

DEFORMATIONS OF DELIGNE'S $\mathcal{S}_t$

CLAUDE OPUS 5.5 WITH AN INTRODUCTION BY NOAH SNYDER

ABSTRACT. Snyder and Spencer showed that every *ribbon* infinitesimal deformation of the trivalent category $\mathcal{S}_t$ underlying Deligne's $\text{Rep}(\mathcal{S}_t)$ comes from varying t . Here we remove the braided hypothesis: every pivotal deformation of $\mathcal{S}_t$, $t \neq 0$, over a local Artinian $\mathbb{C}$ -algebra comes from varying t . The pivotal hypothesis turns out to be automatic, so the same holds for monoidal deformations. Since $\mathcal{S}_t$ is not generated as a pivotal category by its trivalent vertex, the crossing must be added as a generator, and the Reidemeister moves only survive in a deformed form, with lower order terms in the 2- through 6-box spaces. An open condition on rotational eigenvalues lets us lift the crossing in a rotation invariant way. A handful of overlap computations, in the spirit of Liu's classification of Yang–Baxter relation planar algebras, then kill every lower order term up to a change of the chosen crossing.

1. INTRODUCTION

1.1. Noah Snyder's introduction. The goal of the recent paper [SS] was to explain how to classify infinitesimal deformations of categories with skein theoretic descriptions. In particular, we proved the following result through a short and easy calculation:

Theorem 1.1. *Any ribbon infinitesimal deformation of the category of permutation modules for Deligne's $\mathcal{S}_t$ (for $t \neq 0$) comes from varying t .*

The more complicated the skein theory, the more tedious the calculations become, but it should be *routine* to classify deformations when you have a good skein theory. The goal of this paper is to illustrate that this process is routine in the sense that AI can do it easily and quickly in more challenging examples. In particular, we prove the following strengthening of the above result which drops the ribbon assumption. This uses a skein theory that's essentially the trivalent version of Zhengwei Liu's Yang–Baxter relation planar algebras [Liu]. This calculation was done by Claude Opus 5.5 in less than an hour.

Theorem 1.2. *Any infinitesimal monoidal deformation of the category of permutation modules for Deligne's $\mathcal{S}_t$ (for $t \neq 0$) comes from varying t .*

Specifically, we show that any such deformation must be pivotal and then classify pivotal deformations using skein theory.

This Theorem answers Question 2.5 from the IPAM “Open Problems from the IPAM Workshop on Symmetric Tensor Categories and Representation theory” list, asked by Pavel Etingof.

We encourage the reader who is interested in the *ideas* behind this result to instead read [SS]; this paper is mostly of interest as a bare fact, namely that the ideas in [SS] combined with [Liu] allow you to classify all deformations of $\mathcal{S}_t$ (for $t \neq 0$) *without introducing new ideas* but instead doing a moderately long calculation.

1.2. Prompts. This note was written by Claude in response to the following prompts from Noah Snyder, reproduced verbatim.

The first prompt was accompanied by the arXiv versions of [SS] and [Liu]:

I recently wrote the attached paper which classifies *braided* deformations of Deligne’s S_t . I think a similar idea should work without the braided assumption, but you’d need additional assumptions in the recognition theorem going up to the 6-box space in order to get deformed versions of the Reidemeister moves for trivalent graphs (the usual R1, R2, R3, but also the trivalent twist (R1.5) and moving the trivalent vertex through a crossing (R2.5)). These deformed relations will have some kind of lower order terms, but I think it should still be manageable along the lines of Liu’s Yang-Baxter Relation classification (modified for the trivalent setting). Note that one of the key ideas of my paper is that adding rotational eigenvalue constraints (or similar “open” conditions) can let you get the deformation result without going through the harder combinatorics of the full classification. Can you prove that S_t has no pivotal monoidal deformations following this outline?

The second prompt asked for a different exposition:

Could you rewrite this more in the style of my paper? Most importantly, prefer to use string diagrams to algebraic formulas throughout. Make sure that these string diagrams are honest string diagrams, and not a mix of algebraic and string notation (for example, actually concatenate two pictures, rather than writing a composition symbol between two pictures). Take care to make the pictures readable and planar without accidental crossings. Secondly, include any helpful exposition, think of this as aimed at the audience of an advanced graduate student.

The third prompt asked for a referee report:

Please do an adversarial refereeing pass through the paper, producing a referee report. Do not make the edits yet, just write a referee report. Pay special attention to anywhere an error could have crept in and whether there’s highly relevant work that is not cited. (For example, Sikora-Westbury, and Deligne’s S_t paper jump out immediately.)

There followed a longer discussion between us about some details of the referee report, after which the note was revised in response to it. Those prompts are not reproduced here. A final prompt led to Proposition 3.3:

Great, one last question. I’m curious whether the pivotal assumption is essential or just a convenience? For example, since pivotality on each box space is just a rotational eigenvector, a weakened version of pivotality may just be an open condition?

1.3. Claude’s introduction. Let $\mathcal{S}_t$ be the trivalent category which interpolates the irreducible $(n-1)$ -dimensional representations of S_n , so that for $t \neq 0$ its additive and idempotent completion is Deligne’s $\text{Rep}(\mathcal{S}_t)$ [Del], [SS, §4]. Here “trivalent category” is in the sense of [MPS17]; see also [Kup96, MST] for the skein-theoretic point of view. For more on $\text{Rep}(\mathcal{S}_t)$ see [CO, Kno, Har, HS]; the endomorphism algebras $\text{End}(X^{\otimes r})$ of its generating object are the quasi-partition algebras [DO]. In [SS, Cor. 5.3] it is shown that any infinitesimal deformation of $\mathcal{S}_t$ as a ribbon category comes from varying t . The key input there is a Kuperberg-style recognition theorem [SS, Thm. 6] whose hypotheses are “open” conditions on box spaces of size at most 4. These conditions are automatically inherited by deformations, by Nakayama’s lemma.

In this note we prove the analogous statement for *pivotal* deformations, where the braiding is not part of the data.

Theorem 1.3. *Let $t \in \mathbb{C}^\times$ and let $\mathcal{C}$ be a pivotal deformation of $\mathcal{S}_t$ over a local Artinian $\mathbb{C}$ -algebra A (Definition 3.1). Then there is a unique $t' \in A$ lifting t such that $\mathcal{C}$ is equivalent, as a deformation, to $\mathcal{S}_{t'}(A)$.*

Corollary 1.4. *For $t \neq 0$, every infinitesimal, every order n and every formal pivotal deformation of $\mathcal{S}_t$, and hence of $\text{Rep}(\mathcal{S}_t)$, comes from varying t . The same holds for monoidal deformations (Proposition 3.3).*

The proof never uses semisimplicity, so it also applies when $t = n \in \mathbb{Z}_{>0}$. In that case the theorem is about Deligne’s (non-semisimple, Karoubian) category at $t = n$, equivalently $\mathcal{S}_n$. It is not about its semisimple quotient, the category of representations of the group S_n , nor about its abelian envelope. The former is a fusion category, and so has no deformations at all by Ocneanu rigidity [ENO, GHS].

Etingof and Snowden treat braided deformations of more general symmetric tensor categories [ES]. Neither their work nor [SS, Cor. 5.3] implies the pivotal statement, since a deformation of the associativity constraint need not be compatible with any braiding. (For instance, for G a simple Lie group, nonzero elements of $(\Lambda^3 \mathfrak{g})^G$ give first-order deformations of the associator of $\text{Rep}(G)$ that admit no compatible braiding; cf. [Dri90].)

Relation to Davydov–Yetter theory. Deformations of tensor categories are governed by Davydov–Yetter cohomology [Dav, CY, Yet]. Roughly speaking, Theorem 1.3 says that the third Davydov–Yetter cohomology of $\text{Rep}(\mathcal{S}_t)$ is spanned by the class of the t -direction, up to some bookkeeping concerning, for $t \in \mathbb{Z}_{>0}$, possible deformations of the non-semisimple algebras $\text{End}(X^{\otimes r})$. Davydov–Elbehiry compute this cohomology for free symmetric categories using a Schur–Weyl complex [DE], and Gainutdinov–Haferkamp–Schweigert and Faitg–Gainutdinov–Schweigert describe it via the Drinfeld centre, recovering Ocneanu rigidity and relating second cohomology to infinitesimal braidings [GHS, FGS]. It would be interesting to find an alternative proof along these lines, but this seems to require significant new ideas. Those methods are set up for finite tensor categories, whereas $\text{Rep}(\mathcal{S}_t)$ has infinitely many simple objects (and is not abelian for $t \in \mathbb{Z}_{>0}$); and unlike in the fusion case the relevant cohomology does not vanish, so one would need an actual computation in the Drinfeld centre rather than a vanishing argument. The skein-theoretic argument below avoids all of this.

We work throughout with *pivotal* deformations, but this is only a convenience: every monoidal deformation of $\mathcal{S}_t$ is automatically rigid and has a unique pivotal structure (Proposition 3.3). So Theorem 1.3 also classifies monoidal deformations. The key point is that the trivalent vertex rules out nontrivial monoidal automorphisms of the identity functor, and the remaining check is the same 4-box computation used in §4.

Why this is harder than the ribbon case. As a pivotal category, $\mathcal{S}_t$ is *not* generated by its trivalent vertex. Cutting edges of a planar trivalent graph never produces a crossing partition, so planar trivalent graphs only span the non-crossing part of each box space. For example, the planar trivalent graphs with four boundary points span a 3-dimensional subspace of the 4-dimensional space $\mathcal{S}_t^4$ (there are three non-crossing partitions of four points without singletons), and the crossing $\times$ is not in it. Compare Jones’s observation that the planar partition algebra is a Temperley–Lieb algebra [Jon], and the planar partition categories underlying the quantum permutation groups S_n^+ [Wan, Ban].

So in a pivotal deformation the crossing is an independent generator. It has to be lifted by hand, and there is no reason for the lift to satisfy the Reidemeister moves. What one gets instead are *deformed* Reidemeister moves: each of $R1$, $R1.5$, $R2$, $R2.5$ and $R3$ holds up to an error term in the 2-, 3-, 4-, 5- or 6-box space, respectively. Since $R3$ lives in the 6-box space, the argument has to go up to 6-boxes.

Strategy. The proof has the following parts.

- (1) *Lifting the crossing.* Rotation by one click acts on the 4-box space with eigenvalues ± 1 of multiplicities 3 and 1. This is an open condition, so it persists in any deformation, and lets us choose a rotation invariant lift c of the crossing. The same eigenspace decomposition gives the deformed $I = H$ relation, exactly as in [SS, Thm. 6].
- (2) *Linearization.* We pass from A to $A/\mathfrak{m}^{i+1}$ one step at a time. At each step every error term lies in an ideal J with $\mathfrak{m}J = 0$, and all equations become linear.
- (3) *Gauge freedom.* The lift c can be changed by any rotation invariant 4-box with coefficients in J . This changes the error terms in a computable way, and the goal is to show that a suitable change makes them all vanish.
- (4) *Overlaps.* As in Kuperberg’s and Liu’s work [Kup94, Liu], in Bergman’s diamond lemma [Ber78], or in Sikora–Westbury’s confluence theory for graphs [SW], we take a diagram containing two overlapping relations and simplify it in two different orders. The two answers must agree, which gives linear equations between the error terms.

We organize the overlaps by the size of the box space where they live.

- §4 **4-boxes.** Two easy overlaps show that, up to a change of the lift c , the errors of $R1$, $R1.5$ and $R2$ are controlled by a single number. This number is *not* determined by 4-box considerations alone. In the Brauer category, where the crossing is the only generator, the analogous number is realized by the BMW deformation, to first order in $(q - q^{-1})^2$ (Remark 7.2).
- §5 **5-boxes.** Stacking a crossing under the deformed $R2.5$ relation and evaluating in two ways gives an identity that forces this number to be 0. A cap then kills the $R2.5$ error. Intuitively, the trivalent vertex of $S_t \subset O(t - 1)$ obstructs the BMW deformation of the crossing.
- §6 **6-boxes.** The error in $R3$ is a (-1) -eigenvector for rotation. This second open condition cuts it down from the 41-dimensional space S_t^6 to a 5-dimensional space, where two more overlaps kill it.

Every finite computation below is a statement about partitions of at most six points and can be checked by hand. The computations were also checked by computer, together with an independent computation of a large family of first-order consistency conditions (§7.1).

Conventions for pictures. Strands are unoriented and stand for the generating object X .

- Morphisms are read from bottom to top. Composition is stacking, and tensor product is placing side by side.
- The only crossings that appear are the crossing generator $\times$. Any other point where strands meet is a vertex: $\curvearrowright$ is the trivalent vertex, and the hollow nodes of §2.3 stand for partition basis elements.
- Rectangles labelled with a letter are arbitrary morphisms (“coupons”).

We write $\mathcal{C}^k$ for the space of morphisms with k boundary points, the k -box space. By the pivotal structure it does not depend on how the points are split between top and bottom.

2. S_t AS A PIVOTAL CATEGORY WITH A CROSSING

2.1. Rotation. For a coupon with m strands at the bottom and n at the top, the one-click rotation is

$$\rho\left(\begin{array}{c} \text{---} \\ \boxed{f} \\ \text{---} \end{array}\right) = \begin{array}{c} \text{---} \\ \boxed{f} \\ \text{---} \end{array} \quad (\text{drawn for } m = n = 2).$$

The leftmost top strand is bent down to become the leftmost bottom strand, and the rightmost bottom strand is bent up to become the rightmost top strand. List the boundary points counterclockwise, starting at the bottom left, as

$$b_1, \dots, b_m, t_n, \dots, t_1.$$

Then ρ moves every point one step counterclockwise. Since X is symmetrically self-dual, $\rho^{m+n} = 1$. We will freely rotate a k -box and then read it as a morphism with any convenient split of the k points between bottom and top. A morphism is *rotation invariant* if ρ fixes it.

2.2. Generators and relations. Throughout, R is a commutative $\mathbb{C}$ -algebra and $t \in R^\times$.

Definition 2.1. $\mathcal{S}_t(R)$ is the R -linear pivotal category generated by a symmetrically self-dual object X and two morphisms, a trivalent vertex and a crossing, which are rotation invariant:

They are subject to the planar relations

$$(1) \quad \bigcirc = t - 1, \quad \bigcirc = 0, \quad \bigcirc = \frac{t-2}{t} |, \quad \bigcirc - \bigcirc = \frac{1}{t} (| - |),$$

and the Reidemeister type relations

$$(2) \quad \begin{array}{lll} R1 : \text{crossing} = | & R1.5 : \text{vertex} = \text{vertex} & R2 : \text{crossing} = | \\ R2.5 : \text{vertex} = \text{vertex} & R3 : \text{vertex} = \text{vertex} \end{array}$$

We call the left and right sides of $R2.5$ Λ and Λ' , and the left side of $R3$ T . The right side of $R3$ is the one-click rotation $\rho(T)$. We also use the names

$$e = \text{cup}, \quad I = \text{cap}, \quad H = \text{vertex}.$$

Remark 2.2 (Normalization). In [SS, Def. 5.1] the trivalent vertex is normalized so that the bigon is $t - 2$ and $I - H = \text{id} - e$. Our vertex is $t^{-1/2}$ times theirs, i.e. we work with $\mathcal{S}(R, t, 1)$ in the notation of [SS, §4]. Whenever t has a square root in R , the two are equivalent by [SS, Lemma 2.7], and the rescaling makes no difference to the deformation problem. This is the case for all rings used below: $\mathbb{C}$, and local Artinian $\mathbb{C}$ -algebras, in which every unit has a square root by the binomial series. We use this normalization because it matches Deligne's category with the Harman–Snowden basis on the nose (§2.3).

Because the generators are rotation invariant, every relation holds in all of its rotated forms. For example, rotations of $R1$ and $R1.5$ give

$$(3) \quad \bigcirc = \cap, \quad \bigcirc = \cup, \quad \bigcirc = \cup.$$

Since the crossing satisfies $R2$ and is its own rotation, it behaves like a *symmetric* braiding. This is the content of the next lemma.

Lemma 2.3. *Let $\mathcal{C}$ be an R -linear pivotal category with a symmetrically self-dual object X and rotation invariant morphisms $\smile$ and $\bowtie$ satisfying (1) and (2). Then there is a pivotal functor from the ribbon category $\mathcal{S}_t(R)$ of [SS, §4] to $\mathcal{C}$ sending the vertex and the crossing to the given ones.*

Proof. Define a braiding on tensor powers of X by grids of crossings. The hexagon axioms hold by construction, and $R2$ says the braiding is symmetric. Naturality needs to be checked against the generators.

- *Vertex passing through a strand.* There are four versions, according to whether the vertex points up or down and the strand is on the left or the right:

Each is obtained from $R2.5$ by rotating both sides by the same number of clicks.

- *Cup or cap passing through a strand.* The identities

are rotations of $R2$.

- *Crossing passing through a strand.* This is $R3$.

The curl defines the twist, which is trivial by $R1$. So $\mathcal{C}$ is a symmetric ribbon category in which the defining relations of $\mathcal{S}(R, t, 1)$ hold, and the universal property of that ribbon category gives the functor. $\square$

Corollary 2.4. *The pivotal category $\mathcal{S}_t(R)$ of Definition 2.1 and the ribbon category $\mathcal{S}(R, t, 1)$ of [SS, §4] are equivalent, by pivotal functors which are the identity on the vertex and the crossing.*

Proof. Lemma 2.3, applied to $\mathcal{C} = \mathcal{S}_t(R)$, gives a pivotal functor from the ribbon category to the pivotal one. Conversely, in the ribbon category the vertex and the braiding of X with itself are rotation invariant and satisfy (1) and (2), since the braiding is symmetric with trivial twist and is natural. So the presentation of Definition 2.1 gives a pivotal functor in the other direction. Both composites are the identity on the vertex and the crossing, and both categories are generated as pivotal categories by these two morphisms, so both composites are the identity. $\square$

For the partition category itself there is a well-known presentation of this kind, with the symmetric braiding among the generators: it is the free symmetric monoidal category on a special commutative Frobenius algebra of dimension t [Com, Thm. 2.1], a consequence of the description of 2-dimensional cobordisms by Abrams and Kock.

2.3. The partition basis. To compute in $\mathcal{S}_t$ we use the model from [SS, §4]. Let $\text{Cob} = \text{Cob}(\mathbb{C}, t, 1)$ be Deligne's partition category [Del] (see also [KS] for the cobordism point of view, and [Mar, HR] for the partition algebras), with the Harman–Snowden basis [HS]:

- a basis morphism B_π is a partition π of the boundary points;
- composition merges blocks, and every block which closes up entirely contributes a factor of t ;
- the univalent vertex is a singleton block.

The object X is the image of the idempotent

$$P = \left| - \frac{1}{t} \begin{array}{c} \bullet \\ \vdots \\ \bullet \end{array} \right|.$$

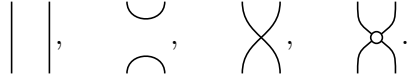
Definition 2.5. For a partition π of the k boundary points with no singletons, let $[\pi] = \text{Res}(B_\pi) \in \mathcal{S}_t^k$ be B_π with P placed on every boundary strand. We draw $[\pi]$ as follows:

- a block of size 2 is a strand;
- a block of size 3 is a trivalent vertex;
- a block of size $k \geq 4$ is a hollow node with k legs.

Strands of different blocks may cross; such a crossing is the crossing generator. Every strand runs from a node to the boundary, or from boundary to boundary.

In the model, the vertex is $[\{1, 2, 3\}]$ and the crossing is $P \otimes P$ composed with the symmetric braiding of Cob ; more generally a hollow node with k legs is $\text{Res}(B_{\{1, \dots, k\}})$. A drawing of $[\pi]$ as above has no strand joining two nodes, so by the cutting rule (Lemma 2.6(3) below) it equals $[\pi]$, however the strands are routed and in whatever order the legs leave each node.

For instance, the four basis elements of $\mathcal{S}_t^4$ are



The identity, e and the crossing are basis elements, as is the trivalent vertex in $\mathcal{S}_t^3$. The hollow node is a new rotation invariant 4-box, which we denote ν_4 .

Lemma 2.6. (1) [SS, Lemma 4.15] *The $[\pi]$ form a basis of $\mathcal{S}_t^k$.*

- (2) *Stacking a crossing on a basis element swaps the labels of the two boundary points involved. Rotation by one click moves every label one step counterclockwise, as in §2.1.*
- (3) (Cutting rule.) *Let D be a diagram built out of black and hollow nodes, crossings, cups and caps, and resolve each crossing into two strands passing through each other. Call a strand internal if neither end is a boundary point. Then*

$$D = \sum_S \left(-\frac{1}{t} \right)^{|S|} t^{c(S)} [\pi_S],$$

where:

- S runs over sets of internal strands;
- π_S records which boundary points are connected after the strands in S are cut;
- $c(S)$ is the number of resulting components that do not reach the boundary;
- $[\pi_S] = 0$ whenever π_S has a singleton.

Proof. (1) is [SS, Lemma 4.15]. (2) holds already for the B_π , and $P \otimes P$ commutes with the crossing.

(3) Regard D as a graph whose vertices are its nodes and boundary points; an edge (“strand”) is a maximal arc of D avoiding the nodes, so it may pass through crossings, cups and caps, and it may be a closed loop or a loop at a single node. In Cob , the diagram D is a composite of block

morphisms, braidings, cups and caps, with a copy of P on each segment. Since P is idempotent, is invariant under rotation, and $P \otimes P$ commutes with the braiding, all the copies of P on one strand can be merged into a single one. On strands with an end on the boundary, this copy is absorbed into the P 's defining $[\pi_S]$. On each internal strand, expand

$$P = \left| - \frac{1}{t} \right| \bullet$$

as “keep the strand” minus t^{-1} times “cut it, leaving two univalent vertices”. For a set S of cut strands, the resulting morphism of Cob is a composite of block morphisms, which equals $t^{c(S)} B_{\pi_S}$: each component not reaching the boundary closes up to a factor of t . Finally, if π_S has a singleton then $\text{Res}(B_{\pi_S}) = 0$, because P composed with a univalent vertex is 0. $\square$

Example 2.7. In I the middle edge is the only internal strand, and in H the rung is. Cutting it gives

$$\begin{array}{c} \text{Y-shape} \\ \text{Y-shape} \end{array} = \begin{array}{c} \text{X-shape} \\ \text{X-shape} \end{array} - \frac{1}{t} \begin{array}{c} \text{cup} \\ \text{cap} \end{array}, \quad \begin{array}{c} \text{Rung} \\ \text{Rung} \end{array} = \begin{array}{c} \text{X-shape} \\ \text{X-shape} \end{array} - \frac{1}{t} \left| \begin{array}{c} \text{strand} \\ \text{strand} \end{array} \right| \left| \begin{array}{c} \text{strand} \\ \text{strand} \end{array} \right|.$$

Subtracting gives back the I – H relation. Similarly, a vertex-free loop is $t - \frac{1}{t} \cdot t = t - 1$.

3. DEFORMATIONS, ERROR TERMS AND GAUGE

3.1. Deformations. Throughout, A is a local Artinian $\mathbb{C}$ -algebra with maximal ideal $\mathfrak{m}$ and residue field $A/\mathfrak{m} = \mathbb{C}$. Typical examples are $\mathbb{C}[\varepsilon]/\varepsilon^2$ and $\mathbb{C}[h]/h^{n+1}$.

Definition 3.1. A *pivotal deformation of $\mathcal{S}_t$ over A* consists of:

- an A -linear pivotal category $\mathcal{C}$ whose objects are the tensor powers of an object X and whose Hom spaces are finitely generated free A -modules;
- a symmetric self-duality of X and a map $\smile$ in $\mathcal{C}$;
- a pivotal equivalence $\mathcal{C} \otimes_A \mathbb{C} \simeq \mathcal{S}_t$ matching X , its self-duality and the vertex.

Two deformations are *equivalent* if there is a pivotal equivalence between them matching X , and matching the self-duality and vertex up to units of A that are $\equiv 1 \pmod{\mathfrak{m}}$, as in [SS, Def. 2.8].

Note that the crossing is *not* part of the data; finding a good lift of it is the whole problem. Deforming the additive idempotent completion $\text{Rep}(\mathcal{S}_t)$ instead gives the same notion: a deformation of $\mathcal{S}_t$ induces one of its completion, because idempotents lift along the nilpotent ideal $\mathfrak{m}$, uniquely up to conjugacy (see e.g. [Lam, §21]); conversely a deformation of $\text{Rep}(\mathcal{S}_t)$ restricts to the full subcategory on the tensor powers of a lift of X .

The basic tool is Nakayama’s lemma in the form of [SS, Lemma 2.2]: a map of finitely generated free A -modules which is an isomorphism modulo $\mathfrak{m}$ is an isomorphism. In particular, any lifts to $\mathcal{C}^k$ of a basis of $\mathcal{S}_t^k$ form a basis of $\mathcal{C}^k$.

The “open” conditions of the introduction survive deformation for the same reason. Suppose an operator θ on $\mathcal{C}^k$ satisfies $\theta^N = 1$. Since $N \in A^\times$, the projectors onto its eigenspaces are polynomials in θ . So the eigenspaces are free direct summands, of the same ranks as modulo $\mathfrak{m}$.

3.2. Monoidal deformations are pivotal. The pivotal hypothesis in Definition 3.1 is only a convenience.

Definition 3.2. A *monoidal deformation of $\mathcal{S}_t$ over A* is an A -linear monoidal category $\mathcal{C}$ whose objects are the tensor powers of an object X and whose Hom spaces are finitely generated free A -modules, together with a monoidal equivalence $\mathcal{C} \otimes_A \mathbb{C} \simeq \mathcal{S}_t$ matching X . Two monoidal deformations are *equivalent* if there is a monoidal equivalence between them matching X .

Replacing $\mathcal{C}$ by its strictification on words in X , we may assume that $\mathcal{C}$ is strict monoidal with objects $X^{\otimes n}$, $n \geq 0$, and that the equivalence $\mathcal{C} \otimes_A \mathbb{C} \simeq \mathcal{S}_t$ is strict and the identity on objects. (Conjugate the equivalence by its structure isomorphisms.) We assume this throughout the proof below.

Proposition 3.3. *Let $\mathcal{C}$ be a monoidal deformation of $\mathcal{S}_t$ over A . Then:*

- (1) $\mathcal{C}$ is rigid, and X is self-dual;
- (2) $\mathcal{C}$ has a unique pivotal structure, and with it X is symmetrically self-dual;
- (3) any monoidal equivalence between two such categories is pivotal.

In particular, together with any lift of the vertex, $\mathcal{C}$ is a pivotal deformation of $\mathcal{S}_t$ in the sense of Definition 3.1, and Theorem 1.3 also holds for monoidal deformations, up to equivalence in the sense of Definition 3.2.

Proof. (1) Lift the cap and cup of $\mathcal{S}_t$ to maps $\cap$ and $\cup$ in $\mathcal{C}$. By Nakayama, $\text{End}(X)$ is free of rank one, so the two zigzags are scalars:

$$\begin{array}{c} \cap \\ \cup \end{array} = u_1 \mid, \quad \begin{array}{c} \cup \\ \cap \end{array} = u_2 \mid, \quad u_1 \equiv u_2 \equiv 1.$$

By the interchange law the two caps below may be slid past each other, so

$$u_1 \cap = \begin{array}{c} \cap \\ \cap \end{array} = \begin{array}{c} \cap \\ \cup \end{array} = u_2 \cap,$$

and $u_1 = u_2$ because the cap is a basis vector. Dividing the cup by u_1 makes both zigzags trivial, so X is its own dual, and $X^{\otimes n}$ is dual to itself via nested cups and caps. In particular $\mathcal{C}$ is rigid.

(2) Using these duality maps, the double dual $F = (-)^{\vee\vee}$ is a strict monoidal functor which is the identity on objects. On a k -box it is the full turn $\rho^{\pm k}$ (the sign depends on conventions), and $F \equiv \text{id} \pmod{\mathfrak{m}}$ because $\mathcal{S}_t$ is pivotal with rotation invariant cup and cap. We show that F is equal to the identity functor.

The dual of the cup is the cap with a zigzag attached, which is the cap; similarly the dual of the cap is the cup. So F fixes the cup and the cap exactly. Suppose F acts as the identity modulo $\mathfrak{m}^i$, for some $i \geq 1$, and let $J = \mathfrak{m}^i / \mathfrak{m}^{i+1}$. Modulo $\mathfrak{m}^{i+1}$ we may write $F = \text{id} + \varepsilon D$, where D is a derivation of $\mathcal{S}_t$ with values in $J \otimes \mathcal{S}_t$, compatible with composition and tensor product. Since $\mathcal{S}_t$ is generated as a monoidal category by the cap, the cup, the vertex and the crossing, D is determined by its values on these. It kills the cup and the cap, and

$$D\left(\begin{array}{c} \cap \\ \cap \end{array}\right) = \gamma \begin{array}{c} \cap \\ \cap \end{array}, \quad D\left(\begin{array}{c} \times \end{array}\right) = w$$

for a scalar γ (that Hom space has rank one) and some 4-box w .

- The upside-down vertex is the one-click rotation of the vertex, built from the vertex and a cup, so D multiplies it by γ too. Hence D multiplies I and H by 2γ and kills id and e , so applying D to the I – H relation gives $2\gamma(I - H) = 0$, hence $\gamma = 0$.
- Applying D to the rotation invariance of the crossing shows that w is rotation invariant. Applying it to $R2$ gives

$$\begin{array}{c} w \\ \cap \\ \cup \end{array} + \begin{array}{c} \cup \\ \cap \\ w \end{array} = 0.$$

The map $w \mapsto wc + cw$ is injective on rotation invariant 4-boxes, by the proof of Lemma 4.3(2) below (which uses only Lemma 2.6(2)). So $w = 0$.

So $D = 0$, and F acts as the identity modulo $\mathfrak{m}^{i+1}$. Since $\mathfrak{m}^N = 0$, induction gives $F = \text{id}$. Hence the identity is a pivotal structure on $\mathcal{C}$. The self-duality is symmetric: rotating the cup by one click gives the cup with a zigzag attached, which is the cup.

For uniqueness, pivotal structures form a torsor over the monoidal automorphisms of the identity functor. Such an automorphism is given by a unit a acting on X , and hence by a^n on $X^{\otimes n}$. Naturality with respect to the vertex $X \otimes X \rightarrow X$ gives $a^2 = a$, so $a = 1$.

(3) A monoidal equivalence carries the pivotal structure of one category to a pivotal structure on the other, which by uniqueness is the given one.

Finally, the same uniqueness argument over $A = \mathbb{C}$ shows that $\mathcal{S}_t$ has a unique pivotal structure, so by (3) the equivalence $\mathcal{C} \otimes_A \mathbb{C} \simeq \mathcal{S}_t$ is pivotal. It matches the self-duality, since the normalized cup lifts the cup of $\mathcal{S}_t$ ($u_1 \equiv 1$), and it matches any lift of the vertex. So $\mathcal{C}$ is a pivotal deformation in the sense of Definition 3.1, and Theorem 1.3 applies. The resulting pivotal equivalence with $\mathcal{S}_{t'}(A)$ is in particular monoidal. Conversely, by (3) any monoidal equivalence of deformations is pivotal, so it preserves the loop value $t' - 1$, and t' is unique. $\square$

Note the role of the trivalent vertex: it rules out the “grading” automorphisms a^n of the identity functor, so the pivotal structure is unique.

3.3. Normalizing the data.

Lemma 3.4. *Let $\mathcal{C}$ be a pivotal deformation of $\mathcal{S}_t$ over A .*

- (1) *The vertex can be chosen rotation invariant. Then $\mathcal{C}^1 = 0$, and the 2- and 3-box spaces $\mathcal{C}^2$ and $\mathcal{C}^3$ are free of rank one, spanned by the cup and by the vertex respectively.*
- (2) *Let $t' \in A$ be one more than the value of the loop. After multiplying the vertex by a unit $\equiv 1$, the planar relations (1) hold in $\mathcal{C}$ with t' in place of t .*
- (3) *There is a rotation invariant $c \in \mathcal{C}^4$ lifting the crossing of $\mathcal{S}_t$.*

Proof. (1) The rotation has order 3 on $\mathcal{C}^3$ and $3 \in A^\times$, so we may average over rotations. The statements about ranks follow from Nakayama.

(2) The lollipop lies in $\mathcal{C}^1 = 0$. On $\mathcal{C}^4$ the rotation satisfies $\rho^4 = 1$. In $\mathcal{S}_t^4$, by Lemma 2.6(2), ρ swaps $\begin{array}{|} \cup \\ \cup \end{array}$ and $\begin{array}{|} \cup \\ \cap \end{array}$ and fixes $\begin{array}{|} \cap \\ \cap \end{array}$ and $\begin{array}{|} \cap \\ \cup \end{array}$, so its (-1) -eigenspace has rank 1. By the discussion above the same holds in $\mathcal{C}^4$. The element $\begin{array}{|} \cup \\ | - \cup \end{array}$ lies in this eigenspace and is part of a basis, so it spans it. Since $I - H$ also lies in it,

$$\begin{array}{|} \cup \\ | - \cup \end{array} - \begin{array}{|} \cup \\ \cap \end{array} = r \left(\begin{array}{|} \cup \\ | - \cup \end{array} \right) \quad \text{for some } r \in A \text{ with } r \equiv t^{-1}.$$

Rescaling the vertex by v multiplies r by v^2 , so $v = (rt')^{-1/2} \equiv 1$ gives $r = 1/t'$. This square root exists in A since $rt' \equiv 1 \pmod{\mathfrak{m}}$ (binomial series). Finally, closing up the right strand of the $I-H$ relation gives the bigon.

(3) Lift the crossing arbitrarily and apply the projector $\frac{1}{4}(1 + \rho + \rho^2 + \rho^3)$ onto the $(+1)$ -eigenspace. $\square$

This is the only place where the eigenvalue condition on 4-boxes is used. It is the same open condition as in [SS, Thm. 6], but here it also provides a rotation invariant crossing.

3.4. Deformed Reidemeister moves. Fix c as in Lemma 3.4(3); from now on $\times$ denotes c . Then $\mathcal{C}$ is generated by the vertex and c , since lifts of the basis diagrams of [SS, Lemma 4.14] form a basis by Nakayama. The Reidemeister type relations (2) hold modulo $\mathfrak{m}$, so they hold in

$\mathcal{C}$ up to error terms:

$$(4) \quad \begin{array}{c} \text{Diagram 1} = (1+a) \text{Diagram 2}, \quad \text{Diagram 3} = (1+b) \text{Diagram 4}, \quad \text{Diagram 5} = \text{Diagram 6} + \boxed{E_2}, \\ \text{Diagram 7} = \text{Diagram 8} + \boxed{E_{2.5}}, \quad \text{Diagram 9} = \text{Diagram 10} + \boxed{E_3}. \end{array}$$

Here $a, b \in \mathfrak{m}$, and the coupons $E_2, E_{2.5}, E_3$ have coefficients in $\mathfrak{m}$.

These are the deformed Reidemeister moves, and the coupons are the “lower order terms”. A priori $E_2, E_{2.5}$ and E_3 could be anything in the 4-, 5- and 6-box spaces, which have ranks 4, 11 and 41. Because the vertex and c are rotation invariant, each relation also holds in every rotated form, with the error coupon rotated along with everything else. In particular, the rotated forms (3) of $R1$ (a cap on a crossing, a crossing on a cup) hold with the same factor $1+a$ as the curl, since all three are obtained by capping two adjacent legs of the rotation invariant c ; similarly for $R1.5$ and $1+b$.

3.5. Linearization and gauge. We will improve c step by step. Suppose $J \subset A$ is an ideal with $\mathfrak{m}J = 0$ and that all errors have coefficients in J . Since $\mathfrak{m}J = 0$:

- an element of $J\mathcal{C}^k$ only depends on its image in $\mathcal{S}_t^k$, so $J\mathcal{C}^k = J \otimes_{\mathbb{C}} \mathcal{S}_t^k$;
- a product of two error terms vanishes.

So we may compute with the error terms inside $\mathcal{S}_t$; Lemma 3.6 below makes this precise.

Notation 3.5. To keep formulas light we write elements of J as ε times a complex number, as if $J = \mathbb{C}\varepsilon$ with $\varepsilon^2 = 0$. Thus $a = \varepsilon\bar{a}$ and $b = \varepsilon\bar{b}$, and the error coupons are ε times

$$\boxed{\bar{E}_2}, \quad \boxed{\bar{E}_{2.5}}, \quad \boxed{\bar{E}_3}$$

in $\mathcal{S}_t$. For general J every statement below holds coordinatewise in a basis of J , since all equations are linear.

Every overlap computation below has the following shape, and this is the only way in which we pass to $\mathcal{S}_t$.

Lemma 3.6 (Overlap principle). *Suppose a diagram D in $\mathcal{C}^k$ is simplified in two ways, using the relations (1) and (4), to*

$$D = M + \varepsilon X \quad \text{and} \quad D = M + \varepsilon Y,$$

where M is the same diagram in $\mathcal{C}$ in both cases, and X, Y are built from error terms. Then $X = Y$, and this equation can be evaluated in $\mathcal{S}_t$, where all of (1) and (2) hold.

Proof. Subtracting gives $\varepsilon(X - Y) = 0$ in $J\mathcal{C}^k$, and $J\mathcal{C}^k = J \otimes_{\mathbb{C}} \mathcal{S}_t^k$. $\square$

The point is that the main term M is never reduced modulo $\mathfrak{m}$: in each overlap the two chains of simplifications end in literally the same diagram, and only the ε -parts are evaluated in $\mathcal{S}_t$.

The lift c is only well defined up to adding εw with w a rotation invariant 4-box of $\mathcal{S}_t$. Such a change preserves everything arranged so far, but changes the errors. For $R2$, for instance, since $\varepsilon^2 = 0$ it changes $\bar{E}_2$ by

$$\boxed{w} + \text{Diagram 1} = \text{Diagram 2} + \boxed{w}.$$

Definition 3.7. A *gauge change* is a replacement of c by $c + \varepsilon w$, where w is a rotation invariant 4-box of $\mathcal{S}_t$ (for general J , with coefficients in J). The *gauge directions* are the 4-boxes $wc + cw$ above, for w rotation invariant: they are exactly the possible changes to $\overline{E}_2$ under a gauge change. We say that something holds *modulo gauge* if it holds modulo the span of the gauge directions.

So if $\overline{E}_2$ is a gauge direction, a suitable gauge change makes $\overline{E}_2 = 0$. The task is to show that a suitable gauge change makes all five errors vanish.

Proposition 3.8 (Main step). *If all errors in (4) have coefficients in an ideal J with $\mathfrak{m}J = 0$, then after a gauge change by some rotation invariant w with coefficients in J , all errors vanish.*

Proof of Theorem 1.3 from Proposition 3.8. Normalize as in Lemma 3.4; then the errors lie in $\mathfrak{m}$. Suppose they lie in $\mathfrak{m}^i$. Apply Proposition 3.8 to $\mathcal{C}/\mathfrak{m}^{i+1}$ with $J = \mathfrak{m}^i/\mathfrak{m}^{i+1}$, and lift the gauge change to $\mathcal{C}$; now the errors lie in $\mathfrak{m}^{i+1}$. Since $\mathfrak{m}^N = 0$ for some N , after finitely many steps (1) (with t') and (2) hold in $\mathcal{C}$.

Lemma 2.3 now gives a pivotal functor $F: \mathcal{S}_{t'}(A) \rightarrow \mathcal{C}$ which reduces to the given equivalence modulo $\mathfrak{m}$. On k -boxes, F sends the basis of [SS, Lemma 4.14, Cor. 4.17] to lifts of a basis of $\mathcal{S}_t^k$, which is a basis of $\mathcal{C}^k$ by Nakayama. So F is an equivalence. The parameter t' is unique because the loop value is an invariant. $\square$

The rest of the paper proves Proposition 3.8. The method is always the same: find a diagram in which two of the relations (4) overlap, simplify it in two different ways, and compare the ε -terms.

4. LEVEL 4: TWO OVERLAPS AND ONE OBSTRUCTION

Lemma 4.1.

$$\boxed{\overline{E}_2} = 2\bar{a} \cap, \quad \boxed{\overline{E}_2} = 2\bar{b} \cap \cap.$$

Proof. Simplify a cap on top of two crossings in two ways. Using the rotated $R1$ of (3) twice, or $R2$ once,

$$(1 + 2\varepsilon\bar{a}) \cap = \cap \cap = \cap + \varepsilon \boxed{\overline{E}_2}.$$

For the second identity, do the same with a vertex on top of two crossings, using $R1.5$ twice or $R2$ once:

$$(1 + 2\varepsilon\bar{b}) \cap \cap = \cap \cap \cap = \cap \cap + \varepsilon \boxed{\overline{E}_2}.$$

$\square$

So $\bar{a}$ and $\bar{b}$ are determined by $\overline{E}_2$, and the real question is how much of $\overline{E}_2$ can be removed by a gauge change. To answer it we introduce a functional that will appear again in §5.

Definition 4.2. Draw 5-boxes with three points at the bottom and two at the top. Let $\phi: \mathcal{S}_t^5 \rightarrow \mathbb{C}$ be the signed sum of the coefficients, in the partition basis, of the six basis elements

$$+ \cap \cap, \quad - \cap \cap, \quad + \cap \cap, \quad - \cap \cap, \quad + \cap \cap, \quad - \cap \cap.$$

For $x \in \mathcal{S}_t^4$ define

$$\psi(x) = \phi \left(\begin{array}{c} \text{diagram with a box labeled } x \text{ and a crossing} \end{array} \right).$$

Lemma 4.3. (1) *By the cutting rule,*

$$\begin{array}{c} \text{diagram 1} = \text{diagram 2}, \quad \text{diagram 3} = \text{diagram 4}, \quad \text{diagram 5} = \text{diagram 6}, \quad \text{diagram 7} = \text{diagram 8} - \frac{1}{t} \text{diagram 9}. \end{array}$$

Hence, on the basis of $\mathcal{S}_t^4$,

$$\psi \left(\begin{array}{c} | \\ | \end{array} \right) = 0, \quad \psi \left(\begin{array}{c} \cup \\ \cup \end{array} \right) = 1, \quad \psi \left(\begin{array}{c} \times \\ \times \end{array} \right) = -1, \quad \psi \left(\begin{array}{c} \circ \\ \circ \end{array} \right) = 0.$$

(2) *The gauge directions (Definition 3.7) are spanned by*

$$\begin{array}{c} \times + \cup, \quad | \quad |, \quad \circ \end{array}$$

and they span exactly $\ker \psi$.

Proof. (1) In the first three diagrams there are no internal strands. In the last one the only internal strand joins the hollow node to the black vertex. Reading off the six coefficients in Definition 4.2 gives the values; for example the third diagram is the fourth basis element there, with sign -1 .

(2) By Lemma 2.6(2), the rotation invariant 4-boxes are spanned by

$$\begin{array}{c} | \quad | + \cup, \quad \times, \quad \circ \end{array}$$

Stacking a crossing on these, above or below, gives respectively

$$\begin{array}{c} \times + \cup, \quad | \quad |, \quad \circ \end{array}$$

again by Lemma 2.6(2). So the gauge directions $wc + cw$ are twice these. The three vectors are independent and lie in $\ker \psi$ by (1). Since $\psi \neq 0$, its kernel is 3-dimensional, so they span it. $\square$

After Lemma 4.1, modulo gauge exactly one number is left at the 4-box level: $\psi(\overline{E}_2)$, the coefficient of $\cup$ minus the coefficient of $\times$ in $\overline{E}_2$. No overlap inside the 4-box space can determine it. For the Brauer category, which has the same 4-box space without ν_4 , this number is a genuine deformation parameter, at least for $t \neq 3$ (Remark 7.2). So to kill it we need to use the trivalent vertex in an essential way.

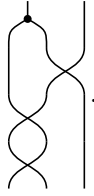
5. LEVEL 5: PUSHING A CROSSING THROUGH $R2.5$

Lemma 5.1 (The key overlap).

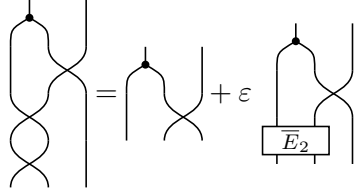
$$\begin{array}{c} \overline{E}_{2.5} \\ \times \end{array} + \begin{array}{c} \text{diagram with } \overline{E}_{2.5} \text{ and a crossing} \end{array} = \begin{array}{c} \text{diagram with } \overline{E}_2 \text{ and a crossing} \end{array}.$$

The second term on the left is the coupon $\overline{E}_{2.5}$ rotated by three clicks.

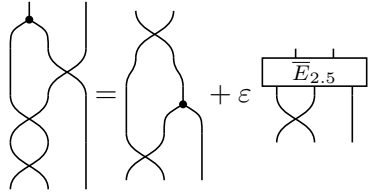
Proof. Consider the diagram Λ with a crossing stacked underneath:



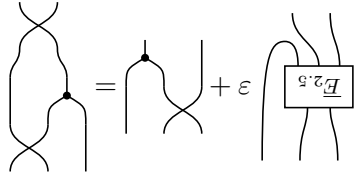
First way. The two lowest crossings form an $R2$ configuration, so



Second way. The upper three layers form Λ , so $R2.5$ gives

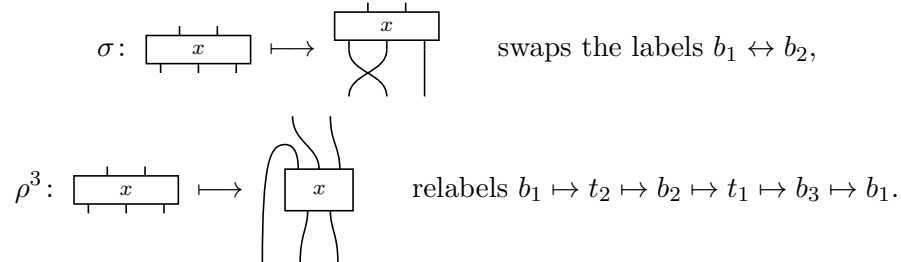


The first diagram on the right is again an $R2.5$ configuration, with two crossings and a vertex bounding a triangle. Rotating Λ by three clicks gives exactly this picture, and rotating Λ' by three clicks gives . So the rotated relation reads



Comparing the two expressions proves the lemma. $\square$

To use this identity we describe the two operations on the left in terms of the partition basis of $\mathcal{S}_t^5$. Label the boundary points b_1, b_2, b_3 (bottom) and t_1, t_2 (top). By Lemma 2.6(2) both operations permute the 11 basis elements:



In this notation, the left side of Lemma 5.1 is $(\sigma + \rho^3) \bar{E}_{2.5}$.

Lemma 5.2. (1) $\phi(\sigma x) + \phi(\rho^3 x) = 0$ for every $x \in \mathcal{S}_t^5$.

(2) If $\sigma x + \rho^3 x = 0$, then x is a multiple of



Proof. Both statements concern the permutations $g = \rho^{-3}\sigma$ and $h = \sigma\rho^{-3}$ of the partition basis. Here products are composites of operators on $\mathcal{S}_t^5$, so σ is applied first in g ; a relabelling f of the boundary points acts on basis elements by $[\pi] \mapsto [f(\pi)]$, and we write f by its effect on labels. The condition in (2) says $gx = -x$, and (1) says $\phi \circ h = -\phi$.

On the boundary points, $g = (b_1 t_2)(b_2 b_3 t_1)$ and $h = (b_1 b_3 t_1)(b_2 t_2)$. On the 11 basis elements, each has one 6-cycle, one 3-cycle and two fixed points. The 6-cycle of g is the sequence of six diagrams defining k_5 , and the 6-cycle of h is the sequence of six diagrams defining ϕ .

For a permutation, the (-1) -eigenspace is spanned by the alternating sums over its even cycles. Here there is a single even cycle in each case, which gives both statements. $\square$

Corollary 5.3. $\psi(\bar{E}_2) = 0$. Consequently, after a gauge change, $\bar{E}_2 = 0$ and $\bar{a} = \bar{b} = 0$.

Proof. Apply ϕ to Lemma 5.1. The left side gives 0 by Lemma 5.2(1), and the right side gives $\psi(\bar{E}_2)$ by definition. By Lemma 4.3(2), $\bar{E}_2$ is then a gauge direction, so a gauge change makes it 0. Lemma 4.1 then gives $\bar{a} = \bar{b} = 0$. $\square$

So from now on $R1$, $R1.5$ and $R2$ hold exactly. Lemma 5.1 now says $\sigma\bar{E}_{2.5} + \rho^3\bar{E}_{2.5} = 0$, so by Lemma 5.2(2), $\bar{E}_{2.5} = \mu k_5$ for some $\mu \in \mathbb{C}$. One more overlap finishes the 5-box level.

Lemma 5.4. (1) Since $\bar{a} = \bar{b} = 0$, we have

$$\begin{array}{c} \boxed{\bar{E}_{2.5}} \\ \cup \\ \text{---} \end{array} = 0.$$

(2) The element k_5 satisfies

$$\begin{array}{c} \boxed{k_5} \\ \cup \\ \text{---} \end{array} = t \begin{array}{c} \text{---} \\ \cup \\ \text{---} \end{array}.$$

Hence $\mu = 0$ (as $t \neq 0$), so $\bar{E}_{2.5} = 0$ and $R2.5$ holds exactly.

Proof. (1) Attach a cup at the bottom left of both sides of $R2.5$. On the left, both strands of the cup run into the same crossing, so by the rotated $R1$ and then the rotated $R1.5$,

$$\begin{array}{c} \text{---} \\ \cup \\ \text{---} \end{array} = (1 + \varepsilon\bar{a}) \begin{array}{c} \text{---} \\ \cup \\ \text{---} \end{array} = (1 + \varepsilon\bar{a})(1 + \varepsilon\bar{b}) \begin{array}{c} \text{---} \\ \cup \\ \text{---} \end{array}.$$

On the right we get a vertex whose legs are crossed, so by $R1.5$,

$$\begin{array}{c} \text{---} \\ \cup \\ \text{---} \end{array} = (1 + \varepsilon\bar{b}) \begin{array}{c} \text{---} \\ \cup \\ \text{---} \end{array}.$$

Since the two diagrams differ by ε times $\bar{E}_{2.5}$ with the same cup attached, we get

$$\varepsilon \begin{array}{c} \boxed{\bar{E}_{2.5}} \\ \cup \\ \text{---} \end{array} = \varepsilon\bar{a} \begin{array}{c} \text{---} \\ \cup \\ \text{---} \end{array} = 0.$$

(2) Attach the cup to each basis element in k_5 and use the cutting rule; the cup creates the only internal strand.

- If b_1 and b_2 form a block, we get a closed loop, so $t - 1$ times the remaining vertex.

- If they lie in a block of size 3, we get a singleton, hence 0.
- If they lie in different blocks, of sizes 2 and 3, the blocks merge with coefficient 1, and cutting produces singletons.

The six terms of k_5 contribute $(t-1)$, 0, +1, -1, +1 and 0 times Ψ , for a total of t . $\square$

6. LEVEL 6: $R3$ VIA A ROTATIONAL EIGENVALUE

From now on $R1$, $R1.5$, $R2$ and $R2.5$ hold exactly, and only $\overline{E}_3$ remains.

Lemma 6.1. *The error $\overline{E}_3$ satisfies:*

$$\begin{aligned}
 (1) \quad & \text{Diagram 1} = - \text{Diagram 2}, \quad \text{i.e. } \overline{E}_3 \text{ is a } (-1)\text{-eigenvector of } \rho; \\
 (2) \quad & \text{Diagram 3} + \text{Diagram 4} = 0; \\
 (3) \quad & \text{Diagram 5} = 0.
 \end{aligned}$$

Proof. (1) This is the open condition at the 6-box level. The diagram T is invariant under rotation by two clicks, and its one-click rotation is the right side of $R3$. Rotating the relation $T = \rho(T) + \varepsilon \overline{E}_3$ by one click therefore gives

$$\text{Diagram 6} = \text{Diagram 7} + \varepsilon \text{Diagram 8}.$$

Adding this to the original relation gives $\varepsilon(\overline{E}_3 + \rho \overline{E}_3) = 0$.

(2) Simplify T with a crossing stacked on top in two ways. Using $R2$ on the top two crossings,

$$\text{Diagram 9} = \text{Diagram 10}.$$

Alternatively, use $R3$ on the bottom three crossings, then $R3$ on the top three crossings of the result, and finally $R2$:

$$\text{Diagram 11} = \text{Diagram 12} + \varepsilon \text{Diagram 13} = \text{Diagram 14} + \varepsilon \text{Diagram 15} + \varepsilon \text{Diagram 16} = \text{Diagram 17} + \varepsilon (\text{Diagram 18} + \text{Diagram 19}).$$

Comparing gives (2).

(3) Simplify T with a vertex attached at the bottom left in two ways. Using $R1.5$ at the bottom and then the one-click rotation of $R2.5$,

The diagram shows a sequence of three tangles connected by equals signs. The first tangle has a crossing and a vertex at the bottom left. The second tangle is the result of applying R1.5 at the bottom. The third tangle is the result of applying the one-click rotation of R2.5.

Using $R3$ first instead, then $R2.5$ on the lower three layers, then $R1.5$ at the top,

The diagram shows a sequence of tangles connected by equals signs and plus signs with epsilon. The first tangle is simplified to a second tangle, then to a third tangle which is a box labeled E3-bar, then to a fourth tangle, then to a fifth tangle, and finally to a sixth tangle which is a box labeled E3-bar.

Comparing gives (3). □

Lemma 6.1(1) is where the rotational eigenvalue earns its keep. The (-1) -eigenspace of ρ on the 41-dimensional space $\mathcal{S}_t^6$ is only 5-dimensional. A basis is given by alternating sums over the rotation orbits of

The diagram shows five tangles separated by commas. Each tangle has a vertex at the bottom left and represents a different rotation orbit.

which are exactly the rotation orbits whose stabilizer does not contain the half-turn ρ^3 .

Label the six boundary points $b_1, b_2, b_3, t_3, t_2, t_1$ counterclockwise as before. Lemma 6.1(1) and (2) are statements about two permutations of the 41 basis elements:

- ρ , the 6-cycle on boundary points;
- the map $\boxed{x} \mapsto \boxed{x}$, which relabels by $s = (b_2 b_3)(t_1 t_2)$.

Stacking $\times$ underneath the right two strands of (2) and using $R2$ shows that (2) says exactly $s\overline{E}_3 = -\overline{E}_3$.

Lemma 6.2. *Let m_6 be the signed sum of the eight perfect matchings with no “antipodal” pair $\{b_1, t_3\}$, $\{b_2, t_2\}$, $\{b_3, t_1\}$:*

The diagram shows the expression m6 = (matching 1) - (matching 2) - (matching 3) + (matching 4) + (matching 5) - (matching 6) - (matching 7) + (matching 8). Each matching is a tangle with four strands and a vertex at the bottom left.

- (1) If $\rho x = -x$ and $sx = -x$, then x is a multiple of m_6 .
- (2) By the cutting rule,

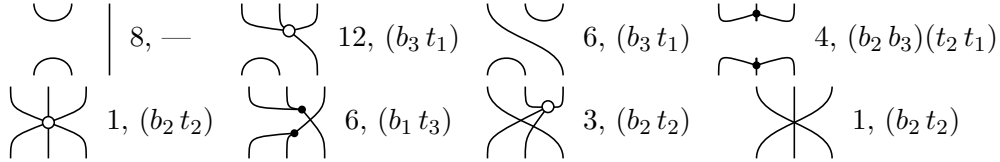
The diagram shows a box labeled m6 with a vertical line passing through it. This is equal to a sum of four tangles, each with a vertex at the bottom left, which is not equal to zero.

Proof. (1) The relabellings ρ and s preserve the antipodal pairings, so they generate a subgroup Γ of the hyperoctahedral group $B_3 = (\mathbb{Z}/2)^3 \rtimes S_3$ of order 48. In fact $\Gamma = B_3$. Indeed, ρ permutes the three pairs cyclically and s swaps two of them, so Γ maps onto S_3 . Moreover $s\rho^{-1}s\rho^2 = (b_3 t_1)$,

as one checks by following each label, and its conjugates by powers of ρ are the transpositions of the other two antipodal pairs; so Γ contains $(\mathbb{Z}/2)^3$.

Let $\chi: B_3 \rightarrow \{\pm 1\}$ be the sign of a permutation of the six points times the sign of the induced permutation of the three antipodal pairs. Then $\chi(\rho) = \chi(s) = -1$, so the hypothesis says $\gamma x = \chi(\gamma)x$ for all $\gamma \in \Gamma$.

The space of such x has a basis indexed by the Γ -orbits of basis elements on whose stabilizers χ is trivial. There are eight orbits. The table lists representatives, orbit sizes, and, where applicable, an element of the stabilizer with $\chi = -1$:



A transposition of an antipodal pair has $\chi = -1$, and so does $(b_2 b_3)(t_2 t_1)$, which swaps two pairs. So only the first orbit contributes, and the corresponding signed sum is m_6 .

(2) Attach the vertex to b_1 and b_2 . If they are matched, the result has a singleton and vanishes. Otherwise the vertex merges the two pairs containing b_1 and b_2 into a block of size 3, and there are no internal strands. Two of the six surviving terms cancel, leaving the four displayed diagrams. $\square$

Corollary 6.3. $\bar{E}_3 = 0$. This completes the proof of Proposition 3.8, and hence of Theorem 1.3.

Proof. By Lemma 6.1(1),(2) and Lemma 6.2(1), $\bar{E}_3 = \lambda m_6$ for some $\lambda \in \mathbb{C}$. By Lemma 6.1(3), attaching a vertex at the bottom left of $\bar{E}_3$ gives 0; by Lemma 6.2(2), doing the same to m_6 does not. So $\lambda = 0$. The only gauge change was the one in Corollary 5.3; the later steps do not change c , so the errors killed earlier stay zero. $\square$

7. REMARKS

Remark 7.1 (Open conditions). The argument uses three facts that are automatic for deformations, by Nakayama's lemma and the invertibility of positive integers in A :

- the rotation eigenspace decomposition of $\mathcal{C}^4$, which gives the I – H relation and a rotation invariant lift of the crossing;
- freeness of $\mathcal{C}^5$ and $\mathcal{C}^6$, so that $\bar{E}_{2.5}$ and $\bar{E}_3$ are honest elements of $\mathcal{S}_t^5$ and $\mathcal{S}_t^6$;
- the fact that the $R3$ error is a rotation eigenvector, which cuts 41 unknowns down to 5.

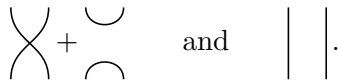
The overlaps in Lemmas 4.1, 5.1, 5.4 and 6.1 play the role of Liu's consistency computations for Yang–Baxter relation planar algebras [Liu]. Because we pass from $A/\mathfrak{m}^i$ to $A/\mathfrak{m}^{i+1}$ one step at a time, they are linear, and no classification of nonlinear solutions is needed. A full recognition theorem along these lines, valid over an arbitrary ring, would require solving the corresponding nonlinear system; we have not attempted this.

Remark 7.2 (The obstruction at level 4 is the BMW direction). For the Brauer category (Deligne's O_{t-1} [Bra, Del], generated by the crossing alone, with loop value $d = t - 1$) the same set-up applies verbatim.

- The 4-box space has basis



- The gauge directions, defined as in Definition 3.7 with w a rotation invariant 4-box of the Brauer category, are spanned by



- So again a single number survives at level 4: the coefficient of $\smile$ in $\overline{E}_2$ modulo these.

For $t \neq 3$ this number is realized by the BMW deformation [BW, Mur] (the quantum orthogonal group [Wen, TW]; compare the Kazhdan–Wenzl classification in type A [KW]), so for the Brauer category it is a genuine deformation parameter. Here is a sketch. Let $g^{\pm 1}$ be the BMW positive and negative crossings, with the Kauffman relations $g - g^{-1} = z(\text{id} - e)$ and $ge = eg = \alpha^{-1}e$, where $z = q - q^{-1}$ and α is chosen so that the loop value $1 + (\alpha - \alpha^{-1})/z$ equals d . One-click rotation exchanges g and g^{-1} , so $c = \frac{1}{2}(g + g^{-1})$ is a rotation invariant lift of the crossing, and the Kauffman relations give exactly

$$\text{crossing} = \text{vertical line} + \frac{z^2}{4} \left(\text{vertical line} + (d-2) \text{cup} \right).$$

Replacing (q, α) by (q^{-1}, α^{-1}) is the mirror image, which exchanges g and g^{-1} and fixes c ; so in terms of c the BMW category depends only on z^2 . To first order in z^2 the error is $\overline{E}_2 = \frac{1}{4}(\text{id} + (d-2)e)$, which is not a gauge direction when $d \neq 2$. Note that the parameter is $(q - q^{-1})^2$, not $q - 1$: to first order in $q - 1$ the BMW category is a trivial pivotal deformation of the Brauer category.

In S_t this number is killed by Lemma 5.1, i.e. by pushing a crossing through a trivalent vertex. Informally, the BMW deformation of the crossing of $O(t-1)$ does not extend to a pivotal deformation of $S_t \subset O(t-1)$.

Remark 7.3 (q -partition algebras). Halverson–Thiem [HT] and Dipper–Thangavelu [DT] study q -partition algebras, which for $q = 1$ recover partition algebras. These are q -analogues in the sense that $GL_n(\mathbb{F}_q)$ is a q -analogue of S_n , rather than deformations of S_t , so there is no tension with Theorem 1.3. Halverson and Thiem define them as centralizer algebras for $GL_n(\mathbb{F}_q)$ with q a prime power, and no tensor product is involved. Dipper and Thangavelu let the Hecke algebra act on tensor space by q -deformed letter permutations, but the Hecke algebra has no compatible coproduct, so this does not come from a monoidal (let alone pivotal) structure.

Remark 7.4 ($t = 0$). We used $t \neq 0$ in the model (the idempotent P) and in Lemma 5.4(2). For $t = 0$ the trivalent category S_0 is not Deligne’s category in any case [SS, Rem. 4.4]. In [SS] there is also a trivalent category $\mathcal{S}(\mathbb{C}, 1, 0)$ with a different normalization, in which P involves no factor t^{-1} ; we have not checked whether the argument adapts to it.

7.1. Computer verification. The following scripts accompany this note. They supplement, but are not needed for, the proofs above.

- `figs.py` generates the pictures, in three ways.
 - Most pictures are drawn from layered words, so each is an honest planar string diagram whose only crossings are the crossing generator.
 - The roughly forty partition diagrams $[\pi]$ are drawn by placing nodes and strands directly (`part.py`).
 - Three pictures are drawn by hand: the two upside-down coupons in §5 and the picture of the second term of P .

For the last two kinds, the absence of unintended crossings and of strands passing through nodes was checked by eye.

- `checks.py` verifies identities among the pictures drawn from layered words: claimed rotations and isotopies, by comparing canonical forms of planar maps, and partition expansions, by evaluating the word diagrams in the model of §2.3. It does not inspect the drawings of partition diagrams. It also evaluates each pictured overlap identity on the error terms of the genuine first-order deformations $c \mapsto c + \varepsilon w$. This is a consistency check

rather than a proof, since an incorrect identity that happens to hold for the error terms of every gauge change would pass; the derivations themselves are the ones given in the text.

- `verify_proof.py` redoes all the finite computations of the proofs, including Lemmas 5.2 and 6.2, with t a formal variable.
- `consistency_check.py` is independent of the hand-chosen overlaps. It generates all first-order consistency equations among a finite set of 53 planar diagrams, namely those reachable from “a cap, vertex or crossing attached to two adjacent legs of $R2$, $R2.5$ or $R3$ ”; there are 984 of them. Elimination over $\mathbb{Q}(t)$, with all divisors recorded and several elimination orders, shows that for $t \notin \{0, 1, 2\}$ their solutions are exactly the error terms produced by the 3-dimensional space of gauge changes; exact rational computations settle $t = 1, 2$. The script also checks that a deliberately mis-rotated error term is detected.
- `brauer_control.py` runs the same machinery on the Brauer category. It finds a 3-dimensional solution space containing the 2-dimensional space of error terms produced by gauge changes, consistent with Remark 7.2; so the machinery does not over-constrain.

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