

Periodic Forcing, Echo Constraints and the Non-Periodicity of the Rule 30 Centre Column: Structural Lemmas, Certified Computations and Barriers

Isaac Kaimfa

October 2026

Abstract

Wolfram’s Rule 30 Prize Problem 1 asks whether the centre column of Rule 30, started from a single lit cell, is ever eventually periodic. We study the problem through a half-line system driven by a periodic boundary signal. Combining a known local identity with the standard reduction to finite seeds, Problem 1 follows from a single statement, the *Echo Lemma*: no finite half-line configuration driven by a non-constant periodic boundary can satisfy, at every lit boundary minute, the identity that Rule 30 forces on the neighbour of the centre. Around this formulation we prove a collection of exact structural results: a dependence-window lemma and a merging criterion for reconstructed columns, triangularity and monotonicity of the finite-support constraints, a forced-creation identity at boundary bounces, a description of the infinite survivors as a full shift for period two, and, for the two-sided period-two problem, exact forced identities near the centre, including a half-column that must vanish identically. For the linear relative Rule 150 we prove the analogue of the Echo Lemma for every odd period by a Frobenius self-similarity argument, and identify exactly the two steps that fail for Rule 30. We report exhaustive and SAT-certified computations excluding counterexamples of bounded width, and document barriers: every exact mechanism found is confined to a bounded neighbourhood of the centre, no additive conserved quantity of small range exists, and the sideways reconstruction has a growing number of distinguishable states. The Echo Lemma, and with it Problem 1, remains open. Every statement is labelled as proved, certified computation or open.

1 Introduction

Rule 30 updates a bi-infinite row by $x_i(t+1) = x_{i-1}(t) \oplus (x_i(t) \vee x_{i+1}(t))$. Started from a single lit cell, its centre column $c(t) = x_0(t)$ passes standard tests of randomness [1, 6]. Problem 1 of the 2019 prizes [9] asks whether c is ever eventually periodic. Jen proved that a finite Rule 30 orbit has at most one eventually periodic column [4]; Condrey proved that no column of a nonzero finite orbit is eventually constant, settling period one, and named eventual period two as the next open case [2].

The formulation in Section 3 uses known ingredients: the local identity of Lemma 3.1 appears in [11], and the reduction to finite seeds, together with uniqueness of the left half given the right half, appears in [2] and in a Lean development [12]. Our contributions are:

- exact structural lemmas for the driven half-line: a dependence-window lemma, a merging criterion certifying exact equivalence of reconstructions, triangularity and monotonicity of the finite-support constraints, and a forced-creation identity (Section 4);
- for the two-sided period-two problem, a golden-mean restriction on boundary inputs and exact forced identities near the centre, including $x_{-4}(2n) \equiv 0$ (Section 5);
- a proof of the Echo Lemma’s analogue for Rule 150 at every odd period, with an account of the two steps that fail for Rule 30 (Section 6);

- certified computations (Section 7) and a catalogue of barriers (Section 8).

We do not claim a solution of Problem 1.

2 Setting and notation

Fix $d : \mathbb{N} \rightarrow \{0, 1\}$, periodic with period p and not constant. A half-line configuration $a_k(t)$, $k \geq 1$, evolves by

$$a_k(t+1) = a_{k+1}(t) \oplus (a_k(t) \vee a_{k-1}(t)), \quad a_0(t) = d(t). \quad (1)$$

A minute t is *lit* if $d(t) = 1$. A start is *finite* of width w if $a_w(0) = 1$ and $a_k(0) = 0$ for $k > w$. Solving (1) for the far cell gives the inverse rule

$$a_{k+1}(t) = a_k(t+1) \oplus (a_k(t) \vee a_{k-1}(t)). \quad (2)$$

Over $\text{GF}(2)$, $u \vee v = u \oplus v \oplus uv$, so (1) is Rule 150 plus the product $a_k a_{k-1}$. We write $\Delta_k(t) = a_k(t) \oplus a_k(t+p)$; since $d(t) = d(t+p)$, $\Delta_0 \equiv 0$.

3 The Echo Lemma formulation

Lemma 3.1 (echo law; known [11]). *In Rule 30, if $c(t) = 1$ then $x_{-1}(t) = 1 \oplus c(t+1)$.*

Proof. $c(t+1) = x_{-1}(t) \oplus (c(t) \vee x_1(t))$ and the OR equals 1. $\square$

Echo Lemma (open). For every non-constant periodic d and every finite start there is a lit minute t with $a_1(t) \neq 1 \oplus d(t+1)$.

Observation 3.2 (combines [2, 11, 12]). The Echo Lemma implies Problem 1.

Proof. If c is eventually constant, [2] gives a contradiction. Otherwise c is eventually periodic with a non-constant pattern of period p from some time T . The cells $a_k = x_{-k}$ obey (1) with $a_0 = c$, and at time T only finitely many are lit. Lemma 3.1 gives the echo identity at every lit minute, contradicting the Echo Lemma. The finite-seed step is the standard same-orbit reduction [2, 12]. $\square$

4 Structural results

Lemma 4.1 (front advance). *If the outermost lit cell at time t is at depth R , then $a_R(t+1) = a_{R+1}(t+1) = 1$ and $a_k(t+1) = 0$ for $k > R+1$.*

Lemma 4.2 (arrival). *If two half-lines agree at depths $1, \dots, m-1$ and differ at depth $m \geq 2$ at time t , then at time $t+1$ they agree at depths $1, \dots, m-2$ and differ at depth $m-1$.*

Lemma 4.3 (dependence window). *$a_k(t)$ depends only on $a_1(t), \dots, a_1(t+k-1)$ and d , and $a_k(0) = a_1(k-1) \oplus F_k(a_1(0), \dots, a_1(k-2))$.*

Proof. Induction on k using (2); the newest value enters only through $a_k(t+1)$, by XOR. $\square$

Under the echo, a_1 is prescribed at lit minutes and free at dark minutes; write $b_k = a_k(0)$.

Proposition 4.4 (triangularity). *If minute $k-1$ is dark, $b_k = 0$ determines the free bit $a_1(k-1)$ uniquely from earlier bits (forcing); if it is lit, $b_k = 0$ constrains earlier bits only (filter).*

Corollary 4.5 (monotonicity). *The sets S_n of surviving prefixes, each extended uniquely through forcing constraints, satisfy $S_{n+1} \subseteq S_n$.*

Lemma 4.6 (merging). *If two traces agree at every minute after N and their reconstructions agree on minutes $0, \dots, N$ at two consecutive depths, they agree at every greater depth and every minute.*

Proof. By Lemma 4.3, values after minute N agree at every depth; the window $0, \dots, N$ propagates by (2), whose boundary term lies after N . $\square$

For song 011111 and support bound $M = 76$, four surviving assignments stall through nine filters; Lemma 4.6 certifies (independently verified) that they coincide from depth 48 on.

Lemma 4.7 (forced creation). *If t is dark, $t + 1$ is lit and the echo holds at $t + 1$, then $[a_1(t) = a_2(t)] = d(t + 2)$; at a bounce, $\Delta_2(t + 1) = \Delta_3(t) \oplus d(t + 2)$.*

Proof. On a dark minute $a_1(t + 1) = a_1(t) \oplus a_2(t)$, and the echo fixes $a_1(t + 1) = 1 \oplus d(t + 2)$. $\square$

With $R_k = a_{k-1}\Delta_k \oplus a_k\Delta_{k-1} \oplus \Delta_k\Delta_{k-1}$ one has $\Delta_k(t + 1) = \Delta_{k+1} \oplus \Delta_k \oplus \Delta_{k-1} \oplus R_k$, and the echo gives $R_1 = d\Delta_1 \equiv 0$.

Proposition 4.8 (front stripes). *Let $F_j(t)$ be the cell j behind the outermost lit cell. Then $F_j(t + 1) = F_{j-2}(t) \oplus (F_{j-1}(t) \vee F_j(t))$; each fixed lag is eventually periodic with period a power of two, doubling exactly when lag $j - 1$ is eventually dark and lag $j - 2$ has odd parity over a period. This matches [10].*

Proposition 4.9 (fiber, period two). *For $d = 1, 0, 1, 0, \dots$ the echo-compatible half-lines correspond bijectively to arbitrary dark-minute sequences, and two steps of (1) act as the shift.*

Lemma 4.10 (finite wallpapers). *For period p , configurations fixed by p steps are determined by their first p cells; there are at most 2^p , each eventually periodic in space.*

5 The two-sided period-two problem

Let $c(2n) = 0$, $c(2n + 1) = 1$. The right half evolves autonomously given c .

Proposition 5.1 (uniqueness; also [12]). *Given a finite right half, $x_{-1}(t) = c(t + 1) \oplus (c(t) \vee x_1(t))$ for all t , and (2) determines the left half.*

Lemma 5.2 (golden-mean restriction). *With $y_j(n) = x_j(2n)$: $y_1(n + 1) = \neg(y_1(n) \vee y_2(n) \vee y_3(n))$; in particular $x_1(2n) x_1(2n + 2) = 0$.*

Proposition 5.3 (forced identities). *With $A_k(n) = x_{-k}(2n)$: $A_2 = \neg A_1$, $A_3(n) = A_1(n + 1)$, $A_4 \equiv 0$, $A_5(n) = [A_1(n + 1) = A_1(n + 2)]$, $A_6 = \neg A_5 \wedge (A_3 \vee A_2)$, $A_7(n) = [A_1(n + 2) = A_1(n + 3)]$.*

The continuation $A_{2m+1}(n) = [A_1(n + m - 1) = A_1(n + m)]$ fails from depth 9. Up to depth 41 the only constant half-columns are x_{-1} at odd minutes and x_{-4} at even minutes; cells -6 to -10 contain no adjacent dark pair at odd minutes in all samples (not proved).

6 A linear comparison: Rule 150

Theorem 6.1. *For $a_k(t + 1) = a_{k+1}(t) \oplus a_k(t) \oplus a_{k-1}(t)$ with $a_0 = d$ of odd period p and a lit phase, no finite nonzero start has $a_1(t) = a_1(t + p)$ at every lit minute.*

Proof. Δ evolves by the same rule with dark boundary and $\Delta(0) \neq 0$. Mirror by $\Delta_{-k} = \Delta_k$; with $\varphi = x^{-1} + 1 + x$ and $P(x) = \sum_k \Delta_k(0)(x^k + x^{-k})$, $\Delta_1(t) = [x^1]\varphi^t P \bmod 2$. Since $\varphi^{2^m} = x^{-2^m} + 1 + x^{2^m} \bmod 2$, for N a sum of distinct widely spaced large powers of two, $\Delta_1(r + N) = \Delta_1(r)$. With o the order of 2 mod p , sums of k powers 2^{M+2io} are $\equiv k2^M$, covering every residue, so $r + N$ can be made lit. Hence $\Delta_1 \equiv 0$, and (2) gives $\Delta \equiv 0$, a contradiction. $\square$

The ingredients are classical [3, 5, 7, 8]. For Rule 90 the central term is absent. For Rule 30 the identity $\Delta_1(r + 2^m) = \Delta_1(r)$ holds in 50.2% of tests (all 268,968 for Rule 150).

7 Certified computations

Statement	Range	Method
Echo Lemma, song 01	widths ≤ 34	exhaustive trees; Z3
Echo Lemma, songs 001, 0001	widths ≤ 20	SAT, two encodings
Echo Lemma, song 011111	widths ≤ 80	exact tables, 2^{19} assignments*
Two-sided, alternating	radius ≤ 10	exhaustive, 2^{21} seeds
Two-sided, alternating	right ≤ 14 , left ≤ 125	all 2^{14} right halves

*An independent run, not included here, reports widths ≤ 97 . Scripts for every other row are in `anc/`.

Maximum steps beyond actual radius R : 6, 1, 3, 3, 4, 4, 3, 7, 8, 7 for $R = 1, \dots, 10$.

8 Barriers

- Single-path XOR identities telescope to their endpoints.
- No bounded quantity can decrease along every finite start; no additive conserved quantity on patterns of length ≤ 5 ; closed global parities are memoryless.
- Exact local forcing terminates within about 9–11 cells of the centre.
- Distinguishable sideways histories grow exponentially (lower bounds $27 \rightarrow 30,301$ and $4 \rightarrow 437$).
- Stall lengths of the surviving set grow with the support bound.
- Rule 30 versus Rule 150 from the same start: neither consistently earlier.

9 Open problems

Problem A. Prove the Echo Lemma (at least as strong as needed for Problem 1).

Problem B. For every finite right half, the left half forced by an alternating centre has infinitely many lit cells (period-two case of Problem 1; open in [2]).

Problem C. Find an exact OR-based constraint reaching arbitrarily far from the centre.

Problem D. Decide whether stalls of single residual behaviours are uniformly bounded.

Acknowledgement

Computational experiments and verification scripts were developed with the assistance of AI systems; certificates in Section 4 were checked by an independent implementation.

References

- [1] S. Wolfram, *A New Kind of Science*, Wolfram Media, 2002.
- [2] D. L. Condrey, Finite configurations cannot generate a constant trace in Rule 30, arXiv:2609.09431, 2026.
- [3] J.-P. Allouche and J. Shallit, *Automatic Sequences*, Cambridge University Press, 2003.
- [4] E. Jen, Global properties of cellular automata, *J. Stat. Phys.* 43 (1986).

- [5] B. Litow and P. Dumas, Additive cellular automata and algebraic series, *Theoret. Comput. Sci.* 119 (1993).
- [6] W. Meier and O. Staffelbach, Analysis of pseudo random sequences generated by cellular automata, EUROCRYPT 1991.
- [7] E. Rowland and R. Yassawi, A characterization of p -automatic sequences as columns of linear cellular automata, *Adv. Appl. Math.* 63 (2015) 68–89.
- [8] O. Martin, A. M. Odlyzko and S. Wolfram, Algebraic properties of cellular automata, *Comm. Math. Phys.* 93 (1984).
- [9] S. Wolfram, Announcing the Rule 30 Prizes, 2019.
- [10] S. Wolfram, notes to *A New Kind of Science*, Rule 30 left-edge periodicities.
- [11] Cellular automata in cryptographic random generators, arXiv:1306.3546, 2013, Proposition 3.4.
- [12] keithadler (GitHub), rule30-seed-gap: Lean reduction of Problem 1 to finite seeds, 2026.