

Sun's supercongruence for an Apéry-like sequence

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Abstract

Let $a_n = \sum_{j=0}^n \binom{n}{j}^2 \binom{2j}{j}$. We prove that, for every prime $p > 3$, the sum

$$(-1)^{(p-1)/2} \sum_{n=0}^{p-1} \binom{2n}{n} a_n / 54^n$$

is congruent modulo p^2 to $4x^2 - 2p$ when $p = x^2 + 3y^2 \equiv 1 \pmod{3}$, and to zero when $p \equiv 2 \pmod{3}$. For the latter primes a quadratic differential invariant of the truncated generating polynomial lifts Sun's congruence modulo p . For the former primes we certify two modular periods directly by identities of weight eight, construct a Hecke polynomial with an explicit degree bound, and specialize a finite polynomial identity at a complex multiplication point. The coefficient at degree p and both parity cases of the complex multiplication matrices are treated explicitly. A corresponding evaluation of the Franel sum with denominator 50 follows by the same arguments.

1 The supercongruence and its two branches

For $n \geq 0$ define

$$a_n = \sum_{j=0}^n \binom{n}{j}^2 \binom{2j}{j}, \quad f_n = \sum_{j=0}^n \binom{n}{j}^3. \quad (1)$$

The integers f_n are the Franel numbers. Write

$$G(t) = \sum_{n \geq 0} \binom{2n}{n} a_n t^n, \quad F(t) = \sum_{n \geq 0} \binom{2n}{n} f_n t^n. \quad (2)$$

For a series $Y(t) = \sum_{n \geq 0} y_n t^n$, write $Y_p(t) = \sum_{n=0}^{p-1} y_n t^n$. Thus all truncations in this article end at $p-1$. The sum $G_p(1/54)$ is p -integral for $p > 3$; the sum $F_p(1/50)$ is used only for $p > 5$. Congruences between p -integral rational numbers are interpreted in $\mathbb{Z}_p$.

Theorem 1.1. *For every prime $p > 3$,*

$$(-1)^{(p-1)/2} G_p(1/54) \equiv \begin{cases} 4x^2 - 2p, & p = x^2 + 3y^2 \equiv 1 \pmod{3}, \\ 0, & p \equiv 2 \pmod{3} \end{cases} \pmod{p^2}. \quad (3)$$

This is Conjecture 3.5 of Zhi-Hong Sun [1, p. 2742]. Every prime $p \equiv 1 \pmod{3}$ admits the indicated representation; the signs of x, y do not affect the expression $4x^2 - 2p$. Theorem 3.5 of the same paper, on p. 2739, concerns the distinct congruence modulo p . Its vanishing statement and proof give the input

$$G_p(1/54) \equiv F_p(1/50) \equiv 0 \pmod{p} \quad (p > 5, p \equiv 2 \pmod{3}). \quad (4)$$

Only this modulo- p result is used in the inert branch below.

The proof separates primes inert and split in $\mathbb{Q}(\sqrt{-3})$. Section 2 proves a finite differential invariant and uses it to lift (4). Sections 3 and 4 identify modular periods, their divisors and

an exact complex multiplication value. Section 5 constructs a Hecke polynomial and proves a polynomial identity modulo p^2 . Section 6 computes a unit root in both split parity cases and completes the proof. The modular polynomial construction follows the method of Beukers [3, Lemma 3.7 and Section 5]; all ingredients needed here at modulus p^2 are proved explicitly. The directly certified periods also identify the Franel parametrization recorded in [3, Section 9.15].

2 A finite differential invariant and the inert primes

We first work with a general integral sequence u_n , parameters $a, b, c \in \mathbb{Z}$, and recurrence

$$u_0 = 1, \quad u_1 = b, \quad (n+1)^2 u_{n+1} = (an(n+1) + b)u_n - cn^2 u_{n-1} \quad (n \geq 1). \quad (5)$$

The sequences a_n and f_n satisfy this recurrence with parameters $(10, 3, 9)$ and $(7, 2, -8)$ respectively [1, pp. 2729–2730]. Exact telescoping proofs are provided in Appendix B. Let $v_n = \binom{2n}{n} u_n$, $V(t) = \sum_{n \geq 0} v_n t^n$ and $\theta = t d/dt$. The ratios of successive central binomial coefficients give

$$n^3 v_n = 2(2n-1)(an(n-1) + b)v_{n-1} - 4c(n-1)(2n-3)(2n-1)v_{n-2} \quad (n \geq 1), \quad (6)$$

where $v_{-1} = 0$. Hence $LV = 0$, with

$$L = \theta^3 - 2t(2\theta + 1)(a\theta^2 + a\theta + b) + 4ct^2(\theta + 1)(2\theta + 1)(2\theta + 3). \quad (7)$$

Put $D(t) = 1 - 4at + 16ct^2$. With coefficients to the left of the operators,

$$\begin{aligned} L &= D\theta^3 + (-6at + 48ct^2)\theta^2 \\ &\quad + (-2(a+2b)t + 44ct^2)\theta - 2bt + 12ct^2. \end{aligned} \quad (8)$$

Its leading coefficient in ordinary derivatives is $t^3 D(t)$.

For any polynomial or formal series Y define

$$I(Y) = D(2Y\theta^2 Y - (\theta Y)^2) + (\theta D)Y\theta Y + 4t(-b + 3ct)Y^2. \quad (9)$$

Lemma 2.1. *The identity*

$$\theta I(Y) = 2YLY \quad (10)$$

holds for every Y . In particular $I(V) = 0$ in $\mathbb{Q}[[t]]$.

Proof. Set $E = 4t(-b + 3ct)$. Applying the product rule to (9) gives

$$2DY\theta^3 Y + 3(\theta D)Y\theta^2 Y + (\theta^2 D + 2E)Y\theta Y + (\theta E)Y^2.$$

This is $2YLY$ by (8). Since $LV = 0$, the series $I(V)$ is constant in characteristic zero. Its constant term is zero, which proves the last assertion. $\square$

Proposition 2.2. *Suppose all u_n in (5) are integers. For any prime $p > 3$, the polynomial $U(t) = \sum_{n=0}^{p-1} v_n t^n$ satisfies*

$$I(U) \equiv 0 \pmod{p^2} \quad \text{in } \mathbb{Z}[t]. \quad (11)$$

Proof. The exact residual of the truncation, computed from (6), is

$$LU = -p^3 v_p t^p + 4cp(2p-1)(2p+1)v_{p-1}t^{p+1}. \quad (12)$$

All coefficients below degree p vanish, the omitted v_p supplies the first term, and the highest possible remaining degree is $p+1$. Since $p \mid \binom{2p-2}{p-1}$ and u_{p-1} is integral, $p \mid v_{p-1}$. Thus $LU \equiv 0 \pmod{p^2}$ and $\theta I(U) \equiv 0 \pmod{p^2}$.

The latter conclusion alone cannot eliminate coefficients whose degrees are divisible by p . Here $\deg I(U) \leq 2p$ and $[t^0]I(U) = 0$. All positive degrees other than $p, 2p$ are units modulo p^2 , so it remains to calculate precisely those two coefficients over $\mathbb{Z}$. Since $V - U = v_p t^p + O(t^{p+1})$ and $I(V) = 0$, comparison at degree p gives

$$[t^p]I(U) = -2p^2 v_p. \quad (13)$$

Indeed the only contribution of the omitted term at this degree pairs the constant $V(0) = 1$ with $\theta^2(v_p t^p)$ in $2V\theta^2 V$; the other terms contain an extra t or a derivative of the constant. At degree $2p$, only the highest term of each factor U contributes. The resulting scalar is

$$16c(p-1)^2 + 32c(p-1) + 12c = 4c(2p-1)(2p+1),$$

so

$$[t^{2p}]I(U) = 4c(2p-1)(2p+1)v_{p-1}^2. \quad (14)$$

Both coefficients are divisible by p^2 . This treats every degree. $\square$

Theorem 2.3 (Lifting at a nonsingular parameter). *Under the hypotheses of Proposition 2.2, let $t_0 \in \mathbb{Z}_p$ and suppose $t_0 D(t_0) \in \mathbb{Z}_p^\times$. Then*

$$U(t_0) \equiv 0 \pmod{p} \implies U(t_0) \equiv 0 \pmod{p^2}. \quad (15)$$

The conclusion holds for every p -integral lift of the same residue class.

Proof. Let $\bar{U}$ be the reduction in $\mathbb{F}_p[t]$. It is nonzero, since $U(0) = 1$, and has degree less than p . At a zero $\bar{t}_0$ of $\bar{U}$, the reduction of (11) gives $-D(\bar{t}_0)(\theta\bar{U}(\bar{t}_0))^2 = 0$. Thus $\theta\bar{U}(\bar{t}_0) = 0$, and $\bar{t}_0 \neq 0$ implies that the zero has multiplicity at least two.

It has multiplicity exactly two. Otherwise write $\bar{U}(t) = w(t - \bar{t}_0)^m + O((t - \bar{t}_0)^{m+1})$ with $w \neq 0$ and $3 \leq m \leq p-1$. Equation (12) implies $\bar{L}\bar{U} = 0$. The coefficient of local degree $m-3$ in this equation is

$$\bar{t}_0^3 D(\bar{t}_0) m(m-1)(m-2)w \neq 0.$$

Lower derivatives have order at least $m-2$ and cannot cancel it. This contradiction proves that $\theta^2 U(t_0)$ is a unit, since its reduction is $\bar{t}_0^2 \bar{U}''(\bar{t}_0) \neq 0$.

Now $U(t_0)$ and $\theta U(t_0)$ are both divisible by p . Evaluating the finite identity (11) at t_0 therefore leaves

$$2D(t_0)\theta^2 U(t_0)U(t_0) \equiv 0 \pmod{p^2}.$$

The coefficient of $U(t_0)$ is a unit, giving (15). All nonvanishing conditions depended only on the residue class of t_0 . $\square$

For a_n and f_n the discriminants specialize to

$$\begin{aligned} D_a(t) &= (1-4t)(1-36t), & D_a(1/54) &= 25/81, \\ D_f(t) &= (1+4t)(1-32t), & D_f(1/50) &= 243/625. \end{aligned} \quad (16)$$

The evaluation points $1/54$ and $1/50$ and the displayed discriminant values are units for $p > 5$. Equations (4) and (15) give

$$G_p(1/54) \equiv F_p(1/50) \equiv 0 \pmod{p^2} \quad (p > 5, p \equiv 2 \pmod{3}). \quad (17)$$

For $p = 5$ the first discriminant is not a unit, so we calculate directly. The initial values $a_0, \dots, a_4$ are 1, 3, 15, 93, 639. The five summands of $G_5(1/54)$ are respectively 1, 14, 15, 15, 5 modulo 25; their sum is $50 \equiv 0 \pmod{25}$. This completes the inert branch of Theorem 1.1.

3 Direct certification of the two modular periods

Let $q = e^{2\pi i\tau}$, $\tau \in \mathfrak{H}$, and let $\eta_d = \eta(d\tau)$, where $\eta(\tau) = q^{1/24} \prod_{m \geq 1} (1 - q^m)$. Define

$$\mathcal{A} = \frac{(\eta_2 \eta_3)^7}{(\eta_1 \eta_6)^5}, \quad \mathcal{B} = \frac{(\eta_1 \eta_6)^7}{(\eta_2 \eta_3)^5}, \quad J = \mathcal{A}\mathcal{B} = (\eta_1 \eta_2 \eta_3 \eta_6)^2, \quad (18)$$

and

$$\Theta_+ = \mathcal{A} + \mathcal{B}, \quad \Theta_- = \mathcal{A} - \mathcal{B}, \quad t_+ = J/\Theta_+^2, \quad t_- = J/\Theta_-^2. \quad (19)$$

These products give $\Theta_{\pm} \in 1 + q\mathbb{Z}[[q]]$ and $t_{\pm} \in q + q^2\mathbb{Z}[[q]]$. The eta criterion and cusp-order formula [4, p. 2] show that $\mathcal{A}, \mathcal{B}$ are holomorphic forms of weight two with trivial character on $\Gamma_0(6)$, and J is a form of weight four. At the four cusps, represented by denominators 1, 2, 3, 6, the respective orders are

$$\text{ord } \mathcal{A} = (0, 1, 1, 0), \quad \text{ord } \mathcal{B} = (1, 0, 0, 1), \quad \text{ord } J = (1, 1, 1, 1). \quad (20)$$

The underlying eta exponent data are in Appendix A. Eta has no zeros in $\mathfrak{H}$. In particular J has no interior zeros, and each $\Theta_{\pm}$ has a nonzero constant term at every cusp: precisely one of $\mathcal{A}, \mathcal{B}$ has order zero there.

Distinguish $\partial = q d/dq = (2\pi i)^{-1} d/d\tau$ from $\theta = t d/dt$. We simultaneously use the two rows

$$(a, b, c, \Theta) = (10, 3, 9, \Theta_+), \quad (a, b, c, \Theta) = (7, 2, -8, \Theta_-). \quad (21)$$

Put $\ell = \partial J/J$ and $W = \Theta\ell - 2\partial\Theta$.

Proposition 3.1. *For each row of (21),*

$$W^2 - \Theta^4 + 4aJ\Theta^2 - 16cJ^2 = 0, \quad (22)$$

$$2\Theta\partial^2\Theta - 3(\partial\Theta)^2 - 4bJ\Theta^2 + 12cJ^2 = 0. \quad (23)$$

Proof. First we verify the space containing the two residuals. For $\gamma = \begin{pmatrix} \alpha & \beta_0 \\ \gamma_0 & \delta_0 \end{pmatrix} \in \Gamma_0(6)$, write $j = \gamma_0\tau + \delta_0$ and $\lambda = \gamma_0/(2\pi i)$. Differentiating the weight-two and weight-four transformation laws yields

$$\begin{aligned} \Theta(\gamma\tau) &= j^2\Theta, \\ (\partial\Theta)(\gamma\tau) &= j^4\partial\Theta + 2\lambda j^3\Theta, \\ (\partial^2\Theta)(\gamma\tau) &= j^6\partial^2\Theta + 6\lambda j^5\partial\Theta + 6\lambda^2 j^4\Theta, \\ \ell(\gamma\tau) &= j^2\ell + 4\lambda j. \end{aligned}$$

Consequently W transforms with weight four, and $2\Theta\partial^2\Theta - 3(\partial\Theta)^2$ transforms with weight eight. Every other term in (22) and (23) has weight eight. There are no interior poles, since the only denominator at this stage is J .

At a cusp choose a scaling matrix σ , and put $y = \Theta|_2\sigma$, $j_0 = J|_4\sigma$. The same differentiation laws give

$$W|_4\sigma = y \frac{\partial j_0}{j_0} - 2\partial y, \quad (2\Theta\partial^2\Theta - 3(\partial\Theta)^2)|_8\sigma = 2y\partial^2 y - 3(\partial y)^2.$$

If the cusp has width w and local parameter $Q = e^{2\pi i\tau/w}$, then $j_0 = Q^r u(Q)$, $u(0) \neq 0$, and $\partial j_0/j_0 = (r + Qu'/u)/w$ has no negative powers. Both y and its derivatives are holomorphic there. Thus the two residuals belong to $M_8(\Gamma_0(6))$.

Their coefficients of $q^0, \dots, q^8$ are zero by the exact product certificate in Appendix A. The index of $\Gamma_0(6)$ in $\text{SL}_2(\mathbb{Z})$ is $6(1 + 1/2)(1 + 1/3) = 12$. The classical valence bound is therefore

$8 \cdot 12/12 = 8$. Indeed, for a nonzero holomorphic form of weight k on $\Gamma_0(6)$, the product of its weight- k slashes over the twelve cosets is a nonzero holomorphic level-one form of weight $12k$. Its order at infinity is at least that of the original form, since all other factors have nonnegative orders. The level-one valence formula bounds that order by $12k/12 = k$. Applying this with $k = 8$, neither residual can have order at least nine unless it is zero. Both residuals vanish identically. $\square$

Theorem 3.2. *The normalized periods satisfy*

$$G(t_+(q)) = \Theta_+(q), \quad F(t_-(q)) = \Theta_-(q). \quad (24)$$

These are identities of formal series and of their analytic germs at $q = 0$.

Proof. Use one row of (21) and $t = J/\Theta^2$. All formal substitutions are invertible because $t = q + O(q^2)$. Set

$$e = \frac{\partial t}{t} = \ell - 2 \frac{\partial \Theta}{\Theta} = \frac{W}{\Theta}.$$

Equation (22) says $e^2 = D\Theta^2$, where $D = 1 - 4at + 16ct^2$. The series e, Θ, D all have constant term one. Since $\partial = e\theta$, differentiation gives

$$2\partial e = (\theta D)\Theta^2 + 2D\Theta\theta\Theta.$$

Substitute $\partial\Theta = e\theta\Theta$ and $\partial^2\Theta = e^2\theta^2\Theta + (\partial e)\theta\Theta$ in (23). Its left side becomes $\Theta^2 I(\Theta)$, with I as in (9). Hence $I(\Theta) = 0$; Lemma 2.1 and the fact that Θ is a formal unit give $L\Theta = 0$.

For each $n \geq 1$ the new coefficient of a solution to $LY = 0$ is multiplied by n^3 , as is explicit in (6). There is a unique solution in $\mathbb{Q}[[t]]$ with constant term one. The binomially defined series G, F satisfy this equation for the respective rows and have that constant term. This proves (24). The eta products and the inverse parameter are analytic near $q = 0$, so the same equalities hold as analytic germs. $\square$

4 The extended group, divisors and the CM parameter

For a real matrix $M = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}$ with positive determinant, use the weight-two slash convention

$$(Y|_2 M)(\tau) = \det(M)(c_0\tau + d_0)^{-2}Y(M\tau). \quad (25)$$

In particular, if $Y|_2 M = \chi(M)Y$ then

$$Y(M\tau) = \frac{\chi(M)}{\det M}(c_0\tau + d_0)^2 Y(\tau). \quad (26)$$

Let Γ be the projective group generated by $\Gamma_0(6)$ and

$$W_3 = \begin{pmatrix} 3 & 1 \\ 6 & 3 \end{pmatrix}, \quad W_6 = \begin{pmatrix} 0 & -1 \\ 6 & 0 \end{pmatrix}. \quad (27)$$

Lemma 4.1. *The forms $\Theta_{\pm}$ are modular of weight two on the extended group Γ , with characters trivial on their restriction to $\Gamma_0(6)$. The quotient $\Gamma/\Gamma_0(6)$ has order four projectively. If W_2 denotes the class of W_6W_3 , the characters are*

e	1	2	3	6
$\chi_+(W_e)$	1	-1	1	-1
$\chi_-(W_e)$	1	1	-1	-1.

(28)

Proof. The transformation law for eta [5, 23.18.5–23.18.7], for $M = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $c_0 > 0$, can be written as

$$\eta\left(\frac{a_0\tau + b_0}{c_0\tau + d_0}\right) = \exp\left(\pi i \left(\frac{a_0 + d_0}{12c_0} - s(d_0, c_0) - \frac{1}{4}\right)\right) (c_0\tau + d_0)^{1/2} \eta(\tau),$$

with the principal square root and Dedekind sum s . For $\tau' = W_3\tau$ and $L_0 = 6\tau + 3$, the transformations of $\eta(\tau'), \eta(2\tau'), \eta(3\tau'), \eta(6\tau')$ arise from

$$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

acting respectively on $3\tau, 6\tau, \tau, 2\tau$. Their phases, in units of πi , are $-1/12, 1/12, -1/12, 1/12$; the Dedekind sums here are zero. The phases cancel in each pair, and the linear factors are positive real multiples of L_0 , so the principal roots multiply without an extra sign. Thus

$$(\eta_2\eta_3)(\tau') = \frac{L_0}{\sqrt{3}}(\eta_1\eta_6)(\tau), \quad (\eta_1\eta_6)(\tau') = \frac{L_0}{\sqrt{3}}(\eta_2\eta_3)(\tau).$$

It follows that $\mathcal{A}(\tau') = L_0^2\mathcal{B}(\tau)/3$ and $\mathcal{B}(\tau') = L_0^2\mathcal{A}(\tau)/3$. Hence $\mathcal{A}|_2W_3 = \mathcal{B}$ and $\mathcal{B}|_2W_3 = \mathcal{A}$. For W_6 the formula $\eta(-1/z) = \sqrt{-iz}\eta(z)$ multiplies each eta pair by $\sqrt{6}(-i\tau)$. Both $\mathcal{A}, \mathcal{B}$ acquire the factor $-6\tau^2$, giving $\mathcal{A}|_2W_6 = -\mathcal{A}$ and $\mathcal{B}|_2W_6 = -\mathcal{B}$.

The generators normalize the base group. Indeed, for $\gamma = \begin{pmatrix} a_0 & b_0 \\ 6c_0 & d_0 \end{pmatrix}$,

$$\begin{aligned} W_6\gamma W_6^{-1} &= \begin{pmatrix} d_0 & -c_0 \\ -6b_0 & a_0 \end{pmatrix}, \\ W_3\gamma W_3^{-1} &= \begin{pmatrix} 3a_0 + 6c_0 - 6b_0 - 2d_0 & -a_0 - 2c_0 + 3b_0 + d_0 \\ 6a_0 + 18c_0 - 12b_0 - 6d_0 & -2a_0 - 6c_0 + 6b_0 + 3d_0 \end{pmatrix}. \end{aligned} \quad (29)$$

Moreover $W_3^2 = 3 \begin{pmatrix} 5 & 2 \\ 12 & 5 \end{pmatrix}$, $W_6^2 = -6I$ and $(W_3W_6)^2 = -18I$. These relations give at most four classes modulo the base group and scalars. They are distinct because their determinant square classes are $1, 3, 6, 2$ in $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$. The transformations just proved yield (28). $\square$

Introduce weight-zero modular functions

$$h = \frac{\eta_2\eta_6^5}{\eta_1^5\eta_3}, \quad h_3 = \frac{\eta_3^{12}}{\eta_1^{12}}, \quad R = \mathcal{A}/\mathcal{B}. \quad (30)$$

Their characters on $\Gamma_0(6)$ are trivial by the eta criterion. The cusp orders of h are $(-1, 0, 0, 1)$, and it has no interior poles. It consequently has one simple pole on the compact curve $X_0(6)$. The resulting map to $\mathbb{P}^1$ has degree one, so h is a coordinate and the function field is $\mathbb{C}(h)$.

Lemma 4.2. *The following identities hold:*

$$h_3(1 + 9h) = h(1 + 8h)^2, \quad (31)$$

$$Rh = (1 + 8h)(1 + 9h) = 1 + 17h + 72h^2. \quad (32)$$

Proof. Let E_1, E_2 be left side minus right side. Exact product expansion gives $E_1 = O(q^9)$ and $E_2 = O(q^{11})$; the coefficient data are in Appendix A. Set

$$M_1 = \eta_1^{17}\eta_3^3, \quad M_2 = \eta_1^{17}\eta_3^2\eta_6^7.$$

Each $M_i E_i$ is a sum of eta monomials with nonnegative exponents, listed in the appendix. Thus it is holomorphic in $\mathfrak{H}$ and at all cusps. Since E_i is invariant, the whole sum has the multiplier of M_i . Eta multipliers have order dividing 24; hence $(M_1 E_1)^{24} \in M_{240}(\Gamma_0(6))$ and $(M_2 E_2)^{24} \in M_{312}(\Gamma_0(6))$ with trivial character. Their orders at infinity are at least

$$24(13/12 + 9) = 242 > 240, \quad 24(65/24 + 11) = 329 > 312.$$

The same index-12 valence bound forces both forms to vanish. Since M_i is nonzero in $\mathfrak{H}$, this proves $E_i = 0$. $\square$

Proposition 4.3. *Each $t_{\pm}$ is a Hauptmodul for Γ . Each $\Theta_{\pm}$ has no cusp zeros, and its interior zeros are exactly the simple zeros above the double poles of $t_{\pm}$ on $X_0(6)$.*

Proof. Equation (32) gives

$$\begin{aligned} t_+ &= \frac{R}{(1+R)^2} = \frac{h(1+8h)(1+9h)}{(1+6h)^2(1+12h)^2}, \\ t_- &= \frac{R}{(R-1)^2} = \frac{h(1+8h)(1+9h)}{(1+16h+72h^2)^2} = \frac{t_+}{1-4t_+}. \end{aligned} \quad (33)$$

The numerator and denominator in each expression are coprime. The quadratic $1+16h+72h^2$ has discriminant $-32 \neq 0$. Thus each rational map has degree four and two double poles on $X_0(6)$. The transformations of $\mathcal{A}, \mathcal{B}$ show that $t_{\pm}$ are Γ -invariant. Since the quotient map has degree four by Lemma 4.1, each $t_{\pm}$ has degree one on the quotient curve and is a Hauptmodul.

There are no elliptic points for $\Gamma_0(6)$ modulo scalars. An elliptic matrix of determinant one would have trace 0 or ± 1 . Trace zero implies $a_0^2 \equiv -1 \pmod{3}$, impossible; odd trace is impossible modulo two, since $a_0 d_0 \equiv 1 \pmod{2}$. Interior local coordinates on $X_0(6)$ are therefore ordinary coordinates. At cusps $\Theta_{\pm}$ are nonzero by (20). The identity $t_{\pm} \Theta_{\pm}^2 = J$ and the absence of zeros of J in $\mathfrak{H}$ imply that above each double pole the corresponding form has a simple zero, and that it has no other interior zeros. $\square$

Lemma 4.4. *Let $\delta = \sqrt{-3}$ have positive imaginary part and let $\beta = \delta/3 = i/\sqrt{3}$. Then*

$$h(\beta) = \frac{\sqrt{3}-1}{24}, \quad t_+(\beta) = 1/54, \quad t_-(\beta) = 1/50. \quad (34)$$

Both $\Theta_+(\beta)$ and $\Theta_-(\beta)$ are nonzero.

Proof. Since $3\beta = -1/\beta$, inversion of eta gives $h_3(\beta) = (-i\beta)^6 = 1/27$. Equation (31) becomes

$$27h(1+8h)^2 - (1+9h) = (6h+1)(288h^2 + 24h - 1) = 0.$$

Eta products are positive on the positive imaginary axis, so $h(\beta) > 0$. This selects the positive root $(\sqrt{3}-1)/24$. The polynomial identity

$$54h(1+17h+72h^2) - (1+18h+72h^2)^2 = -(18h^2 - 6h - 1)(288h^2 + 24h - 1)$$

and (33) give $t_+(\beta) = 1/54$; its denominator is nonzero at positive h . The last expression in (33) yields $t_-(\beta) = 1/50$. Finally $J(\beta) \neq 0$ and the two parameters are finite, so neither form vanishes at β . $\square$

5 The Hecke polynomial and finite specialization

In this section use either pair $(t, \Theta) = (t_+, \Theta_+)$ or (t_-, Θ_-) , and let $Y = G$ or F respectively. In particular, Θ is a weight-two modular form on the extended group Γ with character trivial on $\Gamma_0(6)$, as proved in Lemma 4.1. This extended-group hypothesis is used in the Hecke orbit construction [3, Lemma 3.7].

We spell out that orbit at level six. For $p > 3$ set

$$\mathcal{M}(6, p) = \left\{ \begin{pmatrix} a_0 & b_0 \\ 6c_0 & d_0 \end{pmatrix} : a_0, b_0, c_0, d_0 \in \mathbb{Z}, a_0 d_0 - 6b_0 c_0 = p \right\}.$$

The gcd of the first column is 1 or p . A Bezout identity takes a primitive first column to $(1, 0)^T$ by left multiplication with $\Gamma_0(6)$. If the gcd is p , first divide the column by p ; its lower entry is still divisible by six. The $p + 1$ distinct left classes therefore have representatives

$$\gamma_\infty = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} 1 & k \\ 0 & p \end{pmatrix} \quad (0 \leq k < p). \quad (35)$$

For clarity, when the first-column gcd is g , the lower row of the reducing matrix can be chosen as $(-6c_0/g, a_0/g)$, and a Bezout upper row makes its determinant one. Its lower left entry is divisible by six because p is coprime to six. The resulting upper triangular matrix is $\begin{pmatrix} 1 & k \\ 0 & p \end{pmatrix}$ for

$g = 1$, with k reduced modulo p by a translation; it is $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ for $g = p$, after eliminating its upper right entry. These representatives are distinct by multiplying the triangular inverses.

The conjugation formulas (29) preserve $\mathcal{M}(6, p)$; they do not require determinant one. Thus W_3, W_6 permute its left classes. Put $Q_\gamma = (\Theta|_2 \gamma)/\Theta$. These quotients are independent of the representative because the base character is trivial. For $w \in \Gamma$ and $\gamma' = w^{-1} \gamma w$,

$$Q_\gamma(w\tau) = \frac{\Theta|_2 \gamma w}{\Theta|_2 w} = \frac{\Theta|_2 w \gamma'}{\chi(w)\Theta} = Q_{\gamma'}(\tau).$$

The character cancels, even though it need not be trivial on Γ .

Lemma 5.1. *For every prime $p > 3$ there is a polynomial $\mathcal{T}_p(X, t) = \sum_{m=0}^{p+1} C_m(t) X^m \in \mathbb{Z}[X, t]$ such that*

$$\mathcal{T}_p(X, t(\tau)) = \left(1 - \frac{\Theta(p\tau)}{\Theta(\tau)} X\right) \prod_{k=0}^{p-1} \left(p^2 - \frac{\Theta((\tau+k)/p)}{\Theta(\tau)} X\right). \quad (36)$$

Its coefficients satisfy

$$\deg C_m \leq \lfloor m/2 \rfloor, \quad \mathcal{T}_p(X, t) \equiv C_p(t) X^p + C_{p+1}(t) X^{p+1} \pmod{p^2}, \quad C_{p+1}(t) \equiv 1 \pmod{p}. \quad (37)$$

Proof. The product in (36) equals $p^{2p} \prod_{[\gamma]} (1 - Q_\gamma X/p)$ using (35). Hence its coefficients are Γ -invariant. The numerator of each Q_γ is holomorphic in $\mathfrak{H}$. There are also no poles at cusps: at a cusp represented by σ , the quotient is

$$\frac{\Theta|_2 \gamma \sigma}{\Theta|_2 \sigma}.$$

The denominator has nonzero constant term. Write $\gamma \sigma = \rho u$, where $\rho \in \mathrm{SL}_2(\mathbb{Z})$ and u is upper triangular with positive determinant. The numerator is an expansion of the holomorphic form at a cusp, with the cusp coordinate rescaled by a positive power. It has no negative exponents.

Thus the only possible poles of a coefficient are the zeros of Θ , which lie over the pole of t . Since t is a coordinate on the quotient, each coefficient is a polynomial in t , initially over $\mathbb{C}$. A term of the coefficient of X^m has at most m factors Θ^{-1} . Above $X_0(6)$ its pole order is at most m , while the pole order of t is two. A polynomial of degree d in t has pole order $2d$ there. This proves $\deg C_m \leq \lfloor m/2 \rfloor$.

For integral coefficients, put $s = q^{1/p}$ and $\zeta = \zeta_p$. The product expansions lie in $\mathbb{Z}[\zeta][[s]]$. The symmetry $s \mapsto \zeta s$ removes exponents not divisible by p , and the Galois automorphisms $\zeta \mapsto \zeta^a$ permute the factors. The surviving coefficients are rational algebraic integers, hence integers in $\mathbb{Z}[[q]]$. Because $t = q + q^2\mathbb{Z}[[q]]$, successive subtraction of the lowest terms against powers of t makes all polynomial coefficients integers. The same triangular argument transfers coefficientwise congruences from q to t .

Let $Q = \Theta(q)/\Theta(q^p)$ and $R_k = \Theta(q)/\Theta(\zeta^k s)$. The product is exactly

$$C_{p+1}(t)(X - Q) \prod_{k=0}^{p-1} (X - p^2 R_k).$$

The symmetric functions of the R_k are in $\mathbb{Z}[[q]]$ by the same symmetries. Modulo p^2 , the last product is X^p , which proves the middle congruence in (37). The leading coefficient, with positive sign since $p+1$ is even, is

$$C_{p+1}(t(q)) = \frac{\Theta(q^p) \prod_{k=0}^{p-1} \Theta(\zeta^k s)}{\Theta(q)^{p+1}}.$$

The product in the numerator is integral in q . Modulo $1 - \zeta$ it is $\Theta(s)^p \equiv \Theta(q)$ in characteristic p . Since $(1 - \zeta)\mathbb{Z}[\zeta] \cap \mathbb{Z} = p\mathbb{Z}$, this gives the congruence modulo p on its integer coefficients. Finally $\Theta(q^p) \equiv \Theta(q)^p \pmod{p}$ proves $C_{p+1} \equiv 1 \pmod{p}$. $\square$

Lemma 5.2. *For every prime $p > 3$,*

$$\binom{2p}{p} a_p \equiv 6 \pmod{p^2}, \quad \binom{2p}{p} f_p \equiv 4 \pmod{p^2}. \quad (38)$$

Proof. The exact product $\binom{2p}{p} = 2 \prod_{k=1}^{p-1} (1 + p/k)$ and $\sum_{k=1}^{p-1} k^{-1} \equiv 0 \pmod{p}$ give $\binom{2p}{p} \equiv 2 \pmod{p^2}$. The latter harmonic congruence follows because inversion permutes the nonzero residues modulo p . In a_p all internal terms have the factor $\binom{p}{k}^2$, while the two endpoint terms are 1 and $\binom{2p}{p}$. Thus $a_p \equiv 3 \pmod{p^2}$. For f_p all internal terms are divisible by p^3 , so $f_p \equiv 2 \pmod{p^3}$. Multiplication proves (38). $\square$

Theorem 5.3 (Finite specialization). *Let $p > 3$ be prime. For either pair of period and parameter, the polynomial identity*

$$C_p(t) + C_{p+1}(t)Y_p(t) \equiv 0 \pmod{p^2} \quad (39)$$

holds in $\mathbb{Z}[t]$. If $t_0 \in \mathbb{Z}_p$ and z is an exact root of $\mathcal{T}_p(X, t_0)$ in a finite extension of $\mathbb{Q}_p$ with valuation zero, then $Y_p(t_0) \equiv z \pmod{p^2}$ in its ring of integers.

Proof. The series $Q = \Theta(q)/\Theta(q^p)$ is a root of (36), and is a formal unit with constant term one. Dividing its equation from (37) by the formal unit Q^p gives $C_p + C_{p+1}Q \equiv 0 \pmod{p^2}$. Write $C_{p+1} = 1 + pD_p$. Then

$$Q \equiv K(t) := -(1 - pD_p(t))C_p(t) \pmod{p^2}, \quad \deg K \leq (p-1)/2 + (p+1)/2 = p. \quad (40)$$

The formal inverse $q(t)$ is integral. The period identity (24) expresses Q as $Y(t)/Y(t^\sigma)$, where $t^\sigma = t(q(t)^p) = t^p + O(t^{p+1})$. Hence

$$Q = Y_p(t) + (y_p - y_1)t^p + O(t^{p+1}). \quad (41)$$

Lemma 5.2 makes the coefficient of t^p zero modulo p^2 : the coefficient y_1 is 6 for G and 4 for F . Both K and Y_p have degree at most p , so comparison through degree p proves their congruence as polynomials. Multiplying by C_{p+1} proves (39).

We now specialize only these finite polynomials. The exact equation $\mathcal{T}_p(z, t_0) = 0$ and (37) imply

$$z^p(C_p(t_0) + C_{p+1}(t_0)z) \equiv 0 \pmod{p^2}.$$

Divide by the unit z^p , subtract (39) evaluated at t_0 , and divide by the unit $C_{p+1}(t_0) \equiv 1 \pmod{p}$. The result is the claimed congruence. $\square$

The upper degree p in (40) is needed: comparison through $p - 1$ alone would not exclude an extra multiple of t^p . Equation (38) treats exactly this endpoint. This coefficient comparison follows the mechanism in the proof of [3, Theorem 6.1], here with degree bound p and modulus p^2 . Every evaluation in Theorem 5.3 is of a finite polynomial; it does not require evaluating an infinite series at a p -adic unit. The argument also needs only the modulus p^2 in (37); the additional hypotheses in the stronger modulo- p^3 results of Beukers are not used.

6 The split primes and the Franel corollary

Let $p = x^2 + 3y^2 \equiv 1 \pmod{3}$ be prime, with $x, y > 0$. Then $\gcd(x, 3y) = 1$ and x, y have opposite parity. Keep $\delta = \sqrt{-3}$ and $\beta = \delta/3$. We first construct exact complex roots of the Hecke polynomials at the rational parameters in (34), then choose their p -adic embeddings.

6.1 The case y even

Here $p \equiv 1 \pmod{12}$ and $(-1)^{(p-1)/2} = 1$. Choose $a_0, b_0 \in \mathbb{Z}$ with $a_0x + 3b_0y = 1$ and set

$$M = \begin{pmatrix} a_0 & b_0 \\ -3y & x \end{pmatrix} \in \Gamma_0(6), \quad k = b_0x - a_0y. \quad (42)$$

The determinant is one and direct rationalization yields

$$M\beta = \frac{\beta + k}{p}. \quad (43)$$

Indeed the rationalized numerator is $b_0x - a_0y + (a_0x + 3b_0y)\delta/3 = k + \beta$. The character is trivial on M , so (26) gives

$$p^2 \frac{\Theta_+(\beta)}{\Theta_+(M\beta)} = \frac{p^2}{(x - y\delta)^2} = (x + y\delta)^2. \quad (44)$$

This is an exact root of $\mathcal{T}_p(X, 1/54)$ by the factor with index $k \bmod p$ in (36); period one of Θ_+ justifies reducing the index. The same equality holds with Θ_- , giving an exact root of the corresponding polynomial at $1/50$.

6.2 The case y odd

Here x is even, $p \equiv 7 \pmod{12}$ and $(-1)^{(p-1)/2} = -1$. Choose $a_0, b_0 \in \mathbb{Z}$ with $3a_0y + b_0x = 1$ and put

$$M = \begin{pmatrix} 3a_0 & b_0 \\ -3x & 3y \end{pmatrix}, \quad \det M = 3, \quad k = b_0y - a_0x. \quad (45)$$

Its extended-group class is W_3 , since

$$MW_3^{-1} = \begin{pmatrix} 3a_0 - 2b_0 & b_0 - a_0 \\ -3x - 6y & x + 3y \end{pmatrix} \in \Gamma_0(6). \quad (46)$$

The lower left entry is divisible by six because x is even. Again

$$M\beta = (\beta + k)/p. \quad (47)$$

Here the rationalized numerator is $3k + (b_0x + 3a_0y)\delta = 3k + \delta$, and the denominator norm is $3p$. Using $\chi_+(M) = 1$ and the determinant factor in (26), we obtain the exact root

$$p^2 \frac{\Theta_+(\beta)}{\Theta_+(M\beta)} = \frac{3p^2}{(3y - x\delta)^2} = \frac{(3y + x\delta)^2}{3} = -(x - y\delta)^2 \quad (48)$$

of $\mathcal{T}_p(X, 1/54)$. For Θ_- the character on W_3 is -1 , so the exact root of its polynomial at $1/50$ is instead $(x - y\delta)^2$.

6.3 The unit root and completion of the proof

Since $p \equiv 1 \pmod{3}$, -3 is a square modulo p . The two roots lift by Hensel's lemma, giving the two embeddings $\mathbb{Q}(\delta) \hookrightarrow \mathbb{Q}_p$. In either embedding the factors $x + y\delta$ and $x - y\delta$ are integral and their product is p . They cannot both vanish modulo p , for that would force $p \mid 2x$ and $p \mid 2y$. Thus their valuations are 0 and 1 in some order; the embeddings interchange that order.

In the even case choose the embedding where $x + y\delta$ is a unit; in the odd case choose the one where $x - y\delta$ is a unit. The exhibited root is now a unit before we invoke Theorem 5.3. Its conjugate square is divisible by p^2 , and the exact trace identity is

$$(x + y\delta)^2 + (x - y\delta)^2 = 4x^2 - 2p. \quad (49)$$

Since $1/54 \in \mathbb{Z}_p$, Theorem 5.3 and the roots (44), (48) give

$$G_p(1/54) \equiv \begin{cases} 4x^2 - 2p, & p \equiv 1 \pmod{12}, \\ -(4x^2 - 2p), & p \equiv 7 \pmod{12} \end{cases} \pmod{p^2}. \quad (50)$$

Multiplication by $(-1)^{(p-1)/2}$ proves the split branch of Theorem 1.1. The inert branch, including $p = 5$, was proved in Section 2; hence the theorem holds for all $p > 3$.

The complex period ratios used here are defined because both forms are nonzero at β . The transformations identify those ratios with exact algebraic numbers. Only those algebraic numbers are subsequently embedded in $\mathbb{Q}_p$, so the construction requires no p -adic evaluation of complex eta series at the CM point.

The evaluation in (51) is the statement of Sun's 2018 Conjecture 3.5 [2, Conjecture 3.5], distinct from the 2022 conjecture proved in Theorem 1.1.

Corollary 6.1. *For every prime $p > 5$,*

$$F_p(1/50) \equiv \begin{cases} 4x^2 - 2p, & p = x^2 + 3y^2 \equiv 1 \pmod{3}, \\ 0, & p \equiv 2 \pmod{3} \end{cases} \pmod{p^2}, \quad (51)$$

and

$$G_p(1/54) \equiv \left(\frac{3}{p}\right) F_p(1/50) \pmod{p^2}. \quad (52)$$

Proof. For split primes the roots for Θ_- computed above are the unit square with positive sign in both parity cases. Since $1/50 \in \mathbb{Z}_p$, Theorem 5.3 and (49) give the split value in (51). Equation (17) gives the inert value. For split primes quadratic reciprocity gives $\left(\frac{3}{p}\right) = (-1)^{(p-1)/2}$; for inert primes both sums are zero. This proves (52) as a consequence of their separate evaluations. $\square$

A Exact eta certificates

This appendix specifies the finite calculations used in Proposition 3.1 and Lemma 4.2. They certify identities because the proofs place the residuals in spaces of holomorphic modular forms with explicit valence bounds.

For $E(\tau) = \prod_{d|6} \eta(d\tau)^{r_d}$, the weight is $\frac{1}{2} \sum r_d$. The eta criterion uses $\sum dr_d \equiv \sum (6/d)r_d \equiv 0 \pmod{24}$; for the even weights here, the rational square condition on $\prod d^{r_d}$ gives trivial character. The cusp order at denominator $c \mid 6$ is

$$\frac{6}{24 \gcd(c^2, 6)} \sum_{d|6} \frac{\gcd(c, d)^2}{d} r_d. \quad (53)$$

The data are

Function	(r_1, r_2, r_3, r_6)	Weight	$(\sum dr_d, \sum (6/d)r_d)$	Cusp orders
$\mathcal{A}$	$(-5, 7, 7, -5)$	2	$(0, 0)$	$(0, 1, 1, 0)$
$\mathcal{B}$	$(7, -5, -5, 7)$	2	$(24, 24)$	$(1, 0, 0, 1)$
J	$(2, 2, 2, 2)$	4	$(24, 24)$	$(1, 1, 1, 1)$
h	$(-5, 1, -1, 5)$	0	$(24, -24)$	$(-1, 0, 0, 1)$
h_3	$(-12, 0, 12, 0)$	0	$(24, -48)$	$(-2, -1, 2, 1)$
R	$(-12, 12, 12, -12)$	0	$(-24, -24)$	$(-1, 1, 1, -1)$

In every row, the valuations at 2, 3 of $\prod d^{r_d}$ are even.

All required expansions use the integer product

$$E(q) = q^s \prod_{m \geq 1} (1 - q^m)^{e_m}, \quad s = \frac{1}{24} \sum dr_d, \quad e_m = \sum_{d|m} r_d. \quad (54)$$

For the coefficient of degree at most N , only factors with $m \leq N - s$ contribute. Negative powers are expanded with the integer binomial coefficients for $(1 - q^m)^{-r}$. An equivalent exact recurrence, for $\prod_{m \geq 1} (1 - q^m)^{e_m} = \sum_{n \geq 0} c_n q^n$, is

$$c_0 = 1, \quad nc_n = - \sum_{j=1}^n \left(\sum_{d|j} de_d \right) c_{n-j}. \quad (55)$$

For the period certificate the required coefficients are

n	$[q^n]\Theta_+$	$[q^n]W_+$	$[q^n]\Theta_-$	$[q^n]W_-$	$[q^n]J$
0	1	1	1	1	0
1	6	-8	4	-6	1
2	6	-40	20	-78	-2
3	42	-296	4	-114	-3
4	6	-296	52	-654	4
5	36	-1008	24	-756	6
6	42	-1480	20	-1482	6
7	48	-2752	32	-2064	-16
8	6	-2344	116	-5262	-8

Here $W_{\pm} = \Theta_{\pm} \partial J/J - 2\partial \Theta_{\pm}$. Their coefficients can be obtained without division by a series of zero constant term by first writing $J = qj(q)$, $j(0) = 1$, so that $\partial J/J = 1 + \partial j/j$. In computing through degree eight one expands J through degree nine. Substitution of the table in (22) and (23), using $[q^n]\partial \Theta = n[q^n]\Theta$, gives the zero vector $(0, 0, 0, 0, 0, 0, 0, 0)$ for each of the four residuals.

For the geometric certificate the expansions are

n	$[q^n]h$	$[q^n]h_3$	$[q^n]Rh$
0	0	0	1
1	1	1	17
2	5	12	157
3	19	90	1043
4	61	508	5573
5	174	2391	25422
6	455	9828	102703
7	1112	36428	376600
8	2573	124188	1275445
9	5689	395199	4041161
10	12102	1186344	12096534

These give $[q^0, \dots, q^8]E_1 = 0$ and $[q^0, \dots, q^{10}]E_2 = 0$. After multiplication by M_1, M_2 from Lemma 4.2, the monomials and their coefficients are

Residual	Exponents $(\eta_1, \eta_2, \eta_3, \eta_6)$	Coefficient
M_1E_1	(5, 0, 15, 0)	1
	(0, 1, 14, 5)	9
	(12, 1, 2, 5)	-1
	(7, 2, 1, 10)	-16
	(2, 3, 0, 15)	-64
M_2E_2	(0, 13, 13, 0)	1
	(17, 0, 2, 7)	-1
	(12, 1, 1, 12)	-17
	(7, 2, 0, 17)	-72

Nonnegative eta exponents establish holomorphy of these sums at every cusp. Together with the multiplier argument and bounds $242 > 240$, $329 > 312$, these exact coefficients prove the full identities.

The accompanying software, version 0.1.3, implements the integer products, the recurrence (55), and the residual calculations. The companion repository is

<https://github.com/aconsciousfractal/Sun-3-5-Supercongruence>

The repository README and REPRODUCE.md give the verification and build instructions. Its additional checks at bounded sets of primes are regression checks; the general conclusions follow from the arguments in the article. The evidence inventory is EVIDENCE_SHA256.txt; its SHA-256 is

1699783d715bc7a70a82aae729ef364ff048763b546e11824409f5090d6b0846

This inventory identifies the companion code, tests, configuration and exact evidence, excluding the manuscript and prose that quote the digest.

B Telescoping proofs of the sequence recurrences

For $n \geq 2$, regard binomial terms with lower indices outside their natural range as zero. For the first sequence put $\tau(n, k) = \binom{n}{k}^2 \binom{2k}{k}$ and

$$R_a(n, k) = \frac{k^3(3k - 4n)}{n^2}.$$

The ratio $\tau(n, k+1)/\tau(n, k) = 2(2k+1)(n-k)^2/(k+1)^3$ gives the exact identity

$$\begin{aligned} n^2\tau(n, k) - (10n(n-1) + 3)\tau(n-1, k) + 9(n-1)^2\tau(n-2, k) \\ = R_a(n, k+1)\tau(n, k+1) - R_a(n, k)\tau(n, k). \end{aligned} \quad (56)$$

For example, after dividing by $\tau(n, k)$ the ratios of the two lower rows are $(n-k)^2/n^2$ and $(n-k)^2(n-k-1)^2/(n^2(n-1)^2)$, reducing the identity to a polynomial equality. Summation over $k = 0, \dots, n$ has zero lower boundary because $R_a(n, 0) = 0$, and zero upper boundary because $\tau(n, n+1) = 0$. This proves (5) with $(a, b, c) = (10, 3, 9)$; the initial values are $a_0 = 1, a_1 = 3$.

For the Franel sequence let $\tau(n, k) = \binom{n}{k}^3$ and set

$$\begin{aligned} Q_f(n, k) &= 4k^3 + (6 - 18n)k^2 + (27n^2 - 15n)k - 14n^3 + 10n^2, \\ R_f(n, k) &= \frac{k^3 Q_f(n, k)}{n^3(n-1)}. \end{aligned}$$

Now $\tau(n, k+1)/\tau(n, k) = (n-k)^3/(k+1)^3$, and direct polynomial expansion gives

$$\begin{aligned} n^2\tau(n, k) - (7n(n-1) + 2)\tau(n-1, k) - 8(n-1)^2\tau(n-2, k) \\ = R_f(n, k+1)\tau(n, k+1) - R_f(n, k)\tau(n, k). \end{aligned} \quad (57)$$

The lower-row ratios here are $(n-k)^3/n^3$ and $(n-k)^3(n-k-1)^3/(n^3(n-1)^3)$. Both boundaries again vanish, so summation proves (5) with $(a, b, c) = (7, 2, -8)$. The initial values $f_0 = 1, f_1 = 2$ are immediate. No endpoint certificate has a pole in the summation index: the denominators displayed above depend only on $n \geq 2$.

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