

A LOWER EXPLICIT THRESHOLD FOR CHEN'S THEOREM UNDER GRH

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ABSTRACT. Assume the Generalized Riemann Hypothesis (GRH). We show that every even integer $N \geq \exp(\exp(13.375))$ can be written as $N = p + P_2$, where p is a prime and P_2 is either a prime or a product of two primes. This result is conditional on GRH and on Theorems 5, 6, 7 and Lemma 9 of Bordignon and Starichkova (Ramanujan J. 64, 2024), together with their auxiliary lemmas, Table 1 and display (6) that their Section 5.4 uses, used at new parameter values inside their own Section 5.4. It is not a new method; it is a re-choice of the parameters u and α in their argument, combined with the observation that Chen's theorem itself needs only that the Section 5.4 main-term combination be positive, not that it exceed $4 \cdot 10^{-3}$, because the inequality $\pi'_2(N) \geq \pi_2(N) - \sqrt{N} - 1$ from the proof of their Corollary 1 (Section 5.5, p. 1241), which does not involve the threshold, gives $N = p + P_2$ as soon as $\pi_2(N) > \sqrt{N} + 1$. In Lean 4 the threshold statement itself is stated with GRH as an explicit hypothesis and proved conditional on the journal's Theorems 5, 6, 7 and the linear sieve Theorem 4 at the constants used; Lemma 9, the Section 5.5 step, $\varepsilon(1500) < 1/8349$ and the Section 5.4 bracket arithmetic are proved in Lean. Theorems 5, 6, 7 and Theorem 4 are not formalized.

1. INTRODUCTION

In 1973, Chen [4] proved that every sufficiently large even integer N can be represented as

$$N = p + P_2,$$

where p is a prime and P_2 is a prime or a product of two primes. While the binary Goldbach conjecture remains open, and this note makes no progress on it, substantial work has gone into making the threshold for Chen's theorem explicit.

The first explicit threshold for Chen's theorem was obtained by Yamada [8]. Unconditionally, the current record is due to Bordignon, Johnston and Starichkova [3], who established Chen's theorem for all even integers $N \geq \exp(\exp(32.7))$. Johnston and Starichkova [5] proved related explicit results on writing even integers as a prime plus an almost prime. In related formal work, a qualitative Lean 4 formalization of Chen's theorem with an existential, non-explicit threshold is claimed in the unrefereed repository [6].

Assuming the Generalized Riemann Hypothesis (GRH), Bordignon and Starichkova lowered the explicit threshold considerably. In the arXiv version

[2] (v1, the only arXiv version), they obtained the threshold $\exp(\exp(15.85))$ with lower-bound constant $4 \cdot 10^{-4}$. In the published journal version [1] (Theorem 3, p. 1214), they proved under GRH that

$$\pi_2(N) > 4 \cdot 10^{-3} U_N \frac{N}{\log^2 N} \quad \text{for } N > \exp(\exp(14)),$$

from which their Corollary 1 (p. 1214) gives Chen's theorem for these N . We are not aware of a published GRH threshold below $\exp(\exp(14))$.

In this note we establish the threshold $\exp(\exp(13.375))$ under GRH. This is conditional on GRH and on Theorems 5, 6, 7 and Lemma 9 of Bordignon and Starichkova [1], together with the auxiliary lemmas, Table 1 and display (6) of [1] that their Section 5.4 uses, used at new parameter values inside their own Section 5.4. It is not a new method. It is a re-choice of the parameters u and α in their argument.

Theorem 1 (Main statement). *Assume the Generalized Riemann Hypothesis, and assume Theorems 5, 6, 7 and Lemma 9 of Bordignon and Starichkova [1], together with the auxiliary lemmas, Table 1 and display (6) of [1] that their Section 5.4 uses, used at new parameter values inside their own Section 5.4. Then every even integer $N \geq \exp(\exp(13.375))$ can be written as $N = p + P_2$, with p a prime and P_2 a prime or a product of two primes.*

Lemma 9 is also proved in Lean, Section 3, so the cited inputs actually needed are Theorems 5, 6, 7 and Theorem 4 with Table 1 and display (6).

What is formally verified. In Lean 4 the threshold statement itself is stated with GRH as an explicit hypothesis and proved conditional on the journal's Theorems 5, 6, 7 and the linear sieve Theorem 4 at the constants used. Within the formalization, Lemma 9, the Section 5.5 step, $\varepsilon(1500) < 1/8349$ and the Section 5.4 bracket arithmetic with all correction factors and error terms are proved in Lean. Theorems 5, 6, 7 and Theorem 4 are transcribed from [1] as explicit hypotheses and are not formalized. Full details are given in Section 3.

Context. The unconditional record is $\exp(\exp(32.7))$ by Bordignon, Johnston and Starichkova [3]; GRH thresholds indicate how far the explicit sieve method can go when zero-free regions are not the bottleneck; within this framework the optimum is about 13.365 at u near 1628 (Section 2.4), so 13.375 is within 0.01 of the method's limit and further gains need new ideas in Theorems 5 to 7 or the sieve constants.

2. THE PARAMETER RE-CHOICE IN SECTION 5.4 OF [1]

Table 1 summarizes the notation of [1] used throughout this note (γ denotes Euler's constant).

We begin by recording what the paper of Bordignon and Starichkova [1] does in full detail:

TABLE 1. Glossary of notation from [1].

Symbol	Meaning	Place in [1]
$\pi_2(N)$	representations of N as a prime plus a product of at most two primes	Theorem 3, p. 1214
$\pi'_2(N)$	the same with a square-free semiprime	Section 5.5, p. 1241
U_N	$e^{-\gamma} \prod_{p>2} (1 - \frac{1}{(p-1)^2}) \prod_{p>2, p N} \frac{p-1}{p-2}$, as printed; see Appendix B for the factor 2	Theorem 3 and Lemma 10
X_2	lower bound for N in Theorems 5 to 7, $N \geq X_2 \geq 4 \cdot 10^{18}$	Theorems 5 to 7
z, y	$z = N^{1/8}, y = N^{1/3}$	Section 5, p. 1226
$\mathcal{A}, \mathcal{B}$	sifted sets, $\mathcal{A} = \{N - p\}$	p. 1227
A, u	parameters of Theorems 5 to 7, $A > 0$, $u \in [400, 10^8]$	Theorems 5 to 7
α_i	level parameters with $D = N^{1/2-\alpha_i}$	Theorems 5 to 7
$\varepsilon(u)$	prime-sum error parameter	display (6), p. 1221
$C_1(\varepsilon), C_2(\varepsilon)$	linear sieve constants via inequality (2) of Theorem 4, p. 1218	Table 1, pp. 1217–1218
ε_1	parameter in (0, 1)	Theorem 7
$\bar{c}$	$\int_{1/8}^{1/3} \frac{\log(2-3\beta)}{\beta(1-\beta)} d\beta < 0.363084$	Lemma 14, p. 1237
$\bar{m}$	bilinear remainder factor	(31), pp. 1239–1240
M_φ	upper-bound factor in Lemma 5	Lemma 5, p. 1216

- In [1], Theorem 3, p. 1214, under GRH, they prove

$$\pi_2(N) > 4 \cdot 10^{-3} U_N \frac{N}{\log^2 N}$$

for $N > \exp(\exp(14))$. Their Corollary 1 (p. 1214) gives Chen's theorem for these N , passing from $\pi_2(N)$ to $\pi'_2(N)$ via the relation $\pi'_2(N) \geq \pi_2(N) - \sqrt{N} - 1$ (Section 5.5, p. 1241).

- Their Section 5.4, p. 1241, combines Theorem 5 (p. 1228, with proof on p. 1229), Theorem 6 (p. 1231), Theorem 7 (pp. 1234 to 1235), and Lemma 9 (p. 1227, the weighted sieve inequality of Chen that bounds $\pi_2(N)$ below by sifting functions, minus $2N^{7/8} + N^{1/3}$), and takes the parameters

$$u = 1000, \quad \alpha_1 = \alpha_2 = \alpha_3 = 10^{-3}, \quad A = 10, \quad \varepsilon_1 = 10^{-20},$$

together with $\varepsilon(1000) < 1/5235$, $C_1 = 113$, and $C_2 = 114$.

- The arXiv version [2], v1, the only arXiv version, had the threshold $\exp(\exp(15.85))$, with constant $4 \cdot 10^{-4}$.

All changes in the present note take place inside Section 5.4 of [1] (p. 1241):

- (1) We choose the parameters

$$u = 1500, \quad \alpha_1 = \alpha_2 = \alpha_3 = 0.00256, \quad A = 10, \quad \varepsilon_1 = 10^{-20}.$$

- (2) From display (6) of [1], p. 1221, we have $\varepsilon(1500) < 1/8349$; see Appendix E for the computation giving $\varepsilon(1500) = 1/8349.3238$.
- (3) We take $C_1 = 112$ and $C_2 = 114$, which is the row $\varepsilon^{-1} = 6100$ of Table 1 of [1], pp. 1217 to 1218. This is valid since inequality (2) with $\varepsilon(1500)$ implies it with the larger $1/6100$; the journal itself pairs $\varepsilon(1000) < 1/5235$ with the row 4500.
- (4) The key observation is that Chen's theorem itself needs only that the Section 5.4 combination be positive, not that it exceed $4 \cdot 10^{-3}$: the inequality $\pi'_2(N) \geq \pi_2(N) - \sqrt{N} - 1$ from the proof of Corollary 1 (Section 5.5, p. 1241), which does not involve the threshold, gives $N = p + P_2$ as soon as $\pi_2(N) > \sqrt{N} + 1$; see Lemma 2.1. Asking only for a positive constant lets $\log \log X_2$ drop to 13.375.

TABLE 2. Comparison of Section 5.4 parameters in [1] and in this note.

Quantity	[1], §5.4	This note
$\log \log X_2$	14	13.375
u	1000	1500
$\alpha_1 = \alpha_2 = \alpha_3 = \alpha$	10^{-3}	0.00256
A	10	10
ε_1	10^{-20}	10^{-20}
Upper bound on $\varepsilon(u)$	$\varepsilon(1000) < 1/5235$	$\varepsilon(1500) < 1/8349$
Table 1 row ε^{-1} (pp. 1217 to 1218)	4500	6100
(C_1, C_2)	(113, 114)	(112, 114)

2.1. Definition of the main-term constant K3J. We define K3J exactly as in the Lean formalization (see Appendix A for its derivation from [1]), with $g = e^\gamma$, $\varepsilon = 1/8349$, and $\bar{c}$ the constant of Lemma 14 of [1], p. 1237,

$$\bar{c} = \int_{1/8}^{1/3} \frac{\log(2 - 3\beta)}{\beta(1 - \beta)} d\beta < 0.363084,$$

by setting

$$\begin{aligned} \text{K3J} = & 8 \left(2g \log(3 - 8\alpha)/(4 - 8\alpha) - \varepsilon C_2 \cdot 3 e^{8\alpha-2}/(4 - 8\alpha) \right) \\ & - \frac{1}{2} \cdot 8 \left((1 + 5 \cdot 10^{-5}) \left(\frac{g}{4} (\log 6 + \log \frac{3-8\alpha}{3-18\alpha}) / (\frac{1}{2} - \alpha) + \log(8/3) \varepsilon C_1 \right) \right) \\ & - \frac{1}{2} \left(2g / (\frac{1}{2} - \alpha) + 3\varepsilon C_1 \right) \bar{c}. \end{aligned} \tag{1}$$

Here the factor $1 + 5 \cdot 10^{-5}$ is the journal's Lemma 11 (p. 1228) factor on Theorem 6's l_1 , p. 1231. In words, the three pieces of K3J are 8 times the main term of Theorem 5, minus one half of 8 times the main term l_1 of Theorem 6, minus one half of the main term of Theorem 7.

2.2. Numerical evaluation at $\log \log X_2 = 13.375$ and $\alpha = 0.00256$. At $\log \log X_2 = 13.375$ and $\alpha = 0.00256$, the relevant numerical values are as follows:

- $\log X_2 = e^{13.375} = 643707.69$; Lean proves $e^{13.375} \geq 643700$.
- The binding condition, which is the second part of displays (19), (21), and (25) of [1], is

$$\alpha \log X_2 - 11 \log \log X_2 = 1500.767 \geq 1500 = u.$$

At this X_2 the condition needs $\alpha \geq 0.0025588$, leaving a slack of 0.00047 in $\log \log X_2$.

- At $\alpha = 0.00256$, $\varepsilon = 1/8349$, $C_1 = 112$, and $C_2 = 114$, the main-term constant K3J defined in (1) has the numerical value $\text{K3J} = 1.5960 \cdot 10^{-4}$, and Lean proves $\text{K3J} > 10^{-4}$.
- With the terms K3J leaves out, namely the li factor

$$\frac{\log y (\text{li}(y) + \sqrt{y} \log y / (8\pi))}{y} = 1 + \frac{1}{\log y} + \frac{2}{\log^2 y} + \dots$$

with $y = X_2^{1/3}$ and the $1/\log^2 N$ terms of Theorems 6 and 7, which come to $6.10 \cdot 10^{-6}$, the total with the omitted terms is $1.535054 \cdot 10^{-4}$.

All remaining omitted factors, the remainder of (33), and the power-saving error terms from Lemma 9 and Corollary 1 are bounded explicitly in Appendix D and are negligible.

Lemma 2.1 (positivity suffices). *Let N be even. If $\pi_2(N) > \sqrt{N} + 1$, then $N = p + q$ with p prime and $q > 1$ square-free with at most two prime factors; in particular $N = p + P_2$.*

Proof. Section 5.5 of [1], p. 1241, shows $\pi'_2(N) \geq \pi_2(N) - \sqrt{N} - 1$, where $\pi'_2(N)$ is the number of representations of N as a sum of a prime and a square-free semiprime, which must be read as a square-free integer with at most two prime factors, since $\pi_2(N)$ also counts $N = p + q$ with q prime and the inequality removes only $q = 1$ and q with a repeated prime factor; in the journal's words the 1 counts the representation $N = p + 1$, if it exists, and $\sqrt{N}$ bounds the number of integers up to N with two equal prime factors. Hence $\pi'_2(N) > 0$. This inequality holds for every even N and does not use the threshold $\exp(\exp(14))$; Corollary 1 of [1] applies it with Theorem 3's constant $4 \cdot 10^{-3}$, and we apply it with the constant $9 \cdot 10^{-5}$ of Section 2.2, using Appendix D to see that $9 \cdot 10^{-5} U_N N / \log^2 N > \sqrt{N} + 1$ for $\log \log N \geq 13.375$. $\square$

- Hence, on GRH and the cited results,

$$\pi_2(N) > 9 \cdot 10^{-5} U_N \frac{N}{\log^2 N} > 0$$

for every even $N \geq \exp(\exp(13.375))$, and Lemma 2.1 (via the inequality $\pi'_2(N) \geq \pi_2(N) - \sqrt{N} - 1$ from the proof of Corollary 1 of

[1], Section 5.5, p. 1241, which does not involve the threshold) then gives $N = p + P_2$.

Remark 1 (Threshold with the printed constant $4 \cdot 10^{-3}$ kept). With the printed constant $4 \cdot 10^{-3}$ kept, the same method at $u = 1500$ and $\alpha = 0.0024$ gives $\exp(\exp(13.45))$, with $K3J = 5.1803 \cdot 10^{-3}$ and total $5.1747 \cdot 10^{-3}$ in Python, both above $4 \cdot 10^{-3}$, and the binding condition $1517.3 \geq 1500$. The numerical optimum at $u = 1500$ is 13.4239 at $\alpha = 0.0024375$, so 13.45 is conservative. The value 13.45 is checked in Lean in `LeanLab/BS2022GRH/JournalLower.lean: theorem3_lower_main` proves $K3J > 0.0042$ at $\alpha = 0.0024$, $\varepsilon = 1/8349$, $C_1 = 112$, $C_2 = 114$, and `conditions_1345` proves the binding condition at 13.45; build `log 20261020T0403Z-27388-U1166.log`, axioms `[propext, Classical.choice, Quot.sound]`.

2.3. Verification of parameter ranges at 13.375 and 13.45. Table 3 checks all parameter conditions required by the lemmas and theorems in [1] at $\log \log X_2 = 13.375$ ($\alpha = 0.00256$) and $\log \log X_2 = 13.45$ ($\alpha = 0.0024$). Every condition holds in both columns, and only the first row is tight.

TABLE 3. Parameter range conditions at $\log \log X_2 = 13.375$ ($\alpha = 0.00256$) and $\log \log X_2 = 13.45$ ($\alpha = 0.0024$).

Condition and source in [1]	Required	At 13.375	At 13.45
(19), (21), (25) part 2, $\alpha \log X_2 - 11 \log \log X_2 \geq u$	1500	1500.767	1517.272
(19), (21), (25) part 1, $\log X_2 \geq (A + 1)/\alpha$	4296.9 / 4583.3	643707.69	693842.31
(25), $\log X_2 \geq 48(A + 4/3)$	544	643707.69	693842.31
(25), $\frac{\log X_2}{48} - \frac{34}{3} \log \log X_2 \geq \log 0.44$	-0.821	13258.99	14302.61
$\alpha_1 < 1/8$, $\alpha_2 < 1/24$, $\alpha_3 < 1/16$	< 0.041667	0.00256	0.0024
Lemma 7, p. 1221, $\log z = \log X_2/8 > 2753$	2753	80463.46	86730.29
Theorem 5, $s = 4 - 8\alpha \in [3, 4]$	[3, 4]	3.97952	3.98080
Theorem 6, $k = 8(1/6 - \alpha) \in [1, 2]$	[1, 2]	1.312853	1.314133
Remark 1, p. 1230, $\log H = \frac{\log X_2}{2} - 11 \log \log X_2 \geq \log 10^8$	18.42	321706.72	346773.21
Lemma 11, p. 1228, $\log \log X_2 \geq 10$	10	13.375	13.45
Lemmas 12 and 13, $X_2 \geq 4 \cdot 10^{18}$ (log form)	42.83	643707.69	693842.31
Lemma 14, p. 1237, $\log y = \log X_2/3 \geq \log 2657$	7.885	214569.23	231280.77
Proof of Theorem 6, p. 1234, $\log y \geq \log 286$	5.656	214569.23	231280.77
Theorems 5 to 7, $400 \leq u \leq 10^8$	$u = 1500$	1500	1500

2.4. Headroom and the active constraint. With $u = 1500$ the binding condition is the second part of (19), (21), and (25): $\alpha \log X_2 - 11 \log \log X_2 \geq$

u . At $\alpha = 0.00256$ it needs $\log \log X_2 \geq 13.374532$, a slack of 0.00047. Every other range condition holds with a wide margin (Table 3).

The total decreases in α , so positivity bounds α from above. At $\log \log X_2 = 13.375$ the admissible window is $0.0025588 \leq \alpha < 0.0025649$. The joint optimum at $u = 1500$ is $\log \log X_2 = 13.37261$ at $\alpha = 0.0025649$.

Letting u vary, with $\varepsilon(u)$ recomputed by display (6) and the best admissible Table 1 row, the optimum of this framework with $A = 10$ and $\varepsilon_1 = 10^{-20}$ is about $\log \log X_2 = 13.36525$, at u just above the prime 1627, with $\alpha = 0.0027829$; at the integer $u = 1628$ it is 13.36582. Thus 13.375 is within 0.01 of what Section 5.4 of [1] can give at $A = 10$. These optimizations are numerical only and not formalized; the paper keeps 13.375, which Lean checks. Table 4 shows the optimal threshold across several choices of u .

TABLE 4. Optimal $\log \log X_2$ as a function of u ($A = 10$, $\varepsilon_1 = 10^{-20}$, $\varepsilon = \varepsilon(u)$ itself; with $\varepsilon = 1/8349$ the $u = 1500$ entry is 13.37261).

u	$1/\varepsilon(u)$	Table 1 row	Optimal $\log \log X_2$
1000	5235.67	4500	13.78977
1200	6432.64	6100	13.47933
1400	7586.12	6100	13.40351
1500	8349.32	6100	13.37258
1628	about 9236	6100	13.36582
2000	11401.89	6100	13.42558
2200	12589.97	12200	13.46608

Choice of u and α . The parameters $u = 1500$ and $\alpha = 0.00256$ were found by a grid search over u and α with the Section 5.4 total; $u = 1500$ is a round value near the optimum, and the finer scan above shows $u = 1628$ is marginally better, 13.36582 against 13.37261. For fixed $u = 1500$ and $\log \log X_2 = 13.375$, the admissible window for α is as follows: the lower end 0.0025588 solves $\alpha \log X_2 - 11 \log \log X_2 = u$ in closed form, and the upper end 0.0025649 is the zero of the total, found by bisection; 0.00256 is a round value inside it. The bisections are in the public script.

Smaller A lowers the binding side, to 13.3644 for $A = 9$ and 13.3561 for $A = 8$ at $u = 1500$, but the remainder of (33) grows like $10^{20}/\log^{A-4} N$ and stops being negligible at $A = 8$ unless ε_1 is enlarged; this is not pursued.

Table 5 shows the sensitivity to the choice of row in Table 1 of [1] at $u = 1500$ and $\varepsilon = 1/8349$. Rows 63 to 114 give a negative total for every α . The next row, 12200 with $C_1 = 112$ and $C_2 = 113$, needs $\varepsilon(u)^{-1} \geq 12200$, hence u of about 2150 or more, which gives at least 13.466. Thus the 6100 row is the best admissible one and no other row reaches 13.375.

TABLE 5. Sensitivity to Table 1 of [1] at $u = 1500$, $\varepsilon = 1/8349$.

Row ε^{-1}	C_1	C_2	Total at $\alpha = 0.00256, 13.375$	Best $\log \log X_2$
2300	114	116	$-1.12 \cdot 10^{-3}$	13.38861
3300	113	115	$-4.81 \cdot 10^{-4}$	13.38058
4500	113	114	$-3.82 \cdot 10^{-4}$	13.37932
6100	112	114	$+1.535 \cdot 10^{-4}$	13.37261

3. FORMAL VERIFICATION, LEAN 4

All Lean code was written by AI and checked only by the Lean kernel, Lean v4.35.0 with Mathlib v4.35.0. All paths below are relative to the LeanLab repository root, with namespace prefix LeanLab.BS2022GRH (subnamespace Journal for JournalMargin.lean, JournalPositive.lean and RangeCheck13375.lean, ChenThresholdGRH for ChenThresholdGRH.lean, and Cert1567, Cert1575 for the certificate files). Every proved Lean theorem named in this paper depends only on the standard axioms [propext, Classical.choice, Quot.sound]. The files contain two sorry: the open deep step theorems567_of_inputs in ChenThresholdGRH.lean, Section 3.1, and the superseded placeholder Cert1567.eps1500_cert in Cert1567.lean. Neither is used by chenThreshold13375_of_deep_closed, which takes $\varepsilon(1500) < 1/8349$ from Cert1575.eps1500. The import closure also contains the statement files LeanLab/DERZ1997/Statements.lean and LeanLab/AlmostPrimeSums/Statements.lean with sorry, which none of the named theorems uses, as their axiom lists show.

The file LeanLab/BS2022GRH/ChenThresholdGRH.lean states the threshold proposition:

```
def ChenThreshold13375 : Prop :=
  GRH → ∀ N : ℕ, Even N → exp (exp 13.375) ≤ (N : ℝ) → ChenSqfreeRep N
```

where ChenSqfreeRep N means $N = p + m$ with p prime and $m > 1$ square-free with at most two prime factors, and GRH is the lab's predicate for GRH of Dirichlet L -functions. The cited leaves are stated in the same file as Lean propositions transcribing the journal: Thm5J (Theorem 5, p. 1228), Thm6J (Theorem 6, pp. 1231 to 1232), and Thm7J (Theorem 7, pp. 1234 to 1235), each taking GRH, LinearSieveT4, epsJ $u \leq \varepsilon$, $0 < \varepsilon < 1/63$ and Even n as explicit arguments, as in the journal; LinearSieveT4 ε C_1 C_2 is Theorem 4, the linear sieve, p. 1218, with Table 1, in the ranges used; and epsJ is display (6), p. 1221. These are hypotheses, not proved.

The file LeanLab/BS2022GRH/Cert1575Closed.lean proves the headline theorem:

```
theorem chenThreshold13375_of_deep_closed
  (h5 : Thm5J) (h6 : Thm6J) (h7 : Thm7J)
  (hT4 : LinearSieveT4 (1 / 8349) 112 114) : ChenThreshold13375
```

Thus the full step from Theorems 5, 6, 7 and Theorem 4 to the threshold is checked in Lean.

The file `LeanLab/BS2022GRH/JournalMargin.lean` defines `K3J`:

```
noncomputable def K3J (α g ε C1 C2 cb : ℝ) : ℝ :=
  8 * (2 * g * log (3 - 8 * α) / (4 - 8 * α)
    - ε * C2 * 3 * exp (8 * α - 2) / (4 - 8 * α))
  - 1 / 2 * (8 * ((1 + 5e-5)
    * (g / 4 * (log 6 + log ((3 - 8 * α) / (3 - 18 * α)))
    / (1 / 2 - α) + log (8 / 3) * ε * C1)))
  - 1 / 2 * ((2 * g / (1 / 2 - α) + 3 * ε * C1) * cb)
```

The file `LeanLab/BS2022GRH/JournalPositive.lean` contains the three main-term and parameter results:

```
theorem theorem3_positive_main :
  (0.0001 : ℝ)
  < K3J 0.00256 (exp eulerMascheroniConstant) (1 / 8349) 112 114
  LeanLab.ChenExplicit.BJS.cbar

theorem conditions_13375 :
  (1500 : ℝ) ≤ 0.00256 * exp 13.375 - 11 * 13.375
  ∧ 11 / 0.00256 ≤ exp (13.375 : ℝ)

theorem params_ok_13375 :
  (0.00256 : ℝ) < 1 / 24 ∧ (400 : ℝ) ≤ 1500 ∧ (1500 : ℝ) ≤ 10 ^ 8
  ∧ (0 : ℝ)
  < K3J 0.00256 (exp eulerMascheroniConstant) (1 / 8349) 112 114
  LeanLab.ChenExplicit.BJS.cbar
```

In Lean, e^γ is bracketed by $1.7810699 < e^\gamma < 1.7810725$, `LeanLab.Helfgott2013.GammaFine.exp_gamma_gt` and `exp_gamma_lt`, and $\bar{c} < 0.363084$ is `LeanLab.ChenExplicit.BJS.cbar_lt`.

The file `LeanLab/BS2022GRH/RangeCheck13375.lean` proves eight range lemmas: `lemma7_range`, `display25_first`, `display25_third`, `remark1_H`, `remark1_H_vacuity`, `mono_step`, `mono_range_13375`, and `sieve_ranges_13375`. Two of these statements are:

```
theorem display25_first :
  max ((10 + 1) / (0.00256 : ℝ)) (48 * (10 + 4 / 3)) ≤ exp 13.375

theorem sieve_ranges_13375 :
  (3 : ℝ) ≤ 4 - 8 * 0.00256 ∧ 4 - 8 * (0.00256 : ℝ) ≤ 4
  ∧ (1 : ℝ) < 8 * (1 / 6 - 0.00256)
```

The proved ingredients assembled by `chenThreshold13375_of_deep_closed` are as follows:

- `lemma9J_holds` : `Lemma9J`, the journal's Lemma 9 (Chen's weighted inequality, p. 1227), proved for every $N \geq 2^{24}$ from the lab's `U7153.lemma10`.
- `section55_step`, proving $\pi_2(N) - \sqrt{N} - 1 \leq \pi'_2(N)$ (Section 5.5, p. 1241), and `rep_of_pi2'`, proving that $0 < \pi'_2(N)$ gives the representation.
- `Journal.theorem3_positive_main`, proving $10^{-4} < K3J$ at the new parameters.

- `Cert1567.bracket13375_holds` : `Bracket13375`, which proves that for every $N \geq \exp(\exp 13.375)$,

$$K3J - 10^{-5} \leq 8 \cdot B_5 - \frac{1}{2} \cdot 8 \cdot B_6 - \frac{1}{2} \cdot B_7,$$

where B_5, B_6, B_7 are the journal's exact brackets of Theorems 5, 6, 7 with every correction factor and error term, at $\alpha = 0.00256$, $A = 10$, $\varepsilon = 1/8349$, $C_1 = 112$, $C_2 = 114$, $\varepsilon_1 = 10^{-20}$, and $X_2 = \exp(\exp 13.375)$. The loss is about $6.2 \cdot 10^{-6}$, almost all from the li factor of Theorem 7. This is why the constant $9 \cdot 10^{-5}$ appears: $10^{-4} - 10^{-5}$. It replaces the Python total for the logic; the Python total $1.535054 \cdot 10^{-4}$ stays as a numerical cross-check.

- `Cert1567.mbar_le`, proving $\bar{m}(X_2, 10, X_2^{1/8}) \leq 40$ at $\log \log X_2 = 13.375$, a weaker bound than the paper's hand bound 6.2325 but enough inside `bracket13375_holds`; and `Cert1567.li_factor`, proving that at $\log Y = L/3$ with $L \geq 643700$, $\log Y \cdot (\text{li } Y + Y^{1/2} \log Y / (8\pi)) / Y \leq 1 + 4.6702 \cdot 10^{-6}$.
- `error_13375`, proving that for $\log N \geq e^{13.375}$, $2N^{7/8} + N^{1/3} + \sqrt{N} + 1 < 3.6 \cdot 10^{-5} \cdot N / \log^2 N$; and `two_fifths_le_UNfull`, proving $U_N \geq 2/5$ in the Lean normalisation below.
- `X2_ranges`, `cond19_13375`, and `cond25_13375`, proving the conditions (19), (21), and (25) at the new parameters.
- `Cert1575.eps1500`, proving $\varepsilon(1500) < 1/8349$, display (6) (see Appendix E).

The Lean leaves use $U_N := 2e^{-\gamma} \prod_{p>2} (1 - (p-1)^{-2}) \prod_{p>2, p|N} (p-1)/(p-2)$, the normalisation in which the sieve product $V(z) = \prod_{p<z, p \nmid N} (1 - 1/(p-1))$ is asymptotic to $U_N / \log z$ by Mertens; the journal prints the definition without the factor 2 on pp. 1214 and 1227. The threshold statement does not involve U_N . Thus the Lean U_N is twice the printed one; the bound $U_N \geq 0.37$ of Appendix D refers to the printed normalisation, and `two_fifths_le_UNfull` to the Lean one.

The build logs are recorded in `LeanLab/BS2022GRH/build/`:

- `20261020T0802Z-20406-JournalPositive.log`: Build completed successfully, EXIT 0, sorry count 0; `theorem3_positive_main`, `conditions_13375`, and `params_ok_13375` each depend on axioms `[propext, Classical.choice, Quot.sound]`.
- `20261020T1303Z-22201.log`, a fresh rebuild by a second run: `JournalPositive` and `RangeCheck13375` both EXIT 0, 0 sorry, and all 11 theorems depend only on `[propext, Classical.choice, Quot.sound]`.
- `20261026T2003Z-2973-U-1561.log` and `20261027T0004Z-17012-U-1567.log`: each ends with Build completed successfully and EXIT 0.

- `20261027T0807Z-9485-U-1575.log`: it also records earlier failed attempts; its final entry is a rebuild of `Cert1575Closed` after a docstring fix, Build completed successfully (9101 jobs), EXIT 0, with `Cert1575.eps1500` and `chenThreshold13375_of_deep_closed` depending on axioms `[propext, Classical.choice, Quot.sound]`.

Table 6 maps the principal Lean theorems to the corresponding mathematical inequalities and sections of this paper, and Table 7 summarizes the logical chain leading to Theorem 1.

TABLE 6. Lean theorems and paper inequalities.

Lean name	Statement in mathematical notation	Where used in the paper
<code>theorem3_positive_main</code>	$K3J > 10^{-4}$	(1) and Section 2.2
<code>bracket13375_holds</code>	$8B_5 - 4B_6 - \frac{1}{2}B_7 \geq K3J - 10^{-5}$ for $N \geq X_2$	Appendices A and D, with Lemma 9 and Theorems 5, 6, 7 gives $\pi_2(N) > 9 \cdot 10^{-5} U_N N / \log^2 N - 2N^{7/8} - N^{1/3}$
<code>lemma9J_holds</code>	Lemma 9 of [1]	(A.1)
<code>error_13375</code>	Lemma 9 and Section 5.5 error terms $< 3.6 \cdot 10^{-5} N / \log^2 N$	Appendix D
<code>section55_step_rep_of_pi2'</code>	and $\pi'_2 \geq \pi_2 - \sqrt{N} - 1$ and $\pi'_2 > 0$ gives $N = p + P_2$	Lemma 2.1
<code>eps1500</code>	$\varepsilon(1500) < 1/8349$	Appendix E
<code>mbar_le</code>	$\bar{m} \leq 40$	Appendix D
<code>conditions_13375, cond19_13375, cond25_13375, display25_first, display25_third</code>	parameter ranges	Table 3
<code>chenThreshold13375_of_deep_closed</code>	the main statement from Theorems 5, 6, 7 and Theorem 4	Theorem 1

3.1. Limitations. In the Lean formalization, GRH is an explicit hypothesis. Theorems 5, 6, 7 and Theorem 4 of [1] are transcribed by hand from the journal and stated as Lean hypotheses (`Thm5J`, `Thm6J`, `Thm7J`, and `LinearSieveT4`), so a transcription error there would not be caught by Lean. This risk is real: in the version cited by v1.3 of this note, commit `23bb733`, the three statements omitted the premises $0 < \varepsilon < 1/63$ and N even, and with $\varepsilon = 1$, or with N odd, `Thm5J` could be refuted under GRH, which made the conditional theorem vacuous. Independent audit passes found both gaps on 28 October 2026; the premises were added at commit `2fdf0dc`, the headline still builds with only the standard axioms, and a third audit found no further refutation. They are stated with the

TABLE 7. Logical chain and verification status.

Step	Statement	Checked by
(i)	GRH	assumed, explicit hypothesis in Lean
(ii)	Theorems 5, 6, 7, Theorem 4 with Table 1 of [1]	cited, stated in Lean as hypotheses, not proved
(ii')	Lemma 9 of [1]	Lean, <code>lemma9J_holds</code>
(iii)	Their hypotheses hold at the new parameters, Table 3	Lean, <code>RangeCheck13375.lean</code> , <code>conditions_13375</code> , <code>cond19_13375</code> , <code>cond25_13375</code> , <code>display25_first</code> , <code>display25_third</code> ; the remaining rows by hand
(iv)	$\varepsilon(1500) < 1/8349$	Lean, <code>Cert1575.eps1500</code> , and Python cross-check (<code>scripts/i0088_numerics.py</code> ; Appendix E)
(v)	Row 6100 gives $C_1 = 112$, $C_2 = 114$	read from Table 1 of [1]
(vi)	$K3J > 10^{-4}$	Lean, <code>theorem3_positive_main</code>
(vii)	Bracket combination $\geq K3J - 10^{-5} > 9 \cdot 10^{-5}$	Lean, <code>bracket13375_holds</code> ; Python total $1.535054 \cdot 10^{-4}$ as cross-check (Appendix D)
(viii)	Error terms and the remainder of (33) are negligible	Lean, <code>error_13375</code> and <code>bracket13375_holds</code> ; hand bound in Appendix D as cross-check
(ix)	$\pi_2(N) > \sqrt{N} + 1$ implies $N = p + P_2$	Lean, <code>section55_step</code> , <code>rep_of_pi2'</code> (Lemma 2.1)
(x)	Assembly from (i) to (ix) to <code>ChenThreshold13375</code>	Lean, <code>chenThreshold13375_of_deep_closed</code>

Lean normalisation of U_N of Section 3, twice the printed one, while the printed factor $(e^{-\gamma} \prod_{p>2} (1 - (p-1)^{-2}))^{-1}$ in their negligible error terms is kept as printed; so they are transcriptions in that normalisation, not verbatim copies of the printed statements. Deriving Theorems 5, 6, 7 from GRH through the journal's Lemma 3, Lemma 7, and the linear sieve (Section 4 and Sections 5.1 to 5.3 of [1], the deep part of the argument) is the open step `theorems567_of_inputs`, which is a named `sorry` not used by `chenThreshold13375_of_deep_closed`. Formalizing that derivation together with the linear sieve with explicit constants would be needed for a fully formal proof. The observation that Corollary 1 needs only $K3J > 0$ rather than the printed $4 \cdot 10^{-3}$ applies to the Section 5.4 style assembly of Bordignon, Johnston and Starichkova [3] as well, but the lab has not computed what it gives there.

APPENDIX A. DERIVATION OF K3J FROM [1]

All three sifting bounds in Section 5 of [1] carry the factor $U_N N / \log^2 N$, and K3J is the combined main-term coefficient. Before detailing the individual sources, Table 8 records the line-by-line assembly with explicit signs.

TABLE 8. K3J terms at $\alpha = 0.00256$, $\varepsilon = 1/8349$, $C_1 = 112$, $C_2 = 114$, $g = e^\gamma$.

Term	Expression	Source in [1]	Sign	Value
T_5 main	$8 \cdot \frac{2g \log(3-8\alpha)}{4-8\alpha}$	Theorem 5, p. 1228, inner bracket $f(s)$ at $s = 4 - 8\alpha$	+	+7.818059
T_5 sieve	$8 \cdot \varepsilon C_2 \frac{3e^{8\alpha-2}}{4-8\alpha}$	Theorem 5, p. 1228	−	−0.011375
T_6 main	$\frac{1}{2} \cdot 8(1 + 5 \cdot 10^{-5})^{\frac{(g/4)(\log 6 + \log \frac{3-8\alpha}{3-18\alpha})}{1/2-\alpha}}$	Theorem 6, pp. 1231–1232, display l_1 ; factor $1 + 5 \cdot 10^{-5}$ from Lemma 11, p. 1228	−	−6.446572
T_6 sieve	$\frac{1}{2} \cdot 8(1 + 5 \cdot 10^{-5}) \log(8/3) \varepsilon C_1$	Theorem 6, display l_1	−	−0.052633
T_7 main	$\frac{1}{2} \cdot \frac{2g}{1/2-\alpha} \bar{c}$	Theorem 7, pp. 1234–1235, with (29), p. 1237, and Lemma 14, p. 1237	−	−1.300014
T_7 sieve	$\frac{1}{2} \cdot 3\varepsilon C_1 \bar{c}$	Theorem 7, pp. 1234–1235, with (29), p. 1237, and Lemma 14, p. 1237	−	−0.007306
Sum	K3J	Weights 1, $-1/2$, $-1/2$ from Lemma 9, p. 1227		1.58626 $\cdot 10^{-4}$

Values use $\bar{c} < 0.363084$; with the exact $\bar{c}$ the sum is $1.5960 \cdot 10^{-4}$, as in Section 2.

Each piece arises from [1] as follows:

- By Lemma 9 of [1], p. 1227 (Chen's weighted inequality),

$$\pi_2(N) > S(\mathcal{A}, \mathbb{P}, z) - \frac{1}{2} \sum_{z \leq q < y} S(\mathcal{A}_q, \mathbb{P}, z) - \frac{1}{2} S(\mathcal{B}, \mathbb{P}, y) - 2N^{7/8} - N^{1/3}, \quad (2)$$

with $z = N^{1/8}$ and $y = N^{1/3}$ (Section 5, p. 1226). This gives the weights 1, $-1/2$, and $-1/2$. For a prime $q \mid N$ the set $\mathcal{A}_q$ is empty, since $q \mid N - p$ would force $p = q$, while $\mathcal{A}$ uses only primes $p \nmid N$; so the sum may be restricted to $q \nmid N$, as in Theorem 6.

- By Theorem 5 of [1], p. 1228,

$$S(\mathcal{A}, \mathbb{P}, z) > 8 \frac{U_N N}{\log^2 N} \left(\left(1 - \frac{0.62 \log X_2}{X_2^{1/16}} \right) \times \left(\frac{2e^\gamma \log(3 - 8\alpha_1)}{4 - 8\alpha_1} - \varepsilon C_2(\varepsilon) \frac{3e^{8\alpha_1 - 2}}{4 - 8\alpha_1} \right) - E_5 \right), \quad (3)$$

where E_5 is a term of order $\log^{-8} N$. The factor 8 comes from $V(z)$ and $\log z = \frac{1}{8} \log N$ (Lemma 10), together with $|\mathcal{A}| \geq N/\log N$ (Lemma 11); the inner bracket is $f(s) - \varepsilon C_2 e^2 h(s)$ evaluated at $s = 4 - 8\alpha_1$, where $f(s) = 2e^\gamma \log(s - 1)/s$ and $h(s) = 3s^{-1}e^{-s}$ (proof of Theorem 5, p. 1229).

- By Theorem 6 of [1], pp. 1231 to 1232,

$$\sum_{z \leq q < y, q \nmid N} S(\mathcal{A}_q, \mathbb{P}, z) \leq \frac{8U_N N}{\log^2 N} (l_1 + l_2 / \log^{A-3} N + l_3 / \log^{A-1} N) \quad (4)$$

with

$$\begin{aligned} l_1 &= (1 + 5 \cdot 10^{-5}) \left(1 + \frac{0.62 \log X_2}{X_2^{1/16}} \right) \frac{X_2^{1/8}}{X_2^{1/8} - 1} \\ &\times \left(\frac{e^\gamma}{4} \left(\frac{\log 6 + \log \frac{3-8\alpha_2}{3-18\alpha_2}}{\frac{1}{2} - \alpha_2} + \frac{512}{k_{\alpha_2} \log^2 X_2} \right) \right. \\ &\left. + \left(\log \frac{8}{3} + \frac{64}{\log^2 X_2} \right) \varepsilon C_1(\varepsilon) e^2 h(k_{\alpha_2}) \right). \end{aligned} \quad (5)$$

The factor $1 + 5 \cdot 10^{-5}$ is the upper bound $|\mathcal{A}| \leq (1 + 5 \cdot 10^{-5})N/\log N$ of Lemma 11, p. 1228. Since $k_{\alpha_2} = 8(1/6 - \alpha_2) \in [1, 2]$, we have $h(k_{\alpha_2}) = e^{-2}$ and $e^2 h(k_{\alpha_2}) = 1$.

- By Theorem 7 of [1], pp. 1234 to 1235, and (29), p. 1237, the leading part of the upper bound for $S(\mathcal{B}, \mathbb{P}, y)$ is

$$\frac{U_N N}{\log^2 N} \left(\frac{2e^\gamma}{\frac{1}{2} - \alpha_3} + 3\varepsilon C_1(\varepsilon) \right) \bar{c} \cdot (\text{li factor}) \cdot (1 + \varepsilon_1) \cdots, \quad (6)$$

where $3(F(s) + \varepsilon C_1(\varepsilon) e^2 h(s))$, with $F(s) = 2e^\gamma/s$ and $h(s) = e^{-2}$ at $s = 3/2 - 3\alpha_3 \in [1, 2]$, and $V(y) < 3U_N/\log N$ give the bracket, and $\bar{c}$ enters through Lemma 14.

Setting $\alpha_1 = \alpha_2 = \alpha_3 = \alpha$, dropping the factors listed in Appendix D, and collecting coefficients gives (1). Taking the α_i equal is optimal since the total decreases in each α_i and each α_i has its own copy of the binding condition.

APPENDIX B. THE FACTOR 2 IN U_N

Pages 1214 and 1227 of [1] print $U_N = e^{-\gamma} \prod_{p>2} \left(1 - \frac{1}{(p-1)^2} \right) \prod_{p>2, p|N} \frac{p-1}{p-2}$; that this printed U_N is half the limit of $V(x) \log x$ in Lemma 10 is a lab finding of 27 October 2026, checked on the journal page images. The companion

paper [3], arXiv:2207.09452, defines U_N with $2e^{-\gamma}$. We trace the factor 2 line by line through the five results used in Section 5.4:

- **Lemma 10** (p. 1227). Lemma 10 states that $V(x) = \prod_{p < x, p \nmid N} (1 - \frac{1}{p-1})$ is asymptotic to $U_N / \log x$. Since N is even, $p = 2$ is absent from $V(x)$; using $1 - \frac{1}{p-1} = (1 - \frac{1}{p})(1 - \frac{1}{(p-1)^2})$ and Mertens' theorem, dropping the $p = 2$ factor $1 - 1/2$ doubles the product, so $V(x) \log x$ tends to $\text{UNfull} := 2e^{-\gamma} \prod_{p > 2} (1 - \frac{1}{(p-1)^2}) \prod_{p > 2, p \mid N} \frac{p-1}{p-2}$, twice the printed U_N (numerically, for primes below 10^8 , $V(x) \log x$ divided by the printed U_N is 1.99756, 1.99992, and 1.999992 at $x = 10^4, 10^6, 10^8$ for $N = 2^{20}, 9699690$, and $10^{12} + 2$ alike, and the Lean file `Verify1565.lean` proves the algebraic identities `factor_identity`, `prod_identity`, and `drop_two`).
- **Theorem 5** (p. 1228). Through Theorem 5 the factor 8 comes from $V(z)$ with $\log z = \frac{1}{8} \log N$, and the corrected $V(z)$ carries the factor 2 into the lower bound with UNfull . The printed $(e^{-\gamma} \prod_{p > 2} (1 - (p-1)^{-2}))^{-1}$ in the error terms is twice the needed inverse and only weakens them.
- **Theorem 6** (pp. 1231–1232). Through Theorem 6, which is an upper bound, the printed statement with the smaller U_N is not what its proof gives; the proof gives the upper bound with UNfull , and the printed $(e^{-\gamma} \prod_{p > 2} (1 - (p-1)^{-2}))^{-1}$ in the error terms is twice the needed inverse and only weakens them.
- **Theorem 7** (pp. 1234–1235). Through Theorem 7, which is also an upper bound, the printed statement with the smaller U_N is not what its proof gives; the proof gives the upper bound with UNfull . The printed $(e^{-\gamma} \prod_{p > 2} (1 - (p-1)^{-2}))^{-1}$ in its error terms is likewise twice the needed inverse and only weakens them.
- **Lemma 9** (p. 1227). Through Lemma 9 nothing changes, because Lemma 9 has no U_N .

Accordingly, the Lean leaves `Thm5J`, `Thm6J`, and `Thm7J` use UNfull , equal to 2 times the printed U_N , and we make no claim that the main theorem of [1] is wrong: with the true U_N their Theorem 3 holds as a coefficient statement. Every Section 5.4 bound is a coefficient times $U_N N / \log^2 N$, so `K3J` and the threshold 13.375 do not depend on the normalisation.

APPENDIX C. TRANSCRIPTION OF THE CITED INPUTS

Table 9 records the four cited inputs stated in Lean in `LeanLab/BS2022GRH/ChenThresholdGRH.lean` at public commit `2fd0dc`.

These four transcriptions were checked by AI audit passes only, with Claude reading the journal page images against the Lean text; no human expert has checked them. There were three audits. Two of the three audits found real gaps: the leaves cited by v1.3 of this note, at commit `23bb733`, lacked the premises $0 < \varepsilon < 1/63$ and N even, and Lean proved that the old

TABLE 9. Transcription of the four cited inputs in `ChenThresholdGRH.lean` at commit `2fdf0dc`.

Lean name	Journal result	Premises	Conclusion	Status
Thm5J	Theorem 5, p. 1228	GRH; LinearSieveT4 ε C_1 C_2 ; $0 \leq C_1, 0 \leq C_2$; $\text{epsJ}(u) \leq \varepsilon$ with <code>epsJ display (6)</code> , p. 1221; $0 < \varepsilon < 1/63$; $0 < \alpha < 1/8$; $0 < A$; condition (19); $4 \cdot 10^{18} \leq X_2 \leq N$; N even	$S(\mathcal{A}, \mathbb{P}, z) > 8 \text{UNfull}_{\frac{N}{\log^2 N}} B_5$	MATCH
Thm6J	Theorem 6, pp. 1231–1232	Same as Thm5J , with $0 < \alpha < 1/24$, condition (21) as encoded, $\exp(\exp 10) \leq X_2 \leq N$, N even	$\sum_{z \leq q < y} S(\mathcal{A}_q, \mathbb{P}, z) \leq 8 \text{UNfull}_{\frac{N}{\log^2 N}} B_6$	MATCH
Thm7J	Theorem 7, pp. 1234–1235	Same as Thm5J , with $0 < \varepsilon_1 < 1$, $0 < \alpha < 1/16$, condition (25), $\exp(\exp 10) \leq X_2 \leq N$, N even	$S(\mathcal{B}, \mathbb{P}, y) \leq \text{UNfull}_{\frac{N}{\log^2 N}} B_7$	MATCH
Linear-SieveT4	Theorem 4, p. 1218; Table 1, pp. 1217–1218	Linear sieve inequality (2) for sifting weights with $0 < \varepsilon < 1/63$ and the Mertens-type product condition, used at $\varepsilon = 1/8349$, $C_1 = 112$, $C_2 = 114$	Sifting bounds of Theorem 4 with constants C_1, C_2	MATCH

Thm5J, **Thm6J**, and **Thm7J** each imply $\neg\text{GRH}$, so the conditional headline was vacuous; this was fixed at commit `2fdf0dc`, where Lean proves that each leaf hypothesis other than **GRH** and the sieve can be met at the paper’s parameters. The third audit, on 28 October 2026, rated all four transcriptions **MATCH** and left three open notes: C_1 and C_2 range over every witness of the sieve rather than the Table 1 values; Lemma 7 needs $\log z > 2753$ while Theorems 5 to 7 as printed allow smaller z , which holds at our X_2 since $\log z = 80463$; and the U_N factor 2 of Appendix B.

APPENDIX D. OMITTED TERMS AND ERROR TERMS

Table 10 summarizes the terms omitted from **K3J** and the error terms at $\log X_2 = 643707.69$.

The bounds in the last two rows of Table 10 are obtained as follows:

TABLE 10. Omitted factors and error terms at $\log X_2 = 643707.69$, measured against $U_N N / \log^2 N$.

Term	Source in [1]	Size against $U_N N / \log^2 N$
li factor $\frac{\log y (\text{li}(y) + \sqrt{y} \log y / (8\pi))}{y}$	Theorem 7, Lemma 14	relative $1/\log y = 4.66 \cdot 10^{-6}$, absolute about $6 \cdot 10^{-6}$; largest omitted term, included in the total $1.535054 \cdot 10^{-4}$
$512/(k \log^2 X_2)$ and $64/\log^2 X_2$ in l_1 ; $36/\log^2 X_2$ and $27/\log^2 X_2$ in Theorem 7	Theorems 6 and 7	at most 10^{-9} ; included in the total $1.535054 \cdot 10^{-4}$
$(1 \pm 0.62 \log X_2 / X_2^{1/16})$, $X_2^{1/8} / (X_2^{1/8} - 1)$, $(1 + 0.06 \log X_2 / X_2^{1/6})$, $1 + \varepsilon_1$	Theorems 5 to 7	relative e^{-40000} or 10^{-20}
E_5 , $l_2/\log^7 N$, and $l_3/\log^9 N$	Theorems 5 and 6	below 10^{-40}
Remainder of (33)	Section 5.4, p. 1241	$< 1.7 \cdot 10^{-15}$
$2N^{7/8} + N^{1/3} + \sqrt{N} + 1$	Lemma 9, Corollary 1	ratio to $9 \cdot 10^{-5} U_N N / \log^2 N$ is $\leq$ $6.01 \cdot 10^4 \log^2 N \cdot N^{-1/8} < e^{-80425}$

- From the definition of U_N in Theorem 3 and Lemma 10 of [1],

$$U_N \geq e^{-\gamma} \prod_{p>2} (1 - (p-1)^{-2}) = 0.5614594836 \cdot 0.66016181 \geq 0.370654 > 0.37,$$

since the product over $p \mid N$ is at least 1.

- The journal defines $\bar{m}$ in (31), pp. 1239 to 1240, and bounds $M_\varphi(D^*) \leq 1.08$ on p. 1240; the bound

$$\bar{m}(N, 10, N^{1/8}) \leq 2 \cdot 1.08 (2/\log 2 + 4e^{-13000}) < 6.2325 \leq 8$$

is ours: every other term of (31) carries $\log^{11} N$ or $\log^{12} N$ times a negative power of $w = N^{1/8}$, below e^{-13000} . Consequently, the remainder of (33), p. 1241, where the constant 0.053 is printed in (33), satisfies

$$\begin{aligned} & \frac{1 + \varepsilon_1}{U_N \log(1 + \varepsilon_1) \log^6 N} [0.053 \bar{m}(1 + \varepsilon_1)^{-1/2} + 0.08 N^{-1/8} \log^{11} N] \\ & \leq 1.0001 \cdot \frac{0.053 \cdot 8 \cdot 10^{20}}{0.37 \cdot 7.1 \cdot 10^{34}} < 1.7 \cdot 10^{-15}. \end{aligned}$$

In Lean, the lemma `mbar_1e` proves $\bar{m} \leq 40$ at X_2 without the hand estimate, and `bracket13375_holds` absorbs the remainder of (33) with that weaker bound.

- From Lemma 9 and Corollary 1, comparing $2N^{7/8} + N^{1/3} + \sqrt{N} + 1$ against $9 \cdot 10^{-5} U_N N / \log^2 N$ with $U_N \geq 0.37$ gives a ratio bounded

by

$$\frac{2 + N^{-13/24} + N^{-3/8} + N^{-7/8}}{3.33 \cdot 10^{-5}} \log^2 N \cdot N^{-1/8} \leq 6.01 \cdot 10^4 \log^2 N \cdot N^{-1/8},$$

whose logarithm at $\log N = 643707.69$ is $11.0 + 26.75 - 80463.46 < -80425$, and $2 \log t - t/8$ decreases for $t > 16$.

APPENDIX E. THE VALUE OF $\varepsilon(1500)$

By display (6) of [1], p. 1221,

$$\varepsilon(u_0) = \exp\left(1.131 \cdot 10^{-7} + \left(\frac{u_0}{u_0 - 2} + 0.501\right) \sum_{p \geq u_0} p^{-2}\right) - 1. \quad (7)$$

We sum p^{-2} over primes $1500 \leq p < 10^9$ with a segmented sieve (in `scripts/i0088_numerics.py`), giving $7.96425 \cdot 10^{-5}$, and bound the tail using $\pi(t) < 1.25506 t / \log t$ [7] and partial summation:

$$\begin{aligned} \sum_{p \geq X} p^{-2} &= \int_X^\infty t^{-2} d\pi(t) \leq \int_X^\infty 2\pi(t) t^{-3} dt \\ &\leq 2 \cdot 1.25506 \int_X^\infty \frac{dt}{t^2 \log t} \leq \frac{2 \cdot 1.25506}{X \log X}, \end{aligned}$$

the boundary term $-\pi(X)/X^2$ being negative; at $X = 10^9$ this is $1.2113 \cdot 10^{-10}$. Hence

$$\varepsilon(1500) = 1.1977018 \cdot 10^{-4} = 1/8349.3238 < 1/8349,$$

with a margin of $4.6 \cdot 10^{-9}$. The same method gives $\varepsilon(1000) = 1/5235.667 < 1/5235$, matching [1].

In Lean, `Cert1575.eps1500` proves $\varepsilon(1500) < 1/8349$, display (6): the head sum over primes $1501 \leq p < 9003001$ of $1/p^2$ is at most $7963599791126110 \cdot 10^{-20}$, checked by the kernel with `decide +kernel` in 11 windows `Cert1575W00` to `Cert1575W10`, no `native_decide`, by adding $\lfloor 10^{20}/n^2 \rfloor + 1$ over $n \equiv 1, 5 \pmod{6}$ coprime to $m!$, with $m = 1499$ or 3000 below every prime of the window, an upper bound that needs no primality certificate; the tail over $p \geq H = 9003001$ is at most $(C \cdot H - \theta_0)/(16H^2) + C/(16H)$, $C = 89/80$, $\theta_0 = 8191444$, by summation by parts against Chebyshev's θ with explicit Chebyshev bounds proved in the lab, leaving a margin of about $8.8 \cdot 10^{-10}$ in $1 + 1/8349 - e^x$. This kernel proof replaces the Python sum for the logic, and the two agree.

The conclusion is not sensitive to this margin: at $\alpha = 0.00256$ and $\log \log X_2 = 13.375$ the total stays positive for every $\varepsilon < 1/8331.07$, and the 6100 row of Table 1 needs only $\varepsilon < 1/6100$. Table 11 shows the sensitivity of the total and optimal threshold to $1/\varepsilon$ at $u = 1500$ and row 6100: the threshold 13.375 needs $1/\varepsilon(1500)$ above 8331.07; the computed value is 8349.3238.

TABLE 11. Sensitivity to ε at $u = 1500$, Table 1 row 6100.

$1/\varepsilon$	Total at $\alpha = 0.00256, 13.375$	Best $\log \log X_2$
8349	$1.5351 \cdot 10^{-4}$	13.37261
8340	$7.6548 \cdot 10^{-5}$	13.37357
8331.07	$2.4 \cdot 10^{-8}$	13.37453
8300	$-2.6751 \cdot 10^{-4}$	13.37789
8000	$-2.9576 \cdot 10^{-3}$	13.41228
7000	$-1.3590 \cdot 10^{-2}$	13.56120
6100	$-2.6139 \cdot 10^{-2}$	13.77160

USE OF AI

AI, namely Claude by Anthropic, wrote all Lean code and drafted the text; the Lean of Section 3 was written by Claude in the author’s lab and checked only by the Lean kernel; AI tools were used for the literature search; the statements, the numbers and the references were checked by independent AI checker passes against the Lean source and the published pages of [1]; the author is responsible for the content.

Checker protocol. As done for every version, each draft is checked by a fresh AI session that did not write it, which compares every theorem name and statement against the Lean source, every number against the build logs and the public Python script `i0088_numerics.py`, and every citation against the journal pages, and reports findings that go into the next round; the Lean itself is checked only by the Lean kernel; the author reads every version and is responsible. No human expert has verified the transcriptions of Theorems 4 to 7.

DATA AVAILABILITY

The Lean source of every theorem named in this paper, with its full import closure, the build logs, and the Python script `i0088_numerics.py` with its output, are public at <https://github.com/sthubbar/lean-lab-preprints>, commit 2fdf0dc; the README states exactly what the Lean proves and what it does not. The conditional threshold theorem, with its four cited inputs as explicit hypotheses, is registered in the Palomar registry of Lean-verified mathematics as PALOMAR-2026-10-28-000016, <https://palomar-registry.org/entry.html?id=PALOMAR-2026-10-28-000016&version=1>.

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REFERENCES

- [1] M. Bordignon and V. Starichkova, *An explicit version of Chen's theorem assuming the Generalized Riemann Hypothesis*, Ramanujan J. **64** (2024), no. 4, 1213–1242, <https://doi.org/10.1007/s11139-024-00866-x>.
- [2] M. Bordignon and V. Starichkova, *An explicit version of Chen's theorem assuming the Generalized Riemann Hypothesis*, arXiv:2211.08844, v1, 2022.
- [3] M. Bordignon, D. R. Johnston and V. Starichkova, *An explicit version of Chen's theorem and the linear sieve*, Int. J. Number Theory **21** (2025), 2497–2572, <https://doi.org/10.1142/S1793042125501192>; arXiv:2207.09452.
- [4] J. R. Chen, *On the representation of a larger even integer as the sum of a prime and the product of at most two primes*, Sci. Sinica **16** (1973), 157–176.
- [5] D. R. Johnston and V. V. Starichkova, *Some explicit results on the sum of a prime and an almost prime*, Acta Math. Hungar. **177** (2025), 163–192, <https://doi.org/10.1007/s10474-025-01572-w>; arXiv:2208.01229.
- [6] subfish-zhou, *goldbach-lean*, <https://github.com/subfish-zhou/goldbach-lean>.
- [7] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94.
- [8] T. Yamada, *Explicit Chen's theorem*, arXiv:1511.03409, 2015.

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