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A LOWER EXPLICIT THRESHOLD FOR CHEN'S THEOREM UNDER GRH

STEVE HUBBARD

Abstract. Assume the Generalized Riemann Hypothesis (GRH). We show that every even integer $N \geq \exp(\exp(13.375))$ can be written as $N = p + P_2$, where p is a prime and P_2 is either a prime or a product of two primes. This result is conditional on GRH and on Theorems 5, 6, 7 and Lemma 9 of Bordignon and Starichkova (Ramanujan J. 64, 2024), together with their auxiliary lemmas, Table 1 and display (6) that their Section 5.4 uses, used at new parameter values inside their own Section 5.4. It is not a new method; it is a re-choice of the parameters u and α in their argument, combined with the observation that Chen's theorem itself needs only that the Section 5.4 main-term combination be positive, not that it exceed $4 \cdot 10^{-3}$, because the inequality $\pi_2'(N) \geq \pi_2(N) - \sqrt{N} - 1$ from the proof of their Corollary 1 (Section 5.5, p. 1241), which does not involve the threshold, gives $N = p + P_2$ as soon as $\pi_2(N) > \sqrt{N} + 1$. In Lean 4 the threshold statement itself is stated with GRH as an explicit hypothesis and proved conditional on the journal's Theorems 5, 6, 7 and the linear sieve Theorem 4 at the constants used; Lemma 9, the Section 5.5 step, $\varepsilon(1500) < 1/8349$ and the Section 5.4 bracket arithmetic are proved in Lean. Theorems 5, 6, 7 and Theorem 4 are not formalized.

1. Introduction

In 1973, Chen [4] proved that every sufficiently large even integer N can be represented as

$$N = p + P_2,$$

where p is a prime and P_2 is a prime or a product of two primes. While the binary Goldbach conjecture remains open, and this note makes no progress on it, substantial work has gone into making the threshold for Chen's theorem explicit.

The first explicit threshold for Chen's theorem was obtained by Yamada [8]. Unconditionally, the current record is due to Bordignon, Johnston and Starichkova [3], who established Chen's theorem for all even integers $N \geq \exp(\exp(32.7))$. Johnston and Starichkova [5] proved related explicit results on writing even integers as a prime plus an almost prime. In related formal work, a qualitative Lean 4 formalization of Chen's theorem with an existential, non-explicit threshold is claimed in the unrefereed repository [6].

Assuming the Generalized Riemann Hypothesis (GRH), Bordignon and Starichkova lowered the explicit threshold considerably. In the arXiv version

[2] (v1, the only arXiv version), they obtained the threshold $\exp(\exp(15.85))$ with lower-bound constant $4 \cdot 10^{-4}$. In the published journal version [1] (Theorem 3, p. 1214), they proved under GRH that

$$\pi_2(N) > 4 \cdot 10^{-3} U_N \frac{N}{\log^2 N} \quad \text{for } N > \exp(\exp(14)),$$

from which their Corollary 1 (p. 1214) gives Chen's theorem for these N. We are not aware of a published GRH threshold below $\exp(\exp(14))$.

In this note we establish the threshold $\exp(\exp(13.375))$ under GRH by re-choosing the parameters u and α inside Section 5.4 of [1].

Theorem 1 (Main statement)**.** *Assume the Generalized Riemann Hypothesis, and assume Theorems 5, 6, 7 and Lemma 9 of Bordignon and Starichkova* [1]*, together with the auxiliary lemmas, Table 1 and display (6) of* [1] *that their Section 5.4 uses, used at new parameter values inside their own Section 5.4. Then every even integer $N \geq \exp(\exp(13.375))$ can be written as $N = p + P_2$, with p a prime and P_2 a prime or a product of two primes.*

Lemma 9 is also proved in Lean, Section 3, so the cited inputs actually needed are Theorems 5, 6, 7 and Theorem 4 with Table 1 and display (6).

What is formally verified. In Lean 4 the threshold statement is proved with GRH as an explicit hypothesis and Theorems 5, 6, 7 and Theorem 4 of [1] as explicit unformalized hypotheses, while Lemma 9, the Section 5.5 step, $\varepsilon(1500) < 1/8349$, and the Section 5.4 bracket arithmetic with all correction factors and error terms are kernel-checked (Section 3).

Context. The unconditional record is $\exp(\exp(32.7))$ by Bordignon, Johnston and Starichkova [3]; GRH thresholds indicate how far the explicit sieve method can go when zero-free regions are not the bottleneck; within this framework the optimum is about 13.365 at u near 1628 (Section 2.4), so 13.375 is within 0.01 of the method's limit and further gains need new ideas in Theorems 5 to 7 or the sieve constants.

2. The parameter re-choice in Section 5.4 of [1]

Table 1 summarizes the notation of [1] used throughout this note (γ denotes Euler's constant).

We begin by recording what the paper of Bordignon and Starichkova [1] does:

- In [1], Theorem 3, p. 1214, under GRH, they prove
$$\pi_2(N) > 4 \cdot 10^{-3} U_N \frac{N}{\log^2 N}$$
for $N > \exp(\exp(14))$. Their Corollary 1 (p. 1214) gives Chen's theorem for these N, passing from $\pi_2(N)$ to $\pi_2'(N)$ via $\pi_2'(N) \geq \pi_2(N) - \sqrt{N} - 1$ (Section 5.5, p. 1241).
- Their Section 5.4, p. 1241, combines Theorem 5 (p. 1228, proof p. 1229), Theorem 6 (p. 1231), Theorem 7 (pp. 1234–1235), and Lemma

TABLE 1. Glossary of notation from [1].

Symbol	Meaning	Place in [1]
$\pi_2(N)$	representations of N as a prime plus a product of at most two primes	Theorem 3, p. 1214
$\pi_2'(N)$	the same with a square-free semiprime	Section 5.5, p. 1241
U_N	$e^{-\gamma}\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p>2,\,p\mid N}\frac{p-1}{p-2}$, as printed; see Appendix B for the factor 2	Theorem 3 and Lemma 10
X_2	lower bound for N in Theorems 5 to 7, $N \geq X_2 \geq 4\cdot 10^{18}$	Theorems 5 to 7
z, y	$z = N^{1/8}$, $y = N^{1/3}$	Section 5, p. 1226
$\mathcal{A}, \mathcal{B}$	sifted sets, $\mathcal{A} = \{N - p\}$	p. 1227
A, u	parameters of Theorems 5 to 7, $A > 0$, $u \in [400, 10^8]$	Theorems 5 to 7
α_i	level parameters with $D = N^{1/2-\alpha_i}$	Theorems 5 to 7
$\varepsilon(u)$	prime-sum error parameter	display (6), p. 1221
$C_1(\varepsilon), C_2(\varepsilon)$	linear sieve constants via inequality (2) of Theorem 4, p. 1218	Table 1, pp. 1217–1218
ε_1	parameter in $(0, 1)$	Theorem 7
$\bar{c}$	$\int_{1/8}^{1/3}\frac{\log(2-3\beta)}{\beta(1-\beta)}\,d\beta < 0.363084$	Lemma 14, p. 1237
$\bar{m}$	bilinear remainder factor	(31), pp. 1239–1240
M_φ	upper-bound factor in Lemma 5	Lemma 5, p. 1216

9 (p. 1227, Chen's weighted sieve inequality bounding $\pi_2(N)$ below by sifting functions minus $2N^{7/8} + N^{1/3}$), at

$$u = 1000, \quad \alpha_1 = \alpha_2 = \alpha_3 = 10^{-3}, \quad A = 10, \quad \varepsilon_1 = 10^{-20},$$

with $\varepsilon(1000) < 1/5235$, $C_1 = 113$, and $C_2 = 114$.

- The arXiv version [2] (v1) had $\exp(\exp(15.85))$ with constant $4\cdot 10^{-4}$.

All changes in the present note take place inside Section 5.4 of [1] (p. 1241):

(1) We choose the parameters

$$u = 1500, \quad \alpha_1 = \alpha_2 = \alpha_3 = 0.00256, \quad A = 10, \quad \varepsilon_1 = 10^{-20}.$$

(2) From display (6) of [1], p. 1221, we have $\varepsilon(1500) < 1/8349$; see Appendix E for the computation giving $\varepsilon(1500) = 1/8349.3238$.

(3) We take $C_1 = 112$ and $C_2 = 114$, which is the row $\varepsilon^{-1} = 6100$ of Table 1 of [1], pp. 1217 to 1218. This is valid since inequality (2) with $\varepsilon(1500)$ implies it with the larger $1/6100$; the journal itself pairs $\varepsilon(1000) < 1/5235$ with the row 4500.

(4) The key observation is that Chen's theorem needs only that the Section 5.4 combination be positive, not that it exceed $4\cdot 10^{-3}$: the threshold-free inequality $\pi_2'(N) \geq \pi_2(N) - \sqrt{N} - 1$ (Section 5.5, p. 1241) gives $N = p + P_2$ once $\pi_2(N) > \sqrt{N} + 1$ (Lemma 2.1), letting $\log\log X_2$ drop to 13.375.

TABLE 2. Comparison of Section 5.4 parameters in [1] and in this note.

Quantity	[1], §5.4	This note
$\log\log X_2$	14	13.375
u	1000	1500
$\alpha_1 = \alpha_2 = \alpha_3 = \alpha$	10^{-3}	0.00256
A	10	10
ε_1	10^{-20}	10^{-20}
Upper bound on $\varepsilon(u)$	$\varepsilon(1000) < 1/5235$	$\varepsilon(1500) < 1/8349$
Table 1 row ε^{-1} (pp. 1217 to 1218)	4500	6100
(C_1, C_2)	$(113, 114)$	$(112, 114)$

2.1. **Definition of the main-term constant** K3J**.** We define K3J as in the Lean formalization (Appendix A), with $g = e^{\gamma}$, $\varepsilon = 1/8349$, and $\bar{c}$ from Lemma 14 of [1], p. 1237,

$$\bar{c} = \int_{1/8}^{1/3} \frac{\log(2-3\beta)}{\beta(1-\beta)}\, d\beta < 0.363084,$$

by setting

$$\begin{aligned} \mathrm{K3J} = {} & 8\Big(2g\log(3-8\alpha)/(4-8\alpha) - \varepsilon C_2 \cdot 3\, e^{8\alpha-2}/(4-8\alpha)\Big) \\ & - \tfrac{1}{2}\cdot 8\Big((1+5\cdot 10^{-5})\Big(\tfrac{g}{4}\big(\log 6 + \log\tfrac{3-8\alpha}{3-18\alpha}\big)/\big(\tfrac{1}{2}-\alpha\big) + \log(8/3)\,\varepsilon C_1\Big)\Big) \\ & - \tfrac{1}{2}\Big(2g/\big(\tfrac{1}{2}-\alpha\big) + 3\varepsilon C_1\Big)\,\bar{c}. \end{aligned} \tag{1}$$

Here $1+5\cdot 10^{-5}$ is the Lemma 11 (p. 1228) factor on Theorem 6's l_1, p. 1231; the three pieces of K3J are 8 times the main term of Theorem 5, minus $\frac{1}{2}\cdot 8$ times the main term l_1 of Theorem 6, minus $\frac{1}{2}$ of the main term of Theorem 7.

2.2. **Numerical evaluation at** $\log\log X_2 = 13.375$ **and** $\alpha = 0.00256$**.** At $\log\log X_2 = 13.375$ and $\alpha = 0.00256$, the relevant numerical values are as follows:

- $\log X_2 = e^{13.375} = 643707.69$; Lean proves $e^{13.375} \geq 643700$.
- The binding condition, which is the second part of displays (19), (21), and (25) of [1], is

  $$\alpha \log X_2 - 11 \log\log X_2 = 1500.767 \geq 1500 = u.$$

  At this X_2 the condition needs $\alpha \geq 0.0025588$, leaving a slack of 0.00047 in $\log\log X_2$.
- At $\alpha = 0.00256$, $\varepsilon = 1/8349$, $C_1 = 112$, and $C_2 = 114$, the main-term constant K3J defined in (1) has the numerical value $\mathrm{K3J} = 1.5960 \cdot 10^{-4}$, and the lower bound $\mathrm{K3J} > 10^{-4}$ is kernel-checked in

Lean as `theorem3_positive_main`, so the positivity margin does not rest on the script.

- With the terms K3J leaves out, namely the li factor

$$\frac{\log y\left(\mathrm{li}(y)+\sqrt{y}\log y/(8\pi)\right)}{y}=1+\frac{1}{\log y}+\frac{2}{\log^2 y}+\dots$$

with $y=X_2^{1/3}$ and the $1/\log^2 N$ terms of Theorems 6 and 7, which come to $6.10\cdot10^{-6}$, the total with the omitted terms is $1.535054\cdot10^{-4}$.

All remaining omitted factors, the remainder of (33), and the power-saving error terms from Lemma 9 and Corollary 1 are bounded explicitly in Appendix D and are negligible.

Lemma 2.1 (positivity suffices)**.** *Let N be even. If $\pi_2(N)>\sqrt{N}+1$, then $N=p+q$ with p prime and $q>1$ square-free with at most two prime factors; in particular $N=p+P_2$.*

Proof. Section 5.5 of [1], p. 1241, shows $\pi_2'(N)\ge\pi_2(N)-\sqrt{N}-1$, where $\pi_2'(N)$ counts representations of N as a prime plus a square-free integer $q>1$ with at most two prime factors, since $\pi_2(N)$ also counts prime q and the subtraction removes only $q=1$ (at most 1) and integers up to N with two equal prime factors (at most $\sqrt{N}$). Thus $\pi_2'(N)>0$. This threshold-free inequality applies with the constant $9\cdot10^{-5}$ of Section 2.2, since $9\cdot10^{-5}U_N N/\log^2 N>\sqrt{N}+1$ for $\log\log N\ge13.375$ by Appendix D. □

- Hence, on GRH and the cited results,

$$\pi_2(N)>9\cdot10^{-5}U_N\frac{N}{\log^2 N}>0$$

for every even $N\ge\exp(\exp(13.375))$, and Lemma 2.1 yields $N=p+P_2$.

Remark 1 (Threshold with the printed constant $4\cdot10^{-3}$ kept). With the printed constant $4\cdot10^{-3}$ kept, the same method at $u=1500$ and $\alpha=0.0024$ gives $\exp(\exp(13.45))$, with K3J $=5.1803\cdot10^{-3}$ and total $5.1747\cdot10^{-3}$ in Python, both above $4\cdot10^{-3}$, and the binding condition $1517.3\ge1500$. The numerical optimum at $u=1500$ is 13.4239 at $\alpha=0.0024375$, so 13.45 is conservative. The value 13.45 is checked in Lean in `LeanLab/BS2022GRH/JournalLower.lean`: `theorem3_lower_main` proves K3J >0.0042 at $\alpha=0.0024$, $\varepsilon=1/8349$, $C_1=112$, $C_2=114$, and `conditions_1345` proves the binding condition at 13.45; build log `20261020T0403Z-27388-U1166.log`, axioms `[propext, Classical.choice, Quot.sound]`.

2.3. **Verification of parameter ranges at** 13.375 **and** 13.45**.** Table 3 checks all parameter conditions required by the lemmas and theorems in [1] at $\log\log X_2=13.375$ ($\alpha=0.00256$) and $\log\log X_2=13.45$ ($\alpha=0.0024$). Every condition holds in both columns, and only the first row is tight.

TABLE 3. Parameter range conditions at $\log\log X_2 = 13.375$ ($\alpha = 0.00256$) and $\log\log X_2 = 13.45$ ($\alpha = 0.0024$).

Condition and source in [1]	Required	At 13.375	At 13.45
(19), (21), (25) part 2, $\alpha\log X_2 - 11\log\log X_2 \geq u$	1500	1500.767	1517.272
(19), (21), (25) part 1, $\log X_2 \geq (A+1)/\alpha$	4296.9 / 4583.3	643707.69	693842.31
(25), $\log X_2 \geq 48(A+4/3)$	544	643707.69	693842.31
(25), $\frac{\log X_2}{48} - \frac{34}{3}\log\log X_2 \geq \log 0.44$	-0.821	13258.99	14302.61
$\alpha_1 < 1/8$, $\alpha_2 < 1/24$, $\alpha_3 < 1/16$	< 0.041667	0.00256	0.0024
Lemma 7, p. 1221, $\log z = \log X_2/8 > 2753$	2753	80463.46	86730.29
Theorem 5, $s = 4 - 8\alpha \in [3,4]$	$[3,4]$	3.97952	3.98080
Theorem 6, $k = 8(1/6 - \alpha) \in [1,2]$	$[1,2]$	1.312853	1.314133
Remark 1, p. 1230, $\log H = \frac{\log X_2}{2} - 11\log\log X_2 \geq \log 10^8$	18.42	321706.72	346773.21
Lemma 11, p. 1228, $\log\log X_2 \geq 10$	10	13.375	13.45
Lemmas 12 and 13, $X_2 \geq 4 \cdot 10^{18}$ (log form)	42.83	643707.69	693842.31
Lemma 14, p. 1237, $\log y = \log X_2/3 \geq \log 2657$	7.885	214569.23	231280.77
Proof of Theorem 6, p. 1234, $\log y \geq \log 286$	5.656	214569.23	231280.77
Theorems 5 to 7, $400 \leq u \leq 10^8$	$u = 1500$	1500	1500

2.4. **Headroom and the active constraint.** With $u = 1500$ the binding condition is the second part of (19), (21), and (25): $\alpha\log X_2 - 11\log\log X_2 \geq u$. At $\alpha = 0.00256$ it needs $\log\log X_2 \geq 13.374532$, a slack of 0.00047. Every other range condition holds with a wide margin (Table 3).

The total decreases in α, so positivity bounds α from above. At $\log\log X_2 = 13.375$ the admissible window is $0.0025588 \leq \alpha < 0.0025649$. The joint optimum at $u = 1500$ is $\log\log X_2 = 13.37261$ at $\alpha = 0.0025649$.

Letting u vary, with $\varepsilon(u)$ recomputed by display (6) and the best admissible Table 1 row, the script-only optimum of this framework ($A = 10$, $\varepsilon_1 = 10^{-20}$) is about $\log\log X_2 = 13.36525$ at u just above the prime 1627 with $\alpha = 0.0027829$ (Table 4).

Choice of u and α. The parameters $u = 1500$ and $\alpha = 0.00256$ were found by a grid search over u and α with the Section 5.4 total. Although $u = 1628$ gives 13.36582, about 0.0068 below 13.37261 at $u = 1500$ and 0.0092 below the headline 13.375, the headline keeps $u = 1500$ and 13.375 because every Lean certificate (`eps1500`, `bracket13375_holds`, `mbar_le`, `li_factor`) is proved at $u = 1500$ and $\log\log X_2 = 13.375$; moving to $u = 1628$ would need new kernel certificates for $\varepsilon(1628)$ and the bracket, for a gain under 0.01. For fixed $u = 1500$ and $\log\log X_2 = 13.375$, the lower end 0.0025588 solves $\alpha\log X_2 - 11\log\log X_2 = u$ in closed form and the upper end 0.0025649 is

TABLE 4. Optimal $\log\log X_2$ as a function of u ($A = 10$, $\varepsilon_1 = 10^{-20}$, $\varepsilon = \varepsilon(u)$ itself; with $\varepsilon = 1/8349$ the $u = 1500$ entry is 13.37261).

u	$1/\varepsilon(u)$	Table 1 row	Optimal $\log\log X_2$
1000	5235.67	4500	13.78977
1200	6432.64	6100	13.47933
1400	7586.12	6100	13.40351
1500	8349.32	6100	13.37258
1628	about 9236	6100	13.36582
2000	11401.89	6100	13.42558
2200	12589.97	12200	13.46608

the zero of the total found by bisection in the public script, with 0.00256 a round value inside it.

Smaller A lowers the binding side, to 13.3644 for $A = 9$ and 13.3561 for $A = 8$ at $u = 1500$, but the remainder of (33) grows like $10^{20}/\log^{A-4} N$ and stops being negligible at $A = 8$ unless ε_1 is enlarged; this is not pursued.

Table 5 shows the sensitivity to the choice of row in Table 1 of [1] at $u = 1500$ and $\varepsilon = 1/8349$. Rows 63 to 114 give a negative total for every α. The next row, 12200 with $C_1 = 112$ and $C_2 = 113$, needs $\varepsilon(u)^{-1} \geq 12200$, hence u of about 2150 or more, which gives at least 13.466. Thus the 6100 row is the best admissible one and no other row reaches 13.375.

TABLE 5. Sensitivity to Table 1 of [1] at $u = 1500$, $\varepsilon = 1/8349$.

Row ε^{-1}	C_1	C_2	Total at $\alpha = 0.00256$, 13.375	Best $\log\log X_2$
2300	114	116	$-1.12 \cdot 10^{-3}$	13.38861
3300	113	115	$-4.81 \cdot 10^{-4}$	13.38058
4500	113	114	$-3.82 \cdot 10^{-4}$	13.37932
6100	112	114	$+1.535 \cdot 10^{-4}$	13.37261

3. Formal verification, Lean 4

All Lean code was written by AI and checked only by the Lean kernel (Lean v4.35.0, Mathlib v4.35.0; file paths, namespaces, and build logs are in Appendix F). Every proved Lean theorem named here depends only on the standard axioms `[propext, Classical.choice, Quot.sound]`. The files contain two unused `sorry`: the open deep step `theorems567_of_inputs` in `ChenThresholdGRH.lean` (Section 3.1) and the superseded placeholder `Cert1567.eps1500_cert` in `Cert1567.lean` (replaced in `chenThreshold13375_of_deep_closed` by `Cert1575.eps1500`); unused `sorry` statements in `LeanLab/DERZ1997/Statements.lean` and

`LeanLab/AlmostPrimeSums/Statements.lean` lie in the import closure but are used by none of the named theorems, as their axiom lists show.

Table 6 maps each Lean file to its theorem names and claims, and Table 7 records the verification chain. The core Lean declarations are:

```
def ChenThreshold13375 : Prop :=
  GRH → ∀ N : ℕ, Even N → exp (exp 13.375) ≤ (N : ℝ) → ChenSqfreeRep N

theorem chenThreshold13375_of_deep_closed
    (h5 : Thm5J) (h6 : Thm6J) (h7 : Thm7J)
    (hT4 : LinearSieveT4 (1 / 8349) 112 114) : ChenThreshold13375

noncomputable def K3J (α g ε C₁ C₂ cb : ℝ) : ℝ :=
  8 * (2 * g * log (3 - 8 * α) / (4 - 8 * α)
    - ε * C₂ * 3 * exp (8 * α - 2) / (4 - 8 * α))
  - 1 / 2 * (8 * ((1 + 5e-5)
    * (g / 4 * (log 6 + log ((3 - 8 * α) / (3 - 18 * α)))
    / (1 / 2 - α) + log (8 / 3) * ε * C₁)))
  - 1 / 2 * ((2 * g / (1 / 2 - α) + 3 * ε * C₁) * cb)

theorem theorem3_positive_main :
    (0.0001 : ℝ)
      < K3J 0.00256 (exp eulerMascheroniConstant) (1 / 8349) 112 114
        LeanLab.ChenExplicit.BJS.cbar

theorem conditions_13375 :
    (1500 : ℝ) ≤ 0.00256 * exp 13.375 - 11 * 13.375
      ∧ 11 / 0.00256 ≤ exp (13.375 : ℝ)

theorem params_ok_13375 :
    (0.00256 : ℝ) < 1 / 24 ∧ (400 : ℝ) ≤ 1500 ∧ (1500 : ℝ) ≤ 10 ^ 8
      ∧ (0 : ℝ)
        < K3J 0.00256 (exp eulerMascheroniConstant) (1 / 8349) 112 114
          LeanLab.ChenExplicit.BJS.cbar

theorem display25_first :
    max ((10 + 1) / (0.00256 : ℝ)) (48 * (10 + 4 / 3)) ≤ exp 13.375

theorem sieve_ranges_13375 :
    (3 : ℝ) ≤ 4 - 8 * 0.00256 ∧ 4 - 8 * (0.00256 : ℝ) ≤ 4
      ∧ (1 : ℝ) < 8 * (1 / 6 - 0.00256)
```

Here `ChenSqfreeRep N` means $N = p+m$ with p prime and $m > 1$ squarefree with at most two prime factors, and `GRH` is the lab's predicate for GRH of Dirichlet L-functions. Two further details:

- In `Cert1567.bracket13375_holds : Bracket13375`, for $N \geq X_2 = \exp(\exp 13.375)$ at $\alpha = 0.00256$, $A = 10$, $\varepsilon = 1/8349$, $C_1 = 112$, $C_2 = 114$, $\varepsilon_1 = 10^{-20}$, the exact Theorem 5–7 brackets B_5, B_6, B_7 satisfy
$$\mathrm{K3J} - 10^{-5} \leq 8 \cdot B_5 - \tfrac{1}{2} \cdot 8 \cdot B_6 - \tfrac{1}{2} \cdot B_7$$
(loss $\approx 6.2 \cdot 10^{-6}$, almost all from the Theorem 7 li factor, giving $10^{-4} - 10^{-5} = 9 \cdot 10^{-5}$; Python cross-check $1.535054 \cdot 10^{-4}$).
- In `JournalPositive.lean`, $1.7810699 < e^{\gamma} < 1.7810725$ uses `LeanLab.Helfgott2013.GammaFine.exp_gamma_gt`

and `exp_gamma_lt`, and $\bar{c} < 0.363084$ uses `LeanLab.ChenExplicit.BJS.cbar_lt`.

TABLE 6. Lean files, theorem names, and claims.

Lean file	Theorem names	Claim
`ChenThresholdGRH.lean`	`ChenThreshold13375`, `Thm5J`, `Thm6J`, `Thm7J`, `LinearSieveT4`, `epsJ`, `lemma9J_holds` (`Lemma9J` via `U7153.lemma10`, $N \geq 2^{24}$), `section55_step`, `rep_of_pi2'`, `error_13375`, `X2_ranges`, `cond19_13375`, `cond25_13375`	Threshold proposition and transcribed leaves; Lemma 9 (A.1); $\pi_2' \geq \pi_2 - \sqrt{N} - 1$ and $\pi_2' > 0 \Rightarrow N = p + P_2$ (Lemma 2.1); $2N^{7/8} + N^{1/3} + \sqrt{N} + 1 < 3.6 \cdot 10^{-5} N/\log^2 N$ (Appendix D); conditions (19), (21), (25)
`U7154UN.lean`	`UNfull`, `two_fifths_le_UNfull`	UNfull $\geq 2/5$ (Appendix B)
`Cert1575Closed.lean`	`chenThreshold13375_of_deep_closed`	Headline assembly of Theorem 1 from `Thm5J`, `Thm6J`, `Thm7J`, and `LinearSieveT4`
`JournalMargin.lean`	`K3J`	Main-term constant definition (1)
`JournalPositive.lean`	`theorem3_positive_main`, `conditions_13375`, `params_ok_13375`	K3J $> 10^{-4}$ ((1), Section 2.2) and binding parameter conditions at 13.375
`RangeCheck13375.lean`	`lemma7_range`, `display25_first`, `display25_third`, `remark1_H`, `remark1_H_vacuity`, `mono_step`, `mono_range_13375`, `sieve_ranges_13375`	Eight parameter-range lemmas for Table 3
`Cert1567.lean`	`bracket13375_holds`, `mbar_le`, `li_factor`	$8B_5 - 4B_6 - \frac{1}{2}B_7 \geq \mathrm{K3J} - 10^{-5}$ (gives $\pi_2(N) > 9 \cdot 10^{-5} U_N N/\log^2 N - 2N^{7/8} - N^{1/3}$); $\bar{m}(X_2, 10, X_2^{1/8}) \leq 40$; $\frac{\log Y(\mathrm{li}\, Y + Y^{1/2}\log Y/(8\pi))}{Y} \leq 1 + 4.6702 \cdot 10^{-6}$ at $\log Y = L/3$, $L \geq 643700$ (Appendices A, D)
`Cert1575.lean`, `Cert1575Closed.lean`	`eps1500_proved`, `eps1500`	$\varepsilon(1500) < 1/8349$, display (6) (Appendix E)

TABLE 7. Logical chain and verification status.

Step	Statement	Checked by
(i)	GRH	Assumed (explicit hypothesis in Lean)
(ii)	Theorems 5, 6, 7, Theorem 4 with Table 1 of [1]	Cited (explicit hypotheses in Lean, not proved)
(ii′)	Lemma 9 of [1]	Kernel (`lemma9J_holds`)
(iii)	Their hypotheses hold at the new parameters, Table 3	Kernel for the named lemmas (`RangeCheck13375.lean`, `conditions_13375`, `cond19_13375`, `cond25_13375`, `display25_first`, `display25_third`); Script for the remaining rows (by hand and `scripts/i0088_numerics.py`)
(iv)	$\varepsilon(1500) < 1/8349$	Kernel, script cross-check (`Cert1575.eps1500`; `scripts/i0088_numerics.py` gives $\varepsilon(1500) = 1/8349.3238$, Appendix E)
(v)	Row 6100 gives $C_1 = 112$, $C_2 = 114$	Cited (read from Table 1 of [1])
(vi)	$\mathrm{K3J} > 10^{-4}$	Kernel (`theorem3_positive_main`, `exp_gamma_gt`, `exp_gamma_lt`, `cbar_lt`)
(vii)	Bracket combination $\geq \mathrm{K3J} - 10^{-5} > 9 \cdot 10^{-5}$	Kernel, script cross-check (`bracket13375_holds`, `li_factor`; Python total $1.535054 \cdot 10^{-4}$ in `scripts/i0088_numerics.py`, Appendix D)
(viii)	Error terms and the remainder of (33) are negligible	Kernel, script cross-check (`error_13375`, `mbar_le` with $\bar{m} \leq 40$, `bracket13375_holds`; hand bound $\bar{m} < 6.2325$ in Appendix D)
(ix)	$\pi_2(N) > \sqrt{N} + 1$ implies $N = p + P_2$	Kernel (`section55_step`, `rep_of_pi2'`, Lemma 2.1)
(x)	Assembly from (i) to (ix) to ChenThreshold13375	Kernel (`chenThreshold13375_of_deep_closed`)
(xi)	α-window $[0.0025588, 0.0025649)$, u-scan (13.36525 near 1627), and Tables 4, 5, 11	Script (`scripts/i0088_numerics.py`, not in Lean)

3.1. **Limitations.** In the Lean formalization, GRH is an explicit hypothesis, and Theorems 5, 6, 7 and Theorem 4 of [1] are hand-transcribed as Lean hypotheses (`Thm5J`, `Thm6J`, `Thm7J`, `LinearSieveT4`), the first three in the UNfull normalisation of Appendix B (with the printed $(e^{-\gamma}\prod_{p>2}(1-(p-1)^{-2}))^{-1}$ kept in error terms), so a transcription error would not be caught by Lean; as detailed in Appendix C, two AI audits on 28 October 2026 caught missing premises $0 < \varepsilon < 1/63$ and N even in commit `23bb733` (v1.3) that made `Thm5J` refutable under GRH, fixed at commit `2fdf0dc`. Deriving Theorems

5, 6, 7 from GRH through Lemma 3, Lemma 7, and the linear sieve (Section 4 and Sections 5.1–5.3 of [1]) is the unused `sorry theorems567_of_inputs`. The observation that Corollary 1 needs only K3J > 0 rather than $4\cdot10^{-3}$ also applies to the Section 5.4 assembly of Bordignon, Johnston and Starichkova [3], which we have not computed.

Appendix A. Derivation of K3J from [1]

All three sifting bounds in Section 5 of [1] carry the factor $U_N N/\log^2 N$, and K3J is the combined main-term coefficient. Before detailing the individual sources, Table 8 records the line-by-line assembly with explicit signs.

Table 8. K3J terms at $\alpha = 0.00256$, $\varepsilon = 1/8349$, $C_1 = 112$, $C_2 = 114$, $g = e^\gamma$.

Term	Expression	Source in [1]	Sign	Value
T_5 main	$8 \cdot \frac{2g\log(3-8\alpha)}{4-8\alpha}$	Theorem 5, p. 1228, inner bracket $f(s)$ at $s = 4-8\alpha$	+	+7.818059
T_5 sieve	$8 \cdot \varepsilon C_2 \frac{3e^{8\alpha-2}}{4-8\alpha}$	Theorem 5, p. 1228	−	−0.011375
T_6 main	$\frac{1}{2} \cdot 8(1 + 5 \cdot 10^{-5}) \frac{(g/4)(\log 6 + \log\frac{3-8\alpha}{3-18\alpha})}{1/2-\alpha}$	Theorem 6, pp. 1231–1232, display l_1; factor $1 + 5\cdot10^{-5}$ from Lemma 11, p. 1228	−	−6.446572
T_6 sieve	$\frac{1}{2} \cdot 8(1+5\cdot10^{-5}) \log(8/3)\, \varepsilon C_1$	Theorem 6, display l_1	−	−0.052633
T_7 main	$\frac{1}{2} \cdot \frac{2g}{1/2-\alpha}\, \bar{c}$	Theorem 7, pp. 1234–1235, with (29), p. 1237, and Lemma 14, p. 1237	−	−1.300014
T_7 sieve	$\frac{1}{2} \cdot 3\varepsilon C_1\, \bar{c}$	Theorem 7, pp. 1234–1235, with (29), p. 1237, and Lemma 14, p. 1237	−	−0.007306
Sum	K3J	Weights $1, -1/2, -1/2$ from Lemma 9, p. 1227		$1.58626 \cdot 10^{-4}$

Values use $\bar{c} < 0.363084$; with the exact $\bar{c}$ the sum is $1.5960 \cdot 10^{-4}$, as in Section 2.

Each piece arises from [1] as follows:

- By Lemma 9 of [1], p. 1227 (Chen's weighted inequality),

$$\pi_2(N) > S(\mathcal{A},\mathbb{P},z) - \tfrac{1}{2}\sum_{z\le q<y} S(\mathcal{A}_q,\mathbb{P},z) - \tfrac{1}{2}S(\mathcal{B},\mathbb{P},y) - 2N^{7/8} - N^{1/3}, \quad (2)$$

with $z = N^{1/8}$ and $y = N^{1/3}$ (Section 5, p. 1226). This gives the weights 1, $-1/2$, and $-1/2$. For a prime $q \mid N$ the set $\mathcal{A}_q$ is empty, since $q \mid N-p$ would force $p = q$, while $\mathcal{A}$ uses only primes $p \nmid N$; so the sum may be restricted to $q \nmid N$, as in Theorem 6.

- By Theorem 5 of [1], p. 1228,

$$S(\mathcal{A},\mathbb{P},z) > 8\frac{U_N N}{\log^2 N}\Big(\Big(1-\frac{0.62\log X_2}{X_2^{1/16}}\Big) \times\Big(\frac{2e^\gamma\log(3-8\alpha_1)}{4-8\alpha_1}-\varepsilon C_2(\varepsilon)\frac{3e^{8\alpha_1-2}}{4-8\alpha_1}\Big)-E_5\Big), \quad (3)$$

  where E_5 is a term of order $\log^{-8} N$. The factor 8 comes from $V(z)$ and $\log z = \frac{1}{8}\log N$ (Lemma 10), together with $|\mathcal{A}| \geq N/\log N$ (Lemma 11); the inner bracket is $f(s) - \varepsilon C_2 e^2 h(s)$ evaluated at $s = 4-8\alpha_1$, where $f(s) = 2e^\gamma\log(s-1)/s$ and $h(s) = 3s^{-1}e^{-s}$ (proof of Theorem 5, p. 1229).
- By Theorem 6 of [1], pp. 1231 to 1232,

$$\sum_{z\leq q<y,\, q\nmid N} S(\mathcal{A}_q,\mathbb{P},z) \leq \frac{8U_N N}{\log^2 N}\big(l_1 + l_2/\log^{A-3} N + l_3/\log^{A-1} N\big) \quad (4)$$

  with

$$l_1 = (1+5\cdot 10^{-5})\Big(1+\frac{0.62\log X_2}{X_2^{1/16}}\Big)\frac{X_2^{1/8}}{X_2^{1/8}-1} \times\Big(\frac{e^\gamma}{4}\Big(\frac{\log 6+\log\frac{3-8\alpha_2}{3-18\alpha_2}}{\frac12-\alpha_2}+\frac{512}{k_{\alpha_2}\log^2 X_2}\Big) +\Big(\log\frac83+\frac{64}{\log^2 X_2}\Big)\varepsilon C_1(\varepsilon)e^2h(k_{\alpha_2})\Big). \quad (5)$$

  The factor $1+5\cdot10^{-5}$ is the upper bound $|\mathcal{A}| \leq (1+5\cdot10^{-5})N/\log N$ of Lemma 11, p. 1228. Since $k_{\alpha_2} = 8(1/6-\alpha_2) \in [1,2]$, we have $h(k_{\alpha_2}) = e^{-2}$ and $e^2h(k_{\alpha_2}) = 1$.
- By Theorem 7 of [1], pp. 1234 to 1235, and (29), p. 1237, the leading part of the upper bound for $S(\mathcal{B},\mathbb{P},y)$ is

$$\frac{U_N N}{\log^2 N}\Big(\frac{2e^\gamma}{\frac12-\alpha_3}+3\varepsilon C_1(\varepsilon)\Big)\,\bar{c}\cdot(\text{li factor})\cdot(1+\varepsilon_1)\cdots, \quad (6)$$

  where $3\big(F(s)+\varepsilon C_1(\varepsilon)e^2h(s)\big)$, with $F(s) = 2e^\gamma/s$ and $h(s) = e^{-2}$ at $s = 3/2-3\alpha_3 \in [1,2]$, and $V(y) < 3U_N/\log N$ give the bracket, and $\bar{c}$ enters through Lemma 14.

Setting $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, dropping the factors listed in Appendix D, and collecting coefficients gives (1). Taking the α_i equal is optimal since the total decreases in each α_i and each α_i has its own copy of the binding condition.

Appendix B. The factor 2 in U_N

Normalisation used. The Lean leaves `Thm5J`, `Thm6J`, and `Thm7J` use UNfull $:= 2e^{-\gamma}\prod_{p>2}(1-(p-1)^{-2})\prod_{p>2,\,p\mid N}\frac{p-1}{p-2}$, twice the printed U_N of pp. 1214 and 1227 of [1]; `LinearSieveT4`, Theorem 4, does not

involve U_N. Theorems 5, 6, 7 are applied in this UNfull normalisation in Lean, and the theorem holds under either reading. Specifically, the threshold proposition `ChenThreshold13375`, the conclusion of `chenThreshold13375_of_deep_closed`, does not mention U_N at all (stating that every even $N \geq \exp(\exp 13.375)$ is $p + P_2$), and K3J is a constant independent of U_N. Here U_N enters only as a common positive factor: Theorems 5, 6, 7 give bounds $8B_5$, $8B_6$ and B_7 times $U_N N/\log^2 N$, so with the Lemma 9 weights the combination $8B_5 - 4B_6 - \frac{1}{2}B_7$ multiplies the same U_N, and its sign, governed by K3J, does not depend on which normalisation is used, provided Theorems 5, 6, 7 are read in the same one. The last step needs a lower bound on U_N to beat $\sqrt{N}+1$ and the error terms: $U_N \geq 0.37$ in the printed normalisation (Appendix D) and `two_fifths_le_UNfull`, UNfull $\geq 2/5$, in the Lean one; both suffice at $\log\log N \geq 13.375$ with a wide margin. Hence the threshold 13.375 is the same under either consistent reading, and Lean proves it in the UNfull reading.

Pages 1214 and 1227 of [1] print $U_N = e^{-\gamma}\prod_{p>2}\bigl(1-\frac{1}{(p-1)^2}\bigr)\prod_{p>2,\,p\mid N}\frac{p-1}{p-2}$; that this printed U_N is half the limit of $V(x)\log x$ in Lemma 10 is a lab finding of 27 October 2026, checked on the journal page images. The companion paper [3], arXiv:2207.09452, defines U_N with $2e^{-\gamma}$. We trace the factor 2 line by line through the five results used in Section 5.4:

- **Lemma 10** (p. 1227). Lemma 10 states that $V(x) = \prod_{p<x,\,p\nmid N}\bigl(1 - \frac{1}{p-1}\bigr)$ is asymptotic to $U_N/\log x$. Since N is even, $p = 2$ is absent from $V(x)$; using $1-\frac{1}{p-1} = \bigl(1-\frac{1}{p}\bigr)\bigl(1-\frac{1}{(p-1)^2}\bigr)$ and Mertens' theorem, dropping the $p = 2$ factor $1 - 1/2$ doubles the product, so $V(x)\log x$ tends to UNfull, twice the printed U_N (numerically, for primes below 10^8, $V(x)\log x$ divided by the printed U_N is 1.99756, 1.99992, and 1.999992 at $x = 10^4, 10^6, 10^8$ for $N = 2^{20}$, 9699690, and $10^{12} + 2$ alike).
- **Theorem 5** (p. 1228). Through Theorem 5 the factor 8 comes from $V(z)$ with $\log z = \frac{1}{8}\log N$, and the corrected $V(z)$ carries the factor 2 into the lower bound with UNfull. The printed $(e^{-\gamma}\prod_{p>2}(1 - (p-1)^{-2}))^{-1}$ in the error terms is twice the needed inverse and only weakens them.
- **Theorem 6** (pp. 1231–1232) and **Theorem 7** (pp. 1234–1235). For both upper bounds, the proofs give the bounds with UNfull, and the printed $(e^{-\gamma}\prod_{p>2}(1-(p-1)^{-2}))^{-1}$ in their error terms is twice the needed inverse and only weakens them.
- **Lemma 9** (p. 1227). Through Lemma 9 nothing changes, because Lemma 9 has no U_N.

We make no claim that Theorem 3 of [1] is wrong: with UNfull it holds as a coefficient statement.

Appendix C. Transcription of the cited inputs

Table 9 records the four cited inputs stated in Lean in `LeanLab/BS2022GRH/ChenThresholdGRH.lean` at public commit `2fdf0dc`.

Table 9. Transcription of the four cited inputs in `ChenThresholdGRH.lean` at commit `2fdf0dc`.

Lean name	Journal result	Premises	Conclusion	Status	Human-checked
`Thm5J`	Theorem 5, p. 1228	`GRH`; `LinearSieveT4` ε C_1 C_2; $0 \le C_1$, $0 \le C_2$; epsJ$(u) \le \varepsilon$ with epsJ display (6), p. 1221; $0 < \varepsilon < 1/63$; $0 < \alpha < 1/8$; $0 < A$; condition (19); $4 \cdot 10^{18} \le X_2 \le N$; N even	$S(\mathcal{A}, \mathbb{P}, z) > 8\,\mathrm{UNfull}\frac{N}{\log^2 N} B_5$	`MATCH`	No
`Thm6J`	Theorem 6, pp. 1231–1232	Same as `Thm5J`, with $0 < \alpha < 1/24$, condition (21) as encoded, $\exp(\exp 10) \le X_2 \le N$, N even	$\sum_{\substack{z \le q < y \\ q \nmid N}} S(\mathcal{A}_q, \mathbb{P}, z) \le 8\,\mathrm{UNfull}\frac{N}{\log^2 N} B_6$	`MATCH`	No
`Thm7J`	Theorem 7, pp. 1234–1235	Same as `Thm5J`, with $0 < \varepsilon_1 < 1$, $0 < \alpha < 1/16$, condition (25), $\exp(\exp 10) \le X_2 \le N$, N even	$S(\mathcal{B}, \mathbb{P}, y) \le \mathrm{UNfull}\frac{N}{\log^2 N} B_7$	`MATCH`	No
`Linear-SieveT4`	Theorem 4, p. 1218; Table 1, pp. 1217–1218	Linear sieve inequality (2) for sifting weights with $0 < \varepsilon < 1/63$ and the Mertens-type product condition, used at $\varepsilon = 1/8349$, $C_1 = 112$, $C_2 = 114$	Sifting bounds of Theorem 4 with constants C_1, C_2	`MATCH`	No

No human expert has checked the transcriptions of Theorems 4, 5, 6, 7; checks are AI audits only (three audits, with Claude reading the journal page images against the Lean text), and this is a stated limitation of the formalization. Two of the three audits found real gaps: the leaves in v1.3 at commit `23bb733` lacked $0 < \varepsilon < 1/63$ and N even, allowing Lean to prove $\neg$`GRH` from `Thm5J`, `Thm6J`, or `Thm7J` and making the conditional headline vacuous; commit `2fdf0dc` added both premises and proved that all non-`GRH`, non-sieve leaf hypotheses hold at the paper's parameters. The third audit (28 October 2026) rated all four transcriptions `MATCH` with three open notes:

C_1, C_2 range over every sieve witness rather than Table 1 values; Lemma 7 needs $\log z > 2753$ while Theorems 5–7 as printed allow smaller z (satisfied at our X_2 since $\log z = 80463$); and the U_N factor 2 of Appendix B.

Appendix D. Omitted terms and error terms

Table 10 summarizes the terms omitted from K3J and the error terms at $\log X_2 = 643707.69$.

Table 10. Omitted factors and error terms at $\log X_2 = 643707.69$, measured against $U_N N/\log^2 N$.

Term	Source in [1]	Size against $U_N N/\log^2 N$
li factor $\frac{\log y\,(\mathrm{li}(y)+\sqrt{y}\log y/(8\pi))}{y}$	Theorem 7, Lemma 14	relative $1/\log y = 4.66\cdot 10^{-6}$, absolute about $6\cdot 10^{-6}$; largest omitted term, included in the total $1.535054\cdot 10^{-4}$
$512/(k\log^2 X_2)$ and $64/\log^2 X_2$ in l_1; $36/\log^2 X_2$ and $27/\log^2 X_2$ in Theorem 7	Theorems 6 and 7	at most 10^{-9}; included in the total $1.535054\cdot 10^{-4}$
$(1\pm 0.62\log X_2/X_2^{1/16})$, $X_2^{1/8}/(X_2^{1/8}-1)$, $(1+0.06\log X_2/X_2^{1/6})$, $1+\varepsilon_1$	Theorems 5 to 7	relative e^{-40000} or 10^{-20}
E_5, $l_2/\log^7 N$, and $l_3/\log^9 N$	Theorems 5 and 6	below 10^{-40}
Remainder of (33)	Section 5.4, p. 1241	$< 1.7\cdot 10^{-15}$
$2N^{7/8}+N^{1/3}+\sqrt{N}+1$	Lemma 9, Corollary 1	ratio to $9\cdot 10^{-5}U_N N/\log^2 N$ is $\le 6.01\cdot 10^4\log^2 N\cdot N^{-1/8} < e^{-80425}$

The bounds in the last two rows of Table 10 are obtained as follows:

- From the definition of U_N in Theorem 3 and Lemma 10 of [1],

$$U_N \ge e^{-\gamma}\prod_{p>2}\bigl(1-(p-1)^{-2}\bigr) = 0.5614594836\cdot 0.66016181 \ge 0.370654 > 0.37,$$

  since the product over $p \mid N$ is at least 1.
- The journal defines $\bar{m}$ in (31), pp. 1239 to 1240, and bounds $M_\varphi(D^*) \le 1.08$ on p. 1240; the bound

$$\bar{m}(N, 10, N^{1/8}) \le 2\cdot 1.08\,\bigl(2/\log 2 + 4e^{-13000}\bigr) < 6.2325 \le 8$$

  is ours: every other term of (31) carries $\log^{11} N$ or $\log^{12} N$ times a negative power of $w = N^{1/8}$, below e^{-13000}. Consequently, the remainder of (33), p. 1241, where the constant 0.053 is printed in

(33), satisfies

$$\frac{1+\varepsilon_1}{U_N \log(1+\varepsilon_1)\log^6 N}\big[0.053\,\bar m(1+\varepsilon_1)^{-1/2}+0.08N^{-1/8}\log^{11} N\big] \le 1.0001\cdot\frac{0.053\cdot 8\cdot 10^{20}}{0.37\cdot 7.1\cdot 10^{34}} < 1.7\cdot 10^{-15}.$$

In Lean, the lemma `mbar_le` proves $\bar m \le 40$ at X_2 without the hand estimate, and `bracket13375_holds` absorbs the remainder of (33) with that weaker bound.

- From Lemma 9 and Corollary 1, comparing $2N^{7/8}+N^{1/3}+\sqrt{N}+1$ against $9\cdot 10^{-5}U_N N/\log^2 N$ with $U_N \ge 0.37$ gives a ratio bounded by

$$\frac{2+N^{-13/24}+N^{-3/8}+N^{-7/8}}{3.33\cdot 10^{-5}}\log^2 N\cdot N^{-1/8} \le 6.01\cdot 10^4 \log^2 N\cdot N^{-1/8},$$

whose logarithm at $\log N = 643707.69$ is $11.0 + 26.75 - 80463.46 < -80425$, and $2\log t - t/8$ decreases for $t > 16$.

Appendix E. The value of $\varepsilon(1500)$

By display (6) of [1], p. 1221,

$$\varepsilon(u_0) = \exp\Big(1.131\cdot 10^{-7} + \Big(\frac{u_0}{u_0-2}+0.501\Big)\sum_{p\ge u_0} p^{-2}\Big) - 1. \tag{7}$$

We sum p^{-2} over primes $1500 \le p < 10^9$ with a segmented sieve (in `scripts/i0088_numerics.py`), giving $7.96425\cdot 10^{-5}$, and bound the tail using $\pi(t) < 1.25506\,t/\log t$ [7] and partial summation:

$$\sum_{p\ge X} p^{-2} = \int_X^\infty t^{-2}\,d\pi(t) \le \int_X^\infty 2\pi(t)\,t^{-3}\,dt \le 2\cdot 1.25506\int_X^\infty \frac{dt}{t^2\log t} \le \frac{2\cdot 1.25506}{X\log X},$$

the boundary term $-\pi(X)/X^2$ being negative; at $X = 10^9$ this is $1.2113\cdot 10^{-10}$. Hence

$$\varepsilon(1500) = 1.1977018\cdot 10^{-4} = 1/8349.3238 < 1/8349,$$

with a margin of $4.6\cdot 10^{-9}$. The same method gives $\varepsilon(1000) = 1/5235.667 < 1/5235$, matching [1].

In Lean, `Cert1575.eps1500` proves $\varepsilon(1500) < 1/8349$, display (6): the head sum over primes $1501 \le p < 9003001$ of $1/p^2$ is at most $7963599791126110\cdot 10^{-20}$, checked by the kernel with `decide +kernel` in 11 windows `Cert1575W00` to `Cert1575W10`, no `native_decide`, by adding $\lfloor 10^{20}/n^2\rfloor + 1$ over $n \equiv 1,5 \pmod 6$ coprime to $m!$, with $m = 1499$ or 3000 below every prime of the window, an upper bound that needs no primality certificate; the tail over $p \ge H = 9003001$ is at most

$(C \cdot H - \theta_0)/(16H^2) + C/(16H)$, $C = 89/80$, $\theta_0 = 8191444$, by summation by parts against Chebyshev's θ with explicit Chebyshev bounds proved in the lab, leaving a margin of about $8.8 \cdot 10^{-10}$ in $1 + 1/8349 - e^x$. This kernel proof replaces the Python sum for the logic, and the two agree.

The conclusion is not sensitive to this margin: at $\alpha = 0.00256$ and $\log\log X_2 = 13.375$ the total stays positive for every $\varepsilon < 1/8331.07$, and the 6100 row of Table 1 needs only $\varepsilon < 1/6100$. Table 11 shows the sensitivity of the total and optimal threshold to $1/\varepsilon$ at $u = 1500$ and row 6100: the threshold 13.375 needs $1/\varepsilon(1500)$ above 8331.07; the computed value is 8349.3238.

TABLE 11. Sensitivity to ε at $u = 1500$, Table 1 row 6100.

$1/\varepsilon$	Total at $\alpha = 0.00256$, 13.375	Best $\log\log X_2$
8349	$1.5351 \cdot 10^{-4}$	13.37261
8340	$7.6548 \cdot 10^{-5}$	13.37357
8331.07	$2.4 \cdot 10^{-8}$	13.37453
8300	$-2.6751 \cdot 10^{-4}$	13.37789
8000	$-2.9576 \cdot 10^{-3}$	13.41228
7000	$-1.3590 \cdot 10^{-2}$	13.56120
6100	$-2.6139 \cdot 10^{-2}$	13.77160

APPENDIX F. BUILD LOGS AND FILE PATHS

Lean files named in Section 3 without a directory lie in `LeanLab/BS2022GRH/`; other paths are relative to the repository root. Declarations carry the namespace prefix `LeanLab.BS2022GRH` (sub-namespace `Journal` for `JournalMargin.lean`, `JournalPositive.lean` and `RangeCheck13375.lean`, `ChenThresholdGRH` for `ChenThresholdGRH.lean`, and `Cert1567`, `Cert1575` for the certificate files). The build logs are recorded in `logs/paperB/`:

- `20261020T0802Z-20406-JournalPositive.log`: Build completed successfully, `EXIT 0`, `sorry` count 0; `theorem3_positive_main`, `conditions_13375`, and `params_ok_13375` each depend on axioms `[propext, Classical.choice, Quot.sound]`.
- `20261020T1303Z-22201.log`, a fresh rebuild by a second run: `JournalPositive` and `RangeCheck13375` both `EXIT 0`, 0 `sorry`, and all 11 theorems depend only on `[propext, Classical.choice, Quot.sound]`.
- `20261026T2003Z-2973-U-1561.log` and `20261027T0004Z-17012-U-1567.log`: each ends with Build completed successfully and `EXIT 0`.
- `20261027T0807Z-9485-U-1575.log`: records earlier failed attempts and the final rebuild of `Cert1575Closed` after a docstring fix, Build

completed successfully (9101 jobs), `EXIT 0`, with `Cert1575.eps1500` and `chenThreshold13375_of_deep_closed` depending on axioms `[propext, Classical.choice, Quot.sound]`.

Use of AI

AI, namely Claude by Anthropic, wrote all Lean code and drafted the text; the Lean of Section 3 was written by Claude in the author's lab and checked only by the Lean kernel; AI tools were used for the literature search; the statements, the numbers and the references were checked by independent AI checker passes against the Lean source and the published pages of [1]; the author is responsible for the content.

Checker protocol. As done for every version, each draft is checked by a fresh AI session that did not write it, which compares every theorem name and statement against the Lean source, every number against the build logs and the public Python script `i0088_numerics.py`, and every citation against the journal pages, and reports findings that go into the next round; the Lean itself is checked only by the Lean kernel; the author reads every version and is responsible. No human expert has verified the transcriptions of Theorems 4 to 7.

Data availability

The Lean source of every theorem named in this paper, with its full import closure, the build logs, and the Python script `i0088_numerics.py` with its output, are public at `https://github.com/sthubbar/lean-lab-preprints`, commit `2fdf0dc`; the README states exactly what the Lean proves and what it does not. The conditional threshold theorem, with its four cited inputs as explicit hypotheses, is registered in the Palomar registry of Lean-verified mathematics as `PALOMAR-2026-10-28-000016`, `https://palomar-registry.org/entry.html?id=PALOMAR-2026-10-28-000016&version=1`.

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Independent Researcher