

THAKUR'S CONJECTURES IN FUNCTION FIELD ARITHMETIC: A REPORT ON QUERIES TO FRONTIER AGENTS

ABSTRACT. This report was created on 23 October 2026 from six AI research reports generated with the `frontier-dump` skill on Dinesh Thakur's conjectures in function field arithmetic. Taylor Dupuy initiated those queries during his talk *How to Use AI in Mathematical Research Effectively* at the University of Rochester Number Theory Seminar on that date. The six reports were synthesized with `harvest-dumps` and developed into this manuscript with `using-latexpowers`.

We survey the current status of several conjectures of Dinesh Thakur and his coauthors on the arithmetic of $\mathbb{F}_q[t]$: the degree hypotheses for power sums, the characterisation of zeta-like multizeta values in genus zero and in higher genus, the relations among multizeta values, the distribution of zeros of Goss zeta functions, and the automata-theoretic transcendence questions for special values. For each conjecture we give the precise statement and its status in the literature.

The six computational research dumps report several results and refutations. Among them, the recursion predicted by Thakur's second power-sum hypothesis fails for an explicit family of exponents at thirteen non-prime values of q (a computer-assisted result); the shuffle relations, even with Frobenius and Carlitz's Eulerian evaluations, do not generate all relations among multizeta values; a closed formula for a v -adic valuation difference conjectured by Thakur holds only after a sign change (and, for $d \geq 4$, halving), and the corrected formula is proved, with computer assistance, for degrees $d \leq 11$; and the v -adic Newton polygon of a Goss zeta function over $\mathbb{F}_2[t]$ at a prime of degree two has a slope of multiplicity one, which appears to contradict a result claimed in a 2023 preprint (this conflict needs human verification). The shuffle result and an assembly of published theorems describing all relations are elementary, and we claim no novelty for them. We also write out a proof of a theorem of Wen and Yao on v -adic arithmetic gamma values. Computational evidence is labelled as such throughout, and the open problems are listed.

This manuscript is AI-generated by Claude Code (model `claude-opus-5-5`) from AI-generated research dumps; its proofs received only AI review, and no human has checked them.

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CONTENTS

1. Introduction	2
Audience and conventions	2
Summary	3
2. Notation and standing facts	3
3. Power sums: Thakur's hypotheses H1 and H2	4
3.1. Statements	4
3.2. The recursion fails for non-prime q	5
3.3. Hypothesis H1 for non-prime q	6
4. Zeta-like multizeta values over $\mathbb{F}_q[t]$	7
4.1. The conjectures	7
4.2. What is known	7
4.3. Two elementary results	8
5. Higher genus: the conjectures of Lara Rodríguez and Thakur	9
6. Relations among multizeta values	10

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6.1. Setting	10
6.2. Shuffle-type relations do not suffice	11
6.3. Todd's conjecture and its resolution	12
7. Zeros of Goss zeta functions	13
7.1. Setting	13
7.2. Two lemmas on v -adic power sums	14
7.3. Refutations and corrections	15
7.4. A conjecture and open problems	18
8. Automata and transcendence of special values	18
8.1. The criterion and two corrections	18
8.2. Status	19
8.3. The v -adic arithmetic gamma function	19
9. Summary of open problems	22
10. Software and AI assistance	22
Methods: generation	22
Methods: synthesis and review	23
Methods: computations	23
Methods: citations	23
References	23

1. INTRODUCTION

Thakur's work on $A = \mathbb{F}_q[t]$ has produced a large and precise set of conjectures, many of them found by computation: on the degrees of power sums of monic polynomials, on which multizeta values are rational multiples of a single zeta value, on the relations among multizeta values, on the valuations of v -adic power sums and hence the zeros of Goss's v -adic zeta functions, and on the algebraicity of special values via automata. Some of these conjectures have since been proved, some only in part, and some have turned out to be false. This paper records their status as of October 2026 and adds new partial results, corrections and counterexamples.

A single example shows how prime-field intuition can fail. For q prime, Thakur proved that the degrees $s_d(k) = -\deg \sum_a a^{-k}$ (sum over monic a of degree d) satisfy the recursion $s_d(k) = s_{d-1}(s_1(k)) + s_1(k)$. For $q = 4$ and $k = 75$ one computes $s_1(75) = 96$ and $s_1(96) = 384$, so the recursion predicts $s_2(75) = 480$, while in fact $s_2(75) = 348$ (Example 3.7; this example is Thakur's). Section 3 shows that the failure is systematic.

Audience and conventions. We write for readers who know the Carlitz module, its period $\tilde{\pi}$, the Carlitz zeta values, Thakur's multizeta values and Goss's zeta functions. We recall definitions and fix notation in Section 2, but we do not develop the theory. Every statement carries one of the following statuses.

- *Imported* results are cited at the head of the statement. When we did not read the cited passage ourselves, we say how it was seen (abstract only, or as reported by another source).
- Results announced in *unrefereed preprints* are stated as *claimed results* or in prose, never as theorems. This applies to [CO26], [Hu26], [KMU23], [CWY24] and [DVPC14].
- Results *proved here* have complete proofs in this paper. Some are new, some are elementary assemblies of published theorems, and some write out arguments that the literature omits; we say which.
- *Computer-assisted* results are finite statements certified by an exact computation, and the certificate is described.

- *Computational evidence* (the environment of that name) records finite numerical observations. It never establishes the general claim.
- *Conjectures* are attributed. New conjectures are marked as such.

Results from the six AI-generated research records that this paper harvests are cited as [AI 26f, AI 26g, AI 26c, AI 26b, AI 26e, AI 26d]; certificates name the scripts and data files there. Each section begins with the refutations and corrections it contains and then turns to positive results. Diophantine approximation exponents and continued fractions are outside the scope of this paper.

Summary. Table 1 summarises the status of each conjecture.

Conjecture	Status in the literature	New here
H1, H2 (power sums) [Tha09a]	Recursion for q prime: [Tha09a]; H1, H2 for $q = p$ claimed in the preprint [CO26]	The H2 recursion fails at 13 non-prime q (Thm. 3.6); H1 open (evidence only)
Zeta-like MZVs, $\mathbb{F}_q[t]$ [LRT14]	Weight $\leq q^2$ settled [LND22]; families proved [Che17]; tails Eulerian [CPY19]	Props. 4.5, 4.6; completeness above q^2 open
Zeta-like, higher genus [LRT21]	Both numbered conjectures proved [CNDP23]	Erratum in a display of [CNDP23] (Rem. 5.4)
All MZV relations	Todd's conjecture: B^* form proved [CCM23]; B form claimed in the preprint [Hu26]	Shuffle-type relations do not suffice (Thm. 6.3, elementary)
Goss zeros, v -adic [Tha13]	Ordinary curves: claimed in the preprint [KMU23]	Lemmas 7.6, 7.7; Thakur's formulas proved for $d \leq 11$ with one correction (Thm. 7.11); apparent conflict with [KMU23, Thm. 1.10]
Automata, v -adic Γ [WY02]	Open beyond $\deg v = 1$ and residue 0; rational arguments claimed in the preprint [CWY24]	Proof of [WY02, Thm. 8] written out (Thm. 8.6)

TABLE 1. Status of the conjectures discussed in this paper.

2. NOTATION AND STANDING FACTS

Throughout, $q = p^f$ is a power of the prime p , $A = \mathbb{F}_q[t]$, $K = \mathbb{F}_q(t)$, and $K_\infty = \mathbb{F}_q((1/t))$ is the completion at the infinite place, with completed algebraic closure $\mathbb{C}_\infty$. We write A_+ for the monic elements of A and $A_{+,d}$ for those of degree d . Put $[n] = t^{q^n} - t$, $D_n = \prod_{i=1}^n [i]^{q^{n-i}}$ (so $D_0 = 1$ and D_n is the product of the elements of $A_{+,n}$), and $\ell_n = \prod_{i=1}^n (t - t^{q^i})$. A finite place is a monic irreducible $v \in A$; we write $D = \deg v$ and reserve d for the degree of summation in power sums. K_v is the completion of K at v , and val_v its normalised valuation.

Definition 2.1. Let $k \in \mathbb{Z}$ and $d \geq 0$.

- (1) The *power sum* is $S_d(k) = \sum_{a \in A_{+,d}} a^{-k} \in K$, and $s_d(k) = -\deg S_d(k) = \text{val}_\infty S_d(k)$.
- (2) For a finite place v , the *v -adic power sum* is $S_{d,v}(k) = \sum_{a \in A_{+,d}, v \nmid a} a^{-k} \in K$, and $v_d(k) = \text{val}_v S_{d,v}(k)$ (with $v_d(k) = \infty$ if $S_{d,v}(k) = 0$).

This is the notation of [Tha09a, Tha13]; note that $S_d(k)$ has exponent $-k$.

Definition 2.2. Let $\mathfrak{s} = (s_1, \dots, s_r)$ be a tuple of positive integers.

- (1) The *weight* of $\mathfrak{s}$ is $w(\mathfrak{s}) = s_1 + \dots + s_r$ and its *depth* is r .

- (2) Thakur's *multizeta value* is $\zeta(\mathfrak{s}) = \sum_{d_1 > d_2 > \dots > d_r \geq 0} S_{d_1}(s_1) \cdots S_{d_r}(s_r) \in K_\infty$.
- (3) $\zeta(\mathfrak{s})$ (or $\mathfrak{s}$) is *zeta-like* if $\zeta(\mathfrak{s})/\zeta(w(\mathfrak{s})) \in K$.
- (4) $\zeta(\mathfrak{s})$ is *Eulerian* if $\zeta(\mathfrak{s})/\tilde{\pi}^{w(\mathfrak{s})} \in K$.

Sections 5 and 7 also use a more general base ring.

Definition 2.3. Let X be a smooth, projective, geometrically irreducible curve over $\mathbb{F}_q$ of genus g , and let ∞ be a closed point of X of degree d_∞ .

- (1) The *base ring* is the ring A of functions on X regular away from ∞ . Its fraction field is K , and K_∞ is the completion at ∞ .
- (2) The *class number* of A is the order of its ideal class group, and $H \subset K_\infty$ is the Hilbert class field of A in the sense of [CNDP23] (the maximal unramified abelian extension of K in which ∞ splits completely).
- (3) Suppose $d_\infty = 1$. Fix a uniformiser of K_∞ and the sign function $\text{sgn}: K_\infty^\times \rightarrow \mathbb{F}_q^\times$ it determines [CNDP23, §1.2]. An element $a \in A$ is *monic* if $\text{sgn}(a) = 1$, and $A_{+,d}$ is the set of monic a with $\deg a = -\text{val}_\infty(a) = d$. The power sums $S_d(k)$ and the multizeta values $\zeta_A(\mathfrak{s})$ are defined by the formulas of Definitions 2.1 and 2.2 with these $A_{+,d}$, and $\zeta_A(\mathfrak{s})$ is *zeta-like* if $\zeta_A(\mathfrak{s})/\zeta_A(w(\mathfrak{s})) \in K$.
- (4) With $d_\infty = 1$, let $\tilde{\pi}_A$ be the period of the standard sign-normalised rank-one Drinfeld module attached to A . Following [LRT21] and [CNDP23], $\zeta_A(\mathfrak{s})$ is *Eulerian* if $\zeta_A(\mathfrak{s})/\tilde{\pi}_A^{w(\mathfrak{s})}$ is algebraic over K .
- (5) The *p-rank* of X is the $\mathbb{F}_p$ -dimension of the p -torsion of its Jacobian over $\overline{\mathbb{F}}_q$. X is *ordinary* if its p -rank equals g ; an elliptic curve is *supersingular* if its p -rank is 0.

For $A = \mathbb{F}_q[t]$ the Carlitz module is the sign-normalised module, and the two notions of Eulerian in Definitions 2.2 and 2.3 agree: by Chang's Euler dichotomy [Cha14, Cor. 2.3.3], a multizeta value Z of weight w either has $Z/\tilde{\pi}^w \in K$ or is algebraically independent from $\tilde{\pi}$. When A has class number one, $H = K$. Lara Rodríguez and Thakur state that, apart from the rings $\mathbb{F}_q[t]$, exactly four base rings with $d_\infty = 1$ have class number one [LRT21, §3]:¹

- (i) $\mathbb{F}_2[x, y]/(y^2 + y = x^3 + x + 1)$;
- (ii) $\mathbb{F}_3[x, y]/(y^2 = x^3 - x - 1)$;
- (iii) $\mathbb{F}_4[x, y]/(y^2 + y = x^3 + w)$, where $w^2 + w + 1 = 0$;
- (iv) $\mathbb{F}_2[x, y]/(y^2 + y = x^5 + x^3 + 1)$.

In each case ∞ is the unique point at infinity of the affine model. These are the curves used in the computations of Section 7. The first three curves have genus 1 and the last has genus 2. The three elliptic curves are supersingular (in characteristic 2 the Weierstrass equation has $a_1 = 0$, and in characteristic 3 it has $a_2 = 0$), and the genus-2 curve (iv), an Artin–Schreier curve $y^2 + y = f(x)$, has p -rank 0. We refer to these rings as (i)–(iv) throughout.

We use two standard facts without further comment. Every $\zeta(\mathfrak{s})$ is nonzero [Tha09b]. By Carlitz, $\zeta(n)/\tilde{\pi}^n \in K$ when $(q-1) \mid n$ [Tha04, §5], and by Yu's theorem it is transcendental otherwise [Yu91]. Hence a zeta-like value of weight divisible by $q-1$ is Eulerian. Conversely, an Eulerian value is zeta-like, and its weight is divisible by $q-1$, because $\tilde{\pi}^n \in K_\infty$ only when $(q-1) \mid n$.

3. POWER SUMS: THAKUR'S HYPOTHESES H1 AND H2

3.1. Statements. For $k \geq 0$ and $d \geq 0$, Carlitz's generating-function theorem [Tha09a, 2.3] expresses $S_d(k+1)$ as a finite sum over decompositions $k = \sum_{i=0}^d k_i q^i$ with $k_i \geq 0$. By Lucas's

¹Kramer-Miller and Upton [KMU23, §1.1] speak of “exactly eight such A in all characteristics”; we did not reconcile the two counts, which may use different conventions for ∞ .

theorem only decompositions in which the k_i add without carry in base p contribute. The term of such a decomposition has degree $b(0, d) + \sum_i k_i b(i, d)$, where

$$(1) \quad b(i, d) = -\left[\frac{q^{d+1} - q^{i+1}}{q-1} + iq^i\right] \quad (0 \leq i \leq d),$$

as in [Tha09a, 2.3.1(4)].

Remark 3.1. In a plain-text extraction of [CO26, (4)] the term iq^i appears outside the bracket, which one might take as an alternative convention. In fact the bracketed form (1) is the one that matches the exact degrees in Section 3.2; the discrepancy may be an extraction artefact, and the typeset preprint was not checked.

Conjecture 3.2 (Hypothesis H1, [Tha09a, 2.3.1]). *For every q, d and k , among the carry-free decompositions of k in Carlitz's formula for $S_d(k+1)$ there is exactly one whose term has maximal degree.*

Under H1, $s_d(k+1) = -(b(0, d) + \sum k_i b(i, d))$ for the maximising decomposition. The second hypothesis comes from the expansion $S_d(k) = -\sum_{j \geq 1, (q-1) \mid j} \binom{-k}{j} t^{-(k+j)} S_{d-1}(k+j)$ [Tha09a, 3.2].

Conjecture 3.3 (Hypothesis H2, [Tha09a, 3.2]). *Let q be prime and $k, d \geq 1$. Then the minimum of $s_{d-1}(s_1(k) + j) + s_1(k) + j$ over $j \geq 0$ with $(q-1) \mid j$ and $\binom{s_1(k)+j-1}{k-1} \not\equiv 0 \pmod{p}$ is attained exactly once, at $j = 0$. Consequently*

$$(2) \quad s_d(k) = s_{d-1}(s_1(k)) + s_1(k).$$

Theorem 3.4 ([Tha09a, Thm. 1]). *If q is prime, then (2) holds for all $k \in \mathbb{Z}$ and $d \geq 1$.*

Thakur also states a third hypothesis.

Conjecture 3.5 (Hypothesis H3, [Tha09a, A.10]). *For every $q, d \geq 1$ and $k \geq 1$ with $p \nmid k$, $s_d(k) < s_d(k+1)$.*

The preprint [CO26] announces proofs of H1 and H2 for $q = p$ and of H3 for all q , with Lean formalisations generated by an automated prover. We have not checked these proofs, and the preprint is unrefereed. For non-prime q , H2 was not conjectured, and H1 is open.

3.2. The recursion fails for non-prime q . It is natural to expect that, apart from the location of the minimum, H2 and the recursion (2) survive for $q = p^f$ with $f \geq 2$. In fact they fail for an explicit family of exponents.

Theorem 3.6 (computer-assisted). *Let $q = p^f$ be one of 4, 8, 16, 32, 64, 9, 27, 81, 25, 125, 49, 121, 169, and put $k = q^3 + pq + p^2 - 1$. Then*

$$s_2(k) \neq s_1(s_1(k)) + s_1(k).$$

Moreover, $j = q(q-1)$ is admissible in the minimisation of Conjecture 3.3 with $d = 2$, and its value $s_1(s_1(k) + j) + s_1(k) + j$ equals $s_2(k)$. For $q \in \{4, 8, 9, 16, 25, 27\}$, $j = q(q-1)$ is the unique minimiser.

Certificate. For each listed q , an exact dynamic program over carry-free decompositions finds the maximal degree in Carlitz's formula for $S_2(k)$. It finds that the maximum is attained by a single decomposition whose coefficient is nonzero modulo p . Since Carlitz's formula is an identity, $s_2(k)$ is then minus that degree. The value $s_1(m) = m + j$, with j the least positive multiple of $q-1$ such that $\binom{-m}{j} \not\equiv 0 \pmod{p}$ [Tha09a, 3.1], gives the right-hand side and the value at $j = q(q-1)$ ([AI 26f], `scripts/family.py`, `family_check.json`). For the uniqueness statement, a minimiser

enumerates the admissible j until the lower bound $s_1(s_1(k) + j) + s_1(k) + j \geq 2(s_1(k) + j)$ exceeds the best value found, so its list of minimisers is complete ([AI 26f], `scripts/powersums.py`, function `h2`; run for these six q in [AI 26a], `h2_family_uniqueness.out`). For $q = 4$ the three degrees were also recomputed by an independent direct summation (Example 3.7; [AI 26a], `independent_check.py`). $\square$

Example 3.7. Let $q = 4$ and $k = 75 = 4^3 + 2 \cdot 4 + 2^2 - 1$. Direct summation of the 16 terms a^{-75} , $a \in A_{+,2}$, in $\mathbb{F}_4((1/t))$, together with the analogous sums for $d = 1$, gives

$$s_1(75) = 96, \quad s_1(96) = 384, \quad s_2(75) = 348 \neq 480 = s_1(96) + s_1(75).$$

The minimum in Conjecture 3.3 is at $j = 12 = q(q - 1)$, as Thakur already observed [Tha09a, A.11].

Remark 3.8. Thakur already gave further non-prime counterexamples [Tha09a, A.9, A.11], which we record without re-verification. For $q = 4$ and $d = 2$, among the arguments $k = -m$ with m odd and $m \leq 1000$ (that is, for the power sums $S_2(-m) = \sum_{a \in A_{+,2}} a^m$), the recursion (2) fails exactly for $m = 181, 437, 565, 661, 693, 709, 721, 949$. A naive generalisation to $q = p^f$ of the prime-field formula for $s_d(k + 1)$ fails already for $d = 1$ at $k + 1 = 11$, where it gives 44 and 220 for $d = 1, 2$; the true value $s_1(11) = 32$ was re-verified. And without the binomial condition the minimum in Conjecture 3.3 need not be at $j = 0$ even for prime q (take $q = d = 2$, $k = 4$). Two further expectations fail in Thakur's examples. The inequality $s_d(k) < s_d(k + q - 1) + q - 1$, which would give uniqueness of the minimum and strict growth in d , fails for $q = 2$, $d = 2$, $k = 2^d$ [Tha09a, A.9]. And for non-prime q the two summands u, v that compute $s_d(k + 1)$ in the prime case need not be sub-sums of the base- p expansion of k ; Thakur lists $k + 1 = 11, 43, 47, 59, 75, 107, 139$ and their 2-power multiples for $q = 4$ [Tha09a, A.11].

Remark 3.9. For $q = p \in \{3, 5, 7, 11\}$ and the same k , the binomial condition in Conjecture 3.3 excludes $j = q(q - 1)$, and the recursion holds, as Theorem 3.4 requires. We have not proved that the family fails for every non-prime q . In every range swept, the smallest k violating the recursion for $q = 4, 8, 9, 16, 25, 27$ was the family member $q^3 + pq + p^2 - 1$.

Since H2 is stated only for prime q , its literal extension is false, but uniqueness of the minimum may survive. We record this as a new conjecture.

Conjecture 3.10 (new; Hypothesis H2*). *For every q and $k, d \geq 1$, the minimum in Conjecture 3.3 is attained exactly once.*

If H2* holds, then $s_d(k) = \min_j [s_{d-1}(s_1(k) + j) + s_1(k) + j]$, because the expansion of $S_d(k)$ in terms of the $S_{d-1}(s_1(k) + j)$ then has a unique term of least valuation.

Computational evidence 3.11. Failures of (2) occurred for every non-prime q tested. There were 34, 27, 6 failures for $q = 4$ at $d = 2, 3, 4$; 35, 10 for $q = 8$ at $d = 2, 3$; 35 for $q = 9$ at $d = 2$; and 68, 76, 90 for $q = 16, 25, 27$ near $k \approx q^3$. In all about 1.1×10^4 cases examined the minimum was unique and equal to the exact $s_d(k)$. Prime controls showed no failures. The family members of Theorem 3.6 with $q \in \{32, 49, 64, 81, 121, 125, 169\}$ lie outside the sweeps.

3.3. Hypothesis H1 for non-prime q . It is natural to expect that extension-field effects, such as cancellation among equal-degree terms, break H1 when $f \geq 2$. The computations found no failure.

Computational evidence 3.12. H1 held in about 1.07×10^5 non-prime cases: $q = 4$ ($d \leq 8$, $k \leq 4000$), $q = 8, 9$ ($d \leq 6$, $k \leq 4000$), $q = 16, 25$ ($d \leq 4$, $k \leq 3000$), and 2790 further cases near $k \approx q^3$ for $q = 16, 25, 27$. It also held in about 1.1×10^5 prime-field control cases. The exact degrees were validated against direct summation in 500 cases ($q = 3, 4, 5, 8, 9, 25$) with no mismatch.

Remark 3.13. Thakur exhibits two carry-free decompositions of $k = 2q^6 + 2q^3$ with equal contributions [Tha09a, A.8], which might suggest a failure of H1. In the computations this tie never occurred at the maximum ($q = 3, 4, 5, 7, 8, 9, 16, 25$; $d = 6, 7$), so it is not a counterexample.

Remark 3.14. For $q = p$, one unit p^e placed in slot i of a decomposition lands at position $e + i$ of k , the per-unit costs $\gamma_i = (q^{d+1-i} - q)/(q - 1) + i$ decrease strictly in i , and the greedy-plus-exchange argument claimed in the preprint [CO26] is said to give uniqueness. For $f > 1$ the unit lands at position $e + fi$, each base- p digit holds at most $p - 1$ units, and positions f apart are coupled. Equal-cost exchanges therefore exist (the tie of Remark 3.13 is one). A proof of H1 for $q = p^f$ must show that no equal-cost exchange is available at the optimum.

4. ZETA-LIKE MULTIZETA VALUES OVER $\mathbb{F}_q[t]$

4.1. The conjectures. Since $\zeta(p\mathfrak{s}) = \zeta(\mathfrak{s})^p$, one restricts to *primitive* tuples, those $\mathfrak{s}$ whose entries are not all divisible by p [LRT14, §2]. Lara Rodríguez and Thakur [LRT14] conjectured a description of zeta-like tuples of depth $r \geq 2$. We list their conjectures, numbered as in [LRT14].

Conjecture 4.1 ([LRT14, Conj. 4.1–4.5]). *Let $\mathfrak{s} = (s_1, \dots, s_r)$ with $r \geq 2$.*

- (4.1) *If $\mathfrak{s}$ is zeta-like, then (1) $s_i \leq s_{i+1}$ and $(q - 1)s_i \leq s_{i+1} \leq (q^2 - 1)s_i$ for $1 \leq i < r$, and (2) the tail $(s_2, \dots, s_r)$ is Eulerian and the prefix $(s_1, \dots, s_{r-1})$ is zeta-like.*
- (4.2) *Let $q = 2$. If $(s_1, \dots, s_k)$ and $(s_k, \dots, s_r)$ are zeta-like and $\sum_{i=1}^r s_i$ is a power of 2 or a power of 2 minus one, then $(s_1, \dots, s_r)$ is zeta-like, unless both tuples are $(1, 1)$.*
- (4.3) *(1) Eulerian values of depth $r > 1$ occur only in weights $p^m(q^k - 1)$, and primitive ones only in weights $q(q - 1)$ or $q^n - 1$ if $q > 2$, and $2^n - 1$ or 2^n if $q = 2$. (2) If $q = p$, the weight of a zeta-like, non-Eulerian tuple of depth $r > 1$ is p^m times a number whose base- p digits contain no 0 and at most one 1. (3) The smallest weight of a zeta-like value of depth r is q^{r-1} . (4) If $q > 2$, the smallest weight of an Eulerian value of depth r is $q^r - 1$, $(q - 1)q$ or $q - 1$ according as $r > 2$, $r = 2$ or $r = 1$. (5) The weight q^k is not the weight of a zeta-like primitive tuple of depth r if $k > r > 3$, or if $k > 3$ and $r \in \{2, 3\}$.*
- (4.4) *The primitive zeta-like tuples of depth 2 and weight at most q^2 are exactly $(i, j(q - 1))$ with $1 \leq i \leq q$ and $i \leq j \leq \lfloor (q^2 - i)/(q - 1) \rfloor$.*
- (4.5) *For $q = 2$, the primitive zeta-like tuples of depth 2 are exactly $(1, 1)$, $(1, 3)$, $(3, 5)$, $(2^n - 1, 2^n)$ and $(2^n, 2^{n+1} + 2^n - 1)$ for $n \geq 1$.*

In addition, [LRT14, Conj. 4.6] gives three families of arbitrary depth with explicit ratios; the first is the case $A = \mathbb{F}_q[t]$ of the family in Theorem 5.3 below, and we refer to [LRT14] for the constants. Without the word “primitive”, (4.5) would be false: for $q = 2$ the tuple $(1, 1)$ is zeta-like, hence so is $(2, 2)$, since $\zeta(2, 2)/\zeta(4) = (\zeta(1, 1)/\zeta(2))^2$.

A related completeness statement is the list of primitive Eulerian multizeta values in [CPY19, §6.2], which [CPY19] certify only up to finite weight bounds.

4.2. What is known. It is natural to expect the characterisation to be wide open. In fact much of it is settled; Table 2 summarises.

Theorem 4.2 ([CPY19, Cor. 4.4.3]). *If $\zeta(s_1, \dots, s_r)$ is zeta-like, then each of $\zeta(s_2, \dots, s_r)$, $\zeta(s_3, \dots, s_r)$, $\dots$, $\zeta(s_r)$ is Eulerian. In particular $s_2, \dots, s_r$ are divisible by $q - 1$.*

Remark 4.3. Two attributions are easily confused. The classification of zeta-like values of weight at most q^2 is due to H. H. Le and T. Ngo Dac [LND22]. It is not due to Im–Kim–Le–Ngo Dac–Pham, whose paper [IKL⁺24] concerns the Zagier–Hoffman conjectures. Likewise, arXiv:2205.09929, which proves Thakur’s basis conjecture, is by Chang, Chen and Mishiba [CCM23]. It is not a single-author paper of Chen.

Statement	Status
4.1(2), tail half	Proved: Theorem 4.2
4.4; all zeta-like values of weight $\leq q^2$, any depth	Proved by H. H. Le and T. Ngo Dac [LND22] (we saw only the abstract)
4.6, all three families with their constants	Proved by H.-J. Chen [Che17, Thms. 4.4, 4.5, 5.1, 5.2] (as read in [AI 26g]); [AI 26g] also states, unverified, that this covers the families of [KL16]
4.1(1); 4.1(2), prefix half	Open
4.2, 4.3 beyond weight q^2	Open
4.5	Open (certified up to weight 256 in [CPY19])
Completeness of [CPY19, §6.2]	Open

TABLE 2. Status of Conjecture 4.1. We did not read [KL16] or [CK18], which may settle further depth-two cases.

Remark 4.4. The constant in Chen's version of the second family [Che17, Thm. 4.4] looks different from the constant in [LRT14, 4.6(2)]. In fact they coincide: the exponent of $[j]$ telescopes to q^{n+2-j} , and $\ell_i = (-1)^i L_i$ in the two normalisations. The equality was also confirmed numerically. One might also expect 4.6(2) to need $q > 2$, because [Che17, Thm. 4.4] assumes it. The identity holds numerically for $q = 2$, $n = 0, 1, 2$, and [Che17, Thm. 5.2] covers $q = 2$.

4.3. Two elementary results.

Proposition 4.5. *For every q , the smallest weight of a zeta-like tuple of depth 2 is q , and $(1, q-1)$ is zeta-like.*

This is the case $r = 2$ of Conjecture 4.1(4.3)(3).

Proof. If (s_1, s_2) is zeta-like, then $(q-1) \mid s_2$ by Theorem 4.2, so $s_2 \geq q-1$ and the weight is at least $1 + (q-1) = q$. For the second claim, take $n = 0$, $s_1 = 1$, $s_2 = 0$ in [LRT14, Thm. 3.1(2)]. It gives $a = 1$, $b = q-1$ and $\zeta(1, q-1) = \zeta(q)/\ell_1$, with $\ell_1 = t - t^q \in K^\times$. $\square$

Proposition 4.6. *A zeta-like tuple of depth r has weight at least $1 + (r-1)(q-1)$.*

Proof. By Theorem 4.2, each of $s_2, \dots, s_r$ is a positive multiple of $q-1$, and $s_1 \geq 1$. $\square$

For $r \geq 3$ this is far below the conjectured q^{r-1} . If $q > 2$ and $r \geq 4$, then Conjecture 4.1(4.3)(4) for the Eulerian tail of depth $r-1 \geq 3$ would give weight at least $1 + q^{r-1} - 1 = q^{r-1}$, so (4.3)(3) would follow. For $r = 3$ this argument gives only $1 + q(q-1) < q^2$, and more is needed.

Remark 4.7. Every proved family is consistent with the open constraint 4.1. For example, $(1, q^2-1)$ attains the upper bound $s_2 = (q^2-1)s_1$ exactly, and the hypothesis of [KL16, 2.15(a)] is the lower bound $s_2 \geq (q-1)s_1$. (The second check is taken from [AI 26g], whose source for [KL16, 2.15(a)] we could not trace; it is unverified.)

Computational evidence 4.8. For prime q , $S_d(s)$ was computed exactly to working precision from Carlitz's exponential, and agreed with brute force in 84 cases. A tuple was declared zeta-like when $\zeta(\mathfrak{s})/\zeta(w)$, known to 1500–2500 digits in $1/t$, had a rational approximation from half the digits that persisted to all of them. The heights separated sharply: at most 28 for accepted tuples and at least 261 for the rest. Known identities, including [LRT14, Thms. 3.1(5), 3.2], the formula of [CPY19] for $\zeta(3, 5)$, and the families 4.6(1)–(3) for $q = 2, 3, 5$, were reproduced, and deliberately wrong constants failed. The scans found:

- $q = 2$, depth 2, $w \leq 48$: $(1, 1)$, $(1, 2)$, $(1, 3)$, $(2, 5)$, $(3, 4)$, $(3, 5)$, $(4, 11)$, $(7, 8)$, $(8, 23)$, $(15, 16)$, exactly the primitive list of 4.5 up to weight 48 (the scans skip tuples whose entries are all divisible by p);
- $q = 2$, depth 3, $w \leq 20$: $(1, 1, 2)$, $(1, 2, 4)$, $(1, 2, 5)$, $(1, 3, 4)$, $(3, 4, 8)$, exactly the list of [CPY19]; depth 4, $w \leq 14$: only $(1, 1, 2, 4)$;
- $q = 3$, depth 2, $w \leq 30$: 17 tuples, agreeing with 4.4 for $w \leq 9$, and the Eulerian ones $(2, 4)$, $(2, 6)$, $(6, 20)$, $(8, 18)$ agree with [CPY19]; depth 3, $w \leq 14$: only $(1, 2, 6)$;
- $q = 5$, depth 2, $w \leq 32$: exactly the 15 tuples of 4.4, and none of weight 26–32.

Whenever the data could decide them, 4.1(1), 4.1(2), 4.2 and 4.3(1)–(3) were checked (203 checks), with no violation. The scans cover only prime q and lie inside ranges that [CPY19] already certified.

Question 4.9. The open core of Conjecture 4.1 is completeness above weight q^2 . A proof would seem to need a uniform bound on the solutions of the Frobenius difference systems of [CPY19] and [Che17, Thm. 4.2] in terms of the q -adic shape of $\mathfrak{s}$, and hence control of the Anderson–Thakur polynomials. The prefix half of 4.1(2) has no known motivic mechanism: the argument of [CPY19] reaches only tails.

5. HIGHER GENUS: THE CONJECTURES OF LARA RODRÍGUEZ AND THAKUR

We use the rings (i)–(iv) of Section 2 and Definition 2.3. Following [CNDP23], the displays quoted from that paper write θ and η for the coordinates x and y .

Conjecture 5.1 ([LRT21, Conj. 3.1]). *For every A of class number one with constant field $\mathbb{F}_q$, the values $\zeta_A(q^n - 1, (q - 1)q^n, \dots, (q - 1)q^{n+k})$ are zeta-like.*

Conjecture 5.2 ([LRT21, Conj. 3.3]). *Let $R_n = \zeta_A(q^n - 1, (q - 1)q^n) / \zeta_A(q^{n+1} - 1)$. In case (i),*

$$R_n = \frac{x^{2^{n+1}} + x^2}{y^{2^{n+1}} + y + x^{2^{n+1}+1} + x},$$

and in cases (ii)–(iv), R_n is given by the explicit rational functions of [LRT21, Conj. 3.3].

It is natural to expect these conjectures to be open. In fact both are theorems.

Theorem 5.3 ([CNDP23, Thm. 2]). *Let A be any base ring with $d_\infty = 1$, with Hilbert class field $H \subset K_\infty$, and let $n \geq 1$. For every $k \geq 0$ there is an explicitly computable $\alpha_{n,k} \in H$ such that*

$$\zeta_A(q^n - 1, (q - 1)q^n, \dots, (q - 1)q^{n+k-1}) = \alpha_{n,k} \zeta_A(q^{n+k} - 1).$$

In particular these values are Eulerian.

If A has class number one, then $H = K$, and Theorem 5.3 gives Conjecture 5.1 (with the indexing shifted by one). Chung, Ngo Dac and Pellarin also prove that their $\alpha_{n,1}$ agree with the formulas of Conjecture 5.2 in all four cases; the theorem is stated in [CNDP23] as “Conjecture 3.3 of [LRT21] holds”. An exact comparison of the two formulas in the function field of each curve gives agreement for $n \leq 6$, 4, 3, 6 in cases (i)–(iv) ([AI 26c], `src/algebraic_check.sage`).

Remark 5.4. The second line of the display in [CNDP23] restating case (ii) of Conjecture 5.2 has the numerator $(X - \theta)(Y - \eta)^2 - X^2 + (-\theta^3 + 1)X + \theta^2 - \theta$, where $X = \theta^{3^n}$ and $Y = \eta^{3^n}$. Expanding the first line gives

$$(X - \theta)(-X - \theta^3 - \theta + 1) = -X^2 + (1 - \theta^3)X + \theta^4 + \theta^2 - \theta,$$

so the constant θ^4 is missing from the second line. With the term dropped, the displayed formula disagrees with both [LRT21] and the formula proved in [CNDP23] for $n = 1, \dots, 4$. The later

computation in the proof of [CNDP23] uses the correct expression, so this is a typographical slip and does not affect the theorem.

Remark 5.5. Lara Rodríguez and Thakur state the family only for class number one, and they doubt the existence of single zeta-like values in higher class number. By Theorem 5.3, however, the family is Eulerian for every A with ratio in H . Class number one is needed only to place the ratio in K .

Computational evidence 5.6. It is natural to expect that, as in genus 0, many simple tuples such as $(1, 1)$ are zeta-like in higher genus. The computations below find the family of Theorem 5.3 and nothing else among the tuples tested. For $q = 2$, the values in cases (i) and (iv) were recomputed by summing over orbits of $y \mapsto y + 1$, to t -adic precision 700. For $n = 1, \dots, 4$ the ratio $\zeta(2^n - 1, 2^n)/\zeta(2^{n+1} - 1)$ matched Conjecture 5.2 to that precision. As a control, substituting the $n = 1$ prediction for $n = 2$ fails at valuation 2. Rational reconstruction also found low-degree ratios for $\zeta(1, 2, 4)/\zeta(7)$, $\zeta(3, 4, 8)/\zeta(15)$ and $\zeta(1, 2, 4, 8)/\zeta(15)$ in both cases. In case (i) it suggests

$$x(x+1)(x^8 + x^6 + x^5 + x^3 + 1)\zeta(1, 2, 4) = (x^2 + x + 1)\zeta(7)$$

(degree-(2, 9) reconstruction from 600 coefficients, numerical only). Theorem 5.3 implies that some identity of this shape holds; this explicit form was not compared with the closed formula for $\alpha_{n,k}$. Conversely, no rational ratio of degree up to about 305 exists, in case (i) or case (iv), for any of $\zeta(1, 1)/\zeta(2)$, $\zeta(1, 3)/\zeta(4)$, $\zeta(3, 1)/\zeta(4)$, $\zeta(2, 1)/\zeta(3)$, $\zeta(1, 5)/\zeta(6)$, $\zeta(5, 1)/\zeta(6)$, $\zeta(1, 6)/\zeta(7)$, $\zeta(1, 2, 1)/\zeta(4)$, $\zeta(2, 2, 3)/\zeta(7)$ and $\zeta(3, 4, 1)/\zeta(8)$. This is consistent with the statement of [LRT21] that $\zeta(1, 1)$ is not zeta-like in case (i), but it rules out only ratios of bounded height.

Question 5.7. The following speculations of [LRT21] are not numbered conjectures there, and we know of no work on them after [CNDP23]; our search was not exhaustive.

- (1) Is the family of Theorem 5.3 the only primitive zeta-like family for the rings (i)–(iv)?
- (2) In higher genus, are zeta-like values (rational ratio) and Eulerian values (algebraic ratio) the same?
- (3) In higher class number, when does $\alpha_{n,k} \in H$ lie in K , and which H -linear combinations over ideal classes are Eulerian?
- (4) Are the Frobenius difference equations of [LRT21, Rem. 4] solvable, giving a finite-level mechanism (which [CNDP23] bypass)?
- (5) Are there zeta-like examples for the Dedekind-type relative zeta values of [LRT21, §6]?

6. RELATIONS AMONG MULTIZETA VALUES

6.1. Setting. Let $\mathcal{P} = \overline{K}[X_{\mathfrak{s}}]$ be the polynomial ring in one variable for each tuple $\mathfrak{s}$ (including the empty tuple), graded by weight, and let $\text{ev}: \mathcal{P} \rightarrow \mathbb{C}_{\infty}$ be the $\overline{K}$ -algebra map $X_{\mathfrak{s}} \mapsto \zeta(\mathfrak{s})$, with $\zeta(\emptyset) = 1$. For tuples $\mathfrak{s}, \mathfrak{t}$ there is an explicit $\mathbb{F}_p$ -linear combination $\mathfrak{s} * \mathfrak{t}$ of tuples of weight $w(\mathfrak{s}) + w(\mathfrak{t})$ with $\zeta(\mathfrak{s})\zeta(\mathfrak{t}) = \zeta(\mathfrak{s} * \mathfrak{t})$ [Tha10b]; see also [CCM23, §2.3]. Let V_w be the $\overline{K}$ -span of the $X_{\mathfrak{s}}$ of weight w , and $R_w = \ker(\text{ev}) \cap V_w$ the space of linear relations of weight w . Put $D_1 = t^q - t$. The *fundamental binary relation* is

$$(3) \quad R_1: \quad X_{(q)} + D_1 X_{(1, q-1)}, \quad \text{that is,} \quad \zeta(q) + D_1 \zeta(1, q-1) = 0.$$

It holds by Proposition 4.5, since $\ell_1 = -D_1$.

Theorem 6.1 ([Cha14, Thm. 2.2.1, Cor. 2.2.2]). *Let $w_1, \dots, w_{\ell}$ be distinct positive integers, and for each i let V_i be a finite set of monomials in multizeta values of total weight w_i that is linearly independent over K . Then $\{1\} \cup \bigcup_i V_i$ is linearly independent over $\overline{K}$. Consequently the $\overline{K}$ -algebra $\mathcal{Z}$ generated by all multizeta values is $\overline{K} \oplus \bigoplus_{w \geq 1} \mathcal{Z}_w$, graded by weight; it is defined over K (the*

natural map $\overline{K} \otimes_K \mathcal{Z}^K \rightarrow \mathcal{Z}$ from the K -algebra $\mathcal{Z}^K$ is bijective); and every nontrivial monomial in multizeta values is transcendental over K .

Theorem 6.2 ([CCM23], [IKL⁺24]). *Let d_w be the dimension over K of the K -span of the multizeta values of weight w . Then $\sum_{w \geq 1} d_w x^w = x(1 - x^{q-1})/(1 - 2x + x^{q+1})$. In particular $d_w = 2^{w-1}$ for $w < q$ and $d_q = 2^{q-1} - 1$.*

The two papers proved this independently; the case of weights below a bound is in [ND21]. The values of d_w follow by comparing coefficients in $(1 - 2x + x^{q+1}) \sum d_w x^w = x - x^q$.

6.2. Shuffle-type relations do not suffice. It is natural to expect that all algebraic relations among multizeta values follow from the shuffle (product) relations, perhaps together with the Frobenius relations $\zeta(\mathfrak{s})^p = \zeta(p\mathfrak{s})$ and Carlitz's Eulerian evaluations. In fact this fails in weight q for every q .

Theorem 6.3. *Let $I \subset \mathcal{P}$ be the ideal generated by*

- (1) $X_\emptyset - 1$ and $X_{\mathfrak{s}}X_{\mathfrak{t}} - X_{\mathfrak{s}*\mathfrak{t}}$ for all $\mathfrak{s}, \mathfrak{t}$;
- (2) $X_{\mathfrak{s}}^p - X_{p\mathfrak{s}}$ for all $\mathfrak{s}$;
- (3) *every weight-homogeneous polynomial $G \in \overline{K}[X_{(m(q-1))}] : m \geq 1$ with $\text{ev}(G) = 0$ (the relations given by Carlitz's evaluations $\zeta(m(q-1)) \in K^\times \tilde{\pi}^{m(q-1)}$, written without a variable for $\tilde{\pi}$).*

Then $I \subset \ker(\text{ev})$, but $R_1 \notin I$. In fact $I \cap V_q = 0$.

The argument is elementary, and we claim no novelty for it.

Proof. All generators are relations, so $I \subset \ker(\text{ev})$. Every generator is weight-homogeneous (the generators $X_\emptyset - 1$ only in the sense of a grading in which $X_\emptyset$ has weight 0), so I is a graded ideal and $I \cap V_q$ is spanned by products $m \cdot g$ of monomials m and generators g of weight at most q .

The generators of types (1) and (2) have coefficients in $\mathbb{F}_p$. Let G be a nonzero generator of type (3), of weight w . If $w < 2(q-1)$, then G is a scalar multiple of the single variable $X_{(q-1)}$ (and $w = q-1$), which is impossible because $\zeta(q-1) \neq 0$. So $w \geq 2(q-1)$, which exceeds q when $q \geq 3$. For $q = 2$, the weight-two relations among $\zeta(1), \zeta(2)$ are multiples of $X_{(1)}^2 - X_{(2)}$, which has coefficients in $\mathbb{F}_2$. Hence in all cases the generators of weight at most q are $\mathbb{F}_p$ -rational, and $I \cap V_q = \overline{K} \otimes_{\mathbb{F}_p} W$, where W is the $\mathbb{F}_p$ -span of the $\mathbb{F}_p$ -rational elements of $I \cap V_q$. (Indeed, the weight- $\leq q$ part of I is the scalar extension of the corresponding part of the ideal over $\mathbb{F}_p$ generated by the same generators, and V_q is defined over $\mathbb{F}_p$; the intersection of two subspaces defined over $\mathbb{F}_p$ is the scalar extension of the intersection of their $\mathbb{F}_p$ -forms.)

On the other hand $I \cap V_q \subset R_q$. There are 2^{q-1} tuples of weight q and, by Theorem 6.2 and Theorem 6.1, their values span a space of dimension $2^{q-1} - 1$ over $\overline{K}$. So R_q is a line, and by (3) it is spanned by $X_{(q)} + D_1 X_{(1,q-1)}$. A nonzero $\mathbb{F}_p$ -rational vector on this line would have coordinate ratio $D_1 \in \mathbb{F}_p$, which is false. Hence $W = 0$ and $I \cap V_q = 0$. Since $R_1 \in V_q$ is nonzero, $R_1 \notin I$. $\square$

The proof uses neither associativity of $*$ nor any identity beyond the listed generators. It shows only that the ideal is too small in weight q . Theorem 6.7 below identifies what is missing.

Remark 6.4. The encoding of Carlitz's evaluations matters. If a variable Π for $\tilde{\pi}$ is adjoined and the evaluations are written as $X_{(m(q-1))} - c_m \Pi^{m(q-1)}$ with $c_m \in K^\times$, then already $X_{(q-1)} - c_1 \Pi^{q-1}$ has weight $q-1 < q$ and a coefficient outside $\mathbb{F}_p$, and the argument above does not apply as written. We have not analysed that variant; see Question 6.9.

Remark 6.5. The statement “all relations come from shuffles” is sometimes attributed to Thakur. Thakur says the opposite. In [Tha09b, pp. 2319–2320] he separates shuffle-type relations with $\mathbb{F}_p$ -coefficients from Euler-type relations with $\mathbb{F}_p(t)$ -coefficients, and writes that a simple description of all identities is not known even conjecturally. The only published all-relations conjecture we know is Todd’s [Tod18, Conj. 5.1], which uses one weight- q relation together with operators, not shuffles alone.

Remark 6.6. “Eulerian relation” has two readings. If it means Carlitz’s evaluations of $\zeta(w)$ for $(q-1) \mid w$, then the expectation is false by Theorem 6.3. If it means Thakur’s Euler-type K -linear relations within one weight, such as $\zeta(m, m(q-1)) = \zeta(mq)/\ell_1^m$ [Tha09b, Thm. 5(2)], then the expectation is true but empty until the generating relations are specified. Theorem 6.7 specifies them: R_1 suffices.

6.3. Todd’s conjecture and its resolution. Todd [Tod18] works with *relations of power sums*: families $R = (R(d))_{d \geq 0}$, where $R(d)$ is a K -linear combination of the iterated power sums $S_d(\mathfrak{s}) = \sum_{d=d_1 > d_2 > \dots} S_{d_1}(s_1) \cdots S_{d_r}(s_r)$ of one weight w that vanishes for every d ; summing over d turns such a family into an element of V_w lying in R_w . The binary relation $R_1(d) = S_d(q) + D_1 S_{d+1}(1, q-1)$ is of this kind [Tod18, §3], and its sum over d is (3). Todd defines operators B_U and C_V , indexed by tuples U, V , on such families; for instance $B_U(R)(d) = S_d(U) \sum_{j < d} R(j)$ and $C_V(R)(d) = R(d) S_{< d+1}(V)$. Chang, Chen and Mishiba use a variant B_U^* . We do not reproduce the definitions, for which we refer to [Tod18, §3] and [CCM23, §5]. For $w \geq 1$ let $\mathcal{R}_w$ (resp. $\mathcal{R}_w^*$) be the K -span, in V_w , of the weight- w images of R_1 , $B_U(R_1)$, $C_V(R_1)$ and $B_U C_V(R_1)$ (resp. the same with B_U^*) over all U, V .

Theorem 6.7. *Let $J \subset \mathcal{P}$ be the ideal generated by $X_\emptyset - 1$ and the $X_s X_t - X_{s+t}$.*

- (1) $\ker(\text{ev}) = J + \sum_w \overline{K} \mathcal{R}_w^*$.
- (2) *If the main theorem of the preprint [Hu26] holds, then also $\ker(\text{ev}) = J + \sum_w \overline{K} \mathcal{R}_w$.*

Proof of (1). Clearly $J \subset \ker(\text{ev})$, and the elements of $\mathcal{R}_w^*$ are relations. Conversely, let $F \in \ker(\text{ev})$. Using the generators of J , reduce F modulo J to the form $c_0 + \sum_w \ell_w$ with $c_0 \in \overline{K}$ and $\ell_w \in V_w$ [CCM23, Prop. 2.3.5(1)]. By Theorem 6.1, $c_0 = 0$ and $\text{ev}(\ell_w) = 0$ for each w , and each ℓ_w lies in $\overline{K} \otimes_K R_w^K$, where R_w^K is the space of K -linear relations of weight w . Finally $R_w^K = \mathcal{R}_w^*$ by [CCM23, Thm. 5.4.2]. $\square$

Proof of (2). The same argument applies with $R_w^K = \mathcal{R}_w$, which is [Hu26, Thm. 1.5]. $\square$

The theorem is an assembly of published results, and we claim no novelty for it. Part (1) is unconditional. Part (2) depends on [Hu26], an unrefereed preprint (arXiv v2, October 2026) whose proofs we have not checked. Its abstract states the B form of Todd’s conjecture and credits the B^* form to [CCM23]. In weights $w < q$ there are no linear relations, and in weight q the only one is $K \cdot R_1$. Since $B_U(R)(d)$ and $C_V(R)(d)$ are built from products of power sums, all relations are shuffle-type relations at the level of fixed-degree power sums, provided one starts from the single non-shuffle seed R_1 .

Computational evidence 6.8. One might expect that $\mathbb{F}_p$ -rational linear relations among multizeta values do not occur, so that $\mathbb{F}_p$ -rationality would characterise shuffle-type relations. Truncated-series computations contradict this for $q = 2$. They found $\mathbb{F}_2$ -rational relations in weights 5, 6, 7, spanning spaces of dimensions 1, 4, 13; in weight 5,

$$\zeta(1, 1, 3) + \zeta(1, 2, 2) + \zeta(2, 1, 2) + \zeta(3, 1, 1) + \zeta(4, 1) = 0.$$

The partial sums over $d \leq D$ of this relation have valuations $8, 16, \dots, 2048$ for $D \leq 9$. One weight-7 relation holds exactly at every partial sum:

$$\begin{aligned} &\zeta(2, 1, 1, 3) + \zeta(2, 1, 2, 2) + \zeta(2, 1, 4) + \zeta(2, 2, 1, 2) + \zeta(2, 2, 3) \\ &\quad + \zeta(2, 3, 1, 1) + \zeta(2, 3, 2) + \zeta(2, 4, 1) + \zeta(3, 1, 3) + \zeta(3, 2, 2) = 0. \end{aligned}$$

None were found for $q = 3$ ($w \leq 6$), $q = 4$ ($w \leq 5$) or $q = 5$ ($w \leq 5$); the full relation rank for $q = 5$, $w = 5$ was not recomputed at the precision described below. The relation ranks agreed with $2^{w-1} - d_w$ for $q = 2$ ($w \leq 6$), $q = 3$ ($w \leq 5$) and $q = 4$ ($w \leq 4$). The product formula was checked exactly against partial sums in 1300 cases.

Such computations need care with precision. Deep multizeta values have large valuation; for instance $\text{val}_\infty \zeta(1, 1, 1, 1, 1) = 780$ when $q = 5$. The expansions are also sparse modulo $q - 1$, because $t \mapsto \lambda t$ ($\lambda \in \mathbb{F}_q^\times$) scales $S_d(k)$ by λ^{-dk} . At low precision both effects produce spurious relations. The computations above used precision at least $\max_s \text{val}_\infty \zeta(\mathfrak{s}) + 3(q - 1)N$, where N is the number of unknowns.

- Question 6.9.** (1) Is R_1 a consequence of comparing a harmonic product with the q -shuffle product through a transfer between multizeta values and Carlitz multiple polylogarithms? Todd notes that no shuffle analogue of R_1 is known.
- (2) Are the $\mathbb{F}_2$ -rational relations of Evidence 6.8 explicit images of R_1 under Todd's operators? If $*$ is associative, they do not lie in J .
- (3) Can Theorem 6.7 be extended to include $\tilde{\pi}$? This needs the zeta-like classification of Section 4.

7. ZEROS OF GOSS ZETA FUNCTIONS

7.1. Setting. For a finite place v of degree D , Goss's v -adic zeta function of $A = \mathbb{F}_q[t]$ is

$$\zeta_v(x, y) = \sum_{d \geq 0} x^d \sum_{a \in A_{+,d}, v \nmid a} a^y,$$

an entire function of $x \in \mathbb{C}_v$ for each y in a suitable p -adic space; for an integer $y = -k$ the coefficient of x^d is $S_{d,v}(k)$ [Tha13, 2.3]. The zeros in x are read off the Newton polygon of the points $(d, v_d(k))$: a segment of slope λ and horizontal length m corresponds to m zeros, counted with multiplicity, of valuation $-\lambda$. We call λ a *slope* and m its *multiplicity*. We use the same sign convention at ∞ : Goss's zeta function is $\sum_d x^d \sum_{a \in A_{+,d}} a^y$ up to the normalisation of [Tha13, 2.3], so for an integer $y = m \geq 1$ the coefficients are the power sums $S_d(-m) = \sum_{a \in A_{+,d}} a^m$, which vanish for large d , and the zeros are read off the points $(d, \text{val}_\infty S_d(-m))$. The same applies to a general base ring (Definition 2.3).

Definition 7.1. Let A be as in Definition 2.3, with ∞ of degree d_∞ and residue field of order $r = q^{d_\infty}$, and fix $y \in \mathbb{Z}_p$.

- (1) The multiset of *real parts* $R_A(y)$ is the multiset of the numbers $d_\infty \cdot \text{val}_\infty(x)$ over the zeros x of Goss's zeta function $\zeta_A(x, y)$, counted with multiplicity [KMU23, §1.1.3]. It is determined by the ∞ -adic Newton polygon. Kramer-Miller and Upton write the ∞ -adic zeta function with a^{-y} , so our y at ∞ is their $-y$; at v the two conventions agree. We write $R_{\text{line}}(y)$ for $R_A(y)$ when $A = \mathbb{F}_q[t]$, and drop y when it is clear.
- (2) For a finite place v of A of degree d_v (for $A = \mathbb{F}_q[t]$, $d_v = D$), the *identity component* of Goss's v -adic zeta function is its restriction $\zeta_{A,v}(x, 1, y)$ to the characters with trivial Teichmüller part [KMU23, §1.2.3, Def. 2.17, Rem. 2.18]. For $A = \mathbb{F}_q[t]$ and an integer y divisible by $q^{d_v} - 1$, it equals $\zeta_v(x, y)$.

- (3) A *trivial zero* is a zero forced by the missing Euler factor at ∞ (respectively at v) at the values of y corresponding to the negative integers $-k$ with $(r-1) \mid k$ (respectively $(q^{d_v} - 1) \mid k$); it is *regular* if its order equals the order forced by that factor and *irregular* otherwise [KMU23, §1.1.3, §1.2.2].

Theorem 7.2 (Wan, Diaz-Vargas, Poonen, Sheats; see [She98]). *For $A = \mathbb{F}_q[t]$ and every $y \in \mathbb{Z}_p$, every element of $R_{\text{line}}(y)$ has multiplicity one.*

Kramer-Miller and Upton [KMU23] treat ordinary curves. Their paper is an unrefereed v1 preprint, and Proposition 7.9 below appears to contradict one of its statements, so we record their results as claims.

Claimed result 7.3 ([KMU23, Thm. 1.5], unrefereed). *For each $y \in \mathbb{Z}_p$ and $r = q^{d_\infty}$ there is a (possibly finite) increasing sequence of positive integers $\alpha_{r,1} < \alpha_{r,2} < \dots$, depending only on y and r , such that for every ordinary X , $R_A(y)$ consists of 0 with multiplicity $g-1+d_\infty$ together with $\alpha_{r,i}(r-1)$ with multiplicity d_∞ for each i .*

Claimed result 7.4 ([KMU23, Thm. 1.10], unrefereed). *Let X be ordinary, v a place of A of degree d_v , $r_v = q^{d_v}$, and $y \in \mathbb{Z}_p$. The v -adic Newton slopes of $x \mapsto \zeta_{A,v}(x, 1, y)$ are 0 with multiplicity $g-1+d_v+d_\infty$, together with $\alpha_{r_v,i}/d_v$ with multiplicity d_v for each i .*

In [KMU23], Claimed result 7.4 is deduced from Claimed result 7.3, applied to the ring of functions on X regular away from v , by the comparison [KMU23, Thm. 1.8, Cor. 2.20]. For $A = \mathbb{F}_q[t]$, Claimed result 7.3 contains Theorem 7.2.

7.2. Two lemmas on v -adic power sums.

Lemma 7.5. *Let $W \subset A$ be an $\mathbb{F}_q$ -subspace of dimension $n \geq 1$. Then $\sum_{w \in W} w^m = 0$ for $0 \leq m < q^n - 1$.*

Proof. The polynomial $P(X) = \prod_{w \in W} (X - w)$ is $\mathbb{F}_q$ -linear, $P(X) = \sum_{i=0}^n a_i X^{q^i}$ with $a_n = 1$, and $a_0 \neq 0$ because its roots are simple. Hence $P'(X) = a_0$, and

$$\sum_{m \geq 0} \left(\sum_{w \in W} w^m \right) X^{-m-1} = \sum_{w \in W} \frac{1}{X - w} = \frac{P'(X)}{P(X)} = \frac{a_0}{P(X)} \in a_0 X^{-q^n} (1 + X^{-1} \mathbb{F}_q(t)[[X^{-1}]]).$$

Comparing coefficients of X^{-m-1} with $m+1 < q^n$ gives the claim. $\square$

Lemma 7.6. *Let v have degree D , let $k \in \mathbb{Z}$, and let $d \geq D$. Then $v_d(k) \geq q^{d-D} - 1$.*

The trivial bound of [Tha13, 3.6] grows only linearly in d ; this bound grows exponentially.

Proof. For $d = D$ there is nothing to prove, so let $n = d - D \geq 1$ and let W be the space of polynomials of degree less than n . If $a \in A_{+,d}$ and $v \nmid a$, then every element of $a + vW$ is monic of degree d and prime to v . Conversely, two such elements a, a' with $v \mid a - a'$ satisfy $a - a' \in vW$. So the set summed in $S_{d,v}(k)$ is a disjoint union of cosets $a + vW$. For one coset, in K_v ,

$$\sum_{w \in W} (a + vw)^{-k} = a^{-k} \sum_{w \in W} \left(1 + \frac{v}{a} w \right)^{-k} = a^{-k} \sum_{m \geq 0} \binom{-k}{m} \left(\frac{v}{a} \right)^m \sum_{w \in W} w^m,$$

which converges because $\text{val}_v(v/a) = 1$ and $W \subset A$. By Lemma 7.5, the terms with $m < q^n - 1$ vanish, and the others have valuation at least $q^n - 1$. $\square$

Lemma 7.7. *Let v have degree D and a root $\omega \in \mathbb{F}_{q^D}$. Let $b \geq 1$, $c \geq 0$ and $d \geq 0$ be integers, and put $k_n = bq^n - c$. For $r \in \mathbb{Z}/D\mathbb{Z}$ let*

$$G_r(\varepsilon, \delta) = \sum_{a \in A_{+,d}, v \nmid a} a(\omega + \varepsilon)^c a(\omega^{q^r} + \delta)^{-b} \in \mathbb{F}_{q^D}[\varepsilon][[\delta]],$$

and let $T_{M,J}(r)$ be its coefficient of $\varepsilon^J \delta^M$. If $q^n > cd$ and $r \equiv n \pmod{D}$, then $v_d(k_n) = q^n M + J$, where (M, J) is the lexicographically least pair (first M , then J) with $T_{M,J}(r) \neq 0$, and $v_d(k_n) = \infty$ if there is none.

Proof. Since v is separable, $t \mapsto \omega + \varepsilon$ embeds K_v into $\mathbb{F}_{q^D}((\varepsilon))$ with val_v equal to the ε -adic order. The coefficients of a lie in $\mathbb{F}_q$, so $a(\omega + \varepsilon)^{q^n} = a(\omega^{q^n} + \varepsilon^{q^n}) = a(\omega^{q^r} + \varepsilon^{q^n})$, which is a unit because ω^{q^r} is a root of v and $v \nmid a$. Therefore

$$S_{d,v}(k_n) \mapsto \sum_a a(\omega + \varepsilon)^c a(\omega^{q^r} + \varepsilon^{q^n})^{-b} = G_r(\varepsilon, \varepsilon^{q^n}) = \sum_{M,J} T_{M,J}(r) \varepsilon^{J+q^n M}.$$

Here $0 \leq J \leq cd < q^n$, because $a(\omega + \varepsilon)^c$ has degree cd in ε . So $(M, J) \mapsto J + q^n M$ is injective and increasing for the lexicographic order, and the order of the series is $J + q^n M$ for the least pair with $T_{M,J}(r) \neq 0$. $\square$

Corollary 7.8. *In the setting of Lemma 7.7, there are n_0 and integers M_r, J_r ($r \in \mathbb{Z}/D\mathbb{Z}$) such that $v_d(bq^n - c) = q^n M_r + J_r$ for all $n \geq n_0$ with $n \equiv r \pmod{D}$, unless these valuations are infinite. Hence any conjectured formula of the form $q^n M'_r + J'_r$ holds for all $n \geq n_0$ as soon as it holds for two values of $n \geq n_0$ in each residue class.*

Proof. Take n_0 with $q^{n_0} > cd$. In Lemma 7.7 the coefficients $T_{M,J}(r)$ depend on n only through r , so the least pair (M_r, J_r) does too, which gives the first claim. If $q^n M'_r + J'_r = q^n M_r + J_r$ for two distinct n in one class, subtracting gives $M'_r = M_r$ and then $J'_r = J_r$. $\square$

7.3. Refutations and corrections.

A slope of multiplicity one. It is natural to expect, from Claimed result 7.4 applied to $\mathbb{F}_q[t]$ (genus 0, ∞ of degree 1, and ordinary), that at a place v of degree $d_v = 2$ every nonzero slope of $\zeta_v(x, y)$ on the identity component has even multiplicity. We find a slope of multiplicity one.

Proposition 7.9 (computer-assisted). *Let $q = 2$, $v = t^2 + t + 1$ and $k = 27$. Then*

$$(v_d(27))_{d=0}^6 = (0, 1, 0, 6, 33, 75, 198),$$

and the Newton polygon of $\sum_d S_{d,v}(27)x^d$ begins with the slope 0 of multiplicity 2, followed by the slope 6 of multiplicity 1.

Proof. The seven valuations were computed exactly from the power series of $a(\omega + \varepsilon)^{-27}$ in $\mathbb{F}_4[[\varepsilon]]$, where ω is a root of v (see the proof of Lemma 7.7), by two independent implementations ([AI 26e], `code/vadic.py`; and a separate direct summation, [AI 26a], `independent_check.py`). The points $(0, 0)$, $(1, 1)$, $(2, 0)$ give the slope 0 with multiplicity 2. The line through $(2, 0)$ and $(3, 6)$ is $y = 6(x - 2)$. The points with $4 \leq d \leq 6$ lie strictly above it ($33 > 12$, $75 > 18$, $198 > 24$), and for $d \geq 7$ Lemma 7.6 gives $v_d(27) \geq 2^{d-2} - 1 > 6(d - 2)$. Hence $(3, 6)$ is a vertex, the segment from $(2, 0)$ to $(3, 6)$ is an edge, and the next slope is larger than 6. $\square$

The exact values for $d \leq 11$ in [AI 26e] (`data/vadic_np_certified.json`), combined with Lemma 7.6, certify the initial slopes 0, 0, 6, 27, 42, 123, so the slope 27 also has multiplicity one; this part rests on a single implementation. The point $(3, 6)$ lies below the chord from $(2, 0)$ to $(4, 33)$; the endpoints of the predicted blocks are matched, and only interior vertices dip.

Remark 7.10 (needs human verification). Under our reading of [KMU23], Proposition 7.9 contradicts Claimed result 7.4. The exponent $y = -27$ is divisible by $q^{d_v} - 1 = 3$, so the Teichmüller factors $\omega_v(a)^y$ are trivial and Goss's series coincides with $\zeta_{A,v}(x, 1, -27)$ of [KMU23, Def. 2.17, Rem. 2.18]. Rescaling x by a unit does not change multiplicities. Claimed result 7.4 then requires the slope 0 with multiplicity $g - 1 + d_v + d_\infty = 2$, which is what we find, and every nonzero

slope with multiplicity divisible by $d_v = 2$, which fails for the slope 6. Since [KMU23, Thm. 1.10] follows from [KMU23, Thm. 1.5] applied to the ring of functions regular off v on $\mathbb{P}^1$ (genus 0, with the place at infinity of degree 2), either that case of [KMU23, Thm. 1.5] or the comparison [KMU23, Cor. 2.20] would have to fail. We have not audited the proofs in [KMU23], a v1 preprint, and the conflict should be checked by a human expert before it is relied on.

The same phenomenon occurs (by the same certified method, in a single implementation [AI 26e]) for this v at $k = 3, 6, 12, 15, 51, 54$; for $q = 2$, $v = t^3 + t + 1$ at $k = 7, 14, 21, 42$ (at $k = 21$ a pair occurs where a triple is required); and for $q = 3$, $v = t^2 + 1$ at $k = 8, 16, 24, 40, 48$. So it is not tied to $p \mid d_v$.

Thakur's closed form in [Tha13, 9.1(IV)]. Let $q = 2$ and $v = t^2 + t + 1$, and put $f(n, d) = v_d(5 \cdot 2^n - 1) - v_d(2^{n+1} - 1)$. Thakur conjectures [Tha13, 9.1(IV)] that $f(n, d) = (3d - 6)2^n$ for even d , $f(n, d) = (3d - 15)2^n$ for odd $d > 3$, and, for $d = 3$, $f(n, 3) = 2^n$ for odd n and $2^n - 1$ for even n . It is natural to accept this closed form; in fact the sign is wrong and, for $d \geq 4$, the size is doubled.

Theorem 7.11 (computer-assisted). *Let $q = 2$ and $v = t^2 + t + 1$. For all $n \geq 1$:*

- (1) *for $2 \leq d \leq 11$, Thakur's conjectured formula [Tha13, 3.5] holds: $v_d(2^n - 1) = 2^{n+d-2} - (2^n - 1)\lfloor d/2 \rfloor + (-1)^{d-1}$ for even n , and $v_d(2^n - 1) = 2^{n+d-2} - (2^n - 1)\lfloor d/2 \rfloor - \lfloor (d + 1)/2 \rfloor + \lfloor d/2 \rfloor$ for odd n ;*
- (2) *for $3 \leq d \leq 11$, $w_n = v_d(3 \cdot 2^n - 1)$ satisfies the recursions of [Tha13, 9(ii)];*
- (3) *for $3 \leq d \leq 11$,*

$$f(n, d) = \begin{cases} -(3d - 6)2^{n-1}, & d \text{ even}, \\ -(3d - 15)2^{n-1}, & d \text{ odd}, d \geq 5, \\ -2^n, & d = 3, n \text{ odd}, \\ -(2^n - 1), & d = 3, n \text{ even}. \end{cases}$$

For $d = 12$, the three statements hold for all n with $2^n > 12$.

Certificate. In each case the conjectured value, and both terms of $f(n, d)$, have the form $2^n M_r + J_r$ with $r = n \bmod 2$. For the recursions of 9(ii) this holds because a first-order recursion $x_n = 2x_{n-1} + c_{n \bmod 2}$ has solutions of the form $2^n C + e_{n \bmod 2}$. By Corollary 7.8, with $D = 2$ and $(b, c) \in \{(1, 1), (3, 1), (5, 1), (2, 1)\}$, it suffices to compute the pairs (M, J) of Lemma 7.7 for $r = 0, 1$, compare them with the formulas at four consecutive values $n \geq n_0$ (where $2^{n_0} > d$), and check the finitely many $n < n_0$ by direct computation of v_d ([AI 26e], `code/lemma_linear.py`, `code/prove_vadic.py`, `data/prove_vadic_d12.log`). For $d = 12$ only the part $n \geq n_0$ was run. Part (3) was also checked by an independent direct summation for $d = 3, 4, 5$ and $n = 2, 3, 4$ ([AI 26a], `n4_spotcheck.py`). $\square$

Remark 7.12. Thakur notes that $f(n, 5) = 0$; this survives the correction. The correction does affect his recipe for the transition $B_n \rightarrow B'_n$ [Tha13, 9(iv)], which adds $-f(n, d - i - 1)$, and his assertion [Tha13, 9.1(I)] that these additions are ≤ 0 : with the corrected (negative) f the two cannot both hold as printed, so the sign convention in 9(iv) is in doubt. The full recipes of §7 (708 terms) and §8 (768 terms), the $d = 1, 2, 3$ patterns of §9, the $X_n \rightarrow X'_n$ block recipe (594 values) and the maximum hypothesis of 9.1(II) were verified numerically. The transition recipes 9(iii)–(v) for $d \geq 4$ were not tested.

Thakur's recipe in [Tha13, §7].

Theorem 7.13 (computer-assisted). *Let $q = 3$, $v = t$ and $d = 2$. For every $N \geq 1$,*

$$v_2(2 \cdot 3^N - 1) = 4 \cdot 3^N + 2, \quad v_2(4 \cdot 3^N - 1) = 2 \cdot 3^N + 2.$$

In [Tha13, §7] Thakur writes the sequence of $v_2(k)$, $3 \nmid k$, as blocks separated by entries a_m , the value at $k = 17 + 18(m - 1)$, and conjectures $a_{3^n} = 3^{n+3} + 3^{n+2} + 2$ and $a_{2 \cdot 3^n} = 2 \cdot 3^{n+2} + 2$. With $N = n + 2$ these are the two identities of Theorem 7.13.

Certificate. Apply Lemma 7.7 with $D = 1$, $\omega = 0$, $c = 1$, $d = 2$ and $b = 2$, respectively $b = 4$; its hypothesis $3^N > 2$ holds for $N \geq 1$. The least pairs are $(M, J) = (4, 2)$, respectively $(2, 2)$ ([AI 26e], `code/prove_s7.py`; second run in [AI 26a], `prove_s7_rerun.md`). The direct values at $N = 1$ are 14 and 8, as predicted. $\square$

Further computational findings.

Computational evidence 7.14. In this evidence, the next one and Remark 7.16, R_A and R_{line} are taken at $y = k$ for an integer $k \geq 1$, so that the coefficients are the power sums $S_d(-k) = \sum a^k$. For non-ordinary X , Kramer-Miller and Upton expect [KMU23, Rem. 1.6] that R_A consists of values $s_1, \dots, s_{g-1+d_\infty}$ with $0 \leq s_i \leq \alpha_{r,1}$, together with the values $\alpha_{r,i}(r - 1)$, each with multiplicity d_∞ , of Claimed result 7.3. The computations contradict this.

- (1) For the genus-2, p -rank-0 ring (iv) of Section 5, for every odd $k \leq 511$, $R_A = R_{\text{line}} \sqcup \{\text{the two smallest elements of } R_{\text{line}}\}$. So the second extra value is the second element of R_{line} , which exceeds the expected bound; for $k = 3$, the line has $\{1, 3\}$ and A has $\{1, 1, 3, 3\}$.
- (2) For the genus-1 ring (iii) and $k = 181$, the line has $\{21, 177\}$ and A has $\{21, 21, 189\}$, which is not R_{line} plus one element. The degrees $\deg S_2 = 320$, $\deg S_3 = 312$, $S_4 = 0$ were confirmed by an independent implementation in Singular. The same occurs at $k = 437, 565, 661, 693, 709, 721, 757, 949$ and persists for $k = 181 + j \cdot 2^m$ with $m \leq 14$.
- (3) For $q = 4$ and $k = 1, 5, 17, 65, 257$ both sets are empty, although the expected shape requires $g - 1 + d_\infty = 1$ value s_1 in R_A .

Computational evidence 7.15. Böckle showed that for the ring (i), R_A agrees with R_{line} except that the smallest element has multiplicity two (as reported in [KMU23, §1.1.3]; we could not obtain Böckle's preprint). It is natural to expect the same pattern for the other genus-1 rings. The computations confirm it for $q = 2$ (512 tested $k \leq 1023$) and $q = 3$ (485 of the 486 tested $k \leq 728$, the exception being $k = 1$), but it fails for $q = 4$ at the nine values of Evidence 7.14(2). The guess $\deg S_{d+1}^A = \deg S_d^{\text{line}} + \deg S_1^{\text{line}}$ for the genus-1 rings also fails, in 9 of 150 cases for $q = 2$, 51 of 167 for $q = 3$ and 71 of 130 for $q = 4$.

Remark 7.16. In the tested range $k \leq 1023$, the $\mathbb{F}_4$ anomaly of Evidence 7.14(2) occurs at nine values, $k = 181, 437, 565, 661, 693, 709, 721, 757, 949$. They include all eight values $181, 437, 565, 661, 693, 709, 721, 949$ (odd, at most 1000) at which Thakur found the recursion (2) to fail for $q = 4$ and the power sums $S_2(-k)$ (Remark 3.8); the only additional value is 757. R_{line} was computed by direct summation [AI 26e], so the coincidence is not an artefact of using the recursion. We do not know a mechanism; see Question 7.21.

Computational evidence 7.17. For $q = 2$ and $v = t^2 + t + 1$, every tested pair of equal-slope v -adic zeros consists of two distinct zeros rather than an inseparable double zero. The zeros are those of $x \mapsto \zeta_v(x, -k)$, and the pairs (k, λ) were $(3, 0)$, $(3, 9/2)$, $(9, 0)$, $(9, 39/2)$, $(15, 33/2)$, $(19, 77/2)$, $(5, 1/2)$, $(11, 1/2)$, $(17, 1/2)$, $(17, 7/2)$ and $(9, 3/2)$. A double zero z would force $f_e(z^2) = f_o(z^2) = 0$ for the even and odd parts f_e, f_o ; in each case either f_o has no root of the relevant valuation, or Newton iteration in $\mathbb{F}_4((\varepsilon))$ separates the roots of f_e and f_o . Goss's Conjecture 5 [Gos00] allows finitely many exceptions, so this is evidence against it, not a disproof.

Computational evidence 7.18. It is natural to expect irregular trivial zeros to be sporadic. In the computations they follow an exact rule. At ∞ (with $y = k$ as in Evidence 7.14), a trivial

zero had order 2 exactly when k was among the doubled real parts: $k = 1$ in genus 1, $q = 2$; $k = 3, 9, 21, 33, 69, 81, 129, 261, 273, 321, 513$ in genus 1, $q = 4$; and nine values in genus 2. For $q = 3$ and $k = 3^n + 1$ the real part is also doubled, but the order is 1: the pair consists of two distinct zeros. Also $\zeta_A(-k) \neq 0$ for all 341 tested k with $(q-1) \nmid k$, consistent with Thakur's question. Off the identity component ($q = 2$, $\deg v = 2$) there is one zero of slope 0 instead of two; for $k \equiv 2 \pmod{3}$ the other slopes come in half-integer pairs, while for $k \equiv 1 \pmod{3}$ they are mixed.

Remark 7.19. The literature already contains four failures of naive expectations, which we carry forward without re-verification: Thakur's observation that the naive Riemann hypothesis needs modification at primes of degree two [Tha13]; Goss's near-trivial zeros and Thakur's irregular trivial zeros (as described in [KMU23, §1]); and Böckle's doubled smallest zero (as reported in [KMU23]). The computations re-verified Theorem 7.2 in all 1154 tested cases, with real parts divisible by $q-1$, and the degree-one v -adic Riemann hypothesis claimed in the unrefereed preprint [DVPC14] in every tested case.

7.4. A conjecture and open problems.

Conjecture 7.20 (new). *Let A be a base ring of class number one over a prime field $\mathbb{F}_p$, with $d_\infty = 1$, whose curve has genus $g \geq 1$ and p -rank 0 (so A is one of (i), (ii), (iv)). Then for every integer $k \geq 2$, at $y = k$ (power sums $S_d(-k)$), $R_A = R_{\text{line}} \sqcup \{\text{the } g \text{ smallest elements of } R_{\text{line}}\}$.*

It holds on the tested ranges for $(g, q) = (1, 2), (1, 3), (2, 2)$, and it is false for $(g, q) = (1, 4)$ (Evidence 7.14), which is why we restrict to prime fields.

- Question 7.21.** (1) What is the correct multiplicity law for $d_v \geq 2$? The data suggest that the polygon passes through the endpoints of the blocks predicted by Claimed result 7.4, but that slopes inside a block may split.
- (2) The ordinary case with $d = 1$ and $g \geq 1$ was not tested: there are no ordinary examples of class number one, and non-principal ideals were not implemented.
- (3) Prove Conjecture 7.20 and explain the $\mathbb{F}_4$ anomaly. Is it caused by the failure of the line-side recursion (2) for $q = 4$ (Remark 7.16)?
- (4) Test the transition recipes 9(iii)–(v) of [Tha13] for $d \geq 4$, and settle the sign in 9(iv) (Remark 7.12).
- (5) Does Goss's Conjecture 5 hold for almost all y ?

8. AUTOMATA AND TRANSCENDENCE OF SPECIAL VALUES

8.1. The criterion and two corrections. For a finite field $\mathbb{F}$ of characteristic p , Christol's theorem [CKMFR80] says that $\sum_n c_n x^n \in \mathbb{F}[[x]]$ is algebraic over $\mathbb{F}(x)$ if and only if (c_n) is p -automatic, if and only if its Q -kernel $\{(c_{Q^a n + b})_n : a \geq 0, 0 \leq b < Q^a\}$ is finite for one (or every) power Q of p .

Remark 8.1. It is natural to expect every special value to be transcendental, and hence non-automatic. In fact some are algebraic. The v -adic zeta value $\zeta_v(n)$ vanishes when $(q-1) \mid n$ (Goss; see [CM19, §1.2.1]), the analogue of the vanishing of the Kubota–Leopoldt zeta function at even integers. Thakur's analogue of the Gross–Koblitz formula makes $\Pi_v(n)$ algebraic when the q -adic digits of n are eventually D -periodic [Tha, §4.1], [WY02, §10]. The question is which values are non-automatic.

Remark 8.2. It is also natural to ask whether ∞ -adic values are “automatic in t ”. This is ill-posed: ∞ -adic values ($\zeta(n)$, $\tilde{\pi}$, Γ -values, multizeta values) lie in $K_\infty = \mathbb{F}_q((1/t))$, and Christol's theorem applies to their expansions in $1/t$. An expansion in t makes sense at $v = t$ or after a change

of variable. In the two-variable setting of Anderson–Thakur and Pellarin the coefficients lie in $\mathbb{C}_\infty$, which is not a finite field, so Christol's theorem does not apply. Proposition 8.3 below is an elementary substitute, which is surely known.

In the two-variable setting it is customary to write θ for the image of t in $\mathbb{C}_\infty$ and to use a new variable; we call it z here, to avoid a clash with our t .

Proposition 8.3. *The Anderson–Thakur function $\omega(z) = (-\theta)^{1/(q-1)} \prod_{i \geq 0} (1 - z/\theta^{q^i})^{-1}$, for a fixed choice of $(-\theta)^{1/(q-1)}$, is transcendental over $\mathbb{C}_\infty(z)$.*

Proof. ω is meromorphic on $\mathbb{C}_\infty$ with poles at the infinitely many distinct points θ^{q^i} . If ω satisfied $\sum_{j=0}^m c_j(z)\omega^j = 0$ with $c_j \in \mathbb{C}_\infty[z]$ and $c_m \neq 0$, then at a pole z_0 of ω that is not a zero of c_m , the term $c_m\omega^m$ would have a pole of order strictly larger than every other term, which is impossible. Since c_m has finitely many zeros, this is a contradiction. $\square$

8.2. Status. Table 3 summarises what is known. Entries marked $\dagger$ rest on a reading recorded in [AI 26d] (of an abstract, a summary, a secondary account or the full text), not on our own reading of the source.

Values	Status	Sources
$\tilde{\pi}$	Transcendental (Wade 1941); automata proof by Al-louche	[Tha] † , [WY02] †
$\zeta(n)$, ∞ -adic	Transcendental for all $n \geq 1$; $\zeta(n)/\tilde{\pi}^n$ is in K iff $(q-1) \mid n$, and transcendental otherwise	[Yu91] † , [CM19, §1.2.1] †
$\zeta_v(n)$	0 if $(q-1) \mid n$, transcendental otherwise	[CM19, §1.2.1] †
Arithmetic Γ at ∞ , fractions	Algebraic monomials determined by automata; all algebraic relations with ζ	[Tha, Thm. 4.1] † , [CPTY10] †
Arithmetic Γ at ∞ , $n \in \mathbb{Z}_p \setminus \mathbb{N}$	Transcendental	[MFY97] †
Geometric Γ at ∞ , fractions	All relations determined	[ABP04] †
$\Pi_v(n)$, $n \in \mathbb{Z}_p$	Open in general (Conjecture 8.5); known for $D = 1$ and residue class 0; claimed for rational n	[WY02] † , [CWY24] † , Thm. 8.6
v -adic geometric Γ	Open	[CWY24, Rem. 1.4.3] †
Thakur MZVs at ∞	Transcendental (Remark 8.4)	[Cha14]
v -adic MZVs	Eulerian ones vanish; $D = 1$ criterion; transcendence for $D \geq 2$ in depth ≥ 2 open to our knowledge	[CM21] † , [Che25] †
Tate period (Voloch)	Automata proof known	[Tha10a] †

TABLE 3. Transcendence status of special values.

Remark 8.4. By Theorem 6.1, which includes the constant 1 in weight 0, every multizeta value is transcendental over K [Cha14, Cor. 2.2.2]. By Christol's theorem its $1/t$ -expansion is therefore not automatic.

8.3. The v -adic arithmetic gamma function. Let v have degree D and put $Q = q^D$. For $j \geq D$ let $D_{j,v} = D_j/(v^{q^{j-D}}D_{j-D})$, the product of the elements of $A_{+,j}$ prime to v , and for $j < D$ let $D_{j,v} = D_j$. For $n = \sum_j n_j q^j \in \mathbb{Z}_p$ with digits $0 \leq n_j \leq q-1$, Thakur's v -adic interpolation of the Carlitz factorial is $\Pi_v(n) = \prod_{j \geq 0} (-D_{j,v})^{n_j} \in K_v$.

Conjecture 8.5 (Wen–Yao, [WY02, §10]). $\Pi_v(n)$ is algebraic over K if and only if $n_{j+D} = n_j$ for all sufficiently large j .

Thakur asked whether his sufficient condition is best possible [Tha, §4.1]. The precise conjecture is the one in [WY02, §10], and Thakur's notes credit it to Yao. The “if” direction is Thakur's Gross–Koblitz analogue. The “only if” direction is known in two cases and claimed in a third.

- $D = 1$: [WY02, Thm. 2], [Tha, Thm. 4.5].
- $n \in \mathbb{Q} \cap \mathbb{Z}_p$, claimed: write $n = a/b$ with $\gcd(a, b) = 1$ and $p \nmid b$. The digits of n are eventually periodic with exact period $\text{ord}_b(q)$, so they are eventually D -periodic iff $b \mid q^D - 1$. The case therefore follows from the algebraicity criterion for v -adic arithmetic gamma values at rational arguments claimed in the unrefered preprint [CWY24] (Cor. 1.4.2 there, seen only through a summary recorded in [AI 26d]), via $\Pi_v(n) = \Gamma_v(n + 1)$.
- The digits off the residue class 0 modulo D are eventually constant: Theorem 8.6.

Thakur's survey [Tha10a, p. 16] lists the general case as open. Period methods do not obviously help: [CWY24] works only at rational arguments.

Theorem 8.6 ([WY02, Thm. 8]). *Suppose that for every $r \not\equiv 0 \pmod{D}$ the sequence $(n_{kD+r})_{k \geq 0}$ is eventually constant. Then $\Pi_v(n)$ is algebraic over K if and only if $(n_{kD})_{k \geq 0}$ is eventually constant.*

Wen and Yao leave the proof as an exercise. We give a complete proof, which uses one lemma.

Lemma 8.7. *Let $Q \geq 2$ and $a, b, c \geq 1$ be integers. Then $(Q^c - 1) \mid Q^a(Q^b - 2) + 1$ if and only if $c \mid \gcd(a, b)$.*

Proof. Let A' and B' be the residues of $a + b$ and a modulo c . Modulo $Q^c - 1$, the number $Q^a(Q^b - 2) + 1$ is congruent to $X = Q^{A'} + 1 - 2Q^{B'}$. Since $0 \leq A', B' \leq c - 1$, we have $2 - 2Q^{c-1} \leq X \leq Q^{c-1} - 1$, and both bounds have absolute value less than $Q^c - 1$. So divisibility holds iff $X = 0$, that is, $Q^{A'} + 1 = 2Q^{B'}$. If $B' \geq 1$, the right side is divisible by Q , while the left side is $\equiv 1 \pmod{Q}$ if $A' \geq 1$ and equals 2 if $A' = 0$; the latter is divisible by Q only when $Q = 2$, and then $2 = 2^{B'+1}$ is impossible. So $B' = 0$, and then $A' = 0$. Thus divisibility holds iff $c \mid a$ and $c \mid a + b$, that is, iff $c \mid \gcd(a, b)$. $\square$

Proof of Theorem 8.6. Step 0 (coordinates). Let $\alpha \in \mathbb{F}_Q$ be a root of v and $s = t - \alpha$. Then $t \mapsto \alpha + s$ embeds K_v into $\mathbb{F}_Q((s))$, and an element is algebraic over K iff it is algebraic over $\mathbb{F}_Q(s)$, a finite extension of K . The derivation $' = d/dt = d/ds$ is continuous on $\mathbb{F}_Q((s))$. It maps elements algebraic over $\mathbb{F}_Q(s)$ to algebraic elements, because $\mathbb{F}_Q((s))$ is separable over $\mathbb{F}_Q(s)$.

Step 1 (a homomorphism). For a bounded sequence $c = (c_j)$ of integers put $F(c) = \prod_j (-D_{j,v})^{c_j}$. It converges because $-D_{j,v} \rightarrow 1$ v -adically [Tha, §4]. Then $F(c + c') = F(c)F(c')$ and $F(p^k c) = F(c)^{p^k}$. If c has finite support, then $F(c) \in K^\times$. Moreover $\Pi_v(n) = F(n)$.

Step 2 (reduction to residue 0). Split $n = \sum_r n^{(r)}$, where $n_j^{(r)} = n_j$ for $j \equiv r \pmod{D}$ and 0 otherwise. For $r \neq 0$ the digit sequence $n^{(r)}$ is eventually D -periodic, so $F(n^{(r)})$ is algebraic by the Gross–Koblitz analogue. The sequence $n^{(0)}$ is eventually periodic iff $(n_{kD})_k$ is eventually constant, and in that case $F(n^{(0)})$ is algebraic. It remains to show: if c is supported on $D\mathbb{N}$, takes values in $\{0, \dots, q - 1\}$, and $(c_{kD})_k$ is not eventually constant, then $F(c)$ is transcendental.

Step 3 (normalisation). Put $\delta_k = c_{kD} - c_{(k+1)D}$. The nonzero δ_k occur infinitely often and satisfy $|\delta_k| < q$, so $v_p(\delta_k) < f$. Let κ be the smallest value of $v_p(\delta_k)$ that occurs for infinitely many k . Then $\delta_k \equiv 0 \pmod{p^\kappa}$ for all large k , so $c_{kD} \pmod{p^\kappa}$ is eventually a constant γ . Subtract the periodic sequence $\gamma \cdot \mathbf{1}_{D\mathbb{N}}$ (whose F is algebraic) and change finitely many entries (which changes F by a factor in $K^\times$). The result is $p^\kappa b$ for a bounded integer sequence b supported on

$D\mathbb{N}$, and $F(c)$ is algebraic iff $F(b)^{p^\kappa}$ is, iff $F(b)$ is. By the choice of κ , $\varepsilon_k = b_{kD} - b_{(k+1)D}$ is not divisible by p for infinitely many k .

Step 4 (logarithmic derivative). Since $[m]' = -1$ for $m \geq 1$, and $q^{j-i} \equiv 0$ in characteristic p for $i < j$, we get $\text{dlog } D_j = -1/[j]$ for $j \geq 1$. Hence $\text{dlog } D_{j,v} = 1/[j-D] - 1/[j]$ for $j > D$, and $\text{dlog } D_{D,v} = -1/[D] - v'/v$. For $m = kD$ with $k \geq 1$, $t^{q^m} = \alpha + s^{Q^k}$, so $[kD] = s^{Q^k} - s$ and

$$\frac{1}{[kD]} = -\frac{1}{s} - G_k, \quad G_k = \frac{s^{Q^k-2}}{1-s^{Q^k-1}} = \sum_{m \geq 1} s^{m(Q^k-1)-1}.$$

Thus $\text{dlog } D_{kD,v} = G_k - G_{k-1}$ for $k \geq 2$. The logarithmic derivative of $F(b)$ is the limit of those of its partial products, and Abel summation, with b bounded and $G_k \rightarrow 0$, gives

$$L = \frac{F(b)'}{F(b)} \equiv \sum_{k \geq 2} \varepsilon_k G_k \pmod{\mathbb{F}_Q(s)}.$$

If $F(b)$ were algebraic, L would be algebraic by Step 0, and hence so would $s \sum_{k \geq 2} \varepsilon_k G_k$.

Step 5 (an infinite kernel). We have

$$s \sum_{k \geq 2} \varepsilon_k G_k = \sum_{N \geq 1} c(N) s^N,$$

with $c(N) = \sum_{k \geq 2, (Q^k-1)|N} \varepsilon_k \in \mathbb{F}_p$. For $a \geq 1$ the sequence $u_a(N) = c(Q^a N + 1)$ lies in the Q -kernel of c . By Lemma 8.7, $u_a(Q^h - 2) = \sum_{k \geq 2, k|\gcd(a,h)} \varepsilon_k$ for $h \geq 1$. Let $a > a' \geq 2$ with ε_a and $\varepsilon_{a'}$ nonzero modulo p . Let h be the least divisor of a that does not divide a' and has $\varepsilon_h \not\equiv 0$; it exists because a itself qualifies, and $h \geq 2$. Then

$$u_a(Q^h - 2) - u_{a'}(Q^h - 2) = \sum_{k|h, k \nmid a'} \varepsilon_k = \varepsilon_h \not\equiv 0 \pmod{p},$$

since every other divisor k of h with $k \nmid a'$ is a divisor of a smaller than h , and so has $\varepsilon_k \equiv 0$. Thus the u_a , for the infinitely many a with $\varepsilon_a \not\equiv 0$, are pairwise distinct. The Q -kernel of c is infinite, so by Christol's theorem the series is transcendental over $\mathbb{F}_p(s)$, hence over $\mathbb{F}_Q(s)$. Therefore $F(b)$, and with it $\Pi_v(n)$, is transcendental. $\square$

Remark 8.8. For residue classes $r \neq 0$ the same method stops. Put $\beta_r = \alpha^{q^r} - \alpha \neq 0$. For $j \equiv r \pmod{D}$, $[j] = \beta_r - s + s^{q^j}$ is a v -adic unit, and telescoping gives $\text{dlog } F(b^{(r)}) \equiv \sum_k \varepsilon_k u_{kD+r}$ with $u_j = \sum_{m \geq 1} (-1)^m s^{mq^j} (\beta_r - s)^{-m-1}$. Two steps are missing: (R1) each residue- r series is transcendental when ε is not eventually 0, and (R2) different residue classes cannot cancel. Pure q -power sums such as $\sum s^{Q^k}$ are algebraic, so transcendence must come from the binomial twist. Two natural shortcuts fail. The shift identity $D_{j+1,v} = D_{j,v}^q [j+1]/[j+1-D]$ writes the residue- r factor as $G^{q^r} W$, but $\text{dlog}(G^{q^r}) = 0$, so taking logarithmic derivatives only returns the residue- r series. And the untelescoped series $-\sum_j (n_j - n_{j+D})/[j]$ does not converge v -adically, because $1/[j] \not\rightarrow 0$ at v ; only the grouping used in Step 4 converges.

Computational evidence 8.9. $\Pi_v(n)$ was computed exactly modulo s^P , and the number of distinct p -kernel prefixes of length L was counted. For $(q, v, P, L) = (2, t^2 + t + 1, 8192, 64)$, eventually D -periodic digits gave 2 prefixes; residue-0 non-periodic digits (Theorem 8.6) and residue-1 non-periodic digits (open) both grew to the maximum of about 2^7 . Similar growth appeared for $(3, t^2 + 1, 6561, 27)$ and $(2, t^3 + t + 1, 8192, 64)$. The logarithmic-derivative formula held to $O(s^{398})$, the telescoped identity held exactly to $O(s^{P-2})$ for $(q, D) = (2, 2), (2, 3), (3, 2)$, and Lemma 8.7 had no exceptions for $Q \in \{2, 3, 4, 5, 7, 8, 9, 16, 27\}$, $a, b \leq 8$, $c \leq 12$. As a calibration the algebraic series $\sum s^{Q^k}$ gave bounded counts. Finite windows cannot prove transcendence.

- Question 8.10.** (1) Prove (R1) and (R2), and hence Conjecture 8.5.
 (2) Give a period-method proof of the theorem of Mendès France and Yao [MFY97].
 (3) Are nonzero v -adic multizeta values transcendental when $\deg v \geq 2$?
 (4) Settle the v -adic geometric gamma conjecture [CWY24, Rem. 1.4.3].
 (5) Determine the nature of $\Gamma(0)$ for Thakur's shtuka-theoretic gamma function [Tha, §4.3].

9. SUMMARY OF OPEN PROBLEMS

- (1) H1 for $q = p^f$, $f \geq 2$; Conjecture 3.10; the family of Theorem 3.6 for every non-prime q , and uniqueness of its minimiser beyond $q \leq 27$; Thakur's negative-exponent examples, which were not re-verified (Section 3).
- (2) Completeness of the zeta-like lists above weight q^2 ; the constraints 4.1(1), the prefix half of 4.1(2), 4.2, 4.3 and 4.5 (Question 4.9).
- (3) The higher-genus speculations of Question 5.7; a comparison of the numerical identity of Evidence 5.6 with the closed formula for $\alpha_{n,k}$ in [CNDP23].
- (4) The origin of R_1 , the $\mathbb{F}_2$ -rational relations, a refereed account of the B form of Todd's conjecture, and the inclusion of $\tilde{\pi}$ (Question 6.9). The relation rank for $q = 5$, $w = 5$ was not computed at adequate precision.
- (5) Adjudication of the conflict in Remark 7.10, by a human expert; Conjecture 7.20; Question 7.21; the small- n check for $d = 12$ in Theorem 7.11; an independent audit of the code behind Theorems 3.6, 7.11 and 7.13.
- (6) Conjecture 8.5 beyond residue class 0, and Question 8.10.

10. SOFTWARE AND AI ASSISTANCE

Methods: generation. This AI-generated article originated in a prompt given during Taylor Dupuy's talk *How to Use AI in Mathematical Research Effectively* at the University of Rochester Number Theory Seminar on 23 October 2026. The prompt used `frontier-dump` to investigate selected Thakur conjectures, followed by `harvest-dumps` and `using-latexpowers` to assemble this manuscript. Diophantine approximation exponents and continued fractions were excluded because another researcher was already working on them.

A Claude Code agent [Ant26] using the model `claude-opus-5-5` wrote the article on 23 October 2026 from six research dumps produced the same day by six Claude Code agents using the same model. Taylor Dupuy did not write or check any part of the mathematics. The research agents worked alone and could not delegate.

Their reports are the source of every result here that is not cited to the literature, with two qualifications: the $d = 12$ part of Theorem 7.11 comes from an addendum that the coordinating agent of the research run added to the Goss record after its script finished; and the agent that assembled this paper wrote or re-derived the proofs of Lemmas 7.5, 7.6 and 7.7, Corollary 7.8, Propositions 7.9, 4.5 and 4.6, Theorems 6.3, 6.7 and 8.6, Lemma 8.7 and Proposition 8.3 from the research records.

Agents are not authors and are not the source of any mathematical claim; responsibility for the content rests with the human who prints it.

On 26 October 2026, a Codex desktop agent based on GPT-6 revised this provenance account and the title-page printing credit from Taylor Dupuy's clarification. The exact model identifier was not exposed to the agent. This editorial revision changed no mathematical statement or proof and received no human mathematical review.

On 27 October 2026, a Codex desktop agent based on GPT-6 revised the title, creation-date line, and opening abstract paragraph at Taylor Dupuy's request. It also removed a title-footnote link to a private AI-usage ledger. No mathematical statement or proof changed.

Methods: synthesis and review. The manuscript was assembled with the `harvest-dumps` and `using-latexpowers` skills of `mathcity` [Dup]: a source map, an outline, a filled presentation and an integrated manuscript. Separate Claude Code reviewer agents, which did not write the text they judged, reviewed the outline (accepted after three rounds) and the filled presentation (accepted after three rounds), and a third reviewer checked the integrated manuscript and its compiled PDF; the reports, rounds and verdicts are kept in the directory `latex/thakur-conjectures/harvest/reviews/` of the repository that holds [AI 26a]. All review was by AI agents; no human has checked the proofs, the computations or the citations.

Methods: computations. The research agents used SageMath 10.8 [The25] and Python. Their scripts and data are in the research records [AI 26f, AI 26g, AI 26c, AI 26b, AI 26e, AI 26d]. While writing this paper the coordinating agent independently recomputed Example 3.7, the valuations in Proposition 7.9, and part (3) of Theorem 7.11 for $3 \leq d \leq 5$, $2 \leq n \leq 4$, with its own Python code. It also reran two of the research agents' scripts: the minimiser behind the uniqueness statement of Theorem 3.6, and the computation behind Theorem 7.13. The certificates in Theorems 3.6, 7.11 and 7.13 otherwise rest on the research agents' code, which has not been independently audited. No proof-assistant verification was done. Several computations required care with t -adic precision; see Evidence 6.8.

Methods: citations. Every arXiv identifier cited was checked against the arXiv API, and journal data against Crossref or the publisher's page. Where a pinpoint comes from a research agent's reading of a source that we did not open, the text says so. The preprints [CO26, Hu26, KMU23, CWY24, DVPC14] are unrefereed.

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