\partial_x \psi)$: $g = r$ wherever $\chi = 1$. The cutoff and d_* are fixed independently of the mesh and of the regularization parameters. The stated C^5 regularity suffices for $\psi \in H^4$ and $g \in W^{3,\infty}$; a C^∞ cutoff is also admissible. This includes the piecewise polynomial C^5 cutoff used in the numerical verification.

Lemma 8.1 (Common exact rotation lift). *Suppose*

$$\bar{\varepsilon} + \sqrt{2}h < d_*. \quad (48)$$

Then $g_h \in \text{curl } S_h \subset X_{h,\mathcal{A}}$ for every admissible refinement patch, and

$$\begin{aligned} g_h &= r \quad \text{on } B \text{ and on every regularization support centered on } \Gamma, \\ \gamma_\Gamma g_h &= r, \quad J_\varepsilon g_h = r \quad (0 < \varepsilon \leq \bar{\varepsilon}), \end{aligned} \quad (49)$$

$$\|g_h - g\|_{H^1(\Omega)} \leq C_g h^2. \quad (50)$$

Consequently the nonempty affine constraint space is exactly

$$\omega_n g_h + Z^n = \{v_h \in X_{h,\mathcal{A}^n} : \Pi_{\mathcal{P}^n}(J_{\varepsilon_n} v_h - \omega_n r) = 0\}, \quad \omega_n = \omega(t_n). \quad (51)$$

These assertions require neither independence nor an inf-sup bound for the surface constraints.

Proof. Every coarse cell meeting the $\bar{\varepsilon}$ -neighborhood of Γ lies inside the $(\bar{\varepsilon} + \sqrt{2}h)$ -neighborhood of $\bar{B}$. The same is true of every coarse cell meeting B . On these cells, $\psi = -|x - c|^2/2$, and bicubic Hermite interpolation reproduces this quadratic polynomial exactly. This proves the first assertion and the physical trace equality. All relevant kernel supports are inside Ω , so zero extension does not change the convolution there. Normalization and evenness of ρ imply $\int \rho_\varepsilon(z) dz = 1$ and $\int z \rho_\varepsilon(z) dz = 0$. Hence, for $x \in \Gamma$,

$$J_\varepsilon g_h(x) = \int \rho_\varepsilon(z) r(x - z) dz = r(x) - J \int z \rho_\varepsilon(z) dz = r(x).$$

This is affine reproduction by zeroth and first kernel moments, the continuum counterpart of the familiar IB moment conditions [1, 3]; the displayed equality proves the exact property needed for our lift without a discrete-delta convergence theorem. The interpolation estimate (16), applied to the fixed smooth ψ , gives (50). Finally subtracting $\omega_n g_h$ from either side of (51) gives exactly the homogeneous constraint defining Z^n . $\square$

8.2 Continuous and discrete centered equations

Subtracting a lift is the standard reduction of nonhomogeneous essential boundary data to a homogeneous variational problem; see, for example, Ern and Guermond [28, Chapter 33] for its elliptic form. The time-dependent load and the common discrete lift needed here are specified explicitly below, rather than inferred from that elliptic result. Let $f \in L^2(0, T; U')$, and prescribe initial data in the form $u_0 = \omega(0)g + w_0$, where $w_0 \in H$. Define

$$\langle \mathcal{F}(t), v \rangle = \langle f(t), v \rangle - \omega'(t)(g, v) - \omega(t)a(g, v), \quad v \in U. \quad (52)$$

This functional belongs to $L^2(0, T; U')$. Let w be the solution of (4) with initial value w_0 and load $\mathcal{F}$, and set

$$G(t) = \omega(t)g, \quad u(t) = w(t) + G(t). \quad (53)$$

Then $u \in G + L^2(0, T; V)$, $u \in C([0, T]; L^2)$, $\gamma_\Gamma u = \omega r$ almost everywhere in time, and

$$\langle \partial_t u, v \rangle + a(u, v) = \langle f, v \rangle \quad (v \in V).$$

Conversely any solution in this affine class has the stated form by subtracting G . Existence and uniqueness therefore follow from the homogeneous problem, without solving a time-dependent lifting problem.

Initialize by the true affine projection of the physical initial data:

$$\begin{aligned} u_h^0 &= \omega_0 g_h + P^0(u_0 - \omega_0 g_h), \\ w_h^0 &= P^0(w_0 + \omega_0(g - g_h)). \end{aligned} \quad (54)$$

The difference from the centered initialization $P^0 w_0$ is bounded by $C_g h^2 |\omega_0|$ in the bulk norm. One may instead use that centered initialization; all results below remain valid and the initial perturbation in their proofs is then zero. Put

$$\delta\omega_n = \frac{\omega_n - \omega_{n-1}}{\tau_n} = \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} \omega'(t) dt, \quad \bar{\omega}_n = \frac{1}{\tau_n} \int_{t_{n-1}}^{t_n} \omega(t) dt.$$

The centered backward Euler scheme is

$$\begin{aligned} w_h^n &\in Z^n, \quad u_h^n = w_h^n + \omega_n g_h, \\ \left((w_h^n - P^n w_h^{n-1}) / \tau_n, v_h \right) + a(w_h^n, v_h) &= \langle \mathcal{F}_h^n, v_h \rangle \quad (v_h \in Z^n), \end{aligned} \quad (55)$$

$$\langle \mathcal{F}_h^n, v \rangle = \langle f^n, v \rangle - \delta\omega_n(g_h, v) - \omega_n a(g_h, v). \quad (56)$$

The velocity exists uniquely by positive definiteness on Z^n . It satisfies the exact nonhomogeneous moment constraints (51) at every time level. Although g_h belongs to every unconstrained Cartesian space, it generally does *not* belong to Z^n ; no identity $P^n g_h = g_h$ is used.

Remark 8.2 (Affine projection and a potentially inconsistent reset). For a fixed step the genuine L^2 projection onto the current affine space is

$$\mathcal{P}_n^{\text{aff}} q = \omega_n g_h + P^n(q - \omega_n g_h).$$

Orthogonality gives $(\mathcal{P}_n^{\text{aff}} q - q, v_h) = 0$ for every $v_h \in Z^n$. Consequently (55) is equivalent to the physical velocity equation

$$\left((u_h^n - \mathcal{P}_n^{\text{aff}} u_h^{n-1}) / \tau_n, v_h \right) + a(u_h^n, v_h) = \langle f^n, v_h \rangle, \quad v_h \in Z^n.$$

Indeed, writing $u_h^n = w_h^n + \omega_n g_h$ in this identity produces $\delta\omega_n(g_h, v_h)$ in addition to the centered difference, which moves to the right-hand side of (56). A true affine projection therefore retains the prescribed acceleration.

It is different from the lift-reset map

$$\mathcal{R}_n q = \omega_n g_h + P^n(q - \omega_{n-1} g_h), \quad \mathcal{P}_n^{\text{aff}} q = \mathcal{R}_n q - \tau_n \delta \omega_n P^n g_h.$$

Replacing the affine projection by $\mathcal{R}_n$ in the displayed physical equation, without a compensating acceleration term, omits $-\delta \omega_n(g_h, v_h)$ from the centered load. It therefore defines a different evolution when that functional is nonzero. The centered formulation specifies the transfer and the lifting derivative unambiguously.

8.3 Centered energy and the physical boundary defect

Let

$$\mathcal{D}_\omega = \|w_0\|_0 + \|f\|_{L^2(0,T;U')} + \|\omega'\|_{L^2(0,T)} + \|\omega\|_{L^\infty(0,T)}.$$

Constants below may depend on the fixed lifting, ν , and T , but not on the patch history, panel partitions, regularization widths, or the number of time steps.

Proposition 8.3 (Centered energy and unchanged interface order). *At every step of (55),*

$$\begin{aligned} \|w_h^n\|_0^2 - \|w_h^{n-1}\|_0^2 + \|w_h^n - w_h^{n-1}\|_0^2 + 2\tau_n a(w_h^n, w_h^n) \\ = 2\tau_n \{ \langle f^n, w_h^n \rangle - \delta \omega_n(g_h, w_h^n) - \omega_n a(g_h, w_h^n) \}. \end{aligned} \quad (57)$$

The increment has the orthogonal decomposition

$$\|w_h^n - w_h^{n-1}\|_0^2 = \|w_h^n - P^n w_h^{n-1}\|_0^2 + \|(I - P^n)w_h^{n-1}\|_0^2. \quad (58)$$

Moreover,

$$\max_n \|w_h^n\|_0^2 + \sum_n \tau_n \|w_h^n\|_U^2 + \sum_n \|w_h^n - w_h^{n-1}\|_0^2 \leq C \mathcal{D}_\omega^2. \quad (59)$$

Let $U_h(t) = u_h^n$, $W_h(t) = w_h^n$, and $\omega_\tau(t) = \omega_n$ on $(t_{n-1}, t_n]$. Then

$$\|\gamma_\Gamma U_h - \omega_\tau r\|_{L^2(0,T;L^2(\Gamma))} \leq C(\sqrt{\varepsilon} + \sqrt{\ell}) \mathcal{D}_\omega, \quad (60)$$

$$\|\gamma_\Gamma U_h - \omega r\|_{L^2(0,T;L^2(\Gamma))} \leq C(\sqrt{\varepsilon} + \sqrt{\ell}) \mathcal{D}_\omega + \tau_{\max} \|r\|_{L^2(\Gamma)} \|\omega'\|_{L^2(0,T)}. \quad (61)$$

In particular there is no spatial lifting-error term in these trace bounds.

Proof. Test (55) with w_h^n . Projection orthogonality and the polarization identity give (57), and the two components in (58) are orthogonal. The lift estimate gives a uniform bound on $\|g_h\|_U$ and $\|g_h\|_0$. Jensen's inequality and Poincaré's inequality imply

$$\left(\sum_n \tau_n \|\mathcal{F}_h^n\|_{U'}^2 \right)^{1/2} \leq \|f\|_{L^2(0,T;U')} + C_P \|g_h\|_0 \|\omega'\|_{L^2(0,T)} + \nu \sqrt{T} \|g_h\|_U \|\omega\|_{L^\infty(0,T)}.$$

The initial norm is at most $\|w_0\|_0 + C_g h^2 |\omega_0|$. Summation of the energy identity and Young's inequality now prove (59). Since $\gamma_\Gamma U_h - \omega_\tau r = \gamma_\Gamma W_h$ exactly, Lemma 3.1 and the centered energy bound prove (60). Finally the fundamental theorem of calculus and Cauchy–Schwarz give

$$\|\omega_\tau - \omega\|_{L^2(0,T)} \leq \tau_{\max} \|\omega'\|_{L^2(0,T)}, \quad \|\omega_\tau - \omega\|_{L^\infty(0,T)} \leq \sqrt{\tau_{\max}} \|\omega'\|_{L^2(0,T)}. \quad (62)$$

The triangle inequality proves (61). $\square$

Proposition 8.4 (Background kinetic energy and constraint work). *Set $G_h^n = \omega_n g_h$ and define the work functional at the computed velocity by*

$$\mathcal{W}_\Gamma^n = \left((u_h^n - u_h^{n-1})/\tau_n, G_h^n \right) + a(u_h^n, G_h^n) - \langle f^n, G_h^n \rangle. \quad (63)$$

Then the physical-velocity variable satisfies the exact background-domain identity

$$\begin{aligned} \|u_h^n\|_0^2 - \|u_h^{n-1}\|_0^2 + \|u_h^n - u_h^{n-1}\|_0^2 + 2\tau_n a(u_h^n, u_h^n) \\ = 2\tau_n \langle f^n, u_h^n \rangle + 2\tau_n \mathcal{W}_\Gamma^n. \end{aligned} \quad (64)$$

The value of $\mathcal{W}_\Gamma^n$ is unchanged if G_h^n is replaced by any other lift in the same current affine constraint space. If a multiplier formulation is used with sign convention

$$\left((u_h^n - u_h^{n-1})/\tau_n, v_h \right) + a(u_h^n, v_h) + b_{\varepsilon_n}(v_h, \lambda_h^n) = \langle f^n, v_h \rangle \quad (v_h \in X_{h, \mathcal{A}^n}), \quad (65)$$

then $\mathcal{W}_\Gamma^n = -b_{\varepsilon_n}(G_h^n, \lambda_h^n)$. This scalar is unique even if the multiplier is not.

Proof. Let

$$R_n(v) = \left((u_h^n - u_h^{n-1})/\tau_n, v \right) + a(u_h^n, v) - \langle f^n, v \rangle.$$

The centered scheme and projection orthogonality give $R_n(v_h) = 0$ for $v_h \in Z^n$. Since $u_h^n - G_h^n = w_h^n \in Z^n$, $R_n(u_h^n) = R_n(G_h^n) = \mathcal{W}_\Gamma^n$. Multiplication by $2\tau_n$ and the polarization identity prove (64). Any two lifts in the current affine space differ by an element of Z^n , on which R_n vanishes, proving lifting independence. In finite dimensions, $R_n|_{Z^n} = 0$ also implies that $-R_n$ is in the range of the transpose of the constraint map. Hence a multiplier satisfying (65) exists, without a row-rank assumption. Testing that equation with G_h^n gives the stated formula. The difference of two such multipliers pairs to zero with every vector in $X_{h, \mathcal{A}^n}$, including G_h^n , so it cannot change $\mathcal{W}_\Gamma^n$. $\square$

8.4 Reduction to the homogeneous convergence theorem

The discrete load in (56) is not exactly the time average of (52). The next estimate controls this difference independently of the mesh history.

Lemma 8.5 (Lifting load perturbation). *Let $\bar{\mathcal{F}}^n$ denote the exact time average of $\mathcal{F}$ on the n th interval, and put $\mathcal{E}_h^n = \mathcal{F}_h^n - \bar{\mathcal{F}}^n$. Then*

$$\langle \mathcal{E}_h^n, v \rangle = -\delta\omega_n(g_h - g, v) - \omega_n a(g_h - g, v) - (\omega_n - \bar{\omega}_n) a(g, v), \quad (66)$$

$$\left(\sum_n \tau_n \|\mathcal{E}_h^n\|_{U'}^2 \right)^{1/2} \leq C \left[h^2 \{ \|\omega'\|_{L^2(0,T)} + \|\omega\|_{L^\infty(0,T)} \} + \tau_{\max} \|\omega'\|_{L^2(0,T)} \right]. \quad (67)$$

Proof. Time averaging (52) and subtracting gives (66). Its first two terms are bounded in U' by $C_P h^2 C_g |\delta\omega_n|$ and $\nu C_g h^2 |\omega_n|$. For the last term use

$$\sum_n \tau_n |\omega_n - \bar{\omega}_n|^2 \leq \tau_{\max}^2 \|\omega'\|_{L^2(0,T)}^2,$$

which follows interval by interval from the fundamental theorem of calculus, as in (46). Jensen's inequality for $\delta\omega_n$, the bound $\sum_n \tau_n |\omega_n|^2 \leq T \|\omega\|_{L^\infty}^2$, and the triangle inequality in the weighted sequence norm prove the result. $\square$

Theorem 8.6 (Strong convergence for prescribed fixed-disk rotation). *Let $f \in L^2(0, T; U')$, $\omega \in H^1(0, T)$, and $w_0 \in H$. Use the common lift and initialization (54). Suppose (29) holds and, for all sufficiently large m , (48) holds. The admissible patch, kernel width, and surface panel partition may change at every step. Then the physical velocities satisfy*

$$\|U_m - u\|_{L^2(0, T; U)} \longrightarrow 0, \quad \sup_{0 \leq t \leq T} \|U_m(t) - u(t)\|_0 \longrightarrow 0. \quad (68)$$

Also the sum of the two centered losses in (58) tends to zero. None of these statements requires a regular interface multiplier or a restriction on the number of mesh changes.

Proof. For each run let $y_h^n \in Z^n$ solve the homogeneous scheme (23) with initial value $P^0 w_0$ and exact averaged load $\bar{\mathcal{F}}^n$. Theorem 6.1 applies to this auxiliary computation and the fixed continuous data $(w_0, \mathcal{F})$. Its reconstruction Y_m converges strongly to w in both norms stated in (30), and its total loss tends to zero.

Set $e_h^n = w_h^n - y_h^n$. Subtraction gives

$$\left((e_h^n - P^n e_h^{n-1}) / \tau_n, v_h \right) + a(e_h^n, v_h) = \langle \mathcal{E}_h^n, v_h \rangle, \quad e_h^0 = \omega_0 P^0 (g - g_h).$$

The same energy argument as in Proposition 5.1 yields

$$\max_n \|e_h^n\|_0^2 + \sum_n \tau_n \|e_h^n\|_U^2 + \sum_n \|e_h^n - e_h^{n-1}\|_0^2 \leq C_\nu \left(\|e_h^0\|_0^2 + \sum_n \tau_n \|\mathcal{E}_h^n\|_{U'}^2 \right). \quad (69)$$

The initial error is at most $C_g h^2 |\omega_0|$, and Lemma 8.5 makes the remaining term tend to zero. Thus $W_m \rightarrow w$ in $L^2(0, T; U)$ and uniformly in the bulk norm. Moreover,

$$\sum_n \|w_h^n - w_h^{n-1}\|_0^2 \leq 2 \sum_n \|y_h^n - y_h^{n-1}\|_0^2 + 2 \sum_n \|e_h^n - e_h^{n-1}\|_0^2 \longrightarrow 0,$$

which gives the centered loss assertion by (58).

Finally $U_m = W_m + \omega_{\tau_m} g_{h_m}$, whereas $u = w + \omega g$. The lifting difference satisfies

$$\|\omega_{\tau_m} g_{h_m} - \omega g\|_{L^2(0, T; U)} \leq \sqrt{T} \|\omega\|_{L^\infty} \|g_{h_m} - g\|_U + \|\omega_{\tau_m} - \omega\|_{L^2(0, T)} \|g\|_U,$$

and its supremum bulk norm is bounded by

$$\|\omega\|_{L^\infty} \|g_{h_m} - g\|_0 + \|\omega_{\tau_m} - \omega\|_{L^\infty(0, T)} \|g\|_0.$$

Equations (50) and (62) show that both expressions tend to zero. This proves (68), including time zero with the specified initialization. $\square$

8.5 Quantitative velocity estimate

Suppose in addition that $w = u - \omega g \in H^1(0, T; Y) \cap H^1(0, T; V)$ and that $\lambda \in L^2(0, T; L^2(\Gamma)^2)$ satisfies

$$(\partial_t u, v) + a(u, v) + \int_\Gamma \lambda \cdot \gamma_\Gamma v \, ds = \langle f, v \rangle \quad (v \in U) \quad (70)$$

almost everywhere. Define

$$\begin{aligned} \mathcal{B}_m(w, \lambda) &= \sqrt{h_m + \bar{\varepsilon}_m} (\|w\|_{L^\infty(0, T; Y)} + \|\partial_t w\|_{L^2(0, T; Y)}) \\ &\quad + (\sqrt{\bar{\varepsilon}_m} + \sqrt{\bar{\ell}_m}) \|\lambda\|_{L^2(0, T; L^2(\Gamma))} + \tau_{\max, m} \|\partial_t w\|_{L^2(0, T; U)}. \end{aligned}$$

Theorem 8.7 (Energy error with a prescribed rotation lift). *Under the preceding regularity and geometric assumptions,*

$$\begin{aligned} & \max_n \|u_m^n - u(t_n)\|_0 + \left(\sum_n \tau_m^n \|u_m^n - u(t_n)\|_U^2 \right)^{1/2} \\ & \leq C \left[\mathcal{B}_m(w, \lambda) + h_m^2 \{ \|\omega\|_{L^\infty(0,T)} + \|\omega'\|_{L^2(0,T)} \} + \tau_{\max,m} \|\omega'\|_{L^2(0,T)} \right]. \end{aligned} \quad (71)$$

The constant may depend on the fixed smooth lift, but is independent of the refinement history and of the number of mesh changes.

Proof. Subtracting the exact lift from (70) gives

$$(\partial_t w, v) + a(w, v) + \int_\Gamma \lambda \cdot \gamma_\Gamma v \, ds = \langle \mathcal{F}, v \rangle \quad (v \in U).$$

Thus Theorem 7.1 applies to the auxiliary velocities y_h^n used in the preceding proof, with solution w , load $\mathcal{F}$, and the same λ . It bounds their nodal and dissipative error by $C\mathcal{B}_m(w, \lambda)$. The initial perturbation is at most $C_g h^2 |\omega_0|$. Equations (69) and (67) bound $w_h^n - y_h^n$ by the remaining terms in (71). At each node,

$$u_h^n - u(t_n) = (y_h^n - w(t_n)) + (w_h^n - y_h^n) + \omega_n(g_h - g).$$

The last term has maximum bulk norm at most $C_g h^2 \|\omega\|_{L^\infty}$ and weighted U norm at most $C_g h^2 \sqrt{T} \|\omega\|_{L^\infty}$. The triangle inequality proves the estimate. $\square$

The same right-hand side, with a changed constant, controls $\|U_m - u\|_{L^2(0,T;U)}$: the temporal reconstruction error is at most $\tau_{\max,m} \|\partial_t u\|_{L^2(0,T;U)}$, and $\partial_t u = \partial_t w + \omega'g$. As in the homogeneous theorem, a supremum-in-time error for the piecewise constant reconstruction additionally incurs an $O(\sqrt{\tau_{\max,m}})$ term under the stated H^1 time regularity; the nodal first-order time estimate in (71) does not assert a first-order bound for that reconstruction norm.

8.6 Rigid interior interpretation and the scope of the energy law

The affine background-domain problem describes a rigidly rotating interior at the continuum level if the interior data are chosen consistently. Specifically suppose

$$u_0|_B = \omega(0)r, \quad \langle f(t), v \rangle = \omega'(t) \langle r, v \rangle_{L^2(B)}$$

for every divergence-free $v \in H_0^1(B)^2$, extended by zero to Ω . A bulk force equal to $\omega'r$ in B is one such choice. Then $u|_B = \omega r$. To see this, use those interior test functions in the equation for $w = u - \omega g$. Since $g = r$ in B and r is affine,

$$a_B(r, v) = \nu \int_B \nabla r : \nabla v = 0 \quad (v \in H_0^1(B)^2).$$

Thus $w|_B$ has zero trace, zero initial data, and satisfies the homogeneous interior Stokes evolution. Its energy identity gives $w|_B = 0$. This is a property of the limiting problem; the regularized discrete solution is not asserted to be exactly rigid inside B at a finite resolution.

Equation (57) is an exact energy law for the centered velocity, whereas (64) identifies the work of the prescribed discrete constraint in the background kinetic-energy balance. The latter is not automatically an exterior-fluid torque law. The interior kinetic energy and its matching body-force work must be identified when passing from the background domain to the physical

fluid, and a physical torque requires the appropriate exterior stress and its approximation. For example, a rigid rotation has $a_B(\omega r, \omega r) = 2\nu|B|\omega^2$, whereas its symmetric strain is zero. This local observation does not introduce additional physical dissipation: for every globally divergence-free $u \in H_0^1(\Omega)^2$, integration by parts gives

$$\nu \int_{\Omega} |\nabla u|^2 = 2\nu \int_{\Omega} |D(u)|^2, \quad D(u) = \frac{1}{2}(\nabla u + \nabla u^\top).$$

If $u = \omega r$ inside B , the right-hand side is exactly the physical exterior strain dissipation. Thus local full-gradient contributions must not be interpreted separately as physical heating. A whole-domain reaction such as λ in (70) includes the interface balance between the two background subdomains; it is not identified here with exterior physical traction or a convergent torque. No multiplier estimate is needed for the centered energy or qualitative convergence.

The result covers prescribed rotation of a disk whose geometric interface is fixed. It does not cover translation of the disk, rotation of a nonsymmetric body that changes the interface, an angular velocity determined by a coupled rigid-body equation, or unrestricted changes of the lifting function during a run. Those cases would require additional arguments beyond the two common families constructed here.

9 Manufactured-solution verification

We give a small numerical verification study of the specified conforming method. It checks an implementation against an exact nonstationary solution, following the manufactured-solution principle [29]. The first five subsections concern a stationary body; Section 9.6 then verifies prescribed rotation of the same fixed geometric interface. The tests address bulk error, the physical interface trace, and mesh transfer; they are not benchmarks for a conventional MAC implementation.

9.1 An exact solution with nonzero interface shear

Let B be the disk of radius $R = 0.2$ centered at $c = (1/2, 1/2)$ in $\Omega = (0, 1)^2$. Write

$$X = x - c_1, \quad Y = y - c_2, \quad s = X^2 + Y^2 - R^2, \quad Q = x^2(1-x)^2y^2(1-y)^2, \quad s_+ = \max(s, 0).$$

With $A = 1000$, $\nu = 0.1$, and $T = 0.5$, define

$$\psi_0 = A Q s_+^2, \quad U = \text{curl } \psi_0, \quad u(x, t) = a(t)U(x), \quad a(t) = 1 - e^{-t}, \quad p = 0. \quad (72)$$

The velocity is divergence free, vanishes inside B , and satisfies no-slip on both Γ and the outer square. Its initial value is zero. It is smooth on either side of Γ and globally H^1 , so it has the regularity required by Theorem 7.1.

The ordinary bulk force is obtained by differentiating the exterior polynomial, not from a discrete residual:

$$f(x, t) = e^{-t}U - \nu(1 - e^{-t})\Delta U \quad (x \in F), \quad f = 0 \quad (x \in B). \quad (73)$$

For explicit evaluation, set

$$L = s^2\Delta Q + 8s(XQ_x + YQ_y) + (16s + 8R^2)Q.$$

In F , $\Delta U = A(L_y, -L_x)$. For the normal $n = (\cos \theta, \sin \theta)$ pointing out of the disk, the reaction in (39) is

$$\lambda = \nu \partial_n u = 8\nu A a(t) R^2 Q (\sin \theta, -\cos \theta) \quad \text{on } \Gamma. \quad (74)$$

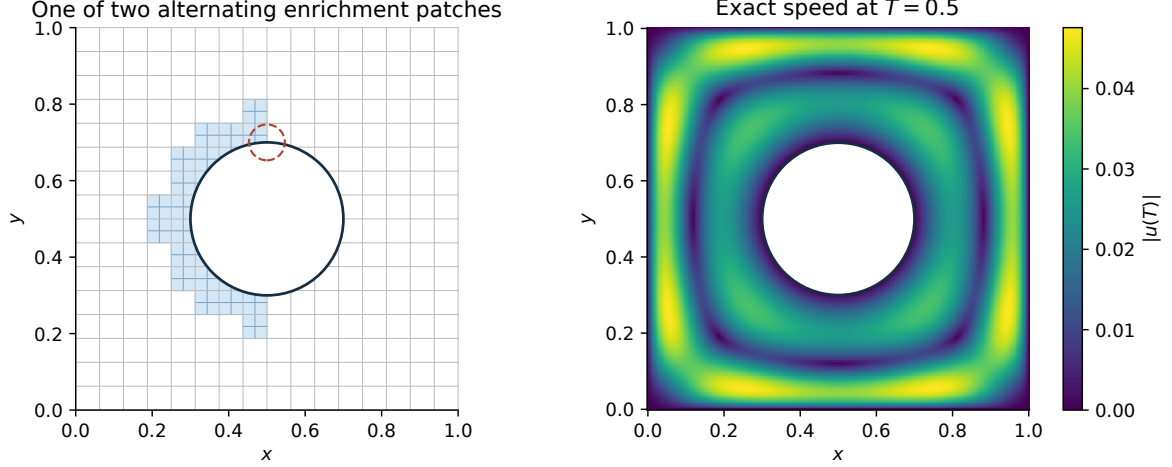


Figure 1: Left: one enrichment pattern for $M = 16$; the second uses the opposite side of the interface. Blue cells have the four additional center degrees of freedom. The dashed circle bounds one kernel support. The background discretization also extends through the white body. Right: speed of the exact solution at $T = 0.5$.

Thus the interface shear is nonzero. The solution does not vanish in an open collar of the interface, and its first spatial derivatives jump across Γ . No surface delta term is added to the bulk load in (73); the constraint supplies the interface reaction. Independent symbolic differentiation and integration by parts check the forcing and the sign in (74).

The load is averaged exactly in time. On $[t_{n-1}, t_n]$, with $d_n = (e^{-t_{n-1}} - e^{-t_n})/\tau_n$, it is $\bar{f}^n = d_n U - \nu(1 - d_n)\Delta U$ in F . The implementation evaluates the exponential difference using `expm1`.

9.2 Discrete realization and numerical accuracy

The implementation uses exactly the spaces of Section 4. All coarse and local enrichment functions are represented in one fixed hierarchical master basis. If z_K is the center of a coarse cell, set

$$\mathcal{R}_h = \{K : ||z_K - c| - R| < 1.5h\},$$

and alternate the two patches

$$\mathcal{A}_0 = \{K \in \mathcal{R}_h : (z_K)_1 < c_1 + h/4\}, \quad \mathcal{A}_1 = \{K \in \mathcal{R}_h : (z_K)_1 > c_1 - h/4\}.$$

Both retain the complete coarse space. Their boundaries intersect the regularization region; the two patterns are not nested.

For the regularizer, let

$$k(z) = \frac{\exp[-1/(1 - z^2)] \mathbf{1}_{|z|<1}}{\int_{-1}^1 \exp[-1/(1 - r^2)] dr}, \quad \rho_\varepsilon(x, y) = \frac{2}{\varepsilon^2} k\left(\frac{\sqrt{2}x}{\varepsilon}\right) k\left(\frac{\sqrt{2}y}{\varepsilon}\right).$$

This is a smooth even kernel supported in a square contained in the radius- ε disk. Unless stated otherwise, $\varepsilon = 0.75h$ and the circle has $P = \lceil 2\pi R/h \rceil$ equal arclength panels, so $\ell = 2\pi R/P \leq h$.

Bulk mass and viscosity matrices use four-point Gauss rules per coordinate on the $h/2$ master cells. These rules integrate the polynomial forms exactly, up to floating-point roundoff.

Table 1: Velocity errors and physical no-slip defect, $N = 400$. Errors E_0, E_1 are expressed as percentages; D_Γ is an absolute norm.

$M = 1/h$	P	$100E_0$	$100E_1$	D_Γ
8	11	17.3205	34.6124	9.232×10^{-4}
16	21	5.3117	10.6920	4.248×10^{-4}
32	41	2.3140	4.1577	1.842×10^{-4}
64	81	1.1353	2.4556	8.417×10^{-5}

The force and error integrals respect the exact circular cut: cells are split into exterior intervals using the circle equation, with additional breaks at circle–cell-edge intersections. Uncut cells use order 10, and cut intervals use order 24. The convolution integral is split at every master-grid line before using order 16 in each coordinate; surface moments use order 10 on each panel. Physical trace integration is independent of the constraint residual and splits the circle into arcs at all master-grid crossings.

The constraints are solved through an orthonormal basis for their row space and a Schur complement. The singular-value cutoff is 10^{-12} relative to the largest singular value. No constraint is discarded in any reported experiment: all matrices have full row rank, with a minimum observed singular-value ratio of 6.47×10^{-4} . Changing the cutoff to 10^{-10} or 10^{-14} at $M = 16$ leaves the solution unchanged. This observation is not a mesh-uniform multiplier stability claim.

The step right-hand side uses the cross-mesh mass pairing with the old velocity. By orthogonality, this is equivalent to the constrained L^2 projection in (23); it does not use nodal interpolation. All energy diagnostics use the same physical mass matrix. The numerical surface integrals are approximations to the exact moments assumed in the theory. Consequently, the computations are a controlled approximate realization, and the quadrature checks below do not replace a quadrature-consistency theorem.

9.3 Velocity convergence and physical slip

Let U_h be the piecewise constant reconstruction. We report

$$E_0 = \frac{\|u_h^N - u(T)\|_{L^2(\Omega)}}{\|u(T)\|_{L^2(\Omega)}}, \quad E_1 = \frac{\|U_h - u\|_{L^2(0,T;U)}}{\|u\|_{L^2(0,T;U)}},$$

$$D_\Gamma = \|\gamma_\Gamma U_h\|_{L^2(0,T;L^2(\Gamma))}.$$

The volume norms include the body interior. In E_1 , the exponential time dependence of the exact solution is integrated analytically on each interval; it is not replaced by its value at the right endpoint.

Table 1 uses $N = 400$ uniform time steps and switches patch at every step. The three spatial lengths h , ε , and ℓ decrease together. Both velocity errors and physical slip decrease. These four fixed-time-step results illustrate convergence; they are not an asymptotic verification of the joint limit in Theorem 6.1, nor do they establish a sharp spatial rate. In particular, a rate faster than the half-order upper bound is not a contradiction of Theorem 7.1.

9.4 Mesh switching, position sensitivity, and energy

At $M = 32$, keep $N = 400$, ε , and the panel partition fixed. Compare a fixed patch with switching every 20, 5, or 1 steps. This changes only the adaptation history. Table 2 shows

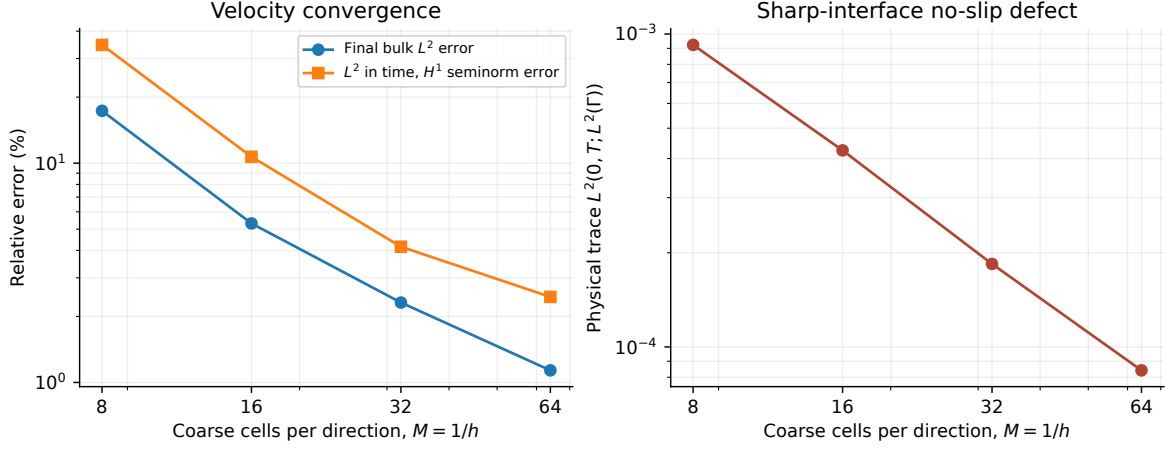


Figure 2: The data in Table 1. Physical slip is evaluated on the sharp circle, rather than inferred from the regularized constraint residual.

Table 2: Switching frequency at fixed $M = 32$ and $N = 400$. The last column is $\sum_n \|(I - P^n)u^{n-1}\|_0^2$.

Mesh changes	$100E_0$	$100E_1$	Projection loss
0	2.31271	4.15536	0.000×10^0
20	2.31272	4.15484	4.093×10^{-10}
80	2.31300	4.15548	1.553×10^{-9}
400	2.31398	4.15772	6.106×10^{-9}

that the final relative velocity error stays between 2.31271% and 2.31398%. The accumulated projection losses are positive: the switches genuinely discard resolved velocity components. The result supports robustness for these histories; it does not assert identical errors for arbitrary histories or replace the event-count-independent theorem.

At $M = 24$, shifting the fixed disk center to $c = (1/2 + \alpha h, 1/2)$ with $\alpha \in \{0, 1/4, 1/2, 3/4\}$ gives Table 3. The manufactured force is recomputed for each center. The errors remain of comparable size, with no anomalous growth in these four alignments. Moving the disk also changes the exact solution relative to the outer walls, so this is a position-sensitivity test, not an isolated test of matrix conditioning.

Across all 26 saved experiment runs, the largest absolute residual of (24) is 1.87×10^{-17} , that of the jump decomposition (25) is 2.17×10^{-19} , and the largest assembled panel-average constraint residual is 5.33×10^{-17} . These algebraic checks would not by themselves establish consistency with the physical boundary condition.

A separate unforced test uses $M = 16$, $N = 200$, initial data equal to the constrained

Table 3: Four fixed interface positions, $M = 24$, $N = 400$.

α (shift αh)	$100E_0$	$100E_1$	D_Γ
0.00	3.1531	5.8654	2.456×10^{-4}
0.25	3.1330	5.7860	2.571×10^{-4}
0.50	3.0754	5.7588	2.523×10^{-4}
0.75	2.9530	5.6123	2.601×10^{-4}

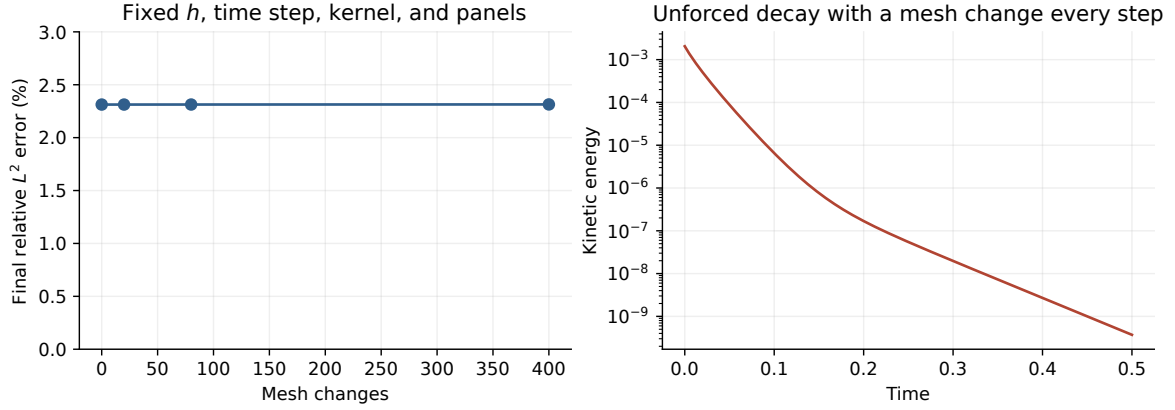


Figure 3: Left: the fixed-time-step switching experiment. Right: kinetic energy in the separate unforced decay test.

projection of U in (72), and a patch change at every step. Its kinetic energy decreases at every step, ending at 1.81×10^{-7} times its initial value. This test has different data from the manufactured problem; no exact-solution error is assigned to it.

9.5 Checks against quadrature and temporal contamination

For the $M = 16$, $N = 200$ dynamic case, increasing the convolution, surface, and cut-cell orders from (16, 10, 24) to (24, 16, 32) changes the final velocity by 5.67×10^{-8} of the exact L^2 norm and 2.17×10^{-7} of the exact H^1 seminorm. These changes are much smaller than the discretization errors. An independently assembled integration-by-parts identity using (74) also checks the body-force assembly.

To isolate time discretization, keep the $M = 32$ patch fixed and use $N = 100, 200, 400, 800$. Relative L^2 differences between successive final-time solutions are 1.9766×10^{-3} , 9.8702×10^{-4} , and 4.9319×10^{-4} . Their ratios, 2.0026 and 2.0013, are consistent with first-order time stepping. These are temporal self-convergence differences, not errors against the exact semidiscrete solution. In the finest dynamic spatial case, doubling N from 400 to 800 changes E_0 from 1.13535% to 1.12074%; thus the reported error is not predominantly temporal. That last comparison also changes the patch-switching frequency and is not used to infer a temporal order.

The companion provides the solver, exact-solution verifier, independent implementation checks, all 26 parameter files, final coefficient vectors, and diagnostic histories, and scripts that regenerate the tables and figures. The results support the implementation and the stated qualitative trends. They do not prove adaptive optimality, multiplier convergence, or general robustness of other IB discretizations.

9.6 Prescribed rotation on fixed geometry

We also impose a nonzero prescribed velocity on the same circular interface. The disk rotates with angular velocity $\omega(t) = a(t) = 1 - e^{-t}$ while its geometric boundary remains fixed. Write

$r(x) = (-Y, X)$ and $\sigma = X^2 + Y^2$. Define the compactly supported streamfunction lift

$$\begin{aligned}\psi_g &= -\frac{1}{2}\sigma\chi(\sigma), & g &= \text{curl } \psi_g, \\ \chi(\sigma) &= 1 - 2772 \int_0^{\eta(\sigma)} q^5(1-q)^5 dq, \\ \eta(\sigma) &= \min \left\{ 1, \max \left\{ 0, \frac{\sigma - 0.35^2}{0.47^2 - 0.35^2} \right\} \right\}.\end{aligned}$$

The cutoff is C^5 , $g = r$ for $|x - c| \leq 0.35$, and $g = 0$ for $|x - c| \geq 0.47$. With the nonzero-shear field U from (72), the exact solution and ordinary bulk force are

$$u(x, t) = a(t)(g + U), \quad f = e^{-t}(g + U) - \nu(1 - e^{-t})(\Delta g + \Delta U). \quad (75)$$

Here ΔU is evaluated separately on each side of the interface, as in (73). In particular, $u = a(t)r$ and $f = e^{-t}r$ inside the disk. The manufactured field is not only the prescribed lift: its centered part aU and its interfacial shear are nonzero. At $T = 0.5$, the $L^2(\Omega)$ norm of the continuous force beyond that generated by ag alone is 1.82845.

Let $g_h = \text{curl } I_h \psi_g$ use only the common coarse Hermite space. For all tested meshes, $R + \varepsilon + \sqrt{2}h < 0.35$; hence every coarse cell meeting a kernel support lies in the rigid region. Polynomial reproduction gives $g_h = r$ on those cells, and also throughout the disk. In particular, every regularization support has the exact rigid lift. The discrete centered unknown is $w_h^n = u_h^n - a(t_n)g_h$. With M_h, K_h denoting the mass matrix and the matrix for $(\nabla v, \nabla z)$, and $b_v, b_{\Delta v}$ denoting the corresponding continuous-field load vectors, its right-hand side uses the force vector

$$d_n(b_U + b_g - M_h g_h) - \nu(1 - d_n)(b_{\Delta U} + b_{\Delta g}) - \nu a(t_n)K_h g_h. \quad (76)$$

The last coefficient is the endpoint value $a(t_n)$, rather than its interval average. Thus both the interpolation mismatch of g_h and the backward Euler lifting term are retained. The force is obtained from (75), not manufactured from the discrete matrices. The same exact mass pairing implements transfer of the centered unknown between the two enrichment patterns.

For this extension only, kernel subdivision contains each master-grid cut and its reflection about the evaluation point. The resulting symmetric nonnegative Gaussian weights are normalized to mass one in each coordinate. This ensures affine moment reproduction without negative correction weights. Diagnostics retain the mass defect before normalization, the normalized mass and first-moment defects, and the assembled discrepancy between $C_h g_h$ and the rigid boundary moments. The archived static solver and its results are unchanged. Lift load and norm integrals are split at the circles of radii 0.2, 0.35, 0.47; they use order 16 curved quadrature. These are still quadrature approximations to the exact-integral method in the theory.

Table 4 uses $M = 16, 32, 64$, $N = 400$, and switches the enrichment pattern at every step. The definitions of E_0, E_1 now use the total field in (75), including the body interior. To distinguish boundary error from time reconstruction error, we report both

$$D_{\Gamma, \text{cont}} = \|\gamma_{\Gamma} U_h - a(t)r\|_{L^2(0, T; L^2(\Gamma))}, \quad D_{\Gamma, \text{node}}^2 = \sum_n \tau_n \|\gamma_{\Gamma} u_h^n - a(t_n)r\|_{L^2(\Gamma)}^2.$$

The latter equals the time-integrated sharp trace of w_h^n . The former is evaluated with analytic exponential time integrals, including the cross term with r ; it does not replace $a(t)$ by $a(t_n)$. The finite-step temporal contribution remains when the mesh is refined at fixed N .

The coarse $M = 16$ lift resolves the narrow cutoff transition poorly; its velocity interpolation error is not suppressed in this test. The final relative total-velocity error decreases from

Table 4: Fixed-geometry prescribed rotation, $N = 400$, with a mesh change at every step. Total-field errors are percentages; both sharp-slip norms are absolute.

M	P	$100E_0$	$100E_1$	$D_{\Gamma,\text{cont}}$	$D_{\Gamma,\text{node}}$
16	21	16.4591	43.3550	4.809×10^{-4}	4.133×10^{-4}
32	41	3.0583	14.8177	2.566×10^{-4}	1.849×10^{-4}
64	81	0.4792	4.0630	1.615×10^{-4}	8.455×10^{-5}

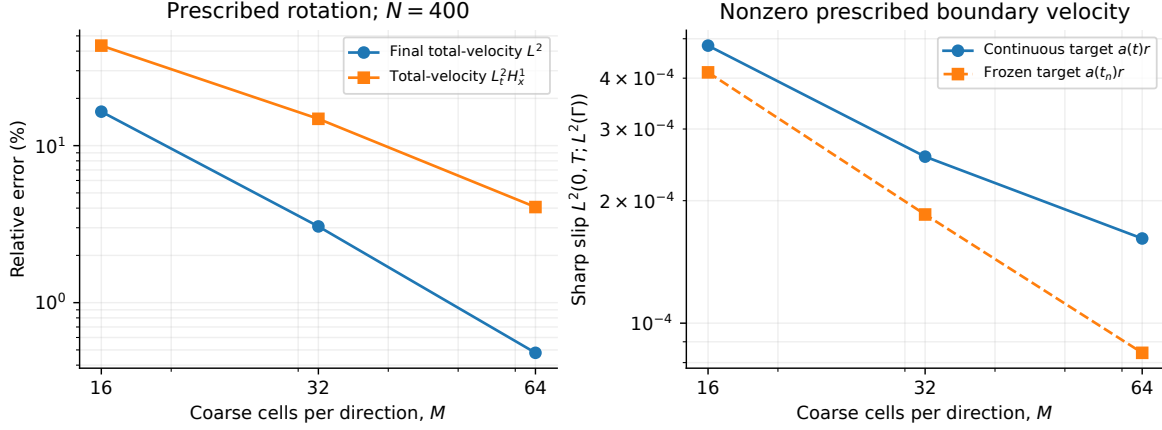


Figure 4: Total-velocity errors and sharp-interface slip for prescribed rotation. The two slip curves distinguish the continuous prescribed motion from its endpoint reconstruction on each time interval.

16.4591% to 3.05830% and 0.479175%. At $M = 64$, the centered final relative L^2 error against $a(T)U$ is 1.41233%. In comparison, the continuous field $a(t)g$ alone would have a 12.4361% relative total-velocity error, independent of resolution. These diagnostics confirm that the computation resolves a nontrivial fluid response beyond prescribing the rigid motion.

At fixed $M = 32$, the cases with 0, 20, 80, 400 mesh changes have final relative L^2 errors between 3.0582811% and 3.0582964%. The centered projection losses are nonzero when the patch changes. Increasing kernel, surface, original cut-load, and lift-load orders from (16, 10, 24, 16) to (24, 16, 32, 24) at $M = 16$ changes the final solution by 5.61×10^{-9} in relative L^2 and 8.95×10^{-9} in relative H^1 .

Across these eight runs, the largest absolute residual of the centered energy identity is 1.94×10^{-17} , and that of the reconstructed full-energy identity is 5.14×10^{-16} . The affine constraint residual is at most 8.10×10^{-15} . Before kernel normalization, the maximum one-dimensional mass defect is 2.06×10^{-8} ; after normalization the mass defect is at most 4.45×10^{-16} and the first centered moment is below 10^{-18} . Independent checks differentiate the analytic lift, verify its rigid and clamped regions, and reconstruct the sharp trace from saved coefficients. This verifies a prescribed nonzero boundary velocity on fixed geometry; it does not test interface translation or fluid-driven rigid-body motion. As in the stationary experiment, finite-resolution trends and quadrature sensitivity do not establish a new asymptotic rate or a quadrature-consistency theorem.

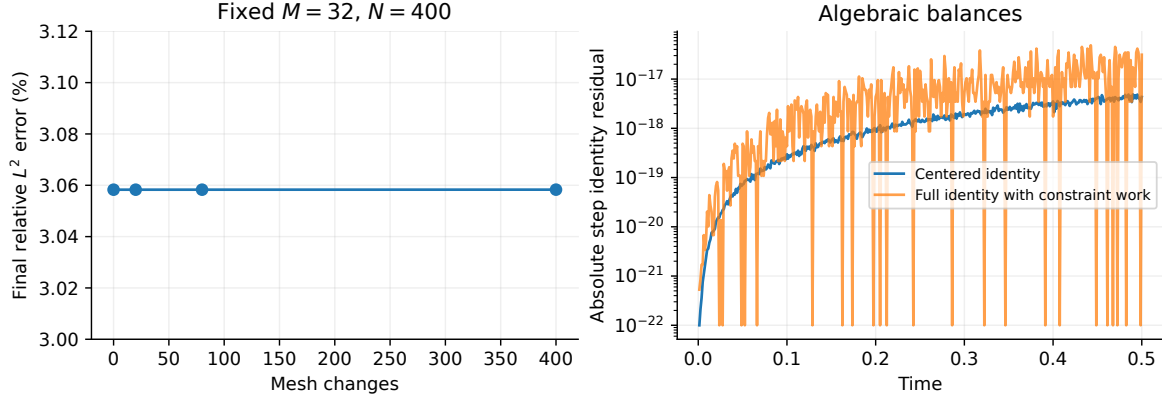


Figure 5: Left: switching frequency at fixed discretization parameters. Right: absolute residuals of the centered energy identity and the full reconstructed-velocity identity, including constraint work. The latter work is an algebraic constraint functional, not a reported hydrodynamic torque.

10 What the result does and does not imply

The estimates permit a completely different set of enriched coarse cells at every time step. Neither nesting in time nor a bound on the number of coarsening events is required. Surface panels and regularization widths may change at the same instants. Their influence enters through their maximal widths and through the common collar used in the proof, rather than through a sum of eventwise consistency errors.

This conclusion is conditional on a concrete structural choice, not on arbitrary interpolation: the full coarse space is retained, the method is conforming in a common physical Hilbert space, and transfer is the exact constrained L^2 projection. General remeshing methods need not share a dense fixed family of test functions. The error accumulation documented in dynamic-mesh analyses [8, 10] is therefore not contradicted.

The half-order spatial estimate does not quantify the gain obtained by an adaptive marking rule. It is a robust upper bound in terms of the coarse grid and a collar construction. It does not establish optimality, a pointwise velocity rate, a uniform-in-time physical trace rate, pressure convergence, or convergence of the multiplier. The results do not imply that a poorly conditioned multiplier system is harmless for finite-precision computation. They show that velocity convergence can be proved without assuming an inf-sup estimate for that system.

The exact-integration qualification is essential. A marker implementation introduces an additional consistency term, for example

$$J_{h,\varepsilon}^{\text{quad}} v_h - J_\varepsilon v_h,$$

after an appropriate surface reconstruction. That term is absent from the exact-integral method and has not been estimated here. Section 9 checks quadrature sensitivity for the reported computations, without adding a quadrature-consistency theorem. The same theoretical limitation applies to approximate projections, changing mass quadrature, solver tolerances, subcycling, and conventional coarse-fine MAC stencils. Proving bounds for these modifications is a separate extension, not a consequence of replacing their matrices by the notation in this paper.

Prescribed rotation of a fixed disk is covered by the common lifting constructed in Section 8. For general prescribed rigid-body motion, the interface itself moves; a collar of the entire swept interface is generally too large to be a dense approximation device. A time-dependent

recovery construction and its transport estimates are therefore needed. Fluid-driven motion also requires consistent momentum and total-energy coupling, including the treatment of fictitious interior fluid. These are future research modules, not consequences asserted by the fixed-geometry theorem.

For example, the prescribed heave and pitch of a plate in Zheng et al. [21] change its geometric interface, unlike rotation of a circular disk about its center. That application therefore requires a moving-geometry analysis even before its free-surface and nonlinear-flow effects are considered. Likewise, the flexible stems in He et al. [20] require structural dynamics beyond the rigid-body scope of this series.

11 Formal verification and reproducibility

The accompanying Lean project checks the algebraic projection and discrete energy layer of the analysis. It was compiled with Lean 4.19.0 [30] and Mathlib [31] at commit `c44e0c8` (the full hash is pinned in the companion). The project contains 26 checked theorems in three source modules. The `Energy.lean` module proves 12 Hilbert-space identities, scalar energy estimates, telescoping statements, and a conditional composition of the two interface bounds. The `Projection.lean` module proves five results using Mathlib’s actual orthogonal projection onto complete subspaces, rather than treating all projection properties as hypotheses. Finite-dimensional spaces such as Z^n are complete and fit that setting. The new `Affine.lean` module adds nine checked statements for actual affine projections, the change of lifting, the centered prescribed-rotation equation, and its energy balances.

Mathematical step	Scope of the checked result
Common functions survive transfer	Subspace inclusion implies that the actual orthogonal projection fixes the common function and preserves its test moment.
Backward Euler energy	The Hilbert-space polarization identity and the tested step equation imply the exact energy balance.
Projection loss	Orthogonality splits the full jump into a time-step increment and a projection loss.
Changing spaces	The exact energy identity telescopes for an arbitrary sequence of complete subspaces; nesting is not assumed.
Residual accumulation	A supplied one-step energy inequality yields the cumulative bound without an additional event-count factor.
Prescribed rotation	The exact affine projector preserves homogeneous test moments; centering the discrete equation retains the angular-acceleration and endpoint viscous terms.
Driven energy	Centered energy and physical kinetic-energy identities include their respective forcing or prescribed-lift work terms; the centered balance telescopes on changing spaces.
Interface estimate	Given the regularization and panel-approximation inequalities, the moment constraint and triangle inequality yield the physical trace bound.

The final build and the explicit `#print axioms` checks passed. The listed foundational axioms are only `propext`, `Classical.choice`, and `Quot.sound`; there is no `sorryAx` and no user-declared axiom. The companion contains the pinned project manifests, source hashes, commands, and compiler logs.

This is not a Lean formalization of Theorems 6.1, 7.1, or the prescribed-rotation extension in their entirety. Zero extension, convolution and trace estimates, fractional panel approximation, the finite element and collar construction, solenoidal density, the evolution equation, and passage to the strong limit are proved in the mathematical text. In particular, the Lean theorem that composes the interface bounds does not establish its analytic hypotheses. The verified layer checks an important part of the argument but does not certify originality or replace review of the remaining analysis.

The manuscript and companion were developed primarily by AI agents; the detailed disclosure at the end of this manuscript states the author’s current understanding of the proofs. The manufactured-solution computations in Section 9 are implementation checks and numerical illustrations, not formal proofs. The reproducibility package contains the complete LaTeX sources, bibliographic database, Lean sources, numerical solver, raw experiment data, and independent implementation checks. Downloaded dependency caches are excluded. The original 17 Lean theorems remain unchanged; nine affine identities were added for version 0.3. None of these theorems certifies the numerical implementation or the analytic construction of the common lifting.

A A kernel obstruction to an unrestricted multiplier assumption

The following elementary calculation motivates avoiding an unproved multiplier stability hypothesis. It is a diagnostic example, not a separate claim of novelty; Fourier analysis of IB kernels and planar boundaries has an established literature [32], and rigid-body mobility conditioning is discussed in Kallemov et al. [7].

On an even periodic row with N aligned markers at spacing h , the integer samples of the standard four-point Peskin kernel are $\phi(0) = 1/2$, $\phi(\pm 1) = 1/4$, and zero at all other integers. The tangential convolution is therefore

$$(A\lambda)_j = \frac{1}{4}\lambda_{j-1} + \frac{1}{2}\lambda_j + \frac{1}{4}\lambda_{j+1}.$$

For $\lambda_j = (-1)^j$, $A\lambda = 0$. A tensor-product spreading operator on a straight aligned interface consequently has a nonzero alternating force mode whose spread is zero. This is a genuine kernel of the coupling map, independent of the downstream fluid solver.

The Fourier symbol is

$$m(\theta) = \frac{1}{2} + \frac{1}{2}\cos\theta = \cos^2(\theta/2).$$

After deleting the exact Nyquist mode, the neighboring mode still has

$$m(\pi - 2\pi/N) = \sin^2(\pi/N) \longrightarrow 0.$$

Thus deleting the exact kernel alone does not produce a uniform ℓ^2 inverse bound for this tangential smoothing map. This calculation does not determine the inf-sup constant of every IB formulation, every marker placement, or every Sobolev-scaled multiplier norm. Nor does a multiplier null mode prevent existence or uniqueness of the constrained velocity. It does rule out an unrestricted claim that all marker-force modes remain uniformly observable for all familiar kernel/grid configurations.

AI use, human understanding, and verification limits

The research direction, scope, and revision requests were provided by Sida He. OpenAI ChatGPT/Codex agents generated most of the mathematical derivations, manuscript text,

numerical implementation, and Lean formalization, and performed automated literature searches and internal checks. This work is therefore disclosed as primarily AI-generated text, rather than merely language editing.

At the preparation date of this version, the author reports that he does not understand the mathematical proofs and has not independently verified them. The intended Hexagon disclosure is *Human understanding: no parts*, with respect to the proofs. No independent human expert verification or external peer review of the proofs is claimed. Internal reviews by additional AI agents do not constitute independent human mathematical review.

The Lean checks described above cover selected algebraic identities and finite-step arguments only. They do not formalize the full PDE convergence theorems or establish their analytic hypotheses. Numerical experiments are implementation checks and illustrations, not proofs. The original proof source, dependency pins, numerical results, and recorded build logs are retained in the companion reproducibility material. This preparation revision does not claim a new Lean compilation or a new numerical experiment. The manuscript is offered for critical scrutiny; its correctness and novelty are not certified by these disclosures or by repository acceptance.

Relationship to companion manuscripts. The other parts of this series are Part II [33], and Part III [34]. They are cited as unpublished companion manuscripts; no public identifier or acceptance is claimed. Each part has its own hypotheses and scope. The series is not asserted to be a chain of special-case theorems.

Availability of reproducibility material. A separate versioned package has been prepared with the Lean source, dependency lockfiles, numerical code, and archived results. No public repository URL or archive identifier has been assigned to that package at the preparation date of this manuscript. The mathematical source submitted to Hexagon does not itself contain the Lean project.

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