

THE UNBALANCED GROUP THEOREM

AXIOM MATH

ABSTRACT. The classification of finite simple groups (CFSG) identifies the basic building blocks of all finite groups. It grew from nineteenth-century work of Galois and Jordan into a collective effort involving many mathematicians, with Gorenstein playing a central organizing role. Its proof is a vast network of interdependent results such as the Feit–Thompson odd-order theorem, the signalizer functor theorem, and Aschbacher’s classical involution theorem. The original papers make up the first-generation proof. The second-generation treatment, initiated by Gorenstein, Lyons, and Solomon, reorganizes and simplifies much of this work.

A key structural result in the first-generation program is the Unbalanced Group Theorem (UGT). Under the inductive hypotheses of the classification, it says that a nontrivial subgroup of odd order normal in the centralizer of an involution forces the ambient simple group to be a group of Lie type in odd characteristic or one of the alternating groups of certain degrees.

While formalizing the first-generation proof, we found counterexamples to a closure assertion used in a published route to UGT. This obstruction concerns that particular argument. UGT also has other proofs. Here we explain the obstruction and develop an alternative argument for the simple-group case using Solomon’s 1976 theory of maximal 2-components, which provides a framework for comparing the simple pieces arising in different involution centralizers.

1. OVERVIEW

The classification of finite simple groups (CFSG) is a monumental achievement in the history of mathematics. A group is *simple* if it is nontrivial and has no normal subgroups other than itself and the identity subgroup. Every finite group can be broken into simple factors by a composition series. Indeed, the Jordan–Hölder theorem says that the resulting factors, counted with multiplicity, do not depend on the series. Classifying these factors does not tell us every way to assemble a finite group, but it tells us which building blocks are available. The CFSG catalogs every finite simple group:

Theorem (Classification of finite simple groups). *Every finite simple group is either cyclic of prime order, an alternating group A_n for $n \geq 5$, a simple group of Lie type, or one of the twenty-six sporadic groups, named after their discoverers apart from the Baby Monster and the Monster.*

The proof grew into a collective enterprise spanning much of the twentieth century. Richard Brauer made centralizers of elements of order two central to the recognition of simple groups. Walter Feit and John Thompson’s odd-order theorem, published in 1963, showed that every nonabelian finite simple group has such an element [11]. Michio Suzuki and Thompson made powerful tools for recovering a group from its subgroups, while Daniel Gorenstein gave the effort a guiding program and an organizing framework. Michael Aschbacher, Richard Lyons, Ronald Solomon, and many others supplied the structural and recognition theorems that carried that program forward. Solomon’s historical account [25] describes both the ideas and the extraordinary collaboration behind them.

What is now referred to as the *first-generation proof* is made up of an extensive web of papers, whose main framework was already established by the early 1980s. Gorenstein’s 1983 volume [16] records much of the relevant work. Work toward completion and verification continued. For example, the substantial remaining case of quasithin groups was resolved by Aschbacher and Stephen Smith in work published in 2004. Aschbacher’s account [3] explains the distinction between the early announcements and the work needed to secure the proof. Beginning in the early 1980s, Gorenstein, Lyons, and Solomon also undertook

a second-generation project, seeking to reorganize and simplify large parts of the proof into a systematic series of volumes. These efforts reflect the scale of the achievement.

CFSG should be thought of as a problem of identification. Suppose there were a finite simple group of smallest order missing from the “known list”. All the simple groups arising as sections of its proper subgroups would already be known. One then examines those subgroups, compares the information they provide, and proves that the supposedly new group must itself be one of the known groups. To this end, group theorists have made great use of centralizers of involutions: the subgroups consisting of all elements that commute with a given element of order two. Such centralizers are proper subgroups of a nonabelian simple group, but their structure can constrain the whole group.

The Unbalanced Group Theorem (UGT) is one of the key results that makes this “local-to-global” strategy work. Roughly speaking, if even one involution centralizer has a nontrivial normal subgroup of odd order, then the ambient simple group belongs to particular familiar families.

To make this precise, we recall some definitions. Throughout, all groups in this paper are finite. Let G be a group. An *involution* is an element of order two. The *centralizer* of a subset A of G is

$$C_G(A) = \{g \in G : ga = ag \text{ for every } a \in A\}.$$

We denote this centralizer by $C_G(t)$ when $A = \{t\}$. The *odd core* $O(G) = O_{2'}(G)$ is the largest normal subgroup of G of odd order. It exists because the product of two normal subgroups of odd order is again a normal subgroup of odd order. The subscript $2'$ means “prime to two.”

Normality depends on the ambient group. In particular, a subgroup can be normal in $C_G(t)$ without being normal in G . Although a nonabelian simple group G has no nontrivial normal subgroup of odd order, an involution centralizer $C_G(t)$ may have one. The unbalanced-group problem asks what this tells us about G .

Problem 1.1 (Unbalanced-group problem: simple-group case). *Which nonabelian finite simple groups G contain an involution t for which $O(C_G(t)) \neq 1$?*

Such a simple group is called *unbalanced*. Generally, a finite group G is unbalanced if $O(C_G(t)) \not\leq O(G)$ for some involution $t \in G$. For a nonabelian simple group G , the Feit–Thompson theorem [11] implies G has even order and hence $O(G) = 1$, so the two conditions agree.

The Unbalanced Group Theorem answers this question under the inductive hypothesis used in the classification program, which we now make precise. A *section* of a group X is a quotient A/N , where $A \leq X$ and $N \trianglelefteq A$. Let $\mathcal{K}$ be the catalog of *known nonabelian finite simple groups*. Namely, they are the alternating groups, the simple groups of Lie type, and the twenty-six sporadic groups. A $\mathcal{K}$ -*group* is a group whose nonabelian simple sections belong to this catalog. A group is $\mathcal{K}$ -*proper* if every proper subgroup is a $\mathcal{K}$ -group. This condition is natural in a classification proof. If G is a nonabelian simple group of least order outside the catalog, every nonabelian simple section of a proper subgroup has smaller order. Consequently G is $\mathcal{K}$ -proper.

Groups of Lie type are the finite groups constructed from algebraic groups over finite fields, together with their twisted analogs. The groups $\text{PSL}_n(q)$ are examples; they are the determinant-one matrices over the finite field of q elements modulo the scalar matrices. Similarly, $\text{PGL}_n(q)$ is the group of all invertible matrices modulo the nonzero scalar matrices. We say that a simple group is of *odd Lie type* if it is isomorphic to a simple group of Lie type in odd characteristic. There are small coincidences such as $A_5 \cong \text{PSL}_2(5)$. We now state the simple-group case of the Unbalanced Group Theorem.

Theorem 1.2 (UGT). *Let G be a nonabelian simple $\mathcal{K}$ -proper group. If $O(C_G(t)) \neq 1$ for some involution $t \in G$, then either G is of odd Lie type, or $G \cong A_{4a+3}$ for some positive integer a .*

Remark 1.3. The version of UGT stated above has been formalized in Lean in the Axiom Math CFSG repository. Our project website provides a link to the formal proof.

Theorem 1.2 gives an important reduction in the classification program. Suppose that G were a nonabelian simple group of smallest order outside the known catalog. Its proper subgroups satisfy the required

inductive hypothesis. If $O(C_G(t)) \neq 1$ for some involution t , the theorem would identify G with a group in the catalog, a contradiction. Therefore, any such counterexample must satisfy

$$O(C_G(t)) = 1 \quad \text{for every involution } t \in G.$$

UGT therefore removes all unbalanced groups from the remaining classification problem.

The theorem also has an important connection with the *B-theorem*. Roughly speaking, the *B-theorem* ensures that the simple pieces visible after removing the odd core of an involution centralizer can be realized by suitable subgroups of the centralizer itself. More precisely, when $O(G) = 1$, every quasisimple component of $C_G(t)/O(C_G(t))$ is the image of a quasisimple component of $C_G(t)$. We define components in Section 2. This lifting property allows one to study actual subgroups of G , compare them across different centralizers, and apply recognition theorems to identify G .

For a smallest counterexample to CFSG, the displayed consequence of UGT makes this lifting property immediate. Namely, there is no odd core to remove. The corresponding *B*-property for its proper subgroups and their sections follows from independently established results for $\mathcal{K}$ -groups. Therefore, the simple-group form of UGT supplies the ambient reduction needed here without requiring an independent proof of the unrestricted *B-theorem* as an additional input. For the historical relationship between the two theorems, see [13, §1], [16, Chapter 3], and [5, §§1.7–1.8].

The restriction to involutions in G distinguishes our statement from some other formulations of UGT.

Remark 1.4. More general versions of UGT treat involutions acting from outside a simple normal subgroup. Theorem 1.2 is the simple-group form required for the classification induction.

For alternating groups, three fixed points of an involution account for the nontrivial odd core.

Example 1.5. As noted above, the groups A_{4a+3} , for $a \geq 1$, are unbalanced. Indeed, for A_7 , let

$$t = (12)(34), \quad U = \langle (567) \rangle.$$

The involution t fixes precisely the three points 5, 6, 7. Every element of $C_{A_7}(t)$ permutes these fixed points, so conjugation preserves U , the alternating group on those three points. Therefore, U is normal in $C_{A_7}(t)$ with order three, and consequently

$$1 \neq U \leq O(C_{A_7}(t)).$$

Hence A_7 is unbalanced, even though simplicity prevents U from being normal in A_7 itself.

The same argument works in A_{4a+3} . Take an involution t consisting of $2a$ disjoint transpositions and fixing the remaining three points. The even number of transpositions ensures that $t \in A_{4a+3}$. The subgroup of order three generated by a 3-cycle on the fixed points is normal in $C_{A_{4a+3}}(t)$, so this centralizer has nontrivial odd core.

1.1. An obstruction to an accepted route to UGT. Our work on formalizing CFSG requires making the hypotheses and dependencies of its many theorems explicit. In this effort, we have assembled a comprehensive errata list. Most of the errors are straightforward to correct: typographical errors, omitted hypotheses, and similar local issues. The difficulty considered here requires a change to an argument.

The first-generation proof of UGT is not contained in a single paper. Gilman–Solomon’s Theorem A [13, p. 59] proves the Unbalanced Group Conjecture for a group whose proper sections satisfy it, subject to Hypothesis S [13, p. 58], a list of standard-component problems, each asking that a group with a standard component of a specified type be identified. These problems were settled in separate papers over the following years, with the $\text{PSL}_3(4)$ case completed by Cheng and Held in 1985 [10, pp. 147–148]. No single reference, as far as we know, records the point at which Hypothesis S was fully settled. Gorenstein’s 1983 volume [16] develops the unbalanced-group argument; the historical overview in [17, pp. 40–41] describes the collective work on the related *B*₂-property. The pattern, a result announced modulo components completed later, is discussed in [3]. Walter’s 1986 memoir [27] gives a treatment organized around the signalizer functor method, and is recognized by Aschbacher [4, §4, p. 10] as a route to UGT and the *B*-conjecture. It is the treatment examined here.

Unexpectedly, our formalization found explicit counterexamples to a closure assertion in Walter's proof of UGT [27], leaving a key step of that argument unjustified. We explain the obstruction and develop an alternative argument for the simple-group case using Solomon's 1976 theory of maximal 2-components.

Remark 1.6 (Other routes to UGT). This difficulty does not obstruct CFSG. In particular, we have Gorenstein's account following an argument of Morton Harris [16, §4.5, p. 291]. There is also a proof of UGT through the second-generation work of Gorenstein, Lyons, and Solomon (GLS). Under the inductive hypothesis of Theorem 1.2, the nontrivial odd-order normal subgroup places the simple group in GLS's odd-type case. Their classification of that case [20, Part III, Chapter 12, Theorem 1.17] identifies it as a known group. Namely, the known-group analysis [19, Theorem 7.7.1], with these corrections [21], then gives the required UGT family restriction. This is a short deduction from substantial second-generation input, not a repair of the obstruction described below. A shorter deduction from the reductions used in this paper is recorded in Theorem 9.5.

We outline the proof route where the difficulty arises. The starting point is a nonabelian simple $\mathcal{K}$ -proper group G with an involution t such that $O(C_G(t)) \neq 1$. The goal is to show that G is in one of the families in Theorem 1.2. The reductions either identify G with one of the required groups or lead to a configuration involving two commuting involutions and a 2-component of one of their centralizers. A 2-component is a perfect subnormal subgroup whose quotient by its odd core is quasisimple; after further quotienting by the center, one obtains a nonabelian simple group (see Section 2).

The task is to find a 2-component whose structure and position in G allow a recognition theorem to identify G . To do this, the proof compares 2-components in different involution centralizers and selects one that is maximal with respect to these comparisons. Results of Thompson–Burgoyne and Aschbacher, together with the alternating-group analyses, handle many of the possibilities. The remaining cases require the work of Foote, Gilman–Solomon, Griess–Solomon, and Harris [2, 5, 9, 12, 13, 22, 23].

The difficulty arises in the comparison stage. The argument works with components whose associated simple quotients belong to a specified family. A later step requires a quotient obtained by taking an involution centralizer to remain in that family. The closure assertion used to justify this can fail. Namely, an involutory automorphism can have a centralizer with a component whose simple quotient lies outside the chosen family. Therefore, the inference placing the resulting component back in the family is unjustified.

This assertion is in John Walter's 1986 memoir [27], in the preparation for Part II, Proposition 5.9(ii). The proposition appears on p. 145, with its proof on pp. 145–148. The passage on p. 136 following (5.2) introduces the family $\mathfrak{L}_0$ consisting of the groups $\mathrm{PSL}_2(q)$ for odd prime powers q other than 9, Fermat primes, and Mersenne primes. A Fermat prime has the form $2^{2^n} + 1$ for an integer $n \geq 0$, and a Mersenne prime has the form $2^n - 1$ for an integer $n \geq 2$.

The obstruction comes from field involutions, automorphisms induced by the order-two automorphism of a quadratic extension of finite fields. For $\mathrm{PSL}_2(r^2)$, this automorphism raises each matrix entry to its r th power. The following proposition shows how a group in $\mathfrak{L}_0$ can have a field-involution centralizer whose component lies outside $\mathfrak{L}_0$.

Proposition 1.7 (Field-involution centralizers). *Let $r \geq 5$ be an odd prime power and let σ be the automorphism of $S = \mathrm{PSL}_2(r^2)$ induced by $x \mapsto x^r$ on matrix entries. Then we have*

$$C_S(\sigma) \cong \mathrm{PGL}_2(r), \quad E(C_S(\sigma)) = L_{2'}(C_S(\sigma)) \cong \mathrm{PSL}_2(r).$$

Here E and $L_{2'}$ denote the ordinary and 2-layers, defined in Section 2.

Proof. Put $k = \mathbf{F}_r$ and $F = \mathbf{F}_{r^2}$, the fields with r and r^2 elements. A projective matrix $[A] \in \mathrm{PGL}_2(F)$ is fixed by σ precisely when $A^\sigma = \lambda A$ for a scalar λ . Divide A by any nonzero entry a . Since $a^r = \lambda a$, the matrix A/a is fixed entrywise, and has entries in k . Hence $C_{\mathrm{PGL}_2(F)}(\sigma) = \mathrm{PGL}_2(k)$.

Every element of $k^\times$ is a square in $F^\times$. For example, if g generates $F^\times$, then $k^\times = \langle g^{r+1} \rangle$, and $r+1$ is even. Therefore, every projective matrix over k has square determinant over F , and the entire fixed group lies in $\mathrm{PSL}_2(F)$. This proves the centralizer assertion.

The subgroup $\mathrm{PSL}_2(r)$ is nonabelian simple, perfect, and normal of index two in $\mathrm{PGL}_2(r)$. Any odd normal subgroup of the latter maps trivially to this quotient of order two, hence lies in $\mathrm{PSL}_2(r)$ and is trivial. The derived subgroup is $\mathrm{PSL}_2(r)$, so this is its unique component and its unique 2-component. $\square$

Theorem 1.7 shows why the proposed closure fails. Start with $S = \mathrm{PSL}_2(49)$, which belongs to $\mathfrak{L}_0$. The number 49 is an odd prime power, is not 9, and is not a prime. The field involution fixes a subgroup isomorphic to $\mathrm{PGL}_2(7)$. The simple component of that subgroup is $\mathrm{PSL}_2(7)$, which does *not* belong to $\mathfrak{L}_0$, because 7 is a Mersenne prime. Therefore, taking the component of an involution centralizer can carry us outside the chosen family. The same calculation gives the following three examples; the middle column records the simple component of the field-involution centralizer.

Group in $\mathfrak{L}_0$	Centralizer component	Reason it is excluded
$\mathrm{PSL}_2(49)$	$\mathrm{PSL}_2(7)$	7 is a Mersenne prime
$\mathrm{PSL}_2(81)$	$\mathrm{PSL}_2(9)$	9 is excluded explicitly
$\mathrm{PSL}_2(289)$	$\mathrm{PSL}_2(17)$	17 is a Fermat prime

Here the involution is an automorphism of S . It can be realized as an element of the larger group $S \rtimes \langle \sigma \rangle$; it is not an involution belonging to S itself.

We explain where the closure assertion enters Walter's proof. The argument compares components in different involution centralizers and needs to show that a newly encountered component still belongs to a specified family. The *type* of a component is the isomorphism type of its nonabelian simple quotient, after removing its odd core and center. Knowing that two components have the same type transfers membership in a family only if one already knows that this common type belongs to that family.

On p. 146 of [27], the relevant family is denoted by $\mathcal{M}_2$ and consists of components of $\mathfrak{L}_0$ -type. Starting from a component L_t , Walter forms

$$L_{t,s} = L(C_{L_t}(s)),$$

the layer of the centralizer of the involution s in L_t , using his notation for the layer. In the branch $M_s^t \neq M_s$, that is, when conjugation by t sends M_s to a different component, the proof identifies the type of M_s with that of $L_{t,s}$ and uses this identification to place M_s in $\mathcal{M}_2$.

The missing justification is the assertion that the type of $L_{t,s}$ still belongs to $\mathfrak{L}_0$. Membership of the starting type in $\mathfrak{L}_0$ does not establish this. Indeed, the calculation above shows that taking an involution centralizer can produce an excluded type. Equality of the types of M_s and $L_{t,s}$ cannot supply the missing membership. For instance, if the resulting type is $\mathrm{PSL}_2(7)$, identifying another component with that type does not put either one back in $\mathfrak{L}_0$. The subsequent argument within $\mathcal{M}_2$ therefore requires an additional justification or a different comparison argument.

The scope of these counterexamples deserves emphasis.

Remark 1.8. The examples above refute the closure assertion used in the proof of Proposition II.5.9(ii). They do not provide groups that violate its conclusion. Namely, we have identified a gap in this inference, not a counterexample to the proposition itself or to UGT.

Here we give an alternative argument. Rather than assuming that the list $\mathfrak{L}_0$ survives every comparison, it keeps track of the actual components and the involutions used to compare them. Under suitable maximality and rank hypotheses, we prove that the relevant comparisons preserve the simple quotient. The precise statement is Theorem 2.3 below, which is the main theorem of this paper. Solomon's theory of maximal 2-components [26] supplies the method for this comparison argument. The maximality reductions come from Gilman–Solomon [13], and the final recognition uses the work of Foote and Harris as described in Section 9. We develop these arguments with the witnessing involutions and the passages through odd-core quotients made explicit.

Our description of the approach as first-generation refers to this component-comparison and recognition strategy. For a systematic exposition of component calculus and known-group calculations, we also cite the

later volumes of Gorenstein, Lyons, and Solomon [18, 19]. The date of an exposition should be distinguished from the provenance of the results it presents. Our GLS citations identify the precise statements used below; they do not constitute an appeal to the separate second-generation odd-type classification route described in Theorem 1.6.

Theorem 2.3 is proved in Sections 3–8, and we apply it to UGT in Section 9. Under its maximality and rank hypotheses, that theorem shows that the allowed comparisons preserve the simple top. In Section 9, we combine this control with the reductions of Gilman–Solomon and the recognition theorems of Foote and Harris to deduce the simple-group case of UGT. This is the route formalized by AxiomProver.

The same reductions also yield a shorter deduction of UGT, since the GS-maximality they establish already satisfies Foote’s hypothesis; see Theorem 9.5. On the other hand, Theorem 2.3 is also stronger than what is needed for UGT and we believe it to be of independent interest. Thus we choose to structure this paper around the proof of the full Theorem 2.3.

2. PRELIMINARIES

The comparison argument uses components, their simple quotients, and the involutions through which they are compared. We introduce these notions, state the comparison theorem, and collect its basic inputs. Throughout, all groups are finite. A group of order a power of two is a *2-group*. We write S_n and A_n for the symmetric and alternating groups on $\{1, \dots, n\}$, respectively. We use $O(X)$ for the largest normal subgroup of X of odd order, and $O_2(X)$ for its largest normal 2-subgroup, called its *2-core*. The trivial group and its identity element are both denoted by 1. For subgroups $A, B \leq X$, we have the commutator subgroup

$$[A, B] = \langle a^{-1}b^{-1}ab : a \in A, b \in B \rangle.$$

For an element b , write $[A, b] = \langle a^{-1}b^{-1}ab : a \in A \rangle$. Write $X' = [X, X]$, and define the derived series by $X^{(0)} = X$ and $X^{(i+1)} = (X^{(i)})'$. Its eventual constant term is the *perfect residual* $X^{(\infty)}$. A group X is *perfect* if $X' = X$ and *solvable* if $X^{(\infty)} = 1$. Every perfect subgroup of X is contained in $X^{(\infty)}$.

A subgroup J is *subnormal* in X if there is a chain $J = J_0 \trianglelefteq J_1 \trianglelefteq \dots \trianglelefteq J_n = X$. We write $Z(X)$ for the center of X . A group J is *quasisimple* if it is perfect and $J/Z(J)$ is nonabelian simple. A *covering group* of a nonabelian simple group T is a quasisimple group Q with $Q/Z(Q) \cong T$. It is a *perfect double cover* if $|Z(Q)| = 2$. An *ordinary component* of X is a subnormal quasisimple subgroup. Distinct ordinary components commute; their product is the *layer* $E(X)$. A group that is a product of commuting quasisimple subgroups is *semisimple*. Such a product is a central product: each factor meets the product of the others in its center.

A *2-component* of X is a perfect subnormal subgroup J for which $J/O(J)$ is quasisimple. A *local 2-component* of G is a 2-component of $C_G(a)$ for some involution a . When distinguishing these from components of G itself, we call the latter *global components*. The product of the 2-components of X is the *2-layer* $L_{2'}(X)$. Each 2-component is normal in the 2-layer. Distinct 2-components commute modulo $O(X)$, although they need not commute in X . Define

$$Z^*(X)/O(X) = Z(X/O(X)) \quad \text{and} \quad \tau(X) = X/Z^*(X).$$

Thus, $Z^*(X)$ is the inverse image of $Z(X/O(X))$. If J is a 2-component, then $\tau(J)$ is nonabelian simple. We call $\tau(J)$ its *simple top*, and the isomorphism class of $\tau(J)$ its *type*. For a group Q , a group M is *odd-by- Q* if $M/O(M) \cong Q$.

The group $O^{2'}(X)$ is the smallest normal subgroup of X with odd-order quotient. Equivalently, it is generated by the elements of X whose orders are powers of two. Such elements are called *2-elements*. The symbols $O^{2'}(X)$ and $O_{2'}(X) = O(X)$ have different meanings.

An *elementary abelian 2-group* has the form C_2^d ; the *2-rank* $m_2(X)$ is the largest rank of such a subgroup of X . A *fourgroup* is a group isomorphic to C_2^2 , and a *line* in an elementary abelian 2-group is a subgroup of order two. We write $\text{Inv}(X)$ for the set of involutions of X and $U^\# = U \setminus \{1\}$. For an abelian 2-group

P , let $\Omega_1(P) = \{x \in P : x^2 = 1\}$. We write $\text{Syl}_2(X)$ for the set of Sylow 2-subgroups of X and $|X|_2$ for the largest power of two dividing $|X|$.

For subsets $A, B \subseteq X$, write $C_X(A, B) = C_X(A) \cap C_X(B)$, allowing either argument to be a single element. Conjugation is written $x^g = g^{-1}xg$ and $M^g = g^{-1}Mg$. We write $N_X(M)$ for the normalizer of M in X . A 2-local subgroup of X is $N_X(Q)$ for a nontrivial 2-subgroup $Q \leq X$. For subgroups A, B , the notation $\langle A^B \rangle$ denotes the group generated by A^b for $b \in B$. In particular, $\langle A^X \rangle$ is the normal closure of A in X when $A \leq X$.

For $M \leq X$, put

$$\mathcal{C}_X(M) = \{g \in N_X(M) : [M, g] \leq O(M)\}.$$

This is the kernel of the action of $N_X(M)$ on $M/O(M)$. For a perfect group M such that $M/O(M)$ is quasisimple, and a 2-subgroup $U \leq \mathcal{C}_G(M)$, write

$$M_U = C_M(U)^{(\infty)} \quad \text{and} \quad M_u = M_{\langle u \rangle}.$$

We will repeatedly use that $M_U O(M) = M$; this is proved in Theorem 2.4. We say that a subgroup $D \leq M$ covers $M/O(M)$ if $DO(M) = M$.

The symbols $\text{Aut}(X)$ and $\text{Inn}(X)$ denote the automorphism group and its subgroup of inner automorphisms, and $\text{Out}(X) = \text{Aut}(X)/\text{Inn}(X)$. A finite group is *nilpotent* if it is the direct product of its Sylow subgroups. The *Fitting subgroup* $F(X)$ is the largest nilpotent normal subgroup of X . The *generalized Fitting subgroup* is $F^*(X) = F(X)E(X)$. We use the standard theorem

$$C_X(F^*(X)) \leq F^*(X)$$

from [18, Theorem 3.6]. In particular, if $O(X) = 1$, then $F(X) = O_2(X)$. Finally, $O_{2',2}(X)$ is given by

$$O_{2',2}(X)/O(X) = O_2(X/O(X)).$$

2.1. Comparison relations. To compare the 2-components arising in different involution centralizers, we adapt the correspondence of Gilman–Solomon (GS) [13, p. 56] and the maximality framework of Foote [12]. Our GS relation uses a containment in place of Gilman–Solomon’s layer equality; Theorem 3.4 explains the passage between these forms where it is needed. The formulations below record the witnessing involutions explicitly and distinguish the pointed and mirror steps used in our argument.

Definition 2.1. An *occurrence* is a pair (a, J) in which a is an involution and J is a 2-component of $C_G(a)$. Suppose that (a, J) and (b, K) are occurrences, a and b commute, and $b \in \mathcal{C}_G(J)$. A *GS correspondence*, with these witnesses, is the inclusion

$$(1) \quad J_b \leq \langle K, K^a \rangle.$$

It is *isomorphic* if $K^a = K$ and $J_b O(K) = K$. The fixed-layer lemma identifies J_b with $L_{2'}(C_J(b))$.

A *pointed Foote step* $(a, J) \rightarrow_F (b, K)$ requires the same hypotheses, $a \neq b$, and the equality

$$(2) \quad \left\langle \left(O_{2'}(C_J(b)O(C_G(b))) \right)^{L_{2'}(C_G(b))} \right\rangle = \langle K, K^a \rangle.$$

An occurrence is *pointed-maximal* if every finite chain of pointed Foote steps starting there preserves the isomorphism type of its simple top. The endpoint occurrence of each step must be the starting occurrence of the next. A local 2-component is *GS-maximal* if every forward chain of GS correspondences preserves its simple top, with the witnessing involutions allowed to be chosen independently at successive steps.

A *mirror step* $J \rightarrow_M K$ has commuting involutions z, s , with J a 2-component of $C_G(z)$, $J^s \neq J$, K a 2-component of $C_G(s)$, and a 2-component D of $C_G(z) \cap C_G(s)$ satisfying

$$D \leq \langle J, J^s \rangle, \quad D \leq \langle K, K^z \rangle.$$

Write $J \rightsquigarrow K$ for a finite chain of GS correspondences, mirror steps, and conjugations, allowing the empty chain. A local 2-component is *mirror-maximal* if every continuation preserves its simple top.

A subgroup of a direct product of groups is *diagonal* on a set of factors if it projects isomorphically onto each of those factors and trivially onto the others. For example, the automorphism exchanging the factors of $T \times T$ fixes the diagonal $\{(x, x) : x \in T\}$, which is isomorphic to either factor but is neither. The *support* of a subgroup of a direct product is the set of factors on which it has nontrivial projection.

The small simple types remaining in the UGT reduction lead to the following hypotheses. Their rank condition will be verified in Section 9.

Hypothesis 2.2. The group G is nonabelian simple and $\mathcal{K}$ -proper. Fix a simple group T isomorphic to A_7 or to $\text{PSL}_2(q)$ for an odd prime power $q \geq 5$, and put

$$\mathcal{F}_T = \{A \leq G : \tau(A) \cong T \text{ and } (z, A) \text{ is pointed-maximal for some } z \in \text{Inv}(G)\}.$$

Assume that every member of $\mathcal{F}_T$ has 2-rank two.

For the comparison argument in the remainder of Sections 2–8, we assume Theorem 2.2, with the following exceptions. The fixed-layer lemma Theorem 2.4 holds under its own stated hypotheses, and the known-group assertion in Theorem 2.7 requires only that H be a $\mathcal{K}$ -group. The correspondence lemma Theorem 3.1, the rank-increase lemma Theorem 3.3, and the maximality criterion Theorem 3.4 do not require the assumption on $\mathcal{F}_T$; we retain the assumption that the ambient G is nonabelian simple and $\mathcal{K}$ -proper, together with the hypotheses stated in each lemma. In particular, the rank-increase lemma requires $m_2(J) \geq 2$ for its source component, and the maximality criterion requires the stated local B -property. These results are therefore available in Section 9 before the family-rank hypothesis is established.

Under the full hypotheses of Theorem 2.2, pointed maximality controls the larger collection of comparisons, including mirror steps.

Theorem 2.3 (Preservation of simple tops). *Assume Theorem 2.2. If $A \in \mathcal{F}_T$ and $A \rightsquigarrow J$, then $\tau(J) \cong T$. In particular, every member of $\mathcal{F}_T$ is mirror-maximal.*

We first show that every member of $\mathcal{F}_T$ can be reached from a GS-maximal component of type T . We then exclude a GS-maximal component of type T that is interchanged with another component by a commuting involution. The latter argument uses a family of subgroups closed under isomorphic correspondences, a signalizer construction, and an embedding theorem.

2.2. Component calculus. The following facts govern passage to odd-core quotients and to centralizers of 2-subgroups. We use the component calculus as presented by Gorenstein, Lyons, and Solomon [18, §5], together with the coprime-action results in Gorenstein’s *Finite Groups* [15, Chapter 5]. If $R = O(X)$, the quotient map gives a bijection between the 2-components of X and the ordinary components of X/R . The inverse image of a component has a unique 2-component, namely its perfect residual. We call this subgroup its *canonical perfect lift*. A subnormal subgroup remains subnormal after intersection with another subgroup, and the image of a 2-component under a surjective homomorphism with odd kernel is a 2-component.

We also use *$L_{2'}$ -balance*: if a 2-group U acts on X , then $L_{2'}(C_X(U)) \leq L_{2'}(X)$. In its comparison form, if a, b are commuting involutions and J is a 2-component of $C_G(a)$ with $b \in C_G(J)$, then there is a 2-component K of $C_G(b)$ such that $J_b \leq \langle K, K^a \rangle$. See [18, Theorem 5.24] and [13, p. 56]. Thus, the comparison relation in Theorem 2.1 always has a target for these witnesses.

An action of U on R is *coprime* if $|U|$ and $|R|$ are relatively prime. For such an action, we use $R = [R, U]C_R(U)$ and $[R, U, U] = [R, U]$. If U is a fourgroup and R has odd order, then

$$R = \langle C_R(u) : u \in U^\# \rangle.$$

We use fixed-point lifting. Namely if a 2-group U acts on X and preserves an odd-order normal subgroup R , then

$$C_X(U)R/R = C_{X/R}(U).$$

This follows from conjugacy of complements to R in $R \rtimes U$ by the Schur–Zassenhaus theorem [15].

By the odd-order theorem [11], odd-order groups are solvable. Consequently, an odd-order perfect group is trivial. These facts also give a useful uniqueness statement: if $P \trianglelefteq X$ is perfect and X/P is solvable, then $X^{(\infty)} = P$. If an automorphism α of a perfect group P acts trivially on $P/Z(P)$, then it is the identity. Indeed, the map $x \mapsto x^{-1}x^\alpha$ is a homomorphism from P to the abelian group $Z(P)$ and is therefore trivial. Also, if a 2-group U acts on a group X of even order, then $C_X(U)$ has even order: every nontrivial U -orbit has even size.

We use the three-subgroups lemma in the following form. Write $[A, B, C] = [[A, B], C]$. If $R \trianglelefteq \langle A, B, C \rangle$ and two of $[A, B, C]$, $[B, C, A]$, and $[C, A, B]$ are contained in R , then so is the third. Fixed-point lifting supplies the perfect subgroups used in our comparisons: they retain the full quotient by the odd core.

Lemma 2.4 (Fixed layers). *Let M be perfect with $M/O(M)$ quasisimple, and let a 2-group U normalize M and act trivially on $M/O(M)$. Then we have*

$$C_M(U)O(M) = M, \quad M_U/O(M_U) \cong M/O(M), \quad C_M(U)/M_U \text{ has odd order.}$$

In particular $M_U = O^{2'}(C_M(U))$ and it is the unique 2-component of $C_M(U)$. If M is a 2-component of $C_G(a)$ and u is an involution commuting with a and in $C_G(M)$, then M_u is a 2-component of $C_G(a) \cap C_G(u)$.

Proof. Put $R = O(M)$ and $C = C_M(U)$. Fixed-point lifting gives $CR = M$. Since M/R is perfect, $M_U R = M$, and hence

$$M_U/(M_U \cap R) \cong M/R, \quad C = M_U(C \cap R).$$

The group M/R has trivial odd core, so $O(M_U) = M_U \cap R$. The quotient C/M_U has odd order; conversely, the perfect group M_U has trivial image in $C/O^{2'}(C)$ by the odd-order theorem. Thus, $M_U = O^{2'}(C)$, and the component correspondence gives uniqueness. Finally, intersecting a subnormal chain for M in $C_G(a)$ with $C_G(u)$ makes $C_M(u)$, and hence M_u , subnormal in $C_G(a) \cap C_G(u)$. $\square$

We first record the covering-group facts independently of the family-rank hypothesis. If Q is perfect, $O(Q) = 1$, and $Q/Z(Q) \cong T$, where $T \cong A_7$ or $\text{PSL}_2(q)$ for odd $q \geq 5$, then Q is either T or its perfect double cover. In the latter case its Sylow 2-subgroups have a unique involution, so $m_2(Q) = 1$; in the former case $m_2(Q) = 2$. The odd central factors in the covers of A_6 and A_7 are excluded by $O(Q) = 1$. These covering-group and Sylow-subgroup calculations are recorded in [19, Section 6.1]. The next lemma uses the family-rank hypothesis to exclude the double-cover possibility.

Lemma 2.5. *If $A \in \mathcal{F}_T$, then $A/O(A) \cong T$. The same conclusion holds for every GS-maximal local 2-component of type T .*

Proof. Quotienting by an odd-order normal subgroup preserves 2-rank: the inverse image of an elementary abelian 2-subgroup has a Sylow 2-subgroup mapping isomorphically onto it. Apply the preceding covering-group facts to $Q = A/O(A)$. The rank-two hypothesis in Theorem 2.2 excludes the double cover, so $A/O(A) \cong T$. Every pointed step is a GS correspondence, directly from its defining equality and Theorem 2.4. Thus, a GS-maximal component of top T is pointed-maximal at every occurrence and belongs to $\mathcal{F}_T$. $\square$

The involutions and fourgroups of T supply the configurations used in the comparison and signalizer arguments.

Lemma 2.6. *The group T has one conjugacy class of involutions, every involution belongs to a fourgroup, and every involution centralizer is solvable. Its outer automorphism group is solvable. If $V \leq T$ is a fourgroup, then $N_T(V)$ acts transitively on $V^\#$.*

Proof. A dihedral group of order $2n$ is the group with generators r, s and relations $r^n = s^2 = 1$ and $srs = r^{-1}$; these groups are solvable. For A_7 , the involutions are the double transpositions. They form one conjugacy class, and their centralizers are subgroups of the product of a group of order eight and S_3 , hence are solvable. For $\text{PSL}_2(q)$ with q odd, the matrix description gives one involution class, dihedral Sylow 2-subgroups, and dihedral involution centralizers. These Sylow subgroups have order at least four.

The Sylow 2-subgroups of A_7 are also dihedral of order eight, so every involution of either group belongs to a fourgroup. Finally, $\text{Out}(A_7)$ has order two. Writing $q = p^f$ with p prime, $\text{Out}(\text{PSL}_2(p^f))$ is a product of a cyclic group of order two and a cyclic group of order f .

It remains to verify the assertion about V . A fourgroup in A_7 has either one orbit of size four and three fixed points, or has three orbits of size two and one fixed point. In the first case the alternating group on the four points supplies an element inducing a 3-cycle on $V^\#$. In the second case an even permutation cyclically permuting the three two-point orbits does so.

Now let $T = \text{PSL}_2(q)$, and write I for the identity matrix. The inverse image of V in the group $\text{SL}_2(q)$ of determinant-one matrices has generators i, j satisfying $i^2 = j^2 = -I$ and $ij = -ji$. Put $k = ij$ and $w = (I + i + j + k)/2$. The displayed relations give $\det(w) = 1$ and show that conjugation by w cyclically permutes the projective images of i, j, k . Thus, the image of w normalizes V and acts transitively on $V^\#$. $\square$

The inductive hypothesis supplies the following local B -property inside proper subgroups. Its use does not require the B -property for G .

Lemma 2.7 (B -property for known groups). *Every $\mathcal{K}$ -group H satisfies*

$$(3) \quad [L_2(N_H(Q)), O(N_H(Q))] \leq O(H)$$

for every nontrivial 2-subgroup $Q \leq H$. If $O(H) = 1$, the 2-components of its 2-local subgroups are ordinary. Consequently, every proper subgroup of the ambient G satisfies (3), and the proper-subgroup B hypothesis in Gilman–Solomon’s Lemma 3.5 is available.

Proof. Suppose first that $O(H) = 1$. The known-group fixed-point result [18, Lemma 5.28], with its hypotheses supplied by [19, Theorem 7.1.9], says that every 2-component of $C_H(Q)$ is quasisimple. Put $M = N_H(Q)$ and let J be a 2-component of M . Modulo $O(M)$ its component image centralizes the normal 2-subgroup $QO(M)/O(M)$. Hence

$$[J, Q] \leq O(M) \cap Q = 1.$$

Intersecting a subnormal chain with $C_H(Q)$ makes J a 2-component of $C_H(Q)$, so the cited result makes J quasisimple. It is therefore an ordinary component of M . Ordinary components centralize the odd core of M , so the commutator in (3) is trivial in this case.

For arbitrary H , quotient by $R = O(H)$. The map

$$N_H(Q) \longrightarrow N_{H/R}(QR/R)$$

is onto with odd kernel. Indeed, an element normalizing QR/R can be adjusted by an element of R to normalize the Sylow 2-subgroup Q of QR . The image of a 2-component under this odd-kernel map is a 2-component, and the image of the odd core is contained in the odd core of the image. The case with trivial odd core gives (3).

Finally every nontrivial 2-local subgroup of a nonabelian simple group is proper, so $\mathcal{K}$ -properness supplies the hypotheses. The right-hand side in (3) is $O(H)$; we have not asserted the global B -property for the unknown group G . $\square$

3. COMPARING COMPONENTS

The maximality argument in this section adapts Gilman–Solomon’s rank-increase lemma and maximality criterion [13, Lemmas 3.4–3.5]. Our goal is to reach each pointed-maximal component from a GS-maximal one. We first relate the GS and pointed Foote correspondences with witnesses fixed, then use mirror steps to reverse the comparisons needed for this choice.

Lemma 3.1. *A GS correspondence with distinct witnesses a, b is a pointed Foote step with the same witnesses. Conversely, every pointed Foote step is a GS correspondence. A GS correspondence with $a = b$ has $J = K$.*

Proof. Put $D = J_b$, $E = L_{2'}(C_G(b))$, $R = O(C_G(b))$, and $W = \langle K, K^a \rangle$. The GS condition is $D \leq W$. The group D is a 2-component of $C_G(a) \cap C_G(b)$, and $C_J(b)/D$ has odd order. Modulo R , its image has noncentral projection on a component represented by W . If $K^a \neq K$, then a fixes D pointwise and interchanges the two factors, so both projections are noncentral. Taking normal closures in the layer, and then the unique 2-component lifts, gives $\langle D^E \rangle = W$. In particular $C_J(b)$ normalizes W . The group R also normalizes every 2-component of $C_G(b)$, hence normalizes W . The quotient $C_J(b)RW/W$ has odd order, since both the image of $C_J(b)/D$ and the normal image of R have odd order. Consequently

$$F = O^{2'}(C_J(b)R) \leq W.$$

On the other hand $D \leq F$, because $D = O^{2'}(D)$. Thus, $\langle F^E \rangle = W$, the required pointed equality.

The converse follows from $D \leq F$. If $a = b$, then $D = J$ and $K^a = K$, so $J \leq K$; two 2-components of the same group cannot be properly nested. $\square$

Equality of the simple quotients permits a return to the source, using a mirror step when two target components are interchanged.

Lemma 3.2. *Suppose $(a, J) \rightarrow_F (b, K)$ and $J/O(J) \cong K/O(K) \cong T$. Then $K \rightsquigarrow J$. If the correspondence is isomorphic, its reverse is also isomorphic.*

Proof. Let $D = J_b$. If $K^a \neq K$, the common-centralizer component D supplies the reverse mirror step, with witnesses (b, a) :

$$D \leq \langle K, K^a \rangle, \quad D \leq J = \langle J, J^b \rangle.$$

If $K^a = K$, then $D \leq K$ and $D \cap O(K) \leq O(D)$. Since $D/O(D) \cong T \cong K/O(K)$, comparison of orders gives $DO(K) = K$ and $D \cap O(K) = O(D)$. Thus, a acts trivially on $K/O(K)$. Intersecting a subnormal chain for D in the common centralizer with $C_K(a)$ shows that D is its unique 2-component. Therefore $K_a = D \leq J$, which is the reverse GS correspondence. Moreover $DO(J) = J$, so this reverse correspondence is isomorphic. $\square$

3.1. Increasing the centralizer rank. The maximal choices below rely on an increase in quotient order or centralizer rank. Starting from a nonisomorphic correspondence, we can obtain such an increase while keeping the source witness a fixed, although the second witness may change. This adapts [13, Lemma 3.4, pp. 68–69], using only $\langle a, b \rangle \leq S$.

Lemma 3.3 (Rank increase). *Let J be a 2-component of $C_G(a)$ with $m_2(J) \geq 2$. Let*

$$S \in \text{Syl}_2(C_G(a) \cap N_G(J)), \quad P = S \cap C_G(J), \quad b \in \text{Inv}(P).$$

If a correspondence from (a, J) at b is not isomorphic, there are $e \in \text{Inv}(P)$ and a correspondent K at (a, e) with $C_S(b) \leq C_S(e)$ such that either

$$|K/O(K)| > |J/O(J)|,$$

or

$$|K/O(K)| = |J/O(J)|, \quad m_2(C_G(e) \cap C_G(K)) > m_2(C_G(a) \cap C_G(J)).$$

Proof. Choose $e \in \text{Inv}(P)$ admitting a nonisomorphic correspondent and such that $Q = C_S(e)$ contains $C_S(b)$ and is maximal among these centralizers. Write K for a nonisomorphic correspondent at e . The correspondence gives $|K/O(K)| \geq |J/O(J)|$; see [12, Lemma 2.1(iv)]. If the inequality is strict, we are done. Suppose equality holds, and put $T_0 = J/O(J)$. Then $K^a \neq K$: otherwise $J_e \leq K$ and order comparison makes the correspondence isomorphic.

Set $M = C_G(e)$ and use bars for images modulo $O(M)$. The two components give

$$\overline{\langle K, K^a \rangle} = \bar{K} \times \bar{K}^a, \quad \bar{K} \cong T_0.$$

These assertions and the diagonal description of $\bar{J}_e$ follow from the equal-order case of the correspondence calculation [13, Lemma 3.4, proof]. To see why the product is direct, a nontrivial common center would contain an involution fixed by a and hence in the kernel of the diagonal map. This would make $J_e/O(J_e)$ a

proper central quotient of $K/O(K)$, contrary to equality of orders. Put $P_0 = P \cap Q = C_P(e)$. This group centralizes $J_e/O(J_e)$, and therefore its permutation action on the two factors has kernel

$$R = C_{P_0}(\bar{K} \times \bar{K}^a), \quad P_0 = \langle a \rangle \times R.$$

Here a is central in S , and the assertion about the kernel follows because an automorphism fixing both factors and the diagonal pointwise fixes both factors pointwise. The opposite factor $\bar{K}^a$ and $\bar{R}$ centralize $\bar{K}$ and intersect trivially: a centralizes $\bar{R}$ and interchanges the two direct factors. Lifting 2-subgroups through the odd kernel gives

$$m_2(C_G(e) \cap C_G(K)) \geq 2 + m_2(R) > m_2(P_0).$$

Thus, it suffices to show $m_2(P_0) = m_2(P)$.

Suppose otherwise. Since $P \trianglelefteq S$, the product PQ is a 2-group, and $P_0 < P$ implies $Q < PQ$. Every proper subgroup of a finite 2-group is properly contained in its normalizer. Applying this fact gives

$$N_{PQ}(Q) > Q, \quad N_{PQ}(Q) = N_P(Q)Q.$$

Consequently $N_P(Q) > P_0$, and $P_0 \trianglelefteq N_P(Q)$. Choose $w \in N_P(Q) \setminus P_0$ whose image in $N_P(Q)/P_0$ is an involution. Then $w \notin Q$ and $w^2 \in P_0$.

We have $|P_0| \geq 4$, since a, e are distinct commuting involutions. If $|P_0| = 4$, use the elementary fact from [13, Lemma 3.4]: a finite 2-group with an involution whose centralizer has order four has 2-rank at most two. This contradicts $m_2(P_0) < m_2(P)$. Thus, $|P_0| \geq 8$. Both R and R^w have index two in P_0 , so their intersection is nontrivial. Also $R \trianglelefteq Q$: the group Q normalizes J_e , hence its normal closure $\langle K, K^a \rangle$ in $L_{2'}(M)$, and therefore the kernel of its action on the quotient in P_0 . Since $w^2 \in P_0$, we also have $R \cap R^w \trianglelefteq \langle Q, w \rangle$. Choose an involution

$$z \in (R \cap R^w) \cap Z(\langle Q, w \rangle).$$

Then $C_S(z)$ properly contains Q . By maximality of Q , every correspondence from (a, J) at z is isomorphic; let V be its single target.

Put $J_0 = J_{\langle e, z \rangle}$. Since z acts trivially on both factors modulo their odd cores, fixed-point lifting shows that $C_{\langle K, K^a \rangle}(z)^{(\infty)}$ still has quotient $T_0 \times T_0$. It is the product of K_z and $(K^a)_z$, 2-components of $C_G(e) \cap C_G(z)$ by Theorem 2.4. The group J_0 projects diagonally onto both factors, so its normal closure in that fixed layer is the whole fixed layer. By $L_{2'}$ -balance this fixed layer lies in $L_{2'}(C_G(z))$. But $J_0 \leq J_z$, and the normal closure of J_z in $L_{2'}(C_G(z))$ is the single component V . This would place an odd-by- $(T_0 \times T_0)$ group inside V , although $V/O(V) \cong T_0$, an impossibility. Hence $m_2(P_0) = m_2(P)$. Finally P is a Sylow 2-subgroup of $C_G(a) \cap C_G(J)$, giving the stated strict rank increase. $\square$

3.2. Choosing a GS-maximal component. To obtain GS-maximality, it suffices to check correspondences at involutions in a chosen Sylow subgroup and a Sylow condition at their targets. The rank increase will make this criterion applicable.

Lemma 3.4 (A criterion for GS-maximality). *Assume the proper-subgroup B-property of Theorem 2.7. Let a be an involution, let J be a 2-component of $C_G(a)$, let $S \in \text{Syl}_2(C_G(a) \cap N_G(J))$, and put $P = S \cap C_G(J)$. Suppose every correspondence from (a, J) at an involution $b \in P$ is isomorphic. Suppose also that, for $b \in \text{Inv}(C_P(S))$ and its target K ,*

$$S \in \text{Syl}_2(C_G(b) \cap N_G(K)).$$

Then every edge of every forward chain of GS correspondences beginning at J is isomorphic. In particular J is GS-maximal.

Proof. The proof of [13, Lemma 3.5, pp. 69–70] gives the assertion for their stronger correspondence

$$J_b = L_{2'}(C_{\langle K, K^a \rangle}(a)).$$

Our assumptions imply theirs, including the proper-subgroup B-property used in their Lemma 3.3.

Suppose X is reached from J by such isomorphic correspondences, and consider one of our correspondences from (c, X) to (d, M) . Balance supplies a stronger correspondence to (d, N) , which is isomorphic

by the cited result. Thus, $X_d O(N) = N = N^c$, whereas $X_d \leq \langle M, M^c \rangle$. Modulo $O(C_G(d))$, the image of X_d is the component represented by N . Since it lies in the product of the components represented by M and M^c , we have $N = M$ or $N = M^c$. The equality $N^c = N$ forces $M = N$ in either case. Induction on the chain length now proves the assertion for our relation. $\square$

Maximizing centralizer rank and then Sylow order supplies a GS-maximal predecessor for each member of $\mathcal{F}_T$.

Lemma 3.5 (GS-maximal predecessors). *For each $A \in \mathcal{F}_T$, there is a GS-maximal 2-component Y of type T with $Y \rightsquigarrow A$.*

Proof. Choose a pointed-maximal occurrence (z, A) . Let $\mathcal{R}$ be its finite closure under pointed Foote steps and simultaneous conjugations of both entries. Each pair in $\mathcal{R}$ remains pointed-maximal of type T , hence has its component in $\mathcal{F}_T$. The rank hypothesis and Theorem 2.5 give $J/O(J) \cong T$ for every $(a, J) \in \mathcal{R}$. Theorem 3.2, together with inverse conjugations, gives $J \rightsquigarrow A$ for all these pairs.

Choose $(a, Y) \in \mathcal{R}$ maximizing lexicographically

$$\rho(a, Y) = m_2(C_G(a) \cap C_G(Y)), \quad \nu(a, Y) = |C_G(a) \cap N_G(Y)|_2.$$

Let $S \in \text{Syl}_2(C_G(a) \cap N_G(Y))$ and $P = S \cap C_G(Y)$. Any GS target with this fixed source occurrence either has the same source and target or belongs to $\mathcal{R}$ by Theorem 3.1. Theorem 3.3 and maximality of ρ therefore make every correspondence at $b \in \text{Inv}(P)$ isomorphic.

If $b \in \text{Inv}(C_P(S))$, then $b \in Z(S)$. The group S normalizes Y_b and its unique containing component K , so $S \leq C_G(b) \cap N_G(K)$. Also P acts trivially on $Y_b/O(Y_b)$, which covers $K/O(K)$. Hence $\rho(b, K) \geq \rho(a, Y)$. The pair (b, K) belongs to $\mathcal{R}$, unless it is the original pair. The first maximum forces equality of ranks, and the second forces $S \in \text{Syl}_2(C_G(b) \cap N_G(K))$. Theorem 3.4 proves the result. $\square$

3.3. A family closed under isomorphic correspondences. Suppose a commuting involution interchanges two components, one of which is GS-maximal. For the contradiction argument, we need a conjugacy-invariant family containing this pair and closed under isomorphic correspondences.

Proposition 3.6. *Suppose some GS-maximal 2-component of type T has a distinct conjugate under an involution commuting with its witnessing involution. Then there is a conjugacy-invariant family $\mathcal{S}$ of actual local 2-components such that*

- (1) $J/O(J) \cong T$ for every $J \in \mathcal{S}$;
- (2) every forward GS correspondence from every occurrence of a member of $\mathcal{S}$ is isomorphic, with target in $\mathcal{S}$;
- (3) some $Y, Y^s \in \mathcal{S}$ are distinct 2-components of $C_G(z)$, where s, z are commuting involutions.

Proof. For this proof, call (a, J, u) an *interchanged triple* when J is GS-maximal of type T , J is a 2-component of $C_G(a)$, and u is an involution commuting with a with $J^u \neq J$. Every forward GS target of such a J is again GS-maximal of type T . Moreover, GS maximality implies pointed maximality at every occurrence by Theorem 3.1. Thus, these components lie in $\mathcal{F}_T$, and the family rank hypothesis gives $J/O(J) \cong T$.

Maximize $\rho(a, J)$ over all interchanged triples, and call the maximum ρ_0 . Choose a maximizing triple (z, Y, s) . If any occurrence (a, J) of a GS-maximal component of type T has $\rho(a, J) \geq \rho_0$, then every correspondence from it is isomorphic. Indeed, for a nonisomorphic correspondence choose the Sylow group in Theorem 3.3 to contain its two witnesses. That lemma gives a target K at e with $K^a \neq K$ and $\rho(e, K) > \rho_0$. The triple (e, K, a) is an interchanged triple, contrary to the definition of ρ_0 .

Isomorphic correspondences reverse by Theorem 3.2. Let $\mathcal{I}(Y)$ be the set of local 2-components joined to Y by a finite chain of isomorphic GS correspondences, where the witnessing involutions may change between edges. Reversibility makes this an equivalence class; every member is GS-maximal of type T . Optimize $(\rho(a, J), \nu(a, J))$ over all $J \in \mathcal{I}(Y)$ and all involutions a for which J is a 2-component of $C_G(a)$. Its first coordinate is at least ρ_0 . The preceding paragraph gives the isomorphic correspondences needed

for Theorem 3.4; exactly the Sylow-order argument in Theorem 3.5 supplies the remaining hypothesis. Thus, every edge in every forward GS chain from this optimized J is isomorphic.

There is an isomorphic path from J back to Y . Appending any forward chain from Y shows that all of its edges are isomorphic. Define

$$\mathcal{S} = \{K^g : g \in G, \quad Y = J_0 \longrightarrow_{\text{GS}} J_1 \longrightarrow_{\text{GS}} \cdots \longrightarrow_{\text{GS}} J_n = K \text{ for some } n \geq 0\}.$$

The witnesses in this definition are chosen separately for each edge. Conjugation preserves the correspondence relation, so the same isomorphic-transport assertion holds from every occurrence of every member. Both Y and Y^s belong to $\mathcal{S}$, and their original interchanged triple supplies the last assertion. $\square$

4. MARKED SUBGROUPS

To carry the comparison information into proper subgroups, we enlarge our family to include perfect subgroups detected by its components. These will be called marked subgroups. This construction organizes the component comparisons needed for the application of Solomon's method [26]; the defining marking condition keeps its connection to the chosen family explicit.

Fix the conjugacy-invariant family $\mathcal{S}$ supplied by Theorem 3.6. Its members have quotient T modulo their odd cores, and every forward GS correspondence from every occurrence of a member is isomorphic and has its target in $\mathcal{S}$. Recall that $M_U = C_M(U)^{(\infty)}$ for $U \leq \mathcal{C}_G(M)$. By Theorem 2.4, this subgroup covers $M/O(M)$. The construction uses the following control on normal closures: under the stated hypotheses, the image of W is a single component.

Lemma 4.1. *Let $\pi : A \rightarrow A/O(A)$, and let $P \leq W \leq A$ be perfect. Suppose $\pi(P) = X$ is an ordinary component of $A/O(A)$ and*

$$W = \langle P^W \rangle R \quad \text{for some odd-order normal subgroup } R \trianglelefteq W.$$

Then $\pi(W) = X$. If $X \cong T$, then $W = PO(W)$ and $W/O(W) \cong T$. Furthermore, a perfect odd-by- T subgroup containing a simple component $X \cong T$ of its ambient group equals X .

Proof. Set $V = \pi(W)$ and $N = \langle X^V \rangle$. The quotient V/N is perfect and odd, hence trivial by the odd-order theorem. Each conjugate of X is an ambient component, so $V = N \leq E(A/O(A))$. Components are normal in the layer; consequently V normalizes X and $V = X$. When X is simple, $W \cap O(A) = O(W)$, which proves the assertions about W .

For the last assertion, write W for the perfect odd-by- T subgroup. The simple subgroup X injects into and covers $W/O(W)$, so $W = XO(W)$. Its normal closure $N = \langle X^W \rangle$ satisfies $W = NO(W)$; perfectness and the odd-order theorem give $W = N$. Again N lies in the ambient layer, in which X is normal. Thus, $W = X$. $\square$

A marking records agreement with a member of $\mathcal{S}$ in an involution centralizer after its odd core has been removed.

Definition 4.2. A perfect subgroup $M \leq G$ is *marked* if $M/O(M) \cong T$ and there are an involution $a \in \mathcal{C}_G(M)$ and a member $S_a \in \mathcal{S}$ such that S_a is a 2-component of $C_G(a)$ and

$$(4) \quad M_a O(C_G(a)) = S_a O(C_G(a)).$$

Write $\mathcal{F}$ for this conjugacy-invariant family. The involution a is a *marking witness*. Every member of $\mathcal{S}$ is marked, using any witnessing involution at which it occurs as a local component.

The choice of a marking witness is flexible: every involution in the action kernel can serve, and nontrivial 2-subgroups of that kernel also give a single containing component.

Lemma 4.3 (Transport of markings). *Let $M \in \mathcal{F}$. Every involution of $\mathcal{C}_G(M)$ is a marking witness. For every nontrivial 2-subgroup $U \leq \mathcal{C}_G(M)$ there is a single marked 2-component S_U of $C_G(U)$ such that*

$$(5) \quad \langle M_U^{L_{2'}(C_G(U))} \rangle = S_U = O(S_U)M_U.$$

Proof. Suppose first that a witnesses the marking, and put $R = O(C_G(a))$. Since $S_a \trianglelefteq S_a R$ and the quotient by S_a is odd, perfectness in (4) gives

$$M_a \leq S_a, \quad S_a = O(S_a)M_a.$$

Let $v \in C_G(M)$ be an involution commuting with a . It normalizes $M_a R$, hence its unique 2-component S_a , and acts trivially on $S_a/O(S_a)$ because M_a covers that quotient. If $v \neq a$, Theorem 3.6 supplies a single isomorphic correspondent $S_v \in \mathcal{S}$ at v . The group $M_{\langle a, v \rangle}$ lies in $(S_a)_v \leq S_v$ and still maps onto T . Its image in $C_G(v)/O(C_G(v))$ is therefore the simple component represented by S_v . The image of M_v is perfect odd-by- T and contains that component. Theorem 4.1 makes its image exactly that component, proving (4) at v . The case $v = a$ needs no correspondence.

For an arbitrary involution $u \in C_G(M)$, choose a Sylow 2-subgroup of $C_G(M)$ containing u , and conjugate a into it by an element of $C_G(M)$. This preserves M and its marking. A central involution of that Sylow subgroup connects the conjugated witness to u by two commuting steps. Hence every such u is a witness.

Now take $1 \neq U \leq C_G(M)$ and $u \in \Omega_1(Z(U))^\#$. In $A = C_G(u)$, put $\bar{A} = A/O(A)$. The image of M_u is a simple component $K \cong T$ of $\bar{A}$, and $\bar{U}$ centralizes K . Intersecting a subnormal chain shows that K is a component of $C_{\bar{A}}(\bar{U})$. Odd-kernel fixed-point lifting gives an epimorphism

$$C_G(U) = C_A(U) \longrightarrow C_{\bar{A}}(\bar{U})$$

with odd kernel. Its canonical 2-component lift S_U contains M_U and satisfies $S_U = O(S_U)M_U$. Indeed, M_U maps onto K , and every perfect subgroup of the full inverse image lies in its perfect residual. Since u centralizes S_U and its image in $\bar{A}$ is K , the original witness u marks S_U as well.

Finally $S_U \trianglelefteq L_{2'}(C_G(U))$. The normal closure of M_U in that layer lies in S_U and covers $S_U/O(S_U)$. Its quotient in S_U is both odd and perfect, and hence trivial. This proves (5). $\square$

The restriction $U \neq 1$ is essential to the use made of this lemma. No assertion that a marked subgroup lies in a global component of G has been obtained.

4.1. Comparisons in proper subgroups. The local B -property gives stronger control inside proper subgroups: components with marked lifts have only identity correspondences in their odd core quotient.

Fix $H < G$, write $\pi : H \rightarrow \bar{H} = H/O(H)$, and use bars for images. A *relative GS correspondence* means a correspondence from Theorem 2.1 with ambient group $\bar{H}$. The local B -property from Theorem 2.7 applies in the $\mathcal{K}$ -group $\bar{H}$, which has trivial odd core. Its local 2-components are therefore ordinary quasisimple components. Their centers may have odd order; in particular, quasisimple central extensions of A_6 and A_7 by a cyclic group of order three must be allowed.

The marking of the canonical lift is the hypothesis that forces this equality of subgroups.

Lemma 4.4 (Relative identity). *Let A be an ordinary component of $C_{\bar{H}}(\bar{a})$, and suppose its canonical perfect lift*

$$P = (\pi^{-1}(A))^{(\infty)}$$

is marked. Every relative GS correspondence $A \longrightarrow K$ in $\bar{H}$ is the identity subgroup correspondence:

$$(6) \quad K = K^{\bar{a}} = A.$$

This holds at every occurrence of A , including after changing its witnessing involution.

Proof. Let commuting involutions $\bar{a}, \bar{b}$ witness the correspondence. An automorphism of a perfect group acting trivially modulo its center is trivial. Thus, $\bar{b}$ centralizes A , and the correspondence gives $A \leq \bar{W} = KK^{\bar{a}}$. Lift $\langle \bar{a}, \bar{b} \rangle$ across the odd kernel to commuting involutions $a, b \in H$. Then $b \in C_G(P)$. Let W be the perfect residual of the inverse image of $\bar{W}$ in $C_H(b)$. Fixed-point lifting gives $P_b \leq W$ and $\pi(P_b) = A$.

The normal closure of A in $\bar{W}$ is $\bar{W}$. In the one-component case this follows from quasisimplicity. In the two-factor case $\bar{a}$ exchanges the factors and fixes A pointwise, so A has nontrivial projection modulo the center on both. Its normal closure covers both simple factors and therefore the perfect central product itself. It follows that

$$W = \langle P_b^W \rangle (W \cap O(H)).$$

By (5), the image of P_b in $C_G(b)/O(C_G(b))$ is a simple component. Apply Theorem 4.1 in $C_G(b)$ to obtain $W = P_b O(W)$. Projecting into $\bar{H}$ gives $\bar{W} = A\pi(O(W))$. The normal odd subgroup $\pi(O(W))$ of the semisimple group $\bar{W}$ is central. Since $\bar{W}$ is perfect, $\bar{W} = A$, proving (6). The proof depends only on the canonical lift of the subgroup A , and hence applies to any choice of its witnessing involution. $\square$

Marked subgroups of H give the relative components to which this identity property applies.

Lemma 4.5. *Let $M \in \mathcal{F}$, with $M \leq H < G$, and let $a \in H \cap C_G(M)$ be an involution. There is an ordinary component $A = \pi(M_a)$ of $C_{\bar{H}}(\bar{a})$ whose canonical perfect lift is marked. Moreover $A/Z(A) \cong T$ and $Z(A)$ has odd order.*

Proof. Choose the marking S_a from Theorem 4.3, and set

$$I = C_H(a) \cap (S_a O(C_G(a))), \quad J = I^{(\infty)}.$$

The product $S_a O(C_G(a))$ is subnormal in $C_G(a)$ because its image modulo $O(C_G(a))$ is a component. Intersecting its subnormal chain with $C_H(a)$ shows that I is subnormal there. Its image in $C_G(a)/O(C_G(a))$ is T , since $M_a \leq I$ covers that component. Consequently J is a 2-component of $C_H(a)$, contains M_a , and is perfect odd-by- T .

The odd-kernel epimorphism $C_H(a) \rightarrow C_{\bar{H}}(\bar{a})$ sends J to a relative 2-component A . Local B in $\bar{H}$ makes A quasisimple; its odd core is its odd center and its simple quotient is T . Since $J = M_a O(J)$, one has $A = \pi(M_a)Z(A)$; perfectness therefore gives $A = \pi(M_a)$. The canonical lift $P = (\pi^{-1}(A))^{(\infty)}$ contains J and is perfect odd-by- T . Since a centralizes J , the group P_a contains J . Its image in $C_G(a)/O(C_G(a))$ is perfect odd-by- T and contains the simple component represented by J . The last clause of Theorem 4.1 makes that image exactly this component. Thus, a marks P , as required. $\square$

5. CONTAINMENT IN COMPONENTS OF PROPER SUBGROUPS

Aschbacher's component theorem [1, Theorem 9.7] and the recognition theorem of Aschbacher–Seitz [6, Theorem, p. 440] provide the main inputs here. Together with relative identity, they place a relative component with a sufficiently large centralizer in the layer of the whole proper quotient. A subgroup Q of a group B is *tightly embedded* if $|Q|$ is even and $|Q \cap Q^g|$ is odd whenever $Q^g \neq Q$. A quasisimple subgroup $A \leq B$ is *standard* if $C_B(A)$ is tightly embedded, $N_B(A) = N_B(C_B(A))$, and $[A, A^g] \neq 1$ for every $g \in B$.

Lemma 5.1 (The rank-three criterion). *In Theorem 4.4, if $m_2(C_{\bar{H}}(A)) \geq 3$, then A is an ordinary component of $\bar{H}$. In fact $A \cong T$.*

Proof. Write $B = \bar{H}$. First, A is a component of $C_B(v)$ for every involution $v \in C_B(A)$. For v commuting with the current witnessing involution, L_{2r} -balance supplies a GS correspondence; (6) makes its target A . For arbitrary v , conjugate the witnessing involution within $C_B(A)$ into a Sylow 2-subgroup containing v , and use a central involution of that Sylow subgroup to connect the witnesses. These conjugations fix A , and its canonical lift remains marked.

We now verify Aschbacher's original maximality [1, §1, p. 88]. His set $\mathcal{L}$ consists of ordinary local components, and $A <^* K$ means that, for an involution t , A is a component of $C_B(t)$, $A \leq K$, and $K = [K, t]$. If such a successor K existed, choose an involution b witnessing K as a local component. Then b centralizes A , so the preceding paragraph makes A another component of $C_B(b)$. Nested components are equal; it follows that $A = K$, contradicting the fact that $K = [K, t]$ and $[A, t] = 1$. Hence A has the maximality required in Aschbacher's theorem.

Choose a fourgroup $U \leq C_B(A)$. Each nonidentity element of U has A as a component in its centralizer. Choosing a witnessing involution in $U^\#$ gives exactly Hypothesis 9.4(2) of [1]. In the notation of that paper, $O_{2',E}(X)$ is the inverse image of $E(X/O(X))$ in X . Hypothesis 9.1 there, namely $O_{2',E}(C_B(t)) = O(C_B(t))E(C_B(t))$ for every involution t , follows from local B in B : every 2-component of an involution centralizer is ordinary. Theorem 9.7 of that paper now applies. Its rank-one alternative is excluded because an odd central cover of T has 2-rank two. Its alternative in which the Sylow 2-subgroups of $C_B(A)$ are dihedral is also excluded, since a dihedral 2-group has 2-rank at most two. Apart from the global component

conclusion, Theorem 9.7 gives its first alternative, including the absence of commuting conjugates of A . We explain explicitly why this makes A standard here. If an involution v centralizes both A and A^g , the identity-correspondence argument above makes both components of $C_B(v)$. They are equal or commute, so they are equal. Thus, a conjugate of $C_B(A)$ meeting it in even order has $A^g = A$. In particular $N_B(C_B(A)) = N_B(A)$, and $C_B(A)$ is tightly embedded. These properties show that A is standard.

Suppose now that A is standard but not normal in B , and put $X = \langle A^B \rangle$. We apply the theorem of Aschbacher–Seitz [6, Theorem, p. 440], which classifies standard subgroups of known simple type in a group with trivial odd core when their centralizer has 2-rank at least two. Here $O(B) = 1$ because $B = H/O(H)$; the preceding argument establishes that A is standard; and $A/Z(A) \cong T$ is a known simple group. Finally, our hypothesis $m_2(C_B(A)) \geq 3$ supplies the required centralizer rank. Thus, this application falls within the scope of that theorem. For these tops, its nonnormal alternatives give the following possible pairs of isomorphism types of A and X :

$$(A_n, A_{n+4}), \quad (A_5, M_{12}), \quad (A_5, J_2),$$

where $n \in \{5, 6, 7\}$, M_{12} is the degree-twelve Mathieu group, and J_2 is the second Janko group. In particular, X is simple. If $C_B(X)$ contained an involution, it would centralize A and every B -conjugate of A . Tight embedding and $N_B(C_B(A)) = N_B(A)$ would then make A normal in B . Thus, $C_B(X)$ has odd order; it is normal in B , so is trivial. Consequently $X \leq B \leq \text{Aut}(X)$.

Choose $Q \in \text{Syl}_2(C_B(A))$ and $a \in \text{Inv}(Z(Q))$, and extend Q to $S \in \text{Syl}_2(C_B(a))$. Then $C_S(A) = Q$, and the first paragraph makes A a component of $C_B(a)$. The centralizer calculation [13, Proposition 2.1(1)] now gives $m_2(Q) \leq 2$, a contradiction. Thus, A is normal in B in the standard case as well, and is a global component in every case. Finally its center is odd, and the product of the centers of its conjugate components is an odd normal subgroup of B . Since $O(B) = 1$, this center is trivial and $A \cong T$. $\square$

Other marked components can supply the required centralizer rank. With only two components, an additional involution in their common action kernel is needed; we record both cases in one definition.

Definition 5.2. For $N \leq G$, let

$$D(N) = \prod \{K : K \text{ is a marked 2-component of } N\}, \quad N_0 = O_{2',2}(N)L_{2'}(N).$$

Let $r(N)$ be the number of marked 2-components of N , including $r(N) = 0$ if there are none. The empty product in the definition of $D(N)$ is 1. We call N $\mathcal{F}$ -balanced if $r(N) \geq 2$ and either $r(N) \geq 3$ or $\mathcal{C}_{N_0}(D(N))$ has even order. Here $\mathcal{C}_{N_0}(D(N))$ is the kernel of the action on $D(N)/O(D(N))$. This condition concerns the marked family; it is different from the negation of the unbalanced-group condition in the overview.

A marked 2-component of a subgroup H has a simple image isomorphic to T in $H/O(H)$: its image is quasisimple, and its odd center generates an odd normal subgroup there under conjugation, so that center is trivial. Its canonical perfect lift is the original 2-component. The usefulness of $\mathcal{F}$ -balancedness is its persistence under enlargement, with distinct marked components remaining distinct.

Lemma 5.3. *If N is $\mathcal{F}$ -balanced and $N_0 \leq H < G$, then each marked component of N lies in a marked component of H . Distinct marked components of N lie in distinct marked components of H , and H is $\mathcal{F}$ -balanced.*

Proof. Write $D(N) = K_1 \cdots K_r$. Each K_i has an elementary abelian 2-group E of rank at least three in the kernel of its action on the quotient inside N_0 . If $r \geq 3$, two opposite factors give rank four modulo the odd core, and odd-kernel lifting supplies E . If $r = 2$, an opposite fourgroup and an involution in the common kernel give rank three. The common-kernel involution is independent of the opposite factor because the latter has centerless simple quotient. The same odd-kernel lifting gives the required elementary abelian subgroup.

Choose $a \in E^\#$ and form the relative component A associated with $(K_i)_a$ by Theorem 4.5. The image of E centralizes A : its action is trivial modulo the center of the perfect group A , hence trivial. The image

of E retains rank at least three across $O(H)$. Theorem 5.1 makes $A \cong T$ a global component of $\bar{H}$. The image of K_i contains A and is perfect odd-by- T , so Theorem 4.1 makes that image A . Thus, K_i lies in the canonical marked component lift of A in H .

Distinct old factors have commutator of odd order. If their images became the same simple factor, that commutator would map onto T , which is impossible. There are therefore at least r marked factors in H . Only the two-factor $\mathcal{F}$ -balancedness condition needs checking. An old common-kernel involution survives in $\bar{H}$ and centralizes both new simple factors. If their common centralizer in $O_2(\bar{H})E(\bar{H})$ had odd order, then $O_2(\bar{H}) = 1$ and there could be no additional component. Consequently $F^*(\bar{H})$ would be exactly those two centerless simple factors. The surviving involution would centralize $F^*(\bar{H})$, contradicting its self-centralizing property. This gives an even common kernel in the preimage H_0 , and H is $\mathcal{F}$ -balanced. $\square$

Within an $\mathcal{F}$ -balanced proper subgroup, a marked subgroup can be placed in a component whenever its action kernel there contains an involution.

Lemma 5.4. *Let $H < G$ be $\mathcal{F}$ -balanced, let $M \in \mathcal{F}$ satisfy $M \leq H$, and suppose $H \cap C_G(M)$ contains an involution. Then M lies in a single marked 2-component of H .*

Proof. Choose such an involution a , and let A be the relative ordinary component supplied by Theorem 4.5. Work in $B = H/O(H)$. By L_2 -balance, $A \leq E(B)$. Modulo $Z(E(B))$, the layer is a direct product of simple groups. We take the support of A in this quotient. Since A is a component of $C_B(\bar{a})$, its support lies in one component of $E(B)$ or in a pair of components interchanged by $\bar{a}$. In the latter case its image modulo the center is diagonal on that pair. The canonical lift of A is marked, so (6) applies.

Suppose a marked global component of B lies outside this support. The witnessing involution permutes the marked components and preserves the support. An outside orbit has length one or two and supplies an involution b fixed by $\bar{a}$. For a two-factor orbit use the diagonal fixed group. For a fixed factor, the parity of the orbit sizes gives an even-order fixed subgroup. This b centralizes the entire support of A . A support component is therefore a component of $C_B(b)$ and is the target of a relative GS correspondence from A . Equation (6) forces A to be that global component. If $b = \bar{a}$, the same conclusion follows directly by nesting the two components of $C_B(\bar{a})$.

If there is no outside marked factor, $\mathcal{F}$ -balancedness leaves exactly two marked global factors, forming the exchanged support of A . The common kernel in $O_2(B)E(B)$ has even order and is normalized by $\bar{a}$. Its fixed subgroup has even order, so it contains an involution b centralizing both factors and commuting with $\bar{a}$. It is distinct from $\bar{a}$, which exchanges the factors. Again a relative correspondence from A to a support factor contradicts (6). Thus, in every case A is a global component. Its center is odd and $O(B) = 1$, so $A \cong T$. The image of M is perfect odd-by- T and contains A through the image of M_a . Theorem 4.1 makes that image exactly A . Perfectness puts M in the canonical perfect lift of A , which is marked by construction. This proves the assertion. $\square$

Both lemmas apply to proper subgroups. Their use of (3) follows from the $\mathcal{K}$ -properness of G .

6. AN EMBEDDING CRITERION

We now show that a proper $\mathcal{F}$ -balanced subgroup $M = N_G(D(M))$ cannot contain the centralizers in G of all involutions belonging to the individual components of $D(M)$. The proof uses an embedding criterion of Aschbacher, which we recall before establishing the lemma.

For an element or a subgroup U , abbreviate $C_G(U)$ to C_U and $L_2(C_U)$ to L_U . A *pure involution* of a product of 2-components means an involution belonging to one of its individual 2-components. We use the following embedding criterion, proved by Aschbacher and stated as [26, Theorem 2.8, p. 572]. Suppose $D \trianglelefteq M \leq G$ is a product $A_1 \cdots A_r$ of 2-components of M , and put $\bar{A}_i = A_i O(D)/O(D)$. Assume that

- (i) if $t, t^g \in M \setminus Z(G)$ are 2-elements, t acts trivially on $\bar{A}_1$, and t^g acts trivially on some $\bar{A}_j$, then $g \in M$;
- (ii) if $X \leq M^g$ and $A_1 \leq O(D)X$, then $g \in M$.

Then $M = G$, or $r = 1$, or $r = 2$ and $m_2(A_1) = m_2(A_2) = 1$. We will verify the stronger symmetric versions of the two hypotheses, with any component in place of A_1 . Since G is nonabelian simple, every nonidentity 2-element of G is noncentral.

The number and 2-rank of our marked components exclude the exceptional conclusions. Containment of all pure-involution centralizers will supply the hypotheses of the criterion.

Lemma 6.1 (Embedding criterion). *Let $M < G$ be $\mathcal{F}$ -balanced, put $D = D(M) = L_1 \cdots L_r$, and suppose $M = N_G(D)$. If we have*

$$(7) \quad C_G(x) \leq M$$

for every $x \in \text{Inv}(L_i)$ and every $1 \leq i \leq r$, then a contradiction follows.

Proof. Here $r \geq 2$, each $L_i/O(L_i) \cong T$, and $m_2(L_i) = 2$. We give the component-control argument of [26, Lemma 3.13, pp. 590–591], retaining the extra details needed for the marked family.

Let $1 \neq t \in \mathcal{C}_M(L_i)$ be a 2-element. The fixed-point subgroup $(L_i)_{\langle t \rangle}$ has quotient T , so it contains a fourgroup B . Coprime generation in the odd group $O(C_t)$ and (7) give

$$O(C_t) = \langle C_{O(C_t)}(b) : b \in B^\# \rangle \leq M.$$

We also have $N_G(B) \leq M$. To see this, put $H = (L_i)_{\langle t \rangle}$ and write bars for images in $H/O(H) \cong T$. Since $O(H)$ has odd order, the map $B \rightarrow \bar{B}$ is an isomorphism. By Theorem 2.6, there is an element of $N_T(\bar{B})$ inducing a 3-cycle on $\bar{B}^\#$. Choose a lift $h \in H$. Then B and B^h are complements to $O(H)$ in $O(H)B$. By the conjugacy part of the Schur–Zassenhaus theorem, we may multiply h by an element of $O(H)$ to obtain $c \in N_H(B)$ inducing this 3-cycle on $B^\#$.

Now let $g \in N_G(B)$. A permutation of the three elements of $B^\#$ either fixes a point or is a 3-cycle. Thus, for some $k \in \{0, 1, 2\}$, the element gc^k fixes an element $b \in B^\#$. Since $b \in L_i$, hypothesis (7) gives $gc^k \in C_G(b) \leq M$. Also $c \in H \leq M$, so $g \in M$, as required.

By Theorem 4.3, the group $(L_i)_{\langle t \rangle}$ covers the quotient by the odd core of a unique component of L_t . The other components of L_t centralize that quotient. Conjugacy of Sylow 2-subgroups across the odd kernel gives

$$O(C_t)L_t = O(C_t)(L_i)_{\langle t \rangle}N_{O(C_t)L_t}(B).$$

Consequently, we have

$$(8) \quad O(C_t)L_t \leq M$$

for every nonidentity 2-element $t \in \mathcal{C}_M(L_i)$.

We next prove the second condition of the embedding criterion. Suppose $X \leq M^g$ and $L_j \leq O(D)X$. Replace X by $(X \cap O(D)L_j)^{(\infty)}$. It is perfect odd-by- T , still covers $L_j/O(L_j)$, and remains in M^g . Every involution of this new X is $O(D)$ -conjugate to an involution of L_j . Hence $C_G(x) \leq M$ for $x \in \text{Inv}(X)$. A fourgroup in X , acting on the odd normal subgroup $O(M^g)$, gives

$$(9) \quad O(M^g) \leq M.$$

Case 1. $X \leq D^g$. The image of X in $D^g/O(D^g) \cong T^r$ is diagonal on a nonempty set of factors. To see this, every nontrivial projection is perfect odd-by- T and has order at most $|T|$, so it is all of T ; the odd core has trivial image in every such projection.

If the support is all factors, the images of $C_{D^g}(x)$, $x \in \text{Inv}(X)$, generate T^r . In fact, for a prescribed coordinate involution there is a diagonal involution with that coordinate, and the coordinate involution centralizes it. Fixed-point lifting across $O(D^g)$ realizes these elements in the displayed centralizers. Each copy of T is generated by its involutions, and (9) absorbs the odd core. Thus, $D^g \leq M$. Theorem 5.4, using kernel involutions from opposite factors, puts every factor of D^g in a marked factor of M . Hence $D^g \leq D$, so equality of orders gives $D^g = D$ and $g \in N_G(D) = M$.

If a factor K_0 of D^g is unused, X acts trivially on $K_0/O(K_0)$. Apply (8) in M^g to $x \in \text{Inv}(X)$. This gives $L_x \leq M^g$. The subgroups defined by taking fixed points and perfect residuals of the factors L_i , $i \neq j$,

lie in L_x and cover their quotients, while X covers the j th quotient. Therefore $D \leq \langle O(D), M^g \rangle$. Choose a fourgroup $U \leq X$ and a fourgroup $E \leq C_{K_0}(U)$; the latter exists by odd-kernel fixed-point lifting. Then $E \leq M$, since it centralizes U , whereas $C_G(e) \leq M^g$ for every $e \in E^\#$ by (7) in M^g . Coprime generation gives $O(D) \leq M^g$. Thus, $D \leq M^g$, and Theorem 5.4 in M^g gives $D \leq D^g$. Again $g \in M$.

Case 2. X permutes the factors of D^g nontrivially. There is a fourgroup $U \leq X$ acting nontrivially on the factors: the image of X is generated by images of involutions, and each involution extends to a fourgroup. Some U -orbit Ω has size two or four. In either case

$$(10) \quad \langle C_{T^\Omega}(u) : u \in U^\# \rangle = T^\Omega.$$

For a two-element orbit, a swapping involution has a diagonal fixed subgroup T . A kernel involution has a nonidentity fixed element in one coordinate: an involution acting on an even-order group has an even-order fixed subgroup. Conjugating this coordinate element by the diagonal generates the coordinate simple group, and then both factors. For a four-element orbit, identify the coordinates along the regular U -action. Centralizers of the three involutions contain diagonals on pairs of coordinates. Commutators of two such diagonals on intersecting pairs isolate one coordinate, and perfection generates all four coordinates. These arguments are unchanged by the initial choices of coordinate identifications.

All the lifted centralizers in (10) lie in M , and (9) absorbs the odd kernels. Thus, the corresponding product of factors of D^g lies in M . By Theorem 5.4, each such factor lies in a marked factor of D and covers its simple quotient. Taking one of them as the new X reduces to Case 1.

Case 3. X normalizes every factor of D^g . Choose a fourgroup $U \leq X$ and a factor K_1 of D^g . The perfect image of X on $K_1/O(K_1)$ lies in $\text{Inn}(T)$, because $\text{Out}(T)$ is solvable by Theorem 2.6. Its odd-by- T structure then forces this image to be either trivial or all of $\text{Inn}(T)$. In the latter case the image of U is a fourgroup. It follows, with odd-kernel lifting in either case, that

$$\langle C_{K_1}(u) : u \in U^\# \rangle \text{ contains a fourgroup } E_1.$$

Every displayed centralizer lies in M , so $E_1 \leq M$. Its pure involution centralizers lie in M^g . Thus, coprime generation gives $O(M) \leq M^g$, and consequently $L_j \leq M^g$.

Every generator used to construct E_1 fixes an involution whose image in $D/O(D)$ has support precisely L_j . Hence it normalizes the actual factor L_j , and E_1 normalizes $C_M(L_j)$. This group is even, as it contains the images of opposite factors. A 2-group acting on an even-order finite group has an even-order fixed subgroup. Choose

$$z \in \text{Inv}(C_{C_M(L_j)}(E_1)).$$

Since z centralizes a pure involution of $E_1 \leq K_1$, it belongs to M^g . Theorem 5.4 in M^g now puts L_j in D^g . Taking $X = L_j$ reduces to Case 1. We have proved

$$(11) \quad X \leq M^g, \quad L_j \leq O(D)X \implies g \in M.$$

Finally suppose $t \in C_M(L_i)$ and $t^g \in C_M(L_j)$ are nonidentity 2-elements. Conjugating (8) at t puts L_{tg} in M^g . The fixed-point subgroup $(L_j)_{\langle t^g \rangle}$ belongs to this layer and covers $L_j/O(L_j)$, by Theorem 4.3. It supplies the subgroup X in (11). Thus, $g \in M$, proving the first condition as well. The embedding criterion contradicts $M < G$, $r \geq 2$, and $m_2(L_i) = 2$. $\square$

7. THE SIGNALIZER CONSTRUCTION

This section develops the signalizer argument in Solomon's maximal-2-component method [26] for the marked family. Goldschmidt's signalizer functor theorem [14] assembles compatible odd-order subgroups of involution centralizers into one odd-order group. The later recovery of this group as the odd core of its normalizer follows Solomon's argument [26, Lemma 3.11], with the component containments written out below.

Suppose a pair of components interchanged by an involution exists. Choose an $\mathcal{F}$ -balanced subgroup N for which the number $r(N)$ of marked 2-components is as large as possible. Every $\mathcal{F}$ -balanced subgroup is proper, since the simple group G has only one global 2-component. Write

$$D(N) = K_1 \cdots K_r, \quad N_0 = O_{2',2}(N)L_{2'}(N), \quad Q_i = C_{N_0}(K_i).$$

Such an N exists: the original centralizer contains the two marked components Y, Y^s , and its central involution supplies the additional common-kernel involution. $\mathcal{F}$ -balancedness gives $m_2(Q_i) \geq 3$.

For $i \neq j$, choose commuting fourgroups $F = V \leq K_i$ and $E = B_0 \leq K_j$. Such a pair exists by fixed-point lifting on K_j under a fourgroup of K_i . Put $Q = C_{N_0}(K_i K_j)$. The selected factors are normal in N_0 , with centerless simple images modulo $O(N)$, so Q is normal in N_0 and $Q \cap K_i K_j = O(K_i K_j)$ is odd. The group Q is even: this is $\mathcal{F}$ -balancedness if $r = 2$, and a third marked component is contained in Q if $r \geq 3$. Choose $S \in \text{Syl}_2(N_0)$ containing EF and then choose

$$z_0 \in \Omega_1(Z(S) \cap Q)^\#, \quad B = E \times \langle z_0 \rangle.$$

The normal subgroup $S \cap Q$ is nontrivial, so this choice exists. Since $z_0 \notin K_i K_j$, the group B has rank three, $B \cap K_j = E$, and $B \leq C_{Q_i}(F)$. Moreover B centralizes $S \cap K_m$ for every $m \neq j$: distinct component factors commute inside S , since their commutators lie in $O(N) \cap S = 1$. These are the admissible pairs used below.

Write $O_t = O(C_G(t))$ in this section. Define

$$(12) \quad \theta_V(b) = [O(C_b), V](O(C_b) \cap O(C_V)), \quad W(B, V) = \langle \theta_V(b) : b \in B^\# \rangle.$$

For commuting elementary abelian 2-subgroups B, V , we use the same formula to define $W(B, V)$, even when B has rank less than three.

A *solvable B -signalizer functor* is a family $(\theta(b))_{b \in B^\#}$ of B -invariant solvable odd-order subgroups $\theta(b) \leq C_G(b)$ such that

$$\theta(b) \cap C_G(a) \leq \theta(a)$$

for all $a, b \in B^\#$. Goldschmidt's 2-signalizer functor theorem [14] states that, if $m_2(B) \geq 3$, then $W = \langle \theta(b) : b \in B^\# \rangle$ has odd order and $C_W(b) = \theta(b)$ for every $b \in B^\#$.

To apply Goldschmidt's theorem, we must establish compatibility on intersections of centralizers. The marked components provide the required control of the local layers.

Proposition 7.1 (Signalizer balance). *For every family $\mathcal{S}$ satisfying Theorem 3.6, form the marked family $\mathcal{F}$ as above. For every admissible pair (B, V) in an $\mathcal{F}$ -balanced subgroup with the maximum possible number of marked components, for all $a, b \in B^\#$, we have*

$$\theta_V(b) \cap C_G(a) \leq \theta_V(a).$$

Proof. An odd-order normal subgroup of H centralizes $E(H)$. If $U \leq H$ is a 2-group, then $O(C_H(U))$ also centralizes $O_2(H)$: it centralizes $C_{O_2(H)}(U)$, so Thompson's $A \times B$ -lemma [24, Theorem 1] applies. When $O(H) = 1$, an odd-order subgroup centralizing $F^*(H)$ is trivial, by $C_H(F^*(H)) \leq Z(F^*(H))$.

We claim that, for every $v \in V^\#$, the image of V centralizes $E(C_v/O_v)$. Transport K_j at v using Theorem 4.3, and call the resulting 2-component of C_v J . The odd order of $C_{K_j}(v)/(K_j)_v$ gives $E \leq (K_j)_v \leq J$. The group V normalizes $(K_j)_v$ and acts trivially modulo its odd core. The normal-closure equality in (5) shows that V normalizes J , and $J = O(J)(K_j)_v$ then shows that it centralizes the image of J in C_v/O_v .

Let $R \neq J$ be another 2-component of C_v , and choose $e \in E^\#$. Distinct component images commute, so $e \in C_G(R)$. By Theorem 2.4, the group $D = R_e$ covers $R/O(R)$ and is a 2-component of $C = C_e \cap C_v$. By $L_{2'}$ -balance, $D \leq L_{2'}(C_e)$. Transporting K_i at e gives a marked 2-component I of C_e containing V . Work for the moment modulo O_e , using bars for images. The centerless simple component $\bar{I} \cong T$ is a direct factor of $E(C_e/O_e)$. The projection of $\bar{D}$ onto $\bar{I}$ is perfect and lies in $C_{\bar{I}}(\bar{v})$, which is solvable by Theorem 2.6. This projection is trivial. All other components commute with $\bar{I}$, so $[D, V] \leq O_e$.

Now V normalizes C , and $[D, V] \leq O_e \cap C \leq O(C)$. The component correspondence in $C/O(C)$ therefore shows that V normalizes the canonical perfect lift D . Consequently $[D, V] \leq D \cap O_e \leq O(D)$. The fixed-layer lemma gives $O(D) = D \cap O(R) \leq O_v$. Since D covers $R/O(R)$, the image of V centralizes the image of R in C_v/O_v . This proves the claim.

Fix $b \in B^\#$ and $v \in V^\#$, and now put $H = C_v/O_v$, again using bars for images. Odd-kernel fixed-point lifting shows that the image $\bar{X}$ of $X = O_b \cap C_v$ is an odd-order normal subgroup of $C_H(\bar{b})$. Thus, $\bar{X}$ centralizes $O_2(H)$. The group V normalizes X and, by the claim, $\bar{V}$ centralizes $E(H)$. Since $C_H(E(H))$ is normal in H , the odd-order group $[\bar{X}, \bar{V}]$ centralizes both $E(H)$ and $O_2(H)$ and is therefore trivial.

Similarly, the image of $O(C_v)$ is an odd-order normal subgroup of $C_H(\bar{V})$. Every component of H is a component of $C_H(\bar{V})$, because $\bar{V}$ centralizes $E(H)$. The observations at the start of the proof show that this image centralizes both $E(H)$ and $O_2(H)$, so it too is trivial. We have proved

$$(13) \quad [O_b \cap C_v, V] \leq O_v, \quad O(C_v) \leq O_v.$$

Put $P = [O_b, V]$. Since $P \leq O_b$, the product $\theta_V(b)$ is an odd-order subgroup. We have $P = [P, V]$, and Goldschmidt's coprime-action lemma [12, Lemma 2.3] gives

$$C_P(V) = \langle C_{[C_P(w), V]}(V) : w \in V^\# \rangle.$$

By (13), each group $[C_P(w), V]$ lies in O_w , so its V -centralizer lies in $O_w \cap C_v \leq O(C_v)$. Thus, $C_P(V) \leq O(C_v)$. Coprime action and (13) now give

$$C_P(v) = [C_P(v), V]C_P(V) \leq O_v.$$

The second factor of $\theta_V(b) = P(O_b \cap O(C_v))$ centralizes V . Hence $C_{\theta_V(b)}(v) \leq O_b \cap O_v$. Conversely, factor the odd-order V -invariant group $O_b \cap O_v$ under V . Its commutators with V lie in P , and its V -centralizer lies in $O_b \cap O(C_v)$. This proves

$$(14) \quad C_{\theta_V(b)}(v) = O_b \cap O_v.$$

Let $a, b \in B^\#$ and $v \in V^\#$. We next show that

$$(15) \quad O_b \cap O_v \cap C_a \leq O_a.$$

This time work in $H = C_a/O_a$, with bars denoting images. Transport K_i at a to a marked 2-component I of C_a . As before, $V \leq I$. Since b centralizes a and acts trivially on $K_i/O(K_i)$, it centralizes $\bar{I}$. The images of $O_b \cap C_a$ and $O_v \cap C_a$ are odd-order normal subgroups of $C_H(\bar{b})$ and $C_H(\bar{v})$, respectively. The first centralizes $\bar{I}$, which is a component of $C_H(\bar{b})$. The second centralizes $O_2(H)$ and every component of H other than $\bar{I}$, since those components commute with $\bar{v} \in \bar{I}$. Thus, the image of $O_b \cap O_v \cap C_a$ centralizes $F^*(H)$ and is trivial. This proves (15).

Finally, $X = \theta_V(b) \cap C_a$ has odd order and is normalized by V . Equations (14) and (15) give

$$C_X(v) = O_b \cap O_v \cap C_a \leq O_a \cap O_v \leq \theta_V(a)$$

for every $v \in V^\#$. Fourgroup generation yields $X = \langle C_X(v) : v \in V^\# \rangle \leq \theta_V(a)$, as required. $\square$

Goldschmidt's theorem now assembles these subgroups into an odd-order group and recovers them as the centralizers of its involutions.

Lemma 7.2. *For every admissible pair (B, V) , the group $W(B, V)$ has odd order, and for all $b \in B^\#$, we have*

$$C_{W(B, V)}(b) = \theta_V(b).$$

Proof. Since $[O(C_b), V] \leq O(C_b)$, each $\theta_V(b)$ is an odd-order subgroup of C_b , hence solvable. The group B normalizes both defining factors. Proposition 7.1 therefore makes θ_V a solvable B -signalizer functor. Since $m_2(B) = 3$, Goldschmidt's theorem [14] applies. $\square$

7.1. The fourgroup calculation. The signalizer argument requires freedom to change the acting subgroups. We begin with a marked component: each involution of an acting fourgroup gives the same theta subgroup as the whole fourgroup.

Lemma 7.3. *Let K be marked, let $A \leq C_G(K)$ be a fourgroup, and let $f \in \text{Inv}(K)$ centralize A . Put $O_t = O(C_G(t))$. Then, for every $u \in A^\#$,*

$$\theta_A(f) \cap C_G(u) \leq O_u, \quad \theta_A(f) = \theta_{\langle u \rangle}(f).$$

Proof. Let J_u be the containing component supplied by Theorem 4.3 for $K_u = C_K(u)^{(\infty)}$. Since $C_K(u)/K_u$ is odd, $f \in K_u$. The group A normalizes K_u and acts trivially on its odd-core quotient. Since $J_u = O(J_u)K_u$, the image of A centralizes the component image of J_u in $C_G(u)/O_u$. Canonical component lifting therefore gives normalization of J_u by A as well. Every other component of this quotient is centralized by f .

Put $X = O_f \cap C_G(u)$ and $H = C_G(u)/O_u$. The image of X is an odd-order normal subgroup of $C_H(\bar{f})$, where bars denote images in H . By [12, Lemma 2.1(iii)], it normalizes every component and centralizes those other than $\bar{J}_u$. It also centralizes $O_2(H)$, by the $A \times B$ -argument in the proof of Theorem 7.1. Since $\bar{A}$ centralizes $\bar{J}_u$, the commutator $[\bar{X}, \bar{A}]$ centralizes $F^*(H)$ and is trivial. Thus,

$$(16) \quad [O_f \cap C_G(u), A] \leq O_u.$$

We also have

$$(17) \quad O_f \cap O(C_G(A)) \cap C_G(u) \leq O_u.$$

Indeed, $\bar{J}_u$ is a component of $C_H(\bar{A})$, so the image of $O(C_G(A))$ centralizes it. The image of $O_f \cap C_G(u)$ centralizes the other components and $O_2(H)$, as above.

Set $P = [O_f, A]$. Coprime action gives $P = [P, A]$. Goldschmidt's coprime lemma [12, Lemma 2.3] gives

$$C_P(A) = \langle C_P(A) \cap [C_P(U), A] : U \text{ a line of } A \rangle.$$

For $U = \langle w \rangle$, equation (16) puts $[C_P(U), A]$ in O_w . Its A -fixed part lies in $O_w \cap C_G(A) \leq O(C_G(A))$. Thus, $C_P(A) \leq O(C_G(A))$. Now $Z = C_P(u)$ satisfies $Z = [Z, A]C_Z(A)$. Equation (16) handles the first factor and (17) handles the second. Hence $C_P(u) \leq O_u$. Finally

$$\theta_A(f) = P(O_f \cap O(C_G(A)))$$

is a product of odd A -invariant subgroups, with the first normal in the product. Fixed points under $\langle u \rangle$ factor across this product by coprime fixed-point lifting. Equation (17) handles the second factor, proving the first assertion.

Factor $\theta_A(f)$ under $\langle u \rangle$. Its commutator part lies in $[O_f, u]$ and its fixed part lies in $O_f \cap O_u$. Thus, $\theta_A(f) \leq \theta_{\langle u \rangle}(f)$. For the reverse inclusion, let $U \leq A$ and factor the odd A -invariant group $O_f \cap O(C_G(U))$ under A . Its commutator part lies in $[O_f, A]$, and its fixed part lies in $O_f \cap O(C_G(A))$, because $C_G(A) \leq C_G(U)$. Thus, $\theta_U(f) \leq \theta_A(f)$. $\square$

Applying this in opposite components makes the construction symmetric in the two fourgroups.

Lemma 7.4. *Let $E \leq K_j$ and $F \leq K_i$ be commuting fourgroups, $i \neq j$. Then we have*

$$W(E, F) = W(F, E).$$

If the original admissible rank-three group B contains E , both groups are equal to the original $W(B, F)$.

Proof. Apply Theorem 7.3 to $K_j, A = F, f = e$ and to $K_i, A = E, f = v$. For $e \in E^\#$ and $v \in F^\#$ it gives

$$C_{\theta_F(e)}(v) = O_e \cap O_v = C_{\theta_E(v)}(e).$$

The reverse containments here follow from $\theta_{\langle v \rangle}(e) \leq \theta_F(e)$ and its swapped version. Fourgroup generation of each individual odd theta group gives

$$\theta_F(e) = \langle O_e \cap O_v : v \in F^\# \rangle, \quad \theta_E(v) = \langle O_v \cap O_e : e \in E^\# \rangle.$$

Taking joins proves the equality.

For the last assertion, the cross-balance estimate from the original rank-three construction gives

$$C_{\theta_F(b)}(e) \leq \theta_F(e)$$

for $b \in B^\#$ and $e \in E^\#$. Fourgroup generation of the odd E -invariant group $\theta_F(b)$ therefore puts it in $W(E, F)$. The opposite containment is immediate. $\square$

This symmetry places odd-order subgroups normalized by both fourgroups, including the relevant odd cores, inside $N_G(W)$.

Lemma 7.5. *Retain the E, F, W above, and put $H = N_G(W)$. Every odd-order subgroup X normalized by E and F lies in H . Consequently, for every involution $s \in C_G(EF)$, we have*

$$O(C_G(s)) \leq H \quad \text{and} \quad O(N) \leq H.$$

Proof. Restriction of the acting fourgroup in Theorem 7.3 and Theorem 7.4 give

$$(18) \quad C_G(E) \cap C_G(v) \leq H \quad \text{and} \quad C_G(F) \cap C_G(e) \leq H$$

for all $v \in F^\#$ and $e \in E^\#$. For example $W = W(E, \langle v \rangle)$, whose defining terms are preserved by every element centralizing E and v .

For $e \in E^\#$, the component supplied by Theorem 4.3 of $(K_i)_e$ gives a centerless simple component S_e of $C_G(e)/O_e$ containing the image of F and centralized by the image of E . The containment uses the odd quotient $C_{K_i}(e)/(K_i)_e$; the kernel action uses the fixed-point and containing component argument at the start of Theorem 7.3.

Double fourgroup generation reduces $X \leq H$ to $Y = C_X(\langle e, v \rangle) \leq H$, for $e \in E^\#$, $v \in F^\#$. This Y is odd and still E, F -invariant. Work modulo O_e . The image $\bar{Y}$ normalizes S_e . Indeed, if an odd group is normalized by a noncentral involution t of a component S , and $S^y \neq S$, the components S, S^y commute. Then $t^y t^{-1}$ is an element of the odd group of order at most two, so $t^y = t$. This puts t in $S \cap S^y \leq Z(S)$, a contradiction. Apply this observation to a nonidentity element of the image of F .

Thus, $[\bar{Y}, \bar{F}] \leq S_e$ and $[[\bar{Y}, \bar{F}], \bar{E}] = 1$. Since E, F commute, the three-subgroups lemma gives

$$U = [[Y, E], F] \leq O_e.$$

Coprime action gives $U = [U, F]$, so $U \leq [O_e, F] \leq W \leq H$. Now factor twice:

$$Y = C_Y(E)[Y, E], \quad [Y, E] = C_{[Y, E]}(F)U.$$

The two fixed factors belong to the two centralizers in (18); the third factor is in W . Hence $Y \leq H$, proving $X \leq H$. For any involution s centralizing EF , its odd centralizer core is E, F -invariant. The same is true of $O(N)$. This proves the last assertions, including involutions outside N_0 . $\square$

If $W = W(B, V) \neq 1$, its odd order and simplicity of G give $H = N_G(W) < G$. Put $E_0 = EF = B_0V$. By Theorem 7.5 and the definition of W , we have

$$C_G(E_0) \leq H, \quad O(N) \leq H, \quad O(C_G(x)) \leq H$$

for every involution $x \in C_G(E_0)$.

7.2. The normalizer of the signalizer subgroup. If W is nontrivial, its normalizer H is proper. The central claim is that $W = O(H)$ and that this odd core can be recovered from other pairs of subgroups in the marked components of H . The necessary component containments are proved along the way.

Lemma 7.6 (Recovery of the odd core). *If $W(B, V) \neq 1$ and $H = N_G(W(B, V))$, then*

$$(19) \quad N_0 \leq H \quad \text{and} \quad L_x \leq H$$

for every $x \in \text{Inv}(K_i)$ and every $1 \leq i \leq r$. Moreover, we have

$$(20) \quad W(B, V) = O(H) \quad \text{and} \quad D(H) = O(D(H))D(N).$$

For every fourgroup B^* in one marked factor of H and every nontrivial elementary abelian subgroup V^* in a different marked factor of H centralizing B^* ,

$$(21) \quad O(H) = \left\langle [O(C_b), V^*](O(C_b) \cap O(C_{V^*})) : b \in (B^*)^\# \right\rangle.$$

In particular, V^* is allowed to have order two.

Proof. Putting the original N_0 in H . Write $E = B_0 \leq K_j$ and $F = V \leq K_i$, and retain the original $W = W(B, V)$ and $H = N_G(W)$. By Theorems 7.4 and 7.5, we have

$$(22) \quad \begin{aligned} W &= W(E, F) = W(F, E), & O(N) &\leq H, & O(C_G(t)) &\leq H, \\ C_G(E, v) &\leq H & \text{and} & & C_G(F, e) &\leq H \end{aligned}$$

for all $t \in \text{Inv}(C_G(EF))$, $v \in F^\#$, and $e \in E^\#$.

Put $Q = \mathcal{C}_{N_0}(K_i K_j)$. The image of N_0 modulo $O(N)$ is the generalized Fitting subgroup of $N/O(N)$. The two selected marked factors have centerless simple images and are normal in this generalized Fitting subgroup. Hence Q is normal in N_0 and $Q \cap K_i K_j = O(K_i K_j)$ is odd. Moreover Q is even: if $r = 2$, this is $\mathcal{F}$ -balancedness; if $r \geq 3$, Q contains a third marked component.

Choose $S \in \text{Syl}_2(N_0)$ containing EF . The nontrivial normal subgroup $S \cap Q$ of S meets $Z(S)$ nontrivially, so choose $z \in \Omega_1(Z(S) \cap Q)^\#$. Thus, $z \notin K_i K_j$, and $E\langle z \rangle$ is an admissible rank-three group with the same original W . More relevantly, $E\langle z \rangle$ centralizes the full Sylow subgroup $S \cap K_m$ for every $m \neq j$: z is central in S , and the commutators of $E \leq K_j \cap S$ with $K_m \cap S$ lie in $O(N) \cap S = 1$. Furthermore, $\langle e, z \rangle$ and $\langle v, z \rangle$ are actual fourgroups in the appropriate opposite action kernels, for $e \in E^\#$ and $v \in F^\#$. Every acting subgroup in these groups normalizes the marked factor and acts trivially modulo its odd core. Theorem 7.3 therefore gives

$$\theta_E(v) = \theta_{\langle e \rangle}(v) = \theta_{\langle e, z \rangle}(v) = \theta_{\langle z \rangle}(v),$$

and the analogous equality $\theta_F(e) = \theta_{\langle z \rangle}(e)$. Consequently

$$W = W(E, \langle z \rangle) = W(F, \langle z \rangle).$$

In particular $C_G(Ez), C_G(Fz) \leq H$. Odd-kernel fixed-point lifting on the components now gives

$$K_i = O(K_i)C_{K_i}(Ez) \leq H, \quad K_j = O(K_j)C_{K_j}(Fz) \leq H.$$

The remaining generalized-Fitting factors of $N/O(N)$ centralize the image of EF . Their full inverse image in N_0 is therefore contained in $O(N)C_{N_0}(EF)$ by odd-kernel fixed-point lifting, and is contained in H by (22). Together with $K_i, K_j \leq H$, this proves $N_0 \leq H$.

Pure-involution local containment. We record explicitly why the containment extends to every pure involution, not just the original E, F . First use the following elementary observation. Suppose $t \in H$ is an involution, $O_t = O(C_G(t)) \leq H$, and an actual 2-component J of $C_G(t)$ lies in H and contains a fourgroup A such that $C_G(t, A) \leq H$. Then

$$(23) \quad O_{2',2}(C_G(t))L_t \leq H.$$

Indeed every other 2-component M has image centralized by A in $C_G(t)/O_t$. Canonical lifting gives actual normalization, and odd-kernel fixed-point lifting gives

$$M = O(M)C_M(A) \leq O_t C_G(t, A) \leq H.$$

The identical argument for $O_{2',2}(C_G(t))$, whose quotient by O_t centralizes the component image of J , proves (23).

For $t = e \in E^\#$, use the component supplied by Theorem 4.3 of $(K_i)_e$, which contains F and lies in H , since $K_i, O_e \leq H$. For $t = v \in F^\#$ interchange the roles. Thus, (23) holds for all involutions in E and F . By Theorem 2.6, the group T has one conjugacy class of involutions. Odd-kernel complement conjugacy gives the same assertion in each K_m . Since $K_i, K_j \leq H$, conjugating by these groups proves (23), and $O_t \leq H$, for every involution in K_i or K_j .

Now let $t \in \text{Inv}(K_m)$ with $m \notin \{i, j\}$. The group EF acts trivially on $K_m/O(K_m)$. Fixed-point lifting, followed by conjugacy of involution lifts over the odd kernel, allows conjugation of t by an element of $O(K_m) \leq H$ so that $[t, EF] = 1$. Then $O_t \leq H$ by (22). For $e \in E^\#$, the group $\langle e, t \rangle$ is a fourgroup in $C_G(K_i)$ centralizing F . Theorem 7.3 gives

$$\theta_E(v) = \theta_{\langle e \rangle}(v) = \theta_{\langle e, t \rangle}(v) = \theta_{\langle t \rangle}(v)$$

for all $v \in F^\#$. Thus, $W = W(F, \langle t \rangle)$ and $C_G(t, F) \leq H$. The component supplied by Theorem 4.3 of $(K_i)_t$ lies in H and contains F . The preceding observation proves (23). Undoing the conjugation covers the original t . We have proved

$$(24) \quad O(C_G(t)) \leq H \quad \text{and} \quad O_{2',2}(C_G(t))L_t \leq H$$

for every $t \in \text{Inv}(K_m)$ and every $1 \leq m \leq r$.

Recovering the odd core. Put $R = O(H)$. For a pure involution t , the odd group $C_R(t)$ is normal in $C_H(t)$. By (24), every component and the $O_{2',2}$ -subgroup of $C_G(t)$ lie in H . Intersecting the component subnormal chains with $C_H(t)$ therefore shows that the image of $C_R(t)$ centralizes every component of $C_G(t)/O(C_G(t))$. It also centralizes the normal 2-core of that quotient. Self-centralization of the generalized Fitting subgroup gives

$$(25) \quad C_R(t) \leq O(C_G(t))$$

for every $t \in \text{Inv}(K_m)$ and every $1 \leq m \leq r$. For any $v \in F^\#$ this implies

$$C_R(F) \leq O_v \cap C_G(F) \leq O(C_G(F)).$$

Hence, for $e \in E^\#$, coprime factorization gives

$$C_R(e) = [C_R(e), F]C_R(e, F) \leq [O_e, F](O_e \cap O(C_G(F))) = \theta_F(e).$$

Conversely $\theta_F(e) \leq W \leq R$ and this group centralizes e , so equality holds. Fourgroup generation under E proves

$$R = \langle C_R(e) : e \in E^\# \rangle = W(E, F) = W.$$

The marked components of H . Theorem 5.3 applies to $N_0 \leq H < G$. It places each marked factor of N in a marked factor of H , with distinct factors remaining distinct, and proves H $\mathcal{F}$ -balanced. Maximality of $r(N)$ permits no additional marked factors. If K'_i is the factor of $D(H)$ containing K_i , both quotients $K_i/O(K_i)$ and $K'_i/O(K'_i)$ are isomorphic to T . Since $K_i \cap O(K'_i) \leq O(K_i)$, comparison of orders gives $K'_i = O(K'_i)K_i$. Multiplying these equalities gives

$$D(H) = O(D(H))D(N).$$

This proves (20). For the rest of this proof, use $K_1, \dots, K_r$ for the marked components of H , retaining N for the original subgroup.

The general core-recovery identity. After $\mathcal{F}$ -balanced inheritance and the maximal choice give $D(H) = O(D(H))D(N)$, every pure involution in an enlarged marked factor is conjugate by its odd core into its original factor. The conjugating element belongs to H , so (24) and (25) remain valid for the enlarged factors. Let $B^* \leq K_m$ be a fourgroup and let $1 \neq V^* \leq C_{K_n}(B^*)$ be elementary abelian, with $m \neq n$. The first containment in the core-recovery calculation of [26, Lemma 3.11] is $\theta_{V^*}(b) \leq R$ for every $b \in (B^*)^\#$. Here is its actual-component justification. The group O_b lies in H and has odd normal image in $C_{H/R}(bR)$: it is normal in $C_H(b)$, and odd-kernel centralizer lifting makes $C_H(b) \rightarrow C_{H/R}(bR)$ surjective. It therefore centralizes all generalized-Fitting factors of H/R except possibly the marked component containing bR . In particular $[O_b, V^*] \leq R$. The component supplied by Theorem 4.3 of $(K_m)_{V^*}$ lies in $L_v \leq H$ for $v \in (V^*)^\#$, by $L_{2'}$ -balance. Its image in H/R is precisely the marked component containing bR : $(K_m)_{V^*}$ covers that component, and a perfect odd-by- T image containing this actual component equals it. Its odd core therefore lies in R . The odd group $O(C_G(V^*))$ centralizes this containing component modulo its odd core. Consequently $O_b \cap O(C_G(V^*))$ also centralizes the remaining marked component image. Self-centralization of $F^*(H/R)$ gives $O_b \cap O(C_G(V^*)) \leq R$, as required.

Conversely, (25) gives $C_R(b) \leq O_b$ and $C_R(V^*) \leq O(C_G(V^*))$: for the latter, intersect $O(C_G(v))$ with $C_G(V^*)$ for any $v \in (V^*)^\#$. Coprime factorization under V^* now gives

$$C_R(b) = [C_R(b), V^*]C_R(b, V^*) \leq \theta_{V^*}(b).$$

Thus, $C_R(b) = \theta_{V^*}(b)$, and fourgroup generation under B^* proves (21). The group V^* may have order two throughout this argument. $\square$

The recovery formula forces $N_G(W)$ to contain all the pure-involution centralizers required by the embedding lemma. A nontrivial W would therefore give a contradiction.

Lemma 7.7 (Vanishing of the signalizer). *Every admissible pair (B, V) in (12) satisfies $W(B, V) = 1$.*

Proof. Suppose $W \neq 1$ and use the enlarged $\mathcal{F}$ -balanced subgroup H from Theorem 7.6. An element $g \in N_G(D(H))$ permutes the marked components of H and carries each pair in (21) to another such pair. Conjugating that equality gives $O(H)^g = O(H)$. Since $H = N_G(O(H))$, we obtain $N_G(D(H)) \leq H$. Conversely, H normalizes $D(H)$ by its definition, so $N_G(D(H)) = H$.

Let x be a pure involution of $D(H)$ and $g \in C_G(x)$. Choose a fourgroup in the component of $C_G(x)$ obtained from a different marked factor by Theorem 4.3. This component and its g -conjugate lie in H , by (19). Theorem 5.4 places them in marked factors of H . Neither can lie in the factor containing x : otherwise its image would cover that simple factor while centralizing the noncentral involution $\bar{x}$. Apply (21) to the fourgroup and $\langle x \rangle$, and to their g -conjugates. This gives $O(H)^g = O(H)$ and hence $g \in H$. Thus, H satisfies Theorem 6.1, a contradiction. $\square$

8. EXCLUDING INTERCHANGED COMPONENTS

We complete the exclusion argument using Solomon's signalizer method and his application of Aschbacher's embedding criterion [26], developed in the preceding sections. Vanishing of the signalizer first makes the selected 2-components ordinary: their odd cores are central, so they are quasisimple. This permits the exact normalizer comparisons needed for the final contradiction.

Lemma 8.1. *The selected layer $D = D(N)$ is semisimple. For every pure involution $b \in D$, the entire group $L_2(C_G(b))$ is semisimple, and each opposite factor of D is an ordinary component of $C_G(b)$.*

Proof. Fix distinct indices i, j , take a fourgroup $B_0 \leq K_j$, and choose a fourgroup $V \leq (K_i)_{B_0}$. Extend B_0 to the admissible rank-three group $B = B_0 \times \langle z_0 \rangle$ using the pair-kernel and central Sylow construction at the start of Section 7. It applies to every pair of distinct marked factors, for $r = 2$ as well as $r \geq 3$, and preserves the commuting fourgroups.

For $b \in B_0^\#$, vanishing says that $O(C_b)$ centralizes V . Let J_b be the unique containing component of $(K_i)_{\langle b \rangle}$. The normal subgroup $C_{J_b}(O(C_b))$ contains V , so its image is a nontrivial normal subgroup of $J_b/O(J_b) \cong T$. It covers this quotient; its quotient in the perfect group J_b is odd, hence trivial. Thus, $[O(C_b), J_b] = 1$, in particular $O(J_b) \leq Z(J_b)$. Perfection and $J_b = O(J_b)(K_i)_{\langle b \rangle}$ give $J_b = (K_i)_{\langle b \rangle}$, which is quasisimple.

The fourgroup B_0 acts trivially on each $(K_i)_b$ modulo its center. To see that it centralizes $(K_i)_b$, put $Q = (K_i)_b$ and take $a \in B_0$. Since $[Q, a] \leq Z(Q)$, the map

$$Q \longrightarrow Z(Q), \quad x \longmapsto x^{-1}x^a$$

is a homomorphism. The group Q is perfect, so every homomorphism from Q to an abelian group is trivial. Thus, a centralizes Q , as required. Put $R = C_{K_i}(B_0)$. For every $b \in B_0^\#$, the perfect subgroup $(K_i)_b$ lies in R , so $(K_i)_b \leq R^{(\infty)}$. Conversely, $R \leq C_{K_i}(b)$ implies

$$R^{(\infty)} \leq C_{K_i}(b)^{(\infty)} = (K_i)_b.$$

Thus, all the groups $(K_i)_b$ coincide with $R^{(\infty)}$; denote this common subgroup by P . Each $C_{O(K_i)}(b)$ normalizes P , and these centralizers generate $O(K_i)$. Hence $P \trianglelefteq K_i$. Since $K_i = O(K_i)P$, the quotient

K_i/P is a quotient of the odd-order group $O(K_i)$ and is therefore solvable by the odd-order theorem. It is also perfect, being a quotient of the perfect group K_i . A solvable perfect group is trivial, so $K_i = P$. Thus, all the K_i are ordinary components.

Vanishing also gives, for pure involutions $b \in K_i$, $v \in K_j$,

$$(26) \quad [O(C_b), K_j] = 1, \quad O(C_b) \cap O(C_v) = 1.$$

For the first assertion, vary the fourgroups in K_j ; their involutions generate K_j . For the second, take a fourgroup $V \leq K_j$ containing v . The intersection in question centralizes V , and $O(C_v) \cap C_V \leq O(C_V)$. It is therefore killed by $\theta_V(b) = 1$. The first assertion and the single-component identity show that the containing component of K_j in C_b is just K_j : its extra odd kernel centralizes the perfect subgroup covering its quotient.

Write

$$L_b = K_j R_b, \quad L_v = K_i R_v,$$

where R_b, R_v are the products of the other 2-components. The ordinary component K_j centralizes R_b , so $R_b \leq C_v$. Intersecting subnormal chains puts its factors in $C_b \cap C_v$; L_{2r} -balance inside C_v , with acting subgroup b , gives $R_b \leq L_v$. Its image in $K_i/Z(K_i) \cong T$ is a perfect subgroup of the involution centralizer $C_T(\bar{b})$, which is solvable by Theorem 2.6, and is therefore trivial. Perfection removes the remaining central image, so $R_b \leq R_v$. Symmetry gives equality. Its odd core lies in both $O(C_b)$ and $O(C_v)$ and is trivial by (26). Thus, R_b is semisimple, and so is L_b . $\square$

Semisimplicity strengthens the earlier containments to equalities, giving control of the normalizer of every nonempty product of selected components.

Lemma 8.2. *For every nonempty product D_0 of components of D , put $H_0 = N_G(D_0)$. Then we have $D(H_0) = D$ and $N_G(D_0) \leq N_G(D)$.*

Proof. The subgroup H_0 is proper, since $1 < D_0 \leq D < G$ and G is simple. Also $N_0 \leq H_0$: each component and the subgroup $O_{2r,2}(N)$ normalize each chosen component of D . Theorem 5.3 and maximality of r give only the preliminary equality $D(H_0) = O(D(H_0))D$.

Each factor A of D_0 is an ordinary component of H_0 , since $D_0 \trianglelefteq H_0$. If K_j lies outside D_0 , let K'_j be the marked component of H_0 containing it. For $b \in \text{Inv}(A)$, the ordinary component A centralizes K'_j , so K'_j is a 2-component of $C_{H_0}(b)$. Theorem 8.1 makes K_j an actual component of $C_G(b)$, and intersection of subnormal chains makes it a component of $C_{H_0}(b)$. Thus, $K_j \leq K'_j$ are two actual 2-components of the same group, so $K_j = K'_j$. The inside factors were already components, and the maximal number r excludes extra marked factors. This proves $D(H_0) = D$. Its conjugacy-invariant definition then implies $H_0 \leq N_G(D)$. $\square$

Applied to the pure-involution centralizers, this normalizer control gives the contradiction to the embedding lemma that excludes interchanged components.

Theorem 8.3 (Exclusion of interchanged components). *Under Theorem 2.2, a GS-maximal local 2-component Y of top T cannot be interchanged with a distinct component by a commuting involution. More precisely, if Y is a 2-component of $C_G(z)$ and s is an involution commuting with z , then $Y^s = Y$.*

Proof. If a pair of components interchanged by an involution existed, the marked construction and the maximizing $\mathcal{F}$ -balanced subgroup above would be available. Put $H = N_G(D)$. By Theorem 8.2, H is proper and $\mathcal{F}$ -balanced and $D(H) = D$. For $b \in \text{Inv}(K_i)$, every opposite factor K_j is an actual ordinary component of $C_G(b)$. Hence its layer normalizes K_j , and Theorem 8.2 gives

$$L_b \leq N_G(K_j) \leq H.$$

Every marked component of $C_G(b)$ therefore lies in H and has the kernel involution b there. Theorem 5.4 puts it in a marked component of H . Both groups are now quasisimple, by Theorem 8.1; containment and

equality of their simple tops force equality. Consequently $D(C_G(b))$ is a nonempty product of components of D . It is normalized by $C_G(b)$, so

$$C_G(b) \leq N_G(D(C_G(b))) \leq H.$$

The last containment is Theorem 8.2. Thus, H meets all the hypotheses of Theorem 6.1, a contradiction. $\square$

8.1. Proof of the comparison theorem. A GS-maximal component admits no outgoing mirror step. Combining this exclusion with closure under GS correspondences and conjugacy proves preservation for the components reached from the GS-maximal predecessors constructed in Theorem 3.5.

Proof of Theorem 2.3. Let $\mathcal{M}$ be the family of all GS-maximal local 2-components of top T . Every member belongs to $\mathcal{F}_T$, and Theorem 2.5 gives its quotient by its odd core as T . The family $\mathcal{M}$ is closed under conjugacy. It is also closed under a forward GS correspondence: if Y is GS-maximal and $Y \rightarrow_{GS} X$, any forward chain starting at X can be prefixed by this edge, so every endpoint has top T and X is itself GS-maximal.

No mirror step starts at a member of $\mathcal{M}$. Such a step would have commuting witnesses z, s with Y a 2-component of $C_G(z)$ and $Y^s \neq Y$, contradicting Theorem 8.3. Thus, $\mathcal{M}$ is closed under the entire relation $\rightsquigarrow$, and every member is mirror-maximal.

Take $A \in \mathcal{F}_T$. Theorem 3.5 supplies $Y \in \mathcal{M}$ with $Y \rightsquigarrow A$. Closure forces $A \in \mathcal{M}$. If also $A \rightsquigarrow J$, the same closure gives $J \in \mathcal{M}$, so $\tau(J) \cong T$. $\square$

9. THE DEDUCTION OF UGT

The deduction of UGT combines Foote's recognition theorem [12] with the reductions of Gilman–Solomon [13, §§1, 3–4], including the recognition result of Foote and Harris recorded in their Theorem 1.7. The comparison theorem supplies the passage from pointed maximality to the maximality required for recognition. We first specify the maximality and core conditions used in this application. Call a pair (x, L) *Foote-maximal* if L is a 2-component of $C_G(x)$ and every chain of the unpointed relation generated by (2) preserves $\tau(L)$. In the unpointed relation the witnessing involution may be chosen anew at each step. Such a pair is pointed-maximal, since every pointed chain is also an unpointed chain.

If $m_2(\mathcal{C}_G(L)) = 1$, then (x, L) is Foote-maximal: the distinct commuting witnesses of any first step would generate a fourgroup in $\mathcal{C}_G(L)$.

An *unbalancing triple* (x, y, L) consists of commuting involutions $x, y \in G$ and a 2-component L of $C_G(x)$ such that

$$[L, y] = L = [L, D], \quad D = O(C_G(y)) \cap C_G(x).$$

These equalities imply that y normalizes L and $x \neq y$.

For a finite group H , let $\mathbf{C}(H)$ denote the following *hereditary core condition*. For every subgroup $U \leq H$, every pair of commuting involutions $a, b \in U$, and every 2-component Y of $C_U(b)$, either $\tau(Y) \cong \mathrm{PSL}_2(q)$ for an odd prime power q , or

$$[(O(C_U(a)) \cap C_U(b))', Y] \leq O(C_U(b)).$$

Here $U = H$ is allowed.

Once its rank hypothesis is verified, the comparison theorem supplies the maximality needed for Foote's recognition theorem. The alternatives requiring further attention are an action kernel of 2-rank one and four explicit exceptional pairs.

Theorem 9.1 (Recognition from pointed maximality). *Let G be nonabelian simple, unbalanced, and $\mathcal{K}$ -proper, and suppose $\mathbf{C}(G)$ holds. Let (x, L) be pointed-maximal, with $L/O(L) \cong A_7$ or $\mathrm{PSL}_2(q)$ for an odd prime power $q \geq 5$. Then one of the following holds:*

- (i) G is of odd Lie type;
- (ii) $m_2(\mathcal{C}_G(L)) = 1$;

(iii) $(G, L/O(L))$ has one of the types

$$(A_9, A_5), \quad (A_{10}, A_6), \quad (A_{11}, A_7), \quad (J_2, A_5).$$

Proof. Put $T = L/O(L)$, and suppose G is not of odd Lie type. We use Aschbacher's rank-one recognition result in the form recorded in [13, Theorem 1.2, p. 57]: if $F^*(G)$ is quasisimple and J is a 2-component of $C_G(t)$ with $m_2(J) = 1$ and $t \in J$, then unbalancedness of G implies that $F^*(G)/Z(G)$ is of odd Lie type. Here $F^*(G) = G$ and $Z(G) = 1$. This formulation includes odd-core quotients isomorphic to $2A_7$; it does not require the quotient to be $\mathrm{SL}_2(q)$.

We first verify the rank hypothesis for the entire family $\mathcal{F}_T$ in Theorem 2.2. By the covering-group facts [19, Section 6.1], recalled before Theorem 2.5 without any family-rank assumption, an occurrence of type T has quotient by its odd core either T , of 2-rank two, or its perfect double cover $\tilde{T}$, of 2-rank one.

Suppose (z, A) is pointed-maximal of type T with $m_2(A) = 1$. We use the rank-one argument in [12, proof of Theorem 3.1, pp. 430–431], retaining the successive witnesses. Choose $a \in \mathrm{Inv}(A)$. Its image is central in $A/O(A) \cong \tilde{T}$, so $a \in C_G(A)$. If $a = z$, then A is a rank-one 2-component of $C_G(a)$ containing a . The rank-one recognition result just stated therefore gives the excluded odd Lie conclusion.

If $a \neq z$, balance and Theorem 3.1 give $(z, A) \rightarrow_F (a, K)$. Pointed maximality gives $\tau(K) \cong T$. The fixed layer $D = A_a$ has quotient $\tilde{T}$, and $a \in D$ because $C_A(a)/D$ has odd order. If $K^z = K$, then $D \leq K$ forces $K/O(K) \cong \tilde{T}$; hence $m_2(K) = 1$. Since K is a 2-component of $C_G(a)$ and $a \in K$, the same rank-one recognition result applies.

Suppose $K^z \neq K$, and put $W = \langle K, K^z \rangle$. Modulo $O(W)$, the factors are interchanged by z and the image of D is fixed. The perfect residual of this fixed group is T unless both factors are double covers with distinct centers. Since $D/O(D) \cong \tilde{T}$, we obtain

$$W/O(W) \cong \tilde{T} \times \tilde{T}.$$

Choose a z -invariant Sylow 2-subgroup Z of $Z^*(W)$ containing a , by taking a Sylow subgroup in $Z^*(W) \rtimes \langle z \rangle$ containing $\langle a, z \rangle$. Then Z is a fourgroup. If $\langle b \rangle = Z \cap K$, the diagonal description of the fixed group gives $a = bb^z$. Thus, $a \neq b$, $[a, b] = 1$, and $b \in C_G(K)$.

Take the next pointed step $(a, K) \rightarrow_F (b, J)$. Its source is the preceding target, so $\tau(J) \cong T$. Both $b \in K_b$ and $b^z \in (K^z)_b$, by the odd-order fixed-layer quotients. Balance places these subgroups in $L_{2'}(C_G(b))$. Hence $a = bb^z$ normalizes J , and the correspondence gives $K_b \leq J$. It follows that $J/O(J) \cong \tilde{T}$ and $b \in J$. Thus, J is a rank-one 2-component of $C_G(b)$ containing b , and the same recognition result again gives the excluded odd Lie conclusion.

Consequently every member of $\mathcal{F}_T$ has rank two. Theorem 2.3 makes L mirror-maximal, and Theorem 3.1 makes it Foote-maximal. Apply [12, Main Theorem, p. 426], whose core hypothesis is exactly $\mathrm{C}(G)$. The normal closure of L in $L_{2'}(G) = G$ is G . Foote's first two alternatives are impossible by simplicity and $|L/O(L)| < |G|$. His fourth gives (ii), since $x \in C_G(L)$; his third gives $G \in \{A_9, A_{10}, A_{11}, J_2\}$. In A_n for $9 \leq n \leq 11$, the only nonsolvable involution centralizers have their unique 2-component A_{n-4} on the fixed points. For J_2 , the component has type A_5 [6, Table 1, p. 441]. These give (iii). $\square$

Of the four exceptional groups, A_{11} is allowed by UGT; the other three are excluded by their trivial local odd cores.

Lemma 9.2. *If $G \cong A_9$, A_{10} , or J_2 , then $O(C_G(t)) = 1$ for every involution $t \in G$.*

Proof. For $G = A_n$, where $n = 9$ or 10 , an involution t has $k = 2$ or 4 transpositions and $f = n - 2k$ fixed points. The action on these orbits gives a homomorphism

$$C_{A_n}(t) \longrightarrow S_k \times S_f$$

with a 2-group kernel and surjective coordinate projections. Here $k \in \{2, 4\}$ and $f \in \{1, 2, 5, 6\}$, so both symmetric groups have trivial odd core. Thus, $O(C_{A_n}(t))$ lies in the 2-group kernel and is trivial.

For J_2 , each involution centralizer has a normal 2-subgroup with quotient A_5 [6, Table 1, p. 441]. Its odd core therefore lies in that 2-subgroup and is trivial. $\square$

To obtain a pointed-maximal component from unbalancedness, we use Gilman–Solomon’s reductions. These involve an unbalanced-group assertion for groups that need not be simple.

For a finite group H , write $U(H)$ for the following assertion. If $t \in \text{Inv}(H)$ and $O(C_H(t)) \not\leq O(H)$, then $O(C_H(t))$ acts nontrivially on $\tau(J)$ for some 2-component J of H , where $\tau(J)$ is of odd Lie type, an alternating group of odd degree, $\text{PSL}_3(4)$, or the Held sporadic group He . This is the general unbalanced-group assertion of [13, p. 55].

For proper sections, the known groups supply both this assertion and the restriction on odd-core commutators needed for $C(G)$. We use the known-group calculations recorded by Gorenstein, Lyons, and Solomon [19, Theorem 7.7.1], together with the reductions of Gilman–Solomon and Foote cited in the proof.

Lemma 9.3. *Let H be a $\mathcal{K}$ -group. The following are true:*

- (i) $U(H)$ holds.
- (ii) *If $a \in \text{Inv}(H)$ and Y is a 2-component of H with $\tau(Y) \not\cong \text{PSL}_2(q)$ for every odd prime power q , then*

$$[O(C_H(a))', Y] \leq O(H).$$

Proof. Let S be a nonabelian simple $\mathcal{K}$ -group, $S \leq X \leq \text{Aut}(S)$, and $a \in \text{Inv}(X)$. The known-group calculation [19, Theorem 7.7.1, pp. 385–388] shows that $O(C_X(a)) \neq 1$ forces S to be of odd Lie type, of odd alternating degree, $\text{PSL}_3(4)$, or He . We also use

$$(27) \quad S \not\cong \text{PSL}_2(q) \ (q \text{ odd}) \implies O(C_X(a)) \text{ is abelian.}$$

The alternating, sporadic, and characteristic-two cases are given by parts (a)–(c) of that theorem. For odd Lie type, let $I \leq \text{Aut}(S)$ be the inner-diagonal automorphism group of [19, Definition 2.5.10], and use the Lie components defined in [19, Definition 4.9.3]. If $O(C_X(a)) = 1$, there is nothing to prove. Otherwise, [19, Theorem 7.7.1(d)(1)] gives $a \in I$. Part (d)(2) says that $O(C_X(a))$ centralizes these components. The automorphism description in [19, Theorem 2.5.12] and the centralizer list in Table 4.5.1 then give $O(C_X(a)) \leq I$, unless $S \cong \text{PSL}_2(q)$. In that automorphism description, an odd-order automorphism with nontrivial field action acts nontrivially on a listed component [19, Proposition 4.2.3(b)]. The remaining order-three automorphisms in type D_4 either move the involution class or permute its components. Since I/S is abelian, $C_{X \cap I}(a)$ is normal in $C_I(a)$. Its odd core is therefore normal in $C_I(a)$ and contains $O(C_X(a))$. Thus, $O(C_X(a)) \leq O(C_I(a))$, which is cyclic by part (d)(3). This proves (27) for every intermediate X , as required.

If (i) fails, choose a $\mathcal{K}$ -group H of least order for which it fails. Its proper sections satisfy U . Lemma 3.1 of [13] gives $F^*(H) = S$ simple. Thus, $S \in \mathcal{K}$ and $S \leq H \leq \text{Aut}(S)$. The preceding known-group calculation gives $U(H)$, since a nontrivial subgroup of H cannot centralize S , a contradiction.

For (ii), put $R = O(C_H(a))$ and pass to $H/O(H)$. The image of Y is quasisimple. If a does not normalize Y , or acts trivially on $Y/O(Y)$, then R acts trivially there [12, Lemma 2.1(iii)]. Otherwise [18, Proposition 5.32(iii), pp. 33–34], applied with $P = \langle a \rangle$, gives a quotient X of $\langle Y, R, a \rangle$ with

$$F^*(X) \cong \tau(Y), \quad \bar{R} \leq O(C_X(\bar{a})).$$

The kernel of this quotient centralizes $Y/O(Y)$, and $\bar{a}$ is an involution. Equation (27) makes $\bar{R}$ abelian. Hence R' centralizes $Y/O(Y)$, which proves (ii). $\square$

The remaining reductions connect unbalancedness to recognition: one produces a pointed-maximal unbalancing triple, and the other handles an action kernel of 2-rank one without requiring such a triple.

Proposition 9.4 (Reductions to recognition). *Let G be a nonabelian simple $\mathcal{K}$ -proper group that is unbalanced and does not satisfy the conclusion of Theorem 1.2. Then the following statements hold.*

- (R1) *If G has least order among such groups, then $C(G)$ holds and there is an unbalancing triple (x, y, L) such that $L/O(L) \cong A_7$ or $\text{PSL}_2(q)$ with $q \geq 5$ odd, and (x, L) is pointed-maximal.*
- (R2) *If $x \in \text{Inv}(G)$ and L is a 2-component of $C_G(x)$ with $L/O(L) \cong A_7$ or $\text{PSL}_2(q)$ for odd $q \geq 5$, and $m_2(C_G(L)) = 1$, then G is of odd Lie type.*

Proof. Every proper section of G is a $\mathcal{K}$ -group, so satisfies $\mathbf{U}$ by Theorem 9.3(i). The standard-component hypothesis $\mathbf{S}$ in [13, p. 58] also holds in every proper section: whenever such a section has a simple generalized Fitting subgroup, that subgroup already belongs to $\mathcal{K}$. This verifies both proper-section hypotheses of their Theorem 1.7.

We prove $\mathbf{R2}$ first. The rank-one condition makes (x, L) Foote-maximal, as observed above. Apply [13, Theorem 1.7], the combined recognition theorem of Foote and Harris. Since G is simple, x acts on $F^*(G) = G$ by an inner automorphism. Its listed possibilities therefore give either odd Lie type or

$$G \in \{A_9, A_{10}, A_{11}, J_1, J_2, M_{12}, HS\}.$$

Here J_1 and HS are the first Janko and Higman–Sims sporadic simple groups, respectively. The sporadic groups in this list have trivial odd cores in all involution centralizers [19, Theorem 7.7.1(b)]; A_9 and A_{10} are excluded by Theorem 9.2. If $G = A_{11}$, the listed component is A_7 on the fixed points of a double transposition. The fourgroup on the other four points centralizes it, contrary to $m_2(\mathcal{C}_G(L)) = 1$. This proves $\mathbf{R2}$.

For $\mathbf{R1}$, first let $U \leq G$ and let a, b be commuting involutions of U . Set

$$M = C_U(b), \quad D = O(C_U(a)) \cap C_U(b).$$

Then $D \trianglelefteq C_M(a)$ has odd order, so $D \leq O(C_M(a))$. Since $M \leq C_G(b) < G$, part (ii) of Theorem 9.3 applied to M gives exactly $\mathbf{C}(G)$.

Next, $\mathbf{U}(G)$ cannot hold. Otherwise G would itself be one of the groups listed in the definition of $\mathbf{U}$, and the known-group calculation [19, Theorem 7.7.1], with an inner involution, would give the conclusion of $\mathbf{UGT}$. In the alternating case $G = A_n$, the involution must fix three points and have an even number of transpositions, so $n \equiv 3 \pmod{4}$. Lemma 1.1 of [13] therefore supplies an unbalancing triple. Together with the proper-section assertion already proved, this makes G a minimal unbalanced group in the sense of that paper.

We now apply its type reduction and maximal-triple argument [13, Theorems 1.2–1.6, Lemma 3.9, and §4]. We spell out the type selection. The first four recognition results, as used in the first paragraph of §4, reduce the quotients of unbalancing components to

$$\mathrm{PSL}_2(q) \ (q \text{ odd}), \quad A_7, \quad \text{covering groups of } \mathrm{PSL}_3(4), \quad He.$$

Here the list describes $J/O(J)$ for every unbalancing component J , not merely its simple top. In the $\mathrm{PSL}_2(q)$ and A_7 alternatives, the quotient is the simple group itself and contains a fourgroup; their double covers are not included in these alternatives. The Held group also contains a fourgroup, as is visible in its involution centralizers [6, Table 1]. For a covering group Q of $\mathrm{PSL}_3(4)$, suppose instead that $m_2(Q) = 1$. A Sylow 2-subgroup P of Q would then be cyclic or generalized quaternion. Every quotient of such a group has 2-rank at most two. But P maps onto a Sylow 2-subgroup of $\mathrm{PSL}_3(4)$, which contains the elementary abelian subgroup of order 16 represented by

$$\left\{ \begin{pmatrix} 1 & u & v \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : u, v \in \mathbf{F}_4 \right\}.$$

These matrices intersect the scalar subgroup trivially, so their images still have order 16, a contradiction. Thus, every quotient in the list has 2-rank at least two. Since quotienting by the odd normal subgroup $O(J)$ preserves 2-rank, every unbalancing component J satisfies $m_2(J) \geq 2$, as required by [13, Lemma 3.9]. The proper-subgroup B -property required by Lemma 3.9 is Theorem 2.7. In its maximization argument, use Theorem 3.3 for the increase of quotient order or centralizer rank. This supplies the containment form of Lemma 3.4 needed there.

If an unbalancing component of one of the last two types occurs, Theorem 1.6 supplies an unbalancing component with simple top $\mathrm{PSL}_2(q)$, $q \geq 27$: its other conclusion would give $G \cong \mathrm{PSL}_3(4)$ or He , both of which have $O(C_G(t)) = 1$ for every $t \in \mathrm{Inv}(G)$. Apply Lemma 3.9 starting with this component. In its sequence of unbalancing triples the type remains $\mathrm{PSL}_2(r)$ with $r \geq 27$. Indeed a correspondence into

A_7 , a covering group of $\text{PSL}_3(4)$, or He cannot have such a source: their involution centralizers have no component with this simple top. The centralizer lists are [19, §4.9 and Table 5.3]; for He see also [6, Table 1]. At a step between groups of PSL_2 type the quotient order cannot decrease. If neither of the last two types occurs, start instead with any unbalancing triple and apply Lemma 3.9 directly. Thus, in either case we obtain an unbalancing triple (x, y, L) of PSL_2 or A_7 type satisfying the maximality hypothesis of [13, Theorem B].

It remains to verify hypothesis (c) of [13, Theorem B]: the relevant standard-component problems in core-free proper sections must have solutions of *restricted type*, as defined there on p. 59. Let H be such a section, so $O(H) = 1$, and let J be a component of $C_H(a)$ of one of the specified types, standard in H in the sense of [13, p. 56]. Put $E = E(H)$ and $X = \langle J^H \rangle$. By Theorem 2.7 and L -balance, $J \leq E$. The components of $C_E(a)$ lie in single components of E or diagonally across pairs interchanged by a . Thus $Y = \langle J^E \rangle$ is a component or a central product of two components of E .

Standardness implies that $J \trianglelefteq C_H(b)$ for every $b \in \text{Inv}(C_H(J))$: if $h \in C_H(b)$, then b centralizes both J and J^h , so $J = J^h$. If $J \trianglelefteq E$, then J is a component of H ; distinct conjugate components commute, so standardness gives $X = J$. Otherwise $E = Y$: an involution in any component outside Y would force $J \trianglelefteq Y$, hence $J \trianglelefteq E$. For the same reason $Z(E)$ and $O_2(H)$ contain no involutions. Since $O(H) = 1$, this gives $Z(E) = 1$, $O_2(H) = 1$, and $F^*(H) = E = X$. Consequently X is simple or is a direct product of two simple groups interchanged by a . In the latter case J is diagonal and each factor has the simple type of J . It therefore suffices to consider X simple, with $X \leq H \leq \text{Aut}(X)$.

Suppose first that $J/O(J) \cong \text{PSL}_2(q)$, q odd, or A_7 . Here J is GS-maximal in H . Indeed, in a correspondence from (a, J) witnessed by b , the condition $[J, b] \leq O(J) \leq Z(J)$ and perfectness give $[J, b] = 1$. Thus, $J \trianglelefteq C_H(b)$. If K is the corresponding component, $J \leq \langle K, K^a \rangle$ forces $J = K$ or $J = K^a$; as a centralizes J , either gives $J = K$. The proper sections of H satisfy U and Hypothesis S, as noted above. Hence [13, Theorem 1.7] places X in the restricted list.

For $J/Z(J) \cong \text{PSL}_3(4)$, use the involution-centralizer lists for the known simple group X . Alternating groups and groups of odd Lie type have no such component; see [19, Table 4.5.1 and §4.9] for the latter. In characteristic two, if $a \in X$, then $F^*(C_X(a)) = O_2(C_X(a))$ by [7, (2.4)], so $E(C_X(a)) = 1$, contrary to the existence of J . Thus, $a \notin X$, and the outer-involution list leaves only $X \cong \text{PSL}_3(16)$, with a inducing a field involution [7, §19], with the correction to (19.1) in [8, p. 136]. For sporadic X , the possibilities are Suz , He , and $O'N$ [19, Table 5.3]; see also [6, Table 1]. These groups, and the simple types in the cases $X = J$ and X a product of two factors, are all restricted in the sense of [13, p. 59, Theorems 1.7–1.8]. This proves hypothesis (c).

Apply Theorem B. Its first alternative is $\text{U}(G)$, already excluded. Its second supplies a local component K with

$$K/O(K) \cong \text{PSL}_2(r^2), \quad \langle b \rangle \in \text{Syl}_2(\mathcal{C}_G(K)).$$

Reduction R2 excludes this alternative without any unbalancing-triple assumption on K . The remaining alternative is Theorem B(3): the correspondences from (x, L) are isomorphic and satisfy its Sylow-normalizer condition. Lemma 3.5 of that paper, or the argument of Theorem 3.4, makes L GS-maximal. By Theorem 3.1, (x, L) is pointed-maximal. $\square$

Applying these reductions to a least counterexample leaves exactly the alternatives already settled by recognition and the odd-core calculations.

Proof of Theorem 1.2. Choose a counterexample G of least order. Reduction R1 gives $\mathcal{C}(G)$ and a pointed-maximal unbalancing triple (x, y, L) of one of the required types. The group G is unbalanced by the hypothesis of UGT, so Theorem 9.1 applies: either G is of odd Lie type, $m_2(\mathcal{C}_G(L)) = 1$, or $G \in \{A_9, A_{10}, A_{11}, J_2\}$. Apply reduction R2 in the rank-one case. Every remaining group is allowed by UGT or excluded by Theorem 9.2, a contradiction. $\square$

Remark 9.5 (A shorter deduction and the formalized route). The reductions above also permit a shorter deduction of UGT. The proof of reduction R1 obtains GS-maximality of L , which implies Foote-maximality

directly by Theorem 3.1 and the definition of maximality in [12, p. 426]. Since $L/O(L)$ is $\mathrm{PSL}_2(q)$ or A_7 , it has 2-rank two. Foote’s Main Theorem therefore applies directly, with the hereditary core condition supplied by the same reduction. The remaining alternatives are handled by reduction R2 and the odd-core calculations above. This deduction does not require Theorem 2.3. We retain the longer argument because AxiomProver formalized this route through Solomon’s maximal-2-component theory. The presentation thus follows the formalized argument and also establishes the stronger mirror-maximality conclusion of Theorem 2.3.

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