
Improved Cauchy-Schwarz-Kaplansky Inequality

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Abstract: We make an improvement and give a new proof for Cauchy-Schwarz-Kaplansky inequality derived by Kaplansky [*Am. J. Math.*, 1953].

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1. INTRODUCTION

Cauchy-Schwarz angle-length inequality, derived in 1821, is the following.

Theorem 1.1. [1, 2, 5, 8, 9, 12, 13] (*Cauchy-Schwarz Angle-Length Inequality*) Let $n \in \mathbb{N}$. Then

$$\left| \sum_{j=1}^n a_j b_j \right| \leq \left(\sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^n |b_k|^2 \right)^{\frac{1}{2}}, \quad \forall (a_j)_{j=1}^n, (b_j)_{j=1}^n \in \mathbb{C}^n.$$

We clearly have the following sequential version of Theorem 1.1.

Theorem 1.2. [1, 2, 5, 8, 9, 12, 13] (*Cauchy-Schwarz Angle-Length Inequality*) Let $\{a_j\}_{j=1}^\infty, \{b_j\}_{j=1}^\infty \subseteq \mathbb{C}$ be such that

$$\sum_{j=1}^\infty |a_j|^2 \in \mathbb{R}, \quad \sum_{j=1}^\infty |b_j|^2 \in \mathbb{R}.$$

Then

$$\left| \sum_{j=1}^\infty a_j b_j \right| \leq \left(\sum_{j=1}^\infty |a_j|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^\infty |b_k|^2 \right)^{\frac{1}{2}}.$$

In 1953, Kaplansky extended Cauchy-Schwarz inequality to commutative C^* -algebras [6].

Theorem 1.3. [6] (*Cauchy-Schwarz-Kaplansky Inequality*) Let $\mathcal{A}$ be a commutative C^* -algebra. Let $n \in \mathbb{N}$. Then

$$\left[\left(\sum_{j=1}^n a_j^* b_j \right) \left(\sum_{k=1}^n a_k^* b_k \right)^* \right]^{\frac{1}{2}} \leq \left(\sum_{j=1}^n a_j^* a_j \right)^{\frac{1}{2}} \left(\sum_{k=1}^n b_k^* b_k \right)^{\frac{1}{2}}, \quad \forall (a_j)_{j=1}^n, (b_j)_{j=1}^n \in \mathcal{A}^n.$$

In particular,

$$\left\| \sum_{j=1}^n a_j^* b_j \right\| \leq \left\| \sum_{j=1}^n a_j^* a_j \right\|^{\frac{1}{2}} \left\| \sum_{k=1}^n b_k^* b_k \right\|^{\frac{1}{2}}.$$

Theorem 1.4. [6] (*Cauchy-Schwarz-Kaplansky Inequality*) Let $\mathcal{A}$ be a commutative C^* -algebra. Let $\{a_j\}_{j=1}^\infty, \{b_j\}_{j=1}^\infty \subseteq \mathcal{A}$ be such that

$$\sum_{j=1}^\infty a_j^* a_j \in \mathcal{A}, \quad \sum_{j=1}^\infty b_j^* b_j \in \mathcal{A}.$$

Then

$$\left[\left(\sum_{j=1}^\infty a_j^* b_j \right) \left(\sum_{k=1}^\infty a_k^* b_k \right)^* \right]^{\frac{1}{2}} \leq \left(\sum_{j=1}^\infty a_j^* a_j \right)^{\frac{1}{2}} \left(\sum_{k=1}^\infty b_k^* b_k \right)^{\frac{1}{2}}.$$

In particular,

$$\left\| \sum_{j=1}^\infty a_j^* b_j \right\| \leq \left\| \sum_{j=1}^\infty a_j^* a_j \right\|^{\frac{1}{2}} \left\| \sum_{k=1}^\infty b_k^* b_k \right\|^{\frac{1}{2}}.$$

In this note, we improve Cauchy-Schwarz-Kaplansky inequality.

2. IMPROVED CAUCHY-SCHWARZ-KAPLANSKY INEQUALITY

Following result is improvement of Theorem 1.1.

Theorem 2.1. (*Improved Cauchy-Schwarz-Kaplansky Inequality*) Let $\mathcal{A}$ be a commutative C^* -algebra. Let $n \in \mathbb{N}$. Then

$$\begin{aligned} \left(\sum_{j=1}^n a_j^* b_j \right) \left(\sum_{k=1}^n a_k^* b_k \right)^* &= \left(\sum_{j=1}^n a_j^* a_j \right) \left(\sum_{k=1}^n b_k^* b_k \right) - \sum_{1 \leq j < k \leq n} (a_j b_k - a_k b_j)^* (a_j b_k - a_k b_j) \\ &\leq \left(\sum_{j=1}^n a_j^* a_j \right) \left(\sum_{k=1}^n b_k^* b_k \right), \quad \forall (a_j)_{j=1}^n, (b_j)_{j=1}^n \in \mathcal{A}^n. \end{aligned}$$

In particular,

$$\begin{aligned} \left[\left(\sum_{j=1}^n a_j^* b_j \right) \left(\sum_{k=1}^n a_k^* b_k \right)^* \right]^{\frac{1}{2}} &= \left[\left(\sum_{j=1}^n a_j^* a_j \right) \left(\sum_{k=1}^n b_k^* b_k \right) - \sum_{1 \leq j < k \leq n} (a_j b_k - a_k b_j)^* (a_j b_k - a_k b_j) \right]^{\frac{1}{2}} \\ &\leq \left[\left(\sum_{j=1}^n a_j^* a_j \right) \left(\sum_{k=1}^n b_k^* b_k \right) \right]^{\frac{1}{2}} \\ &= \left(\sum_{j=1}^n a_j^* a_j \right)^{\frac{1}{2}} \left(\sum_{k=1}^n b_k^* b_k \right)^{\frac{1}{2}}, \quad \forall (a_j)_{j=1}^n, (b_j)_{j=1}^n \in \mathcal{A}^n. \end{aligned}$$

Proof. Our proof is motivated from the arguments for the scalar case by Cauchy using Lagrange identity [3, 4, 7–9, 12, 14]. From Lagrange identity, we have,

$$\begin{aligned} \sum_{1 \leq j < k \leq n} (a_j b_k - a_k b_j)^* (a_j b_k - a_k b_j) &= \frac{1}{2} \sum_{j=1}^n \sum_{k=1}^n (a_j b_k - a_k b_j)^* (a_j b_k - a_k b_j) \\ &= \left(\sum_{j=1}^n a_j^* a_j \right) \left(\sum_{k=1}^n b_k^* b_k \right) - \left(\sum_{j=1}^n a_j^* b_j \right) \left(\sum_{k=1}^n a_k^* b_k \right)^*. \end{aligned}$$

Rearranging, we get

$$\begin{aligned} \left(\sum_{j=1}^n a_j^* b_j \right) \left(\sum_{k=1}^n a_k^* b_k \right)^* &= \left(\sum_{j=1}^n a_j^* a_j \right) \left(\sum_{k=1}^n b_k^* b_k \right) - \sum_{1 \leq j < k \leq n} (a_j b_k - a_k b_j)^* (a_j b_k - a_k b_j) \\ &\leq \left(\sum_{j=1}^n a_j^* a_j \right) \left(\sum_{k=1}^n b_k^* b_k \right). \end{aligned}$$

Since square root respects the order of positive elements in a C^* -algebra, we have the result. $\square$

Following extension of Theorem 2.1 is clear.

Theorem 2.2. (*Improved Cauchy-Schwarz-Kaplansky Inequality*) Let $\mathcal{A}$ be a commutative C^* -algebra. Let $\{a_j\}_{j=1}^\infty, \{b_j\}_{j=1}^\infty \subseteq \mathcal{A}$ be such that

$$\sum_{j=1}^\infty a_j^* a_j \in \mathcal{A}, \quad \sum_{j=1}^\infty b_j^* b_j \in \mathcal{A}.$$

Then

$$\begin{aligned} \left(\sum_{j=1}^\infty a_j^* b_j \right) \left(\sum_{k=1}^\infty a_k^* b_k \right)^* &= \left(\sum_{j=1}^\infty a_j^* a_j \right) \left(\sum_{k=1}^\infty b_k^* b_k \right) - \sum_{1 \leq j < k < \infty} (a_j b_k - a_k b_j)^* (a_j b_k - a_k b_j) \\ &\leq \left(\sum_{j=1}^\infty a_j^* a_j \right) \left(\sum_{k=1}^\infty b_k^* b_k \right). \end{aligned}$$

In particular,

$$\begin{aligned} \left[\left(\sum_{j=1}^\infty a_j^* b_j \right) \left(\sum_{k=1}^\infty a_k^* b_k \right)^* \right]^{\frac{1}{2}} &= \left[\left(\sum_{j=1}^\infty a_j^* a_j \right) \left(\sum_{k=1}^\infty b_k^* b_k \right) - \sum_{1 \leq j < k < \infty} (a_j b_k - a_k b_j)^* (a_j b_k - a_k b_j) \right]^{\frac{1}{2}} \\ &\leq \left[\left(\sum_{j=1}^\infty a_j^* a_j \right) \left(\sum_{k=1}^\infty b_k^* b_k \right) \right]^{\frac{1}{2}} \\ &= \left(\sum_{j=1}^\infty a_j^* a_j \right)^{\frac{1}{2}} \left(\sum_{k=1}^\infty b_k^* b_k \right)^{\frac{1}{2}}. \end{aligned}$$

3. AN OPEN PROBLEM

For non-commutative C^* -algebras, the following result has been obtained independently by Paschke [10] and Rieffel [11].

Theorem 3.1. [10, 11] (*Cauchy-Schwarz-Paschke-Rieffel Inequality*) Let $\mathcal{A}$ be a C^* -algebra. Let $n \in \mathbb{N}$. Then

$$\left[\left(\sum_{j=1}^n a_j^* b_j \right) \left(\sum_{k=1}^n a_k^* b_k \right)^* \right]^{\frac{1}{2}} \leq \left(\sum_{j=1}^n a_j^* a_j \right)^{\frac{1}{2}} \left\| \sum_{k=1}^n b_k^* b_k \right\|^{\frac{1}{2}}, \quad \forall (a_j)_{j=1}^n, (b_j)_{j=1}^n \in \mathcal{A}^n.$$

In particular,

$$\left\| \sum_{j=1}^n a_j^* b_j \right\| \leq \left\| \sum_{j=1}^n a_j^* a_j \right\|^{\frac{1}{2}} \left\| \sum_{k=1}^n b_k^* b_k \right\|^{\frac{1}{2}}, \quad \forall (a_j)_{j=1}^n, (b_j)_{j=1}^n \in \mathcal{A}^n.$$

Note that Cauchy-Schwarz-Paschke-Rieffel inequality is not a direct generalization of Cauchy-Schwarz-Kaplansky inequality. We don't know how to generalize Theorem 3.1 to noncommutative C*-algebras.

Note that the Cauchy-Binet identity

$$\left(\sum_{j=1}^n a_j c_j \right) \left(\sum_{k=1}^n b_k d_k \right) - \left(\sum_{j=1}^n a_j d_j \right) \left(\sum_{k=1}^n b_k c_k \right) = \sum_{1 \leq j < k \leq n} (a_j b_k - b_j a_k)(c_j d_k - d_j c_k),$$

$$\forall (a_j)_{j=1}^n, (b_j)_{j=1}^n, (c_j)_{j=1}^n, (d_j)_{j=1}^n \in \mathbb{C}^n$$

seems not to improve Cauchy-Schwarz-Paschke-Rieffel inequality (Cauchy-Binet identity holds only for commutative rings).

4. CONCLUSIONS

- (1) In 1821, Cauchy extended Lagrange identity and proved Cauchy-Schwarz inequality [7].
- (2) In 1953, Kaplansky extended Cauchy-Schwarz inequality to commutative C*-algebras [6].
- (3) In this note, we improved Cauchy-Schwarz-Kaplansky inequality.

5. ACKNOWLEDGMENTS

AI is used for the literature survey.

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