

VOLUME COMPUTATION AND COMPARISON FOR FOUR-DIMENSIONAL BODIES OF CONSTANT WIDTH

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ABSTRACT. We compute and compare the volumes of explicit four-dimensional bodies of constant width one. The original five-entry comparison consists of the simplex-symmetric projection body U_4 , the canonical raised Meissner pair, the canonical raise of U_3 , the body $\mathcal{B}_5$, and M_4 . We combine exact geometric formulas, numerical integration, interval enclosures and inscribed rational polytopes to obtain a certified strict ordering of their volumes. We also compare U_4 with the direct four-dimensional AMO peabody. The two raised Meissner bodies are proved noncongruent but exactly equal in volume. An introductory account explains why different constructions benefit from different computational methods, and distinguishes approximate volume estimates from inequalities certified by exact or outward-rounded arithmetic.

1. INTRODUCTION

A body has constant width one if the distance between its two parallel supporting hyperplanes is one in every direction. The Blaschke–Lebesgue problem asks how small its volume can be. This note concerns a finite comparison in dimension four: which of several explicit constructions has the smallest volume? It does not assert a global minimizer among all four-dimensional bodies of constant width.

In dimension three, the minimum-volume problem remains open [7, Section 14.2]. The two classical, noncongruent Meissner bodies have the same width-one volume, approximately 0.419860046, and are conjectured to minimize volume; see [7, Section 8.3.3] and [5]. The tetrahedrally symmetric body U_3 has slightly larger volume [2]. A recent preprint proves the universal lower bound $1651/4000 = 0.41275$ for width-one bodies [4, Theorem 1.1]; this still leaves a gap to the Meissner volume.

The five entries in the original comparison are the projection body $U_4^{(1)}$ [2], the canonical raises of the two Meissner bodies [5, 6] (counted as one volume class), the canonical raise K_U of $U_3^{(1)}$ [2, 6], the body $\mathcal{B}_5$ [8], and $M_4^{(1)}$ [1]. The direct four-dimensional peabody $P_{\text{AMO}}^{(1)}$ [3] is

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Human review and verification. The author has reviewed, checked, and understood all parts of the manuscript, including the mathematical arguments, computational methods, and the validation logic supporting the certified numerical results. The scope of computational certificate checking and reproduction is documented in Appendix E and the accompanying repository. This statement does not imply that every full-scale production computation was independently rerun, nor does it claim formal verification or independent peer review.

included as an additional comparison. The two raised Meissner representatives are different bodies: we prove that they are noncongruent and nevertheless have exactly the same volume. All volumes below use width one; changing width by a factor w changes four-dimensional volume by w^4 .

1.1. Main results. We use the rational separator

$$T = \frac{4437}{20000} = 0.22185. \quad (1.1)$$

Theorem 1.1 (Certified comparison). *For the width-one constructions defined in Section 2,*

$$\text{Vol}_4(U_4^{(1)}) < \text{Vol}_4(K_{\text{RM}}) < \text{Vol}_4(K_U) < \text{Vol}_4(\mathcal{B}_5) < \text{Vol}_4(M_4^{(1)}), \quad (1.2)$$

where $\text{Vol}_4(K_{\text{RM}}) = \text{Vol}_4(K_F) = \text{Vol}_4(K_V)$. Moreover,

$$\text{Vol}_4(U_4^{(1)}) < T < \text{Vol}_4(K) \quad \text{for } K \in \{K_F, K_V, K_U, \mathcal{B}_5, P_{\text{AMO}}^{(1)}, M_4^{(1)}\}. \quad (1.3)$$

Theorem 1.2 (Geometry of the raised pair). *The canonical four-dimensional raises of the two classical Meissner bodies satisfy $K_F \not\cong K_V$ and $\text{Vol}_4(K_F) = \text{Vol}_4(K_V)$.*

The first theorem follows by separating validated intervals; its proof is collected in Section 6. Equality in the second theorem is geometric: a decomposition of horizontal slices gives equality above the original hyperplane, and a decomposition of forbidden regions gives equality below it. Noncongruence is a finite rigidity argument.

1.2. How the volumes are computed and compared.

Averaging radial lengths. Looking from an interior point, let $\rho(u)$ be the distance to the boundary in direction $u \in S^3$. Then

$$\text{Vol}_4(K) = \frac{1}{4} \int_{S^3} \rho(u)^4 d\sigma(u) = \frac{\pi^2}{2} \mathbb{E}[\rho(U)^4],$$

where U is uniform on S^3 . This suggests sampling directions. Independently scrambled Sobol samples were used for exploration; their variation measures numerical stability, whereas a deterministic enclosure requires the additional checks below.

Using symmetry before integrating. For $U_4^{(1)}$, an explicit support function and curvature identities reduce the spherical integral to one one-dimensional and two two-dimensional integrals. The reduction in Section 3 is exact; directed interval arithmetic encloses their values.

Filling from inside. Certified interior points form a polytope whose volume is a lower bound. Determinants give its simplex volumes. Membership and nonoverlap require proof: a prescribed fan supplies nonoverlap, while the more efficient raised-Meissner construction uses an exact convexity audit.

Bounding from outside. Keeping finitely many of the defining unit balls enlarges their intersection, giving an upper radial bound. Outward-rounded quadrature with a proved remainder bounds its integral. Body 5 instead uses a boundary integral, and M_4 has a closed formula.

Matching the method to the body. An upper bound for one body below a lower bound for another proves the comparison. These constructions need different computational methods; forcing all of them into the same discretization would add work without strengthening that argument.

1.3. Numerical scale and certificate status. The sharpened U_4 certificate, reproduced by a new production run, gives

$$0.2218357659033942 < \text{Vol}_4(U_4^{(1)}) < 0.2218363538061581 < T. \quad (1.4)$$

This enclosure implies rounding to 0.221836 to six decimal places. The terminal certificate, completed checkpoint and production log are included in the supplement (Appendix E). The weaker enclosure (5.1) already suffices for the strict comparisons below. Table 1 uses deliberately outward-coarsened endpoints for the other bodies. Their wider intervals do not determine six decimal places, but they suffice for every asserted comparison.

TABLE 1. Volume information for width-one bodies. Each interval is a deterministic enclosure from the recorded certificate, not a confidence interval. The AMO row gives a lower bound only.

Body	Certified enclosure or bound	Method
$U_4^{(1)}$	(0.2218357659, 0.2218363539)	Reduced integral, Arb
K_F, K_V	(0.2218691661, 0.2225951796)	Inner polytope; outer balls
K_U	(0.2226282500, 0.2228681169)	Inner simplices; outer balls
$\mathcal{B}_5$	(0.2228776464, 0.2229911765)	Boundary-piece integration
$M_4^{(1)}$	0.2480396734...	Exact closed form
$P_{\text{AMO}}^{(1)}$ (additional)	> 0.2220978270	Certified inner fan

The approximate AMO volume is 0.2244750148, suggesting a position between Body 5 and M_4 . We do not use that approximate placement in the certified chain. Nor does being the smallest in this finite list imply that U_4 minimizes volume among all constant-width bodies: its volume-decreasing deformations are the subject of a separate manuscript.

Guide to the paper. Section 2 specifies the constructions. Sections 3 and 4 explain the U_4 integral and the raised Meissner pair. Section 5 describes the certified endpoints and Section 6 assembles them. The appendices contain the angular calculations, the optional common-fan construction, membership certificates, and reproducibility information.

2. PRELIMINARIES AND THE CANDIDATE BODIES

2.1. Support functions and constant width. Let $K \subset \mathbb{R}^n$ be a convex body. Its support function is

$$h_K(u) = \max_{x \in K} \langle x, u \rangle, \quad u \in S^{n-1}.$$

The width in direction u is $h_K(u) + h_K(-u)$. Hence K has constant width w if

$$h_K(u) + h_K(-u) = w \quad (2.1)$$

for every $u \in S^{n-1}$. We use width one throughout.

A width- w body is the intersection of all radius- w balls centered at its points,

$$K = \bigcap_{a \in K} \overline{B}(a, w); \quad (2.2)$$

see [7, Theorem 6.1.4]. This description is repeatedly used in the endpoint certificates below.

2.2. Canonical dimension raising. Let $H \subset \mathbb{R}^n$ be an affine hyperplane and let $K_0 \subset H$ be a width-one body. For the canonical choice of the auxiliary set in the construction of Lachand–Robert and Oudet [6, Theorem 4.1], define

$$K_+(K_0) = H_+ \cap \bigcap_{q \in K_0} \overline{B}(q, 1), \quad (2.3)$$

$$K_-(K_0) = H_- \cap \bigcap_{x \in K_+(K_0)} \overline{B}(x, 1), \quad (2.4)$$

and

$$\mathcal{R}(K_0) = K_+(K_0) \cup K_-(K_0). \quad (2.5)$$

Then $\mathcal{R}(K_0)$ has constant width one and

$$\mathcal{R}(K_0) \cap H = K_0. \quad (2.6)$$

If $H = \mathbb{R}^{n-1} \times \{0\}$, the positive half has the graph description

$$K_+(K_0) = \{(y, s) : y \in K_0, 0 \leq s \leq (1 - F_{K_0}(y)^2)^{1/2}\}, \quad F_{K_0}(y) = \max_{q \in K_0} \|y - q\|. \quad (2.7)$$

2.3. The simplex-symmetric bodies M_n and U_n . For $x \in \mathbb{R}^n$, let x_+, x_- be its coordinatewise positive and negative parts. The width-two body of [1] is

$$M_n^{(2)} = \left\{ x : \|x_+\|^2 + (\|x_-\| + \sqrt{2})^2 \leq 4 \right\}. \quad (2.8)$$

Let $H_n = \mathbf{1}^\perp \subset \mathbb{R}^{n+1}$ and let P_n be the orthogonal projection onto H_n . The body

$$U_n^{(2)} = P_n(M_{n+1}^{(2)}) \quad (2.9)$$

has constant width two and the full symmetry group of a regular n -simplex [2]. We write

$$U_n^{(1)} = \frac{1}{2}U_n^{(2)}, \quad M_n^{(1)} = \frac{1}{2}M_n^{(2)}.$$

The other comparison bodies are denoted by

$$K_U = \mathcal{R}(U_3^{(1)}), \quad \mathcal{B}_5, \quad P_{\text{AMO}}^{(1)}, \quad M_4^{(1)}.$$

Here $\mathcal{B}_5$ is the width-one normalization of the body constructed in [8], and $P_{\text{AMO}}^{(1)}$ is the width-one normalization of the direct four-dimensional focal-quadric peabody of [3].

3. THE VOLUME OF U_4

We record the analytic reduction used for the U_4 upper bound in Theorem 1.1. Work first in width two. Let

$$H_4 = \left\{ u \in \mathbb{R}^5 : \sum_{i=1}^5 u_i = 0 \right\}, \quad S(H_4) = H_4 \cap S^4.$$

For $u \in S(H_4)$ set

$$r(u) = \|u_+\|, \quad s(u) = \|u_-\|, \quad m(u) = \sum_{u_i > 0} u_i.$$

Since $\|u\| = 1$, one has $r^2 + s^2 = 1$. The support function of the width-two projection body is

$$H(u) = \begin{cases} \sqrt{2}r(u), & r(u) \geq s(u), \\ 2 - \sqrt{2}s(u), & r(u) \leq s(u). \end{cases} \quad (3.1)$$

Let

$$D = \{u \in S(H_4) : r(u) \geq s(u)\}.$$

On a sign cell in D with k positive coordinates, put

$$B = \nabla_{S^3}^2(\sqrt{2}r) + \sqrt{2}rI.$$

Proposition 3.1 (Curvature-radius spectrum). *For $k \geq 2$, the eigenvalues of B on the tangent space are*

$$\underbrace{\alpha, \dots, \alpha}_{k-2}, \quad \beta_k, \quad \underbrace{0, \dots, 0}_{4-k},$$

where

$$\alpha = \frac{\sqrt{2}}{r}, \quad \beta_k = \frac{\sqrt{2}((5-k)r^2 + m^2)}{5r^3}. \quad (3.2)$$

For $k = 1$, $B = 0$.

Proof. Let P be the positive coordinate set and let A_P be coordinate projection onto P followed by orthogonal projection to H_4 . The homogeneous first-branch support function is $F_P(x) = \sqrt{2}\langle A_P x, x \rangle^{1/2}$. Its Euclidean Hessian restricted to the tangent space is

$$\frac{\sqrt{2}}{r} \left(A_P - \frac{A_P u \otimes A_P u}{r^2} \right).$$

The spectrum of A_P on H_4 is

$$1^{[k-1]}, \quad \frac{5-k}{5}, \quad 0^{[4-k]}.$$

The zero-sum directions supported on P and orthogonal to u give $k - 2$ copies of α , while the projections of $\ker A_P$ onto $u^\perp$ give $4 - k$ zero eigenvalues. Indeed the homogeneous Hessian annihilates both $\ker A_P$ and u , and $u \notin \ker A_P$, so projection onto $u^\perp$ is injective on this kernel. The remaining eigenvalue is determined by the trace and equals β_k in (3.2). $\square$

If $u \in D$, then $k \leq 3$ almost everywhere. Indeed, if four coordinates were positive, the unique negative coordinate would have magnitude m and hence $s = m > r$ in the interior of the sign cell. Thus $\det B = 0$ almost everywhere on D .

For a width-two body the antipodal curvature-radius operator is $2I - B$. Put

$$\Delta_k(u) = \det(2I - B(u)).$$

Proposition 3.1 gives

$$\Delta_1 = 8, \quad \Delta_2 = 4(2 - \beta_2), \quad \Delta_3 = 2(2 - \alpha)(2 - \beta_3). \quad (3.3)$$

Pairing antipodal directions in the support-volume formula [10, Section 5.1] and then scaling from width two to width one yields

$$\text{Vol}_4(U_4^{(1)}) = \frac{1}{64} \sum_{k=1}^3 \int_{D \cap \{\#\{u_i > 0\} = k\}} (2 - \sqrt{2}r(u)) \Delta_k(u) d\sigma(u). \quad (3.4)$$

The coefficient $1/64 = (1/4) 2^{-4}$ combines the four-dimensional support-volume factor with the change from width two to width one. The integral can be reduced to one one-dimensional and two two-dimensional integrals. On a fixed sign cell with k positive and $l = 5 - k$ negative coordinates, write

$$u = (p, -q), \quad p = \frac{m}{k} \mathbf{1}_k + \xi, \quad q = \frac{m}{l} \mathbf{1}_l + \eta,$$

with $\xi \perp \mathbf{1}_k$ and $\eta \perp \mathbf{1}_l$. Put

$$z = m\sqrt{\frac{5}{kl}}, \quad x = \|\xi\|, \quad y = \|\eta\|.$$

The variable z measures the common mean component of the two coordinate groups; x, y measure fluctuations within them. Thus θ below allocates the remaining squared norm between these fluctuation spaces. Then $z^2 + x^2 + y^2 = 1$. For $k = 2, 3$ write

$$x = \sqrt{1 - z^2} \cos \theta, \quad y = \sqrt{1 - z^2} \sin \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

The remaining angular integrations are positivity measures

$$A_j(t) = \sigma_{j-2} \{\omega \in S(\mathbf{1}_j^\perp) : \omega_i \geq -t \text{ for all } i\}.$$

With

$$m = z\sqrt{\frac{k(5-k)}{5}}, \quad r^2 = \frac{5-k}{5} z^2 + (1 - z^2) \cos^2 \theta,$$

the exact reduction is

$$\text{Vol}_4(U_4^{(1)}) = I_1 + I_2 + I_3, \quad (3.5)$$

where

$$I_1 = \frac{5}{64} \int_{\sqrt{5}/8}^1 8 \left(2 - \frac{2\sqrt{2}}{\sqrt{5}} z \right) \sqrt{1 - z^2} A_4 \left(\frac{z}{2\sqrt{5}\sqrt{1 - z^2}} \right) dz, \quad (3.6)$$

and, for $k = 2, 3$,

$$I_k = \frac{\binom{5}{k}}{64} \int_0^1 \int_0^{\pi/2} \mathbf{1}_{\{r^2 \geq 1/2\}} x^{k-2} y^{3-k} A_k \left(\frac{m}{kx} \right) A_{5-k} \left(\frac{m}{(5-k)y} \right) (2 - \sqrt{2}r) \Delta_k(r, m) d\theta dz. \quad (3.7)$$

The piecewise formulas for A_2, A_3, A_4 in Appendix A produce finitely many algebraic subdomains. Splitting along those boundaries before interval evaluation removes the discontinuous indicator from the validated quadrature. Directed Arb arithmetic on the resulting finite family gives the enclosure (1.4). The analytic reduction (3.5) is exact; interval arithmetic enters only in the final evaluation.

4. THE TWO CANONICAL RAISED MEISSNER BODIES

Let A, B, C, D be the vertices of a centered regular tetrahedron of edge length one. For the coordinate witnesses below take

$$A = \frac{1}{2\sqrt{2}}(1, 1, 1), \quad B = \frac{1}{2\sqrt{2}}(1, -1, -1), \quad C = \frac{1}{2\sqrt{2}}(-1, 1, -1), \quad D = \frac{1}{2\sqrt{2}}(-1, -1, 1).$$

Their Gram relations are

$$X \cdot Y = \begin{cases} 3/8, & X = Y, \\ -1/8, & X \neq Y, \end{cases} \quad X, Y \in \{A, B, C, D\}. \quad (4.1)$$

Let Γ_{XY} be the short circular edge of the Reuleaux tetrahedron joining X and Y . The two Meissner types may be described by preserving respectively the three arcs

$$E_F = \{AB, AC, AD\}, \quad E_V = \{BC, BD, CD\}. \quad (4.2)$$

Their canonical raises are $K_F = \mathcal{R}(M_F)$ and $K_V = \mathcal{R}(M_V)$.

4.1. Noncongruence. The section identity (2.6) shows at once that K_F and K_V are different subsets in the fixed raising coordinates, because $M_F \neq M_V$. To rule out an arbitrary ambient Euclidean congruence, we use a finite rigidity argument.

Set

$$E = (0, 0, 0, \sqrt{5/8}). \quad (4.3)$$

The five points A, B, C, D, E form a regular four-simplex of edge length one contained in both K_F and K_V . Let

$$O = E/5.$$

Its circumradius is $\sqrt{2/5}$.

Lemma 4.1 (Circumsphere rigidity). *For $K \in \{K_F, K_V\}$ the unique minimum circumball is*

$$\overline{B}_4(O, \sqrt{2/5}),$$

and

$$K \cap \partial B_4(O, \sqrt{2/5}) = \{A, B, C, D, E\}. \quad (4.4)$$

Proof. Both raised bodies have diameter one. Jung's theorem in dimension four gives radius at most $\sqrt{2/5}$, while the contained regular four-simplex already has that circumradius. The center is unique. Indeed, if $P_1, \dots, P_5$ are the simplex vertices, then

$$\frac{1}{5} \sum_{i=1}^5 \|P_i - c\|^2 = \frac{2}{5} + \|c - O\|^2.$$

Hence a radius- $\sqrt{2/5}$ ball containing the simplex is centered at O .

Write $v_i = P_i - O$. Then

$$\sum_i v_i = 0, \quad \|v_i\|^2 = \frac{2}{5}, \quad v_i \cdot v_j = -\frac{1}{10} \quad (i \neq j),$$

and

$$\sum_i v_i v_i^\top = \frac{1}{2} I_4.$$

Let $x \in K$ lie on the circumsphere and put $w = x - O$. Since every P_i belongs to the diameter-one body K ,

$$\|w - v_i\| \leq 1,$$

so $a_i := w \cdot v_i \geq -1/10$. Moreover

$$\sum_i a_i = 0, \quad \sum_i a_i^2 = \frac{1}{2} \|w\|^2 = \frac{1}{5}.$$

With $b_i = a_i + 1/10 \geq 0$, one has

$$\sum_i b_i = \frac{1}{2}, \quad \sum_i b_i^2 = \frac{1}{4}.$$

Thus

$$0 = \left(\sum_i b_i\right)^2 - \sum_i b_i^2 = 2 \sum_{i < j} b_i b_j,$$

so exactly one b_j equals $1/2$ and all others vanish. Hence $w = v_j$ and $x = P_j$. This proves (4.4). $\square$

Every congruence $T(K_F) = K_V$ must therefore fix O and permute the five vertices. If $v_P = P - O$, the orthogonal map associated with a permutation $\pi \in S_5$ is

$$T_\pi(x) = O + L_\pi(x - O), \quad L_\pi = 2 \sum_P v_{\pi(P)} v_P^\top. \quad (4.5)$$

Both bodies have the common symmetry subgroup $G \cong S_3$ that fixes A, E and permutes B, C, D . Consequently the congruence problem depends only on the double coset $G\pi G$. There are seven such double cosets, represented by

$$\text{id}, \quad (DE), \quad (AB), \quad (AB)(DE), \quad (ABCDE), \quad (AEDCB), \quad (AE). \quad (4.6)$$

Put

$$\lambda = \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{4}$$

and define the central-section points

$$P_{AB} = (\lambda, 0, 0, 0), \quad P_{CD} = (-\lambda, 0, 0, 0), \quad P_{BC} = (0, 0, -\lambda, 0), \quad P_{BD} = (0, -\lambda, 0, 0).$$

By the preserved-arc descriptions, $P_{AB} \in K_F$ and $P_{BC}, P_{BD}, P_{CD} \in K_V$. For distinct $X, Y \in \{A, B, C, D\}$ let

$$U_{XY} = \frac{1}{4}X + \frac{3}{4}Y + \frac{\sqrt{3}}{4}e_4. \quad (4.7)$$

These points belong to both raised bodies. Indeed, for $y = \frac{1}{4}X + \frac{3}{4}Y$ and every point q in either Meissner input,

$$\|q - y\|^2 = \frac{1}{4}\|q - X\|^2 + \frac{3}{4}\|q - Y\|^2 - \frac{3}{16}\|X - Y\|^2 \leq \frac{13}{16},$$

so the canonical upper height is at least $\sqrt{3}/4$.

For the seven representatives in (4.6), direct substitution into (4.5) gives witness pairs $p_\rho \in K_F$, $q_\rho \in K_V$ with squared distance strictly greater than one after applying T_ρ . More precisely, set

$$\Delta_0 = \frac{7}{2} - \sqrt{6} > 1, \quad \Delta_1 = \frac{9}{8} + \frac{3\sqrt{5} - \sqrt{30} - \sqrt{6}}{16} > 1.$$

The certificates are

ρ	$p_\rho \in K_F$	$q_\rho \in K_V$	$\ T_\rho(p_\rho) - q_\rho\ ^2$
id	P_{AB}	P_{CD}	Δ_0
(DE)	P_{AB}	U_{DC}	Δ_1
(AB)	P_{AB}	P_{CD}	Δ_0
$(AB)(DE)$	U_{DB}	P_{BC}	Δ_1
$(ABCDE)$	U_{DB}	P_{BD}	Δ_1
$(AEDCB)$	U_{AB}	P_{BC}	Δ_1
(AE)	U_{AC}	P_{BD}	Δ_1

If $T_\rho(K_F) = K_V$, then both $T_\rho(p_\rho)$ and q_ρ would lie in the diameter-one body K_V , contradicting the last column. We have proved:

Theorem 4.2. *The two canonical raised Meissner bodies are not congruent:*

$$K_F \not\cong K_V.$$

4.2. Equality of volume: the positive half. For $0 \leq \rho \leq 1$ define the common four-vertex lens

$$Q_\rho = \bigcap_{Z \in \{A, B, C, D\}} B_3(Z, \rho). \quad (4.8)$$

For a tetrahedral edge XY , define the arc defect

$$D_{XY}(\rho) = \{x \in Q_\rho : \max_{p \in \Gamma_{XY}} \|x - p\| > \rho\}. \quad (4.9)$$

If E is a set of edges, put

$$S_E(\rho) = Q_\rho \setminus \bigcup_{e \in E} D_e(\rho). \quad (4.10)$$

By (2.7), the horizontal slice at height $s \geq 0$ of the canonical raise is

$$K_{M,+} \cap \{x_4 = s\} = S_{E_M}(\sqrt{1 - s^2}) \times \{s\}, \quad M \in \{M_F, M_V\}. \quad (4.11)$$

Write a point in barycentric coordinates

$$x = \lambda_A A + \lambda_B B + \lambda_C C + \lambda_D D, \quad \lambda_A + \lambda_B + \lambda_C + \lambda_D = 1.$$

From (4.1),

$$\|x - X\|^2 = \frac{1}{2} \sum_Z \lambda_Z^2 + \frac{1}{2} - \lambda_X. \quad (4.12)$$

Define

$$c_\rho(x) = \frac{1}{2} \sum_Z \lambda_Z^2 + \frac{1}{2} - \rho^2. \quad (4.13)$$

Then

$$x \in Q_\rho \iff \lambda_X \geq c_\rho(x) \text{ for every } X. \quad (4.14)$$

Lemma 4.3. *If $x \in D_{XY}(\rho)$, then $\lambda_X < 0$ and $\lambda_Y < 0$.*

Proof. By symmetry take $XY = AB$. A point $p \in \Gamma_{AB}$ has barycentric coordinates with $\mu_C = \mu_D = -a$, $a \geq 0$. Put

$$\theta_A = \mu_A + a, \quad \theta_B = \mu_B + a.$$

The circular parametrization gives

$$0 \leq \theta_A, \theta_B \leq 1, \quad \theta_A + \theta_B = 1 + 4a.$$

Using the Gram relations one obtains

$$\rho^2 - \|x - p\|^2 = \theta_A \lambda_A + \theta_B \lambda_B - c_\rho(x). \quad (4.15)$$

If $\lambda_A \geq 0$, the right side is nonnegative in all three possible cases $\lambda_B \geq 0, c_\rho \leq 0$, $\lambda_B \geq 0, c_\rho > 0$, and $\lambda_B < 0$, using respectively positivity, (4.14), and $c_\rho \leq \lambda_B$. Thus no AB arc constraint can fail. The same argument with A, B interchanged proves the claim. $\square$

Lemma 4.4. *If $x \in Q_\rho$ with $0 \leq \rho \leq 1$, at most two barycentric coordinates of x are negative.*

Proof. If, say, $\lambda_A, \lambda_B, \lambda_C < 0$, then $\lambda_D > 1$ and $\sum_Z \lambda_Z^2 > 1$. Hence $c_\rho(x) > 1 - \rho^2 \geq 0$, contradicting (4.14) for a negative coordinate. $\square$

Corollary 4.5. *The six sets $D_e(\rho)$ are pairwise disjoint.*

Proof. For adjacent edges, Lemma 4.3 would force at least three negative barycentric coordinates; for opposite edges it would force all four. Both contradict Lemma 4.4. $\square$

Define

$$C(\rho) = Q_\rho \setminus \bigcup_e D_e(\rho),$$

where the union is over all six tetrahedral edges. Then

$$S_{E_F}(\rho) = C(\rho) \dot{\cup} D_{BC}(\rho) \dot{\cup} D_{BD}(\rho) \dot{\cup} D_{CD}(\rho), \quad (4.16)$$

$$S_{E_V}(\rho) = C(\rho) \dot{\cup} D_{AB}(\rho) \dot{\cup} D_{AC}(\rho) \dot{\cup} D_{AD}(\rho). \quad (4.17)$$

The tetrahedral symmetry group is transitive on the six edges and preserves Q_ρ , so all six defects have equal volume. Equations (4.16)–(4.17) therefore give equal three-dimensional slice volumes for every s . Fubini yields

$$\text{Vol}_4(K_{F,+}) = \text{Vol}_4(K_{V,+}). \quad (4.18)$$

4.3. Equality of volume: the negative half. Lift the positive slice decomposition to four-dimensional defect towers:

$$C^+ = \{(x, s) : 0 \leq s \leq 1, x \in C(\sqrt{1-s^2})\},$$

$$D_e^+ = \{(x, s) : 0 \leq s \leq 1, x \in D_e(\sqrt{1-s^2})\}.$$

Thus, up to null boundaries,

$$K_{F,+} = C^+ \dot{\cup} D_{BC}^+ \dot{\cup} D_{BD}^+ \dot{\cup} D_{CD}^+,$$

$$K_{V,+} = C^+ \dot{\cup} D_{AB}^+ \dot{\cup} D_{AC}^+ \dot{\cup} D_{AD}^+.$$

For $S \subset \mathbb{R}^4$ put

$$\mathcal{B}(S) = \bigcap_{x \in S} B_4(x, 1),$$

and define

$$L_0 = \{x_4 \leq 0\} \cap \mathcal{B}(C^+), \quad (4.19)$$

$$F_e = L_0 \setminus \mathcal{B}(D_e^+). \quad (4.20)$$

Then

$$K_{F,-} = L_0 \setminus (F_{BC} \cup F_{BD} \cup F_{CD}), \quad (4.21)$$

$$K_{V,-} = L_0 \setminus (F_{AB} \cup F_{AC} \cup F_{AD}). \quad (4.22)$$

We prove that the six forbidden regions are pairwise disjoint. The quantity μ_{XY} below is the minimum of a barycentric linear expression along the retained arc. The function U_R is a squared height: it describes the top of a vertical fibre of the relevant tower, where distance from a point in the negative half is largest.

For an edge XY and negative numbers ξ, ζ , set

$$\mu_{XY}(\xi, \zeta) = -\frac{1}{2} \left(\xi + \zeta + \sqrt{9\xi^2 + 6\xi\zeta + 9\zeta^2} \right). \quad (4.23)$$

The arc-gap minimization used in the positive-half proof gives

$$\min_{p \in \Gamma_{XY}} (\theta_X(p)\lambda_X + \theta_Y(p)\lambda_Y) = \mu_{XY}(\lambda_X, \lambda_Y) < \min\{\lambda_X, \lambda_Y\} \quad (4.24)$$

when $\lambda_X, \lambda_Y < 0$.

For a barycentric vector λ of sum one and an endpoint $R \in \{X, Y\}$ of minimal coordinate, define

$$U_R(\lambda) = \frac{1}{2} - \frac{1}{2} \sum_Z \lambda_Z^2 + \lambda_R = 1 - \frac{1}{2} \|\lambda - e_R\|^2. \quad (4.25)$$

If $U_R(\lambda) \geq 0$, the corresponding point

$$p_R(\lambda) = (x(\lambda), \sqrt{U_R(\lambda)})$$

is the top of the relevant defect tower branch. For a lower point, distance to a fixed vertical fibre increases with height, so every lower violation is witnessed at such a top.

Lemma 4.6 (Opposite-edge sign). *If $z = (y(\eta), -\tau) \in F_{XY}$ and ZW is the edge opposite to XY , then*

$$\eta_Z < 0, \quad \eta_W < 0.$$

Proof. We prove the contrapositive for Z . Assume $\eta_Z \geq 0$. Consider a defect-top branch with

$$\lambda_X \leq \lambda_Y < 0, \quad \lambda_X = -a, \quad \lambda_Y = -b, \quad a \geq b > 0,$$

and active endpoint $R = X$. The condition $U_X(\lambda) \geq 0$ implies the reserve inequalities

$$\lambda_Z > 3b, \quad \lambda_W > 3b. \quad (4.26)$$

Indeed, if $\lambda_Z \leq 3b$, writing $a = b + u$ and $\lambda_Z = 3b - v$ with $u, v \geq 0$, the sum-one condition gives $\lambda_W = 1 + u - b + v$ and direct expansion yields $\|\lambda - e_X\|^2 > 2$, contradicting $U_X \geq 0$.

Set

$$d_Z = e_X + e_Y - 3e_Z + e_W, \quad \lambda(t) = \lambda + td_Z, \quad 0 \leq t \leq b.$$

The sum of coordinates and the active X -branch are preserved. By (4.26), the opposite coordinates remain nonnegative. At $t = b$ one has $\lambda_Y(b) = 0$, hence the terminal top lies in the common core C^+ by Lemma 4.3. Moreover

$$\frac{d}{dt}U_X(\lambda(t)) = 4\lambda_Z(t) > 0.$$

For the lower point z , the squared distance to the branch top is

$$\Psi_X(\lambda; z) = \frac{1}{2} \sum_R \eta_R^2 + \tau^2 + \frac{1}{2} + (e_X - \eta) \cdot \lambda + 2\tau \sqrt{U_X(\lambda)}.$$

Since $(e_X - \eta) \cdot d_Z = 4\eta_Z \geq 0$, the distance does not decrease as the path moves toward the terminal core top. The terminal top belongs to C^+ and $z \in L_0$, so its distance from z is at most one. Therefore the original top is also at distance at most one, contradicting $z \in F_{XY}$. The same argument applies to W and to the branch with active endpoint Y . $\square$

Corollary 4.7. *The six forbidden regions F_e are pairwise disjoint.*

Proof. A point of L_0 projects to a point of a lens $Q_{\sqrt{1-\tau^2}}$, so Lemma 4.4 allows at most two negative barycentric coordinates. If two forbidden regions correspond to adjacent edges, Lemma 4.6 forces at least three negative coordinates; if they correspond to opposite edges, it forces all four. Both cases are impossible. $\square$

The tetrahedral group preserves C^+ and L_0 and acts transitively on the six edges. Hence the six forbidden regions have equal four-dimensional volume. Equations (4.21)–(4.22) and Corollary 4.7 give

$$\text{Vol}_4(K_{F,-}) = \text{Vol}_4(K_{V,-}). \quad (4.27)$$

Combining (4.18) and (4.27) proves the second half of Theorem 1.2.

Theorem 4.8. *The two canonical raised Meissner bodies have exactly equal four-dimensional volume:*

$$\text{Vol}_4(K_F) = \text{Vol}_4(K_V).$$

Together with Theorem 4.2, they are noncongruent equal-volume bodies of constant width one.

5. VALIDATED VOLUME CERTIFICATES

5.1. Upper bound for U_4 . For the separator theorem, the available uniform-grid Arb record already gives the sufficient enclosure

$$0.2218338853720453 < \text{Vol}_4(U_4^{(1)}) < 0.2218382994641744 < T. \quad (5.1)$$

The ten algebraic pieces consist of three pieces for I_1 , five for I_2 , and two for I_3 . This run uses `python-flint` 0.9.0 at 60 decimal digits: 2^{20} intervals per one-dimensional piece and 4096^2 boxes per two-dimensional piece. A centred first-derivative enclosure is used where regular; natural interval evaluation covers boundary and derivative-singular boxes. Exact dyadic endpoint records are summed outwards.

The sharper enclosure (1.4) is reproduced by a hybrid refinement at 50 decimal digits: six pieces are integrated in one dimension, three two-dimensional enclosures are reused, and the remaining piece is refined by 4×4 inside fallback cells of a 4096^2 grid. The returned production

records and local verification scope are specified in Appendix E; the stronger decimal precision is not needed for (5.1).

5.2. Raised Meissner lower and upper bounds. A rational inner polytope with 105654 certified vertices and 708899 boundary tetrahedra gives

$$\text{Vol}_4(K_{\text{RM}}) \geq \frac{36415211368949752100591187\sqrt{2}}{192 \cdot 1048576^4} = 0.22186916613238737325 \dots > T. \quad (5.2)$$

Qhull supplies a candidate tetrahedron list. Membership is checked separately against complete circular arcs and negative-half inner balls. The boundary audit checks distinct, nondegenerate tetrahedra; incidence two at every triangle; a connected facet-dual graph; cyclic edge links; and connected vertex links of Euler characteristic two. Together these checks give a closed connected PL three-manifold, rather than merely a list of positive determinants. Integer orientation tests verify local convexity at all 814553 edges; the recorded run has no unsupported planar edge links. The finite realization is bounded, so Rybnikov's local-to-global theorem [9, Theorem 8] identifies it as a convex boundary. An exact positive barycentric witness places the origin inside, and determinant summation gives (5.2).

An independent validated outer computation gives

$$\text{Vol}_4(K_{\text{RM}}) \leq 0.222595179568435. \quad (5.3)$$

5.3. How the raised-body upper bounds are certified. For a unit ball centred at c and containing the origin, the radial upper root in a unit direction u is

$$R_c(u) = c \cdot u + \sqrt{(c \cdot u)^2 + 1 - \|c\|^2}.$$

If the raised body is contained in that ball, its radial function is at most R_c . On the positive half, centres come from the complete defining arcs; on the negative half they are certified points of the positive half. Choosing one valid centre on each parameter box is sufficient. Taking a finite set of constraints enlarges the intersection, so it yields an upper bound even when the selected centre is not the true active centre.

In a spatial symmetry chamber use

$$D = 1 + a^2(1 + s^2), \quad u_{\pm} = \left(\frac{2q(1, a, as)}{(1 + q^2)\sqrt{D}}, \pm \frac{1 - q^2}{1 + q^2} \right),$$

with $(a, s, q) \in [0, 1] \times [-1, 1] \times [0, 1]$. Here $v = (1, a, as)/\sqrt{D}$ is a spatial unit direction and $q = \tan(\phi/2)$, where $u_{\pm} = (\sin \phi v, \pm \cos \phi)$. The area factors are

$$d\sigma_2(v) = \frac{a}{D^{3/2}} da ds, \quad \sin^2 \phi d\phi = \frac{8q^2}{(1 + q^2)^3} dq.$$

The first follows by differentiating v and taking its cross product; the second follows from the half-angle identities. The box integrand has the form

$$F = \frac{\kappa R_c(u_{\pm})^4 a q^2}{D^{3/2}(1 + q^2)^3}.$$

Here $\kappa = 48$ for the 24 spatial chamber images of K_U . For the raised Meissner body four signed chamber representatives are integrated separately, each with $\kappa = 12$ for six permutation images. Thus $\kappa = 24 \cdot 8/4 = 48$ or $6 \cdot 8/4 = 12$, respectively.

On a box with side lengths ℓ_a, ℓ_s, ℓ_q , two-point Gauss quadrature in each variable gives

$$\int_B F \leq Q_B^{\uparrow} + \frac{\ell_s \ell_q \ell_a^5 M_a + \ell_a \ell_q \ell_s^5 M_s + \ell_a \ell_s \ell_q^5 M_q}{4320}, \quad M_j \geq \sup_B |\partial_j^4 F|.$$

The constant is the one-dimensional two-point Gauss remainder constant. Telescoping the three integrations and using positive quadrature weights gives the displayed estimate. Interval Taylor arithmetic bounds the fourth derivatives on the whole box. Nodes, square roots, centres, Jacobians, weights and additions are enclosed outwards; enclosing only the final sum would not

suffice. The dyadic box upper endpoints are accumulated with integer arithmetic, and the box tree is checked to cover its domain.

The recorded hardened upper sums are

$$\begin{aligned} U_F &= \frac{538202119804071468248419}{2417851639229258349412352} < 0.222595179568435, \\ U_U &= \frac{2155448166585847858684765}{9671406556917033397649408} < 0.222868116845350. \end{aligned}$$

These are the endpoints used in the present comparison, replacing the earlier displayed upper estimates in the working record.

5.4. Raised U_3 and Body 5. The hardened comparison certificate for the canonical raise of U_3 gives

$$0.222628250 < \text{Vol}_4(K_U) \leq 0.222868116845350. \quad (5.4)$$

The independent directed-rounding certificate for Body 5 gives, with endpoints rounded outwards,

$$0.222877646477730 < \text{Vol}_4(\mathcal{B}_5) < 0.222991176426990. \quad (5.5)$$

Consequently

$$\text{Vol}_4(K_{\text{RM}}) < \text{Vol}_4(K_U) < \text{Vol}_4(\mathcal{B}_5). \quad (5.6)$$

Indeed, the adjacent certified separations are

$$0.222595179568435 < 0.222628250$$

and

$$0.222868116845350 < 0.222877646477730.$$

Geometry behind the other two enclosures. The raised- U_3 lower certificate uses a cap-and-chamber simplicial fan at mesh $n = 80$, with 800442 endpoint records per half. Positive-half endpoints satisfy all six elliptical arc constraints, including the interval interpolation error between tested arc parameters. Negative-half endpoints satisfy rigorous witness-ball inequalities. Integer ray determinants and rational radii then give a lower volume sum. This inner construction is distinct from the common fan in Appendix B. In the final compact record each endpoint is $sv/2^{42}$, with integer s and a prescribed integer ray v ; the arc grid has 2048 intervals. The older `n80` JSON uses interval determinants and reports a different endpoint. It is not the source of the final bound (5.4). The quartic/Bernstein adapter in Appendix C belongs to the common-fan route, not to this sampled-arc certificate.

For Body 5, the divergence theorem writes volume as a sum over the parametrized boundary pieces:

$$\text{Vol}_4(K) = \frac{1}{4} \int_{\partial K} \langle x, n(x) \rangle dS(x).$$

Using the boundary strata generated by the six spherical patches, four inherited apex arcs, six new base arcs, four base vertices and the apex, respectively, the signed flux decomposition is

$$\text{Vol}_4(\mathcal{B}_5) = \frac{1}{4}(6I_f + 4I_a + 6I_b + 4I_v + I_E). \quad (5.7)$$

The I 's are outward-normal fluxes over the corresponding smooth boundary pieces, not the lower-dimensional generating strata themselves. Their boundaries have zero surface measure. Appendix D gives a representative kernel and the apex remainder bound; the supplementary code supplies all transformed kernels. The recorded certificate uses a 64×64 face grid, 1024 subintervals on each of two arc types, a 128×128 base-vertex grid, and a truncated moment calculation with a bounded tail for the remaining apex term. Basic binary64 operations are expanded outwards; square-root endpoints are checked by exact rational squaring, and arctangent and π are enclosed by explicit series. Thus the deterministic point estimate 0.2229277265 is supplemented by the enclosure (5.5).

5.5. M_4 . For width one,

$$\text{Vol}_4(M_4^{(1)}) = \frac{\pi}{32}[(24 - 11\sqrt{2})\pi - 24] \approx 0.2480396734. \quad (5.8)$$

The common rational fan already gives the independent lower certificate

$$L_{M_4,3} = 0.233244525156270 \dots > T. \quad (5.9)$$

5.6. **The AMO peabody.** A separate interval-domain computation places the width-one AMO body near

$$\text{Vol}_4(P_{\text{AMO}}^{(1)}) \approx 0.22447501476943.$$

More importantly for the common method, the uniform level-5 certificate contains 87552 rays and 524288 cells and has independent-verifier value

$$L_{\text{AMO},5} = 0.222097827053789 \dots > T. \quad (5.10)$$

Hence

$$\text{Vol}_4(U_4^{(1)}) < T < L_{\text{AMO},5} \leq \text{Vol}_4(P_{\text{AMO}}^{(1)}).$$

5.7. **What the common fan adds.** The uniform level-5 records contain 87552 rays and 524288 cells:

K	$L_{K,5}$ (decimal display)	$L_{K,5} > T$
$\mathcal{B}_5$	0.221864682307840	yes
K_U	0.219885140191584	no
K_{RM}	0.219660269719997	no.

Together with M_4 and AMO, Body 5 thus has a common-fan lower bound above T . The two raised entries do not cross T in this computation; their completed comparisons use the body-specific certificates above. Refining a directional mesh does not enlarge a fixed family of interior witness balls. Consequently a modest fan bound does not contradict a sharper lower bound obtained with a different inner construction.

6. PROOF OF THE COMPARISON THEOREM

Proof of Theorem 1.1. Equation (5.1) gives $\text{Vol}_4(U_4^{(1)}) < T$. The raised Meissner lower bound (5.2), the raised- U_3 lower bound in (5.4), the Body 5 lower bound in (5.5), the common AMO certificate (5.10), and the exact formula (5.8) all lie strictly above T . This proves (1.3).

For the stronger chain (1.2), combine the raised Meissner outer bound (5.3) with the lower endpoint in (5.4), then combine the upper endpoint in (5.4) with the lower endpoint in (5.5). Finally the Body 5 upper endpoint is far below the exact value (5.8). The first inequality follows from $\text{Vol}_4(U_4^{(1)}) < T < \text{Vol}_4(K_{\text{RM}})$. $\square$

Remark 6.1 (Numerical ordering beyond the certified chain). The interval and high-accuracy computations suggest a complete numerical ordering of all explicit candidates considered here, including the direct AMO peabody. Only the inequalities proved above are asserted as theorem-level statements; no adjacent placement of $P_{\text{AMO}}^{(1)}$ is inferred merely from decimal approximations.

APPENDIX A. DETAILS OF THE U_4 LOW-DIMENSIONAL REDUCTION

This appendix supplies the calculations compressed in equations (3.5)–(3.7): the sign-cell measure disintegration, the angular positivity factors A_2, A_3, A_4 , and the measure factors in the final reduced integrals.

A.1. Sign-cell coordinates and spherical measure. Fix a positive index set of cardinality k and put $l = 5 - k$. After listing the positive coordinates first, write

$$u = (p, -q), \quad p \in \mathbb{R}_+^k, \quad q \in \mathbb{R}_+^l, \quad \sum_i p_i = \sum_j q_j = m,$$

and decompose

$$p = \frac{m}{k} \mathbf{1}_k + \xi, \quad q = \frac{m}{l} \mathbf{1}_l + \eta, \quad \xi \perp \mathbf{1}_k, \quad \eta \perp \mathbf{1}_l.$$

Define the unit mean direction

$$e_0 = \sqrt{\frac{kl}{5}} \left(\frac{1}{k} \mathbf{1}_k, -\frac{1}{l} \mathbf{1}_l \right).$$

The spaces $\mathbb{R}e_0$, $\mathbf{1}_k^\perp$, and $\mathbf{1}_l^\perp$ are mutually orthogonal, and therefore

$$u = ze_0 + (\xi, -\eta), \quad z = m\sqrt{\frac{5}{kl}}.$$

With $x = \|\xi\|$ and $y = \|\eta\|$, the unit-sphere condition is

$$z^2 + x^2 + y^2 = 1.$$

For $k = 2, 3$ write

$$\xi = x\omega, \quad \eta = y\nu, \quad \omega \in S(\mathbf{1}_k^\perp), \quad \nu \in S(\mathbf{1}_l^\perp),$$

with

$$x = \sqrt{1 - z^2} \cos \theta, \quad y = \sqrt{1 - z^2} \sin \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}.$$

Lemma A.1 (Sign-cell spherical measure). *For $k = 2, 3$,*

$$\boxed{d\sigma = x^{k-2} y^{l-2} dz d\theta d\sigma_{k-2}(\omega) d\sigma_{l-2}(\nu).}$$

For $k = 1$, with $l = 4$ and $y = \sqrt{1 - z^2}$,

$$\boxed{d\sigma = y dz d\sigma_2(\nu).}$$

Proof. Use the orthogonal decomposition

$$H_4 \cong \mathbb{R}e_0 \oplus \mathbf{1}_k^\perp \oplus \mathbf{1}_l^\perp.$$

Euclidean polar coordinates in the last two factors give

$$dV_4 = x^{k-2} y^{l-2} dZ dx dy d\sigma_{k-2}(\omega) d\sigma_{l-2}(\nu).$$

Introduce an ambient radial variable ρ by

$$Z = \rho z, \quad x = \rho \sqrt{1 - z^2} \cos \theta, \quad y = \rho \sqrt{1 - z^2} \sin \theta.$$

The absolute Jacobian of $(\rho, z, \theta) \mapsto (Z, x, y)$ is ρ^2 . Since $k+l = 5$, the factor $x^{k-2} y^{l-2}$ contributes one further power of ρ , so the total radial factor is ρ^3 . Comparison with $dV_4 = \rho^3 d\rho d\sigma_{S^3}$ proves the first formula. For $k = 1$ the unit sphere is $(z, \nu) \mapsto (z, \sqrt{1 - z^2} \nu)$ in $\mathbb{R}e_0 \oplus \mathbb{R}^3$, whose induced area factor is $\sqrt{1 - z^2} = y$. $\square$

A.2. Angular positivity measures. For the explicit threshold formulas we use the same closed endpoint convention as in the main text:

$$A_j(t) = \sigma_{j-2} \{ \omega \in S(\mathbf{1}_j^\perp) : \omega_i \geq -t \text{ for every } i \}.$$

The open and closed sign-cell conventions differ only on spherical null sets.

For $j = 2$, the sphere in $\mathbf{1}_2^\perp$ consists of the two points $\pm(1, -1)/\sqrt{2}$. Both directions satisfy the closed coordinate inequalities exactly when $t \geq 1/\sqrt{2}$, so

$$A_2(t) = 0 \quad (t < 1/\sqrt{2}), \quad A_2(t) = 2 \quad (t \geq 1/\sqrt{2}).$$

For $j = 3$, $S(\mathbf{1}_3^\perp)$ is a unit circle. The condition $\omega_i \leq -t$ cuts out an arc centered at the normalized projection of $-e_i$ to $\mathbf{1}_3^\perp$. Since that projection has norm $\sqrt{2/3}$, the forbidden condition is

$$\omega \cdot v_i \geq \sqrt{3/2}t.$$

The three centers are separated by $2\pi/3$. For $1/\sqrt{6} \leq t < \sqrt{2/3}$ the forbidden arcs are disjoint and each has length $2 \arccos(\sqrt{3/2}t)$. Below $1/\sqrt{6}$ they cover the circle; above $\sqrt{2/3}$ they are empty. Hence

$$A_3(t) = \begin{cases} 0, & 0 \leq t < 1/\sqrt{6}, \\ 2\pi - 6 \arccos(\sqrt{3/2}t), & 1/\sqrt{6} \leq t < \sqrt{2/3}, \\ 2\pi, & t \geq \sqrt{2/3}. \end{cases}$$

The case $j = 4$ is two-dimensional. Put

$$c = \frac{2t}{\sqrt{3}}.$$

The normalized projections

$$v_i = -\frac{P_{\mathbf{1}_4^\perp} e_i}{\|P_{\mathbf{1}_4^\perp} e_i\|}$$

form the vertices of a regular tetrahedron on S^2 : $v_i \cdot v_j = -1/3$ for $i \neq j$. The forbidden set $\{\omega_i \leq -t\}$ is therefore the spherical cap

$$C_i(c) = \{\omega \in S^2 : \omega \cdot v_i \geq c\}.$$

Its area is $2\pi(1 - c)$. Thus $A_4(t)$ is the area of the complement of four congruent tetrahedrally arranged caps.

The complement is nonempty exactly for $c \geq 1/3$. Indeed, if $\omega_i > -t$ for all four coordinates, then the limiting configuration at first contact has three coordinates equal to $-t$ and the fourth equal to $3t$; the unit-norm condition is $12t^2 = 1$, equivalently $c = 1/3$. For $c \geq 1/3$ there are no positive-area triple cap intersections. Two caps intersect precisely when their angular radius $\alpha = \arccos c$ satisfies $2\alpha \geq \gamma$, where $\gamma = \arccos(-1/3)$ is the tetrahedral center distance. Since

$$\cos(\gamma/2) = \sqrt{\frac{1 + \cos \gamma}{2}} = \frac{1}{\sqrt{3}},$$

pairwise lenses occur exactly for $c < 1/\sqrt{3}$.

It remains to compute the area $L(c)$ of one pairwise lens. Let $1/3 \leq c \leq 1/\sqrt{3}$. Spherical trigonometry for the triangle formed by the two cap centers and one intersection point gives

$$\phi = \arccos\left(\frac{\sqrt{2}c}{\sqrt{1 - c^2}}\right), \quad \psi = \arccos\left(\frac{c^2 + 1/3}{1 - c^2}\right).$$

Each boundary arc is an arc of a small circle of Euclidean radius $\sqrt{1 - c^2}$ and has length $2\phi\sqrt{1 - c^2}$. With the orientation induced from the lens, its geodesic curvature has magnitude $c/\sqrt{1 - c^2}$. Hence the two boundary arcs contribute total geodesic curvature $4c\phi$. The two corner contributions in Gauss–Bonnet reduce to the angle term 2ψ , and since the unit sphere has Gaussian curvature one, Gauss–Bonnet yields

$$L(c) = 2\psi - 4c\phi = 2 \arccos\left(\frac{c^2 + 1/3}{1 - c^2}\right) - 4c \arccos\left(\frac{\sqrt{2}c}{\sqrt{1 - c^2}}\right). \quad (\text{A.1})$$

Inclusion–exclusion now gives

$$A_4(t) = 4\pi - 4 \cdot 2\pi(1 - c) + 6L(c) = 8\pi c - 4\pi + 6L(c)$$

for $1/3 \leq c < 1/\sqrt{3}$. When $1/\sqrt{3} \leq c < 1$ the lenses are absent, so $A_4(t) = 8\pi c - 4\pi$; for $c \geq 1$ all forbidden caps are empty and $A_4(t) = 4\pi$. This proves the piecewise formula used in the main text.

APPENDIX B. A COMMON RATIONAL SIMPLICIAL-FAN LOWER BOUND

B.1. The cross-polytope fan. Let

$$C_4 = \text{conv}\{\pm e_1, \pm e_2, \pm e_3, \pm e_4\} \subset \mathbb{R}^4. \quad (\text{B.1})$$

Its boundary has sixteen tetrahedral facets

$$\text{conv}\{\varepsilon_1 e_1, \varepsilon_2 e_2, \varepsilon_3 e_3, \varepsilon_4 e_4\}, \quad \varepsilon_i \in \{\pm 1\}.$$

The positive hull of each facet is the corresponding orthant. Hence the cones over these sixteen facets cover $\mathbb{R}^4$ and have pairwise disjoint interiors.

Each tetrahedron is refined by the standard edge-midpoint subdivision. If its vertices are v_1, v_2, v_3, v_4 , insert the six edge midpoints $m_{ij} = (v_i + v_j)/2$. The four corner tetrahedra are separated from a central octahedron; splitting the octahedron along one fixed opposite-vertex diagonal gives four further tetrahedra. Thus each parent tetrahedron is replaced by eight children. On a common face of two parent tetrahedra, the induced subdivision is the same triangular midpoint subdivision, so the refined triangulation is conforming.

After k refinements, every boundary vertex has denominator $s_k = 2^k$. Write a ray vertex as $\tilde{u}_i = u_i/s_k$ with $u_i \in \mathbb{Z}^4$. The cones over the refined tetrahedra again form a complete simplicial fan.

Proposition B.1 (Fan counts). *Let V_k, E_k, F_k denote the number of vertices, edges, and tetrahedra in the level- k triangulation of ∂C_4 . Then*

$$V_0 = 8, \quad E_0 = 24, \quad F_0 = 16,$$

and

$$V_{k+1} = V_k + E_k, \quad F_{k+1} = 8F_k, \quad E_k = V_k + F_k. \quad (\text{B.2})$$

Consequently

k	V_k	F_k
2	192	1024
3	1408	8192
4	11008	65536
5	87552	524288
6	699392	4194304

Proof. Every old edge contributes one new midpoint vertex and no other new vertices are created, giving $V_{k+1} = V_k + E_k$. Each old tetrahedron gives eight children. For a triangulation of S^3 , every triangular face belongs to exactly two tetrahedra, so $f_2 = 2F_k$. Euler's relation $V_k - E_k + f_2 - F_k = 0$ gives $E_k = V_k + F_k$. $\square$

B.2. The determinant identity. The four-dimensional cross-polytope has volume

$$\text{Vol}_4(C_4) = \frac{2^4}{4!} = \frac{16}{24}. \quad (\text{B.3})$$

Because the coned refined tetrahedra partition C_4 ,

$$\sum_{C=(i_1, i_2, i_3, i_4)} |\det(u_{i_1}, u_{i_2}, u_{i_3}, u_{i_4})| = 16s_k^4. \quad (\text{B.4})$$

This identity is an exact audit of the reconstructed fan. It is not, by itself, a substitute for proving that an arbitrary list of cells is a conforming triangulation; coverage and non-overlap come from the recursive construction above.

B.3. Certified radial endpoints and the lower bound. Let $K \subset \mathbb{R}^4$ be a convex body containing the origin. Let $A : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the linear chart used for the body and set $J = |\det A|$. Fix an integer denominator D . On each fan ray choose an integer radius $r_i \geq 0$ and define

$$p_i = A \left(\frac{r_i}{Ds_k} u_i \right). \quad (\text{B.5})$$

Assume that every p_i has been rigorously certified to belong to K .

Proposition B.2 (Exact inscribed-simplex bound). *For every fan cell $C = (i_1, i_2, i_3, i_4)$,*

$$\text{conv}\{0, p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}\} \subset K.$$

The interiors of these simplices are pairwise disjoint, and

$$L_K = \frac{J}{24(Ds_k)^4} \sum_C |\det(u_{i_1}, u_{i_2}, u_{i_3}, u_{i_4})| \prod_{j=1}^4 r_{i_j} \leq \text{Vol}_4(K). \quad (\text{B.6})$$

Proof. Each simplex is contained in K by convexity. It is also contained in the cone over the corresponding fan cell, and distinct cone interiors are disjoint. Formula (B.6) is the exact four-simplex determinant formula summed over the cells. $\square$

When $J \in \mathbb{Q}$, (B.6) is rational. If $L_K = N/Q$, the comparison $L_K > T$ is the integer inequality

$$20000N - 4437Q > 0. \quad (\text{B.7})$$

When J contains a fixed quadratic radical, the sign is first determined and the final comparison is reduced to an integer inequality after squaring. No rounded decimal is proof data.

B.4. Uniform and adaptive refinements. The preceding construction is the uniform refinement. The adaptive variant refines selected leaf tetrahedra, together with the conforming closure required on shared faces. The verifier reconstructs the resulting nonuniform simplicial complex from the refinement tree, checks face matching, and repeats the same determinant audit. Proposition B.2 then applies without change. The adaptive strategy affects efficiency, not the proof kernel.

APPENDIX C. EXACT ENDPOINT CERTIFICATES

The fan and determinant calculation are common to all bodies. Only the proof of $p_i \in K$ depends on the geometry of the candidate.

C.1. M_4 . Membership in M_4 follows directly from the Cartesian description (2.8). After fixing the sign pattern of the rational endpoint and clearing denominators, the inequality becomes an exact integer comparison. No root isolation or floating convex-hull computation is required.

C.2. Raised Meissner body. For the positive half, the preserved-edge description gives three circular-arc constraints. For a circular parametrization $\gamma(t) = c + a \cos t + b \sin t$ with $a \perp b$ and $\|a\| = \|b\| = R$ on a closed parameter interval,

$$\|x - \gamma(t)\|^2 = \|x - c\|^2 + R^2 - 2(x - c) \cdot (a \cos t + b \sin t).$$

Thus the maximum distance on the arc is attained at an endpoint or at a stationary point of this elementary trigonometric expression. All three preserved arcs must satisfy the radius-one constraint. Replacing this conjunction by an “at least one arc” test is not conservative; it accepts exterior points.

For the negative half, the certificate uses the rigorously contained finite union

$$L_- = \{x_4 \leq 0\} \cap \bigcup_m B_4 \left(m, 1 - \sqrt{\frac{5}{8} + \|m\|^2} \right), \quad (\text{C.1})$$

with rational witness centers m . Endpoint inclusion is therefore reduced to exact ball inequalities. The equal-volume theorem of Section 4 transfers every strict lower bound proved for one canonical raised Meissner representative to the other.

C.3. Raised U_3 . Each positive-half boundary arc is an ellipse of the form

$$\gamma(t) = A \cos t + B \sin t, \quad 0 \leq t \leq \frac{\pi}{2}.$$

The substitution

$$q = \tan(t/2), \quad 0 \leq q \leq 1, \quad (\text{C.2})$$

covers the complete arc, with $q = 0$ and $q = 1$ corresponding to its endpoints. After clearing denominators, every radius-one constraint becomes an integer quartic inequality

$$P(q) \geq 0 \quad (0 \leq q \leq 1). \quad (\text{C.3})$$

This is an alternative adapter for the common fan of Appendix B; it is not the interpolation test used by the specialized $n = 80$ certificate. Writing P in the Bernstein basis certifies positivity whenever all Bernstein coefficients are nonnegative. If this fails on $[0, 1]$, exact dyadic de Casteljau subdivision produces two child intervals and the test is repeated. A fixed depth guard returns failure rather than an uncertified acceptance. The negative half uses the separately frozen rigorous inner model from the earlier raised- U_3 certificate.

C.4. Body 5. The boundary description of [8] splits the relevant support problem into six spherical patches. On each patch the radial support maximum is attained among a finite set of at most seven candidates. In the coefficient chart used for the certificate the volume Jacobian is $1/4$, while the support comparisons lie in $\mathbb{Q}(\sqrt{5})$. Their signs are decided exactly by separating the rational and $\sqrt{5}$ parts and, only after checking the sign needed for squaring, reducing to integer inequalities.

C.5. The direct AMO peabody. The construction in [3] gives a complete boundary assembled from pair, triple, and cap pieces; it does not provide a global membership oracle. The endpoint certificate therefore starts from a rigorous boundary witness rather than pretending that the boundary parametrization itself is a membership test.

Let G be the barycenter of the regular four-simplex. By simplex symmetry the Chebyshev center is G , and the constant-width relation together with Jung's inequality gives the rational inball

$$B(G, 1/3) \subset P_{\text{AMO}}^{(1)}. \quad (\text{C.4})$$

The following elementary lemma converts a boundary enclosure into an interior certificate.

Lemma C.1 (Inward shrink). *Let $B(G, r_0) \subset K$, let $x \in K$, and suppose $\|x - c\| \leq \rho$. For $0 \leq \lambda \leq 1$,*

$$B(G + \lambda(c - G), (1 - \lambda)r_0 - \lambda\rho) \subset K \quad (\text{C.5})$$

whenever the displayed radius is nonnegative. In particular, a rational point p belongs to K if

$$\|p - [G + \lambda(c - G)]\| + \lambda\rho \leq (1 - \lambda)r_0. \quad (\text{C.6})$$

Proof. For every $y \in B(G, r_0)$, convexity gives $(1 - \lambda)y + \lambda x \in K$. The image of the inball is $B(G + \lambda(x - G), (1 - \lambda)r_0)$. Replacing the unknown x by its enclosure center c loses at most $\lambda\rho$ in radius, which yields (C.5). $\square$

In the centered chart $G = 0$ and $r_0 = 1/3$, the implemented test is

$$\|p - \lambda c\| + \lambda\rho \leq \frac{1 - \lambda}{3}. \quad (\text{C.7})$$

The term $\lambda\rho$ is essential. The spherical-cap branch also requires an exact component check excluding the remote positive component; the finite branch-and-bound audit closes on the component containing the regular-simplex direction.

APPENDIX D. BOUNDARY INTEGRATION AND THE APEX REMAINDER FOR BODY 5

The coordinates used for these kernels are those of [8]: $A_i = e_i/\sqrt{2}$ and $E = \varphi(A_1 + \dots + A_4)/2$, with $\varphi = (1 + \sqrt{5})/2$. All normals in (5.7) are outward. In this coordinate origin the apex flux is negative; it must not be replaced by its absolute value.

D.1. A representative face kernel and the quadrature error. Put $z_0 = -1/\sqrt{3}$, $z_1 = (\varphi - 1)/\sqrt{3}$, and, on $[0, 1]^2$,

$$z = z_0 + (z_1 - z_0)x, \quad b = \frac{1 + \sqrt{3}z}{\sqrt{6}\sqrt{1 - z^2}}, \quad q_0 = \frac{b}{\sqrt{1 - b^2}}, \quad q = q_0 + (1 - q_0)y.$$

The face contribution used in the validator is $I_f = \int_{[0,1]^2} f_f$, where

$$f_f(x, y) = \frac{2(z_1 - z_0)(1 - q_0)}{1 + q^2} \left[J_0 - \left(\frac{\sqrt{3}}{2} + \frac{z}{2} \right) J_c \right], \quad J_0 = \frac{5\pi}{12} - \frac{3\sqrt{3}}{4}, \quad J_c = \frac{11}{12} - \frac{\sqrt{3}\pi}{6}.$$

Integrating in y recovers the angular factor $\pi/2 - 2 \arcsin b$, which explains this transformation. In the code, `f_face2`, `f_apex_arc`, `f_base_arc` and `f_base_vertex` give I_f, I_a, I_b, I_v , respectively, by integration over unit intervals or squares. Their grids are $64^2, 1024, 1024, 128^2$.

For a rectangle of area A and half-widths h_x, h_y , a midpoint Taylor enclosure has error at most

$$A \left(\frac{h_x^2}{6} \sup |f_{xx}| + \frac{h_y^2}{6} \sup |f_{yy}| + \frac{h_x h_y}{4} \sup |f_{xy}| \right).$$

Indeed, the linear terms integrate to zero; integrating the absolute second-order remainder gives the displayed factors. The one-dimensional version is $\ell^3 \sup |f''|/24$. Interval automatic differentiation bounds these derivatives over the complete cell, including the parameter maps.

D.2. An exact moment expansion for the apex. Let $\Delta_3 = \{\lambda_i \geq 0 : \sum_{i=1}^4 \lambda_i = 1\}$, with measure $d\lambda_1 d\lambda_2 d\lambda_3$ after eliminating λ_4 ; thus $\int_{\Delta_3} 1 = 1/6$. With $s = \sum_i \lambda_i^2$, $R = \sqrt{(1+s)/2}$ and $c = (5 + \sqrt{5})/8$,

$$I_E = \frac{\sqrt{5}}{4} \int_{\Delta_3} (R^{-4} - cR^{-5}) d\lambda. \quad (\text{D.1})$$

To see the constants, put $v_i = E - A_i$. Then $v_i \cdot v_j = (1 + \delta_{ij})/2$, $|\det(v_1, \dots, v_4)| = \sqrt{5}/4$ and $E \cdot v_i = c$. For $w = \sum_i \lambda_i v_i$ the boundary point is $E - w/R$, its outward normal is $-w/R$, and the spherical Jacobian is $(\sqrt{5}/4)R^{-4}$. This gives (D.1).

Set $a_0 = 13/8$, $q_* = 3/13$ and

$$M_j = \frac{j!}{(2j+3)!} \sum_{m_1+\dots+m_4=j} \prod_{i=1}^4 \frac{(2m_i)!}{m_i!}, \quad \mu_k = \sum_{j=0}^k \binom{k}{j} (-5/8)^{k-j} M_j.$$

These are the exact moments of s^j and $(s - 5/8)^k$ by the Dirichlet monomial formula. Since $1/4 \leq s \leq 1$, $|(s - 5/8)/a_0| \leq q_*$. Define

$$S_1^{(K)} = \sum_{k=0}^K (-1)^k (k+1) \mu_k / a_0^k, \quad S_2^{(K)} = \sum_{k=0}^K (-1)^k \frac{(5/2)_k}{k!} \mu_k / a_0^k.$$

Then (D.1) equals

$$\frac{\sqrt{5}}{4} \left[4a_0^{-2} (S_1^{(K)} + \varepsilon_1) - c 2^{5/2} a_0^{-5/2} (S_2^{(K)} + \varepsilon_2) \right],$$

where, for $n = K + 1$,

$$|\varepsilon_1| \leq \frac{q_*^n ((n+1) - nq_*)}{6(1 - q_*)^2}, \quad |\varepsilon_2| \leq \frac{(5/2)_n q_*^n}{6n! [1 - q_*(n + 5/2)/(n+1)]}.$$

The first bound sums an arithmetico-geometric tail; for the second, successive absolute terms have ratio at most $q_*(n+5/2)/(n+1) < 1$. The production value $K = 20$ gives $|\varepsilon_1| < 2.046 \cdot 10^{-13}$ and $|\varepsilon_2| < 7.393 \cdot 10^{-13}$. Combining this bound with the four quadrature enclosures in (5.7), using outward arithmetic throughout, gives (5.5).

APPENDIX E. CERTIFICATE INDEX AND REPRODUCIBILITY

A replay must check input schemas and hashes, geometric membership, nonoverlap or convexity, and the final exact or outward-rounded sum. A floating candidate, a stored success flag, and a matching hash do not replace these checks. The common fan is reconstructed from its refinement rule; its final threshold test is the integer comparison (B.7).

TABLE 2. Proof dependencies and the status of the accompanying review archive. Full filenames, hashes, commands and logs are in `CERTIFICATES.md`.

ID	Conclusion and record	Review-archive status
U4-G	Uniform-grid enclosure (5.1); Arb grid manifest	Source and exact endpoint records available
U4-S	Sharper hybrid enclosure (1.4); adaptive six-decimal record	New production certificate, completed checkpoint and log available; local audit passed
RM-L	Inner-polytope bound (5.2); membership and PL boundary checks	Compact data and exact membership/convexity verifier available
R-C	Two raised-body upper bounds and the final $n = 80$ lower bound (5.4)	Recovered source/data; complete fresh replay passed
B5	Boundary enclosure (5.5)	Standalone source and component records available
FAN	Common-fan thresholds, including (5.10)	Code archive, level-5 records and AMO log available

Distinct certificate versions. U4-G uses 60-digit uniform integration; U4-S uses 50-digit hybrid refinement. The older `raised_u3_certified_simplicial_lower_n80.json` differs from R-C's rational-ray record in `Manitoba_full_comparison_certificates_20260929_v2.zip:certify_comparison.py`, `raised_comparison_compact.npz` and `rm_lower.npz` (checker digest prefix `b3aadabd3843`). The new U4-S `u4_6dp_adaptive_certificate.json`, supplied on October 21, 2026, has digest prefix `a5b0a863c50d`; its completed checkpoint has prefix `5ccfe5a3c878`. The historical JSON with prefix `b787db8a5fad` remains unrecovered. The supplement preserves full hashes and the original returned files.

Scope of the present draft. The R-C source and both NPZ inputs match their recorded SHA-256 hashes. A complete replay on October 16, 2026 reproduced all four exact bounds; its rational-ray U_3 lower bound differs from the older interval-determinant record. For U4-S, the returned log records a complete adaptive run of about 83 minutes. A local audit checks hashes, checkpoint counters, exact imported endpoints and aggregation; a local checkpoint replay recomputes the six one-dimensional integrals and total. Both passed, without repeating the adaptive two-dimensional integration. Enclosure (1.4) lies inside $(0.2218355, 0.2218365)$, certifying rounding to 0.221836 at six decimals, but not seven. The computational audits and replays described above establish the documented scope of numerical certificate checking and reproduction. The author's human review and understanding of the manuscript are declared separately on the first page. In particular, the complete adaptive two-dimensional production run was not independently repeated.

Code and data availability. Code, compact data and certificates are available at [WangHangang/4d-constant-width-certificates](#). For this revision, use commit `5d69e0be6ea1`, directory `manuscript1/`. Its `CERTIFICATES.md` maps the claims to their inputs and replay scope. The original large raised-Meissner datasets have a separately linked Zenodo archive. For U4-S, the supplement's `docs/U4_S_RETURN_AUDIT.md` gives the provenance, audit scope and commands for checkpoint replay or a fresh adaptive run.

APPENDIX F. EXACT THRESHOLD ARITHMETIC

For a rational common-fan lower bound $L = N/Q$, the separator test is (B.7). For the raised Meissner exact lower bound (5.2), both sides are positive and

$$\frac{S\sqrt{2}}{192D^4} > \frac{4437}{20000}$$

is equivalent to

$$2(20000S)^2 > (4437 \cdot 192D^4)^2,$$

where

$$S = 36415211368949752100591187, \quad D = 1048576.$$

Thus the strict comparison is an integer statement once the inner-polytope containment has been certified.

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