

A UNIFORM CALCULATION OF WEIGHT-PRESERVING INFINITESIMAL BRAID DEFORMATIONS IN TYPE D

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ABSTRACT. Over $\mathbb{Q}(q)$, for every $n \geq 4$, we calculate the full space of classical-weight-preserving solutions of the linearized braid equation at the standard type- D_n vector braiding. Its dimension is $(n+1)(n+2)/2$. A coordinate slice for scalar and parameter variations, diagonal vector-basis changes, and alternating twists reduces the upper bound to a homogeneous vanishing problem. Outer-pair coefficients eliminate the zero-weight block through a recurrence closed by the two central weights. We also give a functorial formal local parametrization over local Artin algebras. The explicit integrable family belongs to earlier multiparameter constructions; an exact parameter dictionary is provided. No bibliographic priority claim is made for the full tangent-space exhaustion. The proof is uniform in n ; finite-rank computations are diagnostics.

1. INTRODUCTION AND MAIN RESULT

Multiparameter vector braidings of classical type are already known [7, 1]. The question considered here is an exhaustion problem: at the standard type- D_n vector operator, do those families account for every classical-weight-preserving infinitesimal braid deformation? We allow every coefficient in the zero-weight block and impose no initial triangular-support, fixed-metric, Hopf-algebra, or BMW-representation condition. The main step is a uniform elimination of that block, using the outer opposite pair and a recurrence closed by the two central weights. The explicit family is identified with prior constructions in Section 6; the computation does not assert a new family of quantum groups or bibliographic priority for the exhaustion statement. Conventions. Fix $n \geq 4$, $N = 2n$, $i' = N + 1 - i$, and $K = \mathbb{Q}(q)$, with q an indeterminate. Set

$$\lambda = q^{-1}, \quad \delta = q^{-1} - q, \quad \rho_i = \begin{cases} n - i & i \leq n, \\ n + 1 - i & i > n. \end{cases}$$

The basis $v_1, \dots, v_N$ of $V = K^N$ has weights $w_a = \varepsilon_a$ and $w_{a'} = -\varepsilon_a$ for $1 \leq a \leq n$. Write $ij = v_i \otimes v_j$ and $ijk = v_i \otimes v_j \otimes v_k$. The operator $Q = Q_n$ is specified completely by

$$\begin{aligned} Q(ii) &= \lambda ii, \\ Q(ij) &= ji + \delta \mathbf{1}_{i>j} ij \quad (i \neq j, j \neq i'), \end{aligned} \tag{1.1}$$

$$\begin{aligned} Q(z_j) &= \sum_{i=1}^N B_{ij} z_i, \quad z_j = jj', \\ B_{ij} &= q \mathbf{1}_{i=j'} + \delta \mathbf{1}_{j>n} \mathbf{1}_{i=j} - \delta \mathbf{1}_{i>j'} q^{-\rho_i - \rho_j}. \end{aligned} \tag{1.2}$$

This is the usual braided vector operator of type D_n . In particular it agrees with [1, Eq. (3.21)]; the final summation there is reindexed using $\rho_{i'} = -\rho_i$. Thus Q satisfies the braid equation. Our convention is $Q = PR$.

Date: 27 October 2026.

2020 Mathematics Subject Classification. 16T25, 17B37.

Key words and phrases. Yang–Baxter equation, quantum groups, type D, infinitesimal braid deformations, formal deformations, exact computation.

Prepared with extensive AI assistance. The scope of human understanding, provenance, and verification is disclosed in Section 7.

Let $\mathcal{W}_n \subseteq \text{End}_K(V^{\otimes 2})$ consist of *all* operators preserving total classical weight. No sparsity condition inside a weight block is imposed. Its dimension is $12n^2 - 6n$. Define

$$\begin{aligned} \mathcal{L}_Q(H) &= H_{12}Q_{23}Q_{12} + Q_{12}H_{23}Q_{12} + Q_{12}Q_{23}H_{12} \\ &\quad - H_{23}Q_{12}Q_{23} - Q_{23}H_{12}Q_{23} - Q_{23}Q_{12}H_{23}, \\ \mathcal{T}_n &= \ker(\mathcal{L}_Q|_{\mathcal{W}_n}), \quad d_n = \frac{(n+1)(n+2)}{2}. \end{aligned} \tag{1.3}$$

Products act from right to left throughout.

Theorem 1.1. *For every $n \geq 4$, the space $\mathcal{T}_n$ is precisely the tangent space of the explicit family in Section 2. In particular,*

$$\boxed{\dim_K \mathcal{T}_n = d_n, \quad \text{rank}(\mathcal{L}_Q|_{\mathcal{W}_n}) = 12n^2 - 6n - d_n.}$$

These are dimensions before taking any quotient by changes of basis or twists. The statement is over $\mathbb{Q}(q)$, and asserts no dimension formula at an exceptional specialization of q .

2. THE INTEGRABLE FAMILY AND ITS COORDINATE SLICE

Put $M = \mathbb{Z}^n \oplus \mathbb{Z}c$ and $\mu_i = w_i + c$. Choose an alternating multiplicative bicharacter χ on M , and set

$$G(ij) = \chi(\mu_i, \mu_j)ij, \quad D = \text{diag}(d_1, \dots, d_{n-1}, 1, \dots, 1).$$

All parameters are invertible. Allowing a scalar a and a parameter r near q , consider

$$\Phi = a(D \otimes D)GQ_n(r)G^{-1}(D \otimes D)^{-1}. \tag{2.1}$$

There are $2 + \binom{n+1}{2} + n - 1 = d_n$ parameters, with base point $a = d_i = \chi = 1, r = q$.

Lemma 2.1. *Every member of (2.1) satisfies the braid equation and preserves total weight.*

Proof. Only the assertion for G needs explanation. On $V^{\otimes m}$ let G_m multiply a basis tensor by $\prod_{s < t} \chi(\mu_{i_s}, \mu_{i_t})$. The sum of the two augmented weights is preserved by $Q_n(r)$, since the classical weight is preserved and each pair contains two copies of c . Bimultiplicativity makes every factor involving a spectator cancel in the ratio of two diagonal entries of G_m . Consequently

$$G_m Q_{k,k+1} G_m^{-1} = (GQ_n(r)G^{-1})_{k,k+1}.$$

Conjugating the braid equation proves the assertion. The operations involving a and D also preserve it. $\square$

Let $\mathcal{E}_n$ be the image of the differential of (2.1); then $\mathcal{E}_n \subseteq \mathcal{T}_n$. For a coefficient of H write $H_{ab \rightarrow cd}$. Define the following coefficient positions for the normalization:

$$\begin{aligned} F^{(1)} &= \{(ij \rightarrow ji) : n+2 \leq i \leq N-1, n+1 \leq j < i\}, \\ F^{(2)} &= \{(N1 \rightarrow jj') : 2 \leq j \leq N-1\}, \\ F^{(3)} &= \{(Nj \rightarrow jN) : n \leq j \leq N-1\}, \\ F^{(4)} &= \{(N(N-1) \rightarrow N(N-1)), (NN \rightarrow NN)\}. \end{aligned} \tag{2.2}$$

They are disjoint, and their union F has $\#F = d_n$.

Lemma 2.2 (Coordinate slice). *Evaluation $\pi_F : \mathcal{E}_n \rightarrow K^F$ is an isomorphism. In particular $\dim_K \mathcal{E}_n = d_n$, and every $H \in \mathcal{T}_n$ has a unique decomposition $H = H_0 + \tilde{H}$ with $H_0 \in \mathcal{E}_n$ and $\pi_F(\tilde{H}) = 0$.*

Proof. Use tangent parameters $a = 1 + \alpha\epsilon$, $r = q(1 + \beta\epsilon)$, $d_a = 1 + \ell_a\epsilon$, and $\chi = 1 + \epsilon\omega$, where $\epsilon^2 = 0$ and ω is alternating. Put $\ell_n = 0$ and set the diagonal parameters at negative indices to zero. In particular β multiplies the *logarithmic* derivative $q\partial_q$, not ∂_q . The full tangent coefficient is

$$\begin{aligned} H_{ij \rightarrow uv} &= \alpha Q_{ij \rightarrow uv} + \beta q \partial_q Q_{ij \rightarrow uv} \\ &\quad + Q_{ij \rightarrow uv}(\omega(\mu_u, \mu_v) - \omega(\mu_i, \mu_j) + \ell_u + \ell_v - \ell_i - \ell_j). \end{aligned}$$

For a tangent vanishing on F , the two positions in $F^{(4)}$ give

$$\begin{pmatrix} \delta & -q^{-1} - q \\ q^{-1} & -q^{-1} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0, \quad \det = 2.$$

Thus $\alpha = \beta = 0$. Write $h_a = \omega(\varepsilon_a, c)$ and $b_{ab} = \omega(\varepsilon_a, \varepsilon_b)$ in this proof only. Both orientations of each absolute index $a = 2, \dots, n$ in $F^{(2)}$, divided by its nonzero base coefficient, give

$$2(h_1 + h_a) + \ell_a - \ell_1 = 0, \quad 2(h_1 - h_a) + \ell_a - \ell_1 = 0.$$

Hence $h_a = 0$ for $a \geq 2$, $\ell_1 = 2h_1$, and $\ell_2 = \dots = \ell_{n-1} = 0$. The positions $F^{(1)}$ now kill all b_{ab} with $2 \leq a < b \leq n$. The two positions of $F^{(3)}$ involving ε_n and $-\varepsilon_n$ give $h_1 = b_{1n} = 0$, and the remaining positions give $b_{1a} = 0$. All tangent parameters vanish. The differential of the parameter space to K^F is therefore injective between two spaces of dimension d_n , proving all assertions. $\square$

3. NONOPPOSITE BLOCKS AND NORMALIZATION

Lemma 3.1 (Two and three letters). *Every $H \in \mathcal{T}_n$ has common scalars x, t and scalars u_{ab} for $a < b$, $b \neq a'$, such that*

$$\begin{aligned} H(aa) &= xaa, \\ H(ab) &= u_{ab}ba, \\ H(ba) &= (t - u_{ab})ab + ((q^2 + 1)x - qt)ba. \end{aligned}$$

Its remaining coefficients form an arbitrary, as yet unconstrained, matrix $A = (A_{ij})$ on $\langle z_j \rangle$, with $H(z_j) = \sum_i A_{ij}z_i$.

Proof. For one nonopposite pair $a < b$, initially write $H(aa) = x_aaa$, $H(bb) = x_bbb$, $H(ab) = pab + uba$, and $H(ba) = vab + dba$. The coefficient $aab \rightarrow aab$ of (1.3) is $(q^{-2} - 1)p$, so $p = 0$. After this substitution, the coefficients $baa \rightarrow baa$ and $bba \rightarrow bba$ are respectively

$$(1 - q^{-2})((q^2 + 1)x_a - q(u + v) - d), \quad -(1 - q^{-2})((q^2 + 1)x_b - q(u + v) - d).$$

Thus $x_a = x_b$ and the asserted diagonal formula holds with $t_{ab} = u + v$. For $a < b < c$ pairwise nonopposite, substitute these formulas. The coefficients $cab \rightarrow cab$ and $bca \rightarrow bca$ are

$$(q - q^{-1})(t_{ab} - t_{ac}), \quad (q - q^{-1})(t_{ac} - t_{bc}).$$

These computations use only (1.1). The graph of distinct nonopposite weights is a complete graph minus a perfect matching. Its vertices are connected and its edges are connected through triangles: two edges sharing a vertex are in a triangle unless their other endpoints are opposite; in that case insert a vertex avoiding these three absolute indices, which exists for $n \geq 4$. Disjoint edges can first be joined through an edge meeting both (choosing endpoints that are not opposite). This gives common x, t . The only remaining weight space is the zero-weight space. $\square$

From now on assume $\pi_F(H) = 0$, subtracting the unique element of $\mathcal{E}_n$ provided by Lemma 2.2. The two positions in $F^{(4)}$ give $x = t = 0$. We can therefore write

$$H(ii) = 0, \quad H(ij) = s_{ij}ji \ (i \neq j, j \neq i'), \quad s_{ij} = -s_{ji}. \quad (3.1)$$

All coefficients between two distinct negative indices vanish, by $F^{(1)}$ and the negative positions in $F^{(3)}$. Also

$$s_{nN} = 0, \quad A_{iN} = 0 \quad (2 \leq i \leq N - 1). \quad (3.2)$$

Let $I = \{2, \dots, N - 1\}$, and set

$$c_i = B_{iN}, \quad c_1 = q, \quad c_i = -\delta q^{n-1-\rho_i} \ (i \in I), \quad c_N = \delta(1 - q^{2(n-1)}). \quad (3.3)$$

Every c_i with $i \in I$ is nonzero. For $2 \leq k \leq n$ put

$$\tau_k = \frac{c_{k'}}{c_k} = q^{2(n-k)}. \quad (3.4)$$

Useful consequences of (1.2), valid for internal indices, are

$$B_{k'k} = q, \quad B_{Nk} = c_k, \quad B_{ik} = -\delta \frac{c_i}{c_{k'}} \mathbf{1}_{i>k'} \quad (i \notin \{k, k'\}), \quad B_{kk} = 0 \quad (k \leq n), \quad B_{n'n'} = 0. \quad (3.5)$$

4. OUTER COLUMNS AND THE UNIFORM COEFFICIENT EQUATIONS

Lemma 4.1 (Outer reduction). *A normalized H satisfies*

$$A_{\bullet 1} = 0, \quad A_{\bullet N} = \gamma e_N, \quad s_{iN} = 0 \quad (i \in I) \quad (4.1)$$

for a scalar $\gamma = A_{NN}$.

Proof. Here and below a displayed coefficient of $\mathcal{L}_Q(H)$ must be zero. Applying (1.3) to $11N$ gives

$$[11N \rightarrow 11N] = (\lambda^2 - q^2)A_{11}, \quad [11N \rightarrow i1i'] = \delta A_{i1} \quad (i \in I).$$

Thus $H(z_1) = \eta z_N$. By (3.2), write $H(z_N) = \zeta z_1 + \gamma z_N$, and put $a_i = s_{iN}$. The next coefficient is

$$[NN1 \rightarrow ii'N] = c_i(\lambda\zeta + a_i + a_{i'}) \quad (i \in I).$$

Take $i = n$: both $a_n = 0$ and $a_{n'} = 0$, so $\zeta = 0$. For every positive internal i , the coefficient $a_{i'}$ is zero because both its indices are negative. Hence every $a_i = 0$. Finally

$$[N1N \rightarrow Nii'] = -\lambda c_i \eta \quad (i \in I)$$

forces $\eta = 0$. This proves (4.1). Each coefficient follows by substituting (1.1)–(1.2) in the six terms of (1.3); none requires a numerical value of n . More explicitly, the six signed contributions to the first two coefficients are

$$(0, \lambda^2 A_{11}, 0, 0, -q^2 A_{11}, 0), \quad (0, \lambda A_{i1}, 0, 0, -q A_{i1}, 0).$$

After the column-1 reduction, the contributions to $NN1 \rightarrow ii'N$ are $(0, \lambda c_i \zeta, 0, c_i a_{i'}, c_i a_i, 0)$. After $\zeta = a_i = 0$, only the sixth term contributes to $N1N \rightarrow Nii'$, and it equals $-\lambda c_i \eta$. $\square$

Put $b_k = s_{1k}$ for $k \in I$. The following equations retain all internal zero-weight coefficients, including the diagonal and reflected entries.

Lemma 4.2 (Outer-pair equations). *Under (3.1) and (4.1), one has, for $i, k \in I$,*

$$b_i + b_{i'} = 0, \quad (4.2)$$

$$\delta \sum_{j \in I} c_j A_{ij} + \lambda \gamma c_i = 0. \quad (4.3)$$

If $i \notin \{k, k'\}$, one further has

$$c_i(b_k + s_{ki} + s_{ki'}) + \delta c_{k'} \mathbf{1}_{i < k'} A_{ik} = 0, \quad (4.4)$$

$$c_{k'} A_{ik} + \delta c_i \mathbf{1}_{i > k'} (s_{ki} + s_{ki'}) = 0. \quad (4.5)$$

For every $k \in I$ one has

$$A_{1k} = 0, \quad (4.6)$$

$$c_{k'}(A_{Nk} + c_k b_k) + \delta \gamma = 0. \quad (4.7)$$

Finally, for every $k \in I$ and $1 \leq r \leq N$,

$$\lambda c_k A_{rk} + c_{k'} B_{kk} A_{r,k'} + \delta \sum_{\substack{j \in I, j < k \\ j \neq k'}} c_j A_{rj} + c_{k'} A_{kk} B_{r,k'} = -b_k c_k \mathbf{1}_{r=k'}. \quad (4.8)$$

Proof. The two coefficients of $N11$ with outputs $1ii'$ and $ii'1$ are respectively $-c_i(b_i + b_{i'})$ and the left side of (4.3). For the first, only terms one and two contribute, giving $-c_i b_i$ and $-c_i b_{i'}$; for the second, only terms one and three contribute, giving $\delta \sum_{j \in I} c_j A_{ij}$ and $\lambda \gamma c_i$. The coefficients of $kN1$ with outputs $ii'k$ and iki' are the negatives of the left sides of (4.4) and (4.5). Before using (3.5), the latter coefficient is

$$c_{k'} B_{ik} s_{ki'} - c_{k'} A_{ik} - \delta c_i \mathbf{1}_{i > k'} s_{ki}.$$

For input $kN1$ and output $1kN$, only the fifth term contributes, giving $-c_{k'}A_{1k}$. For input $kN1$ and output $Nk1$, terms four, five, and six give $-c_{k'}c_k b_k$, $-c_{k'}A_{Nk}$, and $-\delta\gamma$, proving (4.7).

For completeness, project all six terms at input $kN1$ to output $kr r'$. Their contributions, in the order in (1.3), are

$$\begin{aligned} 0, & \quad -b_k c_k \mathbf{1}_{r=k'}, & 0, \\ & -\lambda c_k A_{rk} - c_{k'} B_{kk} A_{r,k'} - \delta \sum_{\substack{j \in I, j < k \\ j \neq k'}} c_j A_{rj}, \\ & -c_{k'} A_{kk} B_{r,k'}, & 0. \end{aligned}$$

For example, in the fourth term Q_{23} first gives $\sum_j c_j k j j'$. The three cases $j = k$, $j = k'$, and $j \notin \{k, k'\}$ give exactly its three displayed summands. In the last case the part of $Q(kj)$ retaining first index k occurs precisely when $j < k$. This proves (4.8) for all r , including its overlapping or central cases.

The other coefficient formulas are obtained by the same three-case separation. For output $ii'k$, the second, fourth and fifth terms contribute respectively $-b_k c_i$, $-c_i s_{ki'}$, and $-c_i s_{ki} - \delta c_{k'} \mathbf{1}_{i < k'} A_{ik}$; the remaining terms contribute zero. For output iki' , the fourth and fifth terms give the expression displayed above. This supplies an index-uniform expansion for the equations, independently of the diagnostic scripts. $\square$

5. ELIMINATION OF THE ZERO-WEIGHT BLOCK

Lemma 5.1 (Off-diagonal entries). *For $i, k \in I$, $i \notin \{k, k'\}$, the equations imply*

$$s_{ki} + s_{ki'} = -b_k, \quad A_{ik} = \delta \frac{c_i}{c_{k'}} b_k \mathbf{1}_{i > k'}. \quad (5.1)$$

Proof. If $i < k'$, (4.5) gives $A_{ik} = 0$ and (4.4) gives the exchange sum. If $i > k'$, use (4.4) first and then (4.5). All divisions are by nonzero $c_i, c_{k'}$. $\square$

Lemma 5.2 (Positive and reflected entries). *For $2 \leq k \leq n$,*

$$A_{kk} = 0, \quad A_{k',k} = -qb_k, \quad A_{k,k'} = qb_k. \quad (5.2)$$

Also $A_{n',n'} = 0$.

Proof. Use (4.8) with a positive k and $r = k$. Its sum involves only positive $j < k$ and hence vanishes by (5.1): the row index k is smaller than j' . Since $B_{kk} = 0$ and $B_{k,k'} = q$, it reduces to

$$(\lambda c_k + q c_{k'}) A_{kk} = 0.$$

The coefficient is nonzero by (3.4). Use (4.8) again with $r = k'$. The earlier-column entries still vanish, and $A_{kk} = 0$, giving $\lambda c_k A_{k',k} = -b_k c_k$.

Next use (4.8) with input index k' and output index $r = k$. The second term contains $A_{kk} = 0$, and the last term contains $B_{kk} = 0$. In its sum, a positive column $j \neq k$ contributes zero because $k < j'$; a negative column $j = a' < k'$ has $a > k$, so again the row index $k < j'$ gives zero. Thus $A_{k,k'} = -qb_{k'} = qb_k$ by (4.2).

Finally set the input and row index to n' in (4.8). The middle diagonal $B_{n'n'}$ is zero, and every preceding column in the sum is positive other than n , so contributes zero. Since $c_{n'} = c_n$, the result is $(\lambda + q)c_n A_{n'n'} = 0$. $\square$

Lemma 5.3 (Recurrence and central boundary). *All b_i and γ vanish.*

Proof. For a positive row $k = 2, \dots, n$, substitute (5.1) and (5.2) in (4.3) and divide by c_k . This gives

$$\lambda\gamma + \delta q \tau_k b_k - \delta^2 \sum_{a=2}^{k-1} \tau_a b_a = 0. \quad (5.3)$$

Indeed, the reflected column k' contributes $c_{k'} qb_k$, and the only other nonzero entries are columns a' with $a < k$, each contributing $-\delta c_k \tau_a b_a$. The positive diagonal and all other columns contribute zero.

For the negative central row n' , the same equation gives

$$\lambda\gamma - \delta qb_n - \delta^2 \sum_{a=2}^{n-1} \tau_a b_a = 0, \quad (5.4)$$

using $A_{n'n} = -qb_n$, $A_{n'n'} = 0$, and $\tau_n = 1$. Subtracting (5.4) from (5.3) at $k = n$ gives $2\delta qb_n = 0$, hence $b_n = 0$.

Subtract (5.3) at k from the same equation at $k + 1$. As $q + \delta = \lambda$ and $\tau_k = q^2 \tau_{k+1}$, the difference is

$$\delta q \tau_{k+1} (b_{k+1} - b_k) = 0 \quad (2 \leq k < n). \quad (5.5)$$

All positive internal b_k are equal to $b_n = 0$. Equation (4.2) kills the negative ones, and (5.3) at $k = 2$ gives $\gamma = 0$. $\square$

Proof of Theorem 1.1. It remains to prove that a normalized H is zero. The preceding lemmas show that $b_i = 0$, $\gamma = 0$, and all outer exchanges vanish. For distinct $a, b \in \{2, \dots, n\}$, apply the exchange equation in (5.1) with $k = b'$ and $i = a'$. The negative-negative exchange $s_{b'a'}$ is zero by normalization, so $s_{b'a} = 0$. Thus all internal mixed-sign nonopposite exchanges vanish. Apply the same equation with $k = a$, $i = b$ to kill s_{ab} . All exchanges in (3.1) have now vanished.

Equation (5.1) kills every internal off-diagonal entry except the reflected ones, which are zero by (5.2). Equations (4.6), (4.7), and (4.1) kill all entries meeting an outer row or column. The positive diagonal is already zero. For a negative index k' with $2 \leq k \leq n$, equation (4.8) with input and row index k' now reduces to

$$(\lambda c_{k'} + q c_k) A_{k',k'} = 0.$$

Its coefficient is nonzero in K , so the remaining diagonal entries vanish. Hence $H = 0$. Lemma 2.2 gives $\mathcal{T}_n = \mathcal{E}_n$, and the asserted dimension and rank follow. $\square$

Corollary 5.4 (Formal local completeness). *Let $\mathcal{Y}_n \subseteq \mathcal{W}_n$ be the affine K -scheme defined by all coefficients of the braid polynomial*

$$\mathcal{B}(X) = X_{12}X_{23}X_{12} - X_{23}X_{12}X_{23}.$$

For every commutative local Artin K -algebra $(A, \mathfrak{m})$ with residue field K , the parameters of (2.1) reducing to their base values biject naturally with

$$\mathcal{D}_Q(A) = \{X \in A \otimes_K \mathcal{W}_n : \mathcal{B}(X) = 0, X \bmod \mathfrak{m} = Q\}.$$

Consequently

$$\widehat{\mathcal{O}}_{\mathcal{Y}_n, Q} \cong K[[t_1, \dots, t_{d_n}]].$$

This is a formal parametrization of the specified operator scheme, before taking any gauge quotient.

Proof. Write the family parameters locally as $a - 1$, $r/q - 1$, $d_i - 1$, and $\chi(e_u, e_v) - 1$ for a basis of M . The map from these parameters to $\pi_F(\Phi - Q)$ has invertible differential by Lemma 2.2. The formal inverse function theorem gives a unique parameter tuple with any specified F -coordinates in $\mathfrak{m}^F$; its power series evaluate in A because $\mathfrak{m}$ is nilpotent.

Given $X \in \mathcal{D}_Q(A)$, choose these parameters for $\pi_F(X - Q)$ and let Y be the corresponding family member. Thus $\Delta = X - Y \in \mathfrak{m} \otimes_K \mathcal{W}_n$ and $\pi_F(\Delta) = 0$. Suppose inductively that $\Delta \in \mathfrak{m}^r \otimes_K \mathcal{W}_n$. Since $X, Y \equiv Q \pmod{\mathfrak{m}}$, expanding the cubic polynomial gives

$$0 = \mathcal{B}(X) - \mathcal{B}(Y) \equiv \mathcal{L}_Q(\Delta) \pmod{\mathfrak{m}^{r+1} \otimes_K \text{End}_K(V^{\otimes 3})}.$$

Its graded class therefore belongs to

$$\overline{\Delta} \in (\mathfrak{m}^r / \mathfrak{m}^{r+1}) \otimes_K \mathcal{T}_n, \quad (\text{id} \otimes \pi_F)(\overline{\Delta}) = 0.$$

Tensoring kernels with a K -vector space is exact. Theorem 1.1 and Lemma 2.2 imply $\overline{\Delta} = 0$, so $\Delta \in \mathfrak{m}^{r+1} \otimes_K \mathcal{W}_n$. Nilpotence now gives $\Delta = 0$. Uniqueness of the parameters follows from their F -coordinates. The construction commutes with homomorphisms of local Artin K -algebras; it thus identifies the represented formal functors and their complete local rings. No analytic or global classification, or assertion at exceptional q , follows from this argument. $\square$

6. A PRECISE DICTIONARY WITH EARLIER MULTIPARAMETER MATRICES

The explicit family should be distinguished from its exhaustion of $\mathcal{T}_n$. Earlier work already provides the former, including the augmented-weight directions. Here is a direct comparison with Isaev–Ogievetsky–Pyatov [7, Eq. (43)]; see also Isaev [8, Sec. 3.10].

Let $Jv_i = v_{i'}$, $\eta(w) = \chi(w, c)$, and $\chi_0 = \chi|_{\mathbb{Z}^n \times \mathbb{Z}^n}$. Bimultiplicativity gives the exact factorization

$$\chi(w_i + c, w_j + c) = \chi_0(w_i, w_j)\eta(w_i)\eta(w_j)^{-1}, \quad G = G_0(A_\eta \otimes A_\eta^{-1}),$$

where $A_\eta v_i = \eta(w_i)v_i$ and $G_0(ij) = \chi_0(w_i, w_j)ij$. Thus the $\binom{n+1}{2}$ alternating parameters split into $\binom{n}{2}$ ordinary weight twists and n character parameters.

For clarity write the parameter in [7] as s and put $\Lambda = s - s^{-1}$. Its type- D formula is

$$\begin{aligned} \widehat{R}_s^f = & \sum_{i,j} s^{\mathbf{1}_{i=j}-\mathbf{1}_{i=j'}} \frac{f_{ij}}{f_{ji}} E_{ij} \otimes E_{ji} + \Lambda \sum_{i < j} E_{ii} \otimes E_{jj} \\ & - \Lambda \sum_{i > j} s^{\rho_i - \rho_j} \frac{f_{i'j}}{f_{jj'}} E_{i'j} \otimes E_{ij'}. \end{aligned}$$

For its admissible diagonal twists it is sufficient that $f_{ij}f_{i'j} = U_j$ and $f_{ij}f_{ij'} = W_i$ be independent of the paired index, with $U_i U_{i'} = W_i W_{i'}$ constant. Choose

$$s = r^{-1}, \quad f_{ij} = \chi(\mu_{i'}, \mu_{j'}), \quad U_j = \eta(w_j)^2, \quad W_i = \eta(w_i)^{-2}.$$

These identities follow from $\mu_i + \mu_{i'} = 2c$; both constant products equal 1. Reversing all basis indices and using $\rho_{i'} = -\rho_i$ gives, term by term,

$$(J \otimes J) \widehat{R}_{r^{-1}}^f (J \otimes J) = G Q_n(r) G^{-1}.$$

For example, its diagonal term $i < j$ becomes $i' > j'$, and its contraction term becomes the last term of (1.2). The remaining D in (2.1) is an ordinary vector-basis change, and a is an overall scalar. Hence our entire explicit family is contained in this earlier construction after these elementary operations.

The additional parameter denoted by κ_i in [8, Eq. (3.10.35)] has the concrete value

$$\kappa_i = \frac{f_{i'i}}{f_{ii'}} = \eta(w_i)^4.$$

In the formal neighborhood of $\chi = 1$, imposing $\kappa_i = 1$ forces $\eta(w_i) = 1$ in characteristic zero. After also removing scalar and diagonal-basis freedoms, the remaining parameter count is $1 + \binom{n}{2}$, consistent with the usual type- D multiparameter count. Neither the extra symbols nor their number gives a new family. The assertion proved here is that the *whole* weight-preserving formal neighborhood, with no triangular-support or fixed-metric equation imposed initially, lies in the known family. Whether this unrestricted exhaustion already has an equivalent statement elsewhere remains a bibliographic question.

Remark 6.1 (Cocycles, gauge, and rack deformation results). Our dimension d_n is a cocycle-space dimension, not a second-cohomology dimension after quotienting by basis gauge. In the standard normalization $X = Q(1 + \epsilon f)$ one has $f = Q^{-1}H$. Infinitesimal weight-preserving vector-basis changes give

$$H = [T \otimes 1 + 1 \otimes T, Q], \quad T = \text{diag}(t_1, \dots, t_N).$$

Writing $\sigma_i = t_i + t_{i'}$, their zero-block coefficient is $(\sigma_i - \sigma_j)B_{ij}$ and all nonopposite-block coefficients vanish. By (3.3), this commutator is zero exactly when all n values σ_i agree. Its image therefore has dimension $n - 1$. The weight-zero second Yang–Baxter cohomology, with this usual basis-gauge quotient, consequently has dimension

$$d_n - (n - 1) = \binom{n+1}{2} + 2.$$

This is a consequence of Theorem 1.1, not an independent novelty claim. Allowing all $T \in \text{End}_K(V)$ gives the same weight-zero coboundaries: the differential preserves weight shifts, so a weight-zero coboundary is already the differential of the weight-zero part of T .

Eisermann’s rack classification results [9, 10] concern linearizations of finite rack permutations. Their eigenvalues are roots of unity. Our nonopposite block has eigenvalues q^{-1} and $-q$, whose ratio is not a root of unity over K . Thus generic Q cannot even be a scalar multiple of an operator similar to such a finite rack permutation. The general cohomological language applies, but those rack-specific classification theorems do not directly calculate the present tangent space.

7. CONTEXT, PROVENANCE, AND AI ASSISTANCE

The standard operator is already explicit in [1, Eq. (3.21)]. Bicharacter conjugation of braided intertwiners and its application to the classical vector matrices are described in [1, Prop. 3.9 and Sec. 3.3]; the two-parameter classical matrices are also treated in [2]. These works are essential comparisons for the construction in Section 2.

Schirmacher [3, pp. L1253–L1254, Eqs. (22)–(31)] constructs multiparameter B, C, D matrices using an ansatz motivated by the standard solutions and compatibility with an orthogonal or symplectic metric. The present ambient space permits every zero-weight-block coefficient and assumes no fixed metric equation. Whether that earlier ansatz, after allowing the changes of coordinates used here, already implies precisely our local exhaustion theorem requires further comparison. No novelty conclusion follows from the difference in presentation.

García-Gavarini [4] establish broad twist and cocycle deformation results for formal multiparameter quantized enveloping algebras. Frønsdal [5, Secs. 12–15] studies first-order and exact deformations of universal R -matrices in specified classes. Zhuang–Hu–Xu [6] derive a two-parameter type- D basic matrix and its affine RLL realization. These algebraic settings must be compared with the full, fixed vector-operator tangent problem before either treating their results as an exhaustion theorem for $\mathcal{W}_n$ or claiming that they leave it open. The mathematical proof above does not settle that bibliographic issue. In particular an integrable twist family is not presented as a new quantum-group construction.

Data availability. The source submission contains verification programs, rank certificates, and machine-readable results under `anc/`. Reproduction commands and the scope of each check are given in Appendix A. The mathematical proof is independent of these finite diagnostics. The same materials are available in the [versioned companion repository](#) (commit `7f0a4e4aa8a3`).
Provenance, AI assistance, and human understanding. This work originated in a computational study carried out by Hangang Wang in January 2026, following preliminary research-direction discussions with Rushu Zhuang. The historical MATLAB implementations and computational experiments were carried out by Wang with AI-assisted programming, using previously published mathematical constructions cited in this manuscript. The uniform deformation analysis reported here was developed subsequently in October 2026 with extensive assistance from OpenAI ChatGPT and Codex in mathematical exploration, proof derivation, exact computation, literature comparison, and manuscript preparation. The January dates do not date the general-rank theorem or its proof.

The human contributor understands the original computational work but does not claim full understanding or independent verification of the complete general-rank proof. Multiple AI-assisted mathematical and computational audits, including separately implemented programs, have found no counterexample to the main theorem. They do not constitute independent verification by a human specialist or a formal proof-assistant verification; neither has been completed for this version.

This version is made available to facilitate independent mathematical scrutiny and reproducibility. The explicit multiparameter family is acknowledged as prior art; bibliographic priority for the full tangent-space exhaustion remains unresolved. Private correspondence and third-party reference books are not included in the submission.

APPENDIX A. COMPUTATIONAL VERIFICATION AND REPRODUCIBILITY

The following computations corroborate the transcription and implementation of the formulas; the general-rank result follows from the written index-uniform proof.

A.1. Finite-rank certificates. The supplied D_4 – D_7 pivot rows and columns were imported as data, and all entries of their minors were independently reconstructed from (1.1)–(1.2) and (1.3). The main certificates at $p = 1000000007$ and $q = 2$ are listed in Table 1.

TABLE 1. Exact modular rank certificates for the full weight-preserving space.

n	$\dim \mathcal{W}_n$	minor size	determinant modulo p	d_n
4	168	153	673269329	15
5	270	249	670446451	21
6	396	368	912656330	28
7	546	510	392961350	36

For D_4 – D_6 the determinants and row-major hashes match the supplied values. The supplied D_7 data contained the row and column choices but no separate determinant value; the displayed determinant is newly computed. The alternate D_4, D_5 certificates at $q = 3$, $p = 1000000009$ also give nonzero determinants. These are exact modular certificates of generic lower bounds on Jacobian rank, not floating-point experiments.

The coefficient identities in Sections 4 and 5 were also expanded independently over $\mathbb{Z}[q, q^{-1}]$ at $n = 4, 5, 6$. The written proof, rather than this finite diagnostic, supplies the assertion for every n .

A.2. Independent coefficient checks and software. Separately implemented, AI-assisted programs provide additional coefficient checks. Their 150 outer-column and 628 internal modular comparisons were replayed successfully, and their five symbolic expressions were checked by explicit equality assertions. An additional 28 comparisons directly check input $kN1$ and output $Nk1$ in (4.7). These checks do not constitute human specialist peer review or a formal proof-assistant verification.

The parameter dictionary in Section 6 was also checked directly over a Laurent ring in all 15 (respectively 21) independent parameters at $n = 4$ (respectively 5), by comparing all 299 coefficient positions and 346 twist-constraint identities across those two ranks. The argument for the dictionary is the written uniform reindexing; these calculations check its transcription.

The accompanying scripts use Python 3.10 or later. The main coefficient, family, minor, and dictionary checks use only the standard library:

```
python anc/code/run_all.py
```

To also replay the independent symbolic audit, install SymPy and run

```
python anc/code/run_all.py --include-independent-audit
```

The submission source ZIP includes the public verification scripts, independent audit implementations, rank certificates, and JSON results under `anc/`. The replay uses temporary copies and labels the additional row- N check separately. The platform does not execute this code; readers can run it locally. Native MATLAB was not rerun for the October verification.

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