

THE REGULAR PENTAGONAL SYSTEM AND THE SMITH–WARD PROBLEM

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ABSTRACT. The three-dimensional function system of the regular pentagon fails the lifting property, giving a counterexample to the operator-system formulation of the Smith–Ward problem with C^* -envelope $\mathbb{C}^5$. We construct unital k -positive maps from this fixed system into matrix algebras where distance from every ucp map is bounded below independently of k .

We also prove that every regular polygon system with at least five vertices detects nuclearity, while its dual characterizes the weak expectation property. These tensor-product characterizations follow from a graph dichotomy: a graph operator system and its dual are C^* -nuclear when the graph is 2-chordal; otherwise they detect nuclearity and WEP, respectively.

1 Introduction

Let μ_n be the n -th roots of unity, and let z be the coordinate function on μ_n . The operator system

$$\mathcal{P}_n = \text{span}\{1, z, \bar{z}\} \subseteq C(\mu_n) \cong \mathbb{C}^n$$

is the complex affine function system of the regular n -gon, with its inherited matrix ordering. The transition from the square to the pentagon changes its lifting and tensor-product properties without changing its dimension.

Theorem A (Polygonal test systems; Theorem 5.3).

The system $\mathcal{P}_5$ fails the lifting property and has C^* -envelope $\mathbb{C}^5$. For every $n \geq 5$ and every unital C^* -algebra A ,

$$\begin{aligned} \mathcal{P}_n \otimes_{\min} A = \mathcal{P}_n \otimes_{\max} A &\iff A \text{ is nuclear,} \\ \mathcal{P}_n^d \otimes_{\min} A = \mathcal{P}_n^d \otimes_{\max} A &\iff A \text{ has the WEP.} \end{aligned}$$

For $n = 3, 4$, both $\mathcal{P}_n$ and $\mathcal{P}_n^d$ are C^* -nuclear.

Kavruk proved that the Smith–Ward problem has an affirmative answer if and only if every three-dimensional operator system has the lifting property [11, Thm. 11.5]. Scherer obtained a negative answer [22] based on work of Harris [8] on four-dimensional systems, and Harris subsequently constructed further counterexamples and a three-dimensional nuclearity detector [9]. Theorem A gives a counterexample within a finite-dimensional commutative C^* -algebra. It also supplies a three-dimensional nuclearity detector whose dual characterizes WEP. These dimensions are minimal: every operator system of dimension at most two is completely order isomorphic to $\mathbb{C}$ or $\mathbb{C}^2$ [11, Prop. 6.13].

We give an independent quantitative construction of the pentagon’s lifting obstruction.

Date: October 26, 2026.

Theorem B (Uniform pentagon separation; Theorem 3.1).

There are a constant $c_0 > 0$, integers $n_k \rightarrow \infty$, and unital $2k$ -positive maps $\phi_k : \mathcal{P}_5 \rightarrow M_{n_k}$ such that

$$\inf_{\psi \in \text{UCP}(\mathcal{P}_5, M_{n_k})} \|\phi_k(z) - \psi(z)\| \geq c_0 \quad (k \geq 1).$$

The maps are k -contractive.

Starting with the malnormal matrices of Mulcahy and the second author [17], we specify the maps by a cyclic block frame and support projections. A trace-norm stability estimate, followed by an application of the Powers–Størmer inequality, converts the commutator gap into a lower bound independent of the target dimension. The proof gives an explicit bound in terms of the malnormality constant. The maps pass arbitrarily high finite-level positivity tests while remaining uniformly separated from ucp maps; they therefore witness failure of CP-stability. Their product-quotient class and block-diagonal representation give nonliftable ucp maps into $\prod_k M_{n_k} / \bigoplus_k M_{n_k}$ and the Calkin algebra.

The tensor-product assertions in Theorem A come from a general graph dichotomy. For an operator system X , write

$$\text{Cl}(X) = \{A \text{ unital } C^* \text{-algebra} : X \otimes_{\min} A = X \otimes_{\max} A\},$$

where equality means that the canonical identity is a complete order isomorphism. Write Nuc and WEP for the classes of nuclear and WEP unital C^* -algebras. Thus X detects nuclearity when $\text{Cl}(X) = \text{Nuc}$, and characterizes WEP when $\text{Cl}(X) = \text{WEP}$.

For a finite simple graph G on n vertices, adjoin the diagonal to its symmetric edge relation and put

$$\mathcal{S}_G = \text{span}\{E_{ij} : (i, j) \in E(G)\} \subseteq M_n, \quad J_G = \text{span}\{E_{ij} : (i, j) \notin E(G)\}.$$

Its dual is completely order isomorphic to M_n/J_G , with compatible order units. Dor-On and Russell proved that $\mathcal{S}_G$ is C^* -nuclear exactly when G is 2-chordal [3], extending the chordal case of [14]. We retain the coefficient algebra in their obstruction arguments and determine the tensor-equality classes on both sides of duality.

Theorem C (Graph dichotomy; Theorem 4.14).

If G is 2-chordal, then $\mathcal{S}_G$ and $\mathcal{S}_G^d$ are C^* -nuclear. Otherwise,

$$\text{Cl}(\mathcal{S}_G) = \text{Nuc}, \quad \text{Cl}(\mathcal{S}_G^d) = \text{WEP}.$$

The WEP assertion uses relative positive completion and Kavruk’s tight Riesz interpolation theorem. A commutant-extension argument then gives nuclearity detection. Consequently $\mathcal{S}_G$ has LP exactly when G is 2-chordal; equivalently, its dual is exact, or every ambiently completable G -partial matrix over the Calkin algebra has a positive completion over the Calkin algebra itself.

The connection to polygons is Cayley transference (Theorem 5.1). For a finite group Γ and a symmetric subset $C \subseteq \Gamma \setminus \{e\}$, the group system $\text{span}\{\lambda_c : c \in C \cup \{e\}\}$ is a ucp retract of the Cayley graph system, and the graph system is a ucp retract of a matrix amplification of the group system. Their tensor-equality classes, and those of their duals, therefore agree. Applying this to $\Gamma = \mathbb{Z}_n$ and $C = \{\pm 1\}$ proves Theorem A.

Section 2 collects the background. We prove the quantitative pentagon theorem in Section 3, the graph dichotomy in Section 4, and Cayley transference and the polygon consequences in Section 5.

Acknowledgments

The second author thanks Roy Araiza, Adam Dor-On, Matthew Kennedy, Larissa Kroell, and Travis Russell for many stimulating conversations on approximation properties of operator systems and the Smith–Ward problem as part of the AIM SQuaRE workshop “Approximation properties for operator systems and matrix convex sets.” The authors would like to thank Adam Dor-On, Sam Harris, and Vern Paulsen for helpful comments and suggestions.

Use of AI tools

The authors selected the problems, provided the relevant literature, and directed an extensive exploration of the Smith–Ward problem with Claude Opus 5.5. Within that exploration, the proofs of Theorems A and C, the Cayley transference theorem, and the lifting and Calkin-completion results were generated by Claude Opus 5.5. The argument was reorganized and redrafted with the assistance of ChatGPT-6 Astra. Additional prompting of ChatGPT-6 Astra produced the construction in Theorem B. ChatGPT-6 Astra was then prompted to formalize the construction detailed in Theorem B in Lean, assuming only the existence of a sequence of malnormal matrices with rank tending to infinity. During the formalization process, several simplifications of the original construction were found, which are incorporated into this manuscript. Sustained dialog with ChatGPT-6 Astra was used to improve the exposition and distill general principles.

The final AI-drafted manuscript was verified and edited for content and clarity by the authors. The authors take full responsibility for the accuracy of the results.

2 Preliminaries

2.1 Operator systems and their tensor products

We will make use of the theory of operator system tensor products as developed by Kavruk, Paulsen, Todorov, and Tomforde [14]. The systems denoted by X and Y are finite-dimensional unless stated otherwise; tensor partners may be arbitrary operator systems. Throughout, A denotes a unital C^* -algebra. We recall:

- (P1) $\min$ is injective [14].
- (P2) $\max$ is associative and symmetric, and $M_m \otimes_{\min} Y = M_m \otimes_{\max} Y = M_m(Y)$ for every operator system Y [14]. Consequently $M_m(X) \otimes_{\alpha} A = M_m(X \otimes_{\alpha} A)$ for $\alpha \in \{\min, \max\}$, and $M_m(X \otimes_{\alpha} A) = X \otimes_{\alpha} M_m(A)$.
- (P3) For a unital C^* -algebra A , $X \otimes_c A = X \otimes_{\max} A$, and $X \otimes_c A \subseteq C_u^*(X) \otimes_{\max} A$ completely order isomorphically [14, §6].
- (P4) If $\varphi : X_1 \rightarrow X_2$ and $\psi : Y_1 \rightarrow Y_2$ are completely positive (not necessarily unital), then $\varphi \otimes \psi$ is completely positive for both $\min$ and $\max$.
For $\min$ this follows from the definition via cp maps into matrix algebras. For $\max$, $\varphi \otimes \psi$ maps the generators $\alpha(P \otimes Q)\alpha^*$ to elements of the same form, and $(\varphi \otimes \psi)(e \otimes e) \leq \|\varphi(e)\| \|\psi(e)\| e \otimes e$, so the Archimedean closure is preserved.
- (P5) If $X \subseteq Y$ is a unital inclusion, then $Y^d \rightarrow X^d$ is a complete quotient of the operator system Y^d [4].

Lemma 2.1 (Retractions, amplification, and duality).

Let X, Y be finite-dimensional operator systems.

- (i) If X is a cp retract of Y , then $\text{Cl}(Y) \subseteq \text{Cl}(X)$. The dual maps make X^d a cp retract of Y^d , so also $\text{Cl}(Y^d) \subseteq \text{Cl}(X^d)$.
- (ii) For every $m \geq 1$, $\text{Cl}(M_m(X)) = \text{Cl}(X)$ and $\text{Cl}(M_m(X)^d) = \text{Cl}(X^d)$.

Proof. If $\psi\varphi = \text{id}_X$ with $\varphi : X \rightarrow Y$ and $\psi : Y \rightarrow X$ cp, functoriality shows that, for $A \in \text{Cl}(Y)$, the composition

$$X \otimes_{\min} A \xrightarrow{\varphi \otimes \text{id}} Y \otimes_{\min} A \xrightarrow{\psi} Y \otimes_{\max} A \xrightarrow{\psi \otimes \text{id}} X \otimes_{\max} A$$

is cp and equals the identity. The dual maps satisfy $\varphi^d \psi^d = \text{id}_{X^d}$, giving the dual assertion.

Amplification preserves tensor equality; the converse follows from the ucp retraction of $M_m(X)$ onto X given by normalized matrix trace. Finite-dimensional duality [4] and nuclearity of M_m give

$$M_m(X)^d \cong M_m^d \otimes_{\max} X^d \cong M_m(X^d)$$

as matrix-ordered spaces, proving the last assertion. $\square$

Remark 2.2 (Order units). The class $\text{Cl}(X)$ depends only on the matrix ordering. If e, e' are Archimedean matrix order units, then $e \leq Ce'$ and $e' \leq Ce$ for some $C > 0$. The Archimedean closures defining max are therefore unchanged, while the cp maps defining min do not depend on the unit. We may consequently use any faithful positive functional as the order unit of a dual system.

2.2 Lifting property, exactness, and WEP

A finite-dimensional operator system X has the *lifting property* (LP) if every ucp map $X \rightarrow B/I$ into a quotient C^* -algebra lifts to a ucp map $X \rightarrow B$. By [11, Prop. 7.4] it suffices to consider $B = B(\ell^2)$ and $I = \mathcal{K}$. We use the following results of Kavruk.

Theorem 2.3 ([11]).

Let X be a finite-dimensional operator system and let $Q = B(\ell^2)/\mathcal{K}$ be the Calkin algebra.

- (K1) X has the LP iff $X \otimes_{\min} B(H) = X \otimes_{\max} B(H)$ for every (equivalently one infinite-dimensional separable) Hilbert space H [11, Thm. 4.3, §4.2].
- (K2) X has the LP iff X^d is exact [11, Thm. 6.6].

Every operator subsystem of M_n is exact, since exactness passes to subsystems [15, 11]. In particular $\mathcal{S}_G$ is exact and $\mathcal{S}_G^d$ has LP by duality.

A unital C^* -algebra $A \subseteq B(H)$ has the WEP if the inclusion $A \rightarrow A^{**}$ extends to a ucp map $B(H) \rightarrow A^{**}$. Equivalently, the canonical map $D \otimes_{\max} A \rightarrow D \otimes_{\max} B(H)$ is injective for every C^* -algebra D ; see [21]. These conditions are independent of the faithful representation.

Remark 2.4. If a finite-dimensional X characterizes the WEP, then $B(H) \in \text{Cl}(X)$ ($B(H)$ is injective), so X has the LP by (K1), and X^d is exact by (K2).

For x, y self-adjoint write $x < y$ if $y - x \geq \delta 1$ for some $\delta > 0$. Following [12], a unital $A \subseteq B(H)$ has the *complete (2, 3)-tight Riesz interpolation property* if, for every p , every $x_1, x_2, y_1, y_2, y_3 \in M_p(A)_{sa}$ and every $b \in M_p(B(H))_{sa}$ with $x_i < b < y_j$ for all i, j , there is $a \in M_p(A)_{sa}$ with $x_i < a < y_j$ for all i, j .

Theorem 2.5 (Kavruk [12, Thm. 0.4]).

A unital C^* -algebra A has the WEP iff it has the complete $(2, 3)$ -tight Riesz interpolation property in $B(H)$. This does not depend on the faithful representation $A \subseteq B(H)$.

We refer the reader to [24, Theorem 4.43] for an alternate proof of this result.

2.3 CP-stability

We recall the concept of CP-stability, as introduced in [7]. For a finite-dimensional operator system X , write $\|\phi\|_k$ for the norm of $\text{id}_{M_k} \otimes \phi$. The system X is *CP-stable* if for every $\varepsilon > 0$ there are k and $\delta > 0$ such that every unital linear map $\phi : X \rightarrow A$ into a unital C^* -algebra with $\|\phi\|_k \leq 1 + \delta$ is within ε , in operator norm, of a ucp map $X \rightarrow A$. The constants must be independent of A .

The equivalence of LP and CP-stability, including ultrapower lifting, is proved in [23, Prop. 2.5]; dual exactness is treated in [23, Prop. 2.6]. We use the formulation below from [24, Thm. 4.19].

Theorem 2.6.

For finite-dimensional X , LP is equivalent to CP-stability and to the following ultraproduct lifting property: for every sequence of unital C^* -algebras A_j and every free ultrafilter $\mathcal{U}$, each ucp map $X \rightarrow \prod_{\mathcal{U}} A_j$ lifts to a ucp map $X \rightarrow \prod_j A_j$. It suffices to test sequences of matrix algebras.

In the model-theoretic formulation, LP is equivalent to definability of the ucp-map locus in the language of C^* -algebras [24, Rem. 4.20]. WEP corresponds instead to positive existential closure: finite systems of strict linear matrix inequalities with parameters in A that are solvable in a containing operatorsystem are solvable in A [24, Prop. 4.33].

2.4 Malnormal matrices

Write $\tau_m = \text{Tr}/m$, and equip M_m with the unnormalized Hilbert–Schmidt norm

$$\|X\|_2 = (\text{Tr}(X^*X))^{1/2}.$$

A matrix $Z \in M_m$ is said to be κ -malnormal if

$$(1) \quad \|[Z, X]\|_2^2 \geq \kappa \|X - \tau_m(X)I\|_2^2 \quad (X = X^* \in M_m).$$

Thus, a self-adjoint matrix which approximately commutes with Z must be close to a scalar. The restriction to self-adjoint matrices is essential, since Z itself belongs to the kernel of its commutator map.

We use the following existence theorem of Mulcahy and the second author.

Theorem 2.7 ([17]).

There are a constant $0 < \kappa \leq 1$, integers $m_j \rightarrow \infty$, and contractions $Z_j \in M_{m_j}$ so that each Z_j is κ -malnormal.

2.5 Fourier analysis on polygonal systems

Lemma 2.8 (Fourier description of polygon cones).

Let $k \geq 3$, put $\zeta = e^{2\pi i/k}$, and let

$$\mathcal{P}_k = \text{span}\{1, z, z^*\} \subseteq C(\mathbb{Z}_k) \cong \mathbb{C}^k, \quad z(j) = \zeta^j,$$

with its inherited matrix ordering. For $N \geq 1$, evaluation identifies $M_N(\mathcal{P}_k)^+$ with the cone

$$\mathcal{K}_N^{(k)} = \left\{ (K_j)_{j=0}^{k-1} \in (M_N^+)^k : \sum_{j=0}^{k-1} \zeta^{-\ell j} K_j = 0 \quad (2 \leq \ell \leq k-2) \right\}.$$

The list of equations is empty when $k = 3$. Since the K_j are self-adjoint, it is equivalent to impose only the equations $2 \leq \ell \leq \lfloor k/2 \rfloor$.

More explicitly, evaluation sends

$$X = A \otimes 1 + B \otimes z + B^* \otimes z^*, \quad A = A^*,$$

to the tuple

$$K_j = A + \zeta^j B + \zeta^{-j} B^* \quad (0 \leq j \leq k-1).$$

Its inverse is given by

$$A = \frac{1}{k} \sum_{j=0}^{k-1} K_j, \quad B = \frac{1}{k} \sum_{j=0}^{k-1} \zeta^{-j} K_j.$$

In particular, $S_K := \sum_j K_j = kA$.

Proof. The inclusion $\mathcal{P}_k \subseteq \mathbb{C}^k$ is a complete order embedding. Thus a self-adjoint $X \in M_N(\mathcal{P}_k)$ is positive if and only if each evaluation $K_j = X(j)$ is positive. Since $k \geq 3$, the functions $1, z, z^*$ are linearly independent, so X has a unique expression of the displayed form. Orthogonality of the characters of $\mathbb{Z}_k$ gives all the stated Fourier equations and the formulas for A and B .

Conversely, suppose $(K_j) \in \mathcal{K}_N^{(k)}$ and define

$$\widehat{K}(\ell) = \frac{1}{k} \sum_{j=0}^{k-1} \zeta^{-\ell j} K_j \quad (\ell \in \mathbb{Z}_k).$$

Self-adjointness of the K_j gives $\widehat{K}(-\ell) = \widehat{K}(\ell)^*$. The defining equations therefore say that the only possibly nonzero Fourier coefficients occur at $0, 1, -1$. Fourier inversion yields

$$K_j = \widehat{K}(0) + \zeta^j \widehat{K}(1) + \zeta^{-j} \widehat{K}(1)^*.$$

Consequently the inverse formula defines an element of $M_N(\mathcal{P}_k)$ whose evaluations are precisely the positive matrices K_j , and hence that element is positive. The same adjoint relation shows that the equations indexed by $2 \leq \ell \leq \lfloor k/2 \rfloor$ imply all the remaining equations. $\square$

Note that for the pentagon, the forbidden Fourier modes form exactly one adjoint pair, so a single complex matrix equation suffices:

$$\mathcal{K}_N^{(5)} = \left\{ (K_j)_{j=0}^4 \in (M_N^+)^5 : \sum_{j=0}^4 \zeta^{-2j} K_j = 0 \right\}, \quad \zeta = e^{2\pi i/5}.$$

2.6 Graphs

For a graph G , the graphs $K_{2,3}$, $K_{2,2,2}$, $2K_1 + P_4$, C_k ($k \geq 5$), $\overline{C_6}$ and $\overline{P_6}$ are McCullough's forbidden induced subgraphs. G is 2-chordal iff it contains none of them as an induced subgraph [18]; see also [16] and [3, Thm. 2.1]. For $m \geq 1$, $G^{(m)}$ denotes the m -clique blow-up of G . An induced minor of G is a graph obtained by vertex deletions and edge contractions. Every graph on at most four vertices is 2-chordal.

3 Uniform separation and nonlifting for the pentagon

We prove Theorem B directly from the pentagon Fourier constraint and Theorem 2.7. The proof constructs a positive functional F that vanishes on a nonzero positive element but is uniformly positive on positive elements with small matrix support. Subtracting a small multiple of the trace functional therefore preserves positivity on small-support tests while destroying positivity on the chosen witness. By the small-support characterization of r -positivity, the shifted functional corresponds to an r -positive map that is not completely positive. Quantitative control of the witness and the unital normalization then gives a uniform distance from the ucp maps.

The main work is to establish the lower bound on small supports. The frame and Fourier constraint turn a positive element with small matrix support and small value under F into a low-rank positive matrix D of controlled mass, with small commutators in trace norm. A commutator estimate derived from Powers–Størmer then transfers this control to $\sqrt{D}$, where malnormality applies. The identity $\|\sqrt{D}\|_2^2 = \text{Tr } D$ makes the resulting obstruction independent of the dimension.

Let $\mathcal{P}_5 = \text{span}\{1, z, z^*\} \subset \mathbb{C}^5$, where $z(j) = e^{2\pi i j/5}$ for $j = 0, \dots, 4$. All Schatten norms use the ordinary, unnormalized trace.

Theorem 3.1.

Suppose $Z \in M_m$ is a contraction satisfying (1). For every integer $r \geq 1$ with $m \geq 10r$, there is a unital r -positive map $\phi : \mathcal{P}_5 \rightarrow M_{5m}$ such that

$$\inf_{\psi \in \text{UCP}(\mathcal{P}_5, M_{5m})} \|\phi(z) - \psi(z)\| \geq \frac{\kappa^2}{2^{51}}.$$

The map is specified below by support projections and unital normalization.

Taking $r = 2k$ and malnormal dimensions $m \geq 20k$ will give Theorem B. We keep track of constants to show explicitly that the separation is independent of both r and the target dimension. No optimization of the constants is intended.

3.1 The frame and Fourier constraint

Put

$$A = 2I + \text{Re } Z, \quad B = 2I + \text{Im } Z.$$

Then $I \leq A, B \leq 3I$. For self-adjoint X ,

$$(2) \quad \|[A, X]\|_2^2 + \|[B, X]\|_2^2 = \|[Z, X]\|_2^2 \geq \kappa \|X - \tau_m(X)I\|_2^2.$$

The equality follows because the Hilbert–Schmidt inner product of the two skew-adjoint commutators is real.

Let $N = 5m$, $\zeta = e^{2\pi i/5}$, and define

$$V = \begin{pmatrix} I & 0 & 0 \\ I & B & -BA^{-1} \\ 0 & I & 0 \\ 0 & I & A \\ A^{-1}B & 0 & I \end{pmatrix}, \quad Q = VV^*, \quad R = \text{diag}(I, \zeta I, \zeta^2 I, \zeta^3 I, \zeta^4 I).$$

The block indices are 0, 1, 2, 3, 4. The matrix V has full column rank: its rows 0, 2, 3 suffice to recover its argument, since A is invertible. Let P be the orthogonal projection onto its range and

$E = I - P$. Set

$$(3) \quad H_j = R^j Q R^{-j}, \quad E_j = R^j E R^{-j}.$$

Direct multiplication gives $Q_{ab} = 0$ when $a - b \equiv \pm 2 \pmod{5}$. Thus $H_j \geq 0$ and $\sum_j \zeta^{-2j} H_j = 0$. Also $\text{Tr } E = 2m$.

Remark 3.2. The motivation for the construction is as follows. A positive matrix C supported on the range of V and satisfying the pentagon zero pattern forces the decomposition $C = V(I_3 \otimes D)V^*$, with D commuting with both A and B . Malnormality then forces D to be scalar. A low rank requirement imposed below excludes a nonzero scalar. The goal of the estimates that follow is to make this incompatibility quantitative.

For later use,

$$\|V\|^2 \leq 41, \quad \text{Tr } Q \geq 5m.$$

For the first bound dominate the block norms of V by the scalar matrix with rows

$$(1, 0, 0), (1, 3, 3), (0, 1, 0), (0, 1, 3), (3, 0, 1),$$

whose Frobenius norm squared is 41. For the second, each diagonal block of Q is at least I .

By Lemma 2.8, we identify $M_N(\mathcal{P}_5)^+$ with

$$\mathcal{K}_N := \mathcal{K}_N^{(5)} = \{K = (K_j)_{j=0}^4 : K_j \geq 0, \sum_j \zeta^{-2j} K_j = 0\}.$$

Put

$$S_K := \sum_j K_j \quad \text{and} \quad F(K) := \sum_j \text{Tr}(E_j K_j).$$

Write $H = (H_j)_{j=0}^4 \in \mathcal{K}_N$. Since $E_j H_j = R^j E Q R^{-j} = 0$ for every j , we have $F(H) = 0$.

Lemma 3.3.

If $K \in \mathcal{K}_N$, $\text{Tr } S_K = 1$, and $\text{rank } S_K \leq r$, then

$$(4) \quad C = \sum_{j=0}^4 R^{-j} K_j R^j$$

satisfies $C \geq 0$, $\text{Tr } C = 1$, $\text{rank } C \leq 5r$, and $C_{ab} = 0$ for $a - b \equiv \pm 2 \pmod{5}$. Moreover $\text{Tr}(EC) = F(K)$.

Proof. Each K_j has range in $\text{ran } S_K$, so its rank is at most r . Ranks are subadditive and conjugation preserves rank. The (a, b) block of (4) is $\sum_j \zeta^{-j(a-b)} (K_j)_{ab}$, which vanishes on the stated non-edges. The trace identity follows by cyclicity. $\square$

Remark 3.4. The functional

$$F(K) = \sum_j \text{Tr}(E_j K_j)$$

is nonnegative on the entire positive cone and satisfies $F(H) = 0$ for the positive witness H constructed above. Thus F itself corresponds to a completely positive map.

The remaining task is to prove that, for a constant $\delta_* > 0$,

$$F(K) \geq \delta_* \text{Tr} \left(\sum_j K_j \right) \quad \text{whenever} \quad \text{rank} \left(\sum_j K_j \right) \leq r.$$

Once this is established, subtracting $\varepsilon \text{Tr}(\sum_j K_j)$, with $0 < \varepsilon < \delta_*$, produces a functional that is nonnegative on all such small-support tests but strictly negative at H . This is the separation from which the desired maps will be obtained.

3.2 Trace-norm stability

Lemma 3.5 (Block stability).

Let C be as in Lemma 3.3, and put $t = \text{Tr}(EC)$ and $\delta = \|C - PCP\|_1$. There are positive matrices $X \in M_3(M_m)$ and $D \in M_m$ such that $PCP = VXV^*$, $\text{rank } D \leq 5r$, $\delta \leq 2\sqrt{t}$, and

$$\begin{aligned} \|X - I_3 \otimes D\|_1 &\leq C_1\delta, \\ \|[A, D]\|_1 + \|[B, D]\|_1 &\leq C_2\delta, \\ 3 \text{Tr } D &\geq (1 - t)/41 - C_1\delta. \end{aligned}$$

One may take $C_1 = 1862$ and $C_2 = 1386$.

Proof. Let C be as in Lemma 3.3, and set

$$t = \text{Tr}(EC), \quad Y = PCP = VXV^*, \quad X = (V^*V)^{-1}V^*CV(V^*V)^{-1} \geq 0.$$

We have $\text{rank } X \leq 5r$ and $\text{Tr } Y = 1 - t$. Write $\delta = \|C - Y\|_1$. Schatten Hölder gives

$$(5) \quad \delta \leq \|EC\|_1 + \|PCE\|_1 \leq 2\sqrt{t}.$$

For example $EC = (EC^{1/2})C^{1/2}$ and $\|EC^{1/2}\|_2^2 = t$; the other term is treated similarly. Every non-edge block of Y has trace norm at most δ .

Write the 3 by 3 block matrix X as

$$X = \begin{pmatrix} D_0 & U & W \\ U^* & D & T \\ W^* & T^* & D_2 \end{pmatrix}.$$

The diagonal blocks are positive and $\text{rank } D \leq 5r$. The $(0, 2)$, $(0, 3)$, $(2, 4)$ blocks of Y are respectively

$$U, \quad U + WA, \quad U^*BA^{-1} + T.$$

Since $\|A^{-1}\|, \|B^{-1}\| \leq 1$ and $\|A\|, \|B\| \leq 3$,

$$(6) \quad \|U\|_1 \leq \delta, \quad \|W\|_1 \leq 2\delta, \quad \|T\|_1 \leq 4\delta.$$

The $(1, 3)$ block is

$$Y_{13} = U + WA + BD + BTA - BA^{-1}T^* - BA^{-1}D_2A.$$

The four off-diagonal terms have total trace norm at most $(1 + 6 + 36 + 12)\delta = 55\delta$. Hence

$$(7) \quad \|D - A^{-1}D_2A\|_1 \leq 56\delta.$$

Taking adjoints gives the analogous estimate with AD_2A^{-1} . Multiplying (7) on the left by A , and its adjoint on the right by A , gives

$$\|AD - D_2A\|_1, \|DA - AD_2\|_1 \leq 168\delta.$$

Since $[A^2, D] = A(AD - D_2A) - (DA - AD_2)A$, it follows that $\|[A^2, D]\|_1 \leq 1008\delta$.

For any matrix L , writing $A = 2I + J$ with $\|J\| \leq 1$ gives

$$4\|L\|_1 \leq \|AL + LA\|_1 + \|JL + LJ\|_1 \leq \|AL + LA\|_1 + 2\|L\|_1.$$

Consequently,

$$2\|L\|_1 \leq \|AL + LA\|_1.$$

Applying this with $L = [A, D]$ and using $A[A, D] + [A, D]A = [A^2, D]$, we obtain $\|[A, D]\|_1 \leq 504\delta$. Moreover,

$$(D_2 - D)A = [A, D] - (AD - D_2A).$$

Since $\|A^{-1}\| \leq 1$, it follows that

$$(8) \quad \|[A, D]\|_1 \leq 504\delta, \quad \|D_2 - D\|_1 \leq (504 + 168)\delta = 672\delta.$$

Likewise,

$$Y_{14} = D_0 B A^{-1} + W + B U^* B A^{-1} + B T - B A^{-1} W^* B A^{-1} - B A^{-1} D_2.$$

The four off-diagonal terms cost at most $(2 + 9 + 12 + 18)\delta = 41\delta$. Thus the two main terms have difference of trace norm at most 42δ . Multiply on the right by A and use (7) to get

$$\|D_0 B - B D\|_1 \leq (126 + 168)\delta = 294\delta.$$

Set $G = B D - D_0 B$, so that $\|G\|_1 \leq 294\delta$. Since B, D, D_0 are self-adjoint,

$$[B^2, D] = B G - G^* B, \quad \|[B^2, D]\|_1 \leq 2\|B\|\|G\|_1 \leq 1764\delta.$$

The preceding elementary bound also applies to B , since $\|B - 2I\| \leq 1$, and therefore $\|[B, D]\|_1 \leq 882\delta$. Finally, $(D_0 - D)B = [B, D] - G$ and $\|B^{-1}\| \leq 1$ give

$$(9) \quad \|[B, D]\|_1 \leq 882\delta, \quad \|D_0 - D\|_1 \leq (882 + 294)\delta = 1176\delta.$$

Combining (6), (8), (9),

$$(10) \quad \|X - I_3 \otimes D\|_1 \leq 1862\delta, \quad \|[A, D]\|_1 + \|[B, D]\|_1 \leq 1386\delta.$$

As $\text{Tr } Y \leq 41 \text{Tr } X$, these inequalities also give

$$3 \text{Tr } D \geq \frac{1-t}{41} - 1862\delta.$$

In particular, if

$$(11) \quad \sqrt{t} \leq \frac{1}{610736} = \frac{1}{8 \cdot 41 \cdot 1862}, \quad \text{then} \quad \text{Tr } D \geq \frac{1}{492}.$$

Indeed, $t < 1/2$, and $1862\delta \leq 1/164$ by (5). □

3.3 Square roots and the uniform gap

We recall the classical Powers–Størmer inequality [20, Lemma 4.1]: for positive matrices S, T , $\|S - T\|_2^2 \leq \|S^2 - T^2\|_1$. Indeed, set $L = S - T$ and $J = \text{sign}(L)$. Trace-norm duality gives

$$\|S^2 - T^2\|_1 \geq \text{Tr } J(SL + LT) = \text{Tr } |L|(S + T) \geq \text{Tr } L^2.$$

The last inequality follows as the difference is $2 \text{Tr}(L_+ T) + 2 \text{Tr}(L_- S) \geq 0$.

We will need the following variation.

Lemma 3.6.

If $I \leq A \leq 3I$ and $D \geq 0$, then

$$(12) \quad \|[A, \sqrt{D}]\|_2^2 \leq 4\|[A, D]\|_1.$$

Proof. For (12), diagonalize A . Its eigenvalue differences belong to $[-2, 2]$, and $|a - b| \leq 2|e^{ia} - e^{ib}|$ on this interval. Consequently

$$\begin{aligned} \|[A, \sqrt{D}]\|_2^2 &\leq 4\|[e^{iA}, \sqrt{D}]\|_2^2 \\ &\leq 4\|e^{iA} D e^{-iA} - D\|_1 \leq 4\|[A, D]\|_1. \end{aligned}$$

The middle step is the Powers–Størmer inequality; the last follows by integrating the derivative of $s \mapsto e^{isA} D e^{-isA}$ for $0 \leq s \leq 1$. □

Apply (2) to $\sqrt{D}$ and use (10), (12), (5):

$$(13) \quad \kappa \left(\text{Tr } D - \frac{(\text{Tr } \sqrt{D})^2}{m} \right) \leq 5544\delta \leq 11088\sqrt{t}.$$

The rank estimate is now dimension-free in the useful direction:

$$(14) \quad (\text{Tr } \sqrt{D})^2 \leq \text{rank}(D) \text{Tr } D \leq 5r \text{Tr } D.$$

If $m \geq 10r$, the parentheses in (13) are at least $\text{Tr } D/2$.

Proposition 3.7.

Under (1), for every $K \in \mathcal{K}_{5m}$ with $\text{Tr } S_K = 1$ and $\text{rank } S_K \leq r \leq m/10$,

$$F(K) \geq \delta_* := \frac{\kappa^2}{2^{48}}.$$

Proof. Suppose $t = F(K) < \kappa^2/2^{48}$. Since $\kappa \leq 1$, $\sqrt{t} < \kappa/2^{24} \leq 1/2^{24} < 1/610736$, so (11) applies. Equations (13)–(14) give

$$\frac{\kappa}{984} \leq 11088\sqrt{t} < \frac{11088\kappa}{2^{24}},$$

which is impossible since $984 \cdot 11088 = 10910592 < 2^{24}$. $\square$

The use of $\sqrt{D}$ in (13)–(14), rather than D , is essential: its Hilbert–Schmidt norm squared is the mass $\text{Tr } D$, which is bounded below.

We have now established the two properties of F described above. Before constructing the separating maps, we isolate the small-support characterization of positivity and the quantitative normalization argument for a general finite-dimensional operator system.

3.4 Small matrix supports and quantitative separation

The rank restriction in Proposition 3.7 comes from the level at which positivity is tested. We isolate this mechanism for finite-dimensional operator systems, together with the quantitative normalization and separation argument needed below.

Let S be a finite-dimensional operator system and let ω be a faithful state on S . For $X \in M_N(S)$, write

$$\rho_\omega(X) = (\text{id}_{M_N} \otimes \omega)(X), \quad \tau_\omega(X) = \text{Tr } \rho_\omega(X).$$

For a linear map $\Phi : S \rightarrow M_N$, put

$$f_\Phi(X) = \langle \Omega_N, (\text{id}_{M_N} \otimes \Phi)(X) \Omega_N \rangle, \quad \Omega_N = \sum_{a=1}^N e_a \otimes e_a.$$

The correspondence $\Phi \mapsto f_\Phi$ is a linear bijection from $\mathcal{L}(S, M_N)$ onto $M_N(S)^*$: explicitly, $\Phi(s)_{ab} = f_\Phi(E_{ab} \otimes s)$.

Lemma 3.8 (Matrix support and positivity).

Fix $1 \leq r \leq N$. A linear map $\Phi : S \rightarrow M_N$ is r -positive if and only if $f_\Phi(X) \geq 0$ for every $X \in M_N(S)^+$ with $\text{rank } \rho_\omega(X) \leq r$. For positive X , this rank is independent of the faithful state: it is the least rank of a projection $p \in M_N$ satisfying $X = (p \otimes 1)X(p \otimes 1)$.

Proof. If $v \in \ker \rho_\omega(X)$, then $v^*Xv \in S^+$ and $\omega(v^*Xv) = 0$, so faithfulness gives $v^*Xv = 0$. In a concrete representation of S , positivity of X implies that X annihilates $v \otimes \mathcal{H}$. Thus the support projection of $\rho_\omega(X)$ supports X ; conversely, any scalar projection supporting X also supports $\rho_\omega(X)$. This proves the assertion about rank.

It follows that the positive X with $\text{rank } \rho_\omega(X) \leq r$ are exactly those of the form $X = L^*YL$, where $Y \in M_r(S)^+$ and $L : \mathbb{C}^N \rightarrow \mathbb{C}^r$. For these matrices,

$$f_\Phi(L^*YL) = \langle (L \otimes I_N)\Omega_N, (\text{id}_{M_r} \otimes \Phi)(Y)(L \otimes I_N)\Omega_N \rangle.$$

Every vector in $\mathbb{C}^r \otimes \mathbb{C}^N$ has the displayed form, so nonnegativity of these expressions is precisely r -positivity. $\square$

Here rank measures support in the scalar matrix factor. In particular, if $S \subseteq \mathbb{C}^d$ and $\omega(f) = d^{-1} \sum_j f(j)$, a positive matrix is a tuple $K = (K_j)$ and

$$\rho_\omega(K) = \frac{1}{d} \sum_j K_j, \quad \text{rank } \rho_\omega(K) = \text{rank} \left(\sum_j K_j \right).$$

The lemma is an elementary form of the duality underlying the super r -maximal operator-system construction; see [25]. For matrix algebras, the cone generated by these small-support positive elements is the cone of positive operators of Schmidt number at most r ; see [10].

Proposition 3.9 (Quantitative separation from a support gap).

Fix $1 \leq r \leq N$. Let $F \in M_N(S)^*$ be Hermitian, and suppose that, for some $\delta > 0$,

$$(15) \quad F(X) \geq \delta \tau_\omega(X) \quad \text{whenever } X \in M_N(S)^+, \quad \text{rank } \rho_\omega(X) \leq r.$$

Suppose $0 \neq H \in M_N(S)^+$ satisfies $F(H) = 0$ and has a decomposition

$$(16) \quad H = A_0 \otimes 1 + \sum_{a=1}^q (A_a \otimes s_a + A_a^* \otimes s_a^*), \quad A_0 = A_0^*, \quad \|s_a\| \leq 1.$$

Set $b_H = 2 \sum_a \|A_a\| > 0$ and $\beta = F(I_N \otimes 1)$. For $0 < \theta < \delta$, define $\Psi : S \rightarrow M_N$ by $f_\Psi = F - \theta \tau_\omega$, and put

$$D_* = \Psi(1), \quad \phi = D_*^{-1/2} \Psi(\cdot) D_*^{-1/2}.$$

Then $D_* \geq (\delta - \theta)I_N$, $\text{Tr } D_* = \beta - \theta N$, and ϕ is unital and r -positive. Every ucp map $\psi : S \rightarrow M_N$ satisfies

$$(17) \quad \max_a \|\phi(s_a) - \psi(s_a)\| \geq \frac{\theta \tau_\omega(H)}{b_H(\beta - \theta N)} \geq \frac{\theta \tau_\omega(H)}{b_H \beta}.$$

In particular, the same lower bound holds for $\|\phi - \psi\|$.

Proof. Lemma 3.8 and (15) show that Ψ is r -positive. Applying the same lemma to $\Psi - (\delta - \theta)\omega(\cdot)I_N$ shows that this map is also r -positive, since its associated functional is $F - \delta \tau_\omega$. Evaluation at 1 gives $D_* \geq (\delta - \theta)I_N > 0$. Also $\text{Tr } D_* = f_\Psi(I_N \otimes 1) = \beta - \theta N$. Congruence by $D_*^{-1/2}$ therefore makes Ψ unital while preserving r -positivity.

For $y = (I_N \otimes D_*^{1/2})\Omega_N$, normalization cancels exactly:

$$(18) \quad \langle y, (\text{id} \otimes \phi)(H)y \rangle = f_\Psi(H) = -\theta \tau_\omega(H), \quad \|y\|^2 = \text{Tr } D_*.$$

Fix a ucp map ψ and write $d = \max_a \|\phi(s_a) - \psi(s_a)\|$. Both maps preserve adjoints, and unitality cancels the constant term in (16). Hence

$$\|(\text{id} \otimes (\phi - \psi))(H)\| \leq b_H d.$$

Since $(\text{id} \otimes \psi)(H) \geq 0$, equation (18) gives

$$\theta \tau_\omega(H) \leq -\langle y, (\text{id} \otimes (\phi - \psi))(H)y \rangle \leq b_H d \text{Tr } D_*.$$

This proves (17); the numerator is positive because ω is faithful and $H \neq 0$. $\square$

Taking $\theta = \delta/2$, the lower bound is at least $\delta \tau_\omega(H)/(2b_H \beta)$. Thus a lower bound for δ and an upper bound for $b_H \beta/\tau_\omega(H)$, both independent of r and N , give uniform separation from ucp maps. If such data exist for arbitrarily large r for a fixed S , then S fails CP-stability: take $r \geq 2k$ and use that unital $2k$ -positive maps are k -contractive. The next application supplies precisely these two bounds.

3.5 Application to the pentagonal system

Return to $S = \mathcal{P}_5$ and $N = 5m$. Choose the faithful state $\omega(f) = \frac{1}{5} \sum_{j=0}^4 f(j)$. For a positive tuple K ,

$$\rho_\omega(K) = \frac{1}{5} S_K, \quad \tau_\omega(K) = \frac{1}{5} \text{Tr } S_K.$$

By homogeneity, Proposition 3.7 says that

$$F(K) = \sum_j \text{Tr}(E_j K_j) \geq \delta_* \text{Tr } S_K = 5\delta_* \tau_\omega(K) \quad \text{if } \text{rank } S_K \leq r.$$

For zero trace the tuple is zero, so this includes all positive tuples with the required support bound. Thus (15) holds with $\delta = 5\delta_*$. This explains the role of the low-rank hypothesis: it is exactly the restriction arising from an r -level positivity test. The twirl turns it into $\text{rank } C \leq 5r$, and the pullback and principal compression give $\text{rank } D \leq 5r$. With $m \geq 10r$, the variance estimate (14) keeps $\sqrt{D}$ away from scalars by a fixed proportion of its mass; malnormality then gives the support gap.

Put $\varepsilon = \delta_*/2 = \kappa^2/2^{49}$ and take $\theta = 5\varepsilon = \delta/2$ in Proposition 3.9. The corresponding map is

$$\Psi(f) = \sum_{j=0}^4 f(j) E_j^\top - \varepsilon \left(\sum_{j=0}^4 f(j) \right) I_N, \quad f \in \mathcal{P}_5,$$

because

$$f_\Psi(K) = F(K) - \varepsilon \text{Tr } S_K = F(K) - 5\varepsilon \tau_\omega(K).$$

In particular,

$$D_* = \Psi(1) = \sum_j E_j^\top - 5\varepsilon I_N \geq 5\varepsilon I_N,$$

and

$$\phi(f) = D_*^{-1/2} \Psi(f) D_*^{-1/2}$$

is unital and r -positive. Since $\text{Tr } E_j = 2m$, $\beta = F(I_N \otimes 1) = 10m$, and

$$\text{Tr } D_* = 10m - 25m\varepsilon \leq 10m.$$

The witness is the positive tuple H of (3). Since $E_j H_j = R^j E Q R^{-j} = 0$, it satisfies $F(H) = 0$, while $\tau_\omega(H) = \text{Tr } Q \geq 5m$. Let Q_{diag} denote the block diagonal part of Q and set

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 & BA^{-1} \\ I & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & A & 0 \end{pmatrix}.$$

The zero pattern of Q gives

$$H = Q_{\text{diag}} \otimes 1 + L \otimes z + L^* \otimes z^*.$$

The matrix L is a block diagonal matrix with blocks BA^{-1}, I, B, I, A times a unitary cyclic permutation matrix, so $\|L\| \leq 3$. Thus (16) holds with the single contraction $s_1 = z$ and $b_H = 2\|L\| \leq 6$. For $y = (I_N \otimes D_*^{1/2})\Omega_N$, the general witness identity becomes

$$\langle y, (\text{id} \otimes \phi)(H)y \rangle = -5\varepsilon \text{Tr } Q, \quad \|y\|^2 = \text{Tr } D_*.$$

Consequently, (17) gives, for every ucp ψ ,

$$\|\phi(z) - \psi(z)\| \geq \frac{5\varepsilon \text{Tr } Q}{6 \text{Tr } D_*} \geq \frac{5\varepsilon}{12} \geq \frac{\varepsilon}{4} = \frac{\kappa^2}{2^{51}}.$$

This proves Theorem 3.1.

Corollary 3.10 (Matrix-target instability).

There is a sequence of unital $2k$ -positive maps $\phi_k : \mathcal{P}_5 \rightarrow M_{n_k}$ with $n_k \rightarrow \infty$ such that

$$\|\phi_k\|_k = 1, \quad \inf_{\psi \in \text{UCP}(\mathcal{P}_5, M_{n_k})} \|\phi_k(z) - \psi(z)\| \geq c_0 := \frac{\kappa^2}{2^{51}}.$$

In particular $\mathcal{P}_5$ fails CP-stability and $\mathcal{P}_5^d$ is not exact.

Proof. Choose malnormal dimensions $m_k \rightarrow \infty$ with $m_k \geq 20k$ and apply Theorem 3.1 with $r = 2k$, initially taking $n_k = 5m_k$. A unital $2k$ -positive map is k -contractive: apply its $2k$ -th amplification to the positive block matrix $\begin{pmatrix} I & x \\ x^* & I \end{pmatrix}$ for $x \in M_k(\mathcal{P}_5)$ of norm at most one. Unitality gives equality of the amplification norm with one. Since $\|z\| = 1$, separation on z also bounds the ordinary map-norm distance from below. This contradicts CP-stability, and Theorem 2.6 gives the remaining qualitative assertions. $\square$

The sequence also gives a direct lifting obstruction, without using the equivalence with CP-stability.

Corollary 3.11 (Nonliftable quotient maps).

Let ϕ_k be the maps above. The map

$$\Phi : \mathcal{P}_5 \longrightarrow \frac{\prod_k M_{n_k}}{\bigoplus_k M_{n_k}}, \quad \Phi(f) = [(\phi_k(f))_k],$$

is ucp and has no ucp lift to $\prod_k M_{n_k}$.

Proof. The maps ϕ_k are contractive. For every fixed matrix level s , their s -th amplifications are positive for all sufficiently large k . Consequently Φ is well-defined and completely positive; it is unital. A ucp lift would have ucp coordinates ψ_k with $\|\phi_k(z) - \psi_k(z)\| \rightarrow 0$, contrary to the uniform separation. $\square$

4 Relative completion and the graph dichotomy

We determine the tensor-equality classes of graph operator systems and their duals. The positive case is the C^* -nuclearity theorem of Dor-On and Russell. For the converse, we extract a fixed-algebra consequence of their obstruction constructions: for every non-2-chordal graph, relative positive completion at all matrix levels characterizes WEP. Commutant transference then gives nuclearity detection by the original graph system.

4.1 Approximation, ultrapowers, and the bidual

Fix a graph G on $[n]$ and a unital C^* -algebra A . A G -partial matrix over A is a family $a = (a_{ij})_{(i,j) \in E}$ in A with $a_{ji} = a_{ij}^*$. A completion of a is a matrix $X \in M_n(A)$ with $X_{ij} = a_{ij}$ for $(i, j) \in E$. Fix a faithful representation $A \subseteq B(K)$. We say a is $B(K)$ -completable if it has a positive completion in $M_n(B(K))$. For $\varepsilon \geq 0$, $a + \varepsilon I$ is the partial matrix with diagonal entries $a_{ii} + \varepsilon 1$.

Definition 4.1.

$\text{RC}(G, A)$ is the statement: for every $B(K)$ -completable G -partial matrix a over A and every $\varepsilon > 0$, $a + \varepsilon I$ has a positive completion in $M_n(A)$.

By Lemma 4.4(i) below, $B(K)$ -completable does not depend on the faithful representation, so neither does $\text{RC}(G, A)$. This is the approximate relative completion problem of [3, Problem 2.6].

Lemma 4.2 (Exact realizations of approximate completion).

Let a be a self-adjoint G -partial matrix over A , and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. The following are equivalent:

- (i) $a + \varepsilon I$ has a positive completion over A for every $\varepsilon > 0$;
- (ii) a has a positive completion over the norm ultrapower $A^{\mathcal{U}} = \ell^\infty(A) / \{(x_k) : \lim_{\mathcal{U}} \|x_k\| = 0\}$;
- (iii) a has a positive completion over A^{**} .

The coefficients of a are embedded diagonally in (ii) and canonically in (iii).

Proof. Choose positive completions X_k of $a + k^{-1}I$. For a positive matrix X , $\|X\| \leq \sum_i \|X_{ii}\|$, so the X_k are uniformly bounded. Their ultrapower class gives (ii), and a weak* cluster point in $M_n(A^{**})$ gives (iii).

Conversely, a positive matrix over $A^{\mathcal{U}}$ has bounded positive representatives $X_k \in M_n(A)$. Its specified entries converge in norm along $\mathcal{U}$ to those of a . Replacing those entries by a_{ij} changes X_k by a self-adjoint matrix of norm tending to zero. Adding εI absorbs the correction, proving (i).

For (iii), Kaplansky density supplies a bounded net of positive matrices over A converging weak* to the given completion in $M_n(A^{**})$. Their specified entries converge weakly, in the finite product of copies of A , to a . The image of the positive cone under restriction to the pattern is convex, so its weak and norm closures agree by Hahn–Banach. Thus the specified entries can be approximated in norm. Correcting them and adding εI proves (i) as before. $\square$

We will also use the elementary fact that finitely many strict linear matrix inequalities over A which have a solution in $A^{\mathcal{U}}$ have a solution in A : choose bounded representatives for the unknowns and retain a coordinate where all inequalities hold with a smaller positive margin. Constants from A are represented by constant sequences.

Proposition 4.3 (WEP and relative completion).

If A has the WEP, then $\text{RC}(G, M_p(A))$ holds for every finite graph G and every $p \geq 1$.

Proof. Represent $A \subseteq B(H)$ faithfully, and let $\Theta : B(H) \rightarrow A^{**}$ be a weak expectation. Apply $\text{id}_{M_p} \otimes \Theta$ to a positive ambient completion. The specified entries are fixed, so this gives an exact completion over $M_p(A^{**}) = M_p(A)^{**}$. Apply Lemma 4.2. $\square$

4.2 The tensor dictionary

Let $\{\delta_{ij}\}_{(i,j) \in E} \subseteq S_G^d$ be the dual basis to $\{E_{ij}\}$, and set $t_a := \sum_{(i,j) \in E} \delta_{ij} \otimes a_{ij} \in S_G^d \otimes A$. For the quotient identification we take $\text{Tr}|_{S_G}$ as the order unit of S_G^d , and Tr as the order unit of M_n^d . These are faithful positive functionals; their normalization does not affect the tensor cones (Remark 2.2). We reserve $\omega = \frac{1}{n} \text{Tr}|_{S_G}$ for the normalized trace state.

Lemma 4.4 (Dictionary).

- (i) $t_a \in (S_G^d \otimes_{\min} A)^+$ iff the map $\Phi_a : S_G \rightarrow A$, $E_{ij} \mapsto a_{ij}$, is completely positive, iff a is $B(K)$ -completable.
- (ii) $t_a \in (S_G^d \otimes_{\max} A)^+$ iff for every $\varepsilon > 0$, $a + \varepsilon I$ has a positive completion in $M_n(A)$.
- (iii) The same holds at matrix level p with A replaced by $M_p(A)$. Consequently

$$A \in \text{Cl}(S_G^d) \iff \text{RC}(G, M_p(A)) \text{ holds for every } p \geq 1.$$

We write $\mathcal{A}_G := \text{Cl}(S_G^d)$.

Proof. (i) For finite-dimensional S and any $\mathcal{T}$, the positive cone of $S^d \otimes_{\min} \mathcal{T}$ consists of the tensors t for which $\Phi_t : S \rightarrow \mathcal{T}$ is completely positive (see [14, 4]). Now Φ_a is cp into A iff it is cp into $B(K)$. By Arveson's extension theorem this happens iff Φ_a extends to a cp map $\Psi : M_n \rightarrow B(K)$, iff (Choi) there is a positive matrix $[\Psi(E_{ij})] \in M_n(B(K))$, which is precisely a positive completion of a .

(ii) Identify M_n^d with M_n via $f \mapsto [f(E_{ij})]_{i,j}$. By Choi's theorem this is a complete order isomorphism carrying Tr to I_n . The restriction map $r : M_n^d \rightarrow S_G^d$ is the dual of the complete order embedding $S_G \subseteq M_n$, hence a complete quotient map, with kernel corresponding to $J_G = \text{span}\{E_{ij} : (i, j) \notin E\} \subseteq M_n$ under this identification [4]. Under the identification, $M_n^d \otimes_{\max} A = M_n(A)$, and $r \otimes \text{id}$ forgets the entries off the pattern of G . By Han's theorem [5], $r \otimes \text{id} : M_n(A) \rightarrow S_G^d \otimes_{\max} A$ is a complete quotient map. By definition of the quotient cone, $t_a \geq 0$ iff for every $\varepsilon > 0$, $t_a + \varepsilon (r(\text{Tr}) \otimes 1)$ has a positive preimage. Since $r(\text{Tr}) \otimes 1$ corresponds to the identity on the diagonal, this preimage is a positive completion of $a + \varepsilon I$.

(iii) By (P2), $M_p(S_G^d \otimes_{\alpha} A) = S_G^d \otimes_{\alpha} M_p(A)$ for $\alpha = \min, \max$, so level p over A is level 1 over $M_p(A)$. Since max-positivity always implies min-positivity, equality of the two operator systems means that every min-positive tensor is max-positive. That is exactly $\text{RC}(G, M_p(A))$ for all p . $\square$

Theorem 4.5 (Dor-On–Russell [3, Thm. 6.4]).

A finite graph G is 2-chordal if and only if S_G is C^* -nuclear. In this case S_G^d is also C^* -nuclear, and $\text{RC}(G, M_p(A))$ holds for every unital C^* -algebra A and every $p \geq 1$.

Proof. The classification is [3, Thm. 6.4]. Finite-dimensional C^* -nuclearity is preserved under duality [13, Thm. 4.1], so the remaining assertions follow from Lemma 4.4. $\square$

4.3 Permanence for a fixed algebra

Lemma 4.6.

Let A be a unital C^* -algebra.

- (a) If H is an induced subgraph of G , then $\text{RC}(G, A) \Rightarrow \text{RC}(H, A)$.
- (b) $\text{RC}(H, M_m(A)) \Rightarrow \text{RC}(H^{(m)}, A)$.
- (c) If $H = G/e$ is obtained by contracting an edge e , then $\text{RC}(G, A) \Rightarrow \text{RC}(H, A)$.

Consequently, if H is an induced minor of a clique blow-up of G , then $\mathcal{A}_G \subseteq \mathcal{A}_H$.

Proof. (a) It suffices to delete one vertex v . Extend an H -partial matrix a by a zero row and column at v to obtain a G -partial matrix $\tilde{a}$; a positive $B(K)$ -completion of a extends by zeros. An A -completion Q of $\tilde{a} + \varepsilon I$ with $Q \geq 0$ compresses to an A -completion of $a + \varepsilon I$ that is still positive.

(b) Order the vertices of $H^{(m)}$ by fibers. An $H^{(m)}$ -partial matrix over A is the same thing as an H -partial matrix over $M_m(A)$:

- diagonal fibers are cliques, so diagonal blocks are fully specified;
- blocks between adjacent fibers are fully specified;
- blocks between non-adjacent fibers are fully unspecified.

Completions, positivity and $B(K)$ -completability correspond.

(c) Let $e = \{0, 1\}$ and let v be the contracted vertex. Given an ambiently completable H -partial matrix a , duplicate its v -row and column on the pattern of G : set

$$\tilde{a}_{00} = \tilde{a}_{01} = \tilde{a}_{10} = \tilde{a}_{11} = a_{vv},$$

and set $\tilde{a}_{ij} = a_{vj}$ for $i \in \{0, 1\}$ whenever $i \sim j$, retaining all other prescribed entries. Duplicating the corresponding row and column of an ambient completion preserves positivity. By $\text{RC}(G, A)$ and Lemma 4.2, there is an exact positive completion Q over $A^{\mathcal{U}}$. Since

$$(e_0 - e_1)^* Q (e_0 - e_1) = 0,$$

positivity gives $Q(e_0 - e_1) = 0$. The two rows and columns coincide. Deleting vertex 1 therefore gives a positive completion of a over $A^{\mathcal{U}}$, including edges inherited only from vertex 1. Another application of Lemma 4.2 gives $\text{RC}(H, A)$.

For the final assertion: if $A \in \mathcal{A}_G$ then $\text{RC}(G, M_q(A))$ holds for all q , hence $\text{RC}(G^{(m)}, M_q(A))$ by (b) applied to $M_{mq}(A)$. Then (a) and (c) give $\text{RC}(H, M_q(A))$ for all q , i.e. $A \in \mathcal{A}_H$. $\square$

4.4 The fixed-algebra consequence of the obstruction arguments

The positive half of the graph dichotomy follows directly from Theorem 4.5. For the converse we use the constructions of Dor-On and Russell. Their proofs of [3, Thms. 5.1 and 5.2] encode a Riesz interpolation obstruction into a relative completion problem over a noninjective von Neumann algebra. We explain why the same constructions apply to a fixed C^* -algebra; the block matrices and the induced-minor models will be used by reference.

Proposition 4.7 (Riesz obstruction).

Let $F \in \{K_{2,3}, 2K_1 + P_4\}$ and let A be a unital C^* -algebra. If $\text{RC}(F, M_q(A))$ holds for every $q \geq 1$, then A has the WEP. Consequently $\mathcal{A}_F \subseteq \text{WEP}$.

Proof. Fix a faithful representation $A \subseteq B(H)$ and a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Suppose that $x'_1, x'_2, y'_1, y'_2, y'_3 \in M_p(A)_{sa}$ and $b \in M_p(B(H))_{sa}$ satisfy $x'_i < b < y'_j$ for every i, j . Choose $\delta > 0$ such that

$$x'_i + \delta 1 \leq b \leq y'_j - \delta 1,$$

and put $x_i = x'_i + \delta 1/2$ and $y_j = y'_j - \delta 1/2$.

Apply the construction in the proof of [3, Thm. 5.1] when $F = K_{2,3}$, and that in the proof of [3, Thm. 5.2] when $F = 2K_1 + P_4$, to these new bounds. In either case it produces an ambiently positively completable F -partial matrix a over $M_{2p}(A)$. The prescribed entries use only the bounds, scalar matrices, algebraic operations, and positive square roots, so they belong to $M_{2p}(A)$; the ambient interpolant b is used to establish ambient completability.

The extraction step in those proofs has the following further property: an exact positive completion over $M_{2p}(B)$, where B is any unital C^* -algebra containing A , gives $h \in M_p(B)_{sa}$ satisfying

$$x_i \leq h \leq y_j \quad (i = 1, 2, j = 1, 2, 3).$$

Indeed, this step uses Schur complements of identity blocks, compression to a corner, and the real part of the missing (v, w) -entry. These operations and inequalities require no von Neumann algebra hypothesis.

By the assumed relative completion property and Lemma 4.2, a has an exact positive completion over $M_{2p}(A^{\mathcal{U}})$. Apply the extraction step with $B = A^{\mathcal{U}}$. The resulting h satisfies

$$x'_i + \frac{\delta}{2} 1 \leq h \leq y'_j - \frac{\delta}{2} 1.$$

A bounded self-adjoint representative of h therefore has a coordinate in $M_p(A)$ satisfying all the original strict inequalities. Thus A has the complete $(2, 3)$ -tight Riesz interpolation property, and has the WEP by Theorem 2.5. The last assertion follows from Lemma 4.4(iii). $\square$

Proposition 4.8 (Induced-minor reduction).

For

$$G \in \{C_k \ (k \geq 5), \overline{C_6}, \overline{P_6}\},$$

the graph $2K_1 + P_4$ is an induced minor of a clique blow-up of G . Consequently $\mathcal{A}_G \subseteq \text{WEP}$.

Proof. For C_5 , $\overline{C_6}$, and $\overline{P_6}$, use the induced-minor models in the proofs of [3, Thms. 6.1 and 6.2]. For $k > 5$, contract consecutive edges of C_k to obtain C_5 . These contractions, followed by the model for C_5 , can also be realized in a clique blow-up of C_k by replacing each contracted vertex with its connected branch set. Now apply Lemma 4.6 and Proposition 4.7. $\square$

Proposition 4.9.

$\mathcal{A}_{K_{2,2,2}} \subseteq \text{WEP}$.

Proof. By [3, Thm. 6.3], the function system

$$W_6 = \{(a, b, c, x, y, z) \in \mathbb{C}^6 : a + b + c = x + y + z\}$$

is completely order isomorphic to a ucp retract of $S_{K_{2,2,2}}$. Let

$$W_5 = \{(a_1, a_2, b_1, b_2, b_3) \in \mathbb{C}^5 : a_1 + a_2 = b_1 + b_2 + b_3\},$$

with its inherited matrix ordering and order unit $(3, 3, 2, 2, 2)$. The maps

$$\begin{aligned}\iota(a_1, a_2, b_1, b_2, b_3) &= (a_1, a_2, 0, b_1, b_2, b_3), \\ \pi(a, b, c, x, y, z) &= (a, b + c, x, y, z)\end{aligned}$$

are restrictions of completely positive maps between diagonal C^* -algebras and satisfy $\pi\iota = \text{id}_{W_5}$. Thus W_5 is a cp retract of W_6 . By duality, W_5^d is a cp retract of $S_{K_{2,2,2}}^d$.

Writing $J = \text{span}\{(1, 1, -1, -1, -1)\} \subseteq \mathbb{C}^5$, we have $W_5 = J^\perp$ and hence $W_5^d \cong \mathbb{C}^5/J$ as matrix-ordered spaces. Kavruk's WEP characterization [12, Thm. 0.1] gives $\text{Cl}(W_5^d) = \text{WEP}$. Lemma 2.1 proves the assertion. $\square$

4.5 The dual dichotomy

Theorem 4.10 (Dual graph dichotomy).

Let G be a finite graph. If G is 2-chordal, then $\text{Cl}(S_G^d)$ contains every unital C^* -algebra. Otherwise $\text{Cl}(S_G^d) = \text{WEP}$.

Proof. The 2-chordal case is Theorem 4.5. For every graph, Proposition 4.3 and the dictionary give $\text{WEP} \subseteq \mathcal{A}_G$.

If G is not 2-chordal, it contains one of McCullough's forbidden graphs F as an induced subgraph. The fixed-algebra permanence lemma gives $\mathcal{A}_G \subseteq \mathcal{A}_F$. Propositions 4.7, 4.8, and 4.9 give $\mathcal{A}_F \subseteq \text{WEP}$ in every case. $\square$

4.6 The commutant transference principle

We now pass from WEP characterization by the dual system to nuclearity detection by the original system. The argument combines the standard commutant-extension technique known as “The Trick” [2, Proposition 3.6.5] with the approximate matrix-factorization argument in the proof of [23, Lemma 2.9]. Kavruk's proofs of Theorems 0.3 and 0.5 in [13] apply this argument to specific test systems; we isolate it as a general principle valid for every finite-dimensional exact operator system.

Proposition 4.11 (Commutant transference).

Let X be a finite-dimensional exact operator system, let A be a unital C^* -algebra, and let $\pi : A \rightarrow B(H)$ be a unital $*$ -representation. If

$$X \otimes_{\min} A = X \otimes_{\max} A,$$

then, writing $N = \pi(A)'$, we have

$$X^d \otimes_{\min} N = X^d \otimes_{\max} N.$$

Both equalities mean that the canonical identity is a complete order isomorphism.

Proof. Fix a concrete realization $X \subseteq B(K)$ and write $\iota : X \rightarrow B(K)$ for the inclusion. By exactness and the extension and perturbation argument in the proof of [23, Lemma 2.9], there are ucp maps

$$\alpha_j : X \longrightarrow M_{n_j}, \quad \gamma_j : M_{n_j} \longrightarrow B(K)$$

such that

$$\|\gamma_j \alpha_j - \iota\| \longrightarrow 0.$$

Indeed, the matricial approximation characterization of exactness supplies ucp bijections from X onto matrix subsystems whose inverses have cb-norm tending to one. Extend these inverses into $B(K)$ and perturb the extensions to ucp maps.

Fix $p \geq 1$, put $\pi_p(a) = I_p \otimes \pi(a)$, and let $\phi : X \rightarrow M_p(N)$ be ucp. Since $\phi(X)$ commutes with $\pi_p(A)$, the product map

$$F : X \otimes_{\max} A \longrightarrow B(H^{\oplus p}), \quad F(x \otimes a) = \phi(x)\pi_p(a),$$

is ucp. By the assumed tensor equality and injectivity of $\min$, we may regard its domain as an operator subsystem of $B(K) \otimes_{\min} A$. Arveson's theorem gives a ucp extension

$$\tilde{F} : B(K) \otimes_{\min} A \longrightarrow B(H^{\oplus p}).$$

Its restriction to $1 \otimes A$ is the $*$ -representation π_p , so $1 \otimes A$ lies in its multiplicative domain. Consequently

$$\Psi(b) := \tilde{F}(b \otimes 1)$$

commutes with $\pi_p(A)$ for every $b \in B(K)$. Thus $\Psi : B(K) \rightarrow M_p(N)$ is a ucp extension of ϕ .

Set $\beta_j = \Psi\gamma_j$. Then $\beta_j : M_{n_j} \rightarrow M_p(N)$ is ucp and

$$\|\beta_j\alpha_j - \phi\| \leq \|\gamma_j\alpha_j - \iota\| \longrightarrow 0.$$

We now translate these factorizations into positive tensors. For a map $\theta : X \rightarrow M_p(N)$, write t_θ for its corresponding tensor in $X^d \otimes M_p(N)$. Complete positivity of β_j gives

$$t_{\beta_j} \in (M_{n_j}^d \otimes_{\min} M_p(N))^+ = (M_{n_j}^d \otimes_{\max} M_p(N))^+.$$

The dual map $\alpha_j^d : M_{n_j}^d \rightarrow X^d$ is completely positive. Functoriality of $\max$ therefore gives

$$t_{\beta_j\alpha_j} = (\alpha_j^d \otimes \text{id})(t_{\beta_j}) \in (X^d \otimes_{\max} M_p(N))^+.$$

Since X is finite-dimensional, norm convergence of the maps implies convergence of their tensors in the maximal tensor norm. Closedness of the positive cone yields

$$t_\phi \in (X^d \otimes_{\max} M_p(N))^+.$$

The same conclusion holds for every cp map $\phi : X \rightarrow M_p(N)$. To see this, choose a state ω on X and, for $\eta > 0$, set

$$\phi_\eta = \phi + \eta\omega(\cdot)1, \quad h_\eta = \phi_\eta(1).$$

The map $h_\eta^{-1/2}\phi_\eta(\cdot)h_\eta^{-1/2}$ is ucp. Apply the preceding argument and conjugate back by $h_\eta^{1/2}$ to obtain maximal positivity of t_{ϕ_η} . Letting $\eta \downarrow 0$ gives maximal positivity of t_ϕ .

Under the canonical correspondence

$$(X^d \otimes_{\min} M_p(N))^+ \cong \text{CP}(X, M_p(N)),$$

we have therefore proved inclusion of the minimal positive cone in the maximal positive cone. The reverse inclusion always holds. Since p was arbitrary, the equality is a complete order equality. $\square$

Corollary 4.12.

Let X be a finite-dimensional operator system. If X^d characterizes the WEP, then X detects nuclearity.

Proof. We note that exactness is automatic: $B(H) \in \text{Cl}(X^d)$ forces X^d to have the LP, so $X = X^{dd}$ is exact by [11, Thm. 6.6]. Suppose $X \otimes_{\min} A = X \otimes_{\max} A$, and let $\pi : A \rightarrow B(H)$ be the universal representation. Proposition 4.11 gives

$$X^d \otimes_{\min} \pi(A)' = X^d \otimes_{\max} \pi(A)'.$$

Since X^d characterizes the WEP, the von Neumann algebra $\pi(A)'$ has the WEP and hence is injective. Injectivity passes to its commutant, so $\pi(A)'' \cong A^{**}$ is injective. Thus A is nuclear [2]. Conversely, every nuclear unital C^* -algebra A satisfies $X \otimes_{\min} A = X \otimes_{\max} A$. $\square$

Remark 4.13. The converse of the corollary fails. Harris's three-dimensional nuclearity detector [9, Thm. 4.2] is not exact, so its dual lacks the LP and cannot characterize WEP. This answers [9, Problem 4.3] negatively.

4.7 Nuclearity detection and the LP

Theorem 4.14 (Full graph dichotomy).

For a finite simple graph G ,

- (i) if G is 2-chordal, both S_G and S_G^d are C^* -nuclear;
- (ii) otherwise, $\text{Cl}(S_G) = \text{Nuc}$ and $\text{Cl}(S_G^d) = \text{WEP}$.

Proof. The first assertion follows from Theorem 4.5. In the second case, Theorem 4.10 says that S_G^d characterizes the WEP. Since $S_G \subseteq M_n$ is exact, Corollary 4.12 shows that S_G detects nuclearity. $\square$

The graph dichotomy gives the lifting and dual-exactness classification through the equivalences in Section 2.

Corollary 4.15 (Lifting property).

For a graph G on n vertices, the following are equivalent:

- (i) G is 2-chordal;
- (ii) S_G is C^* -nuclear;
- (iii) S_G has the LP;
- (iv) S_G is CP-stable;
- (v) $M_n/J_G \cong S_G^d$ is exact.

Proof. The equivalence of (i) and (ii) is the graph classification. C^* -nuclearity implies LP by (K1). Conversely, if G is not 2-chordal, Theorem 4.14 and nonnuclearity of $B(\ell^2)$ give $S_G \otimes_{\min} B(\ell^2) \neq S_G \otimes_{\max} B(\ell^2)$, so S_G fails LP. The remaining equivalences are Theorem 2.6 and (K2). $\square$

5 Cayley transference and polygonal systems

5.1 The transference theorem

Let Γ be a finite group and $C = C^{-1} \subseteq \Gamma \setminus \{e\}$. The Cayley graph $G = \text{Cay}(\Gamma, C)$ has vertex set Γ , with $g \sim h$ iff $g^{-1}h \in C$. We index M_Γ by Γ . Let λ be the left regular representation, τ the canonical trace on $C^*(\Gamma)$ ($\tau(\lambda_s) = \delta_{s,e}$), and

$$S_C(\Gamma) := \text{span}\{\lambda_c : c \in C \cup \{e\}\} \subseteq C^*(\Gamma).$$

Theorem 5.1 (Cayley transference).

- (i) $\mathcal{S}_C(\Gamma)$ is a ucp retract of $\mathcal{S}_G$.
- (ii) $\mathcal{S}_G$ is a ucp retract of $M_\Gamma(\mathcal{S}_C(\Gamma)) = M_\Gamma \otimes_{\min} \mathcal{S}_C(\Gamma)$.

Consequently

$$\text{Cl}(\mathcal{S}_G) = \text{Cl}(\mathcal{S}_C(\Gamma)), \quad \mathcal{A}_G = \text{Cl}(\mathcal{S}_G^d) = \text{Cl}(\mathcal{S}_C(\Gamma)^d),$$

and $\mathcal{S}_G$ has the LP iff $\mathcal{S}_C(\Gamma)$ does.

Proof. (i) For $c \in \Gamma$ put $R_c = \sum_g E_{g,gc} \in M_\Gamma$. Then $R_c R_d = R_{cd}$ and $R_c^* = R_{c^{-1}}$, so $c \mapsto R_c$ is a unitary representation, equivalent to the regular one, and $\theta : C^*(\Gamma) \rightarrow M_\Gamma$, $\lambda_c \mapsto R_c$, is an injective $*$ -homomorphism. Since $g^{-1}(gc) = c$, we have $R_c \in \mathcal{S}_G$ for $c \in C \cup \{e\}$, so θ restricts to a unital complete order embedding $\mathcal{S}_C(\Gamma) \rightarrow \mathcal{S}_G$.

The left translations $L_k = \sum_g E_{kg,g}$ commute with every R_c , and $\{L_k\}' = \theta(C^*(\Gamma))$. Hence $\mathbb{E}(x) = \frac{1}{|\Gamma|} \sum_k L_k x L_k^*$ is a ucp projection onto $\theta(C^*(\Gamma))$. For $g^{-1}h = c$,

$$\mathbb{E}(E_{g,h}) = \frac{1}{|\Gamma|} \sum_k E_{kg,kh} = \frac{1}{|\Gamma|} \sum_m E_{m,mc} = \frac{1}{|\Gamma|} R_c,$$

so $\mathbb{E}(\mathcal{S}_G) \subseteq \theta(\mathcal{S}_C(\Gamma))$, and $\theta^{-1} \circ \mathbb{E}|_{\mathcal{S}_G}$ is a ucp left inverse of θ .

(ii) Let $W = \sum_g E_{g,g} \otimes \lambda_{g^{-1}} \in M_\Gamma \otimes C^*(\Gamma)$, a unitary, and $\varphi(x) = W(x \otimes 1)W^*$. Then $\varphi(E_{g,h}) = E_{g,h} \otimes \lambda_{g^{-1}h}$. So φ is an injective unital $*$ -homomorphism mapping $\mathcal{S}_G$ into $M_\Gamma \otimes \mathcal{S}_C(\Gamma)$, hence a unital complete order embedding $\mathcal{S}_G \rightarrow M_\Gamma \otimes_{\min} \mathcal{S}_C(\Gamma)$.

Let $\psi(y) = (\text{id} \otimes \tau)(W^* y W)$, which is ucp. Then

$$\psi(E_{g,h} \otimes \lambda_s) = \tau(\lambda_g \lambda_s \lambda_{h^{-1}}) E_{g,h} = \delta_{s, g^{-1}h} E_{g,h}.$$

Hence $\psi(M_\Gamma \otimes \mathcal{S}_C(\Gamma)) \subseteq \mathcal{S}_G$ and $\psi\varphi = \text{id}$.

For the consequences, Lemma 2.1 give $\text{Cl}(\mathcal{S}_C(\Gamma)) \supseteq \text{Cl}(\mathcal{S}_G) \supseteq \text{Cl}(M_\Gamma(\mathcal{S}_C(\Gamma))) = \text{Cl}(\mathcal{S}_C(\Gamma))$, and likewise for duals. For the LP assertion, test the equality $\text{Cl}(\mathcal{S}_G) = \text{Cl}(\mathcal{S}_C(\Gamma))$ at $B(\ell^2)$ and apply (K1). $\square$

The amplification and retraction mechanism also appears in the teleportation and super-dense coding constructions of Gao, Harris, and Junge [6, Thm. 2.5 and Cor. 2.6]. Here the maps preserve the specified Cayley supports, giving retractions of the finite-dimensional systems and the corresponding dual comparisons.

Corollary 5.2.

Let Γ be finite and $C = C^{-1} \subseteq \Gamma \setminus \{e\}$. If $\text{Cay}(\Gamma, C)$ is 2-chordal, then $\mathcal{S}_C(\Gamma)$ and $\mathcal{S}_C(\Gamma)^d$ are C^* -nuclear. Otherwise $\mathcal{S}_C(\Gamma)$ detects nuclearity, its dual $\mathcal{S}_C(\Gamma)^d$ characterizes the WEP, and $\mathcal{S}_C(\Gamma)$ fails the LP.

Proof. Combine Theorems 5.1 and 4.14 with Corollary 4.15. $\square$

For abelian Γ , Fourier transform identifies the group system with the span of the corresponding characters in $C(\widehat{\Gamma})$, with its inherited matrix ordering. For cyclic groups this is the usual diagonalization of circulant matrices.

5.2 Polygonal test systems

For $n \geq 3$, let μ_n be the n -th roots of unity and let $z \in C(\mu_n)$ be the coordinate function. Put

$$\mathcal{P}_n = \text{span}\{1, z, \bar{z}\} \subseteq C(\mu_n) \cong \mathbb{C}^n.$$

Under Fourier transform this is $\mathcal{S}_{\{\pm 1\}}(\mathbb{Z}_n)$, and the corresponding Cayley graph is C_n . Its state space is the regular n -gon. Thus $\mathcal{P}_n$ has the minimal operator system structure on the affine function space of that polygon, while $\mathcal{P}_n^d$ has the maximal operator system structure on the cone over the polygon.

Theorem 5.3.

For every $n \geq 5$,

$$\text{Cl}(\mathcal{P}_n) = \text{Nuc}, \quad \text{Cl}(\mathcal{P}_n^d) = \text{WEP}.$$

Moreover, $\mathcal{P}_n$ is exact and fails the LP, whereas $\mathcal{P}_n^d$ has the LP and is not exact. For $n = 3, 4$, both $\mathcal{P}_n$ and $\mathcal{P}_n^d$ are C^* -nuclear. In particular, $\mathcal{P}_5$ is a three-dimensional counterexample to the operator-system formulation of the Smith–Ward problem, with C^* -envelope $\mathbb{C}^5$.

Proof. Theorem 5.1 gives

$$\text{Cl}(\mathcal{P}_n) = \text{Cl}(\mathcal{S}_{C_n}), \quad \text{Cl}(\mathcal{P}_n^d) = \text{Cl}(\mathcal{S}_{C_n}^d).$$

For $n \geq 5$, the cycle C_n is a forbidden induced subgraph for 2-chordality. The asserted tensor classes follow from Theorem 4.14. As $\mathcal{P}_n \subseteq \mathbb{C}^n \subseteq M_n$, it is exact, and hence $\mathcal{P}_n^d$ has the LP. The failure of LP for $\mathcal{P}_n$ follows either from Cayley transference and Corollary 4.15, or by testing its nuclearity-detection property at the nonnuclear algebra $B(\ell^2)$. Duality then shows that $\mathcal{P}_n^d$ is not exact.

Finally, C_3 and C_4 are 2-chordal. Apply the same transference equalities and the positive half of Theorem 4.14. Each vertex of the pentagon is exposed by a real affine function, so every point of μ_5 belongs to the Shilov boundary of $\mathcal{P}_5$. Hence its C^* -envelope is $\mathbb{C}^5$. The Smith–Ward conclusion is [11, Thm. 11.5]. $\square$

By Theorem 2.6, every $\mathcal{P}_n$ with $n \geq 5$ fails CP-stability. For the pentagon, Section 3 gives the explicit matrix-target obstruction and its nonliftable quotient maps.

Remark 5.4. The Cayley graph of S_3 with the three transpositions is $K_{3,3}$. Thus the four-dimensional group system $\text{span}\{1, (12), (13), (23)\} \subseteq C^*(S_3)$ detects nuclearity, fails LP, and has a dual characterizing WEP. This illustrates that the transference theorem applies beyond commutative group algebras.

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