

FINITE TRUNCATED LATTICE-LENGTH DATA ON THREE-DIMENSIONAL FLAT TORI VOLUME NON-IDENTIFIABILITY, EXPLICIT CONSTRUCTIONS, AND IDENTIFICATION-RADIUS BOUNDS

PULIN SHI

ABSTRACT. We study what finite truncations of the lattice-length multiset can determine about a three-dimensional flat torus. For a full-rank lattice $\Lambda \subset \mathbb{R}^3$, let $\lambda_1(\Lambda) \leq \lambda_2(\Lambda) \leq \lambda_3(\Lambda)$ be its successive minima, and let $\mathcal{L}_T(\Lambda)$ record the lengths of all nonzero lattice vectors of norm at most T , counted with vector multiplicity. First, on the class defined by $\lambda_1 = a$ and $\lambda_2 > T$, we prove an exact volume classification: the set of possible covolumes is $(F(a, T), \infty)$, where $F(a, T) = \frac{aT}{2} \sqrt{3T^2 - a^2}$. The lower bound uses Oppenheim's classical determinant-minimization theorem for Minkowski-reduced ternary forms; an explicit perturbed basis proves every larger volume is attained. Second, we give a pair of positive-definite Gram matrices with equal diagonal entries and matching two-coordinate data but different determinants. This yields a quantitative obstruction to uniform finite identification and gives the lower bound $\sqrt{3}$ for the worst-case identification-radius constant. Schiemann's finite theta-series determination theorem supplies the complementary upper bound $\sqrt{7/2}$ after conversion from squared lengths to radii. We also record a cubic sign-product identity for the eight signed basis vectors and explain its limitations. All contributions are separated from classical inputs, and the possible optimality of the constant $\sqrt{3}$ and the broader topological question are left open.

Keywords: lattice geometry; flat tori; truncated length data; successive minima; ternary quadratic forms; theta series; finite identifiability.

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1. INTRODUCTION

A flat three-dimensional torus is a quotient

$$M_\Lambda = \mathbb{R}^3 / \Lambda,$$

where Λ is a full-rank lattice. Its metric geometry is encoded by the Euclidean lengths of lattice vectors. If only vectors up to a finite length threshold are observed, an inverse question arises: how much of the underlying lattice can the data determine?

The present paper studies the finite, multiplicity-sensitive data

$$\mathcal{L}_T(\Lambda) = \{ \|v\| : v \in \Lambda \setminus \{0\}, \|v\| \leq T \},$$

where repetitions are retained. This is a truncated lattice-length multiset, equivalently a truncation of the length data of closed geodesics on the flat torus. It is not, without passing to the dual lattice, the Laplace eigenvalue spectrum; we keep these notions distinct throughout.

Our results have two complementary forms. In a regime where every vector up to the observation threshold lies on the shortest lattice line, we determine exactly the range of possible volumes compatible with the data. Separately, a Gram-matrix sign construction gives indistinguishable lattices whose first three successive minima agree, showing that a threshold proportional only to the shortest scale cannot identify all three-dimensional flat tori.

1.1. Summary of results. The principal statements are as follows.

- (i) If $\lambda_1(\Lambda) = a$ and $\lambda_2(\Lambda) > T \geq a$, then the truncated data are fixed by a, T :

$$\mathcal{L}_T(\Lambda) = \{a, a, 2a, 2a, \dots, Na, Na\}, \quad N = \lfloor T/a \rfloor.$$

- (ii) On this class, the exact set of possible covolumes is

$$\mathcal{V}(a, T) = (F(a, T), \infty), \quad F(a, T) = \frac{aT}{2} \sqrt{3T^2 - a^2}.$$

In particular, $F(a, T)$ is an infimum and is not attained.

- (iii) A sign-flip pair of Gram matrices has $\lambda_1 = \lambda_2 = \lambda_3 = 1$, identical truncated length data whenever $T < \sqrt{3 - 6\delta}$, and different covolumes. Consequently, the worst-case identification-radius constant is at least $\sqrt{3}$.
- (iv) Schiemann's finite determination result for ternary positive-definite quadratic forms gives the upper bound $\sqrt{7/2}$ for the same normalized worst-case constant, with the usual reduction and normalization hypotheses made explicit below.
- (v) A third-order signed-basis identity isolates the product of the three off-diagonal Gram entries. It explains the sign mechanism in the construction, but it is not by itself an invariant of the unlabelled truncated multiset.

1.2. Relation to classical work. The volume lower bound uses Oppenheim's minimum-determinant result for reduced ternary forms; Barnes and Trenerry explicitly record this result in their discussion of determinant minima [10, 1, 2]. Oppenheim's related work on one-sided inequalities for ternary forms gives further classical context [11]. The finite identification upper bound uses Schiemann's finite determination theorem for ternary positive-definite real quadratic forms [12]. Background on lattice reduction, successive minima, and the geometry of numbers is available in Cassels, Mahler, van der Waerden, and Gruber–Lekkerkerker [3, 7, 13, 6]; Conway and Sloane and Martinet provide complementary treatments of lattice geometry [5, 8]. The flat-torus and spectral remarks use standard references including Milnor, Charlap, and Wolf [9, 4, 14].

1.3. Scope and non-claims. The results establish geometric non-identifiability from finite truncated lattice-length data. They do *not* establish topological non-identifiability within the class of three-dimensional flat tori: every quotient $\mathbb{R}^3/\Lambda$ is homeomorphic (indeed diffeomorphic) to $\mathbb{T}^3$. No novelty claim is made for the classical determinant formula or Schiemann's theorem. We make no priority claim for the sign-flip construction or the signed-basis identity; their relationship to prior work remains to be established by a systematic literature search.

Observation model. The observation studied here is the *primal-lattice vector-length multiset*. It is not the Laplace spectrum of the flat torus, which is described using the dual lattice. The results establish metric/geometric non-identifiability only; topology is unchanged throughout this model class.

2. PRELIMINARIES AND OBSERVATION MODEL

2.1. Lattices and successive minima. A full-rank lattice in $\mathbb{R}^3$ is a subgroup $\Lambda = B\mathbb{Z}^3$ for some invertible real 3×3 matrix B . Its covolume is

$$\text{covol}(\Lambda) = |\det B|.$$

The associated flat torus has Riemannian volume $\text{Vol}(M_\Lambda) = \text{covol}(\Lambda)$.

For $j = 1, 2, 3$, the j th successive minimum is

$$\lambda_j(\Lambda) = \inf\{r > 0 : \dim \text{span}_{\mathbb{R}}(\Lambda \cap \overline{B}(0, r)) \geq j\}.$$

In particular, λ_1 is the shortest nonzero vector length. The quantity λ_2 is the least radius containing two linearly independent vectors, and λ_3 is the least radius containing three linearly independent vectors. We use Euclidean lengths, not squared lengths, for the λ_j .

2.2. Finite truncated data.

Definition 2.1. For $T \geq 0$, the truncated lattice-length multiset is

$$\mathcal{L}_T(\Lambda) = \{\|v\| : v \in \Lambda \setminus \{0\}, \|v\| \leq T\},$$

with each vector counted separately. Thus v and $-v$ contribute two copies of the same positive length.

Two lattices have the same T -truncated data if these multisets agree, including every multiplicity. We write $\Lambda \cong \Gamma$ when the corresponding flat tori are isometric; for Euclidean lattices this is equivalent to the existence of an orthogonal transformation carrying one lattice to the other. In particular, unequal covolumes imply non-isometry.

2.3. Identification radius.

Definition 2.2. For a full-rank lattice Λ , define its identification radius by

$$\rho_{\text{id}}(\Lambda) = \inf\left\{T \geq 0 : \forall \Gamma \subset \mathbb{R}^3, \mathcal{L}_T(\Gamma) = \mathcal{L}_T(\Lambda) \implies \Gamma \cong \Lambda\right\}.$$

The infimum is taken over thresholds for which the displayed uniqueness property holds. The property is monotone in T : once it holds at one radius, it holds at every larger radius.

This definition uses the full truncated multiset of lengths, not merely the set of distinct length values. A finite determination theorem for theta series can therefore be applied after converting its cutoff in squared-length variables to a radius cutoff.

3. THE ONE-DIRECTION OBSERVATION REGIME

Theorem 3.1 (Structure of the truncated multiset). *Let $a > 0$ and $T \geq a$. If Λ satisfies*

$$\lambda_1(\Lambda) = a, \quad \lambda_2(\Lambda) > T,$$

then, with $N = \lfloor T/a \rfloor$,

$$\mathcal{L}_T(\Lambda) = \{a, a, 2a, 2a, \dots, Na, Na\}.$$

Proof. Choose a shortest vector $u \in \Lambda$ of length a . Any lattice vector v with $\|v\| \leq T$ must be linearly dependent over $\mathbb{R}$ with u , since otherwise the lattice would contain two linearly independent vectors of length at most T , contrary to $\lambda_2(\Lambda) > T$. As u is primitive (a nontrivial integer multiple of a shorter lattice vector would contradict minimality), the lattice points on its line are exactly $\mathbb{Z}u$. Therefore the observed nonzero vectors are precisely ku with $1 \leq |k| \leq \lfloor T/a \rfloor$. Each positive integer k gives two vectors $\pm ku$, both of length ka . $\square$

Remark 3.2. The hypothesis $\lambda_2 > T$ is strict. At $T = \lambda_2$, a second independent direction may enter the data, and the stated list need not be complete.

4. EXACT CLASSIFICATION OF COMPATIBLE COVOLUMES

For fixed $a > 0$ and $T \geq a$, define

$$\mathcal{C}(a, T) = \{\Lambda : \lambda_1(\Lambda) = a, \lambda_2(\Lambda) > T\}$$

and

$$\mathcal{V}(a, T) = \{\text{covol}(\Lambda) : \Lambda \in \mathcal{C}(a, T)\}.$$

Put

$$F(a, T) = \frac{aT}{2} \sqrt{3T^2 - a^2}.$$

4.1. The classical determinant input. For a Minkowski-reduced ternary positive-definite quadratic form with ordered diagonal coefficients $A \leq B \leq C$, Oppenheim's determinant-minimization result gives

$$\det G \geq ABC - \frac{AB^2}{4} - \frac{A^2C}{4}. \quad (1)$$

The formula for the minimum, including its dependence on the ordered diagonal entries, is recorded in the literature on Minkowski reduction; see Oppenheim and the account by Barnes–Trenerry [10, 2]. In applying it, the diagonal entries are the squares of the successive-minimum lengths of a suitable reduced basis in dimension three. This is a classical input, not a new result of this paper.

Lemma 4.1 (Covolume lower bound in terms of the first two minima). *For every full-rank three-dimensional lattice Λ , with $a = \lambda_1(\Lambda)$ and $b = \lambda_2(\Lambda)$,*

$$\text{covol}(\Lambda) \geq \frac{ab}{2} \sqrt{3b^2 - a^2}. \quad (2)$$

Proof. Choose a reduced basis with Gram diagonal $A = a^2$, $B = b^2$, $C = \lambda_3(\Lambda)^2$, so $A \leq B \leq C$. From (1),

$$\det G \geq ABC - \frac{AB^2}{4} - \frac{A^2C}{4} = AC \left(B - \frac{A}{4} \right) - \frac{AB^2}{4}.$$

Since $B - A/4 > 0$ and $C \geq B$,

$$\det G \geq AB \left(B - \frac{A}{4} \right) - \frac{AB^2}{4} = \frac{AB}{4} (3B - A).$$

The square of the lattice covolume is $\det G$. Taking square roots and substituting $A = a^2$, $B = b^2$ gives (2). $\square$

4.2. The strict lower bound.

Proposition 4.2. *For every $\Lambda \in \mathcal{C}(a, T)$,*

$$\text{covol}(\Lambda) > F(a, T).$$

Proof. Let $b = \lambda_2(\Lambda) > T$. By theorem 4.1,

$$\text{covol}(\Lambda) \geq H_a(b), \quad H_a(x) = \frac{ax}{2} \sqrt{3x^2 - a^2}.$$

For $x \geq a > 0$, the function H_a is strictly increasing. Thus

$$\text{covol}(\Lambda) \geq H_a(b) > H_a(T) = F(a, T).$$

$\square$

4.3. An explicit construction attaining every larger volume.

Theorem 4.3 (Exact volume classification). *For every $a > 0$ and $T \geq a$,*

$$\mathcal{V}(a, T) = (F(a, T), \infty).$$

Consequently, $F(a, T)$ is the infimum of compatible covolumes, but no lattice in $\mathcal{C}(a, T)$ attains it.

Proof. The strict lower inclusion follows from theorem 4.2. For the reverse inclusion, fix any $V > F(a, T)$. Choose $\varepsilon > 0$ sufficiently small and define

$$s^2 = T^2 - \frac{a^2}{4} + \varepsilon, \quad (3)$$

$$c = 1 - \frac{T^2 + \varepsilon}{2s^2}, \quad (4)$$

$$h_0 = s\sqrt{1 - c^2}. \quad (5)$$

Since $T \geq a > 0$, we have $s^2 \geq 3T^2/4 + \varepsilon$ and $0 < c < 1/2$, so h_0 is real and positive. Consider

$$u = (a, 0, 0), \quad v = (a/2, s, 0), \quad w = (a/2, sc, h),$$

where $h \geq h_0$, and let $\Lambda_h = \mathbb{Z}u + \mathbb{Z}v + \mathbb{Z}w$. For integers m, n, k ,

$$\|mu + nv + kw\|^2 = a^2 \left(m + \frac{n+k}{2} \right)^2 + s^2(n + kc)^2 + k^2h^2. \quad (6)$$

We first show that $\lambda_1(\Lambda_h) = a$ and $\lambda_2(\Lambda_h) > T$ whenever $h \geq h_0$. If $k = 0, n = 0$, the vector is a multiple of u and every nonzero such vector has length at least a . If $k = 0$ and $|n| = 1$, the first term in (6) is at least $a^2/4$ and the second equals s^2 , so

$$\|mu + nv\|^2 \geq a^2/4 + s^2 = T^2 + \varepsilon.$$

If $k = 0$ and $|n| \geq 2$, then $|n|s \geq 2s > T$.

If $|k| \geq 2$, then $\|mu + nv + kw\| \geq |k|h \geq 2h_0 > T$, because $c < 1/2$ implies $h_0^2 > s^2(3/4) > 9T^2/16$.

It remains to consider $|k| = 1$. By replacing a vector by its negative, take $k = 1$. Since $h^2 \geq s^2(1 - c^2)$,

$$\|mu + nv + w\|^2 \geq a^2 \left(m + \frac{n+1}{2} \right)^2 + s^2(n + c)^2 + s^2(1 - c^2).$$

For $n = 0$, the first term is at least $a^2/4$ and the last two terms sum to at least s^2 , giving $T^2 + \varepsilon$. For $n = -1$, the first term is nonnegative and the last two terms are at least $2s^2(1 - c) = T^2 + \varepsilon$. For $n \geq 1$, the last two terms are at least $2s^2(1 + c) > T^2 + \varepsilon$. For $n \leq -2$, they are at least $s^2((2 - c)^2 + 1 - c^2) > 2s^2(1 - c) = T^2 + \varepsilon$. Thus every vector not parallel to u has length strictly greater than T . Hence $\lambda_1 = a$ and $\lambda_2 > T$.

The covolume at $h = h_0$ equals

$$V_\varepsilon = ash_0 = \frac{a}{2} \sqrt{(T^2 + \varepsilon)(3T^2 - a^2 + 3\varepsilon)}.$$

As $\varepsilon \downarrow 0$, $V_\varepsilon \rightarrow F(a, T)$. Choose ε small enough that $V_\varepsilon < V$, and now set $h = V/(as)$. Then $h > h_0$, the same successive-minimum conditions hold, and

$$\text{covol}(\Lambda_h) = ash = V.$$

Every $V > F(a, T)$ is therefore realized. Combined with theorem 4.2, this proves the equality. $\square$

Remark 4.4. The condition $\lambda_2 > T$ is essential. The construction approaches the boundary $\lambda_2 = T$ only in the limit $\varepsilon \downarrow 0$; this explains why the lower endpoint is an infimum rather than an attained minimum.

Literature status. The volume classification combines a classical determinant input with the explicit basis construction above. Its priority relative to the existing literature has not been established by a systematic search.

5. A SIGN-FLIP CONSTRUCTION BEYOND THE FIRST COORDINATE SCALE

The preceding theorem uses a regime in which only one independent direction is visible. We now build a different pair: both lattices have all three successive minima equal to 1, yet their truncated data agree beyond the first-minimum scale.

5.1. The Gram matrices. For $0 < \delta \leq 1/6$, let

$$G_+ = \begin{pmatrix} 1 & \delta & \delta \\ \delta & 1 & \delta \\ \delta & \delta & 1 \end{pmatrix}, \quad G_- = \begin{pmatrix} 1 & \delta & \delta \\ \delta & 1 & -\delta \\ \delta & -\delta & 1 \end{pmatrix}. \quad (7)$$

Both matrices are positive definite. The eigenvalues of G_+ are $1 + 2\delta, 1 - \delta, 1 - \delta$, and those of G_- are $1 - 2\delta, 1 + \delta, 1 + \delta$.

Lemma 5.1 (Successive minima). *The lattices represented by G_+ and G_- both satisfy*

$$\lambda_1 = \lambda_2 = \lambda_3 = 1.$$

Proof. The coordinate vectors have squared norm 1, so $\lambda_3 \leq 1$. Any nonzero integer vector with one nonzero coordinate has squared norm at least 1. If at least two coordinates are nonzero, then the Euclidean integer norm squared is at least 2. The smallest eigenvalue of either Gram matrix is at least $1 - 2\delta \geq 2/3$, and hence

$$Q_{\pm}(x) = x^T G_{\pm} x \geq (1 - 2\delta) \|x\|_2^2 \geq 4/3 > 1.$$

Thus the only vectors of length 1 are the six signed coordinate vectors. They span three dimensions, proving all three successive minima equal 1. $\square$

5.2. Matching all vectors supported on at most two coordinates. For $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$, write $Q_{\pm}(x) = x^T G_{\pm} x$. A vector is called supported on a coordinate subset $S \subset \{1, 2, 3\}$ if its coordinates outside S are zero.

Lemma 5.2 (Two-coordinate matching). *For each fixed coordinate subset S with $|S| \leq 2$, the multiset of values $Q_+(x)$ over integer vectors with support exactly S agrees with the corresponding multiset for Q_- .*

Proof. On the $\{1, 2\}$ - and $\{1, 3\}$ -coordinate planes, the two quadratic forms have identical restrictions. On the $\{2, 3\}$ -plane, the restrictions are $x_2^2 + x_3^2 + 2\delta x_2 x_3$ and $x_2^2 + x_3^2 - 2\delta x_2 x_3$; the substitution $x_3 \mapsto -x_3$ is a multiplicity-preserving bijection between the corresponding integer vectors and preserves exact support. The coordinate axes agree trivially. $\square$

5.3. Equal truncated data and the first discrepancy.

Theorem 5.3 (Finite non-identifiability with $\lambda_3 = 1$). *For $0 < \delta \leq 1/6$ and every $T < \sqrt{3 - 6\delta}$,*

$$\mathcal{L}_T(G_+) = \mathcal{L}_T(G_-),$$

where the notation identifies a positive-definite Gram matrix with its associated lattice. Nevertheless, the two lattices have different covolumes and are not isometric.

Proof. By theorem 5.2, all vectors with at most two nonzero coordinates can be matched, preserving length and multiplicity. Consider a vector with all three coordinates nonzero. Then $\|x\|_2^2 \geq 3$. The minimum eigenvalue bounds give

$$Q_-(x) \geq (1 - 2\delta)\|x\|_2^2 \geq 3 - 6\delta,$$

and

$$Q_+(x) \geq (1 - \delta)\|x\|_2^2 \geq 3 - 3\delta > 3 - 6\delta.$$

If one coordinate has absolute value at least 2, then $\|x\|_2^2 \geq 6$, so $Q_\pm(x) \geq 6(1 - 2\delta) \geq 4 > 3 - 6\delta$. Therefore no vector with three nonzero coordinates enters either truncated multiset at radius $T < \sqrt{3 - 6\delta}$. All included vectors are covered by the matching lemma.

Finally, direct calculation gives

$$\det G_+ = 1 - 3\delta^2 + 2\delta^3, \quad \det G_- = 1 - 3\delta^2 - 2\delta^3.$$

Their difference is $4\delta^3 > 0$. The covolumes, which are $\sqrt{\det G_\pm}$, differ, so the lattices cannot be isometric. $\square$

Construction. For each $0 < \delta \leq 1/6$, this pair has $\lambda_3 = 1$, identical multiplicity-sensitive truncated data for every $T < \sqrt{3 - 6\delta}$, and unequal covolumes. It is therefore an explicit obstruction to any universal identification radius smaller than $\sqrt{3}\lambda_3$.

Proposition 5.4 (The first distinguishing length for this pair). *For the pair (7), the truncated data differ as soon as $T \geq \sqrt{3 - 6\delta}$. In particular, the first distinguishing radius is exactly $\sqrt{3 - 6\delta}$.*

Proof. For G_- , the two integer vectors $(1, -1, -1)$ and $(-1, 1, 1)$ satisfy $Q_-(x) = 3 - 6\delta$. By the proof of theorem 5.3, no vector with all coordinates nonzero in the G_+ lattice has squared length at most $3 - 6\delta$. The vectors with support of size at most two have identical length multiplicities under the matching in theorem 5.2; the two displayed full-support vectors therefore create an additional multiplicity of at least two on the G_- side at the threshold, which cannot be matched by full-support vectors of G_+ . Thus the multisets differ at this radius. $\square$

Example 5.5 (A concrete pair). For $\delta = 0.1$, both lattices have $\lambda_3 = 1$, and their truncated data agree for every $T < \sqrt{2.4} \approx 1.54919$. In particular, they agree at $T = 1.5$. Their covolumes are $\sqrt{0.972}$ and $\sqrt{0.968}$, respectively, so they are not isometric.

6. IDENTIFICATION-RADIUS BOUNDS

Let

$$C_3^* = \sup_{\Lambda} \frac{\rho_{\text{id}}(\Lambda)}{\lambda_3(\Lambda)},$$

where the supremum ranges over all full-rank lattices in $\mathbb{R}^3$. Since both numerator and denominator scale linearly under a uniform dilation, the ratio is scale-invariant.

Theorem 6.1 (Current two-sided bound). *The known results discussed here imply*

$$\sqrt{3} \leq C_3^* \leq \sqrt{\frac{7}{2}}.$$

Proof. For the lower bound, use theorem 5.3. Given any $T < \sqrt{3}$, choose

$$\delta(T) = \min \left\{ \frac{1}{6}, \frac{3 - T^2}{12} \right\} > 0.$$

If the minimum is the second term, then $3 - 6\delta(T) = (3 + T^2)/2 > T^2$. If the minimum is $1/6$, then $T^2 \leq 1$ and $3 - 6\delta(T) = 2 > T^2$. Thus for every $T < \sqrt{3}$ there is a pair with equal truncated data through T , identical $\lambda_3 = 1$, and unequal covolumes. Hence no universal identification radius below $\sqrt{3}\lambda_3$ is possible, yielding $C_3^* \geq \sqrt{3}$.

For the upper bound, Schiemann's theorem states that a ternary positive-definite real quadratic form, in a fixed reduced representative, is determined by the initial theta-series data through a bound that may be taken to be $\frac{7}{2}s_3$, where s_3 is the third successive minimum in the quadratic-form (squared-length) normalization [12]. Converting squared length $s_3 = \lambda_3^2$ to a radius gives $T = \sqrt{7/2}\lambda_3$. Therefore $\rho_{\text{id}}(\Lambda) \leq \sqrt{7/2}\lambda_3(\Lambda)$, subject to the reduced-form normalization in Schiemann's theorem, and taking the supremum gives the upper bound. $\square$

Literature-dependent bound. The lower bound is supplied by the explicit sign-flip family. The upper bound depends on applying Schiemann's finite-determination theorem with the normalization stated above; the optimal constant is not determined here.

Remark 6.2 (Status of the constant). The interval in theorem 6.1 is a rigorous target range, not a determination of the optimum. The assertion $C_3^* = \sqrt{3}$ is a natural conjectural target suggested by the lower-bound construction, but it is not proved here, and no claim is made that the question is absent from the literature.

7. A CUBIC IDENTITY FOR THE THREE-WAY SIGN INTERACTION

The matrices in equation (7) differ by the sign of one off-diagonal coefficient. A useful algebraic identity isolates the cyclic product of all three off-diagonal entries.

Proposition 7.1 (Signed-basis cubic identity). *Let*

$$G = \begin{pmatrix} d_1 & g_{12} & g_{13} \\ g_{12} & d_2 & g_{23} \\ g_{13} & g_{23} & d_3 \end{pmatrix}$$

be a real symmetric 3×3 matrix, and put $D = d_1 + d_2 + d_3$. For each $\epsilon \in \{\pm 1\}^3$, let $q_\epsilon = \epsilon^T G \epsilon$. Then

$$\sum_{\epsilon \in \{\pm 1\}^3} (q_\epsilon - D)^3 = 384 g_{12} g_{13} g_{23}. \quad (8)$$

Proof. Expand

$$q_\epsilon - D = 2(g_{12}\epsilon_1\epsilon_2 + g_{13}\epsilon_1\epsilon_3 + g_{23}\epsilon_2\epsilon_3).$$

Upon cubing and summing over the eight sign triples, every monomial containing an odd power of some ϵ_i cancels. The only surviving mixed term uses each of the three pair products exactly once. It occurs in six permutations, has the factor 2^3 , and has the same value for all eight sign choices. Its total contribution is $8 \cdot 8 \cdot 6 g_{12} g_{13} g_{23} = 384 g_{12} g_{13} g_{23}$. $\square$

The determinant itself is

$$\det G = d_1 d_2 d_3 - d_1 g_{23}^2 - d_2 g_{13}^2 - d_3 g_{12}^2 + 2g_{12} g_{13} g_{23}. \quad (9)$$

Thus, when the diagonal entries and the absolute values of the three off-diagonal entries are fixed, the determinant is sensitive to the sign of the same cyclic product detected by (8).

Remark 7.2 (What the identity does not say). The eight signed basis vectors are labelled by a chosen basis. An unlabelled truncated length multiset does not reveal which observed lengths came from which signed basis vectors. Accordingly, (8) is a basis-level algebraic diagnostic; it is not, by itself, an invariant computed from arbitrary truncated data, nor does it prove the universal upper bound $C_3^* \leq \sqrt{3}$.

8. GEOMETRY, TOPOLOGY, AND THE MEANING OF THE OBSERVATION DATA

8.1. What is proved geometrically. If two lattices have different covolumes, the corresponding flat tori have different total Riemannian volumes and therefore cannot be isometric. The volume classification and the sign-flip construction consequently establish metric non-identifiability from the stated finite length observations.

8.2. What is not proved topologically. Every full-rank lattice $\Lambda = B\mathbb{Z}^3$ gives a quotient homeomorphic to $\mathbb{R}^3/\mathbb{Z}^3 = \mathbb{T}^3$: the invertible linear map induced by B identifies these quotient manifolds. This is part of the standard framework for compact flat manifolds; see Charlap and Wolf [4, 14]. The reduction-theoretic statements used earlier are also treated in Mahler, van der Waerden, and Barnes [7, 13, 1]. Our constructions change the metric lattice, not the underlying topological manifold. Thus a claim that the present examples prove non-identifiability of topology would be false. To pursue topology, the comparison class must be enlarged beyond flat three-tori and a finite observation map defined consistently across that larger class.

8.3. Length data versus Laplace spectrum. The length multiset used here comes from the primal lattice. For a flat torus, the Laplace eigenvalues are proportional to squared lengths in the dual lattice (with a convention-dependent factor such as $4\pi^2$). Classical work on isospectral manifolds, including Milnor's example, illustrates why the precise spectral data and dimension matter [9]. Therefore a theorem about the Laplace spectrum should not be transferred to our observation model without explicitly passing to the dual lattice. The relevant role of Schiemann's theta-series result is that a positive-definite ternary quadratic form has a finite determination bound from its own representation data; it is not a claim that the present truncated primal-length data are literally the Laplace spectrum.

9. OPEN PROBLEMS AND RESEARCH PROGRAMME

The results suggest several concrete next tasks.

- P1. Optimize the universal identification constant.** Determine whether $C_3^* = \sqrt{3}$, or find a construction with a strictly larger lower bound. This requires either a sharper general identification theorem or a better counterexample family.
- P2. Improve the sign-flip construction.** The present pair has first distinguishing radius $\sqrt{3 - 6\delta}\lambda_3$, tending to $\sqrt{3}\lambda_3$ from below. Unequal diagonal entries or a different integer-vector re-pairing may change the first distinguishing radius; no such improvement is established here.
- P3. Identify finite data more efficiently.** Schiemann's bound is sufficient but may not be optimal for every lattice. One can ask for lattice-dependent bounds in terms of gaps among $\lambda_1, \lambda_2, \lambda_3$ or the number of short vectors.
- P4. Study other dimensions.** The sign-product mechanism is specific to a three-edge cycle in dimension three. Higher dimensions may admit analogous cycle or graph invariants, but a dimension- d theorem would require a separate proof.
- P5. Formulate a genuine topological question.** A topological non-identifiability theorem would need spaces of different topological types and an observation process defined on all of them; varying the lattice alone cannot do this.
- P6. Complete a priority search.** Search the literature for the exact volume-classification statement, the sign-flip truncated-multiset construction, the threshold constant problem, and the cubic identity before making a novelty claim.

10. CONCLUSION

This paper gives an exact description of the covolumes compatible with a fixed finite length observation in the one-direction regime, using a classical ternary determinant bound and an explicit perturbation construction. It also gives a pair of three-dimensional lattices with all successive minima equal to one and identical truncated data up to a threshold approaching $\sqrt{3}$, despite different covolumes. This yields the interval

$$\sqrt{3} \leq C_3^* \leq \sqrt{7/2}$$

for the worst-case identification-radius constant, with the upper endpoint supplied by Schiemann's finite theta-series determination theorem. The cubic identity records the algebraic sign mechanism that distinguishes the example, while its limitations are stated explicitly. The optimal constant and any extension to topological non-identifiability remain open within this project.

A. PARAMETER CHECKS FOR THE VOLUME CONSTRUCTION

For completeness, the parameter inequalities in the proof of theorem 4.3 are checked here. Since $T \geq a > 0$,

$$s^2 = T^2 - \frac{a^2}{4} + \varepsilon \geq \frac{3T^2}{4} + \varepsilon > 0.$$

Moreover,

$$1 - c = \frac{T^2 + \varepsilon}{2s^2} > 0,$$

so $c < 1$. The condition $c > 0$ is equivalent to

$$2s^2 > T^2 + \varepsilon \iff T^2 - \frac{a^2}{2} + \varepsilon > 0,$$

which holds because $T^2 \geq a^2$. Finally, $c < 1/2$ is equivalent to $T^2 + \varepsilon > s^2$, or $a^2/4 > 0$. Thus $0 < c < 1/2$.

These inequalities imply

$$h_0^2 = s^2(1 - c^2) > \frac{3}{4}s^2 > \frac{9T^2}{16},$$

so $2h_0 > 3T/2 > T$. They also give $T^2 + \varepsilon = 2s^2(1 - c)$, which is the identity used in the cases $n = -1$ and $n \geq 1$ in the main proof.

B. MULTIPLICITY ACCOUNTING FOR THE SIGN-FLIP EXAMPLE

The truncated data are a multiset over all integer vectors, not just primitive vectors or distinct length values. The matching lemma uses bijections on each exact support set: the identity map on supports $\{1, 2\}$ and $\{1, 3\}$, the sign change $x_3 \mapsto -x_3$ on support $\{2, 3\}$, and the identity on each coordinate axis. Since these support sets partition $\mathbb{Z}^3 \setminus \{0\}$ according to the zero pattern, and no vector with support size three is observed below the stated threshold, the union of these bijections is a length-preserving, multiplicity-preserving matching of the entire truncated multisets.

At the threshold $T_*^2 = 3 - 6\delta$, the two vectors $(1, -1, -1)$ and $(-1, 1, 1)$ enter for G_- , while the lower eigenvalue estimate for G_+ excludes any full-support vector at that length or below. Any two-support vectors at that numerical length have already matching multiplicities on both sides and cannot remove the additional count contributed by the full-support vectors of G_- .

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