

PROVING CONJECTURES OF ASHOKE SEN: F-THEORY LIMITS, SMALL BLACK HOLES AND DYON INDICES

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ABSTRACT. *We examine thirty-six statements of Ashoke Sen on duality, dyon indices, tachyon condensation, F-theory and black hole entropy, and decide which are proved, which are supported but unproved, and where each stops. We prove that every degeneration of F-theory on an elliptic Calabi–Yau variety that is weakly coupled at the generic point of the base is, after base change and rescaling, Sen’s family, and that one exists exactly when $-2K_B$ has a section. We prove that the entropy of a two-charge small black hole is fixed by one critical value for every curvature-corrected action, so that the Gauss–Bonnet truncation matches the microscopic count only in four and five dimensions and Prester’s Lovelock completion is the unique one that matches it in every dimension. Exact product expansions reproduce the known negative dyon indices, and the tachyon-dust caustic concerns the pressureless equations, not the rolling solution.*

Note. This paper was first made public on 25 October 2026, under the title *Conjectures of Ashoke Sen: Scope, Exact Tests and Analytic Limits*, as release v1.0.0 of the repository https://github.com/creelie/paper_11 (concept DOI [10.5281/zenodo.23652472](https://doi.org/10.5281/zenodo.23652472), version DOI [10.5281/zenodo.23652473](https://doi.org/10.5281/zenodo.23652473)); it was revised there on 26 October 2026 as v1.0.1 ([10.5281/zenodo.23652641](https://doi.org/10.5281/zenodo.23652641)) and v1.1.0 ([10.5281/zenodo.23656435](https://doi.org/10.5281/zenodo.23656435)). The first version counted two of the thirty-six statements as false and twenty-one negative charge triples; it used a charge cone with the discriminant in place of the linear inequality of Sen’s equation (3.9), and its dust theorem replaced a supremum by a maximum without an attainment hypothesis. The later versions corrected the cone, extended the scan to thirty triples, restated the two verdicts as limits of scope, since the negative indices are full indices that the hair analysis of Chattopadhyaya and David already accounts for and the caustic concerns the pressureless effective equations rather than the homogeneous solution, and added the infrared appendices. The same mathematics was set in a single-column form for submission as release v2.0.0 of the same repository ([10.5281/zenodo.23669248](https://doi.org/10.5281/zenodo.23669248)); the two misprints of Chattopadhyaya and David recorded in Section 5 and Appendix A were added in release v2.0.1 ([10.5281/zenodo.23670581](https://doi.org/10.5281/zenodo.23670581)), with the exact program that computes every table, the Lean 4 certificates of the finite statements, and the scripts that check the explicit constants of the appendices. Release v2.1.0 ([10.5281/zenodo.23746042](https://doi.org/10.5281/zenodo.23746042)) proved claims 23 and 26, which the earlier versions recorded as limits of scope: Theorem 9.5 with Corollaries 9.6 and 9.8, and Theorems 10.3, 10.6 and 10.7, with full proofs in Appendix G. Release v2.2.0 ([10.5281/zenodo.23747112](https://doi.org/10.5281/zenodo.23747112)) defined the three highlights of the introduction, listed in Table 5 what is left unproved, proved Theorem 10.10 on the central charges of the heterotic string in $D \geq 6$, and corrected the footnote on Sen’s normalisation of the Gauss–Bonnet coupling, whose equation (3.15) sets $C = 1$. This version, release v2.2.1 ([10.5281/zenodo.23749469](https://doi.org/10.5281/zenodo.23749469)), corrects errors of that release, none of them in a statement of Sen. Parts (iii) and (iv) of Corollary 9.8 now assume a base-point-free anticanonical system, without which

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part (iii) fails on $\mathbb{F}_{12}$. The base change of Theorem 9.5 has order at most two. The critical radii of Theorem 8.1(i) are $1/\sqrt{2}$ for the brane pair and $\sqrt{2}$ for the non-BPS brane. The caption of Figure 4, the endpoint argument for Corollary 10.5, the indexing of d_h in Section 4, the last sentence of the verdict on claim 18 and the scope of the condition $R_* = \alpha'$ are corrected as well.

1. INTRODUCTION

Ashoke Sen has published conjectures for more than thirty years, and most of them were made in the same way: a computation in one regime, a symmetry that carries it to a regime where nothing can be computed, and a statement about that second regime sharp enough to be refuted. The statement that the heterotic string on a six-torus is invariant under $SL(2, \mathbb{Z})$ required a bound state of two monopoles that nobody had seen [115, 116]. The statement that the tachyon of an unstable D-brane rolls to a vacuum without open strings, at an energy equal to the tension of the brane [119, 126], required an exact solution of open string field theory that did not exist for seven years. The statement that the Fourier coefficients of a Siegel modular form have a definite sign on a cone of charges [100] required an argument about the angular momentum of a black hole. These are predictions in the strict sense, and they can be checked.

This paper examines a selected catalogue, not the complete publications of one author. For each entry we separate the original hypotheses, the result actually derived, and subsequent tests or proofs. An unproved conjecture is not a false theorem. A counterexample to a statement with an assumption removed is not a counterexample to the statement with that assumption retained.

The labels in the catalogue describe scope. PROVED means established in the mathematical or perturbative framework identified in the corresponding section; it does not assert a nonperturbative construction of string theory. CONFIRMED means supported by the cited tests without a general proof supplied here. SCOPE LIMITED means that an unrestricted restatement loses a necessary hypothesis or exceeds the approximation. The two HEURISTIC entries are proposals rather than theorem statements. The catalogue is a reading guide, not a numerical ranking of scientific merit. Claims already settled or qualified in the cited literature are included as context, not presented as newly discovered errors. Three highlights mark what Sen's statements lack, and each is used for nothing else. A **rose highlight** marks an error: a formula or statement printed by Sen that is false as printed, while his conclusion survives its correction. Text in **burnt orange** marks a fatal error: a conclusion of Sen that fails under his own hypotheses. A **yellow highlight** marks what is left unproved: a step, hypothesis or identification on which a claim rests and which neither Sen nor the later literature has proved, or a statement beyond his claims that the text leaves open. Every statement of the catalogue was compared with Sen's own text. None is false as printed and none fails under its hypotheses, so the paper has no rose and no burnt orange passage. The two statements counted as false in the first version are explained in the Note, and the two misprints corrected in Section 5 and Appendix A are in the tables of Chattopadhyaya and David, not in Sen's papers. Table 5 collects the yellow passages. The highlights record the state of the mathematics and attribute no misconduct to an author.

Two exact tests receive detailed treatment. For the freely acting toroidal orbifolds, the negative full dyon indices were already found by Chattopadhyaya and David [16]; their later hair analysis [17] explains why the horizon index is a different quantity. Our contribution is a reproducible finite computation with the linear charge-cone inequality of [100, Eq. (3.9)] retained. The corrected scan contains thirty negative triples in the stated range. The original twenty-one listed triples have a separate finite Lean certificate; the larger scan is checked by the supplied exact Python program. No inference from thirty charges to infinitely many charges is made without the published Fourier–Jacobi argument.

For tachyon matter, the characteristic calculation applies to the pressureless effective equations. Focusing is controlled by a positive eigenvalue of the Hessian of the initial tachyon

profile on the branch $\partial_t T > 0$. Concave profiles have no forward local focusing. Bounded nonconstant profiles have positive curvature somewhere and hence a finite focal time on some characteristic. The first time is a minimum only under an attainment hypothesis. This describes the limit of a classical effective description, not a counterexample to the homogeneous solution in [88, 87]. Indeed [86] also gives static dust with arbitrary nonconstant density, a useful example that must be retained.

The F-theory calculation similarly requires two spaces to be kept distinct: the base, where $\eta^2 - h\chi = 0$ can be smooth, and the orientifold double cover, where its pullback has a double locus. Non-Higgsable gauge symmetry alone is not an obstruction to weak coupling: I_0^* is an exception, as the explicit family below illustrates. Two of Sen’s statements are proved here in general form. Every degeneration of F-theory on an elliptic Calabi–Yau variety that is weakly coupled at the generic point of the base is, after a base change of order at most two and a rescaling, Sen’s family, and one exists exactly when $-2K_B$ has a section (Theorem 9.5, Corollary 9.6). For two-charge small black holes the entropy of every curvature-corrected action is fixed by one critical value; the Gauss–Bonnet truncation gives the ratio of Corollary 10.5, equal to one only in $D = 4, 5$, and the Lovelock completion of Prester is the only one with the statistical entropy in every dimension (Theorems 10.3, 10.6, 10.7). Full proofs are in Appendix G.

Plan. Section 2 is the catalogue, Tables 1, 2, 3 and 4, with the verdict of every claim and the place where it is treated. Sections 3 and 4 treat duality: the monopole conjectures and the counting of dyons, with the exact verification of the wall-crossing formula. Section 5 treats the positivity conjecture and its domain of applicability. Sections 6 to 8 treat the tachyon conjectures, the rolling tachyon with the caustic theorem, and the non-BPS branes. Section 9 treats the Sen limit, Sections 10 and 11 the entropy function, the small black holes and the quantum entropy function, Section 12 string field theory, D-instantons, the soft theorems and matrix theory. Section 13 records the remaining questions. Appendix A describes the computation of the Fourier coefficients and Appendix B completes the proof of the caustic theorem.

The primary-source register in Appendix C identifies the passages used for the principal scope corrections. Appendix E constructs an explicit finite-energy soft model. Appendix F derives a quantitative subtracted tree-emission bound and identifies the uniform loop estimates still needed for transfer to the complete theory. Literature-status statements concern the sources examined, rather than a claim to have read every citing paper.

2. THE CATALOGUE

Tables 1, 2, 3 and 4 list the claims, and Figure 1 shows them by verdict. The first column gives the claim in one line, as a concise restatement, with hypotheses supplied in the corresponding section; the second the paper in which it was made; the third the verdict; the fourth the section of this paper where the claim is examined and, where one exists, the theorem that settles it. The column of verdicts is the summary of the paper, and every entry in it is justified in the section named.¹

The list records selected claims from the earlier manuscript. The short entries must be read with the hypotheses in the indicated section. A result established within an effective action is not automatically a theorem about the full string theory. Later proofs and qualifications remain credited in those sections; they are excluded from the list of proposed new objections. The verdicts *confirmed*, *scope limited* and *heuristic* mean that something is left unproved, and Table 5 names it for each claim; the same words are highlighted in yellow in the section

¹The format of the table, one claim per line with its source, its verdict and the place of the argument, is the barrier ledger of the engine Anaxia (github.com/creelie/eBLLM, release 3.1.0, [doi:10.5281/zenodo.23484160](https://doi.org/10.5281/zenodo.23484160)), described in [11] and used in the same way for the records of [38, 131]. The engine was not run for this paper; only its ledger format is used.

Claim	Source	Verdict	Where
<i>Duality</i>			
1. The heterotic string on T^6 is invariant under $SL(2, \mathbb{Z})$ acting on the axion–dilaton	[115]	confirmed	§3.1
2. The Atiyah–Hitchin manifold carries exactly one L^2 anti-self-dual harmonic two-form, the two-monopole bound state	[116]	proved	§3.2, Thm 3.2
3. The L^2 cohomology of the moduli space of k monopoles, for every k , is as S -duality requires	[116, 84]	confirmed	§3.3
4. A manifestly duality-symmetric action is equivalent to the Maxwell and heterotic actions	[83]	proved	§3.4
5. The Kerr–Sen fields exactly solve the specified leading-order heterotic equations	[114]	proved	§3.4
<i>Quarter-BPS dyons</i>			
6. The CHL product has the stated modularity; its physical counting interpretation uses duality	[56, 34]	proved	§4.1
7. Across a wall of marginal stability the index jumps by $(-1)^{Q \cdot P + 1} Q \cdot P d_h(Q) d_h(P)$	[97, 96]	proved	§4.2, Thm 4.2
8. Negative-discriminant indices in the stated sectors are accounted for by two-centred sums	[99]	proved	§4.3
9. A single-centred index is positive when the stated horizon and hair assumptions hold	[100]	scope limited	§5, Thm 5.4
10. The hair of a quarter-BPS black hole can be identified and removed, leaving a horizon index	[12]	confirmed	§5.3

TABLE 1. The catalogue, claims 1 to 10. The labels refer to the frameworks and assumptions discussed in the text.

where the claim is examined. No verdict calls a statement of Sen wrong or fatal, since none examined here is false as printed or fails under its own hypotheses.

3. DUALITY AND THE MONOPOLE CONJECTURES

3.1. The $SL(2, \mathbb{Z})$ conjecture. The conjecture of [115] (claim 1 of Table 1) states that the heterotic string compactified on T^6 , whose low energy theory is $\mathcal{N} = 4$ supergravity coupled to 22 vector multiplets, is invariant under the group $SL(2, \mathbb{Z})$ generated by the shift of the axion and by the inversion $\lambda \mapsto -1/\lambda$ of the complex coupling $\lambda = a + ie^{-\phi}$, acting on the electric and magnetic charge vectors (Q, P) of the $O(6, 22)$ lattice by the same matrix.² The inversion exchanges weak and strong coupling, so no perturbative computation can test it directly. Sen reduced the test to the spectrum of states that saturate the Bogomolnyi bound, whose masses and degeneracies are protected by supersymmetry: the conjecture requires that for every pair of coprime integers (p, q) there exist states carrying p units of electric and q units of magnetic charge, in the same supermultiplet as the elementary string states with $(1, 0)$.

For $q = 1$ these are the dyons of the $\mathcal{N} = 4$ theory and they were known to exist [73]. For $(p, q) = (1, 2)$ the conjecture requires a bound state of two monopoles and one unit of electric charge, and nothing in the classical theory produces such a state: the static BPS monopoles have no net force, but a normalisable quantum bound state is not supplied by a static separated configuration and there is no classical bound state. The prediction is therefore a genuine one, and it was tested in [116], as described in the next subsection. The

²The conjecture was made in two steps. Schwarz and Sen [82] showed that the equations of motion of the low energy theory, not its action, are invariant under $SL(2, \mathbb{R})$, and that the charge lattice reduces the symmetry to $SL(2, \mathbb{Z})$; [115] then proposed that the full string theory, at every order and beyond perturbation theory, has this symmetry. The $SL(2, \mathbb{R})$ of the equations of motion is a theorem of the supergravity theory; it is the lift to the full string theory that is the conjecture.

Claim	Source	Verdict	Where
<i>Tachyons on unstable branes</i>			
11. The tachyon potential of a D-brane is universal, independent of the background	[126]	proved	§6.1
12. The tachyon vacuum has energy density $-\mathcal{T}_p$ relative to the brane	[119, 125]	proved	§6.2
13. Lower-dimensional D-branes are lump solutions of the tachyon effective theory	[119, 125]	proved	§6.2
14. There are no open string excitations around the tachyon vacuum	[125, 78]	proved	§6.2
15. The three conjectures hold in the superstring field theory of Berkovits	[15]	confirmed	§6.2
16. The homogeneous rolling tachyon conserves energy and its pressure vanishes exponentially	[88]	proved	§7.1
17. Homogeneous rolling has a pressureless late-time limit; inhomogeneous dust can focus	[87, 86]	scope limited	§7.2, Thm 7.4
18. Open–closed duality at tree level: the open string description of the decay encodes the closed strings emitted	[90]	confirmed	§7.3
19. The kink on a non-BPS Dp -brane is the BPS $D(p-1)$ -brane (descent relations)	[120, 125]	proved	§8
20. Type I string theory has a stable non-BPS D-particle, the $SO(32)$ spinor	[123]	proved	§8
21. The world-volume theory of the kink and the vortex is the Dirac–Born–Infeld action of the lower brane	[89]	proved	§8

TABLE 2. The catalogue, claims 11 to 21. Claim 17 requires separate treatment of the effective dust limit.

Claim	Source	Verdict	Where
<i>F-theory</i>			
22. F-theory on an elliptic K3 has a weak coupling limit that is the type IIB orientifold on $T^2/\mathbb{Z}_2$	[117]	proved	§9
23. Every elliptic Calabi–Yau with a section of $-2K_B$ has Sen’s weak coupling limit, and every such limit has Sen’s form	[118]	proved	Thm 9.5
<i>Black hole entropy</i>			
24. The entropy function, extremised over the near-horizon data, gives the Wald entropy of an extremal black hole	[93]	proved	§10.1
25. The entropy function extends to rotating extremal black holes	[1]	proved	§10.1
26. A Gauss–Bonnet truncation stretches small horizons; its entropy agrees with the count in $D=4, 5$ and not in $D \geq 6$	[92, 94]	proved	Thms 10.3, 10.7
27. The quantum entropy function, a path integral over AdS_2 , equals the microscopic index	[98]	confirmed	§11.1
28. The logarithmic correction to the entropy is 0 for $\mathcal{N}=4$ and $-4 \ln A_H$ for $\mathcal{N}=8$	[7]	proved	§11.2
29. For $\mathcal{N}=2$ the correction is $\frac{1}{12}(23 + n_H - n_V) \ln(A_H/G_N)$	[101]	confirmed	§11.2

TABLE 3. The catalogue, claims 22 to 29. Claims 23 and 26 are proved here in the forms of Theorems 9.5 and 10.3; the physical identification of the limit and the complete heterotic action are outside both.

Claim	Source	Verdict	Where
<i>String field theory and other claims</i>			
30. Closed string field theory is background independent under infinitesimal deformations	[128]	proved	§12.1
31. Superstring field theory has a gauge-invariant 1PI effective action and a BV master action, and is unitary	[104, 105, 106, 107]	proved	§12.1
32. The normalisation of D-instanton amplitudes is fixed by string field theory	[109, 112, 111]	confirmed	§12.2
33. The classical logarithmic soft graviton terms are universal	[64, 80]	proved	§12.3
34. The quantum logarithmic soft terms have the conjectured universal form	[80]	confirmed	§12.3
35. A network of (p, q) strings is a BPS configuration of type IIB	[122]	proved	§3.4
36. Every microstate of a single-centred BPS black hole carries zero angular momentum after the fermion zero modes are removed	[21]	confirmed	§5.3
H1. M-theory on T^n is the large- N limit of D0-branes on T^n for every n	[121]	heuristic	§12.4
H2. For $n \geq 6$ the matrix description is a theory without an independent definition	[121]	heuristic	§12.4

TABLE 4. The catalogue, claims 30 to 36 and the two heuristic statements, which receive no verdict.

Claim	What is left unproved	Where
1	$SL(2, \mathbb{Z})$ invariance beyond the protected quantities; it needs a nonperturbative definition of the heterotic string on T^6	§3.1
3	the L^2 cohomology of the k -monopole moduli space for every k and every charge sector	§3.3
9	positivity of the single-centred index in the CHL models, and the identification of the horizon states in the toroidal orbifolds	§5
10	a classification of the hair over the full string spectrum	§5.3
15	analytic lower-dimensional branes in superstring field theory	§6.2
17	the evolution of inhomogeneous tachyon matter through a caustic	§7.2
18	open–closed duality in critical string theory beyond leading order in g_s	§7.3
23	the identification of the weak coupling limit with a type IIB orientifold beyond its geometry	§9
26	the entropy of the complete heterotic action in $D \geq 6$, outside the curvature-only actions (19) for which the condition is $R_* = \alpha'$; it is not a claim of Sen, and the central charges do not supply it (Theorem 10.10)	Remark 10.9
27	a definition of the quantum entropy function beyond the saddle point expansion, and the three steps of its localisation	§11.1
29	a model-independent microscopic comparison for $\mathcal{N} = 2$ in a fixed ensemble	§11.2
32	the normalisation of D-instanton amplitudes in a general background, with the Ramond zero modes	§12.2
34	the one-loop exactness of the quantum $\ln \omega$ terms	§12.3
36	zero angular momentum of every single-centred microstate	§5.3
H1, H2	the matrix description for $n \geq 4$, which has no independent test, and for $n \geq 6$, where the limiting theory has no definition	§12.4

TABLE 5. What is left unproved, claim by claim; each entry is highlighted in yellow in the section named. The other claims are proved as stated. No statement of Sen examined here is wrong as printed or fatal under its hypotheses, so the rose and burnt orange highlights of the introduction do not occur.

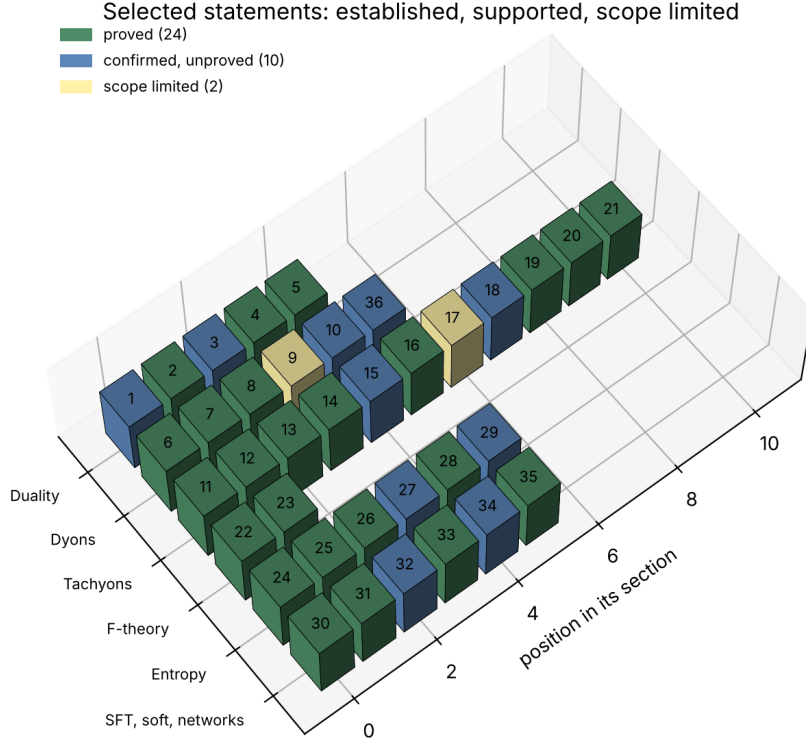


FIGURE 1. The thirty-six claims of Tables 1, 2, 3 and 4, one block per claim, grouped by the section of the paper that treats them and coloured by verdict: established in the stated framework, supported by cited tests, and scope limited. Colours describe statements, not authors.

subsequent tests of the conjecture are the three-string junction states [122, 95], the counting of quarter-BPS dyons (Section 4), whose partition function is a Siegel form invariant under the duality group, the match of the black hole entropy with the index (Section 10), and the equivalent statement for $\mathcal{N} = 4$ super-Yang–Mills theory, where the conjecture of Montonen and Olive [68] is tested by the same bound states [116, 75, 48, 65], by the modular properties of partition functions on four-manifolds [129], and by its embedding into the geometric Langlands programme [57].

Verdict 3.1 (claim 1). Confirmed. Every protected quantity that has been computed is consistent with the $SL(2, \mathbb{Z})$ symmetry, and the strongest test, the exact dyon index of Section 4, is a Siegel form of the duality group. **A proof of the invariance beyond the protected quantities is not known.** It would require a non-perturbative definition of the heterotic string on T^6 , which has no definition at strong coupling other than through such dualities, and the tests cited here do not constitute such a construction.

One point about the statement itself should be made precisely, because it is the kind of place where a conjecture can be “wrong” without its author being wrong. For gauge theories with a non-simply-laced gauge group the duality group of the $\mathcal{N} = 4$ theory is not $SL(2, \mathbb{Z})$: the inversion maps the gauge group G to its Langlands dual $G^\vee$, and the group generated by the shift and the inversion is a Hecke group $\Gamma_0(q)$ -type subgroup, with $q = 2$ or 3 [49, 31, 2]. This is not a correction of [115] or of [116]: both papers treat $SU(2)$ and the heterotic string, where the duality group is $SL(2, \mathbb{Z})$, and [116] is explicit that the test is for $SU(2)$. We mention it because the literature sometimes attributes an $SL(2, \mathbb{Z})$ conjecture “for all groups” to the 1994 papers; they contain no such statement.

3.2. The two-monopole bound state. The moduli space of two $SU(2)$ monopoles of total charge two is $\mathbb{R}^3 \times (S^1 \times \mathcal{M}_0)/\mathbb{Z}_2$, where $\mathcal{M}_0$ is the Atiyah–Hitchin manifold [3], a complete hyperkähler four-manifold with an $SO(3)$ action and a metric known in closed form (for the hypercomplex structures underlying such metrics see also [9]). The low energy dynamics of two monopoles is the supersymmetric quantum mechanics on this moduli space, and a bound state of two monopoles with one unit of electric charge and the right supermultiplet structure exists if and only if the Atiyah–Hitchin manifold carries a harmonic two-form that is square integrable, anti-self-dual (with the orientation fixed by the complex structures), odd under the $\mathbb{Z}_2$, and unique.³ Sen wrote down the form. In the coordinates of [3], with the metric $ds^2 = f^2 d\rho^2 + a^2 \sigma_1^2 + b^2 \sigma_2^2 + c^2 \sigma_3^2$ on left-invariant forms with $d\sigma_i = \frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k$, the form is

$$\begin{aligned} \omega &= F(\rho) \left(d\sigma_1 - \frac{fa}{bc} d\rho \wedge \sigma_1 \right), \\ F(\rho) &= F_0 \exp \left(- \int_{\pi}^{\rho} \frac{fa}{bc} d\rho' \right), \end{aligned} \tag{1}$$

which is anti-self-dual by the choice of the coefficient of $d\rho \wedge \sigma_1$, closed because $dF/d\rho = -(fa/bc)F$, and square integrable because $fa/(bc) \rightarrow \frac{1}{2}$ as $\rho \rightarrow \infty$ (where $f \simeq -1$, $a \simeq b \simeq \rho$, $c \simeq -2$) makes F decay as $e^{-\rho/2}$ while the volume grows polynomially; near the bolt $\rho \rightarrow \pi$, in the coordinates adapted to it, the form is smooth.⁴

Theorem 3.2 (Sen 1994; Hitchin 2000). *The space of L^2 harmonic forms on the Atiyah–Hitchin manifold is one-dimensional. It is spanned by the two-form (1), which is anti-self-dual and odd under the $\mathbb{Z}_2$ that acts on the moduli space of two monopoles.*

Proof. Existence, anti-self-duality, square integrability and the $\mathbb{Z}_2$ parity are the computation of [116] summarised above; we have checked each step. The uniqueness result is taken from [53]. We do not infer it merely by counting a circle fixed set: orientation and the particular cover of the monopole moduli space matter. $\square$

Verdict 3.3 (claim 2). Proved. The bound state predicted by $SL(2, \mathbb{Z})$ exists, and it is the only one in the two-monopole sector, which is also what the duality predicts: the $(1, 2)$ state must be a single short multiplet.

3.3. The Sen conjecture on L^2 cohomology. The argument of [116] applies to any charge. Let $\mathcal{M}_k^0$ denote the reduced moduli space of k monopoles, of real dimension $4k - 4$, and let $\mathcal{M}_k = \mathbb{R}^3 \times (S^1 \times \mathcal{M}_k^0)/\mathbb{Z}_k$. The states with k units of magnetic charge and p units of electric charge, with $\gcd(p, k) = 1$, correspond to L^2 harmonic forms on $\mathcal{M}_k^0$ transforming under the $\mathbb{Z}_k$ by $e^{2\pi i p/k}$; S -duality requires exactly one such form, in the middle degree $2k - 2$, for every p coprime to k , and none for other p (which would give states not in the duality orbit of the elementary string states). This is the *Sen conjecture*; Segal and Selby [84] proved that the image of the compactly supported cohomology in the ordinary cohomology, which is a lower bound for the L^2 cohomology, has exactly the predicted dimension in the middle degree and vanishes elsewhere. The conjecture is thus reduced to the statement that every L^2 harmonic form comes from compactly supported cohomology.

³The electric charge of the state is the momentum along the S^1 factor, and the $\mathbb{Z}_2$ acts on it; the state with one unit of electric charge requires the form on $\mathcal{M}_0$ to be odd under the $\mathbb{Z}_2$ which combines with the half shift on S^1 . The multiplet structure is fixed by the fermion zero modes: the state must be a single short multiplet of $\mathcal{N} = 4$, which is the statement that the harmonic form is unique and of a definite duality type. Sen’s paper carries this out in detail and we do not repeat it.

⁴Equations (16) to (19) of [116]. The form is invariant under the identification $(\rho, \theta, \phi, \psi) \sim (\rho, \pi - \theta, \pi + \phi, -\psi)$ that defines the Atiyah–Hitchin manifold and changes sign under $\psi \mapsto \psi + \pi$, which is the $\mathbb{Z}_2$ of the two-monopole moduli space; this is the parity required for odd electric charge. The uniqueness of the form among L^2 harmonic two-forms was argued in [116] from the symmetries; the complete statement that the space of L^2 harmonic forms of $\mathcal{M}_0$ is one-dimensional is Hitchin’s theorem below. Sen notes that the same form had appeared in work of Gibbons and Ruback for another purpose.

Theorem 3.4 (published partial results and their hypotheses). *The Segal–Selby image has the dimensions predicted by the conjecture [84]. Hitchin’s middle-degree result and the charge-two case are established in [53]. Fritzsche–Kottke–Singer [45, Theorem 1.2 and Section 6] state the coprime electric/magnetic charge case for arbitrary magnetic charge, using their metric compactification. Their introduction announces a separate complete proof of the asymptotic metric description. Kottke–Rochon [60, Theorem A] give the charge-three result provided the monopole metric has the announced quasi-fibered boundary structure. These hypotheses and dependencies are part of the statements, not additional results proved here.*

Proof. This paragraph records the cited results with their own hypotheses. In particular it does not reduce all progress for $k \geq 4$ to a lower bound from compactly supported cohomology: the coprime case is explicitly treated in Section 6 of [45]. Conversely, quoting Theorem A of [60] without its metric qualification would overstate that theorem. $\square$

Verdict 3.5 (claim 3). Confirmed, with substantial established and conditional cases. **The complete statement, for every k and in every charge sector, is not proved.** The non-coprime sector and the cited metric construction dependencies must be checked separately before asserting a general unconditional closure. These are mathematical questions rather than evidence that the original conjecture is false.

3.4. Duality-symmetric actions, the Kerr–Sen black hole and string networks. Three further claims of this period are theorems and we record them briefly.

Proposition 3.6 (claims 4, 5 and 35). *(i) Schwarz and Sen [83] wrote an action with two gauge potentials and a first-order Lagrangian; it is invariant under the duality rotation and has equations of motion equivalent to Maxwell’s equations; the same construction gives a duality-symmetric form of the heterotic effective action in four dimensions.*

(ii) The metric, dilaton, gauge field and axion of [114] solve the equations of motion of the low energy heterotic string in four dimensions; the solution is obtained from the Kerr solution by the $O(1,1)$ twisting transformation of the string background, which maps solutions to solutions.

(iii) A planar network of (p,q) -strings of type IIB with the charges conserved at each junction and the angles fixed by the tensions is supersymmetric, preserving one quarter of the supersymmetry. The proof is in [122] itself, from the supersymmetry algebra of the (p,q) -string; the three-string junction was shown to be a BPS state by [76] and, lifted to a holomorphic membrane in M-theory, by [67]. The statement was a theorem when it was made, and the catalogue records it as proved.

Proof. (i) is a direct computation in the cited paper: the two potentials $A^{(1)}, A^{(2)}$ enter through a first-order Lagrangian whose equations of motion are $F^{(2)} = *F^{(1)}$ up to gauge, so that eliminating $A^{(2)}$ recovers the Maxwell equations; the Lorentz invariance is not manifest but is verified. (ii) is the statement that the twisting transformation of [114], which acts on backgrounds with a Killing direction, is a symmetry of the equations of motion; this was proved for the full string effective action at lowest order in α' , and the Kerr–Sen metric is the image of Kerr. (iii) follows from the supersymmetry algebra: the BPS condition for a network is linear in the charge vectors at each junction, and the tension balance at a junction is the statement that the central charges add; [122] verifies this and [95] uses the same networks to count quarter-BPS dyons. For (iii), “proved” refers to the effective BPS configuration; existence and completeness of the quantum spectrum at finite coupling are separate physical questions. $\square$

4. QUARTER-BPS DYONS

4.1. The partition function and its product formula. Dijkgraaf, Verlinde and Verlinde [37] proposed that the index of quarter-BPS dyons of the heterotic string on T^6 , with electric

charge Q and magnetic charge P in the lattice $\Gamma^{6,22}$, is a Fourier coefficient of the inverse of the Igusa cusp form Φ_{10} of weight ten on $Sp(2, \mathbb{Z})$:

$$d(Q, P) = (-1)^{Q \cdot P + 1} \oint_{\mathcal{C}} d\rho d\sigma dv \frac{e^{-\pi i(\rho Q^2 + \sigma P^2 + 2v Q \cdot P)}}{\Phi_{10}(\rho, \sigma, v)}, \quad (2)$$

the contour $\mathcal{C}$ being a three-torus at fixed, large imaginary parts of (ρ, σ, v) , with the sign of $\text{Im } v$ determined by the asymptotic moduli [97].⁵ Jatkar and Sen [56] proposed the analogous formula for the CHL orbifolds $\mathbb{Z}_N$, $N = 2, 3, 5, 7$, with Φ_{10} replaced by a Siegel form Φ_k of weight $k = 24/(N + 1) - 2$ for a subgroup of $Sp(2, \mathbb{Z})$, and David, Jatkar and Sen [34, 35] derived the product representation

$$\begin{aligned} \tilde{\Phi}(\rho, \sigma, v) &= e^{2\pi i(\rho + \sigma/N + v)} \prod_{b=0}^1 \prod_{r=0}^{N-1} \prod_{\substack{k' \in \mathbb{Z} + r/N, \ l \in \mathbb{Z}, \ j \in 2\mathbb{Z} + b \\ k', l \geq 0; \ j < 0 \text{ if } k' = l = 0}} (1 - e^{2\pi i(k' \sigma + l \rho + j v)})^{c(k', l, j)}, \\ c(k', l, j) &= \sum_{s=0}^{N-1} e^{2\pi i s l / N} c_b^{(r, s)}(4k' l - j^2), \end{aligned} \quad (3)$$

from the twisted elliptic genera

$$F^{(r, s)}(\tau, z) = \sum_b \sum_{j \in 2\mathbb{Z} + b} \sum_{n \in \mathbb{Z}/N} c_b^{(r, s)}(4n - j^2) q^n y^j$$

of K3, by the method of [37]: the dyons are counted in the type IIB frame on $K3 \times T^2$ by the D1–D5 system bound to a Kaluza–Klein monopole, and the three factors of the partition function (the motion of the centre of mass, the Kaluza–Klein monopole and the D1–D5 world-volume theory) each contribute a product. For $N = 1$, (3) with the K3 elliptic genus $8A(\tau, z)$ is the Borcherds product for Φ_{10} .

Theorem 4.1 (claim 6). *For $N \in \{2, 3, 5, 7\}$ the function $\tilde{\Phi}_k$ defined by (3) is a Siegel modular form of weight k for the subgroup $\tilde{G} \subset Sp(2, \mathbb{Z})$ of [56], and it coincides with the form constructed by Borcherds lifting in [27] and with the genus-two partition function of [26].*

Proof. The modularity of the product and its identification with the additive lift are [27, Section 3], who show that (3) is the Borcherds lift of the twisted elliptic genus and that the lift is the Siegel form of [56]; the identification with the genus-two partition function of the heterotic string is [26]. For $N = 1$ the product is the classical Borcherds product of Φ_{10} [50]. We have checked (3) numerically for $N = 1, 2$ by reproducing the tables of [100], see Appendix A. $\square$

The physical content of the conjecture, that this Siegel form counts the quarter-BPS states of the string theory, is not a theorem and cannot be: it is a statement about the strong coupling regime, tested through the duality orbit. The tests are the agreement with the black hole entropy to the first two orders in the large charge expansion [19, 56, 28] and at the logarithmic order (Section 11), the wall-crossing behaviour described next, and the agreement at low charges with the string-junction counting [95].

4.2. Walls of marginal stability and the two-centred states. The Fourier coefficient (2) depends on the contour because $1/\tilde{\Phi}$ has double poles on the divisors $v = 0$ and its images under the duality group. Sen [97] identified the poles with the walls of marginal stability of the quarter-BPS dyon, on which it can decay into a pair of half-BPS states with charges

⁵The sign $(-1)^{Q \cdot P + 1}$ converts the Fourier coefficient into the helicity trace index B_6 of the four-dimensional theory; it was identified in [97, 96] (footnote 2 of [96] records that it was absent from the earlier papers) and traced to the change of statistics between the five- and four-dimensional descriptions in the presence of a Kaluza–Klein monopole. The index d of this paper is $-B_6$ in the notation of [100].

$(Q, 0)$ and $(0, P)$ (and the images under duality), and computed the jump of the index across the wall $v = 0$ from the residue of the double pole. Near $v = 0$,

$$\tilde{\Phi}(\rho, \sigma, v) = -4\pi^2 v^2 f(N\rho) g(\sigma/N) + O(v^4), \quad (4)$$

with $f(\tau)^{-1}$ and $g(\tau)^{-1}$ the generating functions of the half-BPS electric and magnetic states; for $N = 1$, $f = g = \eta^{24}$ and $d_h(Q) := \text{coefficient of } q^{Q^2/2} \text{ in } \eta(\tau)^{-24} = q^{-1} \prod_{m \geq 1} (1 - q^m)^{-24}$, so that $d_h = 1, 24, 324, 3200, 25650, \dots$ for $Q^2/2 = -1, 0, 1, 2, 3, \dots$.

Theorem 4.2 (claim 7; Sen 2007). *Let $N = 1$ and let $d_>(Q, P)$, $d_<(Q, P)$ denote the Fourier coefficients of (2) without the sign $(-1)^{Q \cdot P+1}$, taken over the contours with $\text{Im } v < 0$ and $\text{Im } v > 0$ respectively. Then*

$$d_>(Q, P) - d_<(Q, P) = -Q \cdot P d_h(Q) d_h(P). \quad (5)$$

In particular the index $d = (-1)^{Q \cdot P+1} d_>$ changes across the wall by $(-1)^{Q \cdot P} Q \cdot P d_h(Q) d_h(P)$, whose absolute value $|Q \cdot P| d_h(Q) d_h(P)$ is the index of a two-centred configuration of a half-BPS electric and a half-BPS magnetic centre with angular momentum $(|Q \cdot P| - 1)/2$.

Proof. The difference of the two contour integrals is the residue at $v = 0$. From (4),

$$1/\tilde{\Phi} = -(4\pi^2 v^2)^{-1} f(\rho)^{-1} g(\sigma)^{-1} (1 + O(v^2)),$$

and the residue of $e^{-2\pi i v Q \cdot P} v^{-2}$ at $v = 0$ is $-2\pi i Q \cdot P$; the remaining integrals over ρ and σ pick out $d_h(Q)$ and $d_h(P)$, and the factors of $2\pi i$ cancel against the normalisation of the contour. This is [96, eq. (2.13)]. The sign and the factor $|Q \cdot P|$ are those of a bound state of two centres carrying angular momentum $J = (|Q \cdot P| - 1)/2$ from the electromagnetic field, with $2J + 1 = |Q \cdot P|$ states [96, 30].

We have also verified (5) independently of the residue computation, by expanding the product (3) for $N = 1$ in exact arithmetic on both sides of the wall and comparing the coefficients: for every $Q^2, P^2 \in \{2, 4, 6\}$ and $|Q \cdot P| \leq 4$ the two expansions differ by exactly $-Q \cdot P d_h(Q) d_h(P)$ (Appendix A, Table 7). $\square$

Verdict 4.3 (claim 7). Proved. The formula is a residue computation, and the identification of the jump with the two-centred bound state is confirmed by the exact match of the degeneracy $|Q \cdot P| d_h(Q) d_h(P)$ with the quantisation of the two-centred solution [96], and by the general wall-crossing formulas of Denef and Moore [30] and of Kontsevich and Soibelman [62], of which (5) is the primitive case.

4.3. Negative discriminant states. The discriminant $\Delta = Q^2 P^2 - (Q \cdot P)^2$ must be positive for the regular classical single-centred solutions in the stated sector. The treatment of negative discriminant states uses chamber-dependent sums over two-centred decompositions and bound-state metamorphosis [99, 24]. A single primitive contribution has the form

$$(-1)^{\langle \Gamma_1, \Gamma_2 \rangle + 1} |\langle \Gamma_1, \Gamma_2 \rangle| d_h(\Gamma_1) d_h(\Gamma_2). \quad (6)$$

The full answer requires the allowed decompositions and their identifications. For example $(Q^2, P^2, Q \cdot P) = (2, 2, -2)$ has $\Delta = 0$, not negative discriminant; it must not be used as a negative-discriminant example. The identity $209952 - 648 = 209304$ is an arithmetic wall-crossing check, not a proof of a proposed microscopic decomposition. Similarly $327 = 3 \cdot 109$ does not identify 109 as a half-BPS degeneracy.

Theorem 4.4 (claim 8). *Every Fourier coefficient of $1/\Phi_{10}$ with negative discriminant is given by a finite sum of two-centred contributions of the form (6) over the duality images of the charges. The CHL generalisations require the separate hypotheses and formulas of [99]; no CHL formula is deduced solely from the Φ_{10} theorem.*

Proof. For Φ_{10} this is the main theorem of Chowdhury, Kidambi, Murthy, Reys and Wrase [24], who derive the exact formula for all polar (negative discriminant) coefficients from the mock modular decomposition of [36] and check it against 1650 coefficients; its direct scope

is the $K3 \times T^2$ theory associated with Φ_{10} , and it is not cited as a proof for every CHL form. The physical interpretation, that the states are two-centred, is [99]. $\square$

5. THE POSITIVITY CONJECTURE

5.1. The statement and its hypothesis. A single-centred supersymmetric black hole in four dimensions has a spherically symmetric horizon and carries no angular momentum. The helicity trace index B_6 of a quarter-BPS state is $-\frac{1}{6!} \text{Tr} [(-1)^{2h}(2h)^6]$, and for a configuration whose only degrees of freedom beyond the horizon are the twelve fermion zero modes of the broken supersymmetries, the trace evaluates to -1 times the number of horizon states; the sign is fixed and the index is negative, that is, $d = -B_6 > 0$. Sen [100] turned this into a conjecture about Fourier coefficients. In the region $\mathcal{R}$ of the axion–dilaton moduli space in which the attractor point of the charges (Q, P) lies, bounded by walls of marginal stability, the intended contribution is isolated under the exclusions described in the original argument, and the Fourier coefficient (2) taken with the contour of that region should be positive. For the heterotic string on T^6 the region is cut out by

$$\begin{aligned} Q \cdot P &\geq 0, & Q \cdot P &\leq Q^2, & Q \cdot P &\leq P^2, \\ Q^2 &> 0, & P^2 &> 0, & Q^2 P^2 - (Q \cdot P)^2 &> 0, \end{aligned} \quad (7)$$

(equation (3.7) of [100]), and for the $\mathbb{Z}_2$ CHL model, whose duality group is $\Gamma_1(2)$, by

$$\begin{aligned} Q^2 &> 0, & P^2 &> 0, & Q^2 P^2 &> (Q \cdot P)^2, \\ 2Q^2 + P^2 &\geq 3(Q \cdot P), & 2Q^2 &\geq Q \cdot P, \\ P^2 &\geq Q \cdot P, & Q \cdot P &\geq 0 \end{aligned} \quad (8)$$

(equation (3.9) there, with Q^2 normalised so that $NQ^2/2 \in \mathbb{Z}$). The fourth condition is linear in $Q \cdot P$: it is the image of the wall $Q \cdot P \geq 0$ under the second matrix of Sen’s equation (3.8), $(Q, P) \mapsto (-Q + P, -2Q + P)$, whose inner product is $2Q^2 - 3Q \cdot P + P^2$. Equations (3.25) and (4.7) of [16] print it as $2Q^2 + P^2 - 3(Q \cdot P)^2 \geq 0$; that squared form is a misprint of the same condition, and it would exclude, for instance, the triple $(Q^2, P^2, Q \cdot P) = (3, 2, 2)$ of Table 6. Sen verified the positivity of $d(Q, P)$ on these cones for the heterotic string and for the CHL models $N = 2, 3, 5, 7$ in the range of charges of his Tables 1 to 5, and stated the conjecture for all charges in the cone. We have reproduced his Tables 1 and 2 exactly (Appendix A).

The hypothesis of the argument is explicit in [100]: the conclusion requires “some duality frame in which only single centered black hole solution, whose only hair are the fermion zero modes associated with broken supersymmetry, contributes” to the index. In the heterotic frame the quarter-BPS black hole does have hair beyond the fermion zero modes (the hair modes identified in [12]), but there is a frame, the type IIB frame on $K3 \times T^2$, in which the hair reduces to the fermion zero modes; this is what makes the conjecture consistent there, and what was proved later.

Theorem 5.1 (Bringmann–Murthy 2013). *For type II string theory on $K3 \times T^2$ (equivalently the heterotic string on T^6 , the form $1/\Phi_{10}$) and on T^6 (the $\mathcal{N} = 8$ theory), the single-centred indices in the attractor region are positive for the classes of charges treated in [13]; the proof uses the mock modular decomposition of the Fourier–Jacobi coefficients of $1/\Phi_{10}$ [36].*

The restriction to classes of charges was removed by Rosselló [77], who bounds the tail of the Rademacher expansion of the mock Jacobi forms ψ_m^F of [36] against its leading term.

Theorem 5.2 (Rosselló 2024). *For type II string theory on $K3 \times T^2$, the single-centred index $d^*(m, n, \ell)$, with $m = P^2/2$, $n = Q^2/2$, $\ell = Q \cdot P$, is positive for every $m, n \geq 1$ with $4mn - \ell^2 > 0$: for $n \geq m$ and $0 \leq \ell \leq m$ one has $(-1)^{\ell+1} c_m^F(n, \ell) > 0$, and the general case follows by the S -duality invariance of the single-centred index.*

Verdict 5.3 (claim 9, heterotic and CHL). Proved for the heterotic string on T^6 , by Theorems 5.1 and 5.2; our scan of the single-centred cone up to $Q^2 \leq 12$, $P^2 \leq 6$, $Q \cdot P \leq 4$ (Appendix A) agrees. For the CHL orbifolds no counterexample is known, our scan for $N = 2$ finds only positive indices, while a general proof of positivity in the CHL models is not known: the Rademacher bound of [77] has not been carried out for the mock Jacobi forms of $1/\tilde{\Phi}_k$, and [77] names this as the next case. The conjecture is proved in the model in which it was first stated and confirmed in the CHL models.

5.2. Full and horizon indices in the toroidal orbifolds. Sen’s paper states the conjecture as a consequence of the black hole argument for “the $\mathcal{N} = 4$ supersymmetric string theories”, and the subsequent literature [16, 24] tested it in the full list of $\mathcal{N} = 4$ compactifications with known dyon partition functions. Chattopadhyaya and David [16] found that it fails in two of them: the freely acting $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifolds of type IIB on T^6 , in which the orbifold acts by a reflection of four coordinates of T^4 (or a rotation by $2\pi/3$ of two planes) together with a shift of half (or a third) of a circle of the remaining T^2 . The $\mathbb{Z}_2$ model has the twisted elliptic genera

$$\begin{aligned} F^{(0,0)} &= 0, & F^{(0,1)} &= \frac{8}{3}A - \frac{4}{3}B E_2(\tau), \\ F^{(1,0)} &= \frac{8}{3}A + \frac{2}{3}B E_2(\frac{\tau}{2}), & F^{(1,1)} &= \frac{8}{3}A + \frac{2}{3}B E_2(\frac{\tau+1}{2}), \end{aligned} \quad (9)$$

where $A = \sum_{i=2}^4 \theta_i(\tau, z)^2 / \theta_i(\tau, 0)^2$, $B = \theta_1(\tau, z)^2 / \eta(\tau)^6$, and $E_2(\tau) := 2E(2\tau) - E(\tau)$ is the weight-two form of $\Gamma_0(2)$ built from the Eisenstein series $E(\tau) = 1 - 24 \sum \sigma_1(n)q^n$ (this is the E_N of [34]; it is not the quasi-modular Eisenstein series, and the two are easily confused). The Siegel form is $\tilde{\Phi}_2 = \tilde{\Phi}_6^2 / \tilde{\Phi}_{10}$, of weight two, with the prefactor $e^{2\pi i(\rho+v)}$ in place of $e^{2\pi i(\rho+\sigma/N+v)}$ in (3), and the index is given by (2) with $Q^2 \in \mathbb{Z}$, $P^2 \in 2\mathbb{Z}$. The single-centred region is again cut out by (8), since the duality group is again $\Gamma_1(2)$.

Theorem 5.4 (known negative full indices; finite exact test). *In the $\mathbb{Z}_2$ toroidal orbifold of type IIB on T^6 , the index $d(Q, P)$ computed from (2) with the contour of the attractor region is negative for infinitely many charges satisfying the single-centred conditions (8). In the range $Q^2 \leq 12$, $P^2 \leq 8$, $Q \cdot P \leq 4$ there are exactly thirty such charge triples, listed in Table 6; the first ones are*

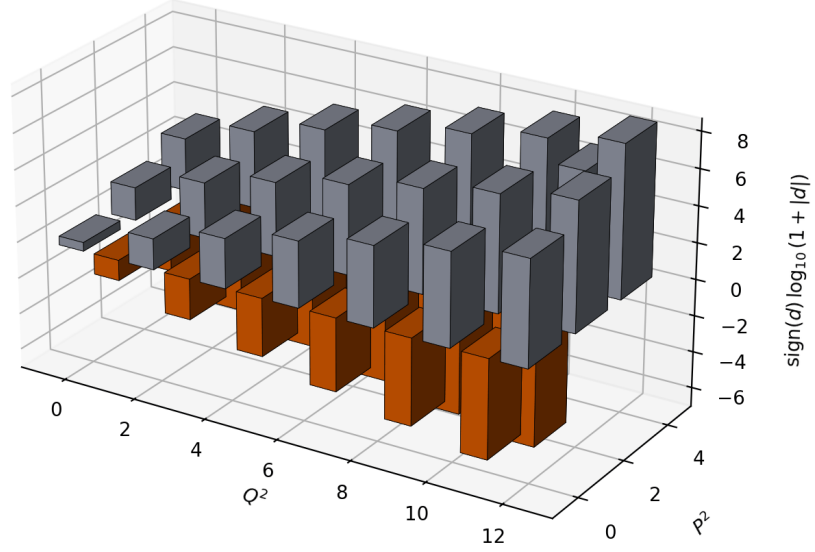
$$\begin{aligned} d(1, 2, 0) &= -224, & d(1, 4, 0) &= -1248, \\ d(2, 2, 1) &= -256, & d(3, 2, 0) &= -3392, \\ d(3, 4, 0) &= -10320, \end{aligned}$$

in the notation $d(Q^2, P^2, Q \cdot P)$. In the first column of Table 8 ($P^2 = 0$, outside the cone) the sign is $(-1)^{Q^2}$ for $Q^2 \leq 12$; Figure 2 draws the table.

Proof. We expanded the product (3) for the genera (9) in exact rational arithmetic, with the Fourier coefficients of the theta functions, of η^{-6} and of the Eisenstein series computed from their definitions, and read off the coefficients in the convention $q_\rho = e^{2\pi i\rho}$, $q_\sigma = e^{2\pi i\sigma/N}$ and $y = e^{2\pi iv}$, keyed by $(NQ^2/2, P^2/2, Q \cdot P)$ in the expansion in which the factor $(1 - e^{-2\pi iv})^{-2}$ at $k' = l = 0$ is expanded in powers of $e^{-2\pi iv}$ (the attractor region for $Q \cdot P \geq 0$). The resulting tables agree entry by entry with Tables 17, 18 and 19 of [16] in their common range. The thirty charge triples of Table 6 satisfy each inequality of (8) by inspection and have negative index. For the infinitude: the values at $P^2 = 2$, $Q \cdot P = 0$ are

$$\begin{aligned} &64, -224, 1152, -3392, 11520, -30336, 83968, \\ &-202560, 496512, -1118496, 2521600, -5374656 \end{aligned}$$

for $Q^2 = 0, 1, 2, \dots, 11$; every odd Q^2 with $P^2 = 2$, $Q \cdot P = 0$ lies in the cone (8), and [16, Section 4.1] show from the Fourier–Jacobi coefficient ψ_1 of $1/\tilde{\Phi}_2$ at $P^2 = 2$ that the finite part of ψ_1 , which is what the single-centred index is, has coefficients of alternating sign in

Full dyon index $d(Q^2, P^2, 0)$ of the $\mathbb{Z}_2$ toroidal orbifold

burnt orange: negative full index; horizon index is a separate quantity; grey: positive

FIGURE 2. The full index $d(Q^2, P^2, 0)$ of the $\mathbb{Z}_2$ toroidal orbifold from Table 8, drawn on the scale $\text{sign}(d) \log_{10}(1 + |d|)$. Every bar in burnt orange is a negative full index; along $P^2 = 2$ the sign alternates with Q^2 , and every odd Q^2 lies inside the cone (8) of Theorem 5.4. A horizon index requires the separate hair analysis.

Q^2	P^2	$Q \cdot P$	d	Q^2	P^2	$Q \cdot P$	d
1	2	0	-224	6	2	1	-33536
1	4	0	-1248	7	2	0	-202560
1	6	2	-224	7	2	2	-30336
1	8	2	-1248	7	4	0	-179022
2	2	1	-256	7	4	4	-10320
2	6	3	-256	8	2	1	-220672
3	2	0	-3392	9	2	0	-1118496
3	2	2	-224	9	2	2	-202560
3	4	0	-10320	9	4	0	-155232
3	6	4	-224	9	4	4	-55424
4	2	1	-3840	10	2	1	-1204992
5	2	0	-30336	11	2	0	-5374656
5	2	2	-3392	11	2	2	-1118496
5	4	0	-55424	11	4	4	-179022
5	4	4	-1248	12	2	1	-5741824

TABLE 6. The thirty negative full-index triples in the linear cone (8), for $0 < Q^2 \leq 12$, $0 < P^2 \leq 8$ and $0 \leq Q \cdot P \leq 4$. The exact Python scan covers this entire box. The separate Lean certificate covers the original twenty-one listed triples with $P^2 \leq 4$ and $Q \cdot P \leq 2$; it does not certify all thirty entries.

Q^2 ; the alternation is visible in our table for $Q^2 \leq 12$. The analogous statement for the $\mathbb{Z}_3$ orbifold is [16, Section 4.2]. $\square$

5.3. Which hypothesis fails. The sign argument requires assumptions about horizon states, hair and other contributions to the trace. The toroidal models of [16] are examples where a negative full index calls for a separate hair analysis. The proposed hair factors and finite positive horizon-index tests of [17] address this distinction. They are not used here as a proof that all horizon indices in all models are positive.

Proposition 5.5 (conditional scope of the sign argument). *For a charge with negative full index, the conjunction of the assumptions leading to a positive full index cannot hold. If the horizon degeneracy is positive and the only external contribution is the universal fermion-zero-mode factor with its prescribed sign, positivity follows. A negative full index therefore rules out this conjunction; by itself it does not identify which physical assumption fails or prove the nonexistence of every conceivable duality frame.*

Proof. The last implication is the contrapositive of the sign argument. Removing hair changes the generating function, not a numerical overall convention. In the toroidal models the proposed hair partition functions and finite positive horizon-index tests are given in [17]. Their interpretation is evidence for nontrivial hair, rather than a classification of all frames proved here. $\square$

Verdict 5.6 (claim 9, generalisation across models). Scope limited. Negative full indices in the toroidal orbifolds are known results of [16]. They test an unrestricted extension of positivity, while [100] explicitly states its hair and single-centre assumptions. The later analysis [17] already addresses the horizon/full-index distinction. We therefore do not count this as a newly discovered false proof by Sen. The computations here reproduce and extend a finite table; the physical identification of horizon states remains subject to the cited assumptions.

Verdict 5.7 (claim 10). Confirmed. The hair modes of the quarter-BPS black hole were identified in the type IIB frame for $K3 \times T^2$ in [12] and for every CHL model and both toroidal orbifolds in [17], and in each case the remaining horizon index is positive. The identification rests on the classification of the supersymmetric deformations of the black hole solution that are supported outside the horizon, within the effective descriptions used in those papers; a classification of the hair over the full string spectrum is not established.

Verdict 5.8 (claim 36). Confirmed. The companion of the positivity conjecture is the statement of [21] that every microstate of a single-centred BPS black hole, once the fermion zero modes of broken supersymmetry are removed, carries zero angular momentum, so that the index counts the states with sign +1 and positivity follows. It was tested in [21] on the quarter-BPS states of type IIB on T^6 at low charges by an explicit enumeration of the supersymmetric configurations of D-branes, and in [51] on multi-centred configurations with pure D-brane charges, and holds in every case computed. It is not proved, and the toroidal orbifolds show what its proof must contain: there the index of the full dyon is negative, so either some microstates carry angular momentum or the hair modes do, and [17] locates the sign in the hair. Later tests at higher charges [22, 23] reproduce the index of the pure D-brane system by algebraic geometry and find no state of nonzero angular momentum. Positivity of the index, now a theorem for $K3 \times T^2$ (Theorem 5.2), does not imply the statement, as [77] notes: an index that is positive can still count fermionic states. The sources examined here do not supply a general proof; the statement for the horizon states remains open and consistent with every test.

Remark 5.9. The numbers of Theorem 5.4 are the index $d = -B_6$ with the sign $(-1)^{Q \cdot P + 1}$ included and without the factor $1/N$ written in front of the contour integral in [16]: the integral over σ runs over a period of length N and produces a factor N that cancels it. The same normalisation reproduces Sen’s Table 2 for the CHL model, which fixes it.

6. THE TACHYON CONJECTURES

6.1. Universality. An unstable D-brane of bosonic string theory, or a brane–antibrane pair or non-BPS brane of type II, carries an open string tachyon. Sen’s conjectures about it [119, 125] are three: the tachyon potential has a minimum at which the negative energy density exactly cancels the tension of the brane; the lump and kink solutions of the tachyon field are the D-branes of lower dimension; and around the minimum there are no perturbative open string states, since there is no brane for them to end on. The first step towards testing them was the observation [126] that the tachyon potential is universal.

Theorem 6.1 (claim 11; Sen 1999). *In the cubic open string field theory of Witten on a D-brane described by a boundary conformal field theory BCFT_0 , the subspace $\mathcal{H}_{\text{univ}}$ of the string field spanned by the states obtained from the $SL(2, \mathbb{R})$ -invariant vacuum by the matter Virasoro generators and the ghost oscillators is closed under the star product and under the BRST operator, and the action restricted to it is $\mathcal{T}_p V_{\text{univ}}$, where $\mathcal{T}_p$ is the tension of the brane and V_{univ} does not depend on BCFT_0 . The same holds for the Berkovits field theory of the brane–antibrane pair and of the non-BPS brane, with the GSO-odd and GSO-even sectors.*

Proof. This is [126, Sections 2 and 3]. The matter part of the action involves only correlators of the matter Virasoro generators and the identity on the disc, which are fixed by the central charge $c = 26$ (or $c = 15$) and the conformal Ward identities, and the ghost part is independent of the background; the normalisation of the disc partition function is the tension. Closure of $\mathcal{H}_{\text{univ}}$ under the star product follows from the fact that the star product is a conformal map of three half-discs into one, which maps Virasoro descendants of the identity to Virasoro descendants of the identity. The equations of motion restricted to $\mathcal{H}_{\text{univ}}$ are therefore the equations of motion in it, and a solution in $\mathcal{H}_{\text{univ}}$ is a solution of the full theory. $\square$

Verdict 6.2 (claim 11). Proved. The universality is used in every subsequent analytic solution: Schnabl’s tachyon vacuum lies in $\mathcal{H}_{\text{univ}}$, and so does the solution of Erler in the superstring.

6.2. The three conjectures. Let Ψ be the open string field of the cubic theory on a Dp-brane with action $S = -\frac{1}{g^2} [\frac{1}{2} \langle \Psi, Q_B \Psi \rangle + \frac{1}{3} \langle \Psi, \Psi * \Psi \rangle]$, normalised so that the tension of the brane is $\mathcal{T}_p = 1/(2\pi^2 g^2)$.

Theorem 6.3 (claims 12, 13, 14). *(i) There is a solution $\Psi_0 \in \mathcal{H}_{\text{univ}}$ of the equation of motion $Q_B \Psi + \Psi * \Psi = 0$ whose action is $S[\Psi_0] = \mathcal{T}_p \text{Vol}$, that is, whose energy density is $-\mathcal{T}_p$ [81].*

(ii) The kinetic operator around Ψ_0 ,

$$Q_{\Psi_0} \Phi = Q_B \Phi + \Psi_0 * \Phi - (-1)^{|\Phi|} \Phi * \Psi_0,$$

has trivial cohomology in every ghost number: there is a string field A with $Q_{\Psi_0} A = 1$, the identity of the star algebra [39].

(iii) For every boundary conformal field theory BCFT_ with the same bulk theory as BCFT_0 and admitting boundary condition changing operators between the two of weight zero with regular operator products, there is a solution Ψ_* of the equation of motion on the Dp-brane whose energy is that of BCFT_* and whose fluctuation spectrum is that of BCFT_* ; the tachyon lump describing a $D(p-1)$ -brane at any compactification radius is of this form [41].*

Proof. (i) is the solution of [81] in the gauge $\mathcal{B}_0 \Psi = 0$, with the energy computed there and confirmed by [72] and by [46], who checked that the equation of motion holds when contracted with the solution itself; the simpler solution of [42], $\Psi = \frac{1}{\sqrt{1+K}} c(1+K) Bc \frac{1}{\sqrt{1+K}}$ in the KBc algebra, has the same energy by a two-line computation. (ii) is [39]: for the Erler–Schnabl solution the homotopy operator is $A = B/(1+K)$, and $Q_{\Psi} A = 1$ is an identity

of the KBc algebra. (iii) is [41, Section 4]: the solution is $\Psi_* = \Psi_{\text{tv}} - \Sigma \Psi_{\text{tv}} \bar{\Sigma}$ with $\Sigma, \bar{\Sigma}$ built from the boundary condition changing operators and a timelike Wilson line, and $\bar{\Sigma}\Sigma = 1$; the energy is the disc partition function of BCFT_* and the cohomology of Q_{Ψ_*} is isomorphic to that of BCFT_* . The hypothesis on the boundary condition changing operators is satisfied by the lump at every radius, by the Wilson line construction of the same paper; the numerical lumps of [63] agree. $\square$

Verdict 6.4 (claims 12, 13, 14). Proved. Each statement is a theorem of cubic open string field theory; the sense in which it is a theorem about string theory depends on cubic string field theory being a complete description of the open string, which is the framework of the conjectures and is not in question in them.

Theorem 6.5 (claim 15; Erler 2013). *In the open superstring field theory of Berkovits on a non-BPS D-brane there is an analytic solution for the tachyon vacuum, built from 2×2 matrices in two GL_2 subgroups of the star algebra, whose action equals the tension of the non-BPS brane, so that the energy density of the vacuum is $-\mathcal{T}_p$.*

Proof. [40]. The energy is computed in closed form from the nonpolynomial action. $\square$

Verdict 6.6 (claim 15). Proved for the energy. The absence of open string states around the superstring vacuum follows from the homotopy operator of the cubic reformulation of the same solution, and **the lower-dimensional branes in the superstring are not yet covered by an analytic solution**; the numerical results of [15], the level truncation to level (2, 4) with 85% of the tension, were the state of the conjecture before [40]. The energy, fluctuation cohomology, and lower-brane solutions are separate assertions; no comprehensive negative claim about all later constructions is made here.

7. THE ROLLING TACHYON AND TACHYON MATTER

7.1. Homogeneous decay. Sen [88] described the time-dependent decay of an unstable Dp -brane by the boundary conformal field theory with the marginal deformation $\lambda \int \cosh X^0$ of the free boson X^0 on the boundary of the world-sheet, and computed the energy-momentum tensor from the boundary state. In the bosonic string, for $0 < \lambda \leq \frac{1}{2}$,

$$\begin{aligned} T_{00} &= \frac{1}{2} \mathcal{T}_p (1 + \cos 2\pi\lambda), & T_{ij} &= -\delta_{ij} \mathcal{T}_p f(x^0), \\ f(x^0) &= \frac{1}{1 + e^{x^0} \sin \pi\lambda} + \frac{1}{1 + e^{-x^0} \sin \pi\lambda} - 1, \end{aligned} \tag{10}$$

and in the superstring the same with x^0 replaced by $\sqrt{2}x^0$ and $\sin \pi\lambda$ by $\sin^2 \pi\lambda$.

Theorem 7.1 (claim 16; Sen 2002). *The energy density (10) is conserved, and the pressure $-\mathcal{T}_p f(x^0)$ tends to zero as $x^0 \rightarrow \pm\infty$, as $e^{-|x^0|}$ in the bosonic string and $e^{-\sqrt{2}|x^0|}$ in the superstring. The endpoint $\lambda = 0$ is the undeformed brane with constant nonzero pressure and is excluded from the decay assertion. For $\lambda = \frac{1}{2}$ the energy density vanishes and the solution is the tachyon vacuum at all times.*

Proof. The boundary state is the product of the Dirichlet boundary state for the spatial directions, the $c = 1$ boundary state of the $\cosh X^0$ deformation, obtained by Wick rotation of the $SU(2)$ rotation of the Neumann boundary state at the self-dual radius, and the ghost boundary state; $T_{\mu\nu}$ is read off from the coefficient of the closed string graviton. As $x^0 \rightarrow +\infty$ the first term of f is $O(e^{-x^0})$ and the second is $1 - \sin \pi\lambda e^{-x^0} + O(e^{-2x^0})$, so $f = O(e^{-x^0})$; the case $x^0 \rightarrow -\infty$ is symmetric. Conservation is $\partial_0 T_{00} = 0$, immediate from the form of T_{00} . This is [88, Section 3]; the computation is exact in α' and at tree level in the string coupling. $\square$

Verdict 7.2 (claim 16). Proved. The statement is a theorem of boundary conformal field theory. The open question is its physical scope, treated in Section 7.3.

7.2. Tachyon matter and the caustic theorem. Sen [87] called the asymptotic state of (10), with constant energy density and vanishing pressure, *tachyon matter*, and [86] proposed an effective field theory for the tachyon near the minimum of its potential whose solutions are this matter:

$$S = - \int d^{p+1}x V(T) \sqrt{1 + \eta^{\mu\nu} \partial_\mu T \partial_\nu T}, \quad (11)$$

$$\eta = \text{diag}(-1, 1, \dots, 1), \quad V(T) \rightarrow 0 \text{ as } T \rightarrow \infty.$$

The late-time pressure statement in [87] concerns the homogeneous rolling solution. The field theory in [86] also has the exact dust solution $T = t$, $\Pi = f(x)$, with arbitrary time-independent nonnegative density f . Thus a nonconstant density alone does not imply focusing. Caustics arise from a compressive velocity gradient. The calculation below determines when that happens in the pressureless equations; it does not refute the existence or the homogeneous asymptotics claimed in those papers.

Near the minimum, $V \rightarrow 0$, the equation of motion of (11) forces $\eta^{\mu\nu} \partial_\mu T \partial_\nu T \rightarrow -1$ wherever the energy density stays finite [86, Section 2], and the energy-momentum tensor takes the form

$$T^{\mu\nu} = \rho u^\mu u^\nu, \quad u^\mu = -\eta^{\mu\nu} \partial_\nu T, \quad u^\mu u_\mu = -1, \quad (12)$$

of a pressureless fluid with velocity u^μ and rest density $\rho = V(T)/\sqrt{1 + \partial T \cdot \partial T} \geq 0$. We write $W = \partial_t T = \sqrt{1 + |\nabla T|^2}$ for the branch in which T increases with time (the other branch is obtained by $T \mapsto -T$), $v = -\nabla T/W$ for the three-velocity u^i/u^0 of the fluid, and $\varepsilon = T^{00} = \rho W^2$ for the energy density.

Lemma 7.3. *Let T be a C^2 solution of $\eta^{\mu\nu} \partial_\mu T \partial_\nu T = -1$ on $[0, t_1) \times \mathbb{R}^p$ with $\partial_t T > 0$, and let $\rho \geq 0$ be C^1 with $\partial_\mu(\rho u^\mu) = 0$. Then:*

- (i) $u^\mu \partial_\mu u^\nu = 0$: *the integral curves of u , equivalently the curves $\dot{X} = v(t, X)$, are straight lines $X(t; x_0) = x_0 + t v_0(x_0)$, $v_0 = v(0, \cdot)$, along which v is constant;*
- (ii) $\partial_t \varepsilon + \partial_i(\varepsilon v^i) = 0$, *and along the lines of (i)*
 $\varepsilon(t, X(t; x_0)) \det(I + t Dv_0(x_0)) = \varepsilon(0, x_0);$
- (iii) $Dv_0 = -W^{-1} P H$ *with $H = \nabla^2 T(0, \cdot)$ the Hessian of the initial profile and $P = I - \nabla T \nabla T^\top / W^2$ symmetric and positive definite; the eigenvalues of $Dv_0(x_0)$ are real, and $Dv_0(x_0)$ has a negative eigenvalue if and only if $H(x_0)$ has a positive eigenvalue.*

Proof. (i) Since $u_\mu = -\partial_\mu T$ is a gradient,

$$u^\mu \partial_\mu u_\nu = u^\mu \partial_\nu u_\mu = \frac{1}{2} \partial_\nu (u^\mu u_\mu) = 0.$$

In components, with $u^0 = W$ and $u^i = W v^i$, the equations $u^\mu \partial_\mu W = 0$ and $u^\mu \partial_\mu (W v^i) = 0$ give $(\partial_t + v \cdot \nabla)v = 0$, so v is constant along the curves $\dot{X} = v$, which are therefore straight. (ii) $\partial_\mu(\rho u^\mu u^0) = u^0 \partial_\mu(\rho u^\mu) + \rho u^\mu \partial_\mu u^0 = 0$ is the continuity equation for $\varepsilon = \rho W^2$ with flux εv . Writing it as $(\partial_t + v \cdot \nabla)\varepsilon = -\varepsilon \nabla \cdot v$ and using

$$\nabla \cdot v(t, X(t; x_0)) = \frac{d}{dt} \ln \det(I + t Dv_0(x_0)),$$

which holds because the Jacobian $D_{x_0} X = I + t Dv_0$ satisfies $\frac{d}{dt} D_{x_0} X = (Dv)(D_{x_0} X)$, gives

$$\frac{d}{dt} [\varepsilon \det(I + t Dv_0)] = 0$$

along the line. (iii) Differentiating $v^i = -\partial_i T/W$ and $W^2 = 1 + |\nabla T|^2$ gives $\partial_j v^i = -W^{-1}(\delta_{ik} - \partial_i T \partial_k T / W^2) \partial_k \partial_j T$. The matrix P has eigenvalue 1 on the orthogonal complement of ∇T and $1 - |\nabla T|^2 / W^2 = W^{-2} > 0$ along it. PH is similar to $P^{1/2} H P^{1/2}$, which is symmetric, so its eigenvalues are real, and Sylvester's law gives the same inertia as H . The additional negative scalar in $Dv_0 = -W^{-1} P H$ reverses positive and negative signs. $\square$

Theorem 7.4 (forward focusing in the dust equations). *Let $T_0 \in C^3(\mathbb{R}^p)$, $v_0 = -\nabla T_0 / \sqrt{1 + |\nabla T_0|^2}$, and ε_0 be a positive C^1 initial density. Put $A(x) = Dv_0(x)$ and assume A is bounded. Its eigenvalues are real. Define*

$$M = \sup_x \max(\{0\} \cup \{-\lambda : \lambda \in \text{spec } A(x)\}),$$

$$t_* = \begin{cases} 1/M & M > 0, \\ +\infty & M = 0. \end{cases} \quad (13)$$

Then the following statements hold.

- (i) *The characteristic map is $X_t(x) = x + tv_0(x)$, with $DX_t = I + tA(x)$. Before its first loss of invertibility,*

$$\varepsilon(t, X_t(x)) = \frac{\varepsilon_0(x)}{\det(I + tA(x))}.$$

For $0 \leq t < t_$ the map is a global diffeomorphism.*

- (ii) *If T_0 is concave, all eigenvalues of A are nonnegative; no forward focal point occurs and the density along each characteristic is nonincreasing. If T_0 has a positive Hessian eigenvalue at some point, a forward focal time is finite on that characteristic.*
- (iii) *If $0 < M < \infty$ is attained at x_* , the first singular Jacobian occurs at $t_* = 1/M$. Its determinant tends to zero from above along x_* and $\varepsilon(t, X_t(x_*)) \rightarrow +\infty$. Without attainment, t_* is only the infimum of the local focal times; a singular characteristic at exactly t_* and a density blow-up there are not asserted.*
- (iv) *In one dimension, $v'_0 = -T''_0 / (1 + T_0'^2)^{3/2}$. Hence compression occurs precisely where $T''_0 > 0$. Every bounded nonconstant C^3 profile on $\mathbb{R}^p$ has a positive Hessian eigenvalue somewhere and therefore a finite local focal time. A periodic profile also attains its finite maximum compression.*

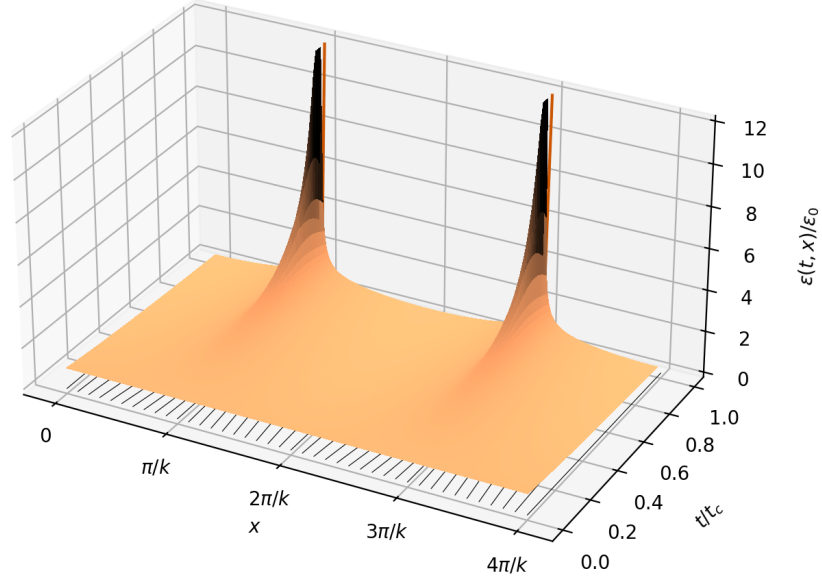
Proof. Lemma 7.3 gives (i) locally. The similarity of PH to the symmetric matrix $P^{1/2}HP^{1/2}$ and the factor $-W^{-1}$ show that a negative eigenvalue of A corresponds to a positive eigenvalue of H , proving (ii). For $t < t_*$ every eigenvalue factor $1 + t\lambda$ is positive. Also $|v_0| \leq 1$, so $|X_t(x)| \geq |x| - t$ and X_t is proper. A proper local diffeomorphism $\mathbb{R}^p \rightarrow \mathbb{R}^p$ is a covering onto the connected target: its image is both open and closed. Since the target is simply connected and the source connected, it is a global diffeomorphism. This proves the global assertion in (i).

At an attained maximiser, some $\lambda = -M$ and all remaining factors are positive for $t < t_*$. Their product approaches zero, while the numerator stays positive, proving (iii). If no maximiser exists, this argument only gives a sequence of local focal times approaching t_* ; passing to a fixed singular characteristic would be unjustified. For (iv), differentiate the displayed velocity. A C^2 function with no positive Hessian eigenvalue anywhere is concave. A concave function on all of $\mathbb{R}^p$ bounded below is constant, by restriction to lines; hence a bounded nonconstant profile cannot have that property. Periodicity supplies compactness for attainment. The dust theory loses a classical single-valued velocity field at focusing; higher-derivative or string effects at that point are outside this theorem. $\square$

Verdict 7.5 (claim 17). Scope limited. Homogeneous rolling has the pressureless asymptotic in the original papers. Inhomogeneous effective dust may develop caustics, as already analysed in [44, 43, 4]. The theorem above states the sign criterion and attainment conditions precisely. It does not disprove Sen's homogeneous result, and the string-theoretic continuation through a caustic is not known. The stationary solution $T = t$, $\Pi = f(x)$ remains a counterexample to any assertion that every spatially varying density must focus.

7.3. Closed string emission and open-closed duality. The decaying brane couples to closed strings, and the question whether the classical open string description of Theorem 7.1 survives their emission was raised in [88] and answered in part by Lambert, Liu and Maldacena [66]: the energy radiated into closed strings by a decaying Dp -brane is infinite at

Tachyon dust from the datum $T_0 = T_* + a \cos kx$: density along the characteristics



the characteristics (floor) converge on the minima of T_0 and the density diverges at $t_c = 1/(ak^2)$

FIGURE 3. Exact relativistic dust characteristics and density for $T_0 = T_* + a \cos kx$, $a = 1/2$, $k = 1$, before $t_c = 1/(ak^2)$. The velocity is $ak \sin(kx)/\sqrt{1 + a^2 k^2 \sin^2(kx)}$. The surface is capped for display; the formula is not capped.

leading order for $p \leq 2$ and finite for $p > 2$; for $p = 0$ the divergence, cut off at energies of order $1/g_s$, carries away an energy of the order of the mass of the brane, which signals the need to control backreaction before inferring an exact energy partition. Sen [90] turned this into the conjecture of *open–closed duality at tree level*: the tree-level open string theory on the decaying brane, with its tachyon matter, is a dual description of the closed string state produced by the decay, so that the two are not to be added.

Verdict 7.6 (claim 18). Confirmed. The conjecture is supported by the computation of [66], in which the closed string state produced is determined by the open string boundary state and its energy, for $p = 0$, is the energy of the brane; by the matching of the energy-momentum tensor of the emitted closed strings with that of tachyon matter in the regime where both can be computed [90]; and by the two-dimensional string theory, where the decay of the D0-brane is the motion of a single fermion of the matrix model and the open and closed descriptions are provably the same [91]. **A general proof in critical string theory is not known**; the missing step is a control of the backreaction of the emitted closed strings on the open string dynamics beyond leading order in g_s . The conjecture is stated for homogeneous decay. In the inhomogeneous case, beyond the caustic time t_c of Theorem 7.4 the single-valued dust description fails; whether a classical open string field theory description continues is not decided by that theorem, and is the open question of claim 17.

8. NON-BPS BRANES AND THE DESCENT RELATIONS

The tachyon conjectures of Section 6 have a static counterpart: the solitons of the tachyon field are themselves D-branes. Sen [119] identified the vortex on a $Dp\text{--}\overline{D}p$ pair of type II

with the BPS $D(p-2)$ -brane, [123] identified the kink on a $D1-\bar{D}1$ pair of type I with a stable non-BPS D-particle transforming as a spinor of $SO(32)$, the state required by the duality with the heterotic string, and [120, 125] organised these into the *descent relations*, which have a bosonic analogue [124]: the non-BPS Dp -brane of type II is the orbifold of the $Dp-\bar{D}p$ pair by $(-1)^{F_L}$, and the BPS $D(p-1)$ -brane is the kink on the non-BPS Dp -brane.

Theorem 8.1 (claims 19, 20). *(i) Let a $Dp-\bar{D}p$ pair of type II be wrapped on a circle of radius R (in units $\alpha' = 1$), with the Wilson line -1 on one of the two branes, so that the tachyon has half-integer momentum and its lowest modes have $m^2 = 1/(4R^2) - 1/2$. At the critical radius $R = 1/\sqrt{2}$ the tachyon kink is an exactly marginal boundary deformation, and the boundary conformal field theory at the end of the deformation is that of a non-BPS $D(p-1)$ -brane. On a non-BPS Dp -brane wrapped on a circle of radius R the tachyon is periodic, with $m^2 = n^2/R^2 - 1/2$. At $R = \sqrt{2}$ its first harmonic is exactly marginal, the deformation ends at a BPS $D(p-1)$ -brane and an anti- $D(p-1)$ -brane at diametrically opposite points of the circle, the kink-antikink pair, and the kink is the BPS $D(p-1)$ -brane.*

(ii) The conserved charges of D-branes and of their tachyon solitons are classified by the K-theory of spacetime: $K(X)$ for type IIB, $K^{-1}(X)$ for type IIA, $KO(X)$ for type I; the kink on the type I $D1-\bar{D}1$ pair carries the class generating $KO^{-1}(\text{pt}) = \mathbb{Z}_2$ and is the $SO(32)$ spinor, stable and non-BPS.

Proof. (i) is the construction of [123, Eqs. (2.13)–(2.14)] and [125, Appendix A] for the pair and of [120] for the non-BPS brane: at the critical radius the tachyon modes have the vertex operators $\cos(X/\sqrt{2})$ and $\sin(X/\sqrt{2})$, with the Chan–Paton factor σ_1 for the pair, which are exactly marginal; the deformation by it can be followed to its end by fermionising the boson X , and the boundary state at the end is the boundary state of the lower branes with Dirichlet condition on X . The two radii are not interchangeable: $1/\sqrt{2}$ belongs to the half-integer momenta of the pair and $\sqrt{2}$ to the periodic tachyon of the non-BPS brane. (ii) is Witten’s theorem [130] (for equivariant classes of this kind in a group-theoretic setting see [8]), which starts from Sen’s identification of the brane–antibrane system with a pair of bundles (E, F) and the tachyon with a map between them and identifies the charge with the class of (E, F) in K-theory; the type I computation $KO^{-1}(\text{pt}) = \mathbb{Z}_2$ gives the $\mathbb{Z}_2$ charge of the spinor, which prevents its decay. $\square$

Verdict 8.2 (claims 19, 20). Proved. Both statements are theorems of boundary conformal field theory and of K-theory respectively; the existence of the spinor as a stable non-BPS state was confirmed by its mass at weak coupling, which matches the heterotic spinor’s at strong coupling up to the expected corrections, and by the identification of the stable non-BPS states of other orbifolds and orientifolds by the same method [125].

Proposition 8.3 (claim 21; Sen 2003). *In the effective action (11) on a non-BPS Dp -brane, the kink solution $T(x) = \lim_{a \rightarrow \infty} ax$ (a step of infinite slope) is singular as a field and regular as a source: its energy-momentum tensor is $\mathcal{T}_{p-1}$ times a delta function at $x = 0$, and the effective action of its collective coordinates and world-volume gauge field is exactly the Dirac–Born–Infeld action of a BPS $D(p-1)$ -brane, to all orders in the fields and for every potential V with $\int V dT$ finite. The same holds for the vortex on the brane–antibrane pair.*

Proof. [89], Sections 3 and 4: for the ansatz $T = f(a(x - \xi(y)))$ with f odd and increasing, in the limit the action evaluated on the ansatz is independent of the profile f and of the scale a , by the change of variables $T = f(\cdot)$, and equals $-\mathcal{T}_{p-1} \int \sqrt{-\det(g_{\alpha\beta} + F_{\alpha\beta})}$ with the induced metric of the embedding ξ ; the equations of motion of the ansatz follow from the full equations because the action is stationary in f and a . $\square$

Verdict 8.4 (claim 21). Proved, within the effective action. The statement is about (11) and is exact there; it says nothing about the string theory corrections to (11), which are not

controlled on the kink, where second derivatives are large. The agreement of the tension and of the form of the world-volume action with those of the $D(p-1)$ -brane is the content of the claim and holds.

9. THE SEN LIMIT OF F-THEORY

Sen [117] showed that F-theory on an elliptic K3 surface, at the point of its moduli space where the Weierstrass model $y^2 = x^3 + f(z)x + g(z)$ has constant modulus τ and four D_4 singular fibres, is the type IIB orientifold on $T^2/\mathbb{Z}_2$ with four orientifold seven-planes and four D7-branes on each, with gauge group $SO(8)^4$ and constant coupling; and that moving away from this point, in a one-parameter family $f = -3h^2 + \varepsilon\eta$, $g = -2h^3 + \varepsilon h\eta - \frac{1}{12}\varepsilon^2\chi$ with $\varepsilon \rightarrow 0$, the coupling becomes weak away from the loci $h = 0$, which are the orientifold planes, while the discriminant $\Delta = -9\varepsilon^2h^2(\eta^2 - h\chi) + O(\varepsilon^3)$ places the D7-branes on $\eta^2 = h\chi$. In [118] the same expansion was applied to F-theory on elliptic Calabi–Yau threefolds and fourfolds over a base B , with h, η, χ sections of $K_B^{-2}, K_B^{-4}, K_B^{-6}$: the statement is that F-theory on such a Calabi–Yau has a weak coupling limit which is type IIB on the double cover of B branched along $h = 0$, with the orientifold projection, O7-planes on $h = 0$ and D7-branes on $\eta^2 = h\chi$.

Theorem 9.1 (scope of the weak-coupling expansion). *For $f = -3h^2 + \varepsilon\eta$ and $g = -2h^3 + \varepsilon h\eta - \varepsilon^2\chi/12$, one has*

$$4f^3 + 27g^2 = -9\varepsilon^2h^2(\eta^2 - h\chi) + \varepsilon^3(4\eta^3 - \frac{9}{2}h\eta\chi) + \frac{3}{16}\varepsilon^4\chi^2. \quad (14)$$

At points with $h(\eta^2 - h\chi) \neq 0$, the modular invariant $j \propto f^3/(4f^3 + 27g^2)$ diverges as ε^{-2} . This is a generic-point limit, not a smooth elliptic fibration at $\varepsilon = 0$: its discriminant vanishes identically there. The perturbative orientifold and D7 interpretation is the construction in [117, 118]. The generic O7 monodromy is $-T^{-4}$ in the convention of [118]; the enhanced O7 with four coincident D7-branes has combined monodromy $-I$.

Proof. The discriminant identity follows by expansion; the leading nonzero coefficient gives the asserted growth. The two monodromies concern different configurations: adding the four D7 monodromies T^4 cancels the T^{-4} factor. A Kodaira type cannot be read from the everywhere-degenerate central fibre as though it were an ordinary smooth family. $\square$

Verdict 9.2 (claim 22). Established in the stated geometric and physical framework. The constant-coupling $SO(8)^4$ model and the generic weak-coupling family are related constructions, not identical fibre configurations.

Proposition 9.3 (base divisor and double-cover divisor). *Suppose h, η are local coordinates near $h = \eta = 0$ on a smooth base and χ is nonzero there. The divisor $D_B = \{\eta^2 - h\chi = 0\}$ on the base is smooth at that point. On the smooth orientifold double cover $X = \{\xi^2 = h\}$ its pullback is*

$$D_X = \{\eta^2 - \xi^2\chi = 0\},$$

which is singular along $\xi = \eta = 0$. Where χ is a unit it has two local branches. At a transverse triple locus $\xi = \eta = \chi = 0$ in a three-dimensional cover, its local equation is the Whitney pinch form $u^2 = v^2w$.

Proof. For $F = \eta^2 - h\chi$, at $h = \eta = 0$ one has $dF = -\chi dh \neq 0$. On the cover, the three derivatives of $G = \eta^2 - \xi^2\chi$ in these coordinates are $2\eta, -2\xi\chi, -\xi^2$, all zero on the stated double locus. A unit χ has a local analytic square root, so $G = (\eta - \xi\sqrt{\chi})(\eta + \xi\sqrt{\chi})$. The transverse triple locus gives the pinch form. This is the distinction used in the D7 analysis of [18]. $\square$

Proposition 9.4 (a non-Higgsable divisor is not a sufficient obstruction). *In a local chart let $s = 0$ be a divisor and take*

$$h = sh_1, \quad \eta = s^2\eta_2, \quad \chi = s^3\chi_3.$$

Then $f = s^2F_\varepsilon$, $g = s^3G_\varepsilon$, with

$$F_\varepsilon = -3h_1^2 + \varepsilon\eta_2, \quad G_\varepsilon = -2h_1^3 + \varepsilon h_1\eta_2 - \varepsilon^2\chi_3/12.$$

If the coefficients are generic units, the divisor has vanishing orders $(2, 3, 6)$ for nonzero ε and j on it grows as ε^{-2} when $h_1(\eta_2^2 - h_1\chi_3) \neq 0$. Thus an I_0^ divisor does not itself force strong coupling in the limit.*

Proof. The common factor s^6 cancels in $f^3/(4f^3 + 27g^2)$, leaving Theorem 9.1 with h_1, η_2, χ_3 . This is a local calculation, not a global existence theorem on an arbitrary base. The allowed I_0^* case is explicitly distinguished from exceptional-fibre obstructions in [132]. $\square$

9.1. Every weak coupling limit is Sen's. Sen's argument for the generality of his family is the following passage of [118], after his equations (4) and (5), $f = C\eta - 3h^2$ and $g = h(C\eta - 2h^2)$:

“Since η is an arbitrary polynomial (subject to the same restrictions as f , and its overall normalization specified separately in the constant C), we see from (4) that by choosing the most general η and C we can get the most general f . Thus we only need to deform g . To this end consider the following form of g : $g = h(C\eta - 2h^2) + C^2\chi$ (21) where χ is a general polynomial subject to the same restrictions as g . This will certainly give us the most general g .”

The limit is then $C \rightarrow 0$ with h, η, χ fixed, and his equation (23) gives $\Delta = -9C^2h^2(\eta^2 + 12h\chi)$; with $C = \varepsilon$ and $\chi \rightarrow -\chi/12$ this is the family of Theorem 9.1.⁶ The passage shows that every Weierstrass model is a member of the family for some value of $C \neq 0$. It does not show the converse that matters for the limit: that a family of Weierstrass models whose coupling becomes weak must approach the orientifold point along a path of this form. Theorem 9.5 proves that converse.

Let B be a normal projective variety and L a line bundle on B ; for a Calabi–Yau total space $L = K_B^{-1}$, which is Sen's equation (29), $c_1(B) = c_1(L)$. A *degeneration* is a pair of formal power series

$$f_\varepsilon \in H^0(B, L^4)[[\varepsilon]], \quad g_\varepsilon \in H^0(B, L^6)[[\varepsilon]], \quad \Delta_\varepsilon = 4f_\varepsilon^3 + 27g_\varepsilon^2 \neq 0,$$

regarded as a Weierstrass curve over the field $K(B)((\varepsilon))$, with $j_\varepsilon = 1728 \cdot 4f_\varepsilon^3/\Delta_\varepsilon$. It is *weakly coupled* if $j_\varepsilon \rightarrow \infty$ at the generic point of B , that is, if $v(j_\varepsilon) < 0$ for the ε -adic valuation v of $K(B)((\varepsilon))$.⁷ Two degenerations are *equivalent* if they differ by a base change $\varepsilon = t^k$ and a rescaling $(f, g) \mapsto (t^{-4r}f, t^{-6r}g)$, $r \in \mathbb{Z}$, which is the isomorphism $(x, y) \mapsto (t^{-2r}x, t^{-3r}y)$ of Weierstrass curves and leaves j unchanged.

Theorem 9.5 (normal form of weak coupling). *(i) Every weakly coupled degeneration is equivalent, through a base change $\varepsilon = t^k$ with $k \in \{1, 2\}$, to a family of Sen's form*

$$f = -3h^2 + t\eta_t, \quad g = -2h^3 + th\eta_t - \frac{1}{12}t^2\chi_t, \quad (15)$$

with $h \in H^0(B, L^2)$ nonzero and independent of t , $\eta_t \in H^0(B, L^4)[[t]]$ and $\chi_t \in H^0(B, L^6)[[t]]$.

*(ii) Conversely, every family (15) with $h \neq 0$ and $\Delta \neq 0$ is weakly coupled, with $v(j) \leq -2$.
(iii) The orientifold divisor is determined by the limit: if (f_0, g_0) is the leading coefficient of the rescaled family, then $4f_0^3 + 27g_0^2 = 0$ and $h = 3g_0/(2f_0)$.*

⁶Sen's χ is $-\frac{1}{12}$ times the χ used here and in [132]; the D7 equation $\eta^2 + 12h\chi_{\text{Sen}} = 0$ of his (23) becomes $\eta^2 - h\chi = 0$.

⁷In the language of elliptic curves over a discretely valued field, the curve has potentially multiplicative reduction [147, Prop. VII.5.5]; $\text{Im } \tau \sim -v(j) \log |\varepsilon|/2\pi$, so the string coupling $g_s = 1/\text{Im } \tau$ goes to zero as $\varepsilon \rightarrow 0$.

The proof, in Appendix G, uses only the valuation of j and the normality of B , which makes a rational section of L^2 whose square is regular itself regular. The t -dependence of η_t and χ_t is the only difference from Sen's ansatz, and it does not change the leading data: by Theorem 9.1 the discriminant is $-9t^2h^2(\eta_0^2 - h\chi_0) + O(t^3)$, so the O7-plane is $h = 0$ and the D7-brane is $\eta_0^2 = h\chi_0$ exactly as Sen states. The D7 equation defines a divisor only when $\eta_0^2 - h\chi_0 \neq 0$, which we assume whenever we speak of the D7 divisor: the constant family $h = \eta = \chi = 1$ has $\Delta = -\frac{1}{2}t^3 + \frac{3}{16}t^4$, so it is weakly coupled, and $\eta_0^2 = h\chi_0$ holds on all of B .⁸ Theorem 9.5 is a statement about Weierstrass curves over $K(B)((\varepsilon))$ whose coupling is weak at the generic point of B . It does not by itself give a Weierstrass model that is minimal along every curve of B , a normal double cover, or a string vacuum.

Corollary 9.6 (existence of a weak coupling limit). *A weakly coupled degeneration over B exists if and only if $H^0(B, L^2) \neq 0$; for a Calabi–Yau total space, if and only if $h^0(B, -2K_B) > 0$. If $h \neq 0$ is a section of L^2 , the family $\eta = h^2$, $\chi = 0$ has*

$$\Delta = h^6 t^2 (4t - 9), \quad j = 1728 \cdot \frac{4(t - 3)^3}{t^2(4t - 9)},$$

constant on B : it is the global weak coupling limit of [132, Eqs. (2.13)–(2.15)]. Here $f = (t - 3)h^2$ and $g = (t - 2)h^3$.

The criterion is a property of the base alone, so it settles the existence question that Sen's paper leaves to the examples. It does not say that the limit is free of strongly coupled regions; that depends on the vanishing of h along special curves. Nor is the witness minimal in general: on $\mathbb{F}_{12}$, where h vanishes doubly on the curve of self-intersection -12 (Proposition 9.7), f and g vanish there to orders at least 4 and 6.

Proposition 9.7 (rigid curves of self-intersection at most -5). *Let B be a smooth surface, $L = K_B^{-1}$, and $C \subset B$ a smooth rational curve with $C^2 = -n$.*

- (i) *Every section of L^k vanishes on C to order at least $\lceil k(n - 2)/n \rceil$. For $n \geq 5$ this gives $\text{ord}_C f \geq 3$ and $\text{ord}_C g \geq 4$.*
- (ii) *If $n \geq 5$ and, for a given $\varepsilon \neq 0$, the Weierstrass model is minimal along C , then the value of j at the generic point of C is 0 or 1728. Along a degeneration minimal along C for $\varepsilon \neq 0$ the coupling on C therefore stays of order one.*
- (iii) *In every weakly coupled degeneration h vanishes on C to order at least 2 if $n \geq 5$, and at least 1 if $n = 3, 4$.*

Part (i) is the intersection argument of Morrison and Taylor [69], and part (ii) restates for every ε the observation of Halverson, Long and Sung that fibres of types IV^* , III^* , II^* carry a fixed coupling [132, Table 1 and Section 2.1]. Part (iii) is new as stated: it places every rigid curve of self-intersection ≤ -3 inside the zero locus of h , the orientifold divisor of the normal form; for $n \geq 5$ the curve lies in a non-reduced part of that divisor, which need not remain a branch curve once the double cover is normalised. For $n = 3, 4$ it reproduces Proposition 9.4; for $n \geq 5$ the branch divisor of the double cover is not reduced along C , so the cover is not normal there. On the Hirzebruch surface $\mathbb{F}_{12}$, for instance, $h^0(-2K) = 51$ and weak coupling exists at the generic point by Corollary 9.6, but every section of $-2K$ vanishes doubly on the curve of self-intersection -12 , where the fibre has type II^* and $j = 0$.⁹

⁸A family $f = -3h^2 + tF_t$, $g = -2h^3 + tG_t$ whose first-order term violates $G_0 = hF_0$ has discriminant $108th^3(hF_0 - G_0) + O(t^2)$ and $v(j) = -1$; the base change $t = s^2$ in the proof brings it to the form (15). This can happen only when the first step of the proof needs no base change: when it does, the rescaled series contain only even powers of t , their first-order terms vanish, and $G_0 = hF_0$ holds. Hence $k \leq 2$.

⁹With the rays $(1, 0)$, $(0, 1)$, $(-1, 12)$, $(0, -1)$ of the fan of $\mathbb{F}_{12}$, the curve of the ray $(0, 1)$ has self-intersection -12 , and the polygon of $-kK$ is $\{(a, b) : a \geq -k, b \leq k, 12b \geq a - k\}$. For $k = 2$ it has $5 + 17 + 29 = 51$ lattice points, all with $b \geq 0$, and the monomial of (a, b) vanishes to order $b + 2 \geq 2$ on the curve; for $k = 4, 6$ the least orders are 4 and 5, the fibre type II^* .

Corollary 9.8 (the orientifold double cover). *Let B be a smooth rational surface or an Enriques surface, $L = K_B^{-1}$, and let a weakly coupled degeneration be given in the form (15). Let $X = \{\xi^2 = h\}$ be the double cover of B branched along $O = \{h = 0\}$, with ξ a section of L , Sen's equation (18).*

- (i) $K_X = \pi^*(K_B + L) = 0$, which is Sen's equation (30). If O is reduced with at most simple singularities, X has rational double points and its minimal resolution is a K3 surface.
- (ii) If B is an Enriques surface, L^2 is trivial, h is a nonzero constant and X is the K3 surface covering B . Then f and g are constants, there is no orientifold plane and no D7-brane, and the coupling is constant over B for every ε .
- (iii) Suppose that $|-K_B|$ has no base points, as for $\mathbb{P}^2$ and the del Pezzo surfaces of degree at least two. For general h, η_0, χ_0 the D7 divisor $D_B = \{\eta_0^2 = h\chi_0\} \in |-8K_B|$ is smooth and meets O only at the $8K_B^2$ points $h = \eta_0 = 0$, where it is tangent to O . Its pullback D_X has a node over each of these points, and the normalisation $\tilde{D} \rightarrow D_B$ is an unramified double cover.
- (iv) Under the hypothesis of (iii), if $Y \rightarrow B$ is the smooth Weierstrass model of general f, g , then

$$\chi(Y) = -60K_B^2, \quad \chi(O) = -2K_B^2, \quad \chi(D_B) = -56K_B^2, \quad \chi(\tilde{D}) = -112K_B^2,$$

$$\text{and } 2\chi(Y) = 4\chi(O) + \chi(\tilde{D}).$$

Part (iv) is the two-dimensional case of the identity $2\chi(Y) = \chi(\bar{D}) + 4\chi(O) - \chi(S)$ of Aluffi and Esole [140, Cor. 4.7], in which the pinch locus S is empty because three general sections h, η_0, χ_0 have no common zero on a surface; here it follows from the adjunction formula and a count of the cusps of the discriminant. Part (iii) shows that the singularity of Proposition 9.3 is, on a surface with base-point-free anticanonical system, a set of nodes, and the Whitney pinch first appears for threefold bases. The hypothesis cannot be dropped. On the rational surface $\mathbb{F}_{12}$, Proposition 9.7(i) gives orders at least 2, 4 and 5 for h, η_0 and χ_0 along the curve C of self-intersection -12 , attained for general sections. Then $h\chi_0$ vanishes on C to order exactly 7 and η_0^2 to order at least 8, so D_B contains $7C$ and is not reduced.

Verdict 9.9 (claim 23). Proved, in the form of Theorem 9.5: up to base change and rescaling, every weak coupling degeneration of F-theory on an elliptic Calabi–Yau, regarded as a family of Weierstrass curves over $K(B)((\varepsilon))$, is Sen's family, with the O7 divisor $h = 0$ fixed by the limit point, and such a degeneration exists exactly when $h^0(-2K_B) > 0$ (Corollary 9.6). Sen calls the reduction formal and says that the auxiliary manifold can be singular; Proposition 9.7 and Corollary 9.8 identify when it is. Rigid curves of self-intersection ≤ -5 keep the coupling of order one and force a non-reduced branch divisor, and on surfaces with base-point-free anticanonical system the D7 singularity is a set of nodes. Beyond the geometry and the monodromies of Theorem 9.1, the physical identification of the limit with a type IIB orientifold is not proved; it remains a statement of string theory.

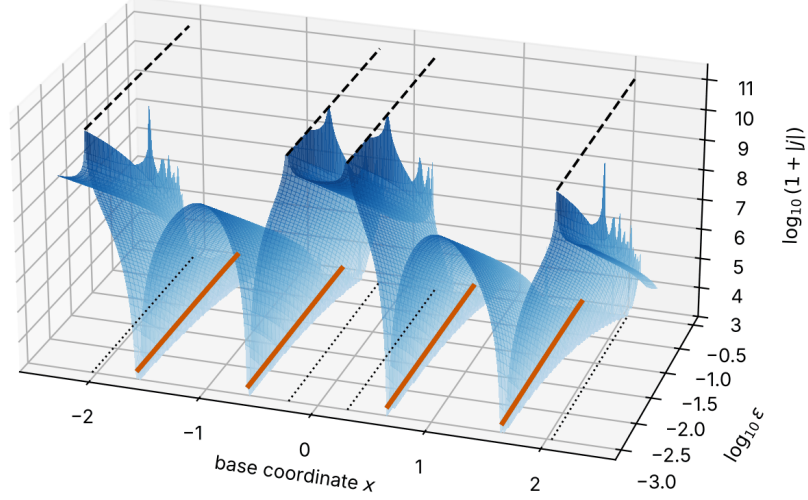
10. BLACK HOLE ENTROPY: THE ENTROPY FUNCTION AND SMALL BLACK HOLES

10.1. The entropy function. For an extremal black hole in a theory of gravity with covariant higher-derivative terms in the framework specified below, whose near-horizon geometry has the symmetry $SO(2, 1) \times SO(D-1)$ of $AdS_2 \times S^{D-2}$ with electric fields e_i and magnetic charges p_a , Sen [93] defined the entropy function

$$\begin{aligned} \mathcal{E}(\vec{u}, \vec{v}, \vec{e}, \vec{q}, \vec{p}) &= 2\pi(e_i q_i - f(\vec{u}, \vec{v}, \vec{e}, \vec{p})), \\ f &= \int_{S^{D-2}} \sqrt{-\det g} \mathcal{L}, \end{aligned} \tag{16}$$

where f is the Lagrangian density integrated over the sphere on the near-horizon ansatz with radii v_1, v_2 of AdS_2 and S^{D-2} and constant scalars u_s .

Sen's weak-coupling limit over $\mathbb{P}^1$: $|j|$ grows like ε^{-2} away from the O7-planes



orange: O7-planes $h = 0$, where $j \rightarrow 1728$; black dashed: D7-branes $\eta^2 = h\chi$, limit of the poles of j as $\varepsilon \rightarrow 0$

FIGURE 4. Sen's K3 limit as a surface over the base coordinate x of $\mathbb{P}^1$ and the parameter ε : the height is $\log_{10}(1 + |j|)$ for $h = x^4 - 3x^2 + 1$, $\eta = x^8 + 2$, $\chi = 3x^{12} + 5$, sections of L^2, L^4, L^6 with $L = \mathcal{O}(2)$; values below 10^3 are drawn on the floor. The coupling becomes weak, $|j| \sim \varepsilon^{-2}$, everywhere except at the four O7-planes $h = 0$ (orange lines), where $j \rightarrow 1728$, and the four real D7-branes $\eta^2 = h\chi$ (dashed), the limit as $\varepsilon \rightarrow 0$ of the poles of j , which for $\varepsilon \neq 0$ lie on $\Delta_\varepsilon = 0$.

Theorem 10.1 (claims 24, 25). (i) *The equations of motion of the near-horizon geometry are the conditions*

$$\partial\mathcal{E}/\partial u_s = \partial\mathcal{E}/\partial v_{1,2} = 0, \quad \partial\mathcal{E}/\partial e_i = 0,$$

the last being the definition of the charges q_i , and the Wald entropy of the black hole is the value of $\mathcal{E}$ at the extremum [93].

(ii) *For rotating extremal black holes, whose near-horizon geometry has the symmetry $SO(2,1) \times U(1)$ and depends on the polar angle through functions $\Omega(\theta), \psi(\theta), \dots$, the same holds with $\mathcal{E}$ a functional of these functions and with an additional term $2\pi\alpha J$ for the angular momentum, the extremisation being over the functions as well as the constants [1].*

Proof. The derivation, including the normalization and curvature-variation convention, is given in [93, Section 2] and its rotating extension in [1]. For a covariant Lagrangian with no derivatives of curvature, the Wald expression is $-2\pi \int_H (\partial\mathcal{L}/\partial R_{\mu\nu\rho\sigma}) \epsilon_{\mu\nu} \epsilon_{\rho\sigma}$ with the induced volume factor and $\epsilon_{\mu\nu} \epsilon^{\mu\nu} = -2$. Curvature derivatives require the corresponding variational derivative, and Chern–Simons terms require separate treatment. The near-horizon equations and the Legendre transform give the entropy function in this framework. Local extremisation does not by itself construct a regular solution extending to prescribed asymptotic data. $\square$

Verdict 10.2 (claims 24, 25). Proved. The theorem is a derivation within classical higher-derivative gravity with the stated symmetry; the attractor mechanism, that the entropy is independent of the asymptotic moduli, is a corollary whenever the extremum is isolated in the scalars. The results agree with the earlier supergravity computations of Cardoso, de Wit and Mohaupt where both apply [93], and the formalism has been used in every later comparison of macroscopic and microscopic entropies.

10.2. Small black holes. A half-BPS state of the heterotic string on T^6 with winding w and momentum n on a circle, with $nw > 0$, has $d_h(nw) \sim e^{4\pi\sqrt{nw}}$ states, while the two-derivative supergravity solution with these charges has a singular horizon of zero area. Sen [92] argued that the higher-derivative terms stretch the horizon to string scale and that the entropy function with the Gauss–Bonnet term, $\mathcal{L} \supset \frac{1}{16\pi G}\phi(S)(R_{\mu\nu\rho\sigma}^2 - 4R_{\mu\nu}^2 + R^2)$ with the one-loop coupling $\phi(S)$, gives

$$S_{\text{BH}} = 4\pi\sqrt{nw} \quad (17)$$

in four dimensions, in agreement with the statistical entropy to leading order; [94] extended the computation to the heterotic string in D dimensions.

Sen states the outcome of the computation of [94] in its abstract:

“The black hole entropy computed this way is finite and has the right dependence on the charges in accordance with the general scaling argument, but the overall numerical factor does not agree with that computed from the statistical entropy except for $D = 4$ and $D = 5$. This is not surprising in view of the fact that we do not use a fully supersymmetric action in our analysis; however this analysis demonstrates that higher derivative corrections are capable of stretching the horizon of a small black hole in arbitrary dimensions.”

His result is [94, Eq. (3.20)], with the Gauss–Bonnet coefficient C of his (3.14),

$$S_{\text{BH}} = a\sqrt{nw}, \quad a = \pi\sqrt{C}\sqrt{\frac{2}{(D-2)(D-3)}}\{8(D-2)(D-3) - 4(D-4)(D-5)\}^{1/2}. \quad (18)$$

Prester [145] then showed that the Lovelock action with the coefficients $\lambda_m = 4^{1-m}/m!$ of his equation (4.27) has, in every dimension, a near-horizon solution with $S_{\text{BH}} = 4\pi\sqrt{nw}$, and that his solution is unique for $D \leq 9$ [145, footnote 9]. The rest of this subsection makes these statements exact for every dimension.

The two-charge horizon. We use the normalisation of [145, Eq. (4.3)], with $n_D = D - 2$ the dimension of the sphere and $\alpha' = 1$ unless α' is written. The near-horizon geometry is $AdS_2 \times S^{n_D}$ with squared radii v_1, v_2 , the string-frame dilaton is u_S , the radius of the circle is u_T , and e_1, e_2 are the electric fields of the momentum and winding charges $q_1 = 2n/\sqrt{\alpha'}$, $q_2 = 2w/\sqrt{\alpha'}$. For a tree-level action whose higher-derivative terms are built from the Riemann tensor alone,

$$f = K u_S v_1 v_2^{n_D/2} \left[\frac{2(u_T^2 e_1^2 + e_2^2 u_T^{-2})}{v_1^2} + Q(v_1, v_2) \right], \quad (19)$$

where $K > 0$ is a constant and Q is the curvature Lagrangian on the product of radii v_1, v_2 ; for the Einstein term $Q = n_D(n_D - 1)/v_2 - 2/v_1$.

Theorem 10.3 (entropy of the two-charge horizon). *Let $R(v_1, v_2) = -v_1^2 Q(v_1, v_2)$. The near-horizon solutions with $v_1, v_2, u_S, u_T, e_i > 0$ are in bijection with the critical points of R in the positive quadrant at which $R > 0$, and at such a point*

$$S_{\text{BH}} = 4\pi\sqrt{nw}\sqrt{R_*/\alpha'}. \quad (20)$$

In particular the black hole entropy equals the statistical entropy $4\pi\sqrt{nw}$ if and only if $R_ = \alpha'$, which is the condition $e_1 e_2 = \alpha'/4$ of [145, Eq. (5.6)].*

The proof is in Appendix G. The theorem reduces the comparison with the statistical entropy, for every action of this class, to a single number: the critical value of a function of two variables. For the Lovelock actions $\mathcal{L} = \sum_m \lambda_m \alpha'^{m-1} \mathcal{L}_m^{GB}$ with $\lambda_1 = 1$ the function (19) is that of [145, Eq. (4.3)], and

$$Q = H(v_2) - \frac{G(v_2)}{v_1}, \quad G(y) = \sum_{m \geq 1} \frac{2m \lambda_m n_D!}{(n_D + 2 - 2m)!} y^{1-m}, \quad H(y) = \sum_{m \geq 1} \frac{\lambda_m n_D!}{(n_D - 2m)!} y^{-m}, \quad (21)$$

the sums running over the terms with nonnegative factorial arguments.

Corollary 10.4 (Lovelock actions). *For a Lovelock action the near-horizon solutions correspond to the critical points y of $\Phi = G^2/H$ on the set where $G > 0$ and $H > 0$, with $v_2 = \alpha' y$, $v_1 = \alpha' G/(2H)$ and $S_{\text{BH}} = 2\pi \sqrt{nw \Phi(y)}$. The statistical value is $\Phi = 4$.*

Corollary 10.5 (the Gauss–Bonnet truncation in closed form). *For $\lambda_2 = 1/8$ and $\lambda_m = 0$ for $m \geq 3$, with $a = (D-2)(D-3)$ and $b = (D-4)(D-5)$, there is exactly one near-horizon solution in every dimension $D \geq 4$. It has $v_2 = \alpha'(a-b)/4$, $v_1 = 2v_2/a$ and*

$$\frac{S_{\text{BH}}}{4\pi \sqrt{nw}} = \sqrt{1 - \frac{(D-4)(D-5)}{2(D-2)(D-3)}}. \quad (22)$$

The ratio is 1 for $D = 4, 5$, it is $\sqrt{11/12}$, $\sqrt{17/20}$, $\sqrt{4/5}$ for $D = 6, 7, 8$, and it decreases to $1/\sqrt{2}$ as $D \rightarrow \infty$. It is the ratio a_D/a_4 of Sen’s coefficients (18), for every value of C .

The corollary turns Sen’s statement into a formula with its domain: the Gauss–Bonnet truncation fixes the horizon in every dimension, the entropy is below the statistical value for every $D \geq 6$, and it never falls below $1/\sqrt{2}$ of it.¹⁰ The solution is the minimum of Φ , and the critical point equation is linear in $1/y$, which is why it can be solved in every dimension.

Theorem 10.6 (Prester’s completion in every dimension). *Let $\lambda_m = 4^{1-m}/m!$ for every $m \geq 1$, and $D \geq 4$.*

- (i) $H = 2(G - 2)$ identically, in every dimension. Equivalently f has the form [145, Eq. (5.3)].
- (ii) On the line $v_1 = \alpha'/2$ one has $R(\alpha'/2, v_2) = \alpha'$ for every v_2 , and every near-horizon solution lies on this line. Hence every solution has $S_{\text{BH}} = 4\pi \sqrt{nw}$.
- (iii) The near-horizon solution is unique: $v_2 = \alpha' y_D$ with y_D the only positive root of $H(y) = 4$. Equivalently $I_{n_D}(1/(2y_D)) = 2$, where

$$I_N(t) = \sum_{m \geq 0} \binom{N}{2m} (2m-1)!! t^m$$

counts the involutions of an N -element set with weight t per transposition, or again $i^{-n_D} \text{He}_{n_D}(i\sqrt{2y_D}) = 2(2y_D)^{n_D/2}$ with He_N the Hermite polynomial.

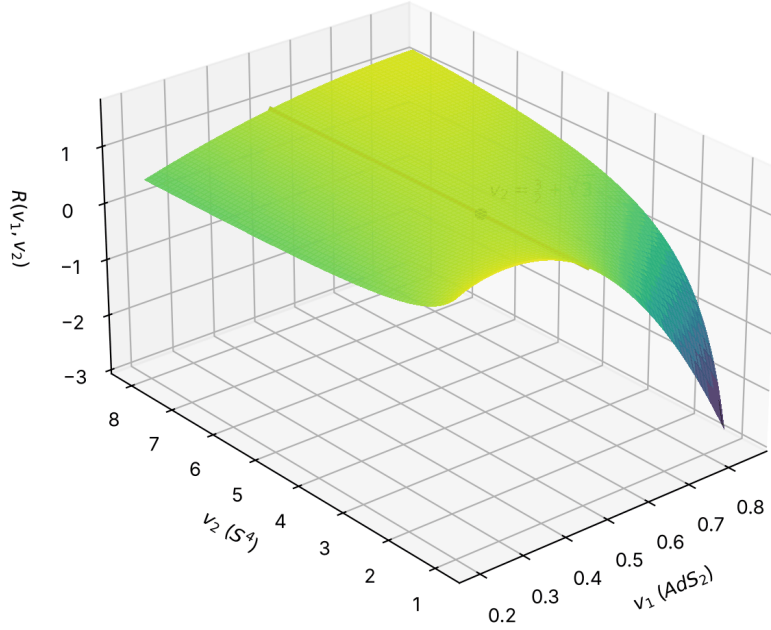
- (iv) As $D \rightarrow \infty$, with $n_D = D - 2$,

$$y_D = \frac{n_D^2}{4 \ln 2} - \frac{(2 \ln 2 + 1) n_D}{4 \ln 2} + O(1).$$

Part (ii) for one solution is Prester’s argument of [145, Section 5], which he gives without solving for v_2 ; parts (i) and (ii) show that it applies to every solution, because $\partial_{v_2} R = v_1 G'(v_2)(1 - 2v_1)$ and $G' < 0$. Part (iii) extends his uniqueness statement from $D \leq 9$ to every dimension, and part (iv) refines the leading term $v_2 \rightarrow D^2 \alpha'/(4 \ln 2)$ of [146, Eq. (20)]. The first values are $y_4 = 1/2$, $y_5 = 3/2$, $y_6 = 3/2 + \sqrt{3}$, and $y_{40} = 489.273 \dots$

¹⁰Sen works in units $\alpha' = 16$ and sets $C = 1$ for the heterotic string in his (3.15). In his Lagrangian density (3.16) the Gauss–Bonnet density then has the coefficient $2C = 2 = \alpha'/8$ relative to the Einstein term, which is the value $\lambda_2 = 1/8$ of the corollary at $\alpha' = 1$, and (18) gives $a_4 = 4\pi$. Since $a \propto \sqrt{C}$, the ratio a_D/a_4 does not depend on the normalisation.

Prester's Lovelock action in $D = 6$: $R \equiv \alpha'$ along $v_1 = \alpha'/2$



every near-horizon solution lies on the orange line, so $S_{\text{BH}} = 4\pi\sqrt{nw}$; the solution is the saddle point

FIGURE 5. The function $R(v_1, v_2)$ of Theorem 10.3 for Prester's coefficients in $D = 6$, in units $\alpha' = 1$. Along the line $v_1 = 1/2$ (orange) the surface has the constant height $R = 1$, the value that gives $S_{\text{BH}} = 4\pi\sqrt{nw}$; the near-horizon solution is the saddle point on this line, at $v_2 = 3/2 + \sqrt{3}$.

Theorem 10.7 (the coefficients are forced). *Fix $D \geq 4$, let $m = \lfloor D/2 \rfloor$, and take $\lambda_k = 4^{1-k}/k!$ for $k < m$ and $\lambda_m = \beta 4^{1-m}/m!$ with $\beta > 0$.*

- (i) *The coefficient λ_m enters G and not H , and $\Phi_\beta = G_\beta^2/H$ attains its minimum on $(0, \infty)$.*
- (ii) *If $\beta > 1$, every near-horizon solution has $S_{\text{BH}} > 4\pi\sqrt{nw}$.*
- (iii) *If $\beta < 1$, the solution at the minimum of Φ_β has $S_{\text{BH}} < 4\pi\sqrt{nw}$.*

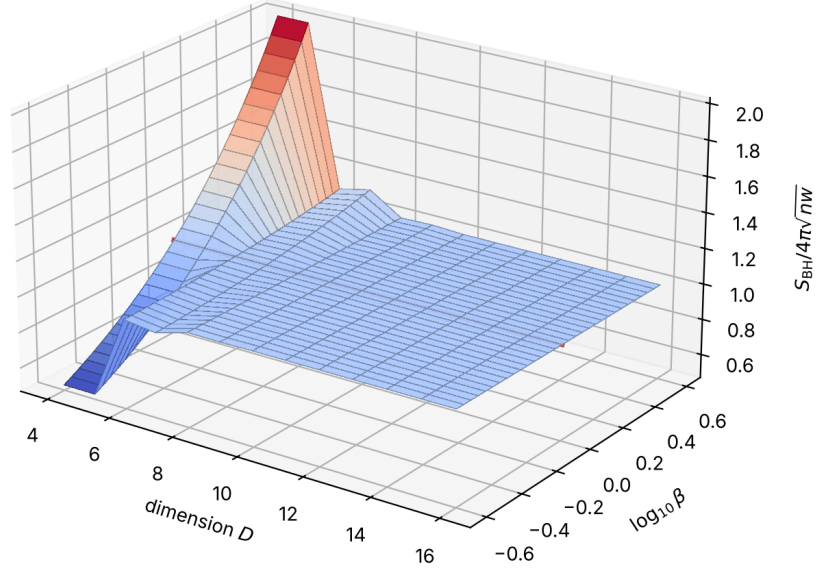
Consequently, if $\lambda_1 = 1$, every λ_k is positive, and the solution at the minimum of Φ has the statistical entropy in every dimension, then $\lambda_k = 4^{1-k}/k!$ for every k .

The induction in the last statement uses that λ_m is the top coefficient both in $D = 2m$ and in $D = 2m + 1$: it is fixed by $D = 2m$, and Theorem 10.6 shows that the same value works in $D = 2m + 1$. This proves the statement of [146] that the matching “fixes the coefficients λ_k to the form (6) uniquely”, which there rests on the computations of [145]. Figures 5 and 6 show the two mechanisms: a line of constant R on which every solution lies, and a minimum that crosses the statistical value only at $\beta = 1$.

Remark 10.8 (four dimensions). In $D = 4$ the agreement extends to all orders in $1/\sqrt{nw}$ when the supersymmetric F -terms are used in place of the Gauss–Bonnet term: [25] evaluated the entropy with the full F -term, and [29] matched it with the asymptotic expansion of d_h up to non-perturbative terms. The Gauss–Bonnet term reproduces the leading term because on the near-horizon geometry the four-dimensional F -term reduces to it at this order.

Remark 10.9 (what remains open for the heterotic string). The heterotic effective action in $D \geq 6$ is not the Lovelock action. It contains no cubic Gauss–Bonnet density, its quartic terms carry $\zeta(3)$, and it has higher-derivative terms in the gauge fields [145, Section 6] and

Top Lovelock coefficient $\lambda_{[D/2]} = \beta 4^{1-m}/m!$: entropy at the minimum of Φ_β



the statistical value 1 is reached only at $\beta = 1$ (orange), in every dimension

FIGURE 6. The entropy at the minimum of Φ_β , divided by $4\pi\sqrt{nw}$, as a surface over the dimension D and the factor β of the top Lovelock coefficient (Theorem 10.7). The level 1 is met only along $\beta = 1$ (orange): above it the entropy exceeds the statistical value, below it the entropy falls short.

gravitational Chern–Simons terms [146, footnote 6]. Theorem 10.3 applies to the actions (19), whose corrections are built from curvature alone, and there the comparison is the single condition $R_* = \alpha'$. Terms with gauge fields, such as F^4 and couplings of curvature to the gauge fields, change the equations for the charges and do not reduce to a function $R(v_1, v_2)$. For the complete action the open question is **to derive and extremise the full entropy function on $AdS_2 \times S^{D-2}$ in every $D \geq 6$** , which requires the full α' expansion, not known. The holographic arguments of [142, 143, 144] avoid the action: they obtain the entropy from the central charges of an AdS_3 near-horizon region, fixed by symmetries and anomalies. In five dimensions this gives the statistical value [142]. Theorem 10.10 works out, step by step, what the same argument gives in $D \geq 6$.¹¹

The heterotic string in $D \geq 6$: what the central charges decide

Theorem 10.10 (the central-charge route). *Let $N = w$ be the winding number, n the momentum and $p = D - 3$, so that $D \geq 6$ means $3 \leq p \leq 7$. Assume, as in [142, 143]:*

- (A1) *the near-horizon region of the N strings is $AdS_3 \times S^p \times T^{7-p}$, dual to a two-dimensional conformal field theory with Virasoro central charges c_L, c_R and an affine $SO(p+1)$ algebra of level k coming from the isometries of S^p ;*

¹¹The question is not one of Sen’s claims. Sen’s statement, claim 26, is the Gauss–Bonnet computation of [94], which Corollary 10.5 proves in every dimension, and Sen attributes the disagreement in $D \geq 6$ to the use of an action that is not fully supersymmetric.

- (A2) the Chern–Simons terms on AdS_3 come only from the Green–Schwarz term of the three-form, so that $c_L - c_R = 12N$ and $k = N$ for $p \geq 3$ [143, Eq. (3.12)];
- (A3) for $n \gg N$ the number of states of momentum n is given by Cardy’s formula, $S = 2\pi\sqrt{c_L n/6}$.

Then the following hold.

- (i) $S = 4\pi\sqrt{nN}$ if and only if $c_L = 24N$, and if and only if $c_R = 12N$.
- (ii) Suppose in addition that the right movers carry the nonlinear superconformal algebra with sixteen supercharges whose R -symmetry has the isometry group of S^p as a simple factor, at level N : $OSp(4^*|4)$ with $Sp(4) \cong Spin(5)$ for $p = 4$, $SU(1, 1|4)$ with $SU(4) \cong Spin(6)$ for $p = 5$, $F(4)$ with $Spin(7)$ for $p = 6$, and $OSp(8|2)$ with $SO(8)$ for $p = 7$. Then

$$\begin{aligned} p = 4 : c_R &= 6N + 6, & p = 5 : c_R &= \frac{3(2N^2 + 9N + 1)}{N + 3}, \\ p = 6 : c_R &= \frac{2N(2N + 11)}{N + 4}, & p = 7 : c_R &= \frac{3N(N + 9)}{N + 5}. \end{aligned}$$

For $p = 4$ and $p = 5$ the algebra has an affine level of negative sign, so no unitary conformal field theory carries it. For $p = 5, 6, 7$, $c_R \neq 12N$ for every $N \geq 1$, and $c_R/(12N)$ tends to $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ as $N \rightarrow \infty$.

- (iii) For $p = 3$, that is $D = 6$, none of the four superconformal algebras with sixteen supercharges has $SU(2) \times SU(2)$, the isometry group of S^3 up to a finite quotient, as its R -symmetry; the route of (ii) has no algebra to use.

Hence, under (A1)–(A3) and the identification of (ii), the central charges do not give Sen’s statistical value $4\pi\sqrt{nw}$ in any dimension $D \geq 6$.

Proof. (i) By (A3), $S = 4\pi\sqrt{nN}$ means $2\pi\sqrt{c_L n/6} = 4\pi\sqrt{nN}$. Both sides are positive, so we may square: $4\pi^2 c_L n/6 = 16\pi^2 nN$. Dividing by $4\pi^2 n$ gives $c_L/6 = 4N$, that is $c_L = 24N$. By (A2), $c_R = c_L - 12N$, so $c_L = 24N$ is the same as $c_R = 12N$.

(ii) A nonlinear superconformal algebra is generated by the stress tensor, the supercurrents and the currents of the R -symmetry, and the operator product of two supercurrents contains products of two currents. The Jacobi identities of these products fix the Virasoro central charge as a function of the affine level k , normalised with long roots of squared length two; the four functions are collected in [143, Table 2], from the classification of the nonlinear algebras cited there. We take them one at a time.

For $OSp(4^*|2m)$ the R -symmetry is $SU(2) \times Sp(2m)$. Let k be the level of $SU(2)$ and k_1 that of $Sp(2m)$. The Jacobi identities give $k_1 = -(k + m + 2)$ and

$$c = -\frac{[6k - (2m + 1)(m - 2)](k + m + 2) - 6k}{k - m + 2}.$$

For $m = 2$ the factor $(2m + 1)(m - 2)$ vanishes, and $c = -(6k(k + 4) - 6k)/k = -6(k + 4) + 6 = -6k - 18$. The isometry group of S^4 is $SO(5)$, whose Lie algebra is that of $Sp(4)$, so (ii) sets $k_1 = N$. Then $k = -k_1 - 4 = -N - 4$ and $c_R = 6(N + 4) - 18 = 6N + 6$. The level $k = -N - 4$ is negative. In a unitary theory the state $J_{-1}^a|0\rangle$ of a current J^a has squared norm equal to the level, up to a positive normalisation, so a negative level is excluded.

For $SU(1, 1|n)$ with $n \neq 2$ the R -symmetry is $U(1) \times SU(n)$ with $SU(n)$ at level k , and

$$c = \frac{3k(n + 2k) + (n - 1)(1 + (n + 1)k)}{k + n - 1}.$$

For $n = 4$ the numerator is $6k^2 + 12k + 3 + 15k = 6k^2 + 27k + 3 = 3(2k^2 + 9k + 1)$ and the denominator is $k + 3$. The isometry group of S^5 is $SO(6)$, with the Lie algebra of

$SU(4)$, so $k = N$, which gives the second formula. The Jacobi identities fix the level of the $U(1)$ current to $-(k+n) = -(N+4) < 0$, which excludes unitarity as before. The equation $c_R = 12N$ reads $3(2N^2 + 9N + 1) = 12N(N+3)$, that is $6N^2 + 9N - 3 = 0$, or $2N^2 + 3N - 1 = 0$. Its roots are $(-3 \pm \sqrt{17})/4$, which are not integers, so $c_R \neq 12N$ for every $N \geq 1$.

For $F(4)$ the R -symmetry is $Spin(7)$, whose Lie algebra is that of $SO(7)$, the isometry group of S^6 ; with $k = N$,

$$c = \frac{2N(2N+11)}{N+4},$$

and $c_R = 12N$ would mean $2(2N+11) = 12(N+4)$, that is $8N = -26$, which has no positive solution.

For $OSp(n|2)$ the R -symmetry is $SO(n)$ and $c = k(6k + n^2 - 10)/(2(k+n-3))$. For $n = 8$, the isometry group of S^7 , this is $k(6k + 54)/(2(k+5)) = 3k(k+9)/(k+5)$, and with $k = N$ the equation $c_R = 12N$ would mean $3(N+9) = 12(N+5)$, that is $9N = -33$, again with no positive solution.

The limits of $c_R/(12N)$ follow from the leading terms $6N$, $4N$ and $3N$ of the last three formulas.

(iii) The supergroups with sixteen supercharges whose bosonic part is the conformal group $SL(2, \mathbb{R})$ times a compact R -symmetry are $OSp(8|2)$, $F(4)$, $SU(1,1|4)$ and $OSp(4^*|4)$, with R -symmetries $SO(8)$, $Spin(7)$, $U(1) \times SU(4)$ and $SU(2) \times Sp(4)$ [143, Table 1]. None of these is $SU(2) \times SU(2)$. The algebra with that R -symmetry, the large $\mathcal{N} = 4$ algebra $D(2, 1; \alpha)$, has eight supercharges.

The final statement combines (i) with (ii) and (iii): in each dimension $D = 6, \dots, 10$ the route either has no algebra, or has one with no unitary representation, or gives $c_R \neq 12N$, while (i) requires $c_R = 12N$. $\square$

The theorem settles the central-charge route under its own hypotheses, and nothing more. In five dimensions the same table shows why the route works there: at $p = 2$, where $k = 2N$, the algebra $OSp(4^*|4)$ gives $c_R = -6k - 18 = -12N - 18$, while the small $\mathcal{N} = 4$ algebra used in [142] gives $c_R = 6k = 12N$ and therefore Sen's value, in agreement with Corollary 10.5.^a The hypotheses (A2) and the choice of algebra in (ii) are not proved: [143] note that the reduction to three dimensions could generate further Chern–Simons terms, and the algebra realised by the near-horizon region in $D \geq 6$ has not been derived. Whether the complete heterotic effective action gives $4\pi\sqrt{nw}$ in $D \geq 6$ therefore remains open.

^aThe superisometry group of the five-dimensional near-horizon solution is $OSp(4^*|4)$ [144]; [143] note the sign of c_R it gives and the non-unitarity of its levels.

Verdict 10.11 (claim 26). Proved, for the statements Sen makes. The Gauss–Bonnet truncation gives a finite horizon in every dimension, with the entropy (22) in closed form, equal to the statistical value exactly for $D = 4, 5$ (Corollary 10.5). For every action whose corrections are built from curvature, the comparison is the single condition $R_* = \alpha'$ (Theorem 10.3); Prester's Lovelock completion satisfies it with a unique solution in every dimension, and its coefficients are the only ones that do (Theorems 10.6 and 10.7). The statement that the complete heterotic effective action gives $4\pi\sqrt{nw}$ in every $D \geq 6$ remains open. It is not a statement of Sen, and Theorem 10.10 shows that the central charges of the near-horizon region do not supply it (Remark 10.9).

11. THE QUANTUM ENTROPY FUNCTION AND THE LOGARITHMIC CORRECTIONS

11.1. The quantum entropy function. Sen [98] proposed that the exact degeneracy of the horizon states of an extremal black hole with charges $\vec{q}$ is the finite part of the Euclidean

string theory path integral on the near-horizon AdS_2 ,

$$d_{\text{hor}}(\vec{q}) = \left\langle \exp \left[-iq_i \oint_{AdS_2} A_\theta^{(i)} d\theta \right] \right\rangle_{AdS_2}^{\text{finite}}, \quad (23)$$

with a Wilson line insertion at the boundary carrying the charges, the boundary conditions fixing the charges rather than the chemical potentials, and the “finite part” defined by subtracting the terms proportional to the length of the boundary; and that this quantity matches the appropriately signed horizon index for quarter-BPS black holes after hair removal and the helicity-trace convention are fixed (Section 5.3).

Verdict 11.1 (claim 27). Confirmed. The proposal passes every test that can be computed: the classical limit of (23) is the exponential of the Wald entropy; the one-loop correction is the logarithmic term, which matches the microscopic index exactly in $\mathcal{N} = 4$ and $\mathcal{N} = 8$ (Theorem 11.2); and the localisation of the path integral in $\mathcal{N} = 2$ supergravity, proposed in [6] and carried out in [32, 33], reproduces the exact Rademacher expansion of the index of $\mathcal{N} = 8$ black holes and the leading Bessel function of the $\mathcal{N} = 4$ one [71, 54, 55]. The last of these results is weaker than its statement in the literature, and Sen’s own later assessment [113] says so: the claim that supersymmetric localisation of the AdS_2 path integral reproduces the Rademacher expansion from first principles rests on three steps that are not established. The Killing spinors of the near-horizon geometry grow at the boundary of AdS_2 and do not act on the space of normalisable fluctuations over which the path integral is defined, so the localisation argument has no proved domain; the measure on the localisation manifold used in [32, 71] is postulated and not derived, and a different measure changes every subleading term; and the one-loop determinants over the transverse directions depend on an ultraviolet cut-off that the computation sets, without justification, at the four-dimensional Planck scale. Sen himself discusses these qualifications in [113]; they are not counted here as errors left unacknowledged by him. There is no proof: the path integral is not defined beyond the semiclassical expansion, the localisation argument assumes an off-shell supersymmetric completion of the higher-derivative terms, and the identification of the subtracted terms with the hair is made case by case. The missing step is a definition of the right-hand side of (23) that does not rely on a saddle point expansion; the conjecture is, in this sense, a proposal for what that definition should give.

11.2. The logarithmic corrections. The one-loop contribution to (23) from the massless fields on $AdS_2 \times S^2$ gives a term $c \ln A_H$ in the entropy, with c determined by the spectrum and the two-derivative action alone. Banerjee, Gupta, Mandal and Sen [7] computed it for quarter-BPS black holes in $\mathcal{N} = 4$ and eighth-BPS black holes in $\mathcal{N} = 8$ string theory, and Sen [101] for half-BPS black holes in $\mathcal{N} = 2$ theories with n_V vector and n_H hypermultiplets.

Theorem 11.2 (claim 28). *The logarithmic correction to the entropy of a quarter-BPS black hole in any $\mathcal{N} = 4$ string compactification is zero, and that of an eighth-BPS black hole in $\mathcal{N} = 8$ string theory on T^6 is $-4 \ln A_H$. Both agree with the microscopic indices: the asymptotic expansion of the Fourier coefficients of $1/\Phi_{10}$ and its CHL analogues has no logarithmic term, and that of the $\mathcal{N} = 8$ index, a Fourier coefficient of $\vartheta_1(\tau, z)^2/\eta(\tau)^6$, has the term $-2 \ln \Delta$ with $\Delta \propto A_H^2$, that is, $-2 \ln \Delta = -4 \ln A_H$ [7].*

Proof. The macroscopic computation is the heat kernel of the quadratic fluctuation operator of all massless fields on $AdS_2 \times S^2$ with the near-horizon flux, including the graviton, the gravitini and the vector fields, with the zero modes of the asymptotic symmetries treated separately; the coefficient is the s^0 term of the heat kernel together with the zero-mode counting, $-(c_{\text{local}} + c_{\text{zm}}) \ln a$ with a the radius. In $\mathcal{N} = 4$ the contributions cancel between the gravity multiplet and the vector multiplets [7, Section 7]; in $\mathcal{N} = 8$ the net result is $-8 \ln a = -4 \ln A_H$ [7, Section 8]. The microscopic expansions are in [7, Section 2], from the saddle point expansion of the inverse Siegel form [28] and of the Jacobi form. The computation was rederived from the on-shell action of the Euclidean solution and from the AdS_2 Schwarzian in [54, 55], with the same coefficients. $\square$

Verdict 11.3 (claim 28). Proved, as a match between two computations each of which is complete: the macroscopic coefficient is a theorem of one-loop quantum field theory on the near-horizon geometry and the microscopic one of the asymptotics of modular forms. The rotating black holes in four and five dimensions [102] and the extremal Kerr and Reissner–Nordström black holes without supersymmetry [101, 103], for which no microscopic count exists, are predictions and remain confirmed in the sense of claim 29.

Proposition 11.4 (claim 29; Sen 2012). *For a half-BPS black hole in an $\mathcal{N} = 2$ supergravity with n_V vector and n_H hypermultiplets, the logarithmic correction is $\frac{1}{12}(23 + n_H - n_V) \ln A_H$, equal to $(2 - \frac{\chi}{24}) \ln A_H$ with $\chi = 2(n_V - n_H + 1)$ the Euler number of the Calabi–Yau threefold in the type IIA convention, with the universal hypermultiplet included; the type IIB Euler-number relation has the opposite sign.*

Proof. [101], by the same heat kernel method, the only difference from $\mathcal{N} = 4$ being the absence of the cancellation; the result was rederived by [59] from the heat kernel of $\mathcal{N} = 2$ supergravity on the same background. $\square$

Verdict 11.5 (claim 29). Confirmed. A universal comparison for the full class of $\mathcal{N} = 2$ theories is not established here; this does not deny microscopic indices in particular $\mathcal{N} = 2$ compactifications. The comparison made in [101] is with the measure in the OSV formula, and agrees with the measure proposed by Denef and Moore if their result is extrapolated to weak topological string coupling. **A model-independent microscopic comparison in a fixed ensemble is not established.** The general framework and the ensemble must be fixed before comparing a logarithmic coefficient.

11.3. A check of the cutoff-scale preprint. The current preprint [134] separates the extended-supergravity cancellation from an unresolved nonzero-Euler-number case. There is a local notation problem in its Eq. (2.7): a contribution to $\Delta \ln Z_n$ is printed as the multiplicative factor $(g_s^{-1} \lambda_S^2)^2$. In the logarithm it must instead read

$$\Delta \ln Z_{\text{rot}} = \ln[(g_s^{-1} \lambda_S^2)^2] = -2 \ln g_s + 4 \ln \lambda_S. \quad (24)$$

This is a notation correction, not a refutation of the later result. Combining this expression with the three translational modes, the fermionic modes, two-form modes and the ensemble terms gives the stated coefficients $(\alpha_{NZ} + 7 - n)$ and $(n_{NS} - 4)$ in its Eq. (2.10). The elementary coefficient sums are checked in the analytic supplement. The current arXiv PDF was inspected for this equation; the INSPIRE record checked for this revision supplied no journal-version or erratum metadata. That search result is not a proof that no correction exists elsewhere.

The cutoff comparison in [134] keeps other inputs fixed. Its Section 4 explicitly allows coordinated changes of duality frame and measure, and its Section 5 leaves the $\chi \neq 0$ discrepancy unresolved. These qualifications prevent the conditional cutoff calculation from being restated as a theorem excluding every other ultraviolet completion. We do not claim to solve its Calabi–Yau discrepancy here.

12. STRING FIELD THEORY, D-INSTANTONS, SOFT THEOREMS AND MATRIX THEORY

12.1. String field theory. Two claims of Sen about string field theory itself are in the catalogue. The first, with Zwiebach [128], is that closed string field theory is background independent under infinitesimal deformations of the conformal field theory: the string field theories formulated around two nearby conformal backgrounds are related by a field redefinition. The second is the construction of superstring field theory [104, 105, 106] and the proof of its unitarity [107]: a gauge-invariant one-particle-irreducible effective action for heterotic and type II strings, free of the divergences of the picture-changing operators of earlier formulations through the use of a second, free string field of picture number $-3/2$; a Batalin–Vilkovisky master action; and the cutting rules, hence unitarity, of the resulting amplitudes.

Verdict 12.1 (claims 30, 31). Proved. Claim 30 is proved in [128] for marginal deformations to first order (and to second order in the companion paper), by constructing the field redefinition from the connection on the space of conformal field theories; its extension to finite deformations is a theorem only within the formal perturbation theory of the string field and is not a claim of the paper. Claim 31 is a construction and is proved in the papers that make it: the 1PI action is gauge invariant by the main identity of the vertices, the master action satisfies the master equation, and the Cutkosky rules are derived in [107] with the $i\epsilon$ prescription of [74]. The reader who wants the construction in one place has the review [127]. None of this depends on a conjecture; the open questions of superstring field theory are not about these statements.

Corrected versions and infrared scope. The published erratum to [74] and the corrected arXiv v3 must be used for the cutting argument; the correction is excluded from the list of unresolved errors. Section 4 of [107] explicitly notes the infrared problem for $D \leq 4$ and that the inclusive-cross-section construction is not carried out there. This is an acknowledged limitation, not a hidden failure of its perturbative cutting proof. A simple leading-soft cancellation and a counterexample to inferring exact unitarity from a formal series are given in Appendix D. Appendix E constructs the finite-energy limit in the explicit positive soft model and states the additional string-amplitude estimate needed for transfer. Appendix F derives a uniform subtracted tree-emission bound from Sen’s soft theorem and demonstrates why overlapping limits require additional estimates. None of these is a construction of the full infrared-finite string S-matrix.

12.2. D-instantons. The contribution of D-instantons to string amplitudes has an overall normalisation that the world-sheet theory does not fix, because the open string path integral on the instanton has zero modes, bosonic and fermionic, whose measure is ambiguous. Sen [109, 112, 111] showed that open string field theory fixes it: the zero modes are treated by changing variables from the string field to the collective coordinates with a Jacobian computed in the field theory, the gauge zero mode of the $U(1)$ on the instanton is handled by a gauge fixing that the field theory provides, and the resulting normalisation is finite and unambiguous.

Verdict 12.2 (claim 32). Confirmed. In two-dimensional string theory the normalisation computed this way [109, 110] agrees with the matrix model to the order compared, including the two-instanton contribution [14]; in type IIB string theory in ten dimensions the normalisation of the D-instanton correction to the four-graviton amplitude [111] agrees with the coefficient predicted by $SL(2, \mathbb{Z})$ invariance, the Green–Gutperle Eisenstein series. The claim that string field theory fixes the normalisation in every background is confirmed by these cases and **not proved in general**: the construction requires the string field theory of the instanton to be well defined, which it is in these examples, and a general statement needs a general treatment of the zero modes of the Ramond sector, which is not established by the examples discussed here.

12.3. Soft theorems. Laddha and Sen [64] proved the classical soft graviton theorem in dimensions $D > 4$, the statement that the low-frequency gravitational radiation from a scattering process is determined by the momenta and angular momenta of the incoming and outgoing objects, including the sub-subleading term, directly from the classical equations [108]; and Sahoo and Sen [80] found that in $D = 4$ the soft expansion acquires logarithmic terms, $\ln \omega$ at the subleading order, whose coefficients are again universal, and conjectured the form of the corresponding quantum terms.

Verdict 12.3 (claims 33, 34). Claim 33 is proved: the classical theorem in $D > 4$ is derived in [64], and the classical logarithmic terms in $D = 4$ in [80], from the classical equations of motion with the long-range forces included. Claim 34 is confirmed: the quantum logarithmic

terms were computed at one loop in [80] for scalar matter coupled to gravity and to electromagnetism and found to have the conjectured form, with the classical term and a quantum correction that depends only on the asymptotic momenta; the universality of the one-loop $\ln \omega$ terms was extended to spinning particles with the non-minimal couplings allowed by gauge and coordinate invariance by Krishna and Sahoo [61], and the classical soft factors have been pushed to all orders in the electromagnetic case [58]. It is not proved that the quantum $\ln \omega$ terms are one-loop exact; the missing step is a proof that the two-loop and higher infrared divergences exponentiate in the way the conjecture requires; [61] states this as open, a complete higher-loop argument is not supplied here.

A recent scope check. The all-orders classical analysis of [133] distinguishes the Newtonian result from its proposed relativistic extension. It also finds that a separate all-orders conjecture of Alessio, Di Vecchia and Heissenberg, consistent with two-particle scattering, does not hold for general initial or final states with at least three particles. This is not a counterexample to Sen’s earlier subleading quantum result. The extension in [133] is explicitly conjectural and is recorded as an open construction, not a wrong proof.

12.4. Matrix theory. Sen [121] argued, independently of Seiberg [85], that the matrix theory description of M-theory in the infinite momentum frame follows from the compactification of M-theory on a light-like circle, taken as a limit of a small space-like circle, and extended it to tori T^n : the theory of N D0-branes on T^n in a limit in which the torus and the string scale shrink. For $n \leq 3$ the limit is a maximally supersymmetric Yang–Mills theory in $n + 1$ dimensions; for $n = 4, 5$ it is the $(2, 0)$ theory on T^5 and the little string theory on T^5 respectively; and for $n \geq 6$ the limit is, as Sen states, “a theory without an independent definition”, the degrees of freedom of the D0-branes not decoupling from gravity. The two statements H1 and H2 of the catalogue are these, and they are recorded without a verdict: the first is the Seiberg–Sen argument, which is a derivation within the assumptions of the infinite momentum frame and has no independent test for $n \geq 4$; the second is a statement about a theory that is not defined, and cannot be tested in any regime. We mention them because the paper is cited as a derivation of matrix theory and the reader should know that its content for $n \geq 6$ is a limit of a non-decoupled system, as its author says.

13. CONCLUSION

The selected catalogue contains results proved within specified frameworks and physical proposals with different degrees of support. The exact computations and analytic tests do not justify a count of “fatal errors” in an author’s work. The toroidal negative full indices and their subsequent hair interpretation are already in the literature. The caustic calculation concerns the classical dust approximation and leaves the homogeneous rolling solution intact. The divisor singularity belongs to the orientifold cover, while a local I_0^* family shows why non-Higgsability alone does not prohibit weak coupling. Two of Sen’s statements now hold in general form: his weak coupling family is the normal form of every degeneration with $v(j) < 0$, and the entropy of a two-charge small black hole is fixed by one critical value, so that the Gauss–Bonnet truncation fails exactly from $D = 6$ on and Prester’s Lovelock completion is forced.

The remaining questions are mathematical and physical construction problems: the identification of the limit of Theorem 9.5 with a type IIB orientifold beyond its geometry, the entropy of the complete heterotic effective action in $D \geq 6$, which lies outside the curvature-only actions of Theorem 10.3 and which the central charges of Theorem 10.10 do not supply, the full monopole L^2 statement beyond the cited cases, general hair identification, a controlled definition of the quantum entropy function, and extensions of the stated soft and string-field results. They are not resolved by the finite tables. A meaningful new criticism would require a precise original statement with all its hypotheses retained and a counterexample within those hypotheses. No new false proof by Sen meeting that criterion is established

in this revision, so the text has no rose and no burnt orange passage, and every yellow one is collected in Table 5.

A reader who wants to repeat the computations of this paper will find them in the archive [10], together with the Lean formalisation of the finite statements: the Fourier coefficients of Tables 7 and 8, the wall-crossing identities of Theorem 4.2 and the twenty-one negative indices of Theorem 5.4, each computed inside the Lean kernel from the definitions of the theta functions. The catalogue and physical interpretations are not formalised by these certificates. Mathematical parts of such subjects can be formalised, but that work must not be inferred from a finite arithmetic check.

APPENDIX A. THE COMPUTATION OF THE FOURIER COEFFICIENTS

The tables of this paper are Fourier coefficients of the inverse of the product (3), computed in exact rational arithmetic. We record the steps so that the numbers can be reproduced without the program, which is in the archive [10].

The genera. The twisted elliptic genera are expanded in $q = e^{2\pi i\tau}$ and $y = e^{2\pi iz}$ from the theta functions $\theta_i(\tau, z)$ and $\eta(\tau)$ as q -series with coefficients in $\mathbb{Z}[y, y^{-1}]$, and the coefficients $c_b^{(r,s)}(4n - j^2)$ are read off from $\sum_{n,j} c_b^{(r,s)}(4n - j^2) q^n y^j$. For K3, $F^{(0,0)} = 8A$ with $A = \sum_{i=2}^4 \theta_i(\tau, z)^2 / \theta_i(\tau, 0)^2$; for the $\mathbb{Z}_2$ CHL model the genera are $F^{(0,0)} = 4A$, $F^{(0,1)} = \frac{4}{3}A - \frac{2}{3}B E_2(\tau)$, $F^{(1,0)} = \frac{4}{3}A + \frac{1}{3}B E_2(\tau/2)$, $F^{(1,1)} = \frac{4}{3}A + \frac{1}{3}B E_2((\tau + 1)/2)$, with $B = \theta_1(\tau, z)^2 / \eta(\tau)^6$ and $E_2(\tau) = 2E(2\tau) - E(\tau)$, where E is the Eisenstein series of weight two normalised to $1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n$; for the $\mathbb{Z}_2$ toroidal orbifold they are (9). The arguments $\tau/2$ and $(\tau + 1)/2$ are handled by expanding in $q^{1/2}$ with the sign $(-1)^n$ on the odd powers, so every series is a Laurent series in $x = q^{1/8}$, which is the smallest fractional power that occurs.

The product. Each factor $(1 - e^{2\pi i(k'\sigma + l\rho + jv)})^{-c}$ of $1/\tilde{\Phi}$ is expanded as a binomial series in the monomial $e^{2\pi i(k'\sigma + l\rho + jv)}$ to the order needed, and the factors are multiplied as polynomials in the three variables $e^{2\pi i\rho}$, $e^{2\pi i\sigma/N}$, $e^{2\pi iv}$ truncated at the exponents $Q^2/2 \leq L$, $NP^2/2 \leq K$ wanted in the tables. The exponent $c = \sum_s e^{2\pi isl/N} c_b^{(r,s)}(4k'l - j^2)$ is an integer for $N \leq 2$, where the root of unity is ± 1 ; for $N = 3, 5, 7$ it would require cyclotomic arithmetic, and we have not computed those models. The one factor with $k' = l = 0$, $(1 - e^{\mp 2\pi iv})^{-2}$ after the prefactor is taken into account, is the only one whose expansion depends on the chamber: in the chamber $\text{Im } v < 0$ it is expanded in powers of $e^{-2\pi iv}$, in the chamber $\text{Im } v > 0$ in powers of $e^{2\pi iv}$. The two expansions give the two columns of Table 7 at $\pm Q \cdot P$, and their difference is the wall-crossing jump of Theorem 4.2, which the program checks for every $Q^2, P^2 \in \{2, 4, 6\}$ and $|Q \cdot P| \leq 4$.

The index. The coefficient of $e^{2\pi i(\rho Q^2/2 + \sigma P^2/2 + v Q \cdot P)}$ in $1/\tilde{\Phi}$, multiplied by $(-1)^{Q \cdot P + 1}$, is the index $d(Q, P)$ of (2). For the orbifolds the prefactor of the product is $e^{2\pi i(\rho + \sigma/N + v)}$ for the CHL models and $e^{2\pi i(\rho + v)}$ for the toroidal orbifolds, as in [16], and the integral over σ runs over a period of length N (Remark 5.9).

The tables. Table 7 is the heterotic index for $Q^2, P^2 \in \{2, 4, 6\}$ and $-2 \leq Q \cdot P \leq 4$; it reproduces Table 1 of [100] entry by entry. Table 8 is the index of the $\mathbb{Z}_2$ toroidal orbifold for $Q^2 \leq 12$, $P^2 \in \{0, 2, 4\}$ and $Q \cdot P \in \{0, 1, 2\}$; it agrees with Tables 17 to 19 of [16] in every entry with $Q^2 \leq 12$ but one, the entry $Q^2 = 4$, $P^2 = 2$, $Q \cdot P = 0$, which [16] prints as $-11\,520$ and which the product formula gives as $+11\,520$, the sign required by the rule $(-1)^{Q^2}$ of [16] for $P^2 = 2$; the misprint lies inside the single-centred cone and would otherwise be a thirty-first negative entry. Table 6 lists the charges in the cone (8) with $Q^2 \leq 12$, $P^2 \leq 8$, $Q \cdot P \leq 4$ whose index is negative. The same scan over the cone of the heterotic string and of the $\mathbb{Z}_2$ CHL model, up to $Q^2 \leq 12$ (heterotic) or $NQ^2/2 \leq 6$ (CHL), $P^2 \leq 6$, $Q \cdot P \leq 4$, finds no non-positive index, and the CHL values reproduce Table 2 of [100].

Q^2	P^2	$d(Q, P)$ for $Q \cdot P = -2, -1, 0, 1, 2, 3, 4$						
2	2	-209 304	130 329	50 064	25 353	648	327	0
4	2	-2 023 536	1 598 376	1 127 472	561 576	50 064	8 376	-648
6	2	-15 493 728	16 844 421	16 491 600	8 533 821	1 127 472	130 329	-15 600
4	4	-16 620 544	28 698 000	32 861 184	18 458 000	3 859 456	561 576	12 800
6	4	-53 249 700	474 507 528	632 078 672	392 427 528	110 910 300	18 458 000	1 127 472
6	6	2 857 656 828	11 890 608 225	16 193 130 552	11 232 685 725	4 173 501 828	920 577 636	110 910 300

TABLE 7. The index $d(Q, P) = (-1)^{Q \cdot P + 1} c(Q, P)$ of quarter-BPS dyons of the heterotic string on T^6 , from $1/\Phi_{10}$, in the chamber $\text{Im } v < 0$ (the expansion in powers of $e^{-2\pi i v}$). The table agrees entry by entry with Table 1 of [100]. Entries with $Q \cdot P < 0$ are the other chamber's values at $|Q \cdot P|$ by the symmetry $v \mapsto -v$; their negativity for $Q \cdot P \leq -1$ is the two-centred contribution of Theorem 4.2.

Q^2	$Q \cdot P = 0$			$Q \cdot P = 1$			$Q \cdot P = 2$		
	$P^2 = 0$	$P^2 = 2$	$P^2 = 4$	$P^2 = 0$	$P^2 = 2$	$P^2 = 4$	$P^2 = 0$	$P^2 = 2$	$P^2 = 4$
0	2	64	816	0	0	-8	0	0	0
1	-12	-224	-1 248	1	96	1 968	0	0	-12
2	48	1 152	18 240	-8	-256	840	0	64	2 592
3	-168	-3 392	-10 320	37	1 376	34 656	-2	-224	2 432
4	528	11 520	200 736	-136	-3 840	16 632	16	1 152	43 392
5	-1 512	-30 336	-55 424	439	13 152	343 152	-72	-3 392	33 720
6	4 032	83 968	1 544 832	-1 288	-33 536	171 152	256	11 520	414 336
7	-10 176	-202 560	-179 022	3 503	92 928	2 476 752	-806	-30 336	302 400
8	24 528	496 512	9 480 000	-8 968	-220 672	1 265 256	2 320	83 968	2 926 080
9	-56 796	-1 118 496	-155 232	21 854	540 416	14 545 584	-6 200	-202 560	2 049 968
10	127 008	2 521 600	49 523 328	-51 080	-1 204 992	7 558 560	15 616	496 512	16 919 712
11	-275 544	-5 374 656	2 684 560	115 154	2 711 616	73 540 080	-37 508	-1 118 496	11 568 000
12	581 952	11 389 440	228 872 064	-251 528	-5 741 824	38 736 600	86 544	2 521 600	84 554 880

TABLE 8. The index $d(Q, P)$ of quarter-BPS dyons in the $\mathbb{Z}_2$ toroidal orbifold of type IIB on T^6 , from the product formula for $1/\tilde{\Phi}_2$ with the twisted elliptic genera (9), in the chamber $\text{Im } v < 0$ and with the normalisation of Remark 5.9. The entries agree with Tables 17, 18 and 19 of [16] in their common range $Q^2 \leq 12$, except the entry $Q^2 = 4, P^2 = 2, Q \cdot P = 0$, printed there as $-11 520$: the product formula gives $+11 520$, and the sign rule $(-1)^{Q^2}$ for $P^2 = 2$ stated in the same section of [16] requires the positive value. The column $P^2 = 0$ lies outside the single-centred cone; its sign is $(-1)^{Q^2}$. The negative entries inside the cone are those of Table 6.

APPENDIX B. THE CAUSTIC FOR A PERIODIC PERTURBATION

This appendix works out Theorem 7.4 for the simplest inhomogeneous initial datum in one space dimension, a periodic perturbation of the homogeneous decay,

$$T_0(x) = T_* + a \cos kx, \quad a > 0, k > 0, \quad (25)$$

on the branch $\partial_t T > 0$, and records the quantities that a numerical solution of the full equation (11) can be compared with.

Velocity and its derivative. With $T'_0 = -ak \sin kx$ and $W_0 = \sqrt{1 + a^2 k^2 \sin^2 kx}$,

$$v_0(x) = \frac{ak \sin kx}{\sqrt{1 + a^2 k^2 \sin^2 kx}}, \quad v'_0(x) = \frac{ak^2 \cos kx}{(1 + a^2 k^2 \sin^2 kx)^{3/2}}. \quad (26)$$

The dust moves away from the maxima of T_0 (where $\cos kx = 1$ and $v'_0 > 0$, a rarefaction) and towards its minima (where $\cos kx = -1$ and $v'_0 = -ak^2 < 0$).

The caustic time. The supremum of $-v'_0$ is attained at the minima $x_m = (2m+1)\pi/k$ of T_0 , so by Theorem 7.4(iv)

$$\mu_- = \sup_x (-v'_0(x)) = ak^2, \quad t_c = \frac{1}{ak^2}. \quad (27)$$

The characteristics $X(t; x_0) = x_0 + t v_0(x_0)$ from a neighbourhood of x_m converge on x_m , which is a fixed point of the flow, and have a singular differential there at t_c ; for $t > t_c$ the map $x_0 \mapsto X(t; x_0)$ folds over near x_m and three characteristics pass through each point of a growing interval around it. The smaller the perturbation, the later the caustic, and no value of $a > 0$ avoids it; the time is finite for every amplitude and every wavelength, and diverges only in the homogeneous limit $a \rightarrow 0$ or in the long-wavelength limit $k \rightarrow 0$ at fixed a .

The density. Along the characteristic from x_0 ,

$$\varepsilon(t, X(t; x_0)) = \frac{\varepsilon_0(x_0)}{1 + t v'_0(x_0)}, \quad \varepsilon(t, x_m) = \frac{\varepsilon_0(x_m)}{1 - t/t_c},$$

and the density at the minimum diverges as $(t_c - t)^{-1}$. Near the caustic, for $x_0 = x_m + \xi$ with ξ small,

$$v'_0(x_0) = -ak^2 \left(1 - \frac{k^2 \xi^2}{2} (1 + 3a^2 k^2) + O(\xi^4) \right),$$

so at $t = t_c$ the linear term of $X(t_c; x_m + \xi) - x_m$ cancels and

$$X(t_c; x_m + \xi) = x_m + c \xi^3 + O(\xi^5), \quad c = \frac{k^2}{6} (1 + 3a^2 k^2),$$

while the density there is $\varepsilon_0(x_m) / (\frac{k^2}{2} (1 + 3a^2 k^2) \xi^2) \propto |x - x_m|^{-2/3}$: the integrable density singularity at the cubic cusp onset, followed by fold points after the first focusing time, with total energy in any interval finite at t_c and infinite density at one point. For $t > t_c$ the three-valued velocity field carries a finite energy density on each sheet and the dust description of [86] has no single-valued continuation.

Comparison with the full equation. Felder, Kofman and Starobinsky [44] integrated (11) numerically with the exponential potential and inhomogeneous initial data, and found that after a transient the solution is approximated by the free-streaming (Hamilton–Jacobi) solution, which is (12), and that the caustics predicted by it form in the full equation, with second derivatives of T growing without bound at the caustic; Barnaby [4] showed the same analytically for several effective actions. The dust formula gives a comparison time after matching effective initial data. It is not an exact prediction of the full Born–Infeld evolution: the transient and higher-derivative corrections depend on the initial data and the effective action. No amplitude-independent transient time or string-scale conclusion is inferred here.

APPENDIX C. INDEPENDENT SCOPE CHECKS AND SOURCE REGISTER

C.1. The sign of compression. For the forward branch $T_t = \sqrt{1 + |\nabla T|^2}$, the characteristic velocity is minus the normalised gradient. In one dimension let $d = (1 + T_0'^2)^{3/2} > 0$. Then

$$v'_0 = -T_0''/d < 0 \quad \Longleftrightarrow \quad T_0'' > 0.$$

This sign is independent of the density and reverses if the time branch is reversed. The scalar equivalence and the identity $1 + (-1/\lambda)\lambda = 0$ for $\lambda < 0$ are separately checked in the analytic Lean supplement. These identities are not a formalisation of the full characteristic PDE theorem. For the periodic datum in Appendix B, compactness ensures attainment. For a general noncompact datum, replacing a supremum by a maximum is an additional assumption.

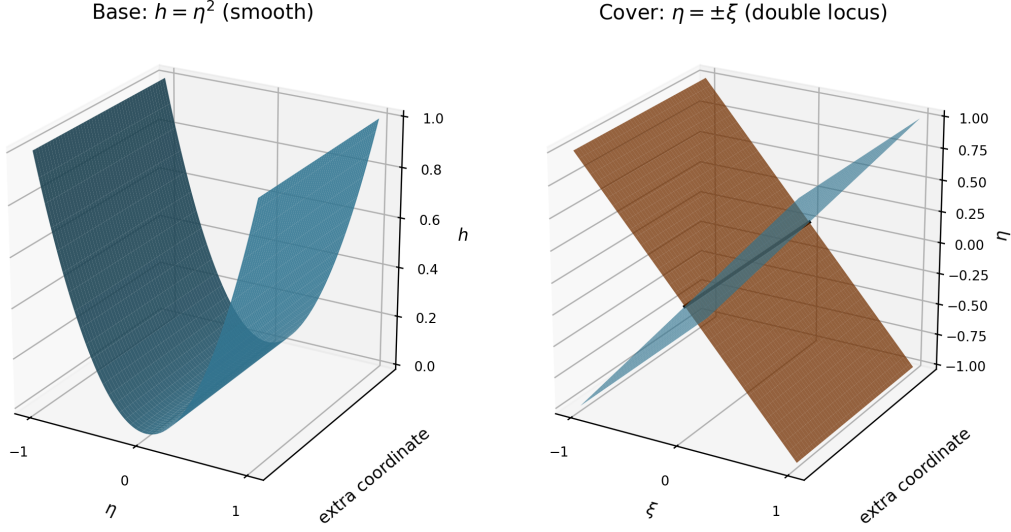


FIGURE 7. Local distinction at $\chi = 1$. The base divisor $h = \eta^2$ is smooth. Its pullback under $h = \xi^2$ consists of the crossing branches $\eta = \pm\xi$. The diagram displays real slices of complex divisors; it is a schematic, not a rendering of the full compactification.

C.2. What an index can and cannot imply. Write a signed index as $I = B - F$ with nonnegative integer counts. The toy example $B = 3, F = 2$ gives $I = 1 > 0$ although fermionic states remain. Thus positivity cannot alone prove that every contributing state has zero angular momentum. Conversely, if assumptions A_1, A_2, A_3 imply $I > 0$ and a computed index has $I < 0$, one can conclude only $\neg(A_1 \wedge A_2 \wedge A_3)$. Selecting a particular failed assumption requires additional physics. This logical distinction is especially relevant when interpreting a negative full index as a statement about hair in every duality frame.

C.3. The charge cone and reproducibility. Equation (3.9) of [100] contains $3(Q \cdot P) \leq 2Q^2 + P^2$. The squared inner product occurs instead in the discriminant constraint. Mixing these two conditions changes the admissible charges. The Python scan uses the linear inequality, integer arithmetic, $Q^2 \leq 12$, $P^2 \leq 8$ and $Q \cdot P \leq 4$ and returns thirty negative triples. The Lean series certificate retains its original table range and verifies the original twenty-one triples with the corrected cone predicate. Its list is a subset, not an exhaustive scan of the larger box. The formal scope report records the exact toolchain, source hashes, successful checks and omissions.

C.4. The base and the cover. The local test $\chi = 1$ already separates the two spaces. On the base, $\eta^2 - h = 0$ has derivative $\partial_h = -1$, hence is smooth. On the cover $h = \xi^2$, the equation becomes $(\eta - \xi)(\eta + \xi) = 0$, a singular union at the origin. Blowing up or normalising the cover divisor may be useful for its invariants, but cannot turn this distinction into an error in the original weak-coupling expansion. The discriminant identity of Theorem 9.1 is checked as a polynomial identity in the analytic Lean supplement; the global geometry is not.

C.5. Primary passages used for scope corrections. The principal comparisons are reproducible from the following sources, with their original qualifications retained.

- [100], introduction, pp. 3–5 and Eq. (3.9): assumptions about hair and other index contributions, and the linear charge cone.

- [16], Sections 4.1–4.2: known negative full indices in toroidal orbifolds; [17], Sections 4–5: proposed hair factors, finite positive horizon-index tests, and remaining mode-identification questions.
- [86], Eqs. (19)–(25): the pressureless Hamiltonian equations and the static solution with arbitrary density. The homogeneous decay assumption is stated in its abstract.
- [118], abstract, weak-coupling expansion and auxiliary double cover: the reduction is formal and singular auxiliary spaces are explicitly discussed. The D7 cover singularities are developed in [18]; global weak-coupling obstructions and the I_0^* exception are in [132].
- [94], abstract and Gauss–Bonnet calculation: the higher-dimensional coefficient mismatch is reported by Sen, rather than discovered here.
- [113]: domain, ultraviolet and flat-direction issues are raised by Sen himself; they are not new unacknowledged objections.
- [45], Theorem 1.2 and Section 6, state the general coprime case using the announced metric compactification; [60], Theorem A: magnetic charge three is covered under the stated QFB-metric hypothesis. [61] separates proved one-loop universality from conjectural higher quantum extensions. [133] labels its beyond-Newtonian extension a conjecture.

This register is not evidence that every reference or every later citing paper has been independently checked. The bibliography supplies further context; new objections require their own exact source and hypothesis check.

APPENDIX D. TWO LIMITS OF PERTURBATIVE REASONING

D.1. An exact leading-soft model. Assume independent soft emissions with intensity $b d\omega/\omega$, $b \geq 0$, an infrared cutoff λ , detector resolution E and hard scale Λ , with $0 < \lambda < E < \Lambda$. Assume also the virtual factor $\exp[-b \ln(\Lambda/\lambda)]$. Summing any number of unobserved emissions below E gives

$$\begin{aligned} P_\lambda &= e^{-b \ln(\Lambda/\lambda)} \sum_{n=0}^{\infty} \frac{[b \ln(E/\lambda)]^n}{n!} \\ &= e^{-b \ln(\Lambda/\lambda) + b \ln(E/\lambda)} = e^{-b \ln(\Lambda/E)}. \end{aligned} \quad (28)$$

The series is absolutely convergent and the last expression is independent of λ . This is a self-contained cancellation in the independent-emission model. It assumes soft factorisation and the virtual factor; it does not prove either for every string-field-theory diagram, include subleading emissions, or construct the general four-dimensional observable. The missing physical and analytic steps must not be concealed by the elementary exponential identity. The finite-exponent cancellation is separately checked in Lean.

D.2. Formal unitarity does not determine a completion.

Proposition D.1. *There exist two smooth scalar amplitudes with the same Taylor series at zero coupling such that one is unitary and the other is not for every positive coupling.*

Proof. Take $S_0(g) = 1$ and $S_1(g) = 1 + f(g)$, where $f(0) = 0$ and $f(g) = e^{-1/g^2}$ for $g \neq 0$. For every integer $N \geq 0$, choose an integer m with $2m > N$. Since $e^t \geq t^m/m!$ for $t > 0$,

$$0 \leq |g|^{-N} f(g) \leq m! |g|^{2m-N} \rightarrow 0.$$

Each derivative of f away from zero is a finite linear combination of powers of g^{-1} times $f(g)$. Induction using the displayed bound extends every derivative continuously to zero with value zero. Thus both amplitudes have the same Taylor series, the constant 1. It satisfies the formal scalar unitarity identity to every order, whereas $S_1(g) > 1$ gives $|S_1(g)|^2 > 1$ for $g > 0$. $\square$

This is a counterexample to a general inference from a perturbative series to an exact completion, not a counterexample to Sen’s perturbative theorem. The flat-function smoothness

argument is proved here in ordinary analysis and is not part of the finite Lean certificate. A nonperturbative completion needs additional input such as a summation prescription, analytic class, boundary conditions or an independently constructed theory.

APPENDIX E. A FINITE-ENERGY INFRARED LIMIT IN AN EXPLICIT SOFT MODEL

The leading-soft cancellation of Appendix D can be strengthened to a construction allowing infinitely many unresolved quanta and a total-energy observable. The theorem below is a statement about an explicitly defined positive emission model. Its relation to a string amplitude requires additional factorisation estimates, stated afterwards. It is not a claimed solution of the full four-dimensional string-field-theory problem.

Theorem E.1 (finite-energy limit and quantitative cutoff removal). *Fix $b \geq 0$ and $0 < E < \Lambda$. For $0 < \lambda < E$, let N_λ be a Poisson random measure on $[\lambda, E]$ with intensity $b d\omega/\omega$, coupled by restriction of one measure on $(0, E]$. Put $S_\lambda = \int_{[\lambda, E]} \omega N(d\omega)$. There is a finite nonnegative random variable S such that $S_\lambda \uparrow S$ almost surely and in L^1 , and*

$$\mathbb{E}S = bE, \quad \mathbb{E}|S - S_\lambda| = b\lambda, \quad (29)$$

$$\mathbb{E}e^{-tS} = \exp \left[b \int_0^E \frac{e^{-t\omega} - 1}{\omega} d\omega \right] \quad (t \geq 0). \quad (30)$$

For a bounded Lipschitz function $F : [0, \infty) \rightarrow \mathbb{R}$, with Lipschitz constant L_F , define the inclusive model functional

$$\mathcal{I}_\lambda(F) = e^{-b \ln(\Lambda/\lambda)} \sum_{n=0}^{\infty} \frac{b^n}{n!} \int_{[\lambda, E]^n} F \left(\sum_{j=1}^n \omega_j \right) \prod_{j=1}^n \frac{d\omega_j}{\omega_j}. \quad (31)$$

Then

$$\mathcal{I}_0(F) = (E/\Lambda)^b \mathbb{E}F(S), \quad (32)$$

$$|\mathcal{I}_\lambda(F) - \mathcal{I}_0(F)| \leq (E/\Lambda)^b L_F b\lambda. \quad (33)$$

If $b > 0$, the total number of emitted quanta is infinite almost surely, although S is finite, and the law of S has no atoms. Consequently the cutoff also has a limit for each total-energy veto $F(s) = \mathbf{1}_{s \leq \delta}$.

Proof. Partition $(0, E]$ into dyadic bands $J_k = (E2^{-k-1}, E2^{-k}]$, $k \geq 0$. On each band take an independent Poisson number with mean $b \ln 2$, and conditional on that number take independent energies with density $1/(\omega \ln 2)$ on the band. This constructs the measure on the product probability space without assuming a finite total particle number. The expectation of the energy in J_k is $b|J_k|$, so the sum of the expectations is bE . The nonnegative total energy is therefore finite almost surely. Monotone convergence gives the almost-sure limit and

$$\mathbb{E}(S - S_\lambda) = \int_0^\lambda \omega b \frac{d\omega}{\omega} = b\lambda.$$

For a finite cutoff, the elementary generating function of a Poisson distribution yields

$$\mathbb{E}e^{-tS_\lambda} = \exp \left[b \int_\lambda^E \frac{e^{-t\omega} - 1}{\omega} d\omega \right].$$

The bound $|e^{-t\omega} - 1| \leq t\omega$ makes the exponent integrable at zero. Dominated convergence proves the Laplace formula for S . Writing the Poisson expectation of $F(S_\lambda)$ as its absolutely convergent sum over particle numbers gives exactly

$$\mathcal{I}_\lambda(F) = (E/\Lambda)^b \mathbb{E}F(S_\lambda).$$

The Lipschitz estimate follows from the common coupling and the preceding L^1 identity.

If $b > 0$, the count on $(\lambda, E]$ has Poisson mean $b \ln(E/\lambda) \rightarrow \infty$. For every fixed integer K , its probability of being at most K tends to zero. Hence the total count is infinite almost surely.

To prove absence of atoms, split at $a \in (0, E)$. The sum of energies in $[a, E]$, conditional on at least one emission, has a continuous distribution, since each energy has a density and convolutions retain that property. It is independent of the energies below a . Thus for every x ,

$$\mathbb{P}(S = x) \leq \mathbb{P}(N([a, E]) = 0) = (a/E)^b.$$

Let $a \downarrow 0$. Finally $S_\lambda \rightarrow S$ almost surely, and the indicator of a fixed total-energy threshold is continuous away from an event of probability zero. Bounded convergence proves the final assertion. For $b = 0$ the measure is concentrated at $S = 0$ and the conclusion is immediate directly. $\square$

This theorem treats total unresolved energy, not just a bound on each individual quantum. It also explains why a finite-energy limit need not be a finite-particle Fock vector. It constructs an energy distribution, not the angular momentum, memory algebra or gauge constraints of a gravitational quantum state.

Proposition E.2 (an explicit transfer criterion). *Let $\Sigma_\lambda(F)$ be a proposed regulated string observable. Suppose, for each observable in a specified class,*

$$\Sigma_\lambda(F) = H\mathcal{I}_\lambda(F) + R_\lambda(F),$$

where H is finite and independent of λ , and $R_\lambda(F) \rightarrow R_0(F)$ is separately proved. Then $\Sigma_\lambda(F)$ has a finite limit, with

$$\Sigma_0(F) = H\mathcal{I}_0(F) + R_0(F).$$

For Lipschitz F , its error is at most $|H|(E/\Lambda)^b L_F b \lambda + |R_\lambda(F) - R_0(F)|$.

Proof. Subtract the two expressions and apply Theorem E.1 and the triangle inequality. The convergence assumption on the remainder is essential; the proposition does not establish it. $\square$

The remaining string-theory problem. Weinberg's treatment [135] establishes the familiar gravitational soft cancellations in its stated setting. More recent algebraic scattering constructions [136, 137] explicitly address the state-space problem. They do not justify silently replacing a constructed string-field evolution by an assumed automorphism. For the full string problem one must supply a specified stable compactification and observable, a common regulator compatible with the gauge identities, factorisation over all relevant degeneration regions, the corresponding virtual factors and measure, and control of collinear regions and overlapping limits. In the transfer proposition these obligations reside in the existence of H and the convergence of R_λ . No proof of those obligations for every four-dimensional superstring background is supplied here. The Poisson construction and the scalar exponential identity are not Lean-formalised as probability theory in this release.

APPENDIX F. A UNIFORM SINGLE-SOFT ESTIMATE AND THE OVERLAPPING-LIMIT OBSTRUCTION

This appendix makes one part of the transfer problem quantitative. The tree soft theorem is an established result of Sen [138]; the estimate below is a consequence on compact sets of massive hard kinematics, not a replacement for that theorem or a new proof of the complete four-dimensional infrared problem. We then give an exact example explaining why the same argument does not establish the required loop remainder.

F.1. Kinematics and the tree pole decomposition. Use signature $(+---)$ and write a future null soft momentum as $q = \omega n$, where $n = (1, \hat{n})$, $|\hat{n}| = 1$. The hard momenta p_i are future timelike; incoming and outgoing legs have signs $\eta_i = -1$ and $+1$, respectively. At the limiting hard configuration, $\sum_i \eta_i p_i = 0$. In the convention with linear coupling $-(\kappa/2)h_{\mu\nu}T^{\mu\nu}$, the leading coefficient for one physical graviton polarisation is

$$a(n) = \frac{\kappa}{2} \mathcal{M}_N \sum_i \eta_i \frac{\epsilon_{\mu\nu}(n) p_i^\mu p_i^\nu}{p_i \cdot n}. \quad (34)$$

Here $\mathcal{M}_N$ is the stripped hard amplitude. A smooth on-shell recoil prescription is fixed when imposing momentum conservation at nonzero q . Its changes at order ω belong in the remainder. Formula (34) is unchanged by $\epsilon_{\mu\nu} \mapsto \epsilon_{\mu\nu} + n_\mu \xi_\nu + \xi_\mu n_\nu$, since the change is $\kappa \mathcal{M}_N \xi \cdot \sum_i \eta_i p_i = 0$.

Choose a compact set of nonsingular hard configurations, away from additional hard-channel poles and thresholds, and restrict to a recoil prescription defined on a neighbourhood of it. Since each hard leg is massive,

$$p_i \cdot n \geq p_i^0 - |\mathbf{p}_i| = \frac{m_i^2}{p_i^0 + |\mathbf{p}_i|} > 0. \quad (35)$$

After decreasing the upper soft energy E if necessary, the same lower bound with a smaller positive constant holds uniformly along the recoil family. A physical helicity basis can be chosen in finitely many angular charts, a finite cover of the sphere of the kind whose combinatorics is studied in [5]; the bounds below concern its invariant norms and do not require a globally smooth polarisation vector on the sphere.

In the tree sectors covered by [138], use the usual meromorphic tree amplitude and its external-leg pole factorisation. On the selected compact neighbourhood it has the form

$$\mathcal{M}_{N+1}(\omega, n) = \frac{1}{\omega} \sum_i f_i(\omega, n) + h(\omega, n), \quad \sum_i f_i(0, n) = a(n), \quad (36)$$

where the f_i are continuously differentiable in ω and h is bounded, uniformly in the hard configurations and the angular charts. These are tree analytic inputs, not assumptions about a loop amplitude. One obtains the f_i by absorbing the nonzero factor $2\eta_i p_i(\omega, n) \cdot n$ into the numerator of each external-leg propagator. Other channels are regular by the choice of neighbourhood. The amplitude may contain infinitely many string poles elsewhere; the statement uses only local regularity away from those poles, rather than a truncation of the string spectrum.

Proposition F.1 (uniform tree remainder). *Let $H = \sup |h|$, $D_i = \sup |\partial_\omega f_i|$, and $C = H + \sum_i D_i$ on the neighbourhood just specified. Then*

$$\mathcal{M}_{N+1}(\omega, n) = \frac{a(n)}{\omega} + r(\omega, n), \quad |r(\omega, n)| \leq C \quad (0 < \omega \leq E). \quad (37)$$

Proof. Subtract the coefficient at $\omega = 0$ in (36). The fundamental theorem of calculus gives

$$r(\omega, n) = h(\omega, n) + \sum_i \int_0^1 \partial_\omega f_i(t\omega, n) dt.$$

The stated bound follows by the triangle inequality. All suprema are finite by the regularity and compactness hypotheses. The lower bound (35) is what makes the estimate uniform in angle for this argument. It supplies no such constant for a massless hard leg. $\square$

F.2. An explicit bound after the real-emission subtraction. Let Ω be the angular and physical-polarisation space, with finite measure $\mu(\Omega)$; for both helicities integrated over the full sphere, $\mu(\Omega) = 8\pi$. Put $A = \sup_\Omega |a|$. The one-particle phase-space convention is

$$d\Phi_q = \frac{\omega d\omega d\mu(n)}{2(2\pi)^3}.$$

Let a real measurement function satisfy $|F(\omega, n)| \leq B$ and $|F(\omega, n) - F(0, n)| \leq L\omega$. Smooth recoil Jacobians and bounded hard-bin weights may be included in F when these two estimates hold.

Theorem F.2 (integrable subtracted tree emission). *Under Proposition F.1, define*

$$T(\omega, n) = F(\omega, n)|\mathcal{M}_{N+1}(\omega, n)|^2 - F(0, n)\frac{|a(n)|^2}{\omega^2}.$$

The signed integral $\mathcal{R}_\lambda(F) = \int_{\lambda < \omega < E} T d\Phi_q$ has a limit as $\lambda \downarrow 0$, and

$$|\mathcal{R}_0(F) - \mathcal{R}_\lambda(F)| \leq \frac{\mu(\Omega)}{2(2\pi)^3} \left[(LA^2 + 2BAC)\lambda + \frac{BC^2}{2}\lambda^2 \right]. \quad (38)$$

Proof. The complex amplitude identity in (37) gives

$$\left| \frac{a}{\omega} + r \right|^2 - \frac{|a|^2}{\omega^2} \leq \frac{2|a||r|}{\omega} + |r|^2.$$

Separate T into $(F - F(0, n))|a|^2/\omega^2$ and F times the difference of the two squared amplitudes. Consequently

$$|T(\omega, n)| \leq \frac{LA^2 + 2BAC}{\omega} + BC^2.$$

Multiplication by the phase-space factor ω leaves an integrable bound at zero. Integrating it over $(0, \lambda)$ proves absolute convergence and (38). $\square$

This proves cutoff removal for the *subtracted single-real-emission tree sector*. It neither assigns a finite value to its unsubtracted logarithmic term nor supplies the corresponding virtual contribution. Those are separate ingredients in an inclusive observable.

The lower bound (35) applies to a real on-shell null q , not to an arbitrary virtual loop momentum. For example $p = (m, 0, 0, 0)$ and $k = (0, \mathbf{k})$ give $p \cdot k = 0$ even when $\mathbf{k} \neq 0$. Hence there is no estimate $|p \cdot k| \geq c\|k\|$ on the complete real loop domain merely because p is massive. A virtual estimate must handle the pole prescription, contour and cancellations; it cannot be obtained by copying the real-emission angular bound.

F.3. Logarithmic remainders: a sufficient quantitative bound. A bounded tree remainder must not be assumed unchanged at four-dimensional loop level. Logarithmic soft terms are known [80]. Their presence alone does not prevent integration after the leading subtraction. For a nonnegative integer m , define

$$h_m(\omega) = (1 + \ln(E/\omega))^m, \quad (39)$$

$$P_m(x) = \sum_{j=0}^m \binom{m}{j} j! x^{m-j}, \quad (40)$$

$$Q_k(x) = \sum_{j=0}^k \binom{k}{j} \frac{j!}{2^j} x^{k-j}. \quad (41)$$

For $0 < \lambda \leq E$ and $x = 1 + \ln(E/\lambda)$, direct integration gives

$$\int_0^\lambda h_m(\omega) d\omega = \lambda P_m(x), \quad (42)$$

$$\int_0^\lambda \omega h_m(\omega)^2 d\omega = \frac{\lambda^2}{2} Q_{2m}(x). \quad (43)$$

Indeed put $\omega = \lambda e^{-u}$, expand $(x + u)^m$, and use $\int_0^\infty e^{-cu} u^j du = j!/c^{j+1}$, proved by j integrations by parts. The exponential decay makes each boundary term vanish.

Proposition F.3 (logarithmic extension). *If a specified amplitude has the decomposition (37) with the weaker bound $|r(\omega, n)| \leq Ch_m(\omega)$, the same subtraction is absolutely integrable and its tail is at most*

$$\frac{\mu(\Omega)}{2(2\pi)^3} \left[LA^2\lambda + 2BAC\lambda P_m(x) + \frac{BC^2\lambda^2}{2} Q_{2m}(x) \right]. \quad (44)$$

If a regulator-dependent family r_λ has this bound with the same C, m and converges pointwise to r_0 , its subtracted integrals converge by dominated convergence. If in addition $|r_\lambda - r_0| \leq \varepsilon(\lambda)h_m$ for $\omega \geq \lambda$, their difference is bounded by the tail (44) plus

$$\frac{\mu(\Omega)}{2(2\pi)^3} \varepsilon(\lambda) \left[2BAEP_m(1) + BCE^2 Q_{2m}(1) \right]. \quad (45)$$

Proof. Repeat the proof of Theorem F.2, replacing C by $Ch_m(\omega)$, and then use (42). These moments give a regulator-independent integrable majorant when C, m are uniform. For the quantitative last assertion,

$$|a/\omega + r_\lambda|^2 - |a/\omega + r_0|^2 \leq \frac{2A\varepsilon(\lambda)h_m}{\omega} + 2C\varepsilon(\lambda)h_m^2.$$

Integrate the right side with the measurement bound B and phase space, extending its domain to $(0, E)$. This proves (45). $\square$

The proposition does not assert that the remainders of every string loop amplitude satisfy its uniform hypothesis. Bounds whose constant diverges with the virtual infrared regulator do not meet it.

F.4. Why separate soft limits do not control a corner.

Proposition F.4 (an overlapping-limit obstruction). *For every fixed $x > 0$, the integral $\int_0^E (x+y)^{-2} dy$ is finite, and the analogous statement holds with x and y exchanged. Nevertheless,*

$$\int_\lambda^E \int_\lambda^E \frac{dx dy}{(x+y)^2} = \ln \frac{(E+\lambda)^2}{4E\lambda} \rightarrow +\infty. \quad (46)$$

Proof. The first assertion follows from the antiderivative $-1/(x+y)$. Integrating first in x gives $1/(\lambda+y) - 1/(E+y)$. Integrating in y yields (46). In coordinates $r = x+y$, $t = x/(x+y)$, the Jacobian has absolute value r , so the density becomes $dr dt/r$. Resolving the corner by this change of variables exposes a logarithmic radial divergence; it does not remove it. $\square$

This is an analytic counterexample to an inference from separate integrability to joint integrability. It is not asserted to be an actual string diagram or a counterexample to a gravitational Ward identity. It shows why a proof about one soft external momentum cannot replace an estimate at every overlapping degeneration.

For clarity, the useful estimate after such a resolution can also be stated exactly. Suppose one *subtracted* sector of an integral is represented by variables $u_j \in (0, 1]$, an angular variable θ , and density

$$G(u, \theta) \prod_{j=1}^s \frac{du_j}{u_j} d\nu(\theta).$$

If there are $\delta_j > 0$ and an integrable $C(\theta) \geq 0$ with

$$|G(u, \theta)| \leq C(\theta) \prod_j u_j^{\delta_j}, \quad (47)$$

then its absolute integral is at most $\|C\|_1 \prod_j \delta_j^{-1}$. Removing the region where any $u_j < t$ changes the integral by at most

$$\|C\|_1 \left(\prod_j \delta_j^{-1} \right) \sum_j t^{\delta_j}. \quad (48)$$

This follows from Tonelli's theorem, $\int_0^t u^{\delta-1} du = t^\delta/\delta$, and a union bound on the removed regions. The argument proves a sufficient estimate once the sector representation and its bound are established. It does not establish (47) for the string integrand or identify the physical regulator with this box cutoff.

F.5. A useful power count for the pure-graviton soft sector. There is also an all-graph homogeneous estimate worth separating from a full convergence theorem. Consider soft graviton propagators connecting massive eikonal lines. Suppose there are I soft propagators, V internal soft interaction vertices of valences $d_v \geq 3$, and E_a attachments to the eikonal lines. Take each interaction to have at least two soft derivatives. Exclude vacuum components and tadpoles; absorb two-point kinetic insertions into the propagator. Assume the V internal momentum-conservation constraints are independent, so the number of independent soft loop momenta is $\ell = I - V$. Incidence counting gives $E_a = 2I - \sum_v d_v$.

Under a uniform rescaling of all these loop momenta by r , the loop measures, soft propagators, eikonal denominators and vertex numerators have total degree at least

$$\gamma = 4\ell - 2I - E_a + 2V = \sum_v (d_v - 2). \quad (49)$$

Thus $V > 0$ implies $\gamma \geq V > 0$, whereas $V = 0$ gives the marginal degree zero of independent exchanges. The equality follows by substitution of the two counting identities; the inequality uses $d_v \geq 3$. If the remaining angular factors are bounded, a radial integral has density bounded by a constant times $r^{\gamma-1} dr$ and converges for $\gamma > 0$.

This is consistent with the established gravitational infrared analysis [139]. It explains the distinguished role of independent eikonal exchanges. It is not a bound on every partial rescaling: if some momenta stay fixed, a vertex need not supply two powers of the momenta being scaled. Nor does the count bound angular pinch surfaces or other massless species. To use it in (47), every nested and overlapping scaling region must be controlled after the relevant Ward-identity cancellations. A favourable homogeneous degree for the whole graph does not prove this last assertion.

F.6. What remains for the full string-field-theory observable. Gravitational soft factorisation and cancellation results extend well beyond a single tree diagram [135, 139]. Those results must retain their scattering assumptions and the cancellations between complete sets of diagrams. They do not identify an arbitrary positive Poisson model with a complete compactified string theory containing its other massless fields.

To apply the transfer proposition of Appendix E, one still needs a specified background, external states and infrared-safe measurement; the common real and virtual regulator; and a subtraction construction compatible with the gauge identities. At each fixed perturbative order that construction must control every simultaneous soft, collinear and degeneration region, the relevant string-state sums, and the remaining hard coefficient. A sufficient route is a uniform integrable bound after all required cancellations, such as (47), on the appropriate contours or cut phase spaces, with control of regulator removal. A rigorously controlled distributional limit could provide another route. Neither route is supplied for the complete observable here. An ordinary nonperturbative S-matrix would require further construction, beyond finiteness at each order.

The tree estimate (38), the logarithmic integration formulas and the corner calculation are established here. A factorisation theorem and convergent remainder for the complete four-dimensional string-field observable have not been proved. The finite algebraic certificates

accompanying this appendix do not formalise the soft theorem, the phase-space integrals or the full string amplitude.

APPENDIX G. PROOFS FOR THE SEN LIMIT AND THE SMALL BLACK HOLES

This appendix proves the results of Sections 9.1 and 10.2. Every step is written out; the only external inputs are the adjunction formula, the standard facts on double covers of surfaces recalled in the proof of Corollary 9.8, and the Euler numbers of the Kodaira fibres I_1 and II .

G.1. The weak coupling limit. Throughout, B is a normal projective variety, L a line bundle on B , and v the ε -adic valuation: for a series $s = \sum_i s_i \varepsilon^i$ with coefficients $s_i \in H^0(B, L^k)$, $v(s) = \min\{i : s_i \neq 0\}$. Since B is irreducible, the ring $\bigoplus_k H^0(B, L^k)$ has no zero divisors, so $v(st) = v(s) + v(t)$ and $v(s + t) \geq \min(v(s), v(t))$, with equality when $v(s) \neq v(t)$.

Proof of Theorem 9.5. Step 1. Neither f nor g vanishes identically: if $f \equiv 0$ then $j \equiv 0$, and if $g \equiv 0$ then $j \equiv 1728$. Put $a = v(f)$, $b = v(g)$, $d = v(\Delta)$, so that $v(j) = 3a - d$.

Step 2. If $3a < 2b$, the term $4f^3$ has the smaller valuation, $d = 3a$ and $v(j) = 0$. If $3a > 2b$, then $d = 2b$ and $v(j) = 3a - 2b > 0$. Neither is weakly coupled, so $3a = 2b$; hence $a = 2c$, $b = 3c$ for an integer $c \geq 0$, and $d > 3a$ because $v(j) < 0$.

Step 3. If c is odd, the base change $\varepsilon = t^2$ doubles every valuation and makes c even. With $c = 2r$, the rescaling $(f, g) \mapsto (t^{-4r}f, t^{-6r}g)$ gives $v(f) = v(g) = 0$ and $v(\Delta) > 0$. The coefficients are still sections of L^4 and L^6 , and no negative powers of t occur because a and b were the least exponents. The order of the base change so far is $k_1 \in \{1, 2\}$.

Step 4. Let f_0, g_0 be the coefficients of t^0 . They are nonzero, and the coefficient of t^0 in Δ gives $4f_0^3 + 27g_0^2 = 0$. The quotient $h = 3g_0/(2f_0)$ is a rational section of L^2 , and

$$h^2 = \frac{9g_0^2}{4f_0^2} = \frac{9}{4f_0^2} \cdot \left(-\frac{4f_0^3}{27}\right) = -\frac{f_0}{3}, \quad h^3 = h \cdot h^2 = -\frac{g_0}{2}.$$

In a trivialisation of L near a point of B , h is a rational function whose square is regular, so it is a root of the monic polynomial $X^2 - h^2$ over the local ring. The local ring of a normal variety is integrally closed, so h is regular. Thus $h \in H^0(B, L^2)$, $h \neq 0$, $f_0 = -3h^2$ and $g_0 = -2h^3$, which also proves part (iii).

Step 5. Write $f = -3h^2 + tF$ and $g = -2h^3 + tG$ with $F \in H^0(L^4)[[t]]$ and $G \in H^0(L^6)[[t]]$. Expanding,

$$\Delta = 4(-3h^2 + tF)^3 + 27(-2h^3 + tG)^2 = 108th^3(hF_0 - G_0) + O(t^2),$$

since $4 \cdot 3 \cdot 9h^4 = 108h^4$ and $27 \cdot 2 \cdot 2h^3 = 108h^3$. If $G_0 = hF_0$, then $G - hF = t\Psi$ for a series Ψ , and $\eta_t = F$, $\chi_t = -12\Psi$ give

$$-2h^3 + th\eta_t - \frac{1}{12}t^2\chi_t = -2h^3 + thF + t^2\Psi = -2h^3 + tG = g,$$

which is (15) with $k = k_1$. If $k_1 = 2$, the rescaled series contain only even powers of t , because the base change $\varepsilon = t^2$ and the rescaling by t^{-4r}, t^{-6r} preserve parity; then $F_0 = G_0 = 0$ and this case applies. If $G_0 \neq hF_0$, therefore $k_1 = 1$; make the base change $t = s^2$ and put $\eta_s = sF(s^2)$, $\chi_s = -12(G(s^2) - hF(s^2))$. Then $-3h^2 + s\eta_s = -3h^2 + s^2F(s^2) = f$ and

$$-2h^3 + sh\eta_s - \frac{1}{12}s^2\chi_s = -2h^3 + s^2hF(s^2) + s^2(G(s^2) - hF(s^2)) = -2h^3 + s^2G(s^2) = g,$$

which is (15) in the parameter s , with $k = 2$. This proves part (i).

Part (ii). If $h \neq 0$, then $f_0 = -3h^2 \neq 0$ and $v(f) = 0$. The identity of Theorem 9.1 is a polynomial identity in t, h, η, χ , so it holds with η_t, χ_t in place of η, χ : Δ is divisible by t^2 . Hence $v(j) = -v(\Delta) \leq -2$. $\square$

Proof of Corollary 9.6. A weakly coupled degeneration produces $h \in H^0(L^2) \setminus 0$ by Theorem 9.5(i). Conversely, for $h \neq 0$ take $\eta = h^2$ and $\chi = 0$. Theorem 9.1 gives $\Delta = -9t^2h^6 + 4t^3h^6 = h^6t^2(4t - 9) \neq 0$, so the family is weakly coupled by Theorem 9.5(ii). Since $f = (t - 3)h^2$, $4f^3 = 4(t - 3)^3h^6$, and the factor h^6 cancels in $j = 1728 \cdot 4f^3/\Delta$. $\square$

Proof of Proposition 9.7. (i) By adjunction $K_B \cdot C + C^2 = 2g(C) - 2 = -2$, so $L \cdot C = -K_B \cdot C = 2 - n$. Let $s \in H^0(L^k)$ vanish to order exactly a on C , and write $s = \sigma^a s'$ with σ a section of $\mathcal{O}(C)$ defining C . Then s' is a section of $L^k(-aC)$ that does not vanish identically on $C \cong \mathbb{P}^1$, so its restriction is a nonzero section of a line bundle of degree $(kL - aC) \cdot C = k(2 - n) + an$. This degree is nonnegative, so $a \geq k(n - 2)/n$. For $n \geq 5$, $4(n - 2)/n \geq 12/5$ and $6(n - 2)/n \geq 18/5$, so $\text{ord}_C f \geq 3$ and $\text{ord}_C g \geq 4$.

(ii) Let $a = \text{ord}_C f \geq 3$ and $b = \text{ord}_C g \geq 4$. If $3a = 2b$, then a is even, so $a \geq 4$ and $b = 3a/2 \geq 6$, and the model is not minimal along C . If $3a < 2b$, then $\Delta = \sigma^{3a} \Delta'$ with $\Delta' \equiv 4f'^3 \pmod{\sigma}$ where $f = \sigma^a f'$, and $j = 1728 \cdot 4f'^3/\Delta'$ equals 1728 at the generic point of C . If $3a > 2b$, then $\Delta = \sigma^{2b} \Delta'$ with $\Delta' \equiv 27g'^2 \pmod{\sigma}$ and $j = 1728 \cdot 4\sigma^{3a-2b}f'^3/\Delta'$ vanishes on C .

(iii) In the normal form h is a section of L^2 , so by part (i) with $k = 2$ it vanishes on C to order at least $\lceil 2(n - 2)/n \rceil$, which is 2 for $n \geq 5$ and 1 for $n = 3, 4$. Base change and rescaling do not change h (Step 4 above). $\square$

Proof of Corollary 9.8. We use the following facts on a double cover $\pi : X \rightarrow B$ of a smooth surface branched along a divisor $O \in |L^2|$, given by $\xi^2 = h$ in the total space of L [141, Chapters I and III]: $\pi_* \mathcal{O}_X = \mathcal{O}_B \oplus L^{-1}$; X is normal if and only if O is reduced; if O is reduced with at most simple singularities, X has at most rational double points; and the dualising sheaf is $\omega_X = \pi^*(K_B \otimes L)$.

(i) With $L = K_B^{-1}$, $\omega_X = \pi^* \mathcal{O}_B = \mathcal{O}_X$. Rational double points are canonical, so the minimal resolution $\tilde{X}$ has trivial canonical bundle. For rational B , $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_B) + h^1(K_B) = 0 + h^1(\mathcal{O}_B) = 0$ by Serre duality, so $\tilde{X}$ is a K3 surface.

(ii) On an Enriques surface $K_B \neq 0$ and $2K_B = 0$, so L^2 , L^4 and L^6 are trivial and their sections are constants. Thus h is a nonzero constant, f and g are series with constant coefficients, and j does not depend on the point of B . The equation $\xi^2 = h$ with ξ a section of K_B defines the unramified double cover attached to the two-torsion class K_B , which is the K3 surface covering B . The discriminant is a nonzero constant for $\varepsilon \neq 0$ small, so there is no seven-brane, and $O = \emptyset$.

(iii) By hypothesis $|-K_B|$ has no base points, so $|-kK_B|$ has none for $k \geq 1$. By Bertini's theorem a general h defines a smooth curve O , and a general η_0 meets it transversally in $(-2K_B) \cdot (-4K_B) = 8K_B^2$ points, at which a general χ_0 does not vanish. Fix such h and η_0 , and consider the linear system spanned by η_0^2 and $hH^0(L^6)$. Its base locus is the finite set $\{h = \eta_0 = 0\}$: at a point with $h \neq 0$ some member $h\chi_0$ does not vanish, and at a point with $h = 0$ and $\eta_0 \neq 0$ the member η_0^2 does not vanish. By Bertini's theorem its general member $D_B = \{\eta_0^2 = h\chi_0\}$ is smooth away from the base locus. At a base point $d(\eta_0^2 - h\chi_0) = -\chi_0 dh \neq 0$, because $\chi_0 \neq 0$ there and O is smooth. Hence D_B is smooth. A point of $D_B \cap O$ has $h = 0$ and hence $\eta_0 = 0$. Near such a point (h, η_0) are local coordinates and D_B is the graph $h = \eta_0^2/\chi_0$, which is tangent to $O = \{h = 0\}$ with intersection multiplicity 2; this accounts for $D_B \cdot O = 16K_B^2 = 2 \cdot 8K_B^2$. On X , (ξ, η_0) are local coordinates and $D_X = \{\eta_0^2 = \xi^2\chi_0\}$ has the two branches $\eta_0 = \pm \xi\sqrt{\chi_0}$, a node. Each branch, parametrised by ξ , maps to D_B , parametrised by $\eta_0 = \pm \xi\sqrt{\chi_0}$, isomorphically. Away from O the map π is unramified. Hence the normalisation $\tilde{D}$ of D_X is an unramified double cover of D_B .

(iv) For a curve Γ on B of arithmetic genus p_a , $2p_a - 2 = \Gamma \cdot (\Gamma + K_B)$. For the smooth curves $O \in |-2K_B|$ and $D_B \in |-8K_B|$ this gives $\chi(O) = -(-2K_B) \cdot (-K_B) = -2K_B^2$ and $\chi(D_B) = -(-8K_B) \cdot (-7K_B) = -56K_B^2$, and an unramified double cover doubles the Euler characteristic, $\chi(\tilde{D}) = -112K_B^2$. For general f, g the discriminant $\Delta \in |-12K_B|$ has

ordinary cusps at the $(-4K_B) \cdot (-6K_B) = 24K_B^2$ points $f = g = 0$, with fibres of type II and Euler number 2, and is smooth elsewhere, with fibres of type I_1 and Euler number 1. Its arithmetic genus satisfies $2p_a - 2 = 132K_B^2$; each cusp lowers the geometric genus by one and has one preimage in the normalisation, so $\chi(\Delta) = 2 - 2(p_a - 24K_B^2) = -84K_B^2$. The smooth fibres have Euler number 0, so

$$\chi(Y) = 1 \cdot (\chi(\Delta) - 24K_B^2) + 2 \cdot 24K_B^2 = -108K_B^2 + 48K_B^2 = -60K_B^2.$$

Finally $4\chi(O) + \chi(\tilde{D}) = -8K_B^2 - 112K_B^2 = -120K_B^2 = 2\chi(Y)$. For $B = \mathbb{P}^2$, $K_B^2 = 9$ and $\chi(Y) = -540$. $\square$

G.2. The small black holes.

Proof of Theorem 10.3. Write $P = 2(u_T^2 e_1^2 + e_2^2 u_T^{-2})$ and $F = v_1 v_2^{n_D/2} [P/v_1^2 + Q] = v_2^{n_D/2} v_1^{-1} (P - R)$, so that $f = Ku_S F$. The equations of the entropy function [145, Section 3] are $\partial f / \partial u_S = \partial f / \partial u_T = \partial f / \partial v_1 = \partial f / \partial v_2 = 0$ and $\partial f / \partial e_i = q_i$, and the entropy is $2\pi(e_1 q_1 + e_2 q_2 - f)$.

The equation for u_S is $F = 0$, that is $P = R$; in particular $f = 0$ on a solution. The equation for u_T is $\partial P / \partial u_T = 0$, that is $u_T^2 e_1^2 = e_2^2 u_T^{-2}$, so $u_T^2 = e_2 / e_1$ and $P = 4e_1 e_2$. The charges are

$$q_1 = Ku_S v_2^{n_D/2} \frac{4u_T^2 e_1}{v_1}, \quad q_2 = Ku_S v_2^{n_D/2} \frac{4e_2}{u_T^2 v_1},$$

so with $\kappa = Ku_S v_2^{n_D/2} / v_1$ one has $e_1 q_1 = e_2 q_2 = 4\kappa u_T^2 e_1^2 = \kappa P$ and $q_1 q_2 = 16\kappa^2 e_1 e_2 = 4\kappa^2 P$. Hence

$$S_{\text{BH}} = 2\pi(e_1 q_1 + e_2 q_2) = 4\pi\kappa P = 2\pi\sqrt{q_1 q_2 P} = 4\pi\sqrt{nw P / \alpha'}.$$

At fixed u_S, u_T, e_i and on the locus $P = R$,

$$\frac{\partial F}{\partial v_1} = -\frac{v_2^{n_D/2}}{v_1} \frac{\partial R}{\partial v_1}, \quad \frac{\partial F}{\partial v_2} = -\frac{v_2^{n_D/2}}{v_1} \frac{\partial R}{\partial v_2},$$

because the derivatives of the prefactor multiply $P - R = 0$. So a solution is a critical point of R with $R = P = 4e_1 e_2 > 0$, and $S_{\text{BH}} = 4\pi\sqrt{nw R_* / \alpha'}$.

Conversely, given a critical point with $R_* > 0$ and charges $n, w > 0$, the equations $P = R_*$, $u_T^2 = e_2 / e_1$ and $q_1 / q_2 = e_2 / e_1$ determine $e_1 e_2 = R_* / 4$ and $e_2 / e_1 = n / w$, hence e_1, e_2, u_T , and then q_1 determines $u_S > 0$. So the correspondence is a bijection. The statistical value requires $R_* = \alpha'$, that is $e_1 e_2 = \alpha' / 4$. $\square$

Proof of Corollary 10.4. With $Q = H - G / v_1$, $R = v_1 G - v_1^2 H$, so $\partial_{v_1} R = G - 2v_1 H$ and $\partial_{v_2} R = v_1 (G' - v_1 H')$. At a critical point $G = 2v_1 H$, and then $R = v_1^2 H$; since $R > 0$ and $v_1 > 0$, $H > 0$ and $G > 0$. Substituting $v_1 = G / (2H)$, $R_* = G^2 / (4H) = \Phi / 4$, and

$$G' - v_1 H' = \frac{2G'H - GH'}{2H} = \frac{H^2 \Phi'}{2GH}, \quad \Phi' = \frac{G(2G'H - GH')}{H^2},$$

so $\partial_{v_2} R = 0$ if and only if $\Phi' = 0$. Conversely a critical point of Φ with $G, H > 0$ gives $v_1 = G / (2H) > 0$ and a critical point of R with $R_* = \Phi / 4 > 0$. By Theorem 10.3, $S_{\text{BH}} = 4\pi\sqrt{nw \Phi / 4} = 2\pi\sqrt{nw \Phi}$. $\square$

Proof of Corollary 10.5. For $\lambda_1 = 1$, $\lambda_2 = 1/8$, (21) gives $G = 2 + a / (2y)$ and $H = (a/y)(1 + b / (8y))$, since $n_D(n_D - 1) = a$ and $n_D(n_D - 1)(n_D - 2)(n_D - 3) = ab$. Put $u = a/y$ and $\rho = b / (8a)$, so that $\Phi = (2 + u/2)^2 / (u(1 + \rho u))$. Both G and H are positive for $y > 0$. The logarithmic derivative in u vanishes when

$$\frac{1}{2 + u/2} = \frac{1}{u} + \frac{\rho}{1 + \rho u} = \frac{1 + 2\rho u}{u(1 + \rho u)},$$

that is $2u(1 + \rho u) = (4 + u)(1 + 2\rho u)$, which reduces to $u(1 - 8\rho) = 4$. Since $8\rho = b/a < 1$, there is exactly one solution, $u = 4a/(a - b)$, so $y = (a - b)/4$. There

$$G = \frac{2(2a - b)}{a - b}, \quad H = \frac{4a}{a - b} \cdot \frac{2a - b}{2(a - b)}, \quad \Phi_* = \frac{G^2}{H} = \frac{2(2a - b)}{a} = 4 - \frac{2b}{a},$$

and $v_1 = G/(2H) = (a - b)/(2a) = 2y/a$. By Corollary 10.4, $S_{\text{BH}}/(4\pi\sqrt{nw}) = \sqrt{\Phi_*/4} = \sqrt{1 - b/(2a)}$. As $u \rightarrow 0$, $\Phi \rightarrow \infty$. As $u \rightarrow \infty$, $\Phi \rightarrow \infty$ if $b = 0$, that is for $D = 4, 5$, and $\Phi \rightarrow 1/(4\rho) = 2a/b$ if $b > 0$, where $2a/b - \Phi_* = 2(a - b)^2/(ab) > 0$. The logarithmic derivative is negative near $u = 0$ and has the single zero found above, so Φ decreases on $(0, 4a/(a - b))$ and increases beyond it, and the critical point is the global minimum. The ratio b/a increases with $D \geq 5$ and tends to 1, which gives the limit $1/\sqrt{2}$. In (18), $a_D/a_4 = \{(2/a)(8a - 4b)\}^{1/2}/4 = \sqrt{1 - b/(2a)}$, independently of C . $\square$

Proof of Theorem 10.6. (i) Shifting the index by one in G ,

$$G = 2\lambda_1 + \sum_{k \geq 1} \frac{2(k+1)\lambda_{k+1}n_D!}{(n_D - 2k)!} y^{-k}, \quad H = \sum_{k \geq 1} \frac{\lambda_k n_D!}{(n_D - 2k)!} y^{-k},$$

both sums running over $1 \leq k \leq \lfloor n_D/2 \rfloor$. So $H = 2(G - 2\lambda_1)$ if and only if $\lambda_k = 4^{1-k}/k!$.

(ii) With $H = 2(G - 2)$, $R = v_1 G - 2v_1^2(G - 2) = 4v_1^2 + v_1 G(1 - 2v_1)$. On $v_1 = 1/2$ this is 1. Moreover $\partial_{v_2} R = v_1 G'(1 - 2v_1)$, and $G' < 0$ because every term of $G - 2$ is a positive multiple of a negative power of y , and the term $k = 1$ is present for $n_D \geq 2$. So a critical point with $v_1 > 0$ has $v_1 = 1/2$. Then $\partial_{v_1} R = 8v_1 + G(1 - 4v_1) = 4 - G$ vanishes when $G = 4$, equivalently $H = 4$, and $R_* = 1$. Theorem 10.3 gives $S_{\text{BH}} = 4\pi\sqrt{nw}$.

(iii) $H(y) = \sum_{k=1}^{\lfloor n_D/2 \rfloor} c_k y^{-k}$ with every $c_k > 0$, so H decreases strictly from $+\infty$ to 0 on $(0, \infty)$ and $H = 4$ has exactly one positive root. Since $\lambda_m n_D!/(n_D - 2m)! = 4(n_D)_{2m}/(m! 4^m)$ with $(N)_j = N(N - 1) \cdots (N - j + 1)$, and $(2m)!/m! = 2^m(2m - 1)!!$,

$$\frac{H}{4} + 1 = \sum_{m \geq 0} \frac{(n_D)_{2m}}{m! (4y)^m} = \sum_{m \geq 0} \binom{n_D}{2m} (2m - 1)!! \left(\frac{1}{2y}\right)^m = I_{n_D} \left(\frac{1}{2y}\right).$$

A set of m disjoint pairs in an N -element set is chosen in $\binom{N}{2m} (2m - 1)!!$ ways, which is the stated count of involutions. With $\text{He}_N(x) = \sum_m (-1)^m \binom{N}{2m} (2m - 1)!! x^{N-2m}$ one has $i^{-N} \text{He}_N(ix) = \sum_m \binom{N}{2m} (2m - 1)!! x^{N-2m} = x^N I_N(x^{-2})$, and $x = \sqrt{2y}$ gives the Hermite form.

(iv) Put $N = n_D$ and $x = N^2/(4y)$, so that $I_N(1/(2y)) = F_N(x)$ with

$$F_N(x) = \sum_{m \leq N/2} \frac{x^m}{m!} \prod_{j=0}^{2m-1} \left(1 - \frac{j}{N}\right).$$

For numbers $a_j \in [0, 1]$ with sum σ , $1 - \sigma \leq \prod(1 - a_j) \leq 1 - \sigma + \sigma^2/2$. Here $\sigma = m(2m - 1)/N$, so the product is $1 - m(2m - 1)/N + O(m^4/N^2)$ with an absolute constant. Since $\sum_m m(2m - 1)x^m/m! = (2x^2 + x)e^x$ and $\sum_m m^4 x^m/m!$ is bounded on $[0, 2]$, and since the terms with $m > N/2$ of these series and of e^x are at most $\sum_{m > N/2} m^4 2^m/m! = O(N^{-2})$ for $x \in [0, 2]$,

$$F_N(x) = e^x \left(1 - \frac{2x^2 + x}{N}\right) + O(N^{-2}) \quad \text{uniformly on } [0, 2].$$

F_N is increasing, $F_N(0) = 1$ and $F_N(2) \rightarrow e^2 > 2$, so the root x_N of $F_N = 2$ lies in $(0, 2)$ for large N , and the derivative of the right-hand side is bounded away from 0 near $\ln 2$. Hence $x_N = \ln 2 + (2 \ln^2 2 + \ln 2)/N + O(N^{-2})$, and

$$y_D = \frac{N^2}{4x_N} = \frac{N^2}{4 \ln 2} \left(1 - \frac{2 \ln 2 + 1}{N} + O(N^{-2})\right) = \frac{N^2}{4 \ln 2} - \frac{(2 \ln 2 + 1)N}{4 \ln 2} + O(1). \quad \square$$

Proof of Theorem 10.7. Write $p_k = 4^{1-k}/k!$ and $N = n_D$.

(i) The coefficient λ_m enters H only if $N - 2m \geq 0$, and $N - 2m = -2$ for $D = 2m$ and -1 for $D = 2m + 1$. It enters G through the term $2m\lambda_m N!/(N + 2 - 2m)!y^{1-m}$, with $N + 2 - 2m = 0$ or 1 . Hence $G_\beta = G_1 + (\beta - 1)cy^{1-m}$ with $c > 0$, and H does not depend on β . All coefficients of G_β and H are positive, so both are positive on $(0, \infty)$. As $y \rightarrow \infty$, $G_\beta \rightarrow 2$ and $H \rightarrow 0^+$, so $\Phi_\beta \rightarrow \infty$. As $y \rightarrow 0$, $G_\beta \sim \beta cy^{1-m}$ and $H \sim c'y^{1-m}$ with $c' > 0$, the leading term of H having index $m - 1$ in both parities, so $\Phi_\beta \sim (\beta c)^2 y^{1-m}/c' \rightarrow \infty$ since $m \geq 2$. The minimum is attained.

(ii) By Theorem 10.6(i), $H = 2(G_1 - 2)$, so $\Phi_1 - 4 = (G_1^2 - 8G_1 + 16)/H = (G_1 - 4)^2/H \geq 0$ for all $y > 0$. If $\beta > 1$, then $G_\beta > G_1 > 0$ and $\Phi_\beta > \Phi_1 \geq 4$ everywhere, so every critical point has $\Phi_\beta > 4$ and, by Corollary 10.4, $S_{\text{BH}} > 4\pi\sqrt{nw}$.

(iii) If $0 < \beta < 1$, then at the root y_D of Theorem 10.6, $0 < G_\beta(y_D) < G_1(y_D) = 4$ and $\Phi_\beta(y_D) < 16/4 = 4$. So the minimum of Φ_β is below 4, and the solution there has $S_{\text{BH}} < 4\pi\sqrt{nw}$.

For the last statement, argue by induction on $m \geq 2$ in $D = 2m$, where $\lambda_1, \dots, \lambda_m$ are the only coefficients present. If $\lambda_k = p_k$ for $k < m$, write $\lambda_m = \beta p_m$ with $\beta > 0$; parts (ii) and (iii) leave only $\beta = 1$. $\square$

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Data availability. The program that computes the Fourier coefficients of Appendix A and every table of this paper, the Lean 4 certificates of its finite statements, the scripts that check the explicit constants of Appendices E and F and the identities of Appendix G, and the source of the paper are in release v2.2.1 of the repository named in the note under the abstract, archived on Zenodo as [10.5281/zenodo.23749469](https://zenodo.org/record/23749469) [10]. The catalogue of Section 2 is kept in the ledger format of the engine Anaxia [11]. The source register in Appendix C records the primary passages used for the scope corrections.

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