

ANALYTIC q -INEQUALITIES AND A PROOF OF BERGERON'S CONJECTURE

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ABSTRACT. For integers $1 \leq a \leq b \leq c \leq d$ with $ad = bc$, Bergeron asked whether the difference of Gaussian binomial coefficients $\binom{b+c}{b}_q - \binom{a+d}{a}_q$ has nonnegative coefficients. We prove that it does, and more generally that $\binom{b+c}{b}_q - \binom{a+d}{a}_q$ has nonnegative coefficients whenever $a \leq b \leq c$ and $ad \leq bc$, a strengthening proposed by Troyka. The proof is analytic rather than bijective. We develop a toolkit for coefficientwise inequalities, including a coefficientwise local limit theorem for Gaussian polynomials based on Irwin–Hall densities, and use it in a Pascal-type induction that proves the case $a \leq 20$. For general a the problem reduces to a two-parameter inequality between linearly interpolated Gaussian polynomials. Near its base case this inequality reduces exactly to the monotonicity of rational q -Catalan numbers. Using the circle method with explicit error terms, we prove the conjecture of Chen, Wang and Wang that the coefficients of MacMahon's q -Catalan numbers $C_n(q)$ are unimodal for every $n \geq 16$, and an analogue for the rational q -Catalan numbers $C_{x,N+1}(q)$ with $x \leq N \leq 2x$. Beyond this range the difference splits into two nonnegative parts, which we compare degree by degree using local limit bounds for Gaussian polynomials that are uniform in both parameters. The proof is computer-assisted: it uses exact integer and rational arithmetic, ball arithmetic, and one floating-point computation with a proved error bound. The scripts and the logs of their runs are provided as ancillary files.

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INTRODUCTION

For integers $x, y \geq 0$ let

$$G(x, y) = \binom{x+y}{x}_q = \prod_{i=1}^x \frac{1 - q^{y+i}}{1 - q^i}$$

be the Gaussian polynomial, the generating function of the partitions whose Young diagram fits in an $x \times y$ box. For power series P, Q with integer coefficients write $P \geq_q Q$ when every coefficient of $P - Q$ is nonnegative. Bergeron asked whether

$$\binom{b+c}{b}_q \geq_q \binom{a+d}{a}_q \quad \text{whenever } 1 \leq a \leq b \leq c \leq d \text{ and } ad = bc$$

(see [22]; the question comes from his q -analogue of Foulkes' conjecture [4]). Both sides are palindromic polynomials of degree $ad = bc$, so the conjecture says that, among boxes of a given area, a box closer to a square contains at least as many partitions of every size. Since $G(x, y)$ is, up to a shift, the character of $S^x(S^y \mathbb{C}^2)$, the conjecture compares weight multiplicities of two SL_2 -modules. Zanello [22] conjectured that the difference is even unimodal, which is the statement that $S^a(S^d \mathbb{C}^2)$ embeds in $S^b(S^c \mathbb{C}^2)$, the two-row case of Vesseney's generalized Foulkes conjecture [20], and he proved this for $a \leq 3$ and $b, c \geq 4$. Amdeberhan and Callan [2] proved the case $c = 2a$, $d = 2b$, and Amdeberhan and Beck [1] proved the inequality at $q = 1$, also for real parameters. Gangl, Gutiérrez and Szwey [9], using a theorem of Raicu, Sam, Weyman and Yang [16] on the Foulkes–Howe map, proved the conjecture, together with unimodality, when a divides b or c , in particular for every prime a . Troyka [19] proposed relaxing $ad = bc$ to $ad \leq bc$; Gangl, Gutiérrez and Szwey also prove this form under their divisibility condition. Our main result settles the conjecture in this stronger form.

Theorem E (Bergeron's conjecture, in Troyka's form). *Let $1 \leq a \leq b \leq c$, and let d be any integer with $ad \leq bc$. Then*

$$\binom{b+c}{b}_q \geq_q \binom{a+d}{a}_q.$$

In particular Bergeron's conjecture holds: $\binom{b+c}{b}_q - \binom{a+d}{a}_q$ has nonnegative coefficients whenever $1 \leq a \leq b \leq c \leq d$ and $ad = bc$.

Theorem E is Corollary 9.29. The smallest case of Bergeron's conjecture that the divisibility results do not cover is $(a, b, c, d) = (4, 6, 6, 9)$, shown in Figure 1.

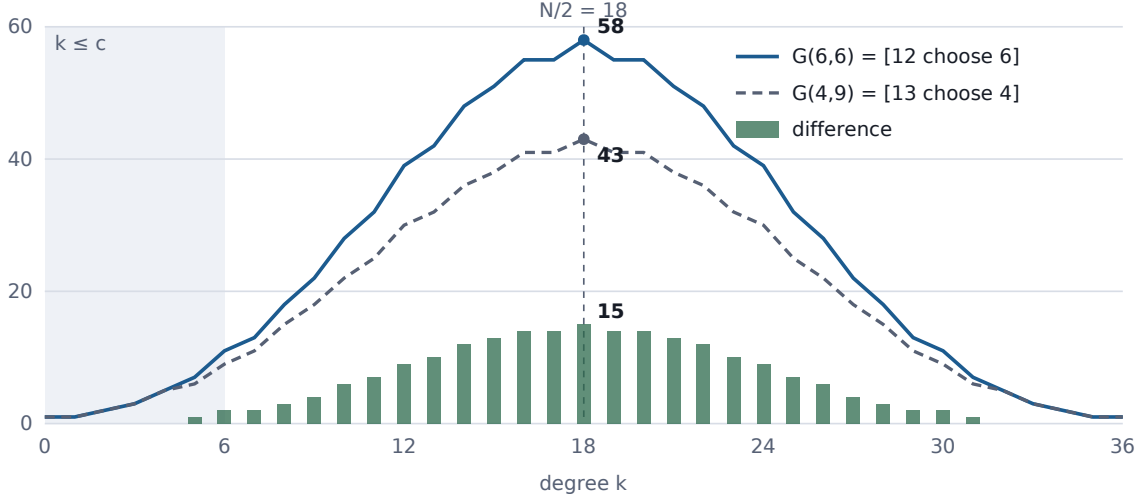


FIGURE 1. Coefficients of $G(6,6)$ (solid) and $G(4,9)$ (dashed), the case $(a, b, c, d) = (4, 6, 6, 9)$, which the divisibility results do not cover. Bars: the difference, zero through degree 4 and largest (15) at the center $N/2 = 18$. The shaded zone $k \leq c$ is where both sides are partition counts $p_{\leq 6}(k) \geq p_{\leq 4}(k)$.

Approach. No injection is constructed. The paper develops methods for proving coefficientwise inequalities $P \geq_q Q$ by analytic means, that is, by identities with visibly nonnegative terms, by estimates, and by evaluations, and Bergeron's conjecture is the test case on which they are developed. The order $\geq_q$ forbids most real-variable tools (Section 1): one cannot cancel a common factor, take roots or logarithms, or differentiate. Sections 2–7 collect substitutes: truncation and half-range arguments, the absorption of one factor by unimodality, a coefficientwise Karamata inequality, a smoothing identity that turns the coefficients of $[n]^m$ into values of the Irwin–Hall density and gives a coefficientwise local limit theorem for Gaussian polynomials, and an induction scheme along Pascal's rule (Section 6). Section 7 summarizes them as a dictionary between real-variable tools and their coefficientwise replacements.

Run on Bergeron's conjecture, this toolkit gives the following.

Theorem A. *Let $1 \leq a \leq 20$, let $a \leq b \leq c$, and let d be any integer with $ad \leq bc$. Then*

$$\binom{b+c}{b}_q \geq_q \binom{a+d}{a}_q.$$

Theorem A' (Section 8.7) proves more for $a \leq 20$: writing $bc = aD + r$ with $0 \leq r < a$,

$$a \binom{b+c}{b}_q \geq (a-r) \binom{a+D}{a}_q + r \binom{a+D+1}{a}_q \quad \text{in degrees } k \leq bc/2.$$

This fractional form has an induction step that involves only the family $G(a, \cdot)$, linearly interpolated between integers. If that step holds for all b and c (Conjecture 8.10), Bergeron's conjecture follows for every a . The step is proved here for $a \leq 20$, and it has no exceptions in any computation. Its tightest case in all computations, $b = c = a + 1$, has an exact formula (Proposition 8.12) that reduces it to a monotonicity property of MacMahon's q -Catalan numbers

$C_n(q) = \frac{1}{[n+1]} \binom{2n}{n}_q$. The circle method of [Section 9](#) settles that case for every a : it proves the conjecture of Chen, Wang and Wang [\[6\]](#) that these q -Catalan numbers are unimodal for all $n \geq 16$ (see also [\[10, Conjecture 1.8\]](#)).

Theorem C' (the conjecture of Chen, Wang and Wang). *For every $n \geq 16$ the coefficients c_k of $C_n(q)$ satisfy $c_{k-1} \leq c_k$ for $2 \leq k \leq n(n-1)/2$. Hence $(c_1, \dots, c_{n(n-1)-1})$ is unimodal.*

The proof is an explicit estimate for $n \geq 1001$, checked in ball arithmetic, together with exact computation below that. As a consequence $(1+q)C_n(q)$ is unimodal for every n ([Corollary 9.14](#)), the case $(n, n+1)$ of a conjecture of Xin and Zhong [\[21\]](#).

The same corner $b = c = a + 1$ is the base case of a second reduction ([Proposition 8.15](#)): the first row $b = a + 1$ of the fractional form, with c allowed to be fractional, already implies Bergeron's conjecture for every a . This two-parameter inequality $R(x, A)$, the fractional first row, holds in every computed case ([Conjecture 8.16](#)). Its first band is proved for every x , which gives the conjecture near the diagonal.

Theorem D. *Let $1 \leq a \leq b \leq c$ with $bc \leq 2a(a+1)$, and let d be any integer with $ad \leq bc$. Then $\binom{b+c}{b}_q \geq_q \binom{a+d}{a}_q$.*

[Theorem D](#) is [Corollary 8.20](#). On the band $x^2 \leq A \leq 2x(x-1)$, the fractional first row reduces exactly to the monotonicity of the rational q -Catalan numbers $C_{x,N+1}(q)$ with $x \leq N \leq 2x-2$ ([Proposition 8.19](#)). [Section 9.5](#) proves this monotonicity for every $x \geq 201$ by the circle method, with a second ball-arithmetic certificate, and exact computation covers $13 \leq x \leq 200$. The same argument extends [Theorem C'](#) to the rational q -Catalan numbers $C_{x,N+1}(q)$ with $x \leq N \leq 2x$ ([Theorem 9.16](#)).

Beyond the band the difference splits into two nonnegative parts whose ratio stays below 0.624 in every computed case ([Proposition 8.23](#) and [Observation 8.24](#)), so what is needed there is analysis with a margin of order one ([Remark 8.25](#)). [Section 9.6](#) carries it out for $x \geq 201$ and [Section 9.7](#) for $22 \leq x \leq 200$. Together with Theorems A and D this proves [Theorem E](#). In more detail, the circle method compares the two parts degree by degree. Their ratio at a real point is bounded by a sum of explicit terms ([Lemma 9.20](#)), evaluated in interval arithmetic on cells, with a closed form for large A/x^2 . The local limit constants of all Gaussian coefficients $\binom{m+n}{m}_q$ with $m \geq 199$ and $n \geq 1.98m$ are bounded uniformly in m and n , for every tilt that occurs, by two more ball-arithmetic certificates: a table in scaled variables for $n \leq 16.25m$, and five regimes for larger n , one of them a transfer to independent summands through the cycle index of the symmetric group; the regimes hold for $m \geq 99$. The degrees just above A/x are settled by a count of partitions ([Lemma 9.19](#) (b)). For $22 \leq x \leq 200$, a sweep in floating-point arithmetic with a proved error bound covers $A \leq 64x^2$, $32x^2$ or $16x^2$, and the same regimes, run for $m \geq 20$ where needed, cover larger A .

For every fixed a the method of [Theorem A](#) reduces the inequality to a finite, explicit computation ([Theorem B](#)). The proofs use q -series identities, Sylvester's unimodality theorem, a smoothing identity that turns coefficients of $[n]^m$ into values of a continuous density, the circle method with explicit error terms, and finite computations in exact, ball or floating-point arithmetic with proved error bounds.

Organization. The paper has three parts. Sections 1–7 set out the toolkit: what the order $\geq_q$ allows, what it forbids, and a small set of lemmas that replace the missing real-variable tools, including a coefficientwise local limit theorem ([Section 5](#)). [Section 8](#) runs the toolkit on Bergeron's conjecture, including the approaches that failed ([Section 8.9](#)). [Section 9](#) adds a second analytic tool, the circle method, which proves an inequality by estimating an integral instead of exhibiting nonnegative terms, and uses it to settle the tightest case of [Section 8](#), the

band behind [Theorem D](#), and the first row beyond the band, which completes the proof of [Theorem E](#). Statements are numbered within sections; the main results carry letters.

Computer assistance. The proof is computer-assisted. Besides the exact verification of finitely many cases, it uses symbolic positivity checks of explicit polynomials ([Proposition 8.5](#)), certificates in ball arithmetic (Arb [11], through python-flint) for the circle-method estimates of Sections 9.4–9.7, and one floating-point computation with a proved error bound ([Proposition 9.26](#)). Each run checks every inequality it relies on and reports failure otherwise. The runs are listed in [Remark 9.30](#); the scripts and the logs of their runs are in the ancillary folder `anc/` ([Appendix A](#)). This paper has not been refereed, and a proof of this kind deserves an independent re-implementation of the certificates.

1. GROUND RULES

For $P, Q \in \mathbb{Z}[[q]]$ write $P \geq_q Q$ when every coefficient of $P - Q$ is nonnegative. Nonnegative series form a semiring, so sums and products of nonnegative series are nonnegative, and

$$P \geq_q Q \geq_q 0, \quad P' \geq_q Q' \geq_q 0 \implies PP' \geq_q QQ' \quad (PP' - QQ' = P(P' - Q') + (P - Q)Q').$$

Three things that drive real-variable proofs are missing.

- (1) **No cancellation.** $(1 - q + q^2)[2][3] = [6]$ has nonnegative coefficients while $1 - q + q^2$ does not. A double-counting proof of $PX \geq QX$ therefore proves nothing about P and Q .
- (2) **No roots or logarithms.** With $P = 1 + q^2$ and $Q = q$ we have $P^2 \geq_q Q^2$ but not $P \geq_q Q$.
- (3) **No division, no derivatives of logarithms.** Monotonicity arguments must be replaced by telescoping identities with nonnegative increments.

Every tool below is an identity, a statement inside the semiring, or a cancellation whose cost is controlled.

Notation.

- $[n] = 1 + q + \cdots + q^{n-1}$, $[n]! = [1][2] \cdots [n]$ and $(q; q)_m = (1 - q)(1 - q^2) \cdots (1 - q^m)$; $[q^k]F$, also written F_k , is the coefficient of q^k in F .
- $G(x, y) = \binom{x+y}{x}_q = \prod_{i=1}^x \frac{1 - q^{y+i}}{1 - q^i}$, the generating function of partitions in an $x \times y$ box, with coefficients $g_{x,y}(k)$; $G(x, y) = 0$ if $x < 0$ or $y < 0$.
- $p_{\leq m}(n) = [q^n] 1/(q; q)_m$, the number of partitions of n into at most m parts (zero for $n < 0$).
- The *darga* of a palindromic polynomial is its lowest plus its highest degree; its *center* is half of that.
- A *string* is $q^j[n - 2j + 1]$ (darga n); S_n is the cone of nonnegative combinations of strings of darga n , i.e. palindromic unimodal polynomials with nonnegative coefficients and darga n , and $S = \bigcup_n S_n$. Since $[m][n]$ is a sum of strings, $S_m S_n \subseteq S_{m+n}$.

Black boxes. (F1) $G(x, y) = G(y, x)$ is palindromic of darga xy . (F2) Pascal: $G(x, y) = G(x - 1, y) + q^x G(x, y - 1)$ and $G(x, y) = G(x, y - 1) + q^y G(x - 1, y)$. (F3) $G(x, y) \in S_{xy}$, that is, Gaussian polynomials are unimodal (Sylvester [18]; see Proctor [15] for a proof by linear algebra and O'Hara [14] and Zeilberger [23] for a constructive one). F3 is the only deep input.

2. TRUNCATION, HALF-RANGE AND NESTING

Definition 2.1. Let C_m be the set of $F \in \mathbb{Z}[[q]]$ with $[q^j]F \geq 0$ for $0 \leq j \leq m$. We say $F \geq_q G$ up to degree m when $F - G \in C_m$.

Lemma 2.2 (truncated semiring). C_m is closed under sums and products and contains $\mathbb{N}[[q]]$. In particular $F \in C_m$ implies $F/(1 - q^k) \in C_m$.

Proof. $[q^j](FG) = \sum_{i \leq j} [q^i]F [q^{j-i}]G$ only uses coefficients of degree at most j . $\square$

So inside C_m one may multiply by rational functions such as $1/(1 - q^k)$; only the truncation level matters. This is where “analysis” enters: rational manipulations are free, as long as the level is tracked.

Lemma 2.3 (half-range). *If F is palindromic of darga n , then $F \geq_q 0$ if and only if $F \in C_{\lfloor n/2 \rfloor}$.*

Proof. $F_k = F_{n-k}$ for every k , and if $k > \lfloor n/2 \rfloor$, then $n - k \leq \lfloor n/2 \rfloor$. $\square$

Lemma 2.4 (nesting). *Let $U \in S_n$ and let V be palindromic of darga $m \leq n$. If $U - V \in C_{\lfloor m/2 \rfloor}$, then $U \geq_q V$.*

Proof. For $\lfloor m/2 \rfloor < k \leq m$: $V_k = V_{m-k} \leq U_{m-k} \leq U_k$, because $m - k < k$, $m - k \leq n - k$, and U is unimodal and palindromic about $n/2$. For $k > m$, $V_k = 0$. $\square$

Lemma 2.5 (low degrees are free). *$G(x, y) \equiv 1/(q; q)_x \pmod{q^{y+1}}$. Hence $g_{x,y}(k) = p_{\leq x}(k)$ for $k \leq y$, $0 \leq g_{x,y} \leq p_{\leq x}$ in every degree, and $G(x, t) \equiv G(x, t - s) \pmod{q^{t-s+1}}$.*

Proof. $(q^{y+1}; q)_x \equiv 1 \pmod{q^{y+1}}$. By (F2), $G(x, y) - G(x, y - 1) = q^y G(x - 1, y) \geq_q 0$, so $G(x, y)$ increases with y towards $1/(q; q)_x$. $\square$

Lemma 2.6 (crossing). *Let $U \in S_n$ and V palindromic of darga n . If $U_k \geq V_k$ for $k < k_0 \leq n/2$ and $U_{k_0} \geq \max_j V_j$, then $U \geq_q V$.*

Proof. For $k_0 \leq k \leq n/2$, unimodality gives $U_k \geq U_{k_0} \geq \max_j V_j \geq V_k$. So $U - V \in C_{\lfloor n/2 \rfloor}$, and Lemma 2.3 applies. $\square$

The picture behind these lemmas: in low degree a Gaussian polynomial is an infinite product (Lemma 2.5), near the middle unimodality controls it (Lemmas 2.4, 2.6), and every real difficulty sits in a window between. Figure 1 shows a concrete case.

3. UNIMODAL ABSORPTION AND THE DIVISION LEMMA

Lemma 3.1. *Let F be palindromic of darga n with nonnegative coefficients. Then $F \in S_n$ if and only if $(1 - q)F \in C_{\lfloor n/2 \rfloor}$, and in that case $(1 - q^r)F \in C_{\lfloor (n+r)/2 \rfloor}$ for every $r \geq 1$.*

Proof. A string $q^j[n - 2j + 1]$ is the indicator of the interval $[j, n - j]$. So $F \in S_n$ if and only if $F_{k-1} \leq F_k$ for $k \leq n/2$, in which case $F = \sum_{j \leq n/2} (F_j - F_{j-1}) q^j [n - 2j + 1]$; and $F_k - F_{k-1} = [q^k](1 - q)F$. For the second claim let $k \leq (n + r)/2$. If $k \leq n/2$, then $F_{k-r} \leq F_k$ by monotonicity. If $k > n/2$, then $F_k = F_{n-k}$ with $k - r \leq n - k < n/2$, so again $F_{k-r} \leq F_k$. $\square$

Lemma 3.2 (master lemma). *Let F be palindromic of darga n . If $F = A \cdot B$ in $\mathbb{Z}[[q]]$ with $A, B \in C_{\lfloor n/2 \rfloor}$ (for instance expansions of rational functions), then $F \geq_q 0$.*

Proof. $F \in C_{\lfloor n/2 \rfloor}$ by Lemma 2.2, and Lemma 2.3 applies. $\square$

Lemma 3.3 (division lemma). *Let $U \in S_n$ and $1 \leq r \leq k$. If $W = U \frac{1 - q^r}{1 - q^k}$ is a polynomial, then $W \geq_q 0$, and W is palindromic of darga $n + r - k$. In particular, $F \cdot [k] \in S$ implies $F \geq_q 0$.*

Proof. $(1 - q^r)U \in C_{\lfloor (n+r)/2 \rfloor}$ by Lemma 3.1; dividing by $1 - q^k$ keeps it there (Lemma 2.2); W is palindromic with center $(n + r - k)/2 \leq (n + r)/2$, so Lemma 2.3 applies. $\square$

Example 3.4 (successive Gaussian polynomials; Butler [5], Andrews [3]). For $1 \leq k \leq n/2$, $\binom{n}{k}_q \geq_q \binom{n}{k-1}_q$.

Proof. Following the decomposition of Amdeberhan and Callan [2], write $n = \alpha k + d$ with $0 \leq d < k$, $\alpha \geq 2$, and $U = \binom{n}{k-1}_q \in S$. From $\binom{n}{k}_q = U \frac{1-q^{n-k+1}}{1-q^k}$,

$$\binom{n}{k}_q - \binom{n}{k-1}_q = q^k U [\alpha - 2]_{q^k} + q^{(\alpha-1)k} U \frac{1-q^{d+1}}{1-q^k}.$$

The first term is nonnegative. For the second, $U \frac{1-q^{d+1}}{1-q^k} = \frac{[n][n-1]\cdots[n-k+2][d+1]}{[k]!}$, and $\{n-k+2, \dots, n, d+1\}$ is a complete residue system mod k ; so for each $j \leq k$ at least $\lfloor k/j \rfloor$ of these integers are multiples of j , and counting cyclotomic factors shows the quotient is a polynomial. Lemma 3.3 with $r = d+1 \leq k$ finishes. $\square$

Limit 3.5. Only one factor can be cancelled this way. $(1-q+q^2)[2][3] = [6] \in S_5$, yet $1-q+q^2 \not\geq_q 0$. For Bergeron's difference $\mathcal{D}$ (Section 8), $\mathcal{D}(1-q)(1-q^2) \notin C_{N/2}$ in 2,648 of the 2,898 cases with $N \leq 500$: there is no usable second-order unimodality.

A monotone series can nevertheless carry a factor $1-q$ against a nonnegative cofactor. This is the discrete form of integrating by parts against a monotone density.

Lemma 3.6 (Abel summation against a monotone base). *Let $B = \sum_m b_m q^m$ with $b_0 = b_2 = 1$, $b_1 = 0$, and let $Y \in C_K$ vanish in degrees below t_0 . If $b_{m-1} \leq b_m$ for $3 \leq m \leq K - t_0$ and $(1-q+q^2)Y \in C_K$, then $(1-q)BY \in C_K$.*

Proof. $(1-q)B = 1-q+q^2 + R$, where R vanishes below degree 3 and has nonnegative coefficients up to degree $K - t_0$. In degrees $\leq K$, RY only uses those coefficients of R , so $RY \in C_K$, and $(1-q)BY = (1-q+q^2)Y + RY$. $\square$

The base is typically a q -Catalan-type series, whose monotonicity is a deep fact (Theorem C' and Section 9.5), and the cofactor Y an explicit rational function. Propositions 8.18 and 8.19 are instances, and Proposition 8.12 is a variant. The condition on b_0, b_1, b_2 is the defect $1-q+q^2$ of Limit 3.5, which costs one extra condition on Y .

4. q -KARAMATA

Lemma 4.1 (Robin Hood identity). *For $1 \leq m \leq n-2$,*

$$[m+1][n-1] - [m][n] = q^m [n-m-1],$$

a string with the same center as $[m][n]$. (For SL_2 this is Clebsch-Gordan: $V_m \otimes V_{n-2} \cong V_{m-1} \otimes V_{n-1} \oplus V_{n-m-2}$.)

Theorem 4.2 (q -Karamata). *Let $x, y \in \mathbb{Z}_{>0}^r$ have equal sums, and let y be majorized by x . Then $\prod_i [y_i] - \prod_i [x_i] \in S_n$ with $n = \sum_i (x_i - 1)$. In particular $\prod [y_i] \geq_q \prod [x_i]$.*

Proof. For integer vectors, y is reached from x by finitely many transfers $(x_i, x_j) \rightarrow (x_i-1, x_j+1)$ with $x_i \geq x_j + 2$ (Muirhead; Hardy, Littlewood and Pólya; see [12]). Each transfer adds $q^{x_j} [x_i - x_j - 1] \prod_{l \neq i,j} [x_l]$, which lies in S_n by Lemma 4.1 and $S \cdot S \subseteq S$. $\square$

Lemma 4.3. *A set of r distinct integers majorizes the interval of r consecutive integers with the same sum.*

Proof. Sort $y_1 > \dots > y_r$ and put $z_i = y_i - (r-i)$. Then z is weakly decreasing, the interval becomes a constant vector, and top- k averages of a decreasing vector are at least its mean. $\square$

Corollary 4.4 (q -AM-GM). *If $\sum_i x_i = rm + s$ with $0 \leq s < r$, then $\prod_i [x_i] \leq_q [m+1]^s [m]^{r-s}$.*

Proof. The vector with s entries $m+1$ and $r-s$ entries m is majorized by every integer vector with the same sum; apply Theorem 4.2. $\square$

Remark 4.5. For fixed $q > 0$, $x \mapsto \log[x]_q$ is concave, so the classical Karamata inequality gives the numerical version at every q . The coefficientwise statement is stronger, and the Robin Hood identity is its certificate. This is the general pattern: a real-variable convexity argument supplies the shadow, and an identity with visibly nonnegative terms supplies the coefficientwise version.

5. THE EVALUATION PRINCIPLE AND SMOOTHING

Principle 5.1. *If $P \geq_q Q$, then $\sum_k w_k [q^k](P - Q) \geq 0$ for every nonnegative weight w . Useful weights: $w_k = q_0^k$ (evaluation at real $q_0 > 0$) and, for palindromic P, Q of the same darga n , the central moments $w_k = (k - n/2)^{2j}$.*

Use these to discard false conjectures quickly, and to find the real-variable shadow of a q -inequality. The shadow can usually be proved by calculus, which tells you the statement is at least plausible and often shows which quantity has to be controlled.

Example 5.2 (continuous q -Bergeron). For $0 < q < 1$ the q -digamma function $\psi_q = \Gamma'_q / \Gamma_q$ satisfies

$$\psi_q''(x) = (\log q)^3 \sum_{n \geq 0} \frac{q^{x+n}(1 + q^{x+n})}{(1 - q^{x+n})^3} < 0.$$

Repeating Amdeberhan and Beck's argument [1] with ψ_q in place of ψ shows that for every $\theta > 0$ the function $x \mapsto \Gamma_q(x + \theta/x + 1) / (\Gamma_q(x + 1)\Gamma_q(\theta/x + 1))$ increases on $(0, \sqrt{\theta}]$. At divisors of θ it equals $G(x, \theta/x)$ evaluated at q . Hence $G(b, c)(q) \geq G(a, d)(q)$ for all real $q > 0$ whenever $ad = bc$ and $a \leq b \leq c$ (for $q > 1$ use palindromy).

Example 5.3 (second moment). Since the coefficient distribution of $G(x, y)$ has variance $xy(x + y + 1)/12$, Bergeron's inequality implies

$$(b + c + 1) \binom{b + c}{b} \geq (a + d + 1) \binom{a + d}{a}.$$

Equality holds at $(1, 2, 2, 4)$, where the difference is q^2 ; there is no failure for $N \leq 6000$.

Smoothing. Paired with the right weights, evaluation does more than test a conjecture: it recovers individual coefficients up to explicit errors. Let $m \geq 2$ and let f_m be the Irwin–Hall density, the density of $U_1 + \cdots + U_m$ for independent U_i uniform on $[0, 1]$:

$$f_m(y) = \frac{1}{(m-1)!} \sum_{0 \leq i \leq y} (-1)^i \binom{m}{i} (y-i)^{m-1} \quad (0 < y < m), \quad f_m(y) = 0 \text{ otherwise.}$$

It is symmetric about $m/2$, log-concave, increasing on $[0, m/2]$, its integer translates sum to 1, and its maximum is $\varphi_m = f_m(m/2)$.

Lemma 5.4 (smoothing identity). *Let $m \geq 2$. For $P = \sum_j p_j q^j$ put $(\mathcal{S}_m P)(z) = \sum_j p_j f_m(z - j)$. Then for every $n \geq 1$*

$$\mathcal{S}_m([n]^m)(z) = n^{m-1} f_m(z/n).$$

Proof. $\int_0^\infty e^{-sz} (\mathcal{S}_m P)(z) dz = P(e^{-s}) \left(\frac{1-e^{-s}}{s} \right)^m$, an evaluation at $q = e^{-s}$. For $P = [n]^m$ it equals $\left(\frac{1-e^{-ns}}{s} \right)^m$, the transform of $n^{m-1} f_m(z/n)$. Both sides are continuous for $m \geq 2$, so they agree everywhere. $\square$

Lemma 5.5 (local bounds). *Let $m \geq 2$ and $w_j \geq 0$, with $w_j = 0$ for $j < 0$, be nondecreasing for $j \leq M$, and let $F(z) = \sum_j w_j f_m(z - j)$. Then (a) $w_N \geq F(N + 1)$ for $0 \leq N \leq M$; (b) $w_N \leq F(N + m - 1)$ if $N + m - 2 \leq M$; (c) if $|w_{j+1} - w_j| \leq \Delta$ for all j , then $\max_j w_j \leq \sup F + \frac{m-1}{2}\Delta$.*

Proof. The weights $f_m(z - j)$ sum to 1 and vanish unless $z - m < j < z$. In (a) this means $N + 1 - m < j \leq N$, where $w_j \leq w_N$; in (b) it means $N \leq j < N + m - 1$, where $w_j \geq w_N$. In (c) take $z = j^* + m/2$ with w_{j^*} maximal: on the support $|j - j^*| \leq \frac{m-1}{2}$, so $w_j \geq w_{j^*} - \frac{m-1}{2}\Delta$. $\square$

Corollary 5.6 (Irwin–Hall bounds). *Let $m \geq 2$ and $W_m(S, N) = [q^N][S + 1]^m$, the number of $x \in \{0, \dots, S\}^m$ with $x_1 + \dots + x_m = N$. Then*

$$\begin{aligned} W_m(S, N) &\geq (S + 1)^{m-1} f_m\left(\frac{N + 1}{S + 1}\right) && \text{if } 0 \leq N \leq \frac{mS}{2}, \\ W_m(S, N) &\leq (S + 1)^{m-1} f_m\left(\frac{N + m - 1}{S + 1}\right) && \text{if } N + m - 2 \leq \frac{mS}{2}, \\ W_m(S, N) &\leq (S + 1)^{m-1} \varphi_m + \frac{m-1}{2}(S + 1)^{m-2} && \text{for every } N. \end{aligned}$$

Proof. $[S + 1]^m \in S_{mS}$, so Lemma 5.5 applies with $M = \lfloor mS/2 \rfloor$, and $F(z) = (S + 1)^{m-1} f_m(z/(S + 1))$ by Lemma 5.4. From $(1 - q)[S + 1]^m = (1 - q^{S+1})[S + 1]^{m-1}$, consecutive coefficients differ by at most $\max_N W_{m-1}(S, N) \leq (S + 1)^{m-2}$. $\square$

Lemma 5.7 (cube sandwich). *For all $m, S \geq 1$,*

$$m! q^{\binom{m}{2}} G(m, S + 1 - m) \leq_q [S + 1]^m \leq_q m! G(m, S).$$

Consequently, for all s and n ,

$$W_m(s, n) \leq m! g_{m,s}(n) \leq W_m\left(s + m - 1, n + \binom{m}{2}\right).$$

Proof. In variables $x_0, \dots, x_S$, $p_1^m = \sum_{|\alpha|=m} \binom{m}{\alpha} x^\alpha$, and each multinomial coefficient lies between 1 and $m!$, with $m!$ exactly when x^α is squarefree. Hence $m! e_m \leq p_1^m \leq m! h_m$ monomial by monomial, and a specialization to nonnegative series keeps this. At $x_i = q^i$ we get $p_1 \mapsto [S + 1]$, $h_m \mapsto G(m, S)$ and $e_m \mapsto q^{\binom{m}{2}} G(m, S + 1 - m)$. $\square$

Remark 5.8. Together, 5.4–5.7 form a local limit theorem in coefficientwise form. In the lower half, $m! g_{m,s}(n)$ lies between $(s + 1)^{m-1} f_m\left(\frac{n+1}{s+1}\right)$ and the upper bound that Corollary 5.6 gives for $W_m(s + m - 1, n + \binom{m}{2})$. Near the middle the two differ by a factor $1 + O(m^2/s)$ (Figure 2). The analysis enters through the density f_m and the evaluation $q = e^{-s}$; positivity comes from unimodality (F3) and from monomial inequalities between symmetric functions, not from a limit.

6. PASCAL–MATCHING INDUCTION

This is the method that proves Theorem A. It applies whenever the larger side satisfies a positive recursion and the smaller side lives in a family with explicit low-degree behaviour.

- (1) **Monotone enlargement.** Enlarge the statement to an order ideal on which it is monotone, so that the smaller instances produced by the recursion are again instances. For Bergeron this means all $d \leq bc/a$, not just $ad = bc$; exact-area instances are not closed under Pascal's rule.
- (2) **Half-range.** By nesting (Lemma 2.4) only the lower half of the smaller side needs checking, while the induction hypothesis may be used in every degree.

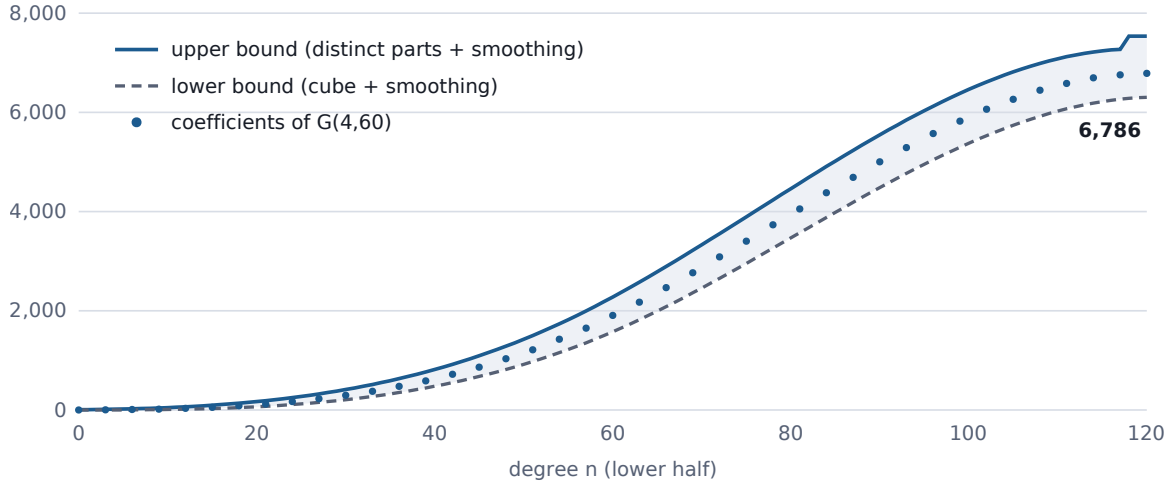


FIGURE 2. Coefficients of $G(4, 60)$ in degrees 0 to 120 (dots), with the upper (solid) and lower (dashed) bounds of Remark 5.8 divided by $4!$. At the center the coefficient is 6,786 and the bounds are 7,538 and 6,303.

- (3) **Target-only step.** After inserting the induction hypothesis, what remains is an inequality inside the target family alone.
- (4) **Matching.** Expand both sides of that inequality by a positive decomposition and pair terms whose truncations agree (Lemma 2.5). Most of the step cancels exactly; the remainder is a defect minus a surplus, and the defect lives only in a window near the middle.
- (5) **Estimates.** The defect counts objects with one more constraint than the surplus, so their ratio improves with size. Crude bounds (composition counting, sums compared with integrals) already prove the step outside a finite set. Sharp bounds come from smoothing (Section 5): both sides become integrals of Irwin–Hall densities, and on each cell of a partition of the degree range the step reduces to a polynomial inequality with rational coefficients.
- (6) **Finite check.** Verify the remaining instances exactly. Kronecker substitution makes this cheap: a polynomial with coefficients below 2^{B-1} is stored as the single integer $\sum_k u_k 2^{kB}$, Pascal steps become one shift and one addition, and a coefficientwise comparison is one subtraction followed by a check of the top bit of every B -bit digit.

7. DICTIONARY

The table summarizes the correspondence between real-variable tools and the coefficientwise substitutes used in this paper; numbers in parentheses refer to the statements and sections where they appear.

Real-variable tool	Coefficientwise substitute
Monotonicity from $f' \geq 0$	Telescoping with nonnegative increments, e.g. $G(a, t) - G(a, t - 1) = q^t G(a - 1, t)$
Taylor expansion, local behaviour	Truncation mod q^{m+1} ; Gaussian polynomials are infinite products in low degree (2.5)
Convexity, Jensen, Karamata	Robin Hood identity and q -Karamata (4.1, 4.2)
Dividing by a positive factor	Forbidden in general; one q -integer may be cancelled from a unimodal product (3.3)
Integration by parts against a monotone density	Abel summation $(1 - q)BY$ with a monotone base B and a nonnegative cofactor Y (3.6, 8.19)
Comparing maxima	Half-range, nesting, crossing (2.3, 2.4, 2.6)
Gamma and digamma arguments	They prove the numerical shadow (5.2); necessary, not sufficient
Local limit theorem, densities approximating counts	Smoothing identity with B-spline weights, unimodality, and $m!e_m \leq p_1^m \leq m!h_m$ (5.4–5.7)
Saddle-point asymptotics with error terms	Circle method at the saddle point of each degree: difference formula, pair decomposition, parity, Bernoulli pairs, a product formula for the modulus (9)
Induction on parameters	Pascal–matching induction with a monotone hypothesis (6)
Extending an integer parameter to real values	Linear interpolation of coefficients, $H_a(x) = \int_0^{x+1} q^{\lfloor s \rfloor} G(a - 1, \lfloor s \rfloor) ds$ (8.7)
Error terms and estimates	Cell-by-cell polynomial inequalities with rational coefficients (8.4); crude defect–surplus bounds for finiteness (8.5); ball arithmetic on cells, made uniform in n by monotonicity (9.4)
Numerical evidence	Exact finite verification by Kronecker substitution

8. TEST CASE: BERGERON'S CONJECTURE

Conjecture (Bergeron). *If $1 \leq a \leq b \leq c \leq d$ and $ad = bc$, then $\binom{b+c}{b}_q \geq_q \binom{a+d}{a}_q$.*

Via $G(x, y) = (\text{shifted}) \text{ character of } S^x(S^y\mathbb{C}^2)$, the conjecture compares weight multiplicities of two SL_2 -modules. Zanello's stronger conjecture [22], that the difference is unimodal, is the statement $S^a(S^d\mathbb{C}^2) \hookrightarrow S^b(S^c\mathbb{C}^2)$, the two-row case of Vessenès' generalized Foulkes conjecture [20].

8.1. Previous results. Table 1 lists the results on Bergeron's conjecture that were known before this paper, together with those proved here.

For the equal-area conjecture the genuinely new cases of [Theorem A](#) are those with composite $a \leq 20$, $a \nmid b$ and $a \nmid c$, for example $(4, 6, 6, 9)$, $(6, 8, 9, 12)$, $(8, 10, 12, 15)$, $(9, 12, 15, 20)$, $(12, 14, 18, 21)$ and $(20, 24, 25, 30)$. [Theorem A](#) also gives an independent elementary proof of the other cases with $a \leq 20$.

8.2. Reformulations.

- **Half-range.** $\mathcal{D} = G(b, c) - G(a, d)$ is palindromic of darga $N = ad = bc$, so it suffices that $g_{b,c}(k) \geq g_{a,d}(k)$ for $k \leq N/2$ ([Lemma 2.3](#)).
- **Low degrees are free.** For $k \leq c$, $g_{b,c}(k) - g_{a,d}(k) = p_{\leq b}(k) - p_{\leq a}(k) \geq 0$ ([Lemma 2.5](#)). Only $c < k \leq N/2$ matters, and the tightest spot is just after c : for $(20, 24, 25, 30)$, $g_{24,25}(26) = 2433$ against $g_{20,30}(26) = 2417$. In the middle the ratio is roughly $\binom{b+c}{b} / \binom{a+d}{a} \cdot \sqrt{(a+d+1)/(b+c+1)}$, about $4/3$ along $(a, a+1, 2a, 2a+2)$.

TABLE 1. Results on Bergeron's conjecture.

Scope	Method	Source
$a \leq 3$, with unimodality of the difference for $b, c \geq 4$	Zeilberger's KOH identity [23]	Zanello [22]
$c = 2a$, $d = 2b$	WZ method and an explicit expansion	Amdeberhan–Callan [2]
$q = 1$, also real parameters	calculus, Gamma function	Amdeberhan–Beck [1]
$a \mid b$ or $a \mid c$ (so every prime a), with unimodality; also with $ad \leq bc$	geometry of the Foulkes–Howe map	Gangl–Gutiérrez–Szwejt [9], using Raicu–Sam–Weyman–Yang [16]
the version with $ad \leq bc$	proposed as a conjecture	Troyka [19]
$a \leq 20$, with $ad \leq bc$	Pascal–matching induction, smoothing estimates, finite check	Theorem A
all a , with $ad \leq bc$	as for Theorem A, and the circle method	Theorem E

- **Monotone form.** Since $G(a, d)$ increases with d , the whole family is equivalent to $T(b, c): G(b, c) \geq_q G(a, \lfloor bc/a \rfloor)$ for $a \leq b \leq c$. This is the form that admits induction.

8.3. What q -Karamata gives on its own.

Theorem 8.1. *Let $a \leq b \leq c$, $ad = bc$, and $\mathcal{D} = G(b, c) - G(a, d)$. Then (i) $\mathcal{D} \cdot [b]! \in S_{N+b(b-1)/2}$; (ii) $\mathcal{D} \cdot [b-1]! \geq_q 0$; (iii) $G(b, c)(q) \geq G(a, d)(q)$ for every real $q > 0$.*

Proof. $G(b, c)[b]! = [c+1][c+2] \cdots [c+b]$ and $G(a, d)[b]! = [a+1] \cdots [b] \cdot [d+1] \cdots [d+a]$. The second multiset consists of b distinct integers with the same sum as the first (this is $ad = bc$), so by Lemma 4.3 it majorizes the interval $\{c+1, \dots, c+b\}$, and Theorem 4.2 gives (i). Part (ii) is the division lemma applied to (i) with $k = b$. Part (iii) is (i) evaluated at q , or Example 5.2. $\square$

This is as far as multiplicative methods go. A double-counting proof of the $q = 1$ inequality has exactly this shape: it proves $P \cdot X \geq Q \cdot X$ and then cancels X . Coefficientwise, one factor of $[b]!$ can be cancelled and the rest cannot (Limit 3.5). Additive methods are needed for the full statement.

8.4. Theorem A and its proof. Fix a . For $a \leq b \leq c$ put

$$D = \left\lfloor \frac{bc}{a} \right\rfloor, \quad D' = \left\lfloor \frac{(b-1)c}{a} \right\rfloor, \quad D'' = \left\lfloor \frac{b(c-1)}{a} \right\rfloor,$$

$$\delta = D - D', \quad k_{\max} = \left\lfloor \frac{aD}{2} \right\rfloor, \quad K = k_{\max} - b,$$

and let $T(b, c)$ be the statement $G(b, c) \geq_q G(a, D)$. Theorem A for (a, b, c) is equivalent to $T(b, c)$, and $T(a, c)$ is trivial.

Lemma 8.2 (Pascal reduction). *Let $a < b \leq c$. Assume $T(b-1, c)$, and $T(b, c-1)$ if $b < c$ or $T(b-1, b)$ if $b = c$. If*

$$(\text{Step}) \quad g_{a,D}(k) \leq g_{a,D'}(k) + g_{a,D''}(k-b) \quad \text{for } 0 \leq k \leq k_{\max},$$

then $T(b, c)$ holds.

Proof. By (F2), $G(b, c) = G(b-1, c) + q^b G(b, c-1)$, and $G(b, c-1) = G(b-1, b)$ when $c = b$. The hypotheses give $g_{b-1, c} \geq g_{a, D'}$ and $g_{b, c-1} \geq g_{a, D''}$ in every degree, so $g_{b, c}(k) \geq g_{a, D'}(k) + g_{a, D''}(k-b) \geq g_{a, D}(k)$ for $k \leq k_{\max}$. As $G(b, c) \in S_{bc}$ and $aD \leq bc$, nesting (Lemma 2.4) gives $T(b, c)$. $\square$

So induction on $b+c$ proves every $T(b, c)$ for which (Step) holds or which is checked directly. Step is a statement about the single family $G(a, \cdot)$.

Lemma 8.3 (matching identity). *For $a \geq 2$ and $a < b \leq c$,*

$$G(a, D) - G(a, D') - q^b G(a, D'') = \sum_{t=D'+1}^D \sum_{v=t-b+1}^t q^{t+v} G(a-2, v) - \sum_{s \in I} q^{s+b} G(a-1, s),$$

where $I = [0, D''] \setminus [D' + 1 - b, D - b]$. Both sums have nonnegative coefficients; write $\text{Def}(k)$ and $\text{Sur}(k)$ for their coefficients of q^k .

Proof. By (F2), $G(a, t) - G(a, t-1) = q^t G(a-1, t)$ and $G(a-1, t) - G(a-1, t-b) = \sum_{v=t-b+1}^t q^v G(a-2, v)$. Hence $G(a, D) - G(a, D') = \sum_{t=D'+1}^D q^t G(a-1, t)$ and $G(a, D'') = \sum_{s=0}^{D''} q^s G(a-1, s)$. The substitution $s = t-b$ maps $(D', D]$ into $[0, D'']$, because $D' \geq b-1$ (as $c \geq b > a$) and $D - D'' \leq \lceil b/a \rceil \leq b$. Subtract. $\square$

In words: $G(a-1, t)$ and $G(a-1, t-b)$ agree up to degree $t-b$ (Lemma 2.5), so the shifted terms cancel exactly in low degree. Every term of the defect has degree $t+v \geq 2D' + 3 - b$, so **Step holds for all $k \leq 2D' + 2 - b$** . If $k_{\max} \leq 2D' + 2 - b$ (an empty window), Step holds outright.

By Lemma 8.3 only the window $2D' + 3 - b \leq k \leq k_{\max}$ remains. By (F2), Step at degree k reads $L(k) \leq R(k)$, where

$$L(k) = g_{a, D}(k) - g_{a, D'}(k) = \sum_{t=D'+1}^D g_{a-1, t}(k-t), \quad R(k) = g_{a, D''}(k-b).$$

Both sides are coefficients of Gaussian polynomials, so Section 5 bounds them by Irwin–Hall densities. The bounds are organised in cells: write $k = \kappa bc/a$, cut the range of κ into intervals $[\kappa_l, \kappa_h]$, and cut the t -sum into J chunks.

Proposition 8.4 (cell inequality). *Fix $a \geq 3$, $J \geq 1$, integers $a < b_0 \leq c_0$ and rationals $0 \leq \kappa_l < \kappa_h \leq a/2$. Let $m = a-1$ and, for $0 \leq j < J$,*

$$\beta_j = b_0 - 1 + \frac{j}{J}, \quad \bar{Y}_j = \frac{\kappa_h b_0}{\beta_j} - 1 + \frac{(a-1)(1 - \frac{j}{J}) + a \binom{a-1}{2} + a - 2}{\beta_j c_0}.$$

Put $\Phi_j = f_m(\max(\bar{Y}_j, 0))$ if $\bar{Y}_j \leq \frac{m}{2}(1 - \frac{a}{\beta_j c_0})$, and $\Phi_j = \varphi_m$ otherwise; let $\varepsilon = 1$ if the second case occurs for some j and $\varepsilon = 0$ if not; and let $\zeta = \max(0, \kappa_l - a/c_0)$. For real b, c put

$$\hat{D} = \frac{bc}{a}, \quad \check{D}' = \frac{(b-1)c - a + 1}{a}, \quad \check{D}'' = \frac{b(c-1) - a + 1}{a},$$

$$\tau_j = \check{D}' + \frac{j}{J}(\hat{D} - \check{D}'), \quad H(x) = \frac{(x+a)^{a-1}}{a-1},$$

$$P(b, c) = f_a(\zeta)(\check{D}'' + 1)^{a-1} - a \sum_{j=0}^{J-1} \Phi_j(H(\tau_{j+1}) - H(\tau_j)) - \frac{a\varepsilon}{2} \left((\hat{D} + a)^{a-2} - (\check{D}' + a)^{a-2} \right).$$

Let Ω be $\{b = b_0, c \geq c_0\}$ or $\{b \geq b_0, c \geq b\}$ (then $c_0 = b_0$). If $P \geq 0$ on Ω , then $L(k) \leq R(k)$ at every integer point $(b, c) \in \Omega$ and every integer k of the window with $\kappa_l bc/a \leq k \leq \kappa_h bc/a$. The window itself lies in $[(2 - \frac{2}{b_0} - \frac{a}{c_0})\frac{bc}{a}, \frac{bc}{2}]$.

Proof. (i) By Lemma 5.7, $(a-1)!g_{a-1,t}(k-t) \leq W_m(t+a-2, k-t + \binom{a-1}{2})$. (ii) As $D' \geq \check{D}'$ and $D \leq \hat{D}$, the index t runs over integers in $(\check{D}', \hat{D}]$; let $\Phi(t) = \Phi_j$ for $\tau_j < t \leq \tau_{j+1}$. For such t and $k \leq \kappa_h \hat{D}$ let $\rho = (k-t + \binom{a-1}{2} + a-2)/(t+a-1)$. If $\rho < 0$, then $W_m = 0$ and there is nothing to prove. Otherwise ρ is at most its value at $k = \kappa_h \hat{D}$, $t = \tau_j$. With $\beta = b-1 + \frac{j}{J}$ and E_j the numerator in $\bar{Y}_j$, that value is $N_j/(\beta c + V_j)$, where $N_j = (\kappa_h b - \beta)c + E_j \geq 0$ and $V_j = (a-1)(a-1 + \frac{j}{J}) > 0$; hence it is at most $N_j/(\beta c) = \kappa_h b/\beta - 1 + E_j/(\beta c)$. This decreases in b and c , so $\rho \leq \bar{Y}_j$. Also $t+a-1 \geq \beta c/a \geq \beta_j c_0/a$, so if $\bar{Y}_j \leq \frac{m}{2}(1 - \frac{a}{\beta_j c_0})$ the condition of the second bound in Corollary 5.6 holds. Either way Corollary 5.6 gives $W_m \leq \Phi(t)(t+a-1)^{a-2} + \varepsilon \frac{a-2}{2}(t+a-1)^{a-3}$. (iii) Φ is nonincreasing ($\bar{Y}_j$ decreases in j while the threshold increases) and $(t+a-1)^{a-2}$ increases, so $\Phi(t)(t+a-1)^{a-2} \leq \int_{t-1}^t \Phi(x)(x+a)^{a-2} dx$. Summing over $D' < t \leq D$ and enlarging the range to $(\check{D}', \hat{D}]$,

$$(a-1)!L(k) \leq \sum_j \Phi_j(H(\tau_{j+1}) - H(\tau_j)) + \frac{\varepsilon}{2}((\hat{D}+a)^{a-2} - (\check{D}'+a)^{a-2}).$$

(iv) In the window $0 \leq k-b \leq aD''/2$, so Lemma 5.7 and Corollary 5.6 give $a!R(k) \geq (D''+1)^{a-1}f_a(\frac{k-b+1}{D''+1})$. Since $f_a(y)/y^{a-1}$ is proportional to the volume of $\{v \in \Delta : v_i \leq 1/y\}$ for the standard simplex Δ , the function $s \mapsto s^{a-1}f_a(x/s)$ is nondecreasing, so D'' may be replaced by $\check{D}''$. Then $\frac{k-b+1}{\check{D}''+1} \in [\kappa_l - \frac{a}{c}, \frac{a}{2}]$, where f_a increases, so $a!R(k) \geq f_a(\zeta)(\check{D}''+1)^{a-1}$. Together, $a!(R(k) - L(k)) \geq P(b, c) \geq 0$. The window bounds follow from $D' \geq \check{D}'$ and $k_{\max} \leq bc/2$. $\square$

Remark. The proof gives slightly more. With $\theta = bc/a - D$, the quantity $L(k) + \theta g_{a-1, D+1}(k - D - 1)$ is also at most $R(k)$, for every $k \leq bc/2$ in the cell, not only in the window. The extra term is one more t -term, at $t = D + 1$, with weight θ . Step (ii) applies to it with $\Phi(D+1) = \Phi_{J-1}$, because ρ decreases in t and $D+a \geq \beta_{J-1}c_0/a$. In step (iii) it is absorbed by $\int_D^{\hat{D}} \Phi(x)(x+a)^{a-2} dx$ (and its ε -part by $\int_D^{\hat{D}} \frac{a-2}{2}(x+a)^{a-3} dx$); this interval does not meet the intervals $[t-1, t]$, $t \leq D$, used for the other terms. Step (iv) needs only $k-b \leq aD''/2$, which holds for every $k \leq bc/2$ since $b \geq a-1$; for $k < b$ both sides vanish. Section 8.7 uses this.

Proposition 8.5 (certified regions). *For $3 \leq a \leq 20$ there are B_1 and thresholds $C_a(b) \geq b$ ($a < b < B_1$) such that $P \geq 0$ holds for every cell of a finite partition of $[\min(2 - \frac{2}{b_0} - \frac{a}{c_0}, \frac{a}{2}(1 - \frac{1}{b_0})), \frac{a}{2}]$, on $\Omega = \{b = b_0, c \geq C_a(b_0)\}$ for each $a < b_0 < B_1$ (with $J = 8$), and on $\Omega = \{b \geq B_1, c \geq b\}$ (with $J = 4$, or $J = 8$ for $a \geq 15$).*

The check is exact. After the shift $c = C_a(b) + y$, respectively $b = B_1 + x$, $c = B_1 + x + y$, each P either has only nonnegative coefficients or, failing that, is shown nonnegative with Sturm sequences: in one variable directly, in two variables through its coefficient polynomials in x (or in y). Each partition starts from 16 equal cells and bisects where a cell fails. The complementary sets $F_a = \{(b, c) : a < b < B_1, b \leq c < C_a(b)\}$ are finite, and $T(b, c)$, in the stronger form FT of Section 8.7, was verified on each by exact integer arithmetic (Table 2).

Proof of Theorem A. $a = 1$: $G(b, c)$ has all coefficients ≥ 1 in degrees $0, \dots, bc$ by unimodality, and $G(1, d) = [d+1]$ with $d \leq bc$. $a = 2$: the window is always empty. If $(b-1)c$ is even, $k_{\max} \leq bc/2 \leq (b-1)c + 2 - b$ is $(b-2)(c-2) \geq 0$. If $(b-1)c$ is odd, then b is even and c odd, $bc/2$ is an integer, and $bc/2 \leq (b-1)c + 1 - b$ is $(b-2)(c-2) \geq 2$, true for $b \geq 4, c \geq 5$. So

TABLE 2. The certified regions of Proposition 8.5 and the finite sets F_a , on which FT was verified exactly. For $a = 2$ the window is always empty (proof of Theorem A below), so no cells are needed and $F_2 = \emptyset$.

a	B_1	$C_a(a+1)$	cells	pairs in F_a	largest bc in F_a	failures
2	—	—	—	0	—	0
3	8	21	81	30	80	0
4	11	37	125	77	180	0
5	15	59	294	157	348	0
6	19	85	954	282	588	0
7	23	116	817	462	920	0
8	27	152	3,418	706	1,359	0
9	31	192	2,555	1,023	1,910	0
10	36	236	3,844	1,429	2,585	0
11	41	286	3,748	1,933	3,420	0
12	46	339	4,791	2,543	4,394	0
13	51	398	7,664	3,270	5,558	0
14	56	461	10,056	4,137	6,900	0
15	61	528	13,504	5,139	8,432	0
16	67	600	14,411	6,299	10,183	0
17	73	676	21,292	7,624	12,150	0
18	79	757	25,172	9,129	14,364	0
19	85	843	31,307	10,822	16,840	0
20	91	933	31,355	12,722	19,572	0

Step holds for every $3 \leq b \leq c$ and induction gives $T(b, c)$ everywhere. This case is a complete proof by hand. $3 \leq a \leq 20$: Lemma 8.3 and Propositions 8.4–8.5 give Step for every $(b, c) \notin F_a$, and T is checked on F_a . Induction on $b + c$ with Lemma 8.2 completes the proof. $\square$

The proof is computer-assisted in two places: symbolic positivity of explicit polynomials (Proposition 8.5) and the finite exact check. The script `bergeron_certificate.py` (Appendix A) reproduces both, and its option `--selftest` compares every bound of Proposition 8.4 with exact coefficients. This is a new argument and has not been refereed; the certificate is there so it can be checked independently.

Two features of Table 2 are worth noting. The finite check is cheap: all 12,722 pairs of F_{20} take about 12 seconds, and the time goes into the symbolic part. And everything grows polynomially: $C_a(a+1)/a^2$ stays between 2.31 and 2.38 for $3 \leq a \leq 20$, the largest area in F_a is about $2.5a^3$, and B_1 grows a little faster than linearly (from $3.4a$ at $a = 8$ to $4.6a$ at $a = 20$). The reason is that the main losses are the factors $1 + O(a^2/D)$ of Remark 5.8. Nothing in the method stops at 20; running it further is a matter of computer time.

8.5. Finiteness for every a .

Theorem B. *For every $a \geq 1$ there is an explicit finite set F_a such that $G(b, c) \geq_q G(a, d)$ holds for all $a \leq b \leq c$, $ad \leq bc$, as soon as it holds for $(b, c) \in F_a$.*

Finiteness already follows from crude bounds, which together with a finite check suffice for [Theorem A](#) when $a \leq 7$. Split Step with the matching identity: at degree k it reads $\text{Def}(k) \leq \text{Sur}(k)$, with δ and K as in 8.4.

Lemma 8.6 (window criterion). *Let $a \geq 3$. If the window $[2D' + 3 - b, k_{\max}]$ is nonempty and*

$$(\star) \quad \delta b p_{\leq a-2}(k_{\max} - 2D' - 3 + b) \leq \sum_{s=\lceil K/a \rceil}^{D'-b} p_{\leq a-1}(\min(as - K, s)),$$

then Step holds.

Proof. By [Lemma 8.3](#), Step at degree k is $\text{Def}(k) \leq \text{Sur}(k)$. Each of the δb terms of $\text{Def}(k)$ is $g_{a-2,v}(k-t-v) \leq p_{\leq a-2}(k-t-v) \leq p_{\leq a-2}(k-2D'-3+b)$, which is at most the left side of $(\star)$ for $k \leq k_{\max}$. For $\text{Sur}(k)$ keep only $s \in [0, D'-b]$; for k in the window, $n = k-b-s > s$. Palindromy and unimodality of $G(a-1, s)$ with [Lemma 2.5](#) give $g_{a-1,s}(n) \geq p_{\leq a-1}(\min(as - (k-b), s))$ whenever $as \geq k-b$. This bound decreases in k , so $\text{Sur}(k)$ is at least the right side of $(\star)$ throughout the window. $\square$

Lemma 8.7 (explicit bounds, $a \geq 4$). *Put*

$$X = \frac{bc}{2} - \frac{2(b-1)c}{a} + b - 1, \quad Y = \frac{(b-1)c - a + 1}{a} - b, \quad Z = \frac{bc/2 - b}{a-1},$$

$$U_m(n) = \frac{1}{m!} \binom{n + \frac{m(m+1)}{2} - 1}{m-1}.$$

If $X \geq 0$ and $Y \geq Z \geq 0$, then the left side of $(\star)$ is at most $\hat{L} = \frac{c+a-1}{a} b U_{a-2}(X)$, and the right side is at least

$$\hat{R} = \frac{Y^{a-1} - Z^{a-1}}{((a-1)!)^2} \geq \frac{(Y-Z)Z^{a-2}}{(a-1)!(a-2)!}.$$

Proof. $\delta \leq \lceil c/a \rceil$; $k_{\max} \leq bc/2$ and $D' \geq ((b-1)c - a + 1)/a$ bound the argument by X ; and $p_{\leq m}(n) \leq U_m(n)$ (shift a partition to distinct parts and count weak compositions). On the right keep $s \geq K/(a-1)$, where the minimum is s ; use $p_{\leq a-1}(s) \geq \binom{s+a-2}{a-2}/(a-1)! \geq s^{a-2}/((a-2)!(a-1)!)$, compare the sum with $\int_{K/(a-1)}^{D'-b} t^{a-2} dt$, and use $D'-b \geq Y$, $K/(a-1) \leq Z$. The last inequality is $Y^{a-1} - Z^{a-1} \geq (a-1)Z^{a-2}(Y-Z)$. $\square$

Proof sketch of Theorem B for $a \geq 4$. Let $\hat{R}_{\text{lin}}$ be the right-hand form in [Lemma 8.7](#). In $\hat{R}_{\text{lin}} - \hat{L}$ the monomial $(bc)^{a-1}$ has the positive coefficient

$$\frac{a-2}{2a(a-1)} \cdot \frac{(2(a-1))^{-(a-2)}}{(a-1)!(a-2)!}.$$

Every other monomial $b^i c^j$ has $i, j \leq a-1$, hence is at most $(bc)^{a-1}/b$ in size when $b \leq c$; so $\hat{R} > \hat{L}$ once b is large. For fixed $b \geq a+1$, $\hat{R}_{\text{lin}} - \hat{L}$ has degree $a-1$ in c with leading coefficient proportional to $b(a-2) - 2(a-1) > 0$, so it is positive for large c . The side conditions behave the same way. Cases $a \leq 3$ are in the proof of [Theorem A](#). $\square$

The crude bounds give sets that grow very fast: 210, 678, 3,840 and 34,026 pairs for $a = 4, \dots, 7$. They lose a factor of roughly $(a-2)!$ by replacing the defect by its maximum and discarding most of the surplus. The smoothing bounds of [Proposition 8.4](#) keep both sides up to factors $1 + O(a^2/D)$, and $|F_7|$ drops from 34,026 to 462.

8.6. Where the method stops. Step is a real restriction, not an artefact of the estimates. Since Step is proved outside F_a , checking it on F_a gives the complete list of pairs where it fails. For $4 \leq a \leq 20$ there are 235 of them (1, 1, 1, 3, 4, 5, 6, 10, 13, 14, 16, 20, 22, 24, 27, 33, 35 for $a = 4, \dots, 20$, roughly $0.09a^2$), all with $b \leq 1.55a$ and $c \leq 2.55a$. The family $(a, a+1, 2a)$ fails for every $7 \leq a \leq 20$. At 225 of the 235 pairs the other Pascal split $G(b, c) = G(b, c-1) + q^c G(b-1, c)$ fails as well; at the remaining 10 it succeeds, so those could leave the list. At the 225, T holds but follows from T at the neighbours by neither split: the obstruction is the induction hypothesis, not the choice of recursion.

What decides Step here is simple. At the failing pairs D is only a small multiple of a , central coefficients are roughly proportional to the totals $\binom{a+D}{a}$, and a good proxy for Step near the middle is the comparison of totals

$$\binom{a+D'}{a} + \binom{a+D''}{a} \geq \binom{a+D}{a}.$$

This criterion predicts the exact outcome at 5,866 of the 5,916 pairs with $4 \leq a \leq 20$ and $c \leq 3a$. Without floors, and with $b = \beta a$, $c = \gamma a$, its large- a limit is $(1 + \frac{1}{\beta\gamma})^{-\gamma} + (1 + \frac{1}{\beta\gamma})^{-\beta} \geq 1$, with equality only at $\beta = \gamma = 1$. So the failures are floor effects: along $(a, a+1, 2a)$, for example, $D - D'' = 2$ although $b/a = 1 + 1/a$.

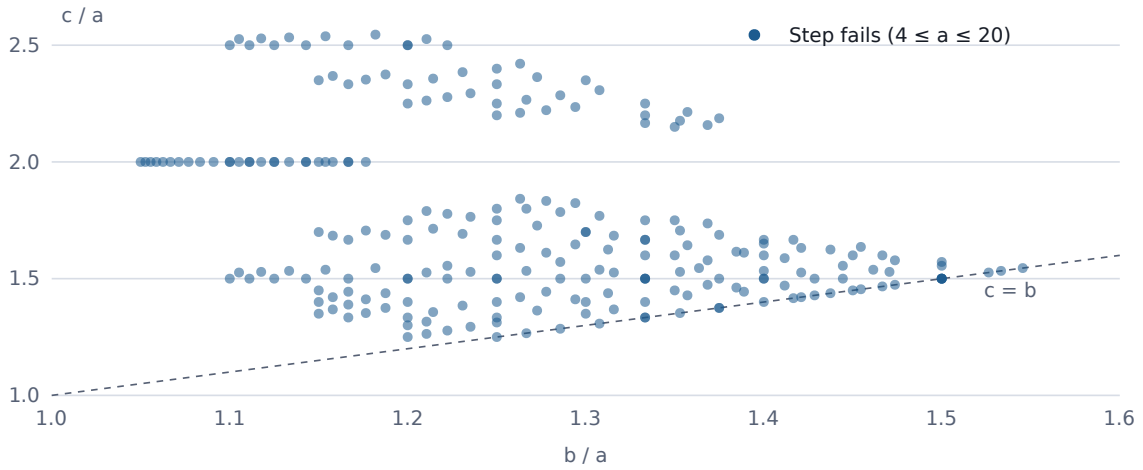


FIGURE 3. The 235 pairs (b, c) with $4 \leq a \leq 20$ at which Step fails, plotted at $(b/a, c/a)$. They stay in a bounded region; the row at $c/a = 2$ contains the family $(a, a+1, 2a)$.

So the exceptional pairs are an artefact of the integer part in $D = \lfloor bc/a \rfloor$: the induction hypothesis forgets the fractional part of bc/a . Section 8.7 keeps it, and the exceptions disappear.

The method also says nothing about Zanello's unimodality conjecture, because Pascal's rule does not preserve a common center.

8.7. A fractional hypothesis. Interpolate the family $G(a, \cdot)$ linearly between consecutive integers. For real $x \geq 0$ put

$$H_a(x) = G(a, \lfloor x \rfloor) + \{x\} q^{\lfloor x \rfloor + 1} G(a-1, \lfloor x \rfloor + 1) = \int_0^{x+1} q^{\lfloor s \rfloor} G(a-1, \lfloor s \rfloor) ds.$$

At integers $H_a(n) = G(a, n)$; between them every coefficient is linear in x , because the difference $G(a, n+1) - G(a, n)$ equals $q^{n+1}G(a-1, n+1)$. The integral form is Pascal's rule $G(a, n) = \sum_{t \leq n} q^t G(a-1, t)$.

Conjecture FT (fractional form). *For $a \leq b \leq c$, $G(b, c) \geq_q H_a(bc/a)$ in degrees $k \leq bc/2$. With $bc = aD + r$, $0 \leq r < a$, this reads*

$$a G(b, c) \geq (a - r) G(a, D) + r G(a, D + 1)$$

in those degrees.

FT implies Troyka's form T , since $H_a(x) \geq_q G(a, \lfloor x \rfloor)$ and T only needs degrees up to $aD/2 \leq bc/2$. The range $k \leq bc/2$ is what the induction uses. FT cannot hold in all degrees: when $r > 0$, $H_a(bc/a)$ has degree $bc + a - r > bc$. In the data it holds well past $bc/2$. Among the 12,899 triples with $3 \leq a \leq 8$, $b < 40$, $bc \leq 1500$, the first failure above $bc/2$ is 5.5 degrees above it, at $(a, b, c) = (4, 5, 5)$; only 29 triples fail within 20 degrees of $bc/2$.

Lemma 8.8 (fractional matching identity). *Let $a \geq 2$, $a < b \leq c$, $x = bc/a$, $x' = (b-1)c/a$, $x'' = b(c-1)/a$. Then*

$$H_a(x) - H_a(x') - q^b H_a(x'') = \mathcal{E} - \mathcal{S},$$

with the defect and the surplus

$$\mathcal{E} = \int_{x'+1}^{x+1} \sum_{v=\lfloor s \rfloor - b + 1}^{\lfloor s \rfloor} q^{\lfloor s \rfloor + v} G(a-2, v) ds,$$

$$\mathcal{S} = \int_I q^{\lfloor s \rfloor} G(a-1, \lfloor s \rfloor - b) ds, \quad I = [b, x'+1) \cup (x+1, x''+b+1].$$

Both have nonnegative coefficients, and $\mathcal{E}$ vanishes in degrees below $2D' + 3 - b$.

Proof. The integral form gives $q^b H_a(x'') = \int_b^{x''+b+1} q^{\lfloor s \rfloor} G(a-1, \lfloor s \rfloor - b) ds$, and $[x'+1, x+1] \subseteq [b, x''+b+1]$ because $c \geq a$ and $b \geq b/a$. On $[x'+1, x+1]$ use $G(a-1, t) - G(a-1, t-b) = \sum_{v=t-b+1}^t q^v G(a-2, v)$. The lowest degree in $\mathcal{E}$ is $2\lfloor x'+1 \rfloor - b + 1 = 2D' + 3 - b$. $\square$

The induction runs as before, starting from $\text{FT}(a, c)$, which is an identity ($r = 0$). FT at $(b-1, c)$ and at $(b, c-1)$ (at $(b-1, b)$ if $c = b$) bounds $G(b-1, c)$ and $G(b, c-1)$ from below in every degree, by palindromy and unimodality. Let $\Lambda_{b-1, c}(k)$ be the largest coefficient of $H_a((b-1)c/a)$ in degrees $j \leq \min(k, (b-1)c - k)$, and 0 if $k < 0$ or $k > (b-1)c$; then $g_{b-1, c}(k) \geq \Lambda_{b-1, c}(k)$ for all k , and similarly for $\Lambda_{b, c-1}$. FT at (b, c) then follows from the fractional step

$$(\text{FStep}) \quad [q^k] H_a(bc/a) \leq \Lambda_{b-1, c}(k) + \Lambda_{b, c-1}(k-b) \quad (0 \leq k \leq bc/2).$$

By Lemma 8.8, FStep holds whenever $k \leq \min(2D' + 2 - b, (b-1)c/2)$.

Observation 8.9 (no exceptions). FStep holds, in exact arithmetic, at every triple with $2 \leq a \leq 40$, $a < b \leq c$, $bc \leq 4000$ (133,683 triples), and with $2 \leq a \leq 24$, $bc \leq 8000$ (289,414 triples). At the 235 exceptional pairs of 8.6 the margin is comfortable. Write FStep as

$$[q^k](H_a(x) - H_a(x')) \leq \Lambda_{b-1, c}(k) - [q^k]H_a(x') + \Lambda_{b, c-1}(k-b);$$

at these pairs the right side is at least 1.13 times the left in every degree of the window (the minimum over the 235 pairs is 1.139, at $(a, b, c) = (20, 24, 26)$, in the central degree). In every

computed case the same ratio is smallest at the corner $(b, c) = (a + 1, a + 1)$, for example 1.017 at $(20, 21, 21)$ and 1.008 at $(30, 31, 31)$. At the corner, FStep is equivalent to

$${}_a C_{a+1}(q) \geq q^2 \binom{2a+2}{a-1}_q \quad \text{in degrees} \leq \frac{a^2-1}{2}, \quad C_n(q) = \frac{1}{[n+1]} \binom{2n}{n}_q,$$

after the shift of degrees $k \mapsto k - a - 1$ (Lemma 8.11 below); $C_n(q)$ is MacMahon's q -Catalan number. In this form the ratio of the two sides is smallest in the last degree, where it is about $1 + 4.5/a$ (checked for $a \leq 60$).

Theorem A'. (i) For every $a \leq 20$, FStep holds for all $a < b \leq c$. (ii) Consequently Conjecture FT holds for every $a \leq 20$ and all $a \leq b \leq c$.

Proof. (i) For $a = 1$, $H_1(n) = [n + 1]$ and FStep is immediate. For $a = 2$ the extra term of $H_2(bc/2)$ lives in degrees $\geq D + 1 > bc/2$. So FStep follows from Lemma 8.8 up to degree $(b - 1)c/2$, and above it from the reflection bound below together with Lemma 8.3, whose matching range reaches $\lfloor bc/2 \rfloor$ by the parity argument in the proof of Theorem A. Let $3 \leq a \leq 20$. Since $H_a \geq_q G(a, \lfloor \cdot \rfloor)$, we have $\Lambda_{b-1,c}(k) \geq g_{a,D'}(k)$ for all $k \leq bc/2$: for $k > (b - 1)c/2$ use the degree $(b - 1)c - k$, which lies in $[aD' - k, k]$ and so is closer than k to the center $aD'/2$ of $G(a, D')$. Likewise $\Lambda_{b,c-1}(k - b) \geq g_{a,D''}(k - b)$. So FStep at k follows from $L(k) + \theta g_{a-1,D+1}(k - D - 1) \leq R(k)$, which the Remark after Proposition 8.4 derives from the cells of Proposition 8.5 for every $(b, c) \notin F_a$ and every k they cover. Together with Lemma 8.8 this covers all $k \leq bc/2$ (the lower end $\frac{a}{2}(1 - \frac{1}{b_0})$ of the cells matters only for $a = 3$, where $2D' + 2 - b$ can exceed $(b - 1)c/2$). On F_a , FStep is checked directly. (ii) For $a \leq 2$, FT is Theorem A. For $3 \leq a \leq 20$, (i) and induction on $b + c$ give FT; it is also checked directly on F_a , as an independent confirmation. $\square$

Conjecture 8.10 (one-family step). FStep holds for all $a < b \leq c$.

By the induction above, Conjecture 8.10 implies FT, hence Troyka's form and Bergeron's conjecture, for every a . It is an inequality about the interpolated family H_a alone, it holds for $a \leq 20$ by Theorem A', and it has no exceptions in the data. As in Theorem B, for each fixed a it reduces to a finite computation. What remains for a proof for every a is two things.

- (1) **Uniformity in a .** The cell inequalities hold for $c \gtrsim 2.3a^2$ in every computed case; a proof valid for all a at once needs explicit bounds on Irwin–Hall densities that do not depend on a .
- (2) **Estimates that stay sharp when D is a small multiple of a .** There the smoothing bounds lose factors $1 + O(a^2/D)$. Saddle-point asymptotics for Gaussian coefficients (Melczer, Panova and Pemantle [13]) with explicit error terms are the natural candidate.

A third difficulty, the corner $(a + 1, a + 1)$, is resolved. There the step is the q -Catalan inequality above, whose relative margin is of order $1/a$ in that form and of order $1/a^2$ in the form of Observation 8.9. Section 8.8 turns it into an exact identity and reduces it to a three-step monotonicity of $C_{a+1}(q)$, and Section 9 proves that monotonicity for every a (Corollary 9.13). Section 8.8 also gives a second route to FT that avoids Conjecture 8.10: by Proposition 8.15, the two-parameter Conjecture 8.16 implies FT for every a , and Proposition 8.19 reduces its first band to a monotonicity property of rational q -Catalan numbers, which Section 9.5 proves.

8.8. The corner, the first row and the band. At $(b, c) = (a + 1, a + 1)$ the fractional step becomes an inequality between a q -Catalan number and a Gaussian coefficient. In every computed case it has the smallest margin, and it has an exact description. Throughout, $n = a + 1$.

Lemma 8.11 (the corner as a q -Catalan inequality). *Let $a \geq 2$ and $0 \leq k \leq (a+1)^2/2$. FStep holds at $(b, c) = (a+1, a+1)$ in degree k if and only if*

$$[q^{k-n}] \left({}_a C_n(q) - q^2 \binom{2n}{n-2}_q \right) \geq 0.$$

So FStep holds at the corner if and only if ${}_a C_n(q) \geq q^2 \binom{2n}{n-2}_q$ in degrees $\leq (a^2 - 1)/2$.

Proof. Here $x' = x'' = a+1$, so both Λ -terms come from $H_a(a+1) = G(a, a+1)$, which is symmetric and unimodal of degree $a(a+1) \geq k$. Hence $\Lambda_{a,a+1}(k) = g_{a,a+1}(k)$ and $\Lambda_{a+1,a}(k-b) = g_{a,a+1}(k-a-1)$, and by Pascal's rule the right side of FStep is $[q^k]G(a+1, a+1)$. Since $x = a+2+1/a$, the left side is $[q^k](G(a, a+2) + a^{-1}q^{a+3}G(a-1, a+3))$. Because $[n+1] - [n] = q^n$,

$$G(a+1, a+1) - G(a, a+2) = \binom{2n}{n}_q \left(1 - \frac{[n]}{[n+1]} \right) = q^n C_n(q),$$

and $G(a-1, a+3) = \binom{2n}{n-2}_q$. So a times (right side minus left side) is the coefficient displayed in the statement. Finally $k \leq (a+1)^2/2$ if and only if $k-n \leq (a^2-1)/2$, and negative degrees contribute nothing. $\square$

Write $c_k = [q^k]C_n(q)$ and $y_k = c_k + c_{k-1}$, with $c_k = 0$ for $k < 0$, so that $\sum_k y_k q^k = (1+q)C_n(q)$.

Proposition 8.12 (corner identity). *For $a \geq 2$, $n = a+1$ and every k ,*

$$[q^k] \left({}_a C_n(q) - q^2 \binom{2n}{n-2}_q \right) = \sum_{i=2}^{a+1} (c_k - c_{k-i}) + \sum_{j \geq 1} (y_{k-j(a+3)} - y_{k-j(a+3)-a}).$$

Proof. The ratio $\binom{2n}{n-2}_q / \binom{2n}{n}_q$ is $[n][n-1]/([n+2][n+1])$, so $\binom{2n}{n-2}_q = C_n[a][a+1]/[a+3]$. As power series $[a+1]/[a+3] = (1-q^{a+1})/(1-q^{a+3})$, and $[a](1-q^2) = (1+q)(1-q^a)$, so

$$\frac{q^2[a](1-q^{a+1})}{1-q^{a+3}} = \sum_{j \geq 0} (q^{j(a+3)+2} - q^{(j+1)(a+3)})[a] = q^2[a] - (1+q)(1-q^a) \sum_{j \geq 1} q^{j(a+3)}.$$

Hence

$${}_a C_n - q^2 \binom{2n}{n-2}_q = (a - q^2[a])C_n + (1 - q^a)(1+q)C_n \sum_{j \geq 1} q^{j(a+3)}.$$

Taking coefficients, with $a - q^2[a] = \sum_{i=2}^{a+1} (1 - q^i)$, gives the formula. $\square$

So the corner inequality holds up to degree k_0 whenever C_n is nondecreasing on degrees $2, \dots, k_0$: every term on the right is then nonnegative (note $c_0 = 1$, $c_1 = 0$ and $c_2 = 1$). Less is needed, because of a theorem of Stanley [17, p. 523]: $\tilde{C}_n(q) = (1+q)C_n(q)/(1+q^n)$ is a symmetric unimodal polynomial. It has degree $(n-1)^2 = a^2$, so $(1+q)C_n = (1+q^n)\tilde{C}_n$ is nondecreasing in degrees $\leq a^2/2$. That makes every y -term nonnegative for $k \leq (a^2-1)/2$, and, since $y_k - y_{k-1} = c_k - c_{k-2}$, it gives $c_k \geq c_{k-2}$ in that range. The terms with even i are therefore harmless, and for odd $i \geq 3$ we have $c_k - c_{k-i} \geq c_k - c_{k-3}$.

Corollary 8.13. *If $c_k \geq c_{k-3}$ for all $k \leq (a^2-1)/2$, then FStep holds at $(b, c) = (a+1, a+1)$. The hypothesis holds for every a (by computation for $a \leq 999$, and by [Theorem 8.14](#) in general), so the corner case of [Conjecture 8.10](#) holds for every a .*

For $a \leq 999$ the hypothesis is checked by computing C_n for $n \leq 1000$ (`corner_check.py`, [Appendix A](#)); the computation checks $c_k \geq c_{k-2}$ as well, so for these a Stanley's theorem is not needed. The three-step condition is weaker than monotonicity. In the range $k \leq (a^2 - 1)/2$, C_n fails to be nondecreasing only for $a = 4, 5, 6, 8$ (first at degrees 7, 7, 13 and 31), yet $c_k \geq c_{k-3}$ holds there. On the whole lower half, $2 \leq k \leq n(n-1)/2$, monotonicity fails exactly for $n \in \{4, \dots, 12, 14, 15\}$ among $n \leq 1000$, and the three-step condition fails only in the middle degree of C_6 and of C_7 .

Theorem 8.14 (three-step monotonicity). *For every $n \geq 2$ and $3 \leq k \leq (n^2 - 2n)/2$, $[q^k]C_n(q) \geq [q^{k-3}]C_n(q)$. The proof is in [Section 9.4](#) ([Corollary 9.13](#)).*

By [Corollary 8.13](#), [Theorem 8.14](#) settles the corner case of [Conjecture 8.10](#) for every a . Chen, Wang and Wang [6] conjectured that $(c_1, \dots, c_{n(n-1)-1})$ is unimodal for all sufficiently large n , and wrote that “an easy computation indicates that $C_n(q)$ are unimodal for $n \geq 16$ ”. Since C_n is palindromic, unimodality here means monotonicity on $[1, n(n-1)/2]$. So their conjecture in the form “for all $n \geq 16$ ” implies [Theorem 8.14](#); the cases $n \leq 15$ are in the computation above. [Section 9](#) proves their conjecture in that form: [Theorem C](#) for all large n , and [Theorem C'](#) for every $n \geq 16$.

Stanley's theorem cannot give the three-step condition by itself, since it constrains C_n only through $(1+q)C_n$. That polynomial stays nondecreasing even when the coefficients of C_n alternate between large and small values. What has to be bounded is the parity oscillation of C_n . Below the middle degree the saddle points of the coefficient integral are $\pm\rho$ with $0 < \rho < 1$, and the oscillation is the contribution of $-\rho$. Xin and Zhong [21] conjecture that the rational q -Catalan polynomials, $C_n(q)$ among them, are parity unimodal, equivalently that $(1+q)C_n(q)$ is unimodal. For C_n it follows from [Theorem C'](#) ([Corollary 9.14](#)); like Stanley's theorem, parity unimodality alone would not bound the oscillation.

The first row suffices. The corner is also the base case of a second reduction. It replaces the induction on $b+c$ by an induction on the smaller side at fixed area, and it needs only the first row $b = a+1$ of FT, with c allowed to be fractional.

Proposition 8.15 (the first row suffices). *For integers $x \geq 2$ and $A \geq x^2$ let $R(x, A)$ be the statement*

$$H_x(A/x) \geq_q H_{x-1}(A/(x-1)) \quad \text{in degrees } k \leq A/2.$$

If $R(x, bc)$ holds for every $a < x \leq b$, then FT holds at (a, b, c) .

Proof. $H_b(bc/b) = G(b, c)$, and $bc \geq b^2 \geq x^2$ for $x \leq b$. The statements $R(b, bc), R(b-1, bc), \dots, R(a+1, bc)$ chain to $G(b, c) \geq_q H_a(bc/a)$ in degrees $\leq bc/2$, which is FT at (a, b, c) . $\square$

Conjecture 8.16 (fractional first row). *$R(x, A)$ holds for all integers $x \geq 2$ and $A \geq x^2$.*

By [Proposition 8.15](#), [Conjecture 8.16](#) implies FT, hence Troyka's form and Bergeron's conjecture, for every a . It has two parameters where [Conjecture 8.10](#) has three. Write $A = xD + r = (x-1)E + s$ with $0 \leq r < x$ and $0 \leq s < x-1$. Then $R(x, A)$ reads

$$(x-1)[(x-r)G(x, D) + rG(x, D+1)] \geq_q x[(x-1-s)G(x-1, E) + sG(x-1, E+1)].$$

Let $\mathcal{D}_A$ be the left side minus the right side. For each degree, both sides are affine in A between consecutive points of $x\mathbb{Z} \cup (x-1)\mathbb{Z}$, the breakpoints. At $A = xc$ the statement is FT at $(x-1, x, c)$, the first row of FT; in between, it is that row with c fractional on the left side as well. The base case $A = x^2$ is the corner. With $a = x-1$, $R(x, x^2)$ reads

$$aG(a+1, a+1) \geq_q (a-1)G(a, a+2) + G(a, a+3) \quad \text{in degrees } \leq (a+1)^2/2,$$

and the identities in the proof of [Lemma 8.11](#) turn this into the inequality displayed there. So [Theorem 8.14](#) proves $R(x, x^2)$ for every x .

Observation 8.17. $R(x, A)$ holds in exact arithmetic, in every degree $k \leq A/2$, for all integers A with $x^2 \leq A \leq Mx^2$, where $M = 40$ for $x \leq 12$, $M = 8$ for $x \leq 20$, $M = 6$ for $x \leq 30$, $M = 2$ for $x \leq 60$ and $M = 1.2$ for $x \leq 100$. The margin is smallest near $A = x^2$ and in the degrees just above A/x ; at $A = x^2$ there is equality in degree $x + 1$.

The difficulty is concentrated near $A = x^2$, and it has the same two layers as at the corner. In the degrees just above A/x , both sides count partitions of about A/x with at most x or $x - 1$ parts, and their difference is an explicit combination of partition numbers with bounded coefficients. At $A = (x - 1)(x + 2)$ and degree $x + 2$, for instance, the two sides of the display differ by exactly $(x - 1)(x - 2)$, while each is about $x^2 p(x + 2)$. In the middle degrees, the relative margin at the breakpoints $x(x + j)$ and $(x - 1)(x + j)$ is of order j/x or smaller. A proof therefore needs exact low-degree information that is uniform in x , as in the proof of [Theorem C'](#), and sharp estimates in the middle degrees. Near the corner both come from one exact identity. It writes $\mathcal{D}_A$ as a q -Catalan-type base times a weight whose expansion has nonnegative partial sums, and so it moves the whole difficulty into the monotonicity of the base. For $A \geq Kx^2$ the margins are of order one.

For the first piece above the corner the base is C_x , and its monotonicity is [Theorem C'](#).

Proposition 8.18 (the first piece). *For every $x \geq 2$, $R(x, A)$ holds for $x^2 \leq A \leq (x - 1)(x + 2)$.*

Proof. For $x \leq 60$ this is contained in [Observation 8.17](#); let $x \geq 16$. On this interval $D = x$ and $r = A - x^2$, and the right side of the display is

$$x[(x - 1 - s)G(x - 1, x + 1) + sG(x - 1, x + 2)], \quad s = A - x^2 + 1;$$

at the right end this is the value $x(x - 1)G(x - 1, x + 2)$ that belongs to $E = x + 2$, $s = 0$. So in each degree the difference of the two sides is affine in A , and it suffices to show that it is nonnegative at both ends, in every degree $k \leq (x^2 + x - 2)/2$.

At $A = x^2$ the difference is x times the coefficient of q^{k-x} in $(x - 1)C_x - q^2 \binom{2x}{x-2}_q$ ([Lemma 8.11](#) with $a = x - 1$). [Proposition 8.12](#) writes this coefficient as a sum of terms $c_{k'} - c_{k'-i}$ and $y_{k'} - y_{k'-a}$ with $k' \leq k - x \leq x(x - 1)/2$, and by [Theorem C'](#) every term is nonnegative.

At $A = (x - 1)(x + 2)$ the difference is $(x - 1)P$ with

$$P = 2 \binom{2x}{x}_q + (x - 2) \binom{2x + 1}{x}_q - x \binom{2x + 1}{x - 1}_q.$$

The three Gaussian coefficients are $\frac{[2x]!}{[x]![x+2]!} = \frac{C_x}{[x+2]}$ times $[x + 1][x + 2]$, $[2x + 1][x + 2]$ and $[2x + 1][x]$. Comparing coefficients gives $[x + 2]P = C_x Q_x$ with

$$Q_x = xq^x + \sum_{k=x+1}^{2x} (4x - 2k)q^k + \sum_{k=2x+2}^{3x-1} (2k - 4x - 4)q^k + (2x - 4)q^{3x} + (x - 2)q^{3x+1},$$

which has nonnegative coefficients $Q_x(i)$. As power series $1/[x + 2] = (1 - q) \sum_{j \geq 0} q^{j(x+2)}$, so

$$[q^k]P = \sum_{j \geq 0} \sum_m Q_x(t_j - m)(c_m - c_{m-1}), \quad t_j = k - j(x + 2).$$

Since $Q_x(i) = 0$ for $i < x$, only $m \leq k - x \leq x(x - 1)/2 - 1$ occur, and the terms with $m \geq 3$ are nonnegative by [Theorem C'](#). As $c_0 = c_2 = 1$ and $c_1 = 0$, the terms with $m \leq 2$ add up to $Q_x(t_j) - Q_x(t_j - 1) + Q_x(t_j - 2)$, which is nonnegative for this Q_x . $\square$

The band $A \leq 2x(x-1)$. At the later breakpoints the base C_x is replaced by a rational q -Catalan number, and the argument of [Proposition 8.18](#) goes through up to $A = 2x(x-1)$, provided the base is monotone. For $x \geq 3$ and $N \geq x$ put

$$\Gamma_{x,N} = \prod_{m=2}^x \frac{1 - q^{N+m}}{1 - q^m} = \frac{1}{[N+1]} \binom{x+N}{x}_q \in \mathbb{Z}[[q]], \quad \mu_{x,N} = \frac{(x-1)N}{2},$$

with coefficients $\gamma_{x,N}(k)$. If $\gcd(x, N+1) = 1$, then $\Gamma_{x,N}$ is the rational q -Catalan polynomial $C_{x,N+1}(q) = \frac{[x+N]!}{[x]![N+1]!}$, palindromic of degree $2\mu_{x,N}$; otherwise it is a power series. Also $\Gamma_{x,x} = C_x(q)$. In degrees $k \leq N+1$, $\gamma_{x,N}(k)$ is the number of partitions of k into parts from $\{2, \dots, x\}$; in particular $\gamma_{x,N}(0), \dots, \gamma_{x,N}(3)$ are $1, 0, 1, 1$.

Proposition 8.19 (the band). *Let $x \geq 3$, and suppose that for every N with $x \leq N \leq 2x-2$*

$$(M_{x,N}) \quad \gamma_{x,N}(k-1) \leq \gamma_{x,N}(k) \quad \text{for } 3 \leq k \leq \mu_{x,N} - 1.$$

Then $R(x, A)$ holds for $x^2 \leq A \leq 2x(x-1)$.

Proof. The coefficients of $\mathcal{D}_A$ are affine in A between breakpoints. So it suffices to show, at each breakpoint A of $[x^2, 2x(x-1)]$, that $[q^k]\mathcal{D}_A \geq 0$ for $k \leq K_A$, where $K_A = \lfloor A^+/2 \rfloor$ with A^+ the next breakpoint, and $K_A = A/2$ at $A = 2x(x-1)$. The breakpoints are the rows $A = x(x+j)$, $0 \leq j \leq x-2$, with $D = x+j$, $r = 0$, $E = x+j+1$, $s = j+1$ (at $j = x-2$ this means $E = 2x$, $s = 0$), and the duals $A = (x-1)(x+1+i)$, $1 \leq i \leq x-2$, with $D = x+i-1$, $r = x-1-i$, $E = x+1+i$, $s = 0$.

Identities. At a row let $N = x+j$. The three Gaussian coefficients that occur are $\Gamma_{x,N}$ times $[x+j+1]$, $[x]$ and $[x][2x+j+1]/[x+j+2]$. With $[x+j+1] - [x] = q^x[j+1]$ and $[2x+j+1] - [x+j+2] = q^{x+j+2}[x-1]$ this gives

$$[x+j+2]\mathcal{D}_A = xq^x\Gamma_{x,N}W_j, \quad W_j = (x-1)[j+1][x+j+2] - (j+1)q^{j+2}[x][x-1].$$

At a dual let $N = x+i$. The Gaussian coefficients are $\Gamma_{x,N}$ times $[x+i][x+i+1]/[2x+i]$, $[x+i+1]$ and $[x]$. With $[x+i] = [x] + q^x[i]$ and $[2x+i] = [x+i+1] + q^{x+i+1}[x-1]$,

$$[2x+i]\mathcal{D}_A = (x-1)q^x\Gamma_{x,N}V_i,$$

$$V_i = (x-1-i)[i+1][2x+i] + (i+1)[i][x+i+1] - (i+1)q^{i+1}[x][x-1].$$

As $[L] = (1 - q^L)/(1 - q)$, in both cases $\mathcal{D}_A = (1 - q)\Gamma_{x,N}Y$, with $Y = xq^xW_j/(1 - q^{x+j+2})$ or $Y = (x-1)q^xV_i/(1 - q^{2x+i})$.

Signs. Let $y_t = [q^t]Y$, and let $c(v) = \min(v+1, x-1, 2x-2-v)$ for $0 \leq v \leq 2x-3$, and $c(v) = 0$ otherwise; these are the coefficients of $[x][x-1]$. At a row, with $L = x+j+2$ and $[L]/(1 - q^L) = 1/(1 - q)$,

$$\frac{1}{x}y_{x+t} = (x-1)\min(t+1, j+1) - (j+1)S(t-j-2),$$

$$S(u) = \sum_{l \geq 0} c(u - lL).$$

As $2x-3 < 2L$, at most two terms of $S(u)$ are nonzero, and where there are two their sum is $(2x-2-u) + (u-L+1) = x-j-3$. So $S \leq x-1$ and $y \geq 0$. At a dual, V_i itself is nonnegative. Its coefficient in degree d is $x(d+1)$ for $d < i$, $(i+1)(x-1+i-d)$ for $i \leq d \leq x+i-1$, 0 for $x+i-1 \leq d \leq x+2i$, $(i+1)(d-x-2i)$ for $x+2i \leq d \leq 2x+i-1$, and $(x-1-i)(2x+2i-d)$ for $2x+i \leq d \leq 2x+2i-1$. The same formulas give $y_t - y_{t-1} + y_{t-2} \geq 0$ for all t . Indeed y never drops by more than $x(j+1)$, resp. $(x-1)(i+1)$, in one step, and two steps before a drop it is at least that large. At a row this holds because S climbs in unit steps from its minimum to its maximum $x-1$ and then falls, so $S(u-2) \leq x-2$ whenever $S(u) > S(u-1)$. At a dual,

y drops only where a copy of V_i is in degrees $i + 1, \dots, x + i - 1$, and there the coefficient two steps back is at least $(x - 1)(i + 1)$.

Abel summation. Let $\gamma = \gamma_{x,N}$. Since $\mathcal{D}_A = (1 - q) \Gamma_{x,N} Y$, $\gamma(0) = \gamma(2) = 1$ and $\gamma(1) = 0$,

$$[q^k] \mathcal{D}_A = (y_k - y_{k-1} + y_{k-2}) + \sum_{m \geq 3} (\gamma(m) - \gamma(m-1)) y_{k-m}.$$

Since $y_t = 0$ for $t < x$, only $m \leq K_A - x$ occur when $k \leq K_A$. At a row, also at $j = x - 2$,

$$K_A - x = \left\lfloor \frac{(x-1)(x+j+2) - 2x}{2} \right\rfloor = \lfloor \mu_{x,N} \rfloor - 1.$$

At a dual, $N \leq 2x - 2$ gives

$$K_A - x = \left\lfloor \frac{x(x+i)}{2} \right\rfloor - x \leq \lfloor \mu_{x,N} \rfloor - 1.$$

So by $(M_{x,N})$ every term is nonnegative; this is [Lemma 3.6](#) with $t_0 = x$. $\square$

Corollary 8.20 (Theorem D). *(i) $R(x, A)$ holds for $x^2 \leq A \leq 2x(x-1)$, for every $x \geq 2$. (ii) Consequently FT, and with it Troyka's form $G(b, c) \geq_q G(a, d)$ for $ad \leq bc$, holds whenever $a \leq b \leq c$ and $bc \leq 2a(a+1)$.*

Proof. (i) For $x \leq 60$ this is contained in [Observation 8.17](#). For $13 \leq x \leq 200$ the script `band.check.py` verifies $(M_{x,N})$ for $x \leq N \leq 2x - 2$ in exact arithmetic, and for $x \geq 201$ [Theorem 9.16](#) (ii) gives $(M_{x,N})$ for $x \leq N \leq 2x$; in both cases [Proposition 8.19](#) applies. (ii) If $a = b$, FT is an identity. Otherwise, for $a < x \leq b$ we have $x^2 \leq bc \leq 2a(a+1) \leq 2x(x-1)$, so (i) gives $R(x, bc)$, and [Proposition 8.15](#) gives FT at (a, b, c) . $\square$

Up to $b \leq 200$, part (ii) already covers 374,194 triples (a, b, c) , of which 350,736 have $a > 20$ and $bc > 8000$, beyond [Theorem A](#) and beyond every direct computation in this paper. All of them come from one family of polynomials, the rational q -Catalan numbers, which the next conjecture describes.

Conjecture 8.21 (monotone rational q -Catalan numbers). *For $x \geq 17$ and $N \geq x$, $\gamma_{x,N}(k-1) \leq \gamma_{x,N}(k)$ for $2 \leq k \leq \mu_{x,N}$.*

For $N = x$ this is [Theorem C'](#), which holds from $x = 16$ on. When $\gcd(x, N+1) = 1$ the conjecture says that $(c_1, \dots, c_{2\mu-1})$ is unimodal for the rational q -Catalan polynomial $C_{x,N+1}$. That implies the parity unimodality that Xin and Zhong conjectured for these pairs, as in [Corollary 9.14](#); they proved it for $m \leq 5$ and checked it for $m, n \leq 180$. In exact arithmetic [Conjecture 8.21](#) holds for $17 \leq x \leq 200$ with $N \leq 2x$, for $17 \leq x \leq 40$ with $N \leq 2x + 80$, and for $41 \leq x \leq 100$ with $N \leq 2x + 60$. The weaker $(M_{x,N})$ holds for all $13 \leq x \leq 200$ and $x \leq N \leq 2x - 2$. For $13 \leq x \leq 16$ the full statement fails in these ranges only in the middle degree $\lfloor \mu_{x,N} \rfloor$, and for each $4 \leq x \leq 12$ some $(M_{x,N})$ with $N \leq 2x - 2$ fails. In degrees $k \leq N + 1$ the conjecture says that the number of partitions of k into parts from $\{2, \dots, x\}$ never decreases; for $7 \leq x \leq 200$ this holds at least up to $k = 2 \cdot 10^4$.

For $x \leq N \leq 2x$ the conjecture is now a theorem, apart from one degree: [Theorem 9.16](#) proves it for every $x \geq 201$ in all degrees below the middle, and the computation above covers $17 \leq x \leq 200$ including the middle degree. What remains open is the middle degree $k = \mu_{x,N}$ when it is an integer and $x \geq 201$, and the range $N > 2x$. The family $\Gamma_{x,N}$ is the natural two-parameter extension of C_n , and the proof of [Theorem C'](#) carries over to it with $\nu = N/x$ as a second parameter ([Section 9.5](#)).

Remark 8.22 (beyond the band). For $A > 2x(x-1)$ the same identities hold with more denominators. From $2x(x-1)$ to about $3x(x-1)$, the quotient $\mathcal{D}_A/\Gamma_{x,D+1}$ has two factors $[t]$ in its denominator, $[x+D+1]$ and $[D+3]$, and beyond that three or more. In the second range the method still works numerically in each breakpoint's own degrees $k \leq A/2$: with the base $\Gamma_{x,D+1}$, $Y \geq 0$, the small- m sums are nonnegative, and the base is monotone where needed (checked for $16 \leq x \leq 22$, $x = 30$ and $x = 40$, up to $2.87x^2$ at $x = 16$ and up to $2.95x^2$ at $x = 40$). What fails is the range between breakpoints: at the duals, the degrees up to $A^+/2$ reach one or two degrees past the middle of the base, where it decreases, and no single base $\Gamma_{x,N}$ works there. So extending Proposition 8.19 needs a positivity proof for weights with two denominators, and a separate argument for the few degrees between $A/2$ and $A^+/2$. Past about $3x(x-1)$ no base $\Gamma_{x,N}$ with $D-1 \leq N \leq D+3$ works even in the breakpoint's own degrees, and in the cases we examined Y itself has negative coefficients. There the positivity has to come from saddle-point estimates, as in Principle 9.1. Proposition 8.23 gives the form in which they apply.

Beyond the band. Past $2x(x-1)$ the four Gaussian coefficients of $\mathcal{D}_A$ share no common base, but the difference still splits into two nonnegative parts, for every A .

Proposition 8.23 (the positive split). *Let $x \geq 3$ and $A = xD + r = (x-1)E + s$ with $0 \leq r < x$, $0 \leq s < x-1$ and $E \geq D+1$ (true for $A \geq x^2$). Then $\mathcal{D}_A = \mathcal{P}_A - \mathcal{N}_A$, where*

$$\mathcal{P}_A = (x-1)q^x[(x-r)G(x, D-1) + rG(x, D)],$$

$$\mathcal{N}_A = \sum_{e=D+1}^{E+1} w_e q^e G(x-2, e),$$

with $w_{D+1} = (x-1)(x-r)$, $w_e = x(x-1)$ for $D+2 \leq e \leq E$, and $w_{E+1} = xs$. Both parts have nonnegative coefficients, the weights w_e add up to A , and $\mathcal{N}_A$ vanishes in degrees $\leq D$.

Proof. By the q -Pascal rule $G(x, n) = G(x-1, n) + q^x G(x, n-1)$, the x -side of $\mathcal{D}_A$ equals $\mathcal{P}_A$ plus

$$(x-1)[(x-r)G(x-1, D) + rG(x-1, D+1)].$$

That sum has the same total weight $x(x-1)$ as the $(x-1)$ -side

$$x[(x-1-s)G(x-1, E) + sG(x-1, E+1)].$$

Writing each $G(x-1, n)$ as $G(x-1, D) + \sum_{e=D+1}^n q^e G(x-2, e)$, by the rule $G(x-1, e) - G(x-1, e-1) = q^e G(x-2, e)$, the two sums differ by $-\mathcal{N}_A$: the term $q^e G(x-2, e)$ occurs $x(x-1)$ times on the $(x-1)$ -side when $D+1 \leq e \leq E$, only xs times when $e = E+1$, and $(x-1)r$ times on the other side when $e = D+1$. The weights add up to $x(x-1)(E-D) - (x-1)r + xs$, which is A because $x(x-1)(E-D) = x(A-s) - (x-1)(A-r)$. $\square$

So $\mathcal{P}_A$ counts, with weight $x-1$ per copy, the partitions with exactly x parts, and $\mathcal{N}_A$ counts the partitions with at most $x-1$ parts whose largest part lies in $(D, E+1]$, roughly between A/x and $A/(x-1)$. The split removes the cancellation that makes the band delicate. In degrees up to D the second part is zero, and in higher degrees the two parts stay apart by a fixed factor.

Observation 8.24. For every A with $2x(x-1) < A \leq Mx^2$, where $M = 40$ for $x \in \{6, 10, 15, 20\}$, $M = 6$ for even $x \leq 30$ and $M = 4$ for $x \in \{35, 40, 45, 50, 60\}$, exact computation gives

$$[q^k]\mathcal{N}_A \leq 0.624[q^k]\mathcal{P}_A \quad \text{for } D < k \leq A/2.$$

The ratio is largest in the middle degree $k = \lfloor A/2 \rfloor$ and at the first A past the band. It falls as $\lambda = A/x^2$ grows: its maximum over k is about 0.62, 0.45, 0.35, 0.15 and 0.04 for λ near 2, 3, 4, 10 and 40. The same holds for the values at real $q = \rho = e^{-s}$: in 113 random cases with $6 \leq x \leq 200$ and $1.95 \leq \lambda \leq 25$, $\mathcal{N}_A(\rho)/\mathcal{P}_A(\rho)$ decreases in s , with largest value 0.6235 at $\rho = 1$. For integer λ and $x \rightarrow \infty$ the value at $\rho = 1$ tends to $f(\lambda) = ((1 + 1/\lambda)^\lambda - 1)/\lambda$, which is 0.625, 0.457 and 0.159 at $\lambda = 2, 3$ and 10.

Remark 8.25 (what a proof beyond the band needs). Let $s > 0$, $\rho = e^{-s}$, and $D < k \leq A/2$. Since $\mathcal{P}_A$ is a positive combination of two shifted Gaussian coefficients, and by Cauchy's formula for each term of $\mathcal{N}_A$,

$$[q^k] \mathcal{P}_A \geq \rho^{-k} \mathcal{P}_A(\rho) \min_{i=0,1} \pi_i, \quad [q^k] \mathcal{N}_A \leq \rho^{-k} \mathcal{N}_A(\rho) \max_e \Lambda_e.$$

Here

$$\pi_i = \frac{\rho^{k-x} [q^{k-x}] G(x, D-1+i)}{G(x, D-1+i)(\rho)}$$

is a point probability of the tilted distribution of a Gaussian coefficient, and

$$\Lambda_e = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|G(x-2, e)(\rho e^{i\theta})|}{G(x-2, e)(\rho)} d\theta$$

bounds every point probability of the tilted distribution of $G(x-2, e)$. At the saddle point both are close to $1/\sqrt{2\pi\kappa_2}$, with κ_2 the variance, which is nearly the same for all these coefficients. In a floating-point test at $x = 22$ and $x = 40$, with λ from the band edge to 10 and several degrees each, $\sqrt{2\pi\kappa_2} \pi_i$ lay in $[0.965, 0.997]$ and $\sqrt{2\pi\kappa_2} \Lambda_e$ in $[0.99, 1.35]$, the larger values only in low degrees, where $\mathcal{N}_A(\rho)/\mathcal{P}_A(\rho) < 0.01$. The resulting sufficient condition held with slack at least 0.36. So the fractional first row beyond the band reduces to two analytic statements with a margin of order one:

- (1) the evaluation inequality $\mathcal{N}_A(\rho) \leq c \mathcal{P}_A(\rho)$ for $0 < \rho \leq 1$, with c near 0.63, a statement about Gaussian coefficients at a real point;
- (2) a local limit theorem for the Gaussian coefficients $G(m, n)$ with $n \geq 1.86m$, uniform in the tilt, with relative error well below $1 - c$.

Neither involves the small margins of the band. Both are within reach of the methods of [Section 9](#), and the second is a classical asymptotic statement (the Mann–Whitney distribution) made explicit. By [Proposition 8.15](#), [Theorem A](#) (which covers $a \leq 20$) and [Theorem D](#), the fractional first row for $x \geq 22$ and $A > 2x(x-1)$ would prove Bergeron's conjecture, in Troyka's form, for every a . [Section 9.6](#) makes this criterion precise ([Lemma 9.19](#)) and proves it for every $x \geq 201$ ([Theorem 9.24](#)), and [Section 9.7](#) completes the range $x \geq 22$ ([Theorem 9.28](#)).

8.9. What did not work. Each of these is the q -analogue of a standard real-variable strategy, and each fails for a structural reason.

- **Ratio test.** With $R = G(b, c)/G(a, d) \in \mathbb{Z}[[q]]$, $R - 1 \in C_{N/2}$ would give $(1 - q)\mathcal{D} \in C_{N/2}$, i.e. unimodality, since $(1 - q)\mathcal{D} = ((1 - q)G(a, d))(R - 1)$. It fails in 1,286 of the 2,130 quadruples with $N \leq 400$, first at $(2, 3, 4, 6)$; even the weaker partial-sum version fails in 752.
- **Second-order absorption.** $\mathcal{D}(1 - q)(1 - q^2) \in C_{N/2}$ fails in 2,648 of 2,898 cases with $N \leq 500$ ([Limit 3.5](#)).
- **Likelihood-ratio order.** $g_{b,c}(k+1)g_{a,d}(k) \geq g_{b,c}(k)g_{a,d}(k+1)$ fails in 868 of 2,130 cases, because of parity effects in low degree.
- **Termwise Durfee comparison.** In $G(x, y) = \sum_k q^{k^2} \binom{x}{k}_q \binom{y}{k}_q$ the k -th terms for (b, c) and (a, d) have different degrees, so no termwise inequality exists.

- **A shape-based fractional hypothesis.** Before the linear interpolation of 8.7, a natural candidate was the generating function $\sum_{j \leq r} q^{j(D+1)} G(a-j, D)$ of partitions inside the near-rectangle $(D+1)^r D^{a-r}$, whose area is exactly $bc = aD + r$. $G(b, c)$ fails to dominate it even in the lower half: 34 failures among the 1,706 triples with $r > 0$, $a \leq 11$ and $bc \leq 400$, the first at $(a, b, c) = (4, 5, 5)$. The shape adds whole partitions with a part $D+1$; the interpolation adds a fraction of all of them.

8.10. **Computations.** Table 3 lists the main computations; unless stated otherwise they are exact. The rows marked “proved” refer to theorems of this paper, and the others record checks.

TABLE 3. Main computations.

Item	Value	Scope
Troyka form checked	281,929	triples $a < b \leq c$, $bc \leq 6000$
Bergeron cases checked	84,993	quadruples, $N \leq 6000$
Unimodality checked	9,929	quadruples, $N \leq 1200$
Cell bounds tested	58,246	degrees k , against exact coefficients
Fractional step checked	133,683	triples, $a \leq 40$, $bc \leq 4000$; no failures
Corner checked	$a \leq 999$	via $C_n(q)$, $n \leq 1000$ (Corollary 8.13)
First row checked	$x \leq 60$	all $x^2 \leq A \leq 2x^2$, and further for smaller x (Observation 8.17)
Band proved	every x	R on $x^2 \leq A \leq 2x(x-1)$; Theorem D (Corollary 8.20)
Bergeron proved	every a	in Troyka's form $ad \leq bc$; Theorem E (Corollary 9.29)
First row proved	$x \geq 22$	$R(x, A)$ for all A ; 3,186 table cells, 29,853 criterion boxes, 646,091 swept breakpoints (9.6–9.7)
Rational q -Catalan monotone	$x \geq 17$	monotone below the middle, $x \leq N \leq 2x$: exact to $x = 200$, 1,790 certified cells beyond (9.5)
q -Catalan unimodal	$n \geq 16$	exact for $n \leq 1000$ and $k \leq 10^4$; 180 certified cells beyond (9.4)

The fractional step was also checked on every F_a , $a \leq 20$ (it is part of the certificate), and at the 289,414 triples with $2 \leq a \leq 24$, $bc \leq 8000$. Also checked: Theorem 8.1 (i)–(ii) for $N \leq 160$; the identity of Lemma 8.3 on random parameters; Step wherever $(\star)$ holds, for $a \leq 5$ with $bc \leq 1500$ and $a = 6$ with $bc \leq 5000$, as a consistency check of Lemma 8.6; Lemma 8.8 on 40 random triples in exact rational arithmetic; Lemma 8.11 for $a \leq 60$, with both sides of FStep computed from their definitions; the identity of Proposition 8.12 in all degrees $\leq (a^2 - 1)/2$ for $a \leq 60$, against an independent computation of $\binom{2n}{n-2}_q$; Stanley's theorem for $n \leq 1000$; and both sides of every bound in Proposition 8.4 against exact coefficients, on 242 random regions of both kinds with $3 \leq a \leq 10$ (the certificate's `--selftest` repeats a smaller version). No failures anywhere. The numerical constants and Lemmas 9.2–9.5 of Section 9 are checked on small cases by `circle.check.py`, which also runs the experiment of Remark 9.8. The proof of Theorem C' for $n \geq 1001$ is `cww.certificate.py`. Observation 8.17 is `row.check.py`, and `corner.check.py` also checks that $(1+q)C_n$ is unimodal for $n \leq 1000$. Corollary 8.20 is `band.check.py` for $x \leq 200$ and `rq.certificate.py` (Theorem 9.16) beyond; Theorem 9.24 beyond the band is `pn.certificate.py` and `pn.families.py`, and Section 9.7 is `row.sweep.py` and `pn.families.py` run on three blocks of x ; for $x \leq 60$ `band.check.py` also checks the identities of Proposition 8.19 as power series up to the needed degree, the signs of Y and of the

small- m sums, and $\mathcal{D}_A \geq 0$ directly. The evidence for [Conjecture 8.21](#) was computed with the same routines. As an independent check of [Corollary 8.20](#) (ii), FT and Troyka's inequality were also verified directly, from the Gaussian coefficients, at $(a, b, c) = (100, 141, 142)$, $(120, 150, 190)$, $(150, 170, 190)$ and $(199, 200, 200)$.

8.11. Next steps.

- (1) [Conjecture 8.16](#) is proved for $x \geq 22$ ([Theorem 9.28](#)), and with [Theorem A](#) that proves Bergeron's conjecture. For $3 \leq x \leq 21$ it holds in every computed case ([Observation 8.17](#)) but is not needed. The proof beyond the band is computational at two points: the uniform local limit constants of [Section 9.6](#), and the sweep of [Section 9.7](#) for $x \leq 200$ near $\lambda = 2$, where the ratio $[q^k]\mathcal{N}_A/[q^k]\mathcal{P}_A$ is about 0.62. A closed-form proof of the evaluation inequality $\mathcal{N}_A(\rho) \leq c\mathcal{P}_A(\rho)$ with some $c < 1$, together with a local limit theorem for Gaussian coefficients with explicit constants for $m \geq 20$, would replace both by analysis. An independent re-implementation of the certificates is the natural next check.
- (2) Make [Proposition 8.5](#) uniform in a , so that [Conjecture 8.10](#) holds for all a when $c \geq Ka^2$ or $b \geq K'a^{3/2}$.
- (3) Replace the cube sandwich by estimates that are sharp for $D = O(a)$; together with (2), and with the corner settled by [Corollary 9.13](#), this would settle [Conjecture 8.10](#), which would give a second proof of Bergeron's conjecture, by induction and without the circle method.
- (4) Run the matching lemma in the string cone S instead of the coefficient order, aiming at Zanello's unimodality conjecture.

9. THE CIRCLE METHOD

Sections 2–7 prove q -inequalities by exhibiting nonnegative terms. The circle method estimates instead. A difference of coefficients is an integral over a circle $|q| = \rho$; near $q = \rho$ the integrand is close to a Gaussian times a positive weight, and elsewhere it is small. Apart from a few low degrees, no identity or injection is needed, and all large parameters are handled at once; the price is a threshold that is only as good as the error terms. It is the saddle-point method of analytic combinatorics (see [8, Ch. VIII]), made explicit. This section sets the method up for q -inequalities and uses it to prove a conjecture of Chen, Wang and Wang [6]: the coefficients of MacMahon's q -Catalan numbers are unimodal from degree 1 on, for every $n \geq 16$. [Section 9.3](#) proves this for all large n ; [Section 9.4](#) makes the proof explicit, with a certificate checked in ball arithmetic. By [Section 8.8](#) this settles the corner of [Conjecture 8.10](#) for every a . [Section 9.5](#) extends Theorems C and C' to rational q -Catalan numbers, which settles the band of [Conjecture 8.16](#) and gives [Theorem D](#). [Section 9.6](#) goes beyond the band: it proves [Conjecture 8.16](#) for every $x \geq 201$, with local limit bounds for Gaussian coefficients that are uniform in both parameters and over the tilts that occur. [Section 9.7](#) brings it down to $x \geq 22$, which proves Bergeron's conjecture ([Theorem E](#)).

9.1. The tilted Cauchy formula.

Principle 9.1. *Let $P = \sum_j p_j q^j$ have nonnegative coefficients, let W be a polynomial with real coefficients, and let $0 < \rho \leq 1$. For every integer k ,*

$$[q^k](WP) = \frac{P(\rho)}{2\pi\rho^k} \int_{-\pi}^{\pi} \psi_k(\theta) W(\rho e^{i\theta}) d\theta, \quad \psi_k(\theta) = \frac{P(\rho e^{i\theta}) e^{-ik\theta}}{P(\rho)}.$$

Here ψ_k is the characteristic function of $X - k$, where $\Pr(X = j) = p_j q^j / P(\rho)$. If ρ is the saddle point, $\mathbb{E}X = k$, then $\psi_k(\theta) = \exp(-\kappa_2\theta^2/2 - i\kappa_3\theta^3/6 + \dots)$ with the cumulants κ_r of

X , and formally

$$[q^k](WP) \approx \frac{P(\rho)}{\rho^k \sqrt{2\pi\kappa_2}} \left(W(\rho) \left(1 + \frac{\kappa_4}{8\kappa_2^2} - \frac{5\kappa_3^2}{24\kappa_2^3} \right) - \frac{(\vartheta^2 W)(\rho)}{2\kappa_2} + \frac{\kappa_3 (\vartheta W)(\rho)}{2\kappa_2^2} + \dots \right), \quad \vartheta = \rho \frac{d}{d\rho}.$$

So an inequality in degree k follows from the sign of W at one real point, the saddle point of degree k , once the errors are controlled. This is a converse to the evaluation principle 5.1, which gives only necessary conditions. Where $W(\rho)$ is zero or small, as for $W = 1 - q$ near the middle degree of a palindromic P , the terms with ϑW and $\vartheta^2 W$ decide.

From now on $P = C_n(q) = \prod_{j=2}^n \frac{1-q^{n+j}}{1-q^j} = \sum_k c_k q^k$, which is palindromic of degree 2μ with $\mu = n(n-1)/2$, and $W = 1 - q$. For $0 < k \leq \mu$ let $s = s(k) \geq 0$ be the saddle point, $\rho = e^{-s}$, and $\psi = \psi_k$. [Principle 9.1](#) and conjugate symmetry give an exact formula.

Lemma 9.2 (difference formula).

$$(c_k - c_{k-1}) \frac{\rho^k}{C_n(\rho)} = \frac{1}{\pi} \int_0^\pi \operatorname{Re} [\psi(\theta) (1 - \rho e^{i\theta})] d\theta.$$

Its Gaussian approximation is

$$\mathcal{M} = \frac{1 - \rho + \rho/(2\kappa_2)}{\sqrt{2\pi\kappa_2}},$$

which is positive in every degree $k \leq \mu$, the middle one ($\rho = 1$) included. It is the leading term only up to a relative error $O(1/N)$, with N as below; the work is to show that everything else is smaller.

9.2. Three tools. Pair decomposition. With $\widehat{C}(w) = \prod_{j=2}^n \sinh \frac{(n+j)w}{2} / \sinh \frac{jw}{2}$ we have $C_n(e^{-w}) = e^{-\mu w} \widehat{C}(w)$. $\widehat{C}$ is even and has no zeros for $\operatorname{Re} w > 0$ or $|w| < \pi/n$; let $L = \log \widehat{C}$ there. The tilted law at $\rho = e^{-s}$ has mean $\mu - L'(s)$ and cumulants $\kappa_r = (-1)^r L^{(r)}(s)$. Also $\psi = e^f$ with $f(\theta) = i(\mu - k)\theta + L(s - i\theta) - L(s)$, so $f^{(r)}(\theta) = (-i)^r L^{(r)}(s - i\theta)$ for $r \geq 2$. Put $h(y) = \log \frac{\sinh y}{y}$ and $\ell(y) = \log(1 - e^{-2y})$. For $r \geq 2$

$$L^{(r)}(w) = \sum_{j=2}^n \left[\left(\frac{n+j}{2} \right)^r h^{(r)} \left(\frac{(n+j)w}{2} \right) - \left(\frac{j}{2} \right)^r h^{(r)} \left(\frac{jw}{2} \right) \right],$$

and the same holds with ℓ in place of h . Indeed $h(y) - \ell(y) = y - \log 2 - \log y$, and $(m/2)^r$ times the derivative of order r of $\log y$ at $y = mw/2$ does not depend on m , so it cancels within each pair $(j, n+j)$. The scale of the problem is $N = \min(n, 1/s)$. The pair decomposition and [Lemma 9.3](#) (with $\pi/(2n)$ replaced by $\pi/\max m$) apply to any product of ratios of q -integers, and [Lemma 9.5](#) to any Gaussian coefficient.

Lemma 9.3 (derivative bounds). *Let $r \geq 2$ and $M = \{2, \dots, n\} \cup \{n+2, \dots, 2n\}$.*

(a) *If $s' > 0$, then $|L^{(r)}(s' - it)| \leq \sum_{m \in M} m^r \operatorname{Li}_{1-r}(e^{-ms'})$ for all real t , where $\operatorname{Li}_{1-r}(x) = \sum_{l \geq 1} l^{r-1} x^l$.*

(b) *If $s' \geq 0$ and $|t| \leq \pi/(2n)$, then $|L^{(r)}(s' - it)| \leq (r-1)! H_r \sum_{m \in M} (m/2)^r$, where $H_r = 2 \sum_{l \geq 1} (\pi l - \pi/2)^{-r}$.*

Consequently, for $n \geq 100$ and $s \geq 0$, with $A_r = \sup_{x>0} x^{r+1} \operatorname{Li}_{-r}(e^{-x})$ and

$$D_r = \max \left(\zeta(2) A_r, \frac{(r-1)! H_r 2.01^{r+1}}{2^r (r+1)} \right),$$

we have $|L^{(r)}(s - it)| \leq D_r N^{r+1}$ whenever $|t| \leq \pi/(2N)$; and if $s \leq 1/n$, then $|L^{(r)}(s' - it)| \leq D_r n^{r+1}$ for all $0 \leq s' \leq s$ and $|t| \leq \pi/(2n)$.

Proof. (a) Use the ℓ -form: $\ell(y) = -\sum_l e^{-2ly}/l$ gives $|\ell^{(r)}(y)| \leq 2^r \text{Li}_{1-r}(e^{-2\text{Re } y})$. (b) Use the h -form: $h'(y) = \coth y - 1/y = \sum_{l \neq 0} (y - i\pi l)^{-1}$ (summed symmetrically), so $|h^{(r)}(y)| \leq (r-1)! \sum_{l \neq 0} |y - i\pi l|^{-r}$, which is at most $(r-1)! H_r$ when $\text{Re } y \geq 0$ and $|\text{Im } y| \leq \pi/2$. For the consequence, if $s \geq 1/n$ use (a) and

$$s^{r+1} \sum_{m \geq 1} m^r \text{Li}_{1-r}(e^{-ms}) = \sum_{l \geq 1} l^{-2} (ls)^{r+1} \text{Li}_{-r}(e^{-ls}) \leq \zeta(2) A_r;$$

if $s \leq 1/n$ use (b) and $\sum_{m \leq 2n} (m/2)^r \leq (2n+1)^{r+1}/(2^r(r+1))$. $\square$

Lemma 9.4 (parity). *For real $|t| < \pi/n$, $L^{(r)}(-it)$ is real for even r and purely imaginary for odd r . Hence, if $s \leq 1/n$ and $|t| \leq \pi/(2n)$,*

$$|\kappa_3| \leq s \sup_{0 \leq s' \leq s} |L^{(4)}(s')|, \quad |\text{Re } L^{(5)}(s - it)| \leq s \sup_{0 \leq s' \leq s} |L^{(6)}(s' - it)|.$$

Proof. L is even with real Taylor coefficients at 0, and the series converges for $|w| < \pi/n$; each $(-it)^{2m-r}$ lies in $i^r \mathbb{R}$. Now integrate $L^{(4)}$ and $L^{(6)}$ along $[0, s]$, using $L'''(0) = 0$ and $\text{Re } L^{(5)}(-it) = 0$. $\square$

This is what makes the middle degrees work. There $1 - \rho \approx s$ is tiny, and the odd part of f , which is what can make the integrand of Lemma 9.2 negative, is proportional to s as well.

Lemma 9.5 (Bernoulli pairs). *Let $0 \leq r \leq M'$, $\rho = e^{-s} \leq 1$ and $z > 0$. Let $\xi_0, \dots, \xi_{M'-1}$ be independent with $\Pr(\xi_i = 1) = p_i = \frac{z\rho^i}{1+z\rho^i}$ and $\Pr(\xi_i = 0) = 1 - p_i$, and put $S = \sum_i \xi_i$. Let $d \geq 1$, and let I be a set of indices such that the pairs $\{i, i+d\}$, $i \in I$, are disjoint subsets of $\{0, \dots, M'-1\}$. Then for every real θ*

$$\frac{\left| \binom{M'}{r}_{\rho e^{i\theta}} \right|}{\binom{M'}{r}_{\rho}} \leq \frac{1}{\Pr(S=r)} \prod_{i \in I} (1 - \pi_i(1 - R)),$$

where $\pi_i = \Pr(\xi_i + \xi_{i+d} = 1)$ and $R = (1 - \sin^2(d\theta/2)/\cosh^2(ds/2))^{1/2}$.

Proof. Since $\sum_{|A|=r} q^{\sum A} = q^{\binom{r}{2}} \binom{M'}{r}_q$ over $A \subseteq \{0, \dots, M'-1\}$, and $\Pr(\xi = 1_A)$ is proportional to $z^{|A|} \rho^{\sum A}$, the left side equals $|\mathbb{E}[e^{i\theta Y} \mid S=r]|$ with $Y = \sum_i i \xi_i$. Condition on all ξ_j outside the pairs and on the pair sums $\xi_i + \xi_{i+d}$; this fixes S . What remains random is, for each pair with sum 1, which of its two elements is chosen. These choices are independent, and the element $i+d$ is chosen with probability $\beta = \rho^d/(1+\rho^d)$. Such a pair contributes the factor $e^{i\theta i}(1 - \beta + \beta e^{id\theta})$, of modulus R because $4\beta(1-\beta) = \cosh^{-2}(ds/2)$. Hence, by independence of the pairs,

$$|\mathbb{E}[e^{i\theta Y}; S=r]| \leq \mathbb{E} \prod_{i \in I} R^{[\xi_i + \xi_{i+d}=1]} = \prod_{i \in I} (1 - \pi_i(1 - R)).$$

$\square$

This is the q -analogue of a classical trick for sums of independent variables. When $d\theta \approx \pi$, a pair that splits nearly cancels in the characteristic function. The lemma bounds Gaussian coefficients on the whole circle away from $q = \rho$ at once, with no roots of unity to treat one at a time.

Lemma 9.6 (far from the saddle point). *Let $0 \leq s \leq 1/4$, $N = \min(n, 1/s) \geq 64$, $0 < \gamma \leq 1$ and $\gamma/N \leq \theta \leq \pi$. Then*

$$\frac{|C_n(\rho e^{i\theta})|}{C_n(\rho)} \leq 16 n^{3/2} \exp\left(-\frac{N \sin^2(\gamma/16)}{33}\right).$$

Proof. Write $C_n = G - qG'$ with $G = \binom{2n}{n}_q$ and $G' = \binom{2n}{n-1}_q$. Since $C_n(\rho) = G(\rho)/[n+1]_\rho$ and $G'(\rho) = G(\rho)[n]_\rho/[n+1]_\rho$,

$$\frac{|C_n(\rho e^{i\theta})|}{C_n(\rho)} \leq [n+1]_\rho \frac{|G(\rho e^{i\theta})|}{G(\rho)} + \rho [n]_\rho \frac{|G'(\rho e^{i\theta})|}{G'(\rho)}.$$

Apply [Lemma 9.5](#) with $M' = 2n$, $z = \rho^{-(2n-1)/2}$ and $r = n, n-1$. Then $p_i + p_{2n-1-i} = 1$, so S is symmetric about n . Its law is log-concave (Newton's inequalities), hence unimodal with mode n , and Chebyshev's inequality gives $\Pr(S = n) \geq 3/(4(4\sqrt{V} + 1))$, where $V = \text{Var } S \leq n/2$. If $x = \Pr(S = n \pm 1)/\Pr(S = n)$ were at most $1/2$, log-concavity would give $\Pr(S = n \pm l) \leq x^l \Pr(S = n)$ and $V \leq 2x(1+x)/(1-x)^3 \leq 12$. We will have $V > 12$, so $\Pr(S = n-1) \geq \Pr(S = n)/2$. Next, $p_i(1-p_i) = \frac{1}{4} \cosh^{-2} \frac{s(i-n+1/2)}{2} \geq 0.196$ whenever $s|i-n+\frac{1}{2}| \leq 1$, which holds for at least N indices; so $V \geq 0.196N > 12$.

For the pairs take $I = \{i : i \bmod 2d < d, i+d \leq 2n-1\}$, which leaves at most d indices unpaired. As p_i decreases in i , $\pi_i \geq p_i(1-p_{i+d})$, which is at least $p_i(1-p_i)$ and at least $p_{i+d}(1-p_{i+d})$; so $\sum_{i \in I} \pi_i \geq \frac{1}{2}(V - \frac{d}{4})$. If $\theta \geq 4\pi/N$, let d be the integer nearest to π/θ ; then $\sin^2(d\theta/2) \geq \frac{1}{2}$ and $d \leq N/4 + \frac{1}{2}$. Otherwise let $d = \lfloor N/4 \rfloor \geq N/8$; then $d\theta \in [\gamma/8, \pi]$ and $\sin^2(d\theta/2) \geq \sin^2(\gamma/16)$. In both cases $ds \leq 3/8$, so $1-R \geq \sin^2(d\theta/2)/(2 \cosh^2 \frac{3}{16})$, which is at least $0.48 \sin^2(\gamma/16)$, while $\sum_{i \in I} \pi_i \geq 0.065N$. So the product in [Lemma 9.5](#) is at most $\exp(-0.031N \sin^2(\gamma/16))$, and the prefactor is at most $([n+1]_\rho + 2[n]_\rho) \cdot \frac{4}{3}(4\sqrt{n/2} + 1) \leq 16n^{3/2}$. $\square$

9.3. Unimodality of q -Catalan numbers.

Theorem C. *There is n_0 such that for every $n \geq n_0$ the coefficients c_k of the q -Catalan number $C_n(q) = \frac{1}{[n+1]} \binom{2n}{n}_q$ satisfy $c_{k-1} \leq c_k$ for $2 \leq k \leq \mu$, with strict inequality for $n+2 \leq k \leq \mu$. Hence $(c_1, \dots, c_{2\mu-1})$ is unimodal; this is Conjecture 4.1 of Chen, Wang and Wang [6]. Their threshold $n \geq 16$ ([10, Conjecture 1.8]) is [Theorem C'](#) in [Section 9.4](#).*

Proof. Degrees $k \leq n+1$. The numerator of C_n does not reach these degrees, so c_k counts the partitions of k into parts from $\{2, \dots, n\}$. Adding 1 to a largest part maps the partitions of $k-1$ into parts ≥ 2 bijectively onto the partitions of k into parts ≥ 2 whose largest part is unique and at least 3. So $c_{k-1} \leq c_k$ for $2 \leq k \leq n$. In c_{n+1} the one-part partition $(n+1)$ is missing, but for $n \geq 7$ the image also misses $(2, \dots, 2)$ if $n+1$ is even and $(3, 3, 3, 2, \dots, 2)$ if $n+1$ is odd, so $c_n \leq c_{n+1}$.

The saddle point is small. The tilted mean is

$$\sum_{j=2}^n \frac{j}{e^{js} - 1} - \sum_{m=n+2}^{2n} \frac{m}{e^{ms} - 1} < \int_0^\infty \frac{x dx}{e^{xs} - 1} = \frac{\pi^2}{6s^2},$$

because $x/(e^{xs} - 1)$ decreases. So $k \geq n+2$ forces $s \leq \pi/\sqrt{6k} < 1.29/\sqrt{n}$; hence $s \leq 1/4$ and $N \geq 0.77\sqrt{n}$ once $n \geq 27$.

Variance. For $0 \leq s \leq 1/4$, $N^3/26 \leq \kappa_2 \leq D_2 N^3$. The upper bound is [Lemma 9.3](#). For the lower one, $\kappa_2 = s^{-2} \sum_{j=2}^n (g(js) - g((n+j)s))$ with $g(x) = (x/2)^2 / \sinh^2(x/2)$, which decreases. If $0 < s \leq 1/n$, then $-g'(x) \geq x/10$ on $[0, 2]$ gives $\kappa_2 \geq \sum_j \frac{(n+j)^2 - j^2}{20} = \frac{n^3 - n}{10}$ (and $s = 0$ follows by continuity). If $s > 1/n$, every term with $2 \leq j \leq 1/s$ is at least $g(js) - g(js+1) \geq 0.079$, so $\kappa_2 \geq 0.079 s^{-2}(1/s - 2) \geq s^{-3}/26$.

Taylor expansion. Let $0 < \gamma \leq 1$, to be fixed, and $0 \leq \theta \leq \gamma/N$. Taylor's formula with fifth-order remainder, $f^{(5)}(t) = -iL^{(5)}(s - it)$, and Lemmas 9.3 and 9.4 give

$$|\operatorname{Re} f(\theta) + \tfrac{1}{2}\kappa_2\theta^2| \leq EN^5\theta^4, \quad |\operatorname{Im} f(\theta)| \leq Es(N^5\theta^3 + N^7\theta^5)$$

with an absolute constant E . For the real part, $|\kappa_4| \leq D_4N^5$ and the remainder is at most $D_5N^6\theta^5/120$. For the imaginary part, $|\kappa_3| \leq D_3N^4 = D_3sN^5$ and $|L^{(5)}| \leq D_5sN^7$ when $s \geq 1/n$; when $s \leq 1/n$, Lemma 9.4 supplies the factor s . Fix $\gamma = \min(1, (104E)^{-1/2})$. Then $EN^5\theta^4 \leq \kappa_2\theta^2/4$, so $\operatorname{Re} f(\theta) \leq -\kappa_2\theta^2/4$ on $[0, \gamma/N]$.

Main part. Let $\lambda = \sqrt{20 \log n}$ and $T = \lambda/\sqrt{\kappa_2}$; then $T \leq \gamma/N$ once $N \geq 520 \log n/\gamma^2$, which holds for large n . On $[0, T]$, with $\varphi = \operatorname{Im} f$,

$$\begin{aligned} \operatorname{Re} [\psi(1 - \rho e^{i\theta})] &= e^{\operatorname{Re} f} [(1 - \rho \cos \theta) \cos \varphi + \rho \sin \theta \sin \varphi] \\ &\geq e^{\operatorname{Re} f} [(1 - \rho \cos \theta)(1 - \tfrac{1}{2}\varphi^2) - \rho \theta |\varphi|]. \end{aligned}$$

In the variable $u = \theta\sqrt{\kappa_2} \in [0, \lambda]$ the bounds above read $e^{-u^2/2-676Eu^4/N} \leq e^{\operatorname{Re} f} \leq e^{-u^2/4}$ and $|\varphi| \leq 26^{5/2}Es\sqrt{N}(u^3 + u^5)$. Integrate, with $1 - \rho \cos \theta \geq 1 - \rho + \rho\theta^2/2 - \theta^4/24$. The first term gives at least $(1 - O(1/N))\mathcal{M}$; the Gaussian tail beyond λ and the θ^4 term cost only relative errors $O(n^{-9})$ and $O(1/\kappa_2)$. The φ^2 term is $O(s^2N)\mathcal{M}$, which is $O(s)\mathcal{M}$ as $sN \leq 1$. The $\theta|\varphi|$ term is $O(s/N^{5/2})$, which is $O(1/N)\mathcal{M}$ because $\mathcal{M} \geq (1 - \rho)/\sqrt{2\pi\kappa_2} \geq s/(2\sqrt{2\pi D_2}N^{3/2})$. So this part is at least $(1 - O(n^{-1/2}))\mathcal{M}$.

The rest of the circle. On $[T, \gamma/N]$ the integrand is at most $e^{-\kappa_2\theta^2/4}(1 - \rho + \theta)$, whose integral is at most $(\sqrt{\pi}(1 - \rho)/\sqrt{\kappa_2} + 2/\kappa_2)e^{-\lambda^2/4}$, which is $O(n^{-7/2})\mathcal{M}$. On $[\gamma/N, \pi]$, Lemma 9.6 (which needs $N \geq 64$, true for $n \geq 6909$) bounds this part by $32n^{3/2} \exp(-0.77\sqrt{n} \sin^2(\gamma/16)/33)$, while

$$\mathcal{M} \geq \frac{\rho}{2\kappa_2\sqrt{2\pi\kappa_2}} \geq \frac{1}{4\sqrt{2\pi}D_2^{3/2}n^{9/2}}.$$

For large n the rest is below $\mathcal{M}/2$, and Lemma 9.2 gives $c_k > c_{k-1}$. By palindromy, $(c_1, \dots, c_{2\mu-1})$ is unimodal. $\square$

Corollary 9.7. *For $n \geq n_0$, $c_k \geq c_{k-3}$ for $3 \leq k \leq \mu$ (for $k \geq 5$ by Theorem C; $c_3 = c_0 = 1$ and $c_4 \geq c_1 = 0$), so Theorem 8.14 holds for these n , and by Corollary 8.13 FStep holds at the corner $(b, c) = (a + 1, a + 1)$ for every $a \geq n_0 - 1$. Corollary 9.13 removes the restriction.*

Remark 9.8 (how far from explicit). Every constant above is explicit, but carried through they give an astronomically large n_0 , beyond 10^{19} : the rate $\sin^2(\gamma/16)/33$ in Lemma 9.6 is tiny, and $N^3/26 \leq \kappa_2 \leq D_2N^3$ is crude. The method itself is sharp. In floating point at $n = 1001$, bounding $[0, T]$ by Taylor estimates of the same kind as in the proof (exact $\kappa_2, \kappa_3, \kappa_4$; per-pair minima of the two bounds of Lemma 9.3 for the higher derivatives; the better of a fourth- and a sixth-order expansion; λ chosen per point) and $[T, \pi]$ by a grid sum of the exact modulus $|\psi|$, the right side of Lemma 9.2 stays above $0.96\mathcal{M}$ at all 41 saddle points of a grid spanning $n + 2 \leq k \leq \mu$; at $n = 2000$ the minimum is $0.97\mathcal{M}$. The smallest values occur near $k = n$, where $N \approx 0.8\sqrt{n}$ is smallest. Section 9.4 turns this experiment into a proof, by making the estimates rigorous and uniform in n from $n = 1001$ on; the computation of Section 8.8 shows that for $n \leq 1000$ monotonicity on $[2, \mu]$ fails only for $n \in \{4, \dots, 12, 14, 15\}$.

9.4. Every $n \geq 16$. The proof of Theorem C is effective but wasteful: carried through, it needs n beyond 10^{19} . This subsection makes it explicit from $n = 1001$ on. Together with the computation of Section 8.8, that gives the threshold $n \geq 16$ that Chen, Wang and Wang proposed. The structure of the proof is unchanged, but every $O(\cdot)$ becomes an explicit bound. Two of the bounds are new: an exact product formula for $|\psi|$, which is sharp on the middle

part of the circle, and a sharper form of [Lemma 9.6](#). What remains is a finite list of numerical inequalities between elementary functions, and a script checks them in ball arithmetic.

Theorem C'. *For every $n \geq 16$ the coefficients of $C_n(q)$ satisfy $c_{k-1} \leq c_k$ for $2 \leq k \leq \mu$. So $(c_1, \dots, c_{2\mu-1})$ is unimodal for every $n \geq 16$, which is the conjecture of Chen, Wang and Wang with their threshold. For $n \geq 1001$ the inequality is strict for $n+2 \leq k \leq \mu$.*

Four lemmas turn [Lemma 9.2](#) into checkable inequalities. Throughout, $\sigma = ns$ and $\vartheta = n\theta$.

Lemma 9.9 (positivity criterion). *Let $s > 0$, $0 < \theta_Z \leq \pi$, and $|\operatorname{Im} f| \leq \Phi$ on $[0, \theta_Z]$. Put $b(\theta) = \min(\Phi^2/2, 2) + \theta\Phi/s + \theta^2\Phi^2/(4s)$ and*

$$G_0 = \int_0^{\theta_Z} e^{\operatorname{Re} f} (1-b) d\theta, \quad G_2 = \int_0^{\theta_Z} (1-\cos \theta) e^{\operatorname{Re} f} d\theta, \quad I_j = \int_{\theta_Z}^{\pi} \theta^j |\psi| d\theta.$$

Then

$$\pi(c_k - c_{k-1}) \frac{\rho^k}{C_n(\rho)} \geq (1-\rho)(G_0 - I_0) + \rho(G_2 - I_1).$$

So $c_k > c_{k-1}$ if (a) $G_0 > I_0$ and $G_2 > I_1$, or (b) $G_0 > I_0 + I_1/(1-\rho)$. If $s = 0$, then $\pi(c_k - c_{k-1})/C_n(1) \geq G_2 - I_1$.

Proof. Write $\varphi = \operatorname{Im} f$. On $[\theta_Z, \pi]$ the integrand of [Lemma 9.2](#) is at least $-|\psi||1 - \rho e^{i\theta}| \geq -|\psi|(1 - \rho + \rho\theta)$. On $[0, \theta_Z]$, split $1 - \rho \cos \theta = (1 - \rho) + \rho(1 - \cos \theta)$ and $\cos \varphi = 1 - (1 - \cos \varphi)$: $(1 - \rho \cos \theta) \cos \varphi + \rho \sin \theta \sin \varphi \geq (1 - \rho) \cos \varphi + \rho(1 - \cos \theta) - \rho(1 - \cos \theta)(1 - \cos \varphi) - \rho\theta|\varphi|$. Now $1 - \cos \varphi \leq \min(\varphi^2/2, 2)$, $1 - \cos \theta \leq \theta^2/2$ and $\rho \leq (1 - \rho)/s$, so the right side is at least $(1 - \rho)(1 - b) + \rho(1 - \cos \theta)$. Integrate. For (b) drop $\rho G_2 \geq 0$ and use $\rho \leq 1$. If $s = 0$, the tilted law is symmetric about $k = \mu$, so $\psi = e^f$ is real and positive on $[0, \theta_Z]$, and the integrand is $(1 - \cos \theta)\psi$; on $[\theta_Z, \pi]$ use $1 - \cos \theta \leq \theta$. $\square$

Lemma 9.10 (variance). *Let $\Psi(y) = 1 - y^2/\sinh^2 y$, which increases, and*

$$\bar{\Psi}(y) = \int_0^y \Psi = y + y^2 \coth y - y^2 - 2y \log(1 - e^{-2y}) + \operatorname{Li}_2(e^{-2y}) - \frac{\pi^2}{6}.$$

For $s > 0$ (and, by continuity, for $s = 0$),

$$\begin{aligned} \kappa_2 &\geq \frac{2}{s^3} [\bar{\Psi}(\sigma) - 2\bar{\Psi}(\frac{\sigma+s}{2}) + \bar{\Psi}(s)], \\ \kappa_2 &\leq \frac{2}{s^3} [\bar{\Psi}(\sigma + \frac{s}{2}) - \bar{\Psi}(\frac{\sigma}{2} + s) - \bar{\Psi}(\frac{\sigma}{2}) + \bar{\Psi}(\frac{s}{2})]. \end{aligned}$$

If $\sigma \geq 6$, then $\pi^2/3 - 2s - 2G_6 \leq s^3 \kappa_2 \leq \pi^2/3$ with $G_6 = 50e^{-6}/(1 - e^{-6})^2 < 0.13$.

Proof. The h -form of the pair decomposition and $y^2 h''(y) = \Psi(y)$ give

$$\kappa_2 = \frac{1}{s^2} \left[\sum_{m=n+2}^{2n} \Psi(ms/2) - \sum_{j=2}^n \Psi(js/2) \right].$$

Since Ψ increases, each sum lies between the integrals of $\Psi(ms/2)$ over $m \in [a-1, b]$ and over $m \in [a, b+1]$, where a and b are its first and last indices; this gives both bounds. For $\sigma \geq 6$ use the $\sinh$ form instead:

$$s^3 \kappa_2 = s \sum_{j=2}^n g(js) - s \sum_{m=n+2}^{2n} g(ms), \quad g(x) = \frac{(x/2)^2}{\sinh^2(x/2)},$$

where g decreases, is at most 1, and has $\int_0^\infty g = \pi^2/3$. Compare with integrals, and use $g(x) \leq x^2 e^{-x} (1 - e^{-6})^{-2}$ for $x \geq 6$, so that $\int_\sigma^\infty g \leq G_6$. $\square$

Lemma 9.11 (the modulus as a product). *For real θ put $X_m = \sin^2(m\theta/2)/\sinh^2(ms/2)$. Then*

$$|\psi(\theta)|^2 = \prod_{j=2}^n \frac{1 + X_{n+j}}{1 + X_j}.$$

(a) Let $S(y) = \left(\frac{\sinh(y\sigma/2)}{y\sigma/2}\right)^2$, $T(y) = \left(\frac{\sinh(y\vartheta/2)}{y\vartheta/2}\right)^2$ and

$$F(x) = \log \frac{\sigma^2 S(1+x) + \vartheta^2 T(1+x)}{S(1+x)} - \log \frac{\sigma^2 S(x) + \vartheta^2 T(x)}{S(x)}.$$

Then $\log |\psi(\theta)|^2 = \sum_{j=2}^n F(j/n)$, also for $s = 0$ if $\vartheta < 2\pi$. If $F \leq M_c$ on $[\frac{c}{C}, \frac{c+1}{C}]$ for $0 \leq c < C$, and $F \geq L$ on $[0, \frac{1}{C}]$, then for $n \geq C$

$$\log |\psi(\theta)|^2 \leq \frac{n}{C} \sum_{c=0}^{C-1} M_c + \sum_{c=1}^{C-1} |M_c - M_{c-1}| + \max(0, -L).$$

(b) If $\sigma \geq 6$, then $\log |\psi(\theta)| \leq -\frac{1}{2}D + \frac{2}{s(e^6-1)}$, where $D = \sum_{j=2}^n \log(1 + X_j)$ satisfies

$$D \geq \frac{1}{s} \int_{2s}^Y \log \left(1 + \frac{\sin^2(c_1 y/2)}{\sinh^2(y/2)}\right) dy \quad \text{if } c_1 s \leq \theta \leq c_2 s, \quad Y = \min\left(\frac{\pi}{c_2}, 6\right),$$

$$D \geq \frac{1}{2s} \int_{\pi/c+2s}^6 \log \left(1 + \frac{\sin^2(\pi/8)}{\sinh^2(y/2)}\right) dy \quad \text{if } cs \leq \theta \leq \pi.$$

Proof. $|\psi| = |\widehat{C}(s - i\theta)|/\widehat{C}(s)$, and $|\sinh(a - ib)|^2 = \sinh^2 a + \sin^2 b$. (a) With $y = m/n$, $1 + X_m = (\sigma^2 S(y) + \vartheta^2 T(y))/(\sigma^2 S(y))$, so $\log(1 + X_{n+j}) - \log(1 + X_j) = F(j/n)$. Let $\delta_c = \#\{j \geq 1 : j/n < \frac{c+1}{C}\} - \frac{n(c+1)}{C}$, which lies in $[-1, 0)$. Put every $j \leq n$ in the cell of j/n (the last cell closed). Summation by parts gives

$$\sum_{j=1}^n F(j/n) \leq \sum_c \#\{j : j/n \in \text{cell } c\} M_c = \frac{n}{C} \sum_c M_c + \sum_{c=0}^{C-2} \delta_c (M_c - M_{c+1}).$$

The last sum is at most $\sum_c |M_{c+1} - M_c|$. Then remove $j = 1$, using $F(1/n) \geq L$.

(b) The factors with $m = n + j$ contribute at most

$$\frac{1}{2} \sum_{m \geq n+2} \frac{1}{\sinh^2(ms/2)} \leq \frac{1}{2s} \int_{(n+1)s}^{\infty} \frac{dx}{\sinh^2(x/2)} \leq \frac{2}{s(e^6 - 1)}.$$

First bound: for $y \leq \pi/c_2$, $\sin^2(cy/2)$ increases in $c \in [c_1, c_2]$, and $\sin(c_1 y/2)/\sinh(y/2)$ decreases in y , because its logarithmic derivative is $\frac{c_1}{2} \cot \frac{c_1 y}{2} - \frac{1}{2} \coth \frac{y}{2} \leq \frac{1}{y} - \frac{1}{y}$. So each term with $js < Y$ is at least $\frac{1}{s}$ times the integral over $[js, \min((j+1)s, Y)]$, and $Y \leq 6 \leq (n+1)s$. Second bound: let $d = \lfloor \pi/\theta \rfloor$. Then $d\theta \in [\pi/2, \pi]$, so the angles $j\theta/2$ and $(j+d)\theta/2$ differ by an amount at distance at least $\pi/4$ from $\pi\mathbb{Z}$, and one of them is at distance at least $\pi/8$. Hence

$$\log(1 + X_j) + \log(1 + X_{j+d}) \geq \log \left(1 + \frac{\sin^2(\pi/8)}{\sinh^2((j+d)s/2)}\right).$$

Summing over $2 \leq j \leq n - d$ counts each index at most twice, so $2D$ is at least the sum of the right side, which is at least $\frac{1}{s} \int_{(d+2)s}^6$ of the integrand; and $(d+2)s \leq \pi/c + 2s$. $\square$

Part (a) is a Riemann sum with an explicit error, and it applies uniformly in n : the bounds M_c and L depend only on σ and ϑ . If $\frac{1}{C} \sum_c M_c = A < 0$, then $|\psi| \leq e^{(nA+B)/2}$, with B the remaining two terms. Part (b) covers the regime $\sigma \geq 6$, where $1/s$ is the scale and n plays no role.

Lemma 9.12 (explicit far bound). *Let $s \geq 0$, $\vartheta_1 > 0$ with $\pi n/\vartheta_1 + \frac{1}{2} \leq n$, and $\vartheta_1/n \leq \theta \leq \pi$. Put $\omega = \frac{1}{4} \cosh^{-2}(\frac{1}{2}(\pi\sigma/\vartheta_1 + s/2))$ and $V_- = \frac{1}{s} \tanh \frac{(n-1/2)s}{2} - \frac{1}{4}$ (read as $\frac{n-1}{2}$ if $s = 0$), and assume $V_- > 12$. Then*

$$|\psi(\theta)| \leq (3n+1) \cdot \frac{4}{3} (4\sqrt{n/2} + 1) \cdot \exp\left(-\omega(V_- - \frac{\pi n}{4\vartheta_1} - \frac{1}{8})\right).$$

Proof. Follow the proof of Lemma 9.6, with two changes. First, $\pi_i = p_i + p_{i+d} - 2p_i p_{i+d}$ exceeds $p_i(1-p_i) + p_{i+d}(1-p_{i+d})$ by $(p_i - p_{i+d})^2$, so $\sum_{i \in I} \pi_i \geq V - \frac{d}{4}$ (Lemma 9.6 used half of this). Second, always take d to be the integer nearest to π/θ . Then $\sin^2(d\theta/2) \geq \cos^2(\theta/4) \geq \frac{1}{2}$ and $ds \leq \pi\sigma/\vartheta_1 + s/2$, so $1-R \geq \frac{1}{2} \sin^2(d\theta/2) \cosh^{-2}(ds/2) \geq \omega$, and $d \leq \pi n/\vartheta_1 + \frac{1}{2}$. Also $V = \sum_{i < 2n} \frac{1}{4} \cosh^{-2} \frac{(i-n+1/2)s}{2} \geq V_-$ (compare the sum with an integral), and the prefactor $([n+1]_\rho + 2[n]_\rho)/\Pr(S=n)$ is at most $(3n+1) \cdot \frac{4}{3} (4\sqrt{n/2} + 1)$. $\square$

Proof of Theorem C. For $n \leq 1000$ this is the computation of Section 8.8. Degrees $k \leq K_0 = 10^4$ are also settled by exact computation. Modulo q^{K_0+1} , C_n is the same polynomial for all $n \geq K_0$, because its numerator starts in degree $n+2$. So computing $C_n \bmod q^{K_0+1}$ for $16 \leq n \leq K_0$ covers every $n \geq 16$. The computation finds $c_{k-1} \leq c_k$ for $2 \leq k \leq \min(K_0, \mu)$, strictly when $k \geq n+2$.

Now let $n \geq 1001$ and $K_0 < k \leq \mu$. By the tilted-mean bound in the proof of Theorem C, the saddle point satisfies $s < \pi/\sqrt{6(K_0+1)} < 0.01283$. There are two regimes. In each, the certificate `cow_certificate.py` checks criterion (a) or (b) of Lemma 9.9 in ball arithmetic (Arb [11], through python-flint), cell by cell. An inequality counts only if it holds on the whole cell. It is convenient to measure θ in the unit $1/\sqrt{K}$, where K is the lower bound for κ_2 from Lemma 9.10: with $u = \theta\sqrt{K}$, the integrals $\hat{G}_0 = \sqrt{K} G_0$ and $\hat{G}_2 = K^{3/2} G_2$ are close to $\sqrt{\pi/2}$ and $\sqrt{\pi/2}/2$, and the criteria read $\hat{G}_0 > \sqrt{K} I_0$, $\hat{G}_2 > K^{3/2} I_1$, and $\hat{G}_0 > \sqrt{K}(I_0 + I_1/(1-\rho))$.

Regime F: $\sigma \leq 6$ and every $n \geq 1001$ at once. The interval $[0, 6]$ is cut into 179 cells: $[0, 10^{-3}]$, then cells with ratio 1.25 up to 0.3, then ratio 1.02 up to 6. On a cell all bounds are uniform in $1/n \in [0, 1/1001]$.

- $K = n^3 a_-$, where $a_- \leq \kappa_2/n^3 \leq a_+$ comes from Lemma 9.10 (for $\sigma < 1.2$ through the Taylor series of $\bar{\Psi}$).
- The Taylor zone is $\theta \leq \theta_Z = \vartheta_0/n$, with $\vartheta_0 = 1.2$ for $\sigma < 3$ and $\vartheta_0 = 0.4\sigma_+$ otherwise, where σ_+ is the upper end of the cell. Taylor's formula gives $|\operatorname{Re} f + \frac{1}{2}\kappa_2\theta^2| \leq E$ and $|\operatorname{Im} f| \leq \Phi$, where

$$E = |\kappa_4|\theta^4/24 + \sup |L^{(5)}| \theta^5/120,$$

$$\Phi = |\kappa_3|\theta^3/6 + \sup |\operatorname{Re} L^{(5)}| \theta^5/120,$$

with the suprema over the segment $[s - i\theta, s]$. The derivatives are bounded by Lemma 9.3(b) for $|t| \leq \vartheta_0/n$ instead of $\pi/(2n)$ (the proof only needs $|\operatorname{Im} y| \leq \vartheta_0 < \pi$, with H_r replaced by $2 \sum_{l \geq 1} (\pi l - \vartheta_0)^{-r}$), by Lemma 9.4 for the factor s (its proof works for all $s \geq 0$ and $|t| < \pi/n$), and for $\sigma > \frac{1}{2}$ also by Lemma 9.3(a), with $A_r/r! \leq 1.011, 1.039, 1.08, 1.13$ for $r = 3, \dots, 6$. The certificate checks $E \leq \frac{1}{4} K \theta^2$ on the zone.

- $\hat{G}_0$ and $\hat{G}_2$ are bounded below by sums over u -cells of width about 0.02 on $[0, U]$, $U = \min(18, \vartheta_0 \sqrt{1001 a_-})$, using $-\frac{a_+}{a_-} \frac{u^2}{2} - E \leq \operatorname{Re} f \leq -\frac{u^2}{2} + E$. On the rest of the Taylor zone,

$\operatorname{Re} f \leq -u^2/4$, so the integrand of $\widehat{G}_0$ is at least $-e^{-u^2/4}b$, whose integral beyond U is bounded in closed form.

- I_0 and I_1 are bounded by [Lemma 9.11\(a\)](#) with $C = 200$ on ϑ -cells of width 0.02 to 0.25 from ϑ_0 to $\vartheta_1 = \max(5, 5\sigma)$, with M_c from ball evaluation or a mean-value bound on each box, and by [Lemma 9.12](#) on $[\vartheta_1/n, \pi]$.

The bounds for I_0 and I_1 are evaluated at $n = 1001$, and they also hold for every larger n . A ϑ -cell of [Lemma 9.11\(a\)](#) contributes at most $cn^pe^{nA/2}$ to $\sqrt{K}I_0$, $K^{3/2}I_1$ and $\sqrt{K}I_1/(1-\rho)$, with $p = \frac{1}{2}, \frac{5}{2}, \frac{1}{2}$ and c independent of n . For the last one, $1/(1-\rho) \leq n/(\sigma_-(1-\sigma_+/(2n)))$, and the second factor decreases in n . These decrease in n once $A < -5/n$. In [Lemma 9.12](#) the exponent is $-\omega(nX - \frac{3}{8})$ with $X = (V_- + \frac{1}{4})/n - \pi/(4\vartheta_1)$. Both ω and X increase with n , so if $1001X > \frac{3}{8}$ at $n = 1001$, the exponent is at most $-rn + \frac{3}{8}\omega$, with $r = \omega X$ and ω evaluated at $n = 1001$. The remaining factors are powers of n , $3n+1$ and $4\sqrt{n/2}+1$, with logarithmic derivative at most $6/n$, so these contributions decrease once $r > 6/n$. The certificate checks $A < -5/1001$, $r > 6/1001$ and $1001X > \frac{3}{8}$ on every cell.

Regime P: $\sigma \geq 6$ and $0 < s \leq 0.01283$, for every n . Here $1/s$ is the scale. [Lemma 9.3\(a\)](#) gives $|L^{(r)}(s-it)| \leq \zeta(2)A_r s^{-r-1}$ for all t , and [Lemma 9.10](#) gives $K = (\pi^2/3 - 2s - 2G_6)s^{-3}$. The Taylor zone is $\theta \leq s/2$. Beyond it, [Lemma 9.11\(b\)](#), with c -cells $[\frac{1}{2} \cdot 1.1^i, \frac{1}{2} \cdot 1.1^{i+1}]$ up to $c = 20$ and the second bound beyond, gives $\log |\psi| \leq -0.309/s$. Every ingredient is monotone in s . Put $k(s) = \pi^2/3 - 2s - 2G_6$, which decreases. In the variable u , the bounds for E , Φ , $\theta\Phi/s$ and $\theta^2\Phi^2/(4s)$ are sums of positive powers of s times powers of $1/k(s)$. The ratio of the two bounds for κ_2 , and the condition $E \leq \frac{1}{4}K\theta^2$, depend on s only through $k(s)$. The end of the Taylor zone, $u = \frac{1}{2}\sqrt{k(s)/s}$, decreases. So the lower bound for $\widehat{G}_0$ at $s = 0.01283$ holds for all smaller s . The constant 0.309 only improves as s decreases, since the lower limits $2s$ and $\pi/c + 2s$ of the integrals shrink. The far terms, bounded with the factor $\sqrt{\pi^2/3} s^{-3/2} \geq \sqrt{\kappa_2}$ in place of $\sqrt{K}$, are sums of terms $s^{-p}e^{-\kappa/s}$ with $p \leq 5/2$, some divided by $1-s/2$. These increase on $(0, 0.01283]$ because $\kappa/s > 3$ there. So a single evaluation at $s = 0.01283$ covers the regime.

TABLE 4. The parts of the proof of Theorem C'.

Range	Method	Result
$16 \leq n \leq 1000$	exact $C_n(q)$ (Section 8.8)	nondecreasing on $[2, \mu]$
$n \geq 1001, k \leq 10^4$	exact, modulo q^{10^4+1}	strict for $k \geq n+2$
F: $\sigma \leq 6, n \geq 1001$	179 cells: (a) on 164, (b) on 15	(a): $\widehat{G}_0 - \sqrt{K}I_0 \geq 1.096$, $\widehat{G}_2 \geq 1.23 K^{3/2}I_1$; (b): margin ≥ 1.078
P: $\sigma \geq 6, s \leq 0.01283$	1 cell: (b)	$\widehat{G}_0 \geq 1.154$, far terms $\leq 2 \cdot 10^{-5}$

Table 4 summarizes the parts of the proof. The conditions for uniformity in n hold with room: $A \leq -0.033$ against $-5/1001$, and $r \geq 0.031$ against $6/1001$. The whole check takes about 10 CPU-minutes.

In every cell of both regimes the criterion holds, so $c_k > c_{k-1}$ for $K_0 < k \leq \mu$. By palindromy, $(c_1, \dots, c_{2\mu-1})$ is unimodal. $\square$

Corollary 9.13 (Theorem 8.14). *For every $n \geq 2$ and $3 \leq k \leq (n^2 - 2n)/2$, $c_k \geq c_{k-3}$. Hence FStep holds at the corner $(b, c) = (a+1, a+1)$ for every $a \geq 2$ ([Corollary 8.13](#)), and the corner case of [Conjecture 8.10](#) is settled.*

Proof. For $n \geq 16$ and $5 \leq k \leq \mu$, [Theorem C'](#) gives $c_k \geq c_{k-1} \geq c_{k-2} \geq c_{k-3}$; also $c_3 = c_0 = 1$ and $c_4 \geq c_1 = 0$. For $n \leq 15$ it is part of the computation of [Section 8.8](#). $\square$

Corollary 9.14 (parity unimodality). *For every $n \geq 1$ the polynomial $(1+q)C_n(q)$ is unimodal. This is the case $(m, m+1)$ of the conjecture of Xin and Zhong [21] that $(1+q)C_{m,m'}(q)$ is unimodal for the rational q -Catalan polynomials $C_{m,m'}(q) = \frac{1}{[m+m']_q} \binom{m+m'}{m}_q$, $\gcd(m, m') = 1$; they proved it for $m \leq 5$. [Conjecture 8.21](#) is the corresponding extension of [Theorem C](#).*

Proof. $C_{n,n+1} = C_n$. The coefficients of $(1+q)C_n$ are $y_k = c_k + c_{k-1}$, and the polynomial is palindromic of degree $2\mu + 1$. For $n \geq 16$, [Theorem C'](#) gives $c_{k-1} \leq c_k$ for $2 \leq k \leq \mu$, so y_k is nondecreasing for $2 \leq k \leq \mu$, and $y_0 = y_1 = y_2 = 1$. For $n \leq 15$, and in fact for $n \leq 1000$, it is a direct computation. $\square$

Remark 9.15 (what the certificate needs). The local integral is never the problem: $\widehat{G}_0 \geq 1.09$ in every cell, against the Gaussian value $\sqrt{\pi/2} \approx 1.25$, and $\sqrt{K}I_0 < 10^{-4}$. The tight spots are the terms with I_1 . Near $\sigma = 4.5$, where criterion (a) is still used, $K^{3/2}I_1$ reaches $0.81\widehat{G}_2$. Beyond that, criterion (b) is used, and $\sqrt{K}(I_0 + I_1/(1-\rho))$ stays below 0.03. What costs effort is making the bounds uniform, not accuracy. The derivative bounds of [Lemma 9.3\(b\)](#) grow as ϑ_0 approaches π , which forces a short Taylor zone for $\sigma < 3$. The pair product of [Lemma 9.11\(a\)](#) then has to be sharp from $\vartheta = 1.2$ on, and the Riemann-sum form makes that sharpness hold for all n at once. The Bernoulli pairs cannot take over there: [Lemma 9.12](#) needs pairs at distance $d \approx \pi/\theta \leq n$, so it starts near $\vartheta = \pi$, and it is weak until ϑ is several times larger. The low-degree computation lets the analysis stop at $s = 0.0128$. For larger s the bound $\log|\psi| \leq -0.309/s$ no longer beats the factor $s^{-5/2}$ in criterion (b), and the regime-P check fails from about $s = 0.026$. The threshold $n = 1001$ is only where the computation of [Section 8.8](#) stops; no attempt was made to lower it.

9.5. Rational q -Catalan numbers. The proof of [Theorem C](#) uses only the shape of C_n as a product of ratios $(1 - q^{n+j})/(1 - q^j)$. It therefore applies to the series $\Gamma_{x,N}$ of [Section 8.8](#), which are the rational q -Catalan numbers $C_{x,N+1}$ when $\gcd(x, N+1) = 1$. In this subsection we write y for that N , because N denotes the scale. So $\Gamma_{x,y} = \prod_{j=2}^x \frac{1-q^{y+j}}{1-q^j} = \sum_k \gamma(k)q^k$, with $\mu = (x-1)y/2$.

Theorem 9.16 (rational q -Catalan numbers). *(i) If $x \geq 7$ and $y \geq x$, then $\gamma(k-1) \leq \gamma(k)$ for $2 \leq k \leq x+1$. (ii) For every $x \geq 201$ and $x \leq y \leq 2x$, $\gamma(k-1) \leq \gamma(k)$ for $2 \leq k < \mu$, strictly for $5000 < k < \mu$. Together with the computations of [Section 8.8](#), $(M_{x,y})$ holds for all $x \geq 13$ and $x \leq y \leq 2x-2$, and [Conjecture 8.21](#) holds for $x \leq y \leq 2x$ in every degree below the middle one.*

Proof of (i). In degrees $k \leq x+1 \leq y+1$ the numerator of $\Gamma_{x,y}$ does not contribute, so $\gamma(k)$ counts the partitions of k into parts from $\{2, \dots, x\}$. The first paragraph of the proof of [Theorem C](#) applies with n replaced by x . $\square$

For all large x , part (ii) follows the proof of [Theorem C](#), with the scale $N = \min(x, 1/s)$ and these changes. The pair decomposition holds with the pairs $(j, y+j)$, $2 \leq j \leq x$: $\Gamma_{x,y}(e^{-w}) = e^{-\mu w} \widehat{\Gamma}(w)$ with $\widehat{\Gamma}(w) = \prod_{j=2}^x \sinh \frac{(y+j)w}{2} / \sinh \frac{jw}{2}$, which is even and has no zeros for $\operatorname{Re} w > 0$ or $|w| < 2\pi/(x+y)$. When $\gcd(x, y+1) > 1$, $\Gamma_{x,y}$ is a power series with poles on the unit circle, but the identity holds for $\operatorname{Re} w > 0$, and $k < \mu$ keeps the saddle point at $s > 0$, since $L'(s) = \mu - k$. Lemmas 9.3 and 9.4 hold with $M = \{2, \dots, x\} \cup \{y+2, \dots, y+x\} \subseteq [2, 3x]$, in [Lemma 9.3\(b\)](#) for $|t| \leq \pi/(x+y)$; [Lemma 9.10](#) holds with the same pairs, and [Lemma 9.11](#) with $F(t) = \log(1 + X(\nu + t)) - \log(1 + X(t))$, $\nu = y/x$. [Lemma 9.6](#) is replaced by the

following lemma, whose extra factor is at most polynomial in x . That gives some threshold x_0 , astronomically large as in [Remark 9.8](#); the explicit proof from $x = 201$ on follows [Section 9.4](#).

Lemma 9.17 (far range). *Let $y \geq x \geq 2$, $M' = x + y$, $\rho = e^{-s}$ with $s > 0$, and $0 < \theta \leq \pi$. Let $z > 0$ be such that $p_i = z\rho^i/(1 + z\rho^i)$, $0 \leq i < M'$, satisfy $\sum_i p_i = x$, and put $V = \sum_i p_i(1 - p_i)$. For every integer $1 \leq d < M'$ with $V > d/4$,*

$$\frac{|\Gamma_{x,y}(\rho e^{i\theta})|}{\Gamma_{x,y}(\rho)} \leq Q(\theta) \cdot \frac{4}{3}(2\sqrt{M'} + 1) e^{-(1-R_d)(V-d/4)}, \quad R_d = \left(1 - \frac{\sin^2(d\theta/2)}{\cosh^2(ds/2)}\right)^{1/2},$$

where

$$Q(\theta) = \frac{[y+1]_\rho}{|[y+1]_{\rho e^{i\theta}}|} \leq \frac{2(y+1)}{|1 - \rho^{y+1} e^{i(y+1)\theta}|}.$$

On an interval of length ℓ ,

$$\int Q d\theta \leq 2 \left(\left\lceil \frac{(y+1)\ell}{2\pi} \right\rceil + 1 \right) \frac{\pi}{\sqrt{R'}} \operatorname{asinh} \frac{2\sqrt{R'}}{1 - R'}, \quad R' = \rho^{y+1}.$$

Proof. $\Gamma_{x,y} = G/[y+1]$ with $G = \binom{M'}{x}_q$. [Lemma 9.5](#) with $r = x$ and the pairs $\{i, i+d\}$, $i \bmod 2d < d$, of the proof of [Lemma 9.12](#) give the exponential factor, since $\sum_{i \in I} \pi_i \geq V - d/4$. By Darroch's theorem [\[7\]](#) the mode of a sum of independent Bernoulli variables differs from its mean by less than 1, so the mode of S is x , and Chebyshev's inequality with $t = 2\sqrt{V}$ gives $\Pr(S = x) \geq 3/(4(4\sqrt{V} + 1))$, which is at least $3/(4(2\sqrt{M'} + 1))$ as $V \leq M'/4$. For Q , $[y+1]_\rho \leq y+1$ and $|1 - \rho e^{i\theta}| \leq 2$. For the integral substitute $\varphi = (y+1)\theta$; the interval meets at most $\lceil (y+1)\ell/2\pi \rceil + 1$ periods, and on one period $|1 - R' e^{i\varphi}|^2 \geq (1 - R')^2 + 4R'\varphi^2/\pi^2$. $\square$

The lemma is applied on blocks of θ . Up to $\theta = 1$ the pair distance is $d = \text{round}(\pi/\theta)$, so that $\sin^2(d\theta/2) \geq \cos^2(\theta/4)$ and $d \leq \pi/\theta + \frac{1}{2}$; on $[1, \pi]$ a fixed $d \in \{1, 2, 3\}$ is used on each of eight sub-intervals. Then $1 - R_d$ is close to 1, where [Lemma 9.12](#) used the bound $\frac{1}{4} \cosh^{-2}$. The center c of the p_i , where $z\rho^c = 1$, is located by comparing $\sum_i p_i$ with integrals of $t \mapsto 1/(1 + e^{s(t-c)})$: $sc \in [\lambda_+ - s, \lambda_+]$, where λ_+ solves $\log(1 + e^\lambda) - \log(1 + e^{\lambda - (x+y)s}) = xs$ and lies between $-\log(y/x)$ and $-\log(y/x) + (x+y)s$. Since $p_i(1 - p_i) = \frac{1}{4} \cosh^{-2} \frac{s(i-c)}{2}$ is unimodal in i , this gives

$$V \geq \frac{1}{2s} \left[\tanh \frac{sc}{2} + \tanh \frac{s(x+y-1-c)}{2} \right] - \frac{1}{4}, \quad V \geq (x+y) \min_i p_i(1 - p_i).$$

Integrating Q instead of bounding it pointwise matters near the middle degree. There s is of order x^{-3} , and Q is as large as $2/s$ near the $(y+1)$ -th roots of unity, where $\Gamma_{x,y}$ has its poles when $\gcd(x, y+1) > 1$; but $\int Q$ grows only like $\log(1/s)$.

Proof of (ii). The certificate `rq_certificate.py` follows [Section 9.4](#), with $\nu = y/x \in [1, 2]$ as a second parameter.

Exact low degrees. Degrees $k \leq 5000$ are settled exactly. Modulo q^{5001} , $\Gamma_{x,y}$ depends only on $\min(x, 5000)$ and on the numerator factors below degree 5001. For $y \geq 2498$ at most one of them is there, and

$$\gamma(k) - \gamma(k-1) = \Delta z(k) - (z(k-y-2) - z(k-y-x-1)),$$

where $z(k)$ counts the partitions of k into parts from $\{2, \dots, x\}$. So it suffices that $\Delta z(k) \geq \max_{v \leq k-2500} z(v)$ for $3 \leq k \leq 5000$, which is checked for every $x \geq 201$. For $201 \leq y \leq 2497$ the 1,540,949 series $\Gamma_{x,y} \bmod q^{5001}$, $\lceil y/2 \rceil \leq x \leq y$, are computed and checked.

Regime P. For $k > 5000$ the saddle point satisfies $s < \pi/\sqrt{6 \cdot 5001} < 0.0182$, and $k < \mu$ gives $L'(s) \geq \frac{1}{2}$. The regime $\sigma = xs \geq 6$, $s \leq 0.0182$, is the check of [Section 9.4](#) verbatim, since its

analysis uses only that the numerator exponents $y + j$ are at least $x + 2$. It passes with margin 1.12.

Regime F. The regime $\sigma \leq 6$ uses the 179 σ -cells of [Section 9.4](#) times ten ν -cells of width 0.1 (the script would refine a failing cell to widths 0.02 and 0.005; none failed). Each cell is checked for every $x \geq x_c = \max(201, \sigma_-/0.0182)$; for smaller x its degrees are at most 5000. The changes from [Section 9.4](#) are these.

- κ_2/x^3 is bracketed by [Lemma 9.10](#) with the pairs $(j, y + j)$. The Taylor zone is $\vartheta \leq \vartheta_0$, with $(1 + \nu)\vartheta_0/2 = 1.2$ for $\sigma < 3$, where [Lemma 9.3\(b\)](#) is used, and $\vartheta_0 \leq 0.45\sigma_-$ for $\sigma \geq 3$, where only [Lemma 9.3\(a\)](#) is used.
- The Riemann sums of [Lemma 9.11\(a\)](#) run over $\vartheta \leq \max(5, 5\sigma)$. For $\sigma \geq 0.3$, F is evaluated as $\log(1 + X(\nu + t)) - \log(1 + X(t))$; the form of [Lemma 9.11](#) splits each X into two factors $\sinh$, whose independent enclosures cost up to 0.1 in the average A near $\sigma = 6$.
- Beyond that, [Lemma 9.17](#) on the blocks described above. In the first σ -cell the bound $\sigma \geq 1/(2a_+x^2)$, with $a_+ \geq \kappa_2/x^3$, comes from $L'(s) \geq \frac{1}{2}$.
- Uniformity in $x \geq x_c$. The Riemann part of $\sqrt{K}I_0$ and of $\sqrt{K}I_1/(1 - \rho)$ grows like $x^{1/2}e^{xA/2}$, and that of $K^{3/2}I_1$ like $x^{5/2}e^{xA/2}$. The part from [Lemma 9.17](#) grows at most like x^7e^{-rx} , where r is its exponential rate. The certificate checks $A < -1/x_c$ or $A < -5/x_c$, according to the criterion used, and $rx_c > 7$.

TABLE 5. The parts of the proof of Theorem 9.16 (ii).

Part	Range	Result
exact	$k \leq 5000, x \geq 201$	1,540,949 series, and the one-factor criterion for $y \geq 2498$
P	$\sigma \geq 6, s \leq 0.0182$	1 cell, margin 1.12
F	$\sigma \leq 6, 1 \leq \nu \leq 2$	1790 cells, all at ν -width 0.1; $\widehat{G}_0 - \sqrt{K}I_0 \geq 0.687$ where criterion (a) is used, margin ≥ 0.751 where only (b) holds; $A \leq -0.048, rx_c \geq 23.1$

Table 5 summarizes the parts of the proof. In every cell the criterion holds, so $\gamma(k) > \gamma(k - 1)$ for $5000 < k < \mu$. The last statement of the theorem combines this with `band_check.py` for $13 \leq x \leq 200$ and with the computation reported after [Conjecture 8.21](#). $\square$

Remark 9.18 (why 201 is enough). [Theorem C'](#) needed $n \geq 1001$ for its analytic part; here the threshold is 201. Three changes make the difference: the exact factor $1 - R_d$ in [Lemma 9.17](#), close to 1 for $\theta \geq 5/x$, instead of $\frac{1}{4} \cosh^{-2}(\cdot)$; the integrated pole factor; and the direct evaluation of F . The case $\nu = 1$ is $\Gamma_{x,x} = C_x$, so [Theorem 9.16 \(ii\)](#) also contains [Theorem C'](#) for $x \geq 201$, below the middle degree, with a threshold five times smaller. A floating-point experiment at $x = 201$ ($y = 201, 300, 400$) shows the Gaussian term $\mathcal{M}$ within 1% of the difference integral of [Lemma 9.2](#) at every tested saddle point, so the remaining losses are in the bounds, not in the method.

9.6. Beyond the band. [Proposition 8.23](#) splits $\mathcal{D}_A$ as $\mathcal{P}_A - \mathcal{N}_A$, and [Remark 8.25](#) reduces positivity to an evaluation inequality at a real point and local limit bounds for Gaussian coefficients. This subsection proves both for every $x \geq 201$ and every $A > 2x(x - 1)$, which with [Theorem D](#) gives [Conjecture 8.16](#) for $x \geq 201$. Here $s \geq 0$ is always the tilt, $\rho = e^{-s}$, and we write $A = xD + r = (x - 1)E + r'$ (so r' is the remainder called s in [Proposition 8.23](#)). For a Gaussian coefficient $G(m, n) = \binom{m+n}{m}_q$ put

$$\pi(j) = \frac{\rho^j [q^j] G(m, n)}{G(m, n)(\rho)}, \quad \Lambda = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|G(m, n)(\rho e^{i\theta})|}{G(m, n)(\rho)} d\theta,$$

the point probabilities of the tilted distribution and the bound $\pi(j) \leq \Lambda$ from Cauchy's formula. Let κ_1, κ_2 be the mean and variance of the tilted distribution, and $\sigma = ms$, $\tau = ns$, $\nu = n/m$. A local limit theorem says that $\sqrt{2\pi\kappa_2}\pi(j)$ and $\sqrt{2\pi\kappa_2}\Lambda$ are close to 1 for j near κ_1 ; what is needed is a lower bound p for the first and an upper bound L for the second, valid for all parameters at once.

Lemma 9.19 (the criterion). *Let $x \geq 3$ and $A > 2x(x-1)$. (a) Let $D < k \leq A/2$, and let $s \geq 0$ be the tilt at which $G(x, D-1)$ has mean $k-x$. Suppose that $\sqrt{2\pi\kappa_2}\pi(k-x) \geq p$ for $G(x, D-1)$ and for $G(x, D)$, and $\sqrt{2\pi\kappa_2}\Lambda \leq L$ for every $G(x-2, e)$ with $D < e \leq E+1$, each with its own κ_2 at this s . If*

$$\frac{\mathcal{N}_A(\rho)}{\mathcal{P}_A(\rho)} < \frac{p}{L}\sqrt{1-\ell}, \quad \ell = \frac{\min(2/s^2, (D+x)^2/6)}{\kappa_2(G(x, D-1))},$$

then $[q^k]\mathcal{D}_A > 0$. (b) If $D < k < D+x$, then $[q^k]\mathcal{N}_A \leq [q^k]\mathcal{P}_A$. So (a) is needed only for $k \geq D+x$, and there the tilt satisfies $s < \pi/\sqrt{6D}$.

Proof. (a) The tilt exists: the untilted mean is $x(D-1)/2 \geq k-x$, because $k \leq (xD+r)/2$ and $r < x$, and the tilted mean decreases to 0. [Remark 8.25](#) gives

$$[q^k]\mathcal{D}_A \geq \rho^{-k}(\mathcal{P}_A(\rho) \min_i \pi_i - \mathcal{N}_A(\rho) \max_e \Lambda_e).$$

As in [Lemma 9.10](#), the h -form gives

$$\kappa_2(G(m, n)) = \frac{1}{s^2} \sum_{j=1}^m \left[\Psi\left(\frac{(n+j)s}{2}\right) - \Psi\left(\frac{js}{2}\right) \right],$$

with Ψ increasing and $0 \leq \Psi(y) \leq \min(1, y^2/3)$. For $n' \in \{D-1, D\}$ and $e \geq D+1$, first drop the pairs $j = x-1, x$ from $G(x, n')$, each of which contributes at most $\min(1/s^2, (D+x)^2/12)$, and then raise n' to e , which only increases κ_2 . This gives $\kappa_2(G(x-2, e)) \geq \kappa_2(G(x, n')) - \delta$ with $\delta = \min(2/s^2, (D+x)^2/6)$. Since $\kappa_2(G(x, D)) \geq \kappa_2(G(x, D-1))$, this is at least $(1-\ell)\kappa_2(G(x, n'))$, so $\min_i \pi_i \geq \frac{p}{L}\sqrt{1-\ell} \max_e \Lambda_e$.

(b) Let $p_m(t)$ be the number of partitions of t into at most m parts. Since $A > 2x(x-1)$, $D \geq 2x-2$. For $k < D+x$ we have $k-x \leq D-1$, so the bound on the parts of $G(x, D-1)$ and $G(x, D)$ is not active in degree $k-x$, and $[q^k]\mathcal{P}_A = x(x-1)p_x(k-x)$. For $e \geq D+1$ we have $k-e \leq x-2 < e$, so $[q^{k-e}]G(x-2, e) = p_{x-2}(k-e)$, and every weight w_e is at most $x(x-1)$. Hence

$$[q^k]\mathcal{N}_A \leq x(x-1) \sum_{t=0}^{k-D-1} p_{x-2}(t).$$

Adding the part $k-x-t$ to a partition of $t \leq k-D-1$ into at most $x-2$ parts gives a partition of $k-x$ into at most $x-1$ parts, in which the new part is the largest, since it is at least $D-x+1 \geq x-1 > t$; so it determines t , and the sum is at most $p_{x-1}(k-x) \leq p_x(k-x)$. Finally, for $k \geq D+x$ the tilted mean $k-x$ of $G(x, D-1)$ is at least D , and it is at most $\sum_{j \geq 1} j/(e^{js}-1) < \pi^2/(6s^2)$, as in the proof of [Theorem C](#). $\square$

Beyond the band $D \geq 2x-2$, and $D \geq \lambda x-1$ with $\lambda = A/x^2$. So in the degrees where (a) is used, $s < \pi/\sqrt{12(x-1)}$ and $\sigma = xs < \pi x/\sqrt{6(\lambda x-1)}$; the second bound makes σ small when λ is large compared with x . In the lowest degrees the ratio $[q^k]\mathcal{N}_A/[q^k]\mathcal{P}_A$ is small, but the tilt is large and the local limit bounds are poor; (b) removes exactly those degrees.

Lemma 9.20 (the evaluation bound in closed form). *Let $x \geq 3$, $\lambda = A/x^2 \geq 2$ and $0 < \rho = e^{-s} \leq 1$. Then*

$$\frac{\mathcal{N}_A(\rho)}{\mathcal{P}_A(\rho)} \leq \left(\frac{\sinh(xs/2)}{\sinh(Ds/2)} \right)^2 \left(1 + \frac{\lambda x}{x-1} e^{1+1/\lambda} \right), \quad \left(\frac{\sinh(xs/2)}{\sinh(Ds/2)} \right)^2 \leq \frac{x^2}{D^2}.$$

For $x \geq 101$ and $\lambda \geq 16$ the right side is at most 0.187, and at most $5 \cdot 10^{-38}$ when $xs \geq 6$.

Proof. Write $[a]$ for $[a]_\rho$. Then

$$\frac{G(x-2, D+1)}{G(x, D-1)} = \frac{[x][x-1]}{[D][D+1]}, \quad \frac{G(x-2, e)}{G(x-2, D+1)} = \prod_{i=D+2}^e \frac{[x-2+i]}{[i]} \leq \left(\frac{[x+D]}{[D+2]} \right)^{e-D-1},$$

since $[x-2+i]/[i]$ decreases in i . Also $G(x, D)/G(x, D-1) = [x+D]/[D] \geq 1$, so $\mathcal{P}_A(\rho) \geq x(x-1)\rho^x G(x, D-1)(\rho)$. With the weights of [Proposition 8.23](#) and $z = \rho[x+D]/[D+2]$,

$$\frac{\mathcal{N}_A(\rho)}{\mathcal{P}_A(\rho)} \leq \rho^{D+1-x} \frac{[x][x-1]}{[D][D+1]} \left(1 - \frac{r}{x} + \sum_{u=1}^{E-D-1} z^u + \frac{r'}{x-1} z^{E-D} \right).$$

The prefactor is at most $\rho^{D-x}(1-\rho^x)^2/(1-\rho^D)^2$, which is $(\sinh(xs/2)/\sinh(Ds/2))^2$, and $\sinh(ky) \geq k \sinh y$ for $k \geq 1$ gives the bound $(x/D)^2$. Next,

$$\log z \leq \frac{\rho^{D+2} - \rho^{D+x}}{1 - \rho^{D+2}} \leq \frac{(x-2)s}{e^{(D+2)s} - 1} \leq \frac{x-2}{D+2}.$$

The sum $E-D-1+r'/(x-1)$ equals $W-1+r/x \leq W$, with $W = A/(x(x-1)) = \lambda x/(x-1)$. As $D+2 \geq \lambda x+1$, every z^u with $u \leq E-D \leq W+1$ is at most $\exp((W+1)\frac{x-2}{\lambda x+1}) \leq e^{1+1/\lambda}$, so the bracket is at most $1 + We^{1+1/\lambda}$. The bound decreases in λ and in x , so the numbers follow from its value at $\lambda = 16$, $x = 101$, with $D/x \geq \lambda - 1/x$; for $xs \geq 6$ use the first form, which decreases in s . $\square$

The true ratio is smaller: by [Observation 8.24](#) it is about 0.10 at $\lambda = 16$, and it tends to $f(\lambda) \approx 1.72/\lambda$. For $\lambda \geq 16$ the closed form already leaves a factor of more than three against the local limit constants below. Near $\lambda = 2$ it does not (at $\lambda = 4$ it gives 0.94), and there the sum in the display is used instead, on cells of $(\sigma, \lambda, r/x)$ with $\sigma = xs$: it increases in z and in the fractional part $r'/(x-1)$, so interval arithmetic bounds it on each cell. At $\sigma = 0$ and $\lambda = 2$ it is 0.66, against the true 0.62.

For $\lambda \geq 16$ the local limit bounds must hold for all $m \geq 99$, all $\nu \geq 15.98$ and all tilts that occur. No single scaling is uniform over that range. Two new tools handle the regimes that [Section 9.5](#) did not meet: a Bernoulli bound that needs no pairs, and a transfer to independent summands when n is much larger than m^2 .

Lemma 9.21 (Bernoulli bound). *Let $G = G(m, n)$, $M' = m + n$ and $s > 0$. Let $p_i = 1/(1 + e^{s(i-c)})$, $0 \leq i < M'$, with c chosen so that $\sum_i p_i = m$, and put $V = \sum_i p_i(1 - p_i)$ and $\pi^* = \max_i p_i(1 - p_i)$. For $0 < |\theta| \leq \pi$,*

$$\frac{|G(\rho e^{i\theta})|}{G(\rho)} \leq \frac{4}{3}(4\sqrt{V} + 1) \exp\left(-V + \frac{\pi^*}{\sin(|\theta|/2)}\right).$$

Moreover $p_0 \leq ms/(1 - e^{-ns})$ and $V \geq m(1 - p_0)$.

Proof. Let ξ_i be independent with $\Pr(\xi_i = 1) = p_i$, $S = \sum_i \xi_i$ and $T = \sum_i i \xi_i$. As in the proof of [Lemma 9.17](#), $|G(\rho e^{i\theta})|/G(\rho)$ equals $|\mathbf{E}[e^{i\theta T}; S = m]|/\Pr(S = m)$, and $\Pr(S = m) \geq$

$3/(4(4\sqrt{V} + 1))$. Writing the event $S = m$ as an integral over α ,

$$|\mathbf{E}[e^{i\theta T}; S = m]| \leq \max_{\alpha} \prod_i |1 - p_i + p_i e^{i(\theta i + \alpha)}| \leq \max_{\alpha} e^{-\sum_i \pi_i (1 - \cos(\theta i + \alpha))} = e^{-V + |Z|},$$

with $\pi_i = p_i(1 - p_i)$ and $Z = \sum_i \pi_i e^{i\theta i}$. Summation by parts gives

$$2|\sin(\theta/2)| |Z| \leq \pi_0 + \pi_{M'-1} + \sum_i |\pi_i - \pi_{i-1}| = 2\pi^*,$$

since $\pi_i = \frac{1}{4} \cosh^{-2} \frac{s(i-c)}{2}$ is unimodal in i . Finally $p_i/p_0 \geq \rho^i$, so $m \geq p_0(1 - \rho^{M'})/(1 - \rho) \geq p_0(1 - e^{-ns})/s$, and $V = m - \sum_i p_i^2 \geq m - p_0 m$. $\square$

Unlike [Lemma 9.17](#) it needs no pairs of indices, and it is good exactly when the occupation is low: if $p_0 \leq \frac{1}{2}$, then $\pi^* = \pi_0 \leq p_0$, and the exponent is about $-m(1 - p_0 - 2p_0/\pi)$ already at $\theta = \pi/m$. Its weakness is the factor $\sqrt{\kappa_2}$, of order $n\sqrt{m}$, that it has to beat. That is what the transfer is for.

Lemma 9.22 (good residues). *Let $m \geq 2$ and $\pi/m \leq \theta \leq \pi$. Then at least $\frac{7m}{16} - \frac{5}{8}$ of the integers $1 \leq j \leq m$ satisfy $|\sin(j\theta/2)| \geq \sin(\pi/8)$.*

Proof. Put $\alpha = \theta/2\pi \in [\frac{1}{2m}, \frac{1}{2}]$; the condition is $\|j\alpha\| \geq \frac{1}{8}$, with $\|\cdot\|$ the distance to $\mathbb{Z}$. If $\alpha \geq \frac{1}{4}$, two consecutive j cannot both fail, since $\|\alpha\| \leq \|j\alpha\| + \|(j+1)\alpha\|$; so at least $\lfloor m/2 \rfloor$ succeed. If $\alpha < \frac{1}{4}$, put $M = m\alpha \in [\frac{1}{2}, \frac{m}{4}]$. A failing j has $j\alpha$ within $\frac{1}{8}$ of an integer t with $0 \leq t \leq M + \frac{1}{8}$. There are fewer than $\frac{1}{8\alpha}$ such j for $t = 0$ and at most $\frac{1}{4\alpha} + 1$ for each $t \geq 1$, so at most $\frac{m}{4} + \frac{5m}{32M} + M + \frac{1}{8}$ fail. This is convex in M , so it is at most its value $\frac{9m}{16} + \frac{5}{8}$ at $M = \frac{1}{2}$ (at $M = \frac{m}{4}$ it is $\frac{m}{2} + \frac{3}{4}$). $\square$

Lemma 9.23 (transfer to independent summands). *Let $U = [n+1]_q = \sum_{i=0}^n q^i$, and suppose that $U(\rho) \geq c_0 m^2$ with $c_0 \geq 4$. Write π_{U^f} and Λ_{U^f} for the quantities above with U^f in place of $G(m, n)$, the law of a sum of f independent truncated geometric variables. Then for every j*

$$\pi(j) \geq e^{-1/(2c_0)} \pi_{U^m}(j), \quad \Lambda \leq \Lambda_{U^m} + (e^{1/(2c_0)} - 1) \Lambda_{U^{\lceil m/2 \rceil}} + \frac{(m/\sqrt{U(\rho)})^{\lfloor m/2 \rfloor + 1}}{1 - c_0^{-1/2}}.$$

Proof. $G(m, n) = h_m(1, q, \dots, q^n)$, and by the cycle index of the symmetric group $h_m = \frac{1}{m!} \sum_{\sigma \in S_m} \prod_c p_{|c|}$, where c runs over the cycles of σ and the power sums are $p_k(1, q, \dots, q^n) = U(q^k)$. Every term has nonnegative coefficients, and $U(\rho^k) \leq U(\rho)$. The identity term gives $[q^j]G \geq [q^j]U^m/m!$ and $G(\rho) \geq U(\rho)^m/m!$. Bounding every term by $U(\rho)^{\#\text{cycles}}$, and using $\sum_{\sigma} u^{\#\text{cycles}(\sigma)} = u(u+1) \cdots (u+m-1)$, gives

$$G(\rho) \leq \frac{U(\rho)^m}{m!} \prod_{i < m} \left(1 + \frac{i}{U(\rho)}\right) \leq \frac{U(\rho)^m}{m!} e^{m^2/(2U(\rho))},$$

which proves the first claim. For the second, a permutation with f fixed points contributes at most $|U(\rho e^{i\theta})|^f U(\rho)^{\#\text{cycles}-f}$ to $m! |G(\rho e^{i\theta})|$. Divide by $U(\rho)^m \leq m! G(\rho)$ and integrate. The permutations other than the identity have total weight $\sum_{\sigma} U^{\#\text{cycles}-m} - 1 \leq e^{1/(2c_0)} - 1$, and for $f \geq m/2$, $\Lambda_{U^f} \leq \Lambda_{U^{\lceil m/2 \rceil}}$. There are at most m^k permutations that move $k > m/2$ points, and each has weight at most $U(\rho)^{-k/2}$, since its moved points lie in cycles of length at least 2. $\square$

Theorem 9.24 (the first row for $x \geq 201$). *For every $x \geq 201$ and every $A \geq x^2$, $R(x, A)$ holds. Consequently FT, and with it Troyka's form $G(b, c) \geq_q G(a, d)$ for $ad \leq bc$ and Bergeron's conjecture, holds whenever $200 \leq a \leq b \leq c$; this is the case $a \geq 200$ of [Theorem E](#).*

Proof. For $A \leq 2x(x-1)$ this is [Corollary 8.20](#). Let $A > 2x(x-1)$ and $\lambda = A/x^2$, so $\lambda > 2 - 2/x$. For $k \leq D$, $[q^k] \mathcal{D}_A = [q^k] \mathcal{P}_A \geq 0$ by [Proposition 8.23](#). For $D < k < D+x$ we use [Lemma 9.19](#) (b), and for $D+x \leq k \leq A/2$ [Lemma 9.19](#) (a), whose tilt then satisfies $0 \leq s < \pi/\sqrt{12(x-1)} \leq 0.0642$. Its Gaussian coefficients are $G(m, n)$ with $m \in \{x-2, x\}$, so $m \geq 199$, and $\nu \geq \lambda - 2/x$; on the side of $\mathcal{N}_A$, $\nu \leq \lambda x^2/((x-1)(x-2)) + 1/(x-2)$ and σ is between $(1 - 2/x)xs$ and xs . Two certificates bound p (at offsets up to the shift between the means of $G(x, D-1)$ and $G(x, D)$) and L , for every $m \geq 199$ and every such tilt; the second one works for every $m \geq 99$.

(i) $\lambda \leq 16$: *pn_certificate.py*. A table over $0 \leq \sigma \leq 6$ and $1.98 \leq \nu \leq 16.25$, in the scaled variables of [Section 9.5](#): 69 base cells in σ (up to 0.001, then ratio 1.25 up to 0.05 and 1.1 beyond) times 44 in ν (ratio 1.05), a failing cell split into four. On each cell, κ_2/m^3 is bracketed by the formula in the proof of [Lemma 9.19](#) with integrals of Ψ ; the Taylor zone uses [Lemma 9.3](#); the pair factors of [Lemma 9.11](#)(a) are summed with its Riemann-sum error while [Lemma 9.21](#) is weak, that is, until its exponent per m is below -0.05 (here j runs from 1, so only the positive variation $\sum_c (M_c - M_{c-1})_+$ enters, by the same summation by parts); and [Lemma 9.21](#) covers the rest of $[0, \pi]$, on blocks of θ . The certificate checks that every error term decreases in m , as in [Section 9.5](#). Then the criterion of [Lemma 9.19](#) is checked on boxes in $(\sigma, \lambda, r/x)$: σ in the same 69 cells, λ from $2 - 2/201$ to 16 in ratio 1.05, r/x in tenths, a failing box split in λ and r/x , and from its second split on also in σ ; $\mathcal{N}_A/\mathcal{P}_A$ is bounded by the sum in the proof of [Lemma 9.20](#), and p, L are the worst table values over the ranges above. For $\sigma \geq 6$, regime P of [Section 9.4](#) in its value version, with $s < \pi/\sqrt{12(x-1)} \leq 0.0642$, $\sigma \geq 5.85$ and $\nu \geq 1.96$ for all four coefficients, gives $p/L \geq 0.742$, and the sum is at most 0.0202.

TABLE 6. Part (i) of the proof of Theorem 9.24.

Part	Range	Result
table	$\sigma \leq 6, 1.98 \leq \nu \leq 16.25$	3,186 cells (50 base cells split once), none failed; $p/L \geq 0.675$, median 0.963; $p \geq 0.754, L \leq 1.151$
criterion	$\sigma \leq 6, \lambda \leq 16$	29,853 boxes (61 split once), none failed; smallest margin > 0.0006
P	$\sigma \geq 6$	13 s -cells; $\mathcal{N}_A/\mathcal{P}_A \leq 0.0202 < 0.727$

Table 6 summarizes part (i). The smallest margins occur in unsplit boxes with $\lambda \leq 2.6$ and $0.37 \leq \sigma \leq 1.56$, where the interval bound for the sum is 0.71 to 0.86; at single points the same bound is 0.52 at $\sigma = 0.5$ and 0.41 at $\sigma = 1$ (with $\lambda = 2$). A box is split only when it fails, so margins near 0 mean only that the box was not refined further.

(ii) $\lambda \geq 16$: *pn_families.py*. By [Lemma 9.20](#), $\mathcal{N}_A(\rho)/\mathcal{P}_A(\rho) \leq 0.187$, and $\leq 5 \cdot 10^{-38}$ when $xs \geq 6$. The certificate is run with $m_c = 99$, so it covers every $x \geq 101$, which [Section 9.7](#) uses. Here $\nu \geq 15.98$, and by [Lemma 9.19](#) (b) $s < \pi/\sqrt{6(16x-1)} \leq 0.032$. Five regimes cover all these tilts:

- (N) $\tau \leq 8$ and $n \leq 20m^2/E(\tau)$, where $E(\tau) = (1 - e^{-\tau})/\tau$. The pairs behave like m independent truncated exponential variables on $[0, n]$, and every bound is taken in $\tau, \mu = m/n \leq 1/15.98$ and $\vartheta' = n\theta$. By the formula for κ_2 in the proof of [Lemma 9.19](#), $\kappa_2/(mn^2)$ lies between $\frac{1}{4}(g(\tau/2) - \mu^2/3)$ and $\frac{1}{4}(1 + \mu)^2 g(\tau/2)$, with $g(y) = \Psi(y)/y^2$ decreasing from $g(0) = \frac{1}{3}$. The Taylor zone $\vartheta' \leq \vartheta'_0$ uses [Lemma 9.3](#)(b) on the strip $|\operatorname{Im}| \leq (1 + \mu)\vartheta'_0/2 < \pi$. For $\theta \leq \pi/m$, every pair factor of [Lemma 9.11](#) is bounded over $j/n \in [0, \mu]$ in closed form, with $\sin$ and $\sinh$ written as one-variable functions $\operatorname{sinc}, \operatorname{sinhc}$; the bound depends on θ only through ϑ' , so it holds for every n , and a tail bound $e^F \leq K(\pi/2\vartheta')^2$ covers large ϑ' . For

$\theta \geq \pi/m$, [Lemma 9.21](#) with $p_0 \leq \mu\tau/(1 - e^{-\tau}) \leq 0.501$ gives $|\varphi| \leq \frac{4}{3}(4\sqrt{m} + 1)e^{-0.33m}$, and the factor $n \leq 20m^2/E(\tau)$ that it costs is harmless.

- **(C)** $\tau \leq 8$ and $n \geq 20m^2/E(\tau)$, so that $U(\rho) \geq nE(\tau) \geq 20m^2$. [Lemma 9.23](#) reduces to U^m , whose bounds are those of (N) with the single pair $(1, n+1)$. There the closed-form pair bound holds on all of $[\vartheta'_0, n\pi]$, so no Bernoulli bound and no factor n occur. The variances of $G(m, n)$ and U^m differ by a relative amount $O(m/n)$, and their means by at most $m(m-1)/4$, which is $O(m^{-1/2})$ standard deviations. In the certificate the threshold between (N) and (C) is $20m^2/E(\tau_+)$, with τ_+ the upper end of the τ -cell; (N) and (C) use the same cells, which the certificate checks.
- **(Sm)** $0.5 \leq \sigma \leq 6$, so $\tau \geq 7.99$: the table of part (i) with $\nu = \infty$ allowed. The upper bracket for κ_2 uses $\sum_j \Psi((n+j)s/2) \leq m$; the numerator terms of L''' are bounded by the decayed form or termwise by [Lemma 9.3\(a\)](#); each pair factor is at most $2 \log \coth((\nu+t)\sigma/2) - \log(1+X(t))$; and the center of the Bernoulli profile lies between $\log(e^\sigma - 1)$ and $\lambda_+(\sigma, 15.98)$.
- **(Ss)** $\sigma \leq 0.5$ and $\tau \geq 8$: sums of nearly exponential variables, in the scale $1/s$. Here $s^2\kappa_2/m \in [\psi(\sigma) - \psi(\tau), 1]$ with $\psi(y) = (y/2)^2/\sinh^2(y/2)$, $|L'''| \leq m(A_3 + \tau^4 \text{Li}_{-3}(e^{-\tau}))s^{-4}$ for all θ , with A_3 as in [Lemma 9.3](#), and $|\kappa_3| \leq 2m/s^3$. For $\theta = cs \leq \pi/m$ the numerators contribute at most $2m \log \coth(\tau/2)$ to $\log |\varphi|^2$, and the denominators at least

$$m \int_0^1 \log \left(1 + \frac{c^2 \text{sinc}^2(c\sigma t/2)}{\sinh^2(\sigma t/2)} \right) dt - \log(1 + c^2),$$

because the integrand decreases in t . For $\theta \geq \pi/m$, [Lemma 9.22](#) gives at least $\frac{7m}{16} - \frac{5}{8}$ denominator factors of modulus at most $\sinh(\sigma/2)/\sin(\pi/8) \leq 0.661$. All bounds are uniform as $\sigma \rightarrow 0$.

- **(P)** $\sigma \geq 6$, which needs $x > 350$ because $\sigma < \pi x/\sqrt{6(16x-1)}$: regime P as in part (i), now with $s < 0.032$.

With $\sigma \leq 6$, either $\tau \leq 8$, which is (N) or (C), or $\tau \geq 8$, which is (Ss) or (Sm). Each regime is checked at $m = 99$, together with the monotonicity in m of its error terms; the results are in [Table 7](#).

TABLE 7. The regimes of part (ii) of the proof of Theorem 9.24, run with $m_c = 99$.

Regime	Range	Cells	min p	max L
N	$\tau \leq 8, n \leq 20m^2/E(\tau)$	256	0.8358	1.0806
C	$\tau \leq 8, U(\rho) \geq 20m^2$	64	0.8579	1.1015
Sm	$0.5 \leq \sigma \leq 6$	54	0.2155	1.4499
Ss	$\sigma \leq 0.5, \tau \geq 8$	2	0.9646	1.0197
P	$\sigma \geq 6$	8 s -cells	$(p/L)\sqrt{1-\ell} \geq 0.888$	

Cells count the two sides (with and without the offset) separately. In [Lemma 9.19](#), $\ell \leq 0.0568$. For $xs \leq 1/2$ all four coefficients lie in (N), (C) or (Ss), and the right side of the criterion is at least $\frac{0.8358}{1.1015}\sqrt{1-\ell} \geq 0.736 > 0.187$. For $1/2 \leq xs \leq 6$ all regimes give at least $\frac{0.2155}{1.4499}\sqrt{1-\ell} \geq 0.144$, while [Lemma 9.20](#) in its first form gives $\mathcal{N}_A/\mathcal{P}_A \leq 0.005$ there. For $xs \geq 6$ the right side is at least $0.888 > 5 \cdot 10^{-38}$.

This proves $R(x, A)$ for all $A \geq x^2$. For the consequence, if $200 \leq a < b \leq c$, then every x with $a < x \leq b$ has $x \geq 201$, and [Proposition 8.15](#) applies; for $a = b$ there is nothing to prove. $\square$

Remark 9.25 (accuracy and cost). The certified constants are close to the truth where it matters. A floating-point evaluation of the exact integrals for $m = 199$ at seven points spread over (N), (C), (Sm) and (Ss) gives $\sqrt{2\pi\kappa_2} \pi(\kappa_1)$ between 0.996 and 0.9995 and $\sqrt{2\pi\kappa_2} \Lambda$ between 0.9993 and 1.013. In part (i) the certified p is lowest at σ near 6, where the ratio of the two parts is below 0.03; near $\lambda = 2$ and small σ , where the ratio is largest, the table gives p/L about 0.97. In part (ii), run with $m_c = 99$, p drops to 0.21 near $\sigma = 6$, where the ratio is below 0.005. The run of `pn_certificate.py` takes about 1.7 CPU-hours, almost all of it for the table; `pn_families.py` takes about a minute. The threshold 201 in part (i) is where the table works. In a run of the same certificate with $m_c = 99$, the table's bounds fall below what the criterion needs in 3,679 of the 29,670 criterion boxes: for $\sigma \leq 0.3$ with λ up to 16, for $0.3 \leq \sigma \leq 2.6$ with $\lambda \leq 7$, mostly near $\lambda = 2$, and near $\sigma = 6$, where some lower bounds p are negative; 47% of the sub-boxes fail again after one refinement and 28% after a second. Section 9.7 replaces the table by a direct computation for $x \leq 200$. Without Lemma 9.19 (b), regime P would also stop at $x = 201$, since the tilt in the lowest degrees reaches $\pi/\sqrt{6(x-1)}$; with it, $s < 0.0642$ in regime P, which gives $p/L \geq 0.742$, against 0.138 for $s < 0.0907$.

9.7. Down to $x = 22$: Bergeron's conjecture. For $22 \leq x \leq 200$ the table of Theorem 9.24 (i) is too weak: run with $m_c = 99$, its error terms of order $1/m$ push p/L below the ratio in many boxes, mostly near $\lambda = 2$, where $\mathcal{N}_A/\mathcal{P}_A$ is largest, and for small σ (Remark 9.25). There the inequality is checked directly instead, for λ up to a bound Λ_x . This is a finite computation for each x , and a cheap one, because it adds only nonnegative numbers. For $\lambda \geq \Lambda_x$ the ratio is small, and by Lemma 9.19 (b) so is σ ; part (ii) of the proof of Theorem 9.24 holds for $m \geq 99$, and its regimes go through for m down to 20 on finite ranges of x .

Proposition 9.26 (the sweep). *Let $\Lambda_x = 64$ for $22 \leq x \leq 40$, $\Lambda_x = 32$ for $41 \leq x \leq 60$ and $\Lambda_x = 16$ for $61 \leq x \leq 200$. Then $R(x, A)$ holds for $2x(x-1) \leq A \leq \Lambda_x x^2$.*

Proof. Every coefficient of $\mathcal{D}_A$ is affine in A between consecutive breakpoints (proof of Proposition 8.19). So it suffices to show that $[q^k] \mathcal{D}_A \geq 0$ for every breakpoint A from $2x(x-1)$ to the first breakpoint $A_{\max} \geq \Lambda_x x^2$, in every degree $k \leq \lfloor A^+/2 \rfloor$, where A^+ is the next breakpoint (and $A^+ = A$ at $A_{\max}$). For $k \leq D$ this holds by Proposition 8.23. For $k > D$ the script `row_sweep.py` computes $[q^k] \mathcal{P}_A$ and $[q^k] \mathcal{N}_A$ from Proposition 8.23 in double precision, with the Gaussian coefficients $G(m, n)$, $m \leq x$, obtained from $G(m, 0) = 1$ by the recurrence $G(m, n+1) = G(m, n) + q^{n+1} G(m-1, n+1)$. Every step adds two nonnegative numbers or multiplies by an integer below 2^{16} . By induction on $m+n$, each computed value is therefore the exact one times $1+\delta$ with $|\delta| \leq (1+u)^{m+n+W} - 1$, where $u = 2^{-53}$ and $W \leq E - D + 12$ counts the further operations in $\mathcal{P}_A$ and $\mathcal{N}_A$; here $|\delta| < 5 \cdot 10^{-13}$. For $x \geq 178$ all values are multiplied by one power 2^{-S} , $S \leq 124$, which is exact and keeps them below 10^{300} ; every nonzero value stays at least 2^{-S} , far above the underflow threshold. The script checks $[q^k] \mathcal{N}_A < 0.999 [q^k] \mathcal{P}_A$, which is far more than the rounding needs. $\square$

TABLE 8. The sweep of Proposition 9.26.

x	Λ_x	Breakpoints	$\max [q^k] \mathcal{N}_A / [q^k] \mathcal{P}_A$	Rounding bound	Largest value
22–40	64	73,131	0.6303 ($x = 23$)	$3.1 \cdot 10^{-13}$	$4.7 \cdot 10^{87}$
41–60	32	60,700	0.6280 ($x = 41$)	$2.3 \cdot 10^{-13}$	$9.6 \cdot 10^{114}$
61–100	16	90,360	0.6270 ($x = 61$)	$2.0 \cdot 10^{-13}$	$4.8 \cdot 10^{163}$
101–200	16	421,900	0.6262 ($x = 101$)	$3.9 \cdot 10^{-13}$	$7.0 \cdot 10^{328}$

Table 8 summarizes the sweep, with the values as the script prints them; the rounding bound is the largest over the block, and the last column is the largest value computed, before scaling. At every x the largest ratio, between 0.6256 and 0.6303, occurs at the band edge $A = 2x(x-1)$, in the degrees $A/2 < k \leq \lfloor A^+/2 \rfloor$ that the interpolation to the next breakpoint needs; these values exceed the bound 0.624 of [Observation 8.24](#) only because they are taken at the band edge and in degrees just above $A/2$, outside the range of that observation (at $x = 22$, for $2x(x-1) < A \leq 2x(x-1) + 4x$ and $k \leq A/2$, the largest ratio is 0.6204). So the inequality has a margin of more than a third in every degree above D , which is why double precision suffices. The sweep takes about 3.3 CPU-hours, almost all of it for $x > 100$.

Theorem 9.27 (large λ for $22 \leq x \leq 200$). *For $22 \leq x \leq 200$ and $A \geq \Lambda_x x^2$, $R(x, A)$ holds.*

Proof. For $101 \leq x \leq 200$ this is part (ii) of the proof of [Theorem 9.24](#), which holds for every $m \geq 99$. Let $22 \leq x \leq 100$. For $k \leq D$ use [Proposition 8.23](#), for $D < k < D + x$ [Lemma 9.19](#) (b), and for $D + x \leq k \leq A/2$ [Lemma 9.19](#) (a). There $s < \pi/\sqrt{6D}$ and $D \geq \Lambda_x x - 1$, so $\sigma = xs < \pi x/\sqrt{6(\Lambda_x x - 1)}$, which on the three blocks of [Proposition 9.26](#) below 101 is at most 1.015, 1.757 and 3.208; regime (P) does not occur. The script `pn_families.py`, run with `--xc` and `--xmax` the ends of a block and `--lam0` Λ_x , checks the regimes (N), (C), (Sm) and (Ss) of the proof of [Theorem 9.24](#) (ii) for every $m \geq x_{\min} - 2$ (zone B of (Ss) for $x_{\min} - 2 \leq m \leq x_{\max}$, which contains every m that occurs), with $\nu \geq \Lambda_x - 2/x_{\min}$ and σ up to that bound. Three changes make them work for m down to 20. In (Sm), pair products are used until the Bernoulli exponent per m is below $-\max(0.05, 4.5/m_c)$, so that the check that every error term decreases in m passes. In (C), the permutations with at least $\lceil m/2 \rceil$ fixed points enter only through an upper bound, for which a Taylor zone of any width will do. In zone B of (Ss), a good residue j contributes at most $\sinh(js/2)/\sin(\pi/8) \leq (j/m)r$, $r = \sinh(\sigma/2)/\sin(\pi/8)$, because $\sinh$ is convex; so the $g_0 = \lceil 7m/16 - 5/8 \rceil$ good residues of [Lemma 9.22](#) contribute at most $r^{g_0} \prod_{j=m-g_0+1}^m j/m$, which is evaluated for each $m \leq x_{\max}$. For the first block the boundary between (Ss) and (Sm) is moved from $\sigma = 1/2$ to $\sigma = 1/4$. As in [Theorem 9.24](#) (ii), the ratio of the regimes that cover σ below the boundary is compared with [Lemma 9.20](#) at $\lambda = \Lambda_x$, and the ratio of all regimes with [Lemma 9.20](#) at $\lambda = \Lambda_x$ and σ at the boundary. $\square$

TABLE 9. The runs of `pn_families.py` for Theorem 9.27.

x	Λ_x	$\sigma <$	Cells N, C, Sm, Ss	Below the boundary	Above it
22–40	64	1.015	256, 64, 30, 2	0.4419/1.4369: 0.279 > 0.046	0.0626/1.7910: 0.031 > $1.4 \cdot 10^{-6}$
41–60	32	1.757	256, 64, 28, 2	0.6518/1.2623: 0.492 > 0.091	0.5505/1.3440: 0.390 > $2.8 \cdot 10^{-6}$
61–100	16	3.208	256, 64, 40, 2	0.7209/1.1851: 0.588 > 0.189	0.4560/1.4353: 0.307 > 0.005

In Table 9, each entry gives $\min p/\max L$ over the regimes concerned, then the right side $(p/L)\sqrt{1-\ell}$ of the criterion against the bound of [Lemma 9.20](#) for $\mathcal{N}_A/\mathcal{P}_A$. The boundary is $\sigma = 1/4$ for the first block and $1/2$ for the others; below it the regimes are (N), (C) and (Ss). The runs take about a minute each.

Theorem 9.28 (the first row for $x \geq 22$). *For every $x \geq 22$ and every $A \geq x^2$, $R(x, A)$ holds. So [Conjecture 8.16](#) holds for $x \geq 22$.*

Proof. For $A \leq 2x(x-1)$ this is [Corollary 8.20](#). Beyond the band it is [Theorem 9.24](#) for $x \geq 201$, and [Proposition 9.26](#) and [Theorem 9.27](#) for $22 \leq x \leq 200$. $\square$

Corollary 9.29 (Bergeron's conjecture; Theorem E). *Let $1 \leq a \leq b \leq c$, and let d be an integer with $ad \leq bc$. Then*

$$\binom{b+c}{b}_q \geq_q \binom{a+d}{a}_q.$$

In particular $\binom{b+c}{b}_q - \binom{a+d}{a}_q$ has nonnegative coefficients whenever $1 \leq a \leq b \leq c \leq d$ and $ad = bc$.

Proof. For $a \leq 20$ this is [Theorem A](#). Let $a \geq 21$. If $a = b$, then $d \leq c$, and $G(a, c) - G(a, d) = \sum_{d < t \leq c} q^t G(a-1, t)$ by Pascal's rule. If $a < b$, every x with $a < x \leq b$ has $x \geq 22$ and $bc \geq x^2$, so $R(x, bc)$ holds by [Theorem 9.28](#). [Proposition 8.15](#) then gives FT at (a, b, c) , and FT implies Troyka's form ([Section 8.7](#)). $\square$

Remark 9.30 (what the proof rests on). The proof of [Theorem E](#) is computer-assisted. Besides the lemmas of Sections 2–9, it uses the following runs, each of which checks every inequality it relies on and reports failure otherwise ([Appendix A](#) lists the commands and the logs): `bergeron_certificate.py` ([Theorem A](#); exact rational arithmetic); `row_check.py` 60 2, `band_check.py` for $3 \leq x \leq 200$ and `rq_certificate.py`, which imports `cww_certificate.py` ([Corollary 8.20](#); exact and ball arithmetic); `pn_certificate.py --xc 201` and `pn_families.py --xc 101` ([Theorem 9.24](#); ball arithmetic); `row_sweep.py` for $22 \leq x \leq 200$ ([Proposition 9.26](#); floating point with the error bound above); and the three runs of `pn_families.py` for [Theorem 9.27](#). Some of these runs were split into pieces that can run in parallel ([Table 10](#)). Together they take about 10 CPU-hours. [Theorem C'](#) uses in addition `corner_check.py` 1000 60 and `cww_certificate.py`. No injection is constructed anywhere. This paper has not been refereed, and a proof of this kind deserves an independent re-implementation of the certificates.

USE OF AI

This paper was written with substantial use of an AI system. The mathematical arguments, the certificate scripts and the text were produced in an extended interactive session with Claude (Anthropic), prompted and directed by the author. The computer-assisted parts of the proofs can be re-run from the ancillary files. Supplementary information on the use of AI is provided with the submission.

APPENDIX A. ANCILLARY FILES

The folder `anc/` contains the ten scripts described below, the log of every run used in the paper, and one data file. The scripts need Python 3 and, depending on the script, `sympy`, `numpy` or `python-flint`, the Python interface to Arb [11]. The runs were made with Python 3.13.17, `python-flint` 0.9.0, `numpy` 2.5.3 and `sympy` 1.14.0, on a virtual machine with two cores and 7 GB of memory. To repeat a run, put the scripts in one directory and run the command of [Table 10](#) there: `rq_certificate.py` imports `cww_certificate.py`, and `pn_certificate.py` and `pn_families.py` import both. Each log begins with the command, the start time, the versions of Python and of the libraries, and the SHA-256 digests of all ten scripts, which are those of the files in `anc/`; it ends with the output of `time` and the finishing time. Every certificate checks each inequality it relies on, counts an inequality checked in ball arithmetic only if it holds on the whole ball, and states its result at the end of its output.

The scripts.

TABLE 10. The runs and their log files (in `anc/`, with the extension `.log`). CPU times are those of the logged runs; `pn_certificate.py` ran with two processes, all other runs with one.

Command	Log	Proves or checks	CPU time
<code>bergeron_certificate.py</code>	<code>bergeron_certificate</code>	Proposition 8.5 , FStep and FT on F_a ; Theorems A and A'	18 min
<code>corner_check.py 1000 60</code>	<code>corner_check_1000_60</code>	Corollary 8.13 for $a \leq 999$; Theorem C' and Corollary 9.14 for $n \leq 1000$; Lemma 8.11 and Proposition 8.12 for $a \leq 60$	7 min
<code>circle_check.py</code>	<code>circle_check</code>	numerical checks for Section 9 and the experiment of Remark 9.8 (not part of the proof)	1 min
<code>cww_certificate.py</code>	<code>cww_certificate</code>	Theorem C' for $n \geq 1001$	10 min
<code>row_check.py 60 2</code>	<code>row_check_60_2</code>	Observation 8.17 for $x \leq 60$, $A \leq 2x^2$; Corollary 8.20 (i) for $x \leq 60$	3 min
<code>band_check.py 175 60</code> , <code>band_check.py 200 60</code> 176	<code>band_check_175_60</code> , <code>band_check_200_60_from176</code>	$(M_{x,N})$ for $x \leq 200$; Corollary 8.20 (i) for $61 \leq x \leq 200$	34 min, 43 min
<code>rq_certificate.py</code> <code>--parts PF</code>	<code>rq_certificate_PF</code>	Theorem 9.16 (ii): regimes P and F	23 min
<code>rq_certificate.py</code> <code>--parts L --lsplit i/4</code> , $i = 0, 1, 2, 3$	<code>rq_certificate_Li of 4</code>	Theorem 9.16 (ii): the exact low degrees; with the previous run, Corollary 8.20 (i) for $x \geq 201$	43–47 min each
<code>pn_certificate.py --xc 201 --jobs 2</code>	<code>pn_certificate_xc201</code>	Theorem 9.24 , part (i); writes <code>pn_table_201.json</code>	1.7 h
<code>pn_families.py --xc 101</code>	<code>pn_families_xc101</code>	Theorem 9.24 , part (ii), for $x \geq 101$	1 min
<code>pn_families.py --xc 61</code> <code>--xmax 100 --lam0 16</code>	<code>pn_families_xc61_xmax100</code>	Theorem 9.27 for $61 \leq x \leq 100$	1 min
<code>pn_families.py --xc 41</code> <code>--xmax 60 --lam0 32</code>	<code>pn_families_xc41_xmax60</code>	Theorem 9.27 for $41 \leq x \leq 60$	1 min
<code>pn_families.py --xc 22</code> <code>--xmax 40 --lam0 64</code> <code>--sig-ss 0.25</code>	<code>pn_families_xc22_xmax40</code>	Theorem 9.27 for $22 \leq x \leq 40$	1 min
<code>row_sweep.py 22-40:64</code> <code>41-60:32 61-105:16</code>	<code>row_sweep_22_105</code>	Proposition 9.26 for $22 \leq x \leq 105$	13 min
<code>row_sweep.py 106-150:16</code> , <code>151-175:16</code> , <code>176-190:16</code> , <code>191-200:16</code>	<code>row_sweep_106_150</code> , <code>row_sweep_151_175</code> , <code>row_sweep_176_190</code> , <code>row_sweep_191_200</code>	Proposition 9.26 for $106 \leq x \leq 200$	35, 53, 51, 43 min

`bergeron_certificate.py`: Python 3 with sympy. Proves the cell inequalities of [Proposition 8.5](#) for $3 \leq a \leq 20$ and verifies FStep and FT (hence T) on the finite sets F_a , which proves Theorems A and A'. The option `--amax A` stops earlier, `--selftest` compares every bound of [Proposition 8.4](#) with exact coefficients on random small cases, and `--brute N` adds an independent check of all $bc \leq N$. The thresholds $C_a(b)$ and B_1 are stored in the script; they were found by bisection with the same code.

`corner_check.py N A`: Pure Python 3. Computes $C_n(q)$ for $n \leq N$. It checks $c_k \geq c_{k-2}$ and $c_k \geq c_{k-3}$ up to degree $(n^2 - 2n)/2$ ([Corollary 8.13](#)), Stanley's theorem, the monotonicity of C_n on $[2, \mu]$ for $16 \leq n \leq N$ ([Theorem C'](#) for these n) and the unimodality of $(1+q)C_n$

(Corollary 9.14), and it records where C_n fails to be monotone. For $a \leq A$ it checks Lemma 8.11 and Proposition 8.12 against independently computed Gaussian coefficients.

circle_check.py **N**: Python 3 with numpy. Checks the two numerical constants in the proof of Theorem C, Lemma 9.5 against exact values of $\binom{2n}{n}_q$ and $\binom{2n}{n-1}_q$ for $n = 40$, Lemmas 9.3 and 9.4 on segments for $n = 60$, and Lemma 9.2 for $n = 12$. It then runs the floating-point experiment of Remark 9.8 at $n = N$ (default 1001). These are checks and experiments, not parts of the proof.

cww_certificate.py: Python 3 with python-flint. Proves Theorem C' for $n \geq 1001$, in the four parts described in Section 9.4: the constants A_r , the exact low degrees $k \leq 10^4$, regime P (one cell) and regime F (179 cells). The option `--parts` selects parts and `--jobs` sets the number of processes.

row_check.py **X M**: Pure Python 3. Checks $R(x, A)$ of Proposition 8.15 for $2 \leq x \leq X$ and all integers $x^2 \leq A \leq Mx^2$, in every degree $k \leq A/2$, in exact integer arithmetic, and reports the smallest relative margin.

band_check.py **X XI XS**: Python 3 with python-flint. Checks, for $XS \leq x \leq X$ (default $XS = 3$), the hypothesis $(M_{x,N})$ of Proposition 8.19 for $x \leq N \leq 2x - 2$ in exact arithmetic, and the signs of Y and of the small- m sums at every breakpoint of the band. For $x \leq XI$ it also checks the identities $\mathcal{D}_A = (1 - q) \Gamma_{x,N} Y$ modulo q^{K_A+1} and that $\mathcal{D}_A \geq 0$ in degrees $\leq K_A$.

rq_certificate.py: Python 3 with python-flint; needs **cww_certificate.py**. Proves Theorem 9.16 (ii): the exact low degrees, regime P and the regime-F cells in (σ, ν) described in Section 9.5. The option `--parts` selects parts, `--lsplit i/n` runs one of n pieces of the exact part, and `--yrange a:b` restricts a piece to $a \leq y < b$ (`--yrange single` runs only the one-factor criterion for $y \geq 2498$), so that the exact part can be spread over processes or run in chunks.

pn_certificate.py: Python 3 with python-flint; needs **cww_certificate.py** and **rq_certificate.py**. With `--xc 201` (the default) it proves part (i) of the proof of Theorem 9.24: the local limit table for Gaussian coefficients $\binom{m+n}{m}_q$ in the variables (σ, ν) , valid for every $m \geq 199$ (part T, saved as **pn_table_201.json**); the criterion of Lemma 9.19 on boxes, with the summed evaluation bound of Lemma 9.20 (part C); and regime P (part P). The options `--parts CP --table pn_table_201.json` rerun parts C and P from the saved table in about a minute, and `--test sa,sb,na,nb` prints one table cell.

pn_families.py: Python 3 with python-flint; needs **cww_certificate.py**, **rq_certificate.py** and **pn_certificate.py**. With `--xc 101` it proves part (ii) of the proof of Theorem 9.24: it checks the five regimes (N), (C), (Sm), (Ss) and (P) for every $m \geq 99$, and compares the resulting ratios with Lemma 9.20 at $\lambda = 16$. With `--xmax` it treats only $x \leq \text{XMAX}$, where Lemma 9.19 (b) bounds σ ; `--lam0` sets the lower end of λ and `--sig-ss` the boundary between (Ss) and (Sm). The three runs with `--xmax` prove Theorem 9.27 for $x \leq 100$.

row_sweep.py: Python 3 with numpy. Proves Proposition 9.26: for each x it runs the positive recurrence for $G(m, n)$, $m \leq x$, in double precision (scaled by a power of 2 for large x) and checks $[q^k] \mathcal{N}_A < 0.999 [q^k] \mathcal{P}_A$ at every breakpoint, in the degrees that the affine interpolation needs, and it reports the largest ratio and the bound on the relative rounding error. The arguments are ranges **X1-X2:LAM**; **python3 row_sweep.py X1 X2 LAM** runs one range.

The data file `pn_table_201.json` is the table of part T of `pn_certificate.py`, written by the logged run; it lists, for every leaf cell (σ, ν) , the certified bounds used by part C.

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