

Risk Aversion and Information in Robust Contracting

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Abstract

We study hidden-action contracting with a risk-averse agent and partial knowledge of actions in a compact convex envelope of distribution–cost pairs. With finitely many outcomes and bounded deterministic wages, the principal’s worst-case payoff admits an exact dual representation with an endogenous f -divergence governed by transfer utility and an incentive multiplier. Under strictly concave utility, finite-multiplier branch optima follow a truncated likelihood-ratio rule, including at zero-probability outcomes. To address nonattainment induced by adversarial tie-breaking, we characterize the infinite-multiplier limit as a restricted zero-sum game. For polyhedral envelopes, we establish two-sided $O(1/\lambda)$ convergence when inverse utility is Lipschitz. Positive tolerance for suboptimal agent responses restores existence and confines the multiplier search to an explicit compact interval. A finite optimal multiplier gives a linear upper bound on tolerance loss; an optimal infinite-multiplier endpoint in a polyhedral envelope gives a square-root upper bound. A two-action example attains the square-root rate asymptotically. Restricting contracts to a coarse performance statistic contracts the endogenous divergence. We characterize equality in data processing under wage bounds, bracket the value of finer contractible information using fine and coarse dual minimizers, and derive quantitative bounds for approximately sufficient statistics. Numerical experiments demonstrate strict gains over utility-affine contracts and distinguish technological knowledge from the richness of contractible performance signals.

Keywords. Robust contracts; risk aversion; unknown actions; convex duality; sufficient statistics; approximate best responses.

JEL classification. D82; D86; C61.

1 Introduction

In hidden-action contracting, a principal may know several feasible production methods without knowing every action available to the agent. She may also have bounds on the output distributions and effort costs of the unknown actions. Robust contracting evaluates wages against every technology consistent with this information. With a risk-averse agent, the principal must provide incentives while limiting the cost of wage risk, and the least-favorable technology can change with the contract.

We study this problem on a finite outcome space with bounded deterministic wages and a known, strictly increasing, concave transfer utility. The principal knows a compact convex envelope of distribution–cost pairs and finitely many reference actions. Any compact convex subset of the envelope containing those actions is an admissible technology. An action can be an exact best response under some admissible technology if and only if its utility reaches the threshold set by the best reference action. We also allow ε -approximate responses, which can fall short of that threshold by at most ε utility units. In both cases, response selection is adverse to the principal.

Theorem 4.1 reduces unrestricted contract design to convex minimization over the action envelope, indexed by a reference action and a nonnegative incentive multiplier λ . For transfer utility u and wage cap $\bar{w}$, let

$$C_u(z) = \max_{0 \leq w \leq \bar{w}} \{zu(w) - w\}$$

denote the capped wage conjugate, and set $f_\lambda(t) = C_u(\lambda(t - 1))$. The adversary minimizes expected revenue plus the effort-cost difference scaled by λ and the f_λ -divergence between

the reference and adverse output distributions. Optimizing wages separately at each outcome produces this divergence. Its generator therefore depends on utility curvature and the wage bounds.

Under strictly concave utility, any dual minimizer determines a finite-multiplier branch optimum (Theorem 4.3). Where utility is differentiable and the wage is interior, inverse marginal utility equals λ times the excess likelihood ratio of the reference distribution relative to the adverse distribution. A positive wage requires a likelihood advantage, but that advantage need not suffice when marginal utility at zero is finite. An outcome with positive reference probability and zero adverse probability receives the wage cap when $\lambda > 0$. More generally, every contract is weakly dominated by one supported on a single reference distribution (Proposition 4.2).

Exact best responses can lead to nonattainment: contracts approach a positive supremum, but their limit induces an unfavorable tie. Theorem 6.1 treats the unbounded multiplier range directly. An implementability index determines whether a reference branch tends to $-\infty$ or to the value of a restricted zero-sum game. For polyhedral envelopes, an optimal exact-response contract exists if and only if the global dual maximum occurs at a finite multiplier. If inverse utility is also Lipschitz, the endpoint has a two-sided $O(1/\lambda)$ approximation bound (Theorem 7.2).

Positive response tolerance restores continuity of the worst-case payoff and existence of an optimal contract, while restricting the multiplier search to an explicit compact interval (Theorem 5.1). The value is convex and nonincreasing in ε . A finite optimal multiplier at zero tolerance gives an $O(\varepsilon)$ upper bound on value loss; an optimal infinite-multiplier endpoint in a polyhedral envelope gives an $O(\sqrt{\varepsilon})$ bound. Proposition 7.4 gives a two-action example with square-root loss and computes its leading coefficient.

Performance information enters through the same divergence. Restricting wages to a coarse statistic aggregates the output distributions in the divergence term, while preserving expected revenue and effort costs. Divergence gaps at the fine and coarse dual minimizers give two-sided value comparisons. Pooling distinct likelihood ratios can preserve divergence when every outcome in a cell calls for the same boundary wage. For approximately sufficient statistics, total-variation reconstruction errors perturb wage costs and response tolerances. The resulting information-loss bounds are linear at a finite optimal multiplier and of square-root order at an optimal infinite-multiplier endpoint in a polyhedral envelope.

The numerical analysis combines primal programs, feasible dual mixtures, and analytical tail bounds. It exhibits gains over utility-affine contracts and separates the effects of narrower cost intervals, additional known actions, and finer contractible signals.

Section 2 discusses related work. Sections 3–7 develop the model, dual representation, existence results, and endpoint bounds. Sections 8 and 9 study performance information and numerical examples. The appendices examine utility-affine restrictions and grid discretization.

2 Related Literature

Robust moral hazard. The unknown-action formulation follows Carroll (2015), in which the principal knows a subset of the agent’s actions. Walton and Carroll (2022) relate optimal contract form to structural restrictions on the response correspondence. Our contribution is an unrestricted dual representation for a compact convex action envelope with risk-averse utility and bounded wages, together with results on nonattainment, response tolerance, and contractible information. Distributional and moment ambiguity provide other models of technological uncertainty; see Li and Wang (2025) and Zuo’s *Distributional Robustness of Linear Contracts* (2026, arXiv:2604.24732v1).

Utility coordinates, likelihood ratios, and bounded transfers. Promised-utility coordinates make the principal’s wage cost convex (Grossman and Hart 1983), and likelihood ratios characterize incentive payments in known-technology models (Holmström 1979). Section 3.4

of Carroll’s 21 December 2012 working-paper version of Carroll (2015) establishes optimality of contracts that align the agent’s utility with an affine function of the principal’s utility from net revenue. With a risk-neutral principal, this relation is $u(w(y)) = a[y - w(y)] + b$. We allow arbitrary bounded deterministic wages. The resulting likelihood-ratio rule compares a reference distribution with an endogenous adverse distribution. Utility-affine contracts $u(w(y)) = \alpha + \beta y$ serve as a restricted comparison class in the numerical examples and Appendix A.

Several known-technology models also study unrestricted contract form. Georgiadis et al. (2024) allow the agent to choose output distributions subject to a smooth cost function. Bonham (2024) studies productive activity, measurement, and compensation, obtaining contracts with a floor, a hurdle bonus, an interior incentive region, and a ceiling. Opp’s *Moral Hazard and the Quest for Linear Contracts* (2025, working paper) gives conditions involving a risk-neutral principal, logarithmic agent utility, and a one-parameter exponential family with a linear sufficient statistic under which the first-order approach yields affine contracts. Here the adverse technology is chosen jointly with the likelihood ratios governing wages.

Variational divergence representations. The duality argument uses convex conjugacy, minimax interchange, and variational representations of f -divergences (Csiszár 1967; Rockafellar 1970; Sion 1958; Nguyen et al. 2010). Utility–divergence relations also arise in risk measurement and decision theory (Ben-Tal and Teboulle 2007; Jose et al. 2008). Moral-hazard models also use divergence-based costs of choosing output distributions (Hébert 2018; Mattsson and Weibull 2023). The divergence here arises from the optimization of wage payments, with a generator determined by transfer utility, wage bounds, and the incentive multiplier.

To analyze exact best responses when the incentive multiplier grows without bound, we specialize the saddle-value envelope theorem of Milgrom and Segal (2002, Theorem 5), which generalizes the classical matrix-game perturbation analysis of Mills (1956). In our contracting problem, this limit identifies a restricted zero-sum game over implementability-preserving contracts and maximal-threat actions, replacing ad hoc multiplier truncation with an exact endpoint characterization. Combining this limit game with the polyhedral error bound of Hoffman (1952) yields explicit two-sided $O(1/\lambda)$ tail estimates when inverse utility is Lipschitz.

Approximate best responses and quantitative stability. Approximate best responses have been studied in principal–agent problems (Bacchiocchi et al. 2025; Lin and Chen 2025) and robust bilevel optimization (Besançon et al. 2024). Carroll and Meng (2016) establish local robustness bounds, including a risk-averse extension, by perturbing utility payments toward the principal’s payoff. In our model, positive response tolerance gives a uniform multiplier bound and contract existence. The finite-multiplier and polyhedral endpoint cases yield different loss bounds, with an exact square-root leading term in the two-action example.

Performance information and approximate sufficiency. The information comparisons build on Blackwell (1953), the informativeness principle (Holmström 1979), and approximate sufficiency (Le Cam 1964). Restricting wages to a coarse statistic contracts the endogenous divergence. Wage bounds can create flat or affine regions in its generator, allowing outcomes with distinct likelihood ratios to be pooled without divergence loss. For approximate sufficiency, reconstruction errors affect both expected wage costs and the effective response tolerance; the multiplier bounds control the resulting information loss.

Interpretation of reference actions. The reference actions in our model specify both distributions and effort costs. The technology-completion class imposes no further restrictions linking unknown actions. In data-based approaches, observed responses to historical contracts instead restrict the unknown technology through revealed preference (Antic and Georgiadis 2026;

Burkett and Rosenthal 2024). Our results are conditional on the calibrated distribution–cost information and do not identify those costs from observed output.

3 Model

Consider a contracting environment with a finite set of outcomes $Y = \{1, \dots, n\}$, associated with nonnegative revenues $r_j \geq 0$. The principal offers a deterministic wage schedule taking values in $[0, \bar{w}]$, where $\bar{w} > 0$ is an exogenous wage cap. The agent evaluates monetary transfers according to a continuous, strictly increasing, and concave utility function $u : [0, \bar{w}] \rightarrow \mathbb{R}$ normalized by $u(0) = 0$. Let $U = u(\bar{w}) > 0$, $J = [0, U]$, and $H = u^{-1} : J \rightarrow [0, \bar{w}]$ denote the inverse utility function, which is continuous, strictly increasing, and convex with $H(0) = 0$ and $H(U) = \bar{w}$. It is convenient to represent a contract by its promised-utility vector $x = (x_1, \dots, x_n) \in \mathcal{X} := J^n$, which induces monetary transfers $w_j = H(x_j)$.

An action is a pair $k = (\mu, c) \in \Delta_n \times \mathbb{R}_+$ consisting of an output probability distribution μ over Y and an effort cost c measured in utility units. The principal knows that all potentially feasible actions lie within a nonempty, compact, convex envelope $\mathcal{K} \subseteq \Delta_n \times \mathbb{R}_+$, and knows that a finite set of reference actions $\mathcal{K}_0 = \{k_1, \dots, k_m\} \subseteq \mathcal{K}$, with $k_i = (\nu_i, a_i)$, is available to the agent.¹ To anchor global payoff bounds, we include a null action $k_1 = (e_1, 0) \in \mathcal{K}_0$ that generates outcome 1 with certainty at zero cost, where $r_1 = 0$; the local recovery and infinite-multiplier endpoint results in Sections 4 and 6 hold independently of this normalization.

Given a contract $x \in \mathcal{X}$ and an action $k = (\mu, c) \in \mathcal{K}$, the agent’s net utility (score), the reference utility threshold, and the principal’s expected payoff are given by

$$s_x(k) = \mu \cdot x - c, \quad \sigma_0(x) = \max_{1 \leq i \leq m} s_x(k_i), \quad p_x(k) = \sum_{j=1}^n \mu_j \{r_j - H(x_j)\}.$$

For a response tolerance $\varepsilon \geq 0$, measured in utility units, define the reduced-form admissible response set, the worst-case payoff under contract x , and the optimal robust value by

$$\mathcal{R}_\varepsilon(x) = \{k \in \mathcal{K} : s_x(k) \geq \sigma_0(x) - \varepsilon\}, \quad v_\varepsilon(x) = \min_{k \in \mathcal{R}_\varepsilon(x)} p_x(k), \quad V_\varepsilon = \sup_{x \in \mathcal{X}} v_\varepsilon(x).$$

Because $\mathcal{K}_0 \subseteq \mathcal{K}$, the set $\mathcal{R}_\varepsilon(x)$ is nonempty and compact for every $x \in \mathcal{X}$ and $\varepsilon \geq 0$. The minimization defining $v_\varepsilon(x)$ reflects adversarial selection among all ε -optimal responses, including exact ties when $\varepsilon = 0$. Write $t_+ = \max\{t, 0\}$ and $\text{TV}(\mu, \nu) = \frac{1}{2} \|\mu - \nu\|_1$. Throughout, we also use

$$D = (\max_j r_j - \min_j r_j) + \bar{w}, \quad \bar{\rho}_0 = \max_{1 \leq i \leq m} \nu_i \cdot r,$$

so that the oscillation of $p_x(k)$ over $\mathcal{X} \times \mathcal{K}$ is bounded above by D .

3.1 Unknown technologies and the reference threshold

To microfound the response correspondence $\mathcal{R}_\varepsilon(x)$, interpret $\mathcal{K}$ as the set of actions that nature might make available and $\mathcal{K}_0$ as the actions known to be present. An admissible technology is any compact convex set of actions containing the references and contained in the envelope:

$$\mathfrak{T} = \{T \subseteq \mathcal{K} : T \text{ is compact and convex, } \text{co}(\mathcal{K}_0) \subseteq T\}.$$

When facing technology $T \in \mathfrak{T}$ under contract x , an ε -optimizing agent may select any action in

$$B_\varepsilon(x, T) = \left\{ k \in T : s_x(k) \geq \max_{\ell \in T} s_x(\ell) - \varepsilon \right\}.$$

¹Replacing $\mathcal{K}_0$ by its convex hull $\text{co}(\mathcal{K}_0)$ leaves all subsequent results unchanged.

Proposition 3.1 (Exact reduction over technology completions). *For every contract $x \in \mathcal{X}$ and tolerance $\varepsilon \geq 0$,*

$$\bigcup_{T \in \mathfrak{T}} B_\varepsilon(x, T) = \mathcal{R}_\varepsilon(x), \quad \inf_{T \in \mathfrak{T}} \min_{k \in B_\varepsilon(x, T)} p_x(k) = v_\varepsilon(x).$$

Proof. Because every $T \in \mathfrak{T}$ contains $\mathcal{K}_0$, we have $\max_{\ell \in T} s_x(\ell) \geq \sigma_0(x)$, which implies $B_\varepsilon(x, T) \subseteq \mathcal{R}_\varepsilon(x)$ for all $T \in \mathfrak{T}$. Conversely, given any $k \in \mathcal{R}_\varepsilon(x)$, let $T = \text{co}(\mathcal{K}_0 \cup \{k\}) \in \mathfrak{T}$. By affinity of s_x , the maximum of s_x on T is $\max\{\sigma_0(x), s_x(k)\}$, which is at most $s_x(k) + \varepsilon$. Thus $k \in B_\varepsilon(x, T)$, establishing the set equality. Taking the minimum of $p_x(k)$ over both sides yields the equality of worst-case payoffs. $\square$

The reference actions specify both output distributions and effort costs. Such information may come from engineering specifications or external calibration; output observations alone do not supply the cost component. Proposition 3.1 also uses the admissibility of every completion $\text{co}(\mathcal{K}_0 \cup \{k\})$. Additional restrictions linking actions would have to be imposed on these completions.

The null action guarantees the agent at least zero utility under exact best responses and at least $-\varepsilon$ under ε -approximate responses. The zero contract guarantees the principal $v_\varepsilon(0) \geq 0$ because revenues are nonnegative. The wage bounds are state-independent, so wages can exceed realized revenue. A constraint $w_j \leq r_j$ would instead give an outcome-specific wage conjugate.

4 The Unrestricted Dual and Wage Recovery

For each reference action $i \in \{1, \dots, m\}$, multiplier $\lambda \geq 0$, and action $k = (\mu, c) \in \mathcal{K}$, define the relative score deficit and Lagrangian payoff by

$$A_i(x, k) = s_x(k_i) - s_x(k) = (\nu_i - \mu) \cdot x + c - a_i, \\ F_i(x, k; \lambda) = p_x(k) + \lambda A_i(x, k), \quad Q_i(\lambda) = \max_{x \in \mathcal{X}} \min_{k \in \mathcal{K}} F_i(x, k; \lambda).$$

Theorem 4.1 (Unrestricted contract representation). *For every $\varepsilon \geq 0$, the optimal robust value satisfies*

$$V_\varepsilon = \sup_{1 \leq i \leq m, \lambda \geq 0} \{Q_i(\lambda) - \lambda \varepsilon\}. \quad (4.1)$$

For each fixed (i, λ) , the maximum defining $Q_i(\lambda)$ is attained, and minimax interchange yields

$$Q_i(\lambda) = \min_{(\mu, c) \in \mathcal{K}} \left\{ \mu \cdot r + \lambda(c - a_i) + \sum_{j=1}^n \sup_{z \in J} [\lambda(\nu_{ij} - \mu_j)z - \mu_j H(z)] \right\}. \quad (4.2)$$

The minimization in (4.2) is a convex program that attains its minimum on $\mathcal{K}$. Moreover, if $x \in \mathcal{X}$ attains the maximum defining $Q_i(\lambda)$, then $v_\varepsilon(x) \geq Q_i(\lambda) - \lambda \varepsilon$.

Proof. Fix a contract $x \in \mathcal{X}$. Evaluating $v_\varepsilon(x)$ requires minimizing the continuous affine functional $p_x(k)$ over the compact convex set $\mathcal{K}$ subject to the single affine inequality $\sigma_0(x) - s_x(k) - \varepsilon \leq 0$. When there exists $k \in \mathcal{K}$ satisfying this inequality strictly, Slater's condition holds and strong Lagrangian duality yields an attained finite multiplier $\lambda \geq 0$. When no strictly feasible action exists, every $k \in \mathcal{R}_\varepsilon(x)$ satisfies $s_x(k) = \sigma_0(x) - \varepsilon = \max_{\mathcal{K}} s_x$, so $\mathcal{R}_\varepsilon(x)$ is an exposed face of $\mathcal{K}$. As $\lambda \rightarrow \infty$, the penalized minima

$$\min_{k \in \mathcal{K}} \{p_x(k) + \lambda[\sigma_0(x) - \varepsilon - s_x(k)]\}$$

increase monotonically to $\min_{k \in \mathcal{R}_\varepsilon(x)} p_x(k) = v_\varepsilon(x)$ by compactness of $\mathcal{K}$. In either case,

$$v_\varepsilon(x) = \sup_{\lambda \geq 0} \min_{k \in \mathcal{K}} [p_x(k) + \lambda(\sigma_0(x) - s_x(k) - \varepsilon)].$$

Because $\lambda\sigma_0(x) = \max_{1 \leq i \leq m} \lambda s_x(k_i)$ is independent of k , we may interchange the suprema over $x \in \mathcal{X}$, $\lambda \geq 0$, and $i \in \{1, \dots, m\}$. For each fixed (i, λ) , the objective $F_i(x, k; \lambda)$ is continuous, concave in $x \in \mathcal{X}$, and affine in $k \in \mathcal{K}$. By Sion's minimax theorem (Sion 1958), $\max_{x \in \mathcal{X}} \min_{k \in \mathcal{K}} F_i(x, k; \lambda) = \min_{k \in \mathcal{K}} \max_{x \in \mathcal{X}} F_i(x, k; \lambda)$. Maximizing over the Cartesian product $\mathcal{X} = J^n$ separates across outcomes $j = 1, \dots, n$, giving (4.2). The inner supremum is a pointwise supremum of affine functions of (μ, c) and is continuous on $\mathcal{K}$, so the outer minimization is convex and attains its minimum. Finally, if x maximizes $\min_{k \in \mathcal{K}} F_i(x, k; \lambda)$, replacing $s_x(k_i)$ by $\sigma_0(x) \geq s_x(k_i)$ shows that $v_\varepsilon(x) \geq Q_i(\lambda) - \lambda\varepsilon$. $\square$

Before examining the dual objective in detail, we note a structural simplification of the primal contract space that holds regardless of the curvature of u .

Proposition 4.2 (Reference-support reduction). *For any contract $x \in \mathcal{X}$ and tolerance $\varepsilon \geq 0$, there exists a contract $\tilde{x} \in \mathcal{X}$ such that $v_\varepsilon(\tilde{x}) \geq v_\varepsilon(x)$ and $\tilde{x}_j = 0$ for all outcomes j outside the support of a single reference distribution ν_i .*

Proof. Select an index i that attains $\sigma_0(x) = s_x(k_i)$, and define $\tilde{x}_j = x_j$ if $\nu_{ij} > 0$ and $\tilde{x}_j = 0$ if $\nu_{ij} = 0$. This modification leaves $\tilde{s}_x(k_i) = s_x(k_i)$ unchanged while weakly reducing $\tilde{s}_x(k) \leq s_x(k)$ for every $k \in \mathcal{K}$. Consequently, $\sigma_0(\tilde{x}) = \sigma_0(x)$ and $\mathcal{R}_\varepsilon(\tilde{x}) \subseteq \mathcal{R}_\varepsilon(x)$. Because H is increasing, $\tilde{p}_x(k) \geq p_x(k)$ for every $k \in \mathcal{K}$, and therefore $v_\varepsilon(\tilde{x}) \geq v_\varepsilon(x)$. $\square$

To express (4.2) in terms of statistical divergence, recall the capped wage conjugate $C_u(z) = \max_{0 \leq w \leq \bar{w}} \{zu(w) - w\}$, which is finite, convex, nonnegative, nondecreasing, and satisfies $C_u(0) = 0$ and $\lim_{z \rightarrow \infty} C_u(z)/z = U$. For any $\lambda \geq 0$, the function $f_\lambda(t) = C_u(\lambda(t-1))$ on $t \geq 0$ is convex and satisfies $f_\lambda(1) = 0$. Its closed-perspective f -divergence between $\nu \in \Delta_n$ and $\mu \in \Delta_n$ is

$$D_{f_\lambda}(\nu \parallel \mu) = \sum_{j: \mu_j > 0} \mu_j C_u \left(\lambda \left(\frac{\nu_j}{\mu_j} - 1 \right) \right) + \lambda U \sum_{j: \mu_j = 0} \nu_j. \quad (4.3)$$

With this notation, the branch dual (4.2) takes the compact form

$$Q_i(\lambda) = \min_{(\mu, c) \in \mathcal{K}} \{ \mu \cdot r + \lambda(c - a_i) + D_{f_\lambda}(\nu_i \parallel \mu) \}.$$

When the agent is strictly risk averse, any dual minimizer directly pins down an optimal branch contract through a truncated likelihood-ratio rule, even if the minimizing distribution assigns zero probability to certain outcomes.

Theorem 4.3 (Wage recovery from dual minimizers). *Suppose u is strictly concave. Fix a reference i and multiplier $\lambda > 0$, and let $k^* = (\mu^*, c^*) \in \mathcal{K}$ be any minimizer of (4.2). Define the wage vector $w^* \in [0, \bar{w}]^n$ coordinatewise as follows:*

(i) if $\mu_j^* > 0$, let w_j^* be the unique maximizer

$$w_j^* = \arg \max_{0 \leq w \leq \bar{w}} \left\{ \lambda \left(\frac{\nu_{ij}}{\mu_j^*} - 1 \right) u(w) - w \right\};$$

(ii) if $\mu_j^* = 0 < \nu_{ij}$, set $w_j^* = \bar{w}$;

(iii) if $\mu_j^* = \nu_{ij} = 0$, set $w_j^* = 0$.

Then $x^* = u(w^*)$ maximizes $\min_{k \in \mathcal{K}} F_i(x, k; \lambda)$, and (x^*, k^*) forms a saddle pair for $F_i(\cdot, \cdot; \lambda)$. At $\lambda = 0$, the zero contract $w^* = 0$ is a branch maximizer.

Furthermore, if u is differentiable on $(0, \bar{w})$ and $w_j^* \in (0, \bar{w})$ at an outcome with $\mu_j^* > 0$, then

$$\frac{1}{u'(w_j^*)} = \lambda \left(\frac{\nu_{ij}}{\mu_j^*} - 1 \right). \quad (4.4)$$

In particular, whenever $\mu_j^* > 0$, paying a strictly positive wage $w_j^* > 0$ requires $\nu_{ij} > \mu_j^*$.

Proof. In utility coordinates, the inner maximization for outcome j in (4.2) is $\max_{z \in J} [\lambda(\nu_{ij} - \mu_j^*)z - \mu_j^* H(z)]$. When $\mu_j^* > 0$, strict concavity of u implies strict convexity of H , so this objective has a unique maximizer $z_j^* = u(w_j^*)$. When $\mu_j^* = 0 < \nu_{ij}$, the objective reduces to $\lambda \nu_{ij} z$, which is uniquely maximized on J at $z_j^* = U = u(\bar{w})$. By Sion's minimax theorem, $F_i(\cdot, \cdot; \lambda)$ possesses at least one primal optimal contract $x^\circ \in \mathcal{X}$, and every pair of primal and dual optima forms a saddle point. Consequently, any primal optimum x° must coincide with x^* on all coordinates where $\mu_j^* > 0$ or $\nu_{ij} > 0$. On the remaining coordinates where $\mu_j^* = \nu_{ij} = 0$, setting $x_j = 0$ leaves $F_i(x, k^*; \lambda)$ unchanged and weakly increases $F_i(x, k; \lambda)$ for every $k \in \mathcal{K}$, since the j th coordinate enters F_i as $-\mu_j[H(x_j) + \lambda x_j] \leq 0$. Thus x^* is also primal optimal and (x^*, k^*) is a saddle pair. Differentiating the coordinate objective yields the interior first-order condition (4.4); when $\nu_{ij} \leq \mu_j^*$, the objective is strictly decreasing on $[0, \bar{w}]$ and is uniquely maximized at $w_j^* = 0$. $\square$

4.1 Utility curvature, wage bounds, and incentives

Closed-form expressions for C_u illustrate how risk aversion and the wage cap jointly shape the endogenous divergence. Under square-root utility $u(w) = \sqrt{w}$, we have $H(x) = x^2$ and $U = \sqrt{\bar{w}}$, which yields

$$C_u(z) = \begin{cases} 0, & z \leq 0, \\ z^2/4, & 0 < z < 2\sqrt{\bar{w}}, \\ z\sqrt{\bar{w}} - \bar{w}, & z \geq 2\sqrt{\bar{w}}. \end{cases} \quad (4.5)$$

When the wage cap is slack across all outcomes and ν_i is absolutely continuous with respect to μ , the divergence simplifies to the one-sided chi-square discrepancy

$$D_{f_\lambda}(\nu_i \| \mu) = \frac{\lambda^2}{4} \sum_{j: \mu_j > 0} \frac{(\nu_{ij} - \mu_j)_+^2}{\mu_j}.$$

The wage cap replaces the quadratic growth of $C_u(z)$ for $z \geq 2\sqrt{\bar{w}}$ with an affine tail of slope $\sqrt{\bar{w}}$, keeping the divergence finite even when ν_i is not absolutely continuous with respect to μ .

Similarly, for normalized constant absolute risk aversion (CARA) utility $u_a(w) = (1 - e^{-aw})/a$ with $a > 0$, which satisfies $u_a(0) = 0$ and $u'_a(0) = 1$, the conjugate is

$$C_{u_a}(z) = \begin{cases} 0, & z \leq 1, \\ (z - 1 - \log z)/a, & 1 < z < e^{a\bar{w}}, \\ z(1 - e^{-a\bar{w}})/a - \bar{w}, & z \geq e^{a\bar{w}}. \end{cases} \quad (4.6)$$

Setting $z_j = \lambda(\nu_{ij}/\mu_j^* - 1)$ for $\mu_j^* > 0$, the optimal wage is 0 for $z_j \leq 1$, $(\log z_j)/a$ for $1 < z_j < e^{a\bar{w}}$, and $\bar{w}$ for $z_j \geq e^{a\bar{w}}$. As $a \downarrow 0$, u_a converges to risk-neutral utility $u_0(w) = w$, and $C_{u_a}(z)$ converges to $C_{u_0}(z) = \bar{w}(z - 1)_+$.

Capped conjugates vanish on $(-\infty, 0]$, and on $(-\infty, 1]$ when $u'(0) = 1$. If $u'_-(\bar{w}) > 0$, then $C_u(z) = zU - \bar{w}$ for $z \geq 1/u'_-(\bar{w})$, as in both examples. The resulting generator need not be strictly convex. In particular, a flat region containing likelihood ratios on both sides of one can make $D_{f_\lambda}(\nu \| \mu) = 0$ for distinct distributions. Flat and affine regions also permit lossless pooling of distinct likelihood ratios, as shown in Section 8.

Holding effort costs fixed in utility units gives the following comparison.

Proposition 4.4 (Comparative statics in utility curvature). *Holding the action envelope $\mathcal{K}$, reference set $\mathcal{K}_0$, revenues, and wage cap fixed, if two transfer utilities satisfy $u_2(w) \leq u_1(w)$ for all $w \in [0, \bar{w}]$ with $u_1(0) = u_2(0) = 0$, then $V_\varepsilon(u_2) \leq V_\varepsilon(u_1)$ for every $\varepsilon \geq 0$.*

Proof. Any feasible utility promise vector x under u_2 lies in $[0, u_2(\bar{w})]^n \subseteq [0, u_1(\bar{w})]^n$ and is therefore also feasible under u_1 . Offering the same utility vector x under u_1 induces identical action scores $s_x(k)$ and response sets $\mathcal{R}_\varepsilon(x)$, while requiring pointwise lower monetary transfers

$u_1^{-1}(x_j) \leq u_2^{-1}(x_j)$ at every outcome. Taking the minimum over $k \in \mathcal{R}_\varepsilon(x)$ and the supremum over x yields $V_\varepsilon(u_2) \leq V_\varepsilon(u_1)$. $\square$

For the normalized CARA family $u_a(w) = (1 - e^{-aw})/a$, an increase in a lowers $u_a(w)$ at every $w > 0$ while keeping the marginal utility of money at zero wage equal to one. Proposition 4.4 therefore implies that $V_\varepsilon(u_a)$ is nonincreasing in a .

5 Response Tolerance and Attained Optimization

Allowing a strictly positive response tolerance $\varepsilon > 0$ regularizes the principal's problem by eliminating fragile tie-breaking discontinuities and bounding the shadow price of incentives.

Theorem 5.1 (Existence, multiplier bounds, and envelope properties under positive tolerance). *For every $\varepsilon > 0$, the worst-case payoff function $x \mapsto v_\varepsilon(x)$ is continuous on $\mathcal{X}$, the supremum V_ε is attained by an optimal contract $x_\varepsilon \in \mathcal{X}$, and*

$$V_\varepsilon = \max_{1 \leq i \leq m} \max_{0 \leq \lambda \leq (\nu_i \cdot r)/\varepsilon} \{Q_i(\lambda) - \lambda\varepsilon\}. \quad (5.1)$$

As $\varepsilon \downarrow 0$, $V_\varepsilon \uparrow V_0$, and any family of tolerant optimal contracts $(x_\varepsilon)_{\varepsilon>0}$ satisfies $v_0(x_\varepsilon) \geq V_\varepsilon \rightarrow V_0$. Moreover, $\varepsilon \mapsto V_\varepsilon$ is convex and nonincreasing on $[0, \infty)$. If λ_1, λ_2 are optimal multipliers at tolerances $0 < \varepsilon_1 < \varepsilon_2$, then $\lambda_1 \geq \lambda_2$, and at any point of differentiability, $V'_\varepsilon = -\lambda_\varepsilon$.

Proof. Fix $\varepsilon > 0$ and $x \in \mathcal{X}$, and let $p_{\min}$ and $p_{\max}$ be the minimum and maximum of $p_x(k)$ over $\mathcal{X} \times \mathcal{K}$. Any reference action k_i achieving $\sigma_0(x) = s_x(k_i)$ belongs to $\mathcal{K}$ and yields a constraint slack of $\sigma_0(x) - \varepsilon - s_x(k_i) = -\varepsilon < 0$. Evaluating the inner Lagrangian at k_i bounds the dual objective above by $p_{\max} - \lambda\varepsilon$, whereas evaluating at $\lambda = 0$ yields at least $p_{\min}$. Thus restricting λ to the compact interval $[0, D/\varepsilon]$ preserves the pointwise supremum defining $v_\varepsilon(x)$ for every $x \in \mathcal{X}$. Because the Lagrangian objective is jointly continuous in (x, k, λ) on the compact domain $\mathcal{X} \times \mathcal{K} \times [0, D/\varepsilon]$, Berge's maximum theorem implies that $v_\varepsilon(x)$ is continuous on $\mathcal{X}$ and attains its maximum V_ε .

To obtain the sharper branch-specific upper bound in (5.1), observe that evaluating (4.2) at $k_i = (\nu_i, a_i)$ gives $Q_i(\lambda) \leq \nu_i \cdot r$ for all $\lambda \geq 0$. Because the null action $k_1 = (e_1, 0) \in \mathcal{K}$ has $r_1 = 0$ and $r_j \geq 0$, the zero contract guarantees $V_\varepsilon \geq Q_i(0) = \min_{\mathcal{K}} \mu \cdot r = 0$. Whenever $\lambda > (\nu_i \cdot r)/\varepsilon$, we have $Q_i(\lambda) - \lambda\varepsilon < 0 \leq V_\varepsilon$, so such multipliers are strictly suboptimal.

For any fixed $x \in \mathcal{X}$, the nested compact sets $\mathcal{R}_\varepsilon(x)$ decrease to $\mathcal{R}_0(x)$ as $\varepsilon \downarrow 0$, so continuity of p_x yields $v_\varepsilon(x) \uparrow v_0(x)$. Consequently, $V_\varepsilon \leq V_0$ and $\lim_{\varepsilon \downarrow 0} V_\varepsilon = \sup_{\varepsilon>0} \sup_{x \in \mathcal{X}} v_\varepsilon(x) = \sup_{x \in \mathcal{X}} \sup_{\varepsilon>0} v_\varepsilon(x) = V_0$, establishing value convergence even when v_0 is not upper semicontinuous. Since $\mathcal{R}_0(x_\varepsilon) \subseteq \mathcal{R}_\varepsilon(x_\varepsilon)$, any tolerant optimizer x_ε guarantees $v_0(x_\varepsilon) \geq v_\varepsilon(x_\varepsilon) = V_\varepsilon$.

Finally, (4.1) expresses V_ε as the pointwise supremum of affine functions $\varepsilon \mapsto Q_i(\lambda) - \lambda\varepsilon$ with nonpositive slopes $-\lambda$, which establishes convexity and monotonicity. If (i_t, λ_t) attains V_{ε_t} for $t = 1, 2$ with $\varepsilon_1 < \varepsilon_2$, optimality implies

$$Q_{i_1}(\lambda_1) - \varepsilon_1 \lambda_1 \geq Q_{i_2}(\lambda_2) - \varepsilon_1 \lambda_2, \quad Q_{i_2}(\lambda_2) - \varepsilon_2 \lambda_2 \geq Q_{i_1}(\lambda_1) - \varepsilon_2 \lambda_1.$$

Summing these inequalities gives $(\varepsilon_2 - \varepsilon_1)(\lambda_1 - \lambda_2) \geq 0$, so $\lambda_1 \geq \lambda_2$. Since $-\lambda_\varepsilon$ belongs to the subdifferential ∂V_ε , differentiability at ε implies $V'_\varepsilon = -\lambda_\varepsilon$. $\square$

Theorem 5.1 directly yields a finite grid-search guarantee for $\varepsilon > 0$. Let $C = \max_{\mathcal{K}} c - \min_{\mathcal{K}} c$. Since $|(\nu_i - \mu) \cdot x| \leq U \text{TV}(\nu_i, \mu) \leq U$, each branch function $Q_i(\lambda)$ is $(U + C)$ -Lipschitz, and $\lambda \mapsto Q_i(\lambda) - \lambda\varepsilon$ is $(U + C + \varepsilon)$ -Lipschitz. Discretizing $[0, (\nu_i \cdot r)/\varepsilon]$ with mesh size at most $\eta/(U + C + \varepsilon)$ and solving the branch saddle problem to accuracy η at each grid point requires at most

$$m \left(2 + \frac{\bar{\rho}_0(U + C + \varepsilon)}{\varepsilon \eta} \right)$$

convex optimization oracle evaluations, where the primal certificate at the best grid point delivers a feasible contract with global regret at most 2η . When $\mathcal{K} = \text{co}\{(\mu^\ell, c^\ell)\}_{\ell=1}^N$ is a polytope generated by N vertices, the primal problem at each fixed (i, λ) reduces to the finite-dimensional convex program

$$\max_{x \in J^n, t \in \mathbb{R}} t \quad \text{subject to} \quad t \leq \mu^\ell \cdot (r - H(x)) + \lambda[(\nu_i - \mu^\ell) \cdot x + c^\ell - a_i], \quad \ell = 1, \dots, N.$$

6 The Infinite-Multiplier Endpoint

When $\varepsilon = 0$, the supremum in (4.1) ranges over the unbounded half-line $\lambda \in [0, \infty)$. To characterize the asymptotic behavior of $Q_i(\lambda)$ as $\lambda \rightarrow \infty$, we adapt the saddle-value envelope approach of Milgrom and Segal (2002, Theorem 5) and Mills (1956) to our contracting game.

Fix a reference action $k_i = (\nu_i, a_i)$ and define its capped implementability index by

$$\kappa_i = \min_{(\mu, c) \in \mathcal{K}} \{c - a_i + U \text{TV}(\nu_i, \mu)\} \leq 0, \quad (6.1)$$

where the inequality follows from testing $(\mu, c) = k_i \in \mathcal{K}$. Since $\max_{x \in J^n} (\nu_i - \mu) \cdot x = U \text{TV}(\nu_i, \mu)$, bilinear minimax interchange on the compact convex sets $\mathcal{X}$ and $\mathcal{K}$ yields

$$\kappa_i = \max_{x \in \mathcal{X}} \min_{k \in \mathcal{K}} A_i(x, k).$$

Economically, $\kappa_i = 0$ if and only if there exists a feasible contract $x \in \mathcal{X}$ that makes reference action k_i weakly incentive compatible ($s_x(k_i) \geq s_x(k)$) against every action k in the envelope $\mathcal{K}$.

Whenever $\kappa_i = 0$, let

$$\begin{aligned} \mathcal{X}_i &= \{x \in \mathcal{X} : A_i(x, k) \geq 0 \text{ for all } k \in \mathcal{K}\}, \\ \mathcal{E}_i &= \arg \min_{(\mu, c) \in \mathcal{K}} \{c - a_i + U \text{TV}(\nu_i, \mu)\} \end{aligned}$$

denote the nonempty compact convex sets of optimal primal and dual strategies in the zero-value game A_i . Every $x \in \mathcal{X}_i$ satisfies $A_i(x, k) \geq 0$ for all $k \in \mathcal{K}$, and every $k \in \mathcal{E}_i$ satisfies $A_i(x, k) \leq 0$ for all $x \in \mathcal{X}$.

Theorem 6.1 (Exact endpoint characterization and compactified duality). *For each reference branch $i \in \{1, \dots, m\}$:*

(i) *If $\kappa_i < 0$, then $Q_i(\lambda) \leq \max_j r_j + \lambda \kappa_i \rightarrow -\infty$ as $\lambda \rightarrow \infty$.*

(ii) *If $\kappa_i = 0$, then $Q_i(\lambda)$ converges as $\lambda \rightarrow \infty$ to the finite limit*

$$\lim_{\lambda \rightarrow \infty} Q_i(\lambda) = Q_i(\infty) := \max_{x \in \mathcal{X}_i} \min_{k \in \mathcal{E}_i} p_x(k) = \min_{k \in \mathcal{E}_i} \max_{x \in \mathcal{X}_i} p_x(k). \quad (6.2)$$

Consequently, $V_0 = \max_{1 \leq i \leq m} \max_{\lambda \in [0, \infty]} Q_i(\lambda)$ on the compactified interval $[0, \infty]$, where $Q_i(\infty) = -\infty$ when $\kappa_i < 0$. If the global maximum is attained at a finite multiplier $\lambda < \infty$, the associated primal branch maximizer attains V_0 . Conversely, when $\mathcal{K}$ is a polytope, an optimal contract for V_0 exists if and only if some branch attains V_0 at a finite multiplier $\lambda < \infty$.

Proof. If $\kappa_i < 0$, evaluating $F_i(x, k; \lambda)$ at an action $k^* \in \mathcal{K}$ that attains κ_i gives $\min_{k \in \mathcal{K}} F_i(x, k; \lambda) \leq \max_j r_j + \lambda \kappa_i$ for every $x \in \mathcal{X}$, proving (i).

Now suppose $\kappa_i = 0$, and let (x_λ, k_λ) be a saddle pair for $F_i(\cdot, \cdot; \lambda)$. Let $p_{\min}$ and $p_{\max}$ denote the minimum and maximum of $p_x(k)$ on $\mathcal{X} \times \mathcal{K}$, with $D_p = p_{\max} - p_{\min} \leq D$. Testing any $x \in \mathcal{X}_i$ and $k \in \mathcal{E}_i$ in the saddle inequalities shows that $p_{\min} \leq Q_i(\lambda) \leq p_{\max}$ for all $\lambda \geq 0$. Define the nonnegative optimality gaps for the zero-sum game A_i :

$$g_X(x) = \max_{k \in \mathcal{K}} \{-A_i(x, k)\}, \quad g_K(k) = \max_{x \in \mathcal{X}} A_i(x, k) = c - a_i + U \text{TV}(\nu_i, \mu).$$

Because (x_λ, k_λ) is a saddle pair for $F_i(\cdot, \cdot; \lambda)$, we have

$$p_{\min} \leq Q_i(\lambda) \leq p_{\max} - \lambda g_X(x_\lambda), \quad p_{\max} \geq Q_i(\lambda) \geq p_{\min} + \lambda g_K(k_\lambda),$$

which implies $g_X(x_\lambda) \leq D_p/\lambda$ and $g_K(k_\lambda) \leq D_p/\lambda$. By continuity of g_X and g_K , every cluster point of (x_λ, k_λ) as $\lambda \rightarrow \infty$ belongs to $\mathcal{X}_i \times \mathcal{E}_i$. Moreover, for any $x \in \mathcal{X}_i$ and $k \in \mathcal{E}_i$, the sign conditions $A_i(x, k_\lambda) \geq 0$ and $A_i(x_\lambda, k) \leq 0$ imply

$$p_x(k_\lambda) \leq F_i(x, k_\lambda; \lambda) \leq Q_i(\lambda) \leq F_i(x_\lambda, k; \lambda) \leq p_{x_\lambda}(k).$$

Passing to the limit along any convergent subsequence $(x_{\lambda_r}, k_{\lambda_r}) \rightarrow (x_*, k_*) \in \mathcal{X}_i \times \mathcal{E}_i$ with $Q_i(\lambda_r) \rightarrow q$ gives

$$\min_{k \in \mathcal{E}_i} \max_{x \in \mathcal{X}_i} p_x(k) \leq \max_{x \in \mathcal{X}_i} p_x(k_*) \leq q \leq \min_{k \in \mathcal{E}_i} p_{x_*}(k) \leq \max_{x \in \mathcal{X}_i} \min_{k \in \mathcal{E}_i} p_x(k).$$

By Sion's minimax theorem on $\mathcal{X}_i \times \mathcal{E}_i$, the extreme terms are equal, establishing (ii).

Finally, Theorem 4.1 implies that any finite global maximizer (i, λ) yields a contract attaining V_0 . Conversely, suppose $\mathcal{K} = \text{co}\{k^1, \dots, k^N\}$ is a polytope and $x \in \mathcal{X}$ attains $V_0 = v_0(x)$. Let i be an index achieving $\sigma_0(x) = s_x(k_i)$. In the linear program $\min_{k \in \mathcal{K}} \{p_x(k) : s_x(k) \geq \sigma_0(x)\}$, if some vertex k^ℓ satisfies $s_x(k^\ell) > \sigma_0(x)$, strong linear programming duality provides a finite multiplier $\lambda \geq 0$ such that $Q_i(\lambda) \geq \min_{k \in \mathcal{K}} F_i(x, k; \lambda) = v_0(x) = V_0$. If instead $\max_\ell s_x(k^\ell) = \sigma_0(x)$, any finite multiplier satisfying

$$\lambda \geq \max_{\ell: s_x(k^\ell) < \sigma_0(x)} \frac{[v_0(x) - p_x(k^\ell)]_+}{\sigma_0(x) - s_x(k^\ell)}$$

ensures $p_x(k^\ell) + \lambda[\sigma_0(x) - s_x(k^\ell)] \geq v_0(x)$ for every vertex k^ℓ , with equality at the worst-case vertex in $\mathcal{R}_0(x)$, again yielding $Q_i(\lambda) = V_0$ at a finite λ . If the displayed index set is empty, take $\lambda = 0$. $\square$

When k_i is the unique action in $\mathcal{E}_i$, the endpoint game reduces to standard minimum-cost implementation against the action envelope.

Corollary 6.2 (Endpoint as minimum-cost weak implementation). *If $\kappa_i = 0$ and $\mathcal{E}_i = \{k_i\}$, then*

$$Q_i(\infty) = \nu_i \cdot r - \min_{x \in J^n} \left\{ \sum_{j=1}^n \nu_{ij} H(x_j) : s_x(k_i) \geq s_x(k) \text{ for all } k \in \mathcal{K} \right\}.$$

7 Endpoint Error Bounds and Sharp Tolerance Loss

When $\mathcal{K}$ is a polytope, polyhedral error bounds (Hoffman 1952) convert the $O(1/\lambda)$ decay of the game gaps $g_X(x_\lambda)$ and $g_K(k_\lambda)$ into explicit convergence rates for $Q_i(\lambda) \rightarrow Q_i(\infty)$. The following elementary lemma provides a self-contained construction of the relevant Hoffman constant.

Lemma 7.1 (Piecewise-affine polyhedral error bound). *Let P be a compact polytope in a finite-dimensional normed space, and let $g : P \rightarrow [0, \infty)$ be continuous and affine on each cell of a finite polytopal subdivision of P . Suppose $Z = \{z \in P : g(z) = 0\}$ is nonempty. If g is not identically zero, let $\Delta > 0$ denote the minimum positive value of g across the finitely many vertices of the subdivision. Then*

$$\text{dist}(z, Z) \leq \frac{\text{diam}(P)}{\Delta} g(z) \quad \text{for all } z \in P.$$

If $g \equiv 0$, then $Z = P$ and the error bound holds with constant zero.

Proof. If $g \equiv 0$, the conclusion is immediate. Otherwise, fix $z \in P$, select a subdivision cell containing z , and write $z = \sum_v \alpha_v v$ as a convex combination of that cell's vertices. Since g is affine on the cell, $g(z) = \sum_{v:g(v)>0} \alpha_v g(v) \geq \Delta \sum_{v:g(v)>0} \alpha_v$. If the cell contains a vertex $v_0 \in Z$, replacing every positive-value vertex in the convex combination by v_0 produces a point $\bar{z} \in Z$ satisfying $\|z - \bar{z}\| \leq \text{diam}(P) \sum_{v:g(v)>0} \alpha_v \leq \frac{\text{diam}(P)}{\Delta} g(z)$. If the cell contains no vertex in Z , then $g(z) \geq \Delta$, so for any $z_0 \in Z$ we have $\text{dist}(z, Z) \leq \|z - z_0\| \leq \text{diam}(P) \leq \frac{\text{diam}(P)}{\Delta} g(z)$. $\square$

Suppose $\mathcal{K}$ is a polytope and $\kappa_i = 0$. Subdividing $\mathcal{X}$ along the hyperplanes where two envelope vertices have equal scores makes $g_X(x) = \max_{k \in \mathcal{K}} \{-A_i(x, k)\}$ piecewise affine, and subdividing $\mathcal{K}$ along the hyperplanes $\mu_j = \nu_{ij}$ makes $g_K(k) = c - a_i + U \text{TV}(\nu_i, \mu)$ piecewise affine. Lemma 7.1 therefore yields finite constants $h_X, h_K \geq 0$ such that

$$\text{dist}_\infty(x, \mathcal{X}_i) \leq h_X g_X(x) \quad (x \in \mathcal{X}), \quad \text{dist}_1(k, \mathcal{E}_i) \leq h_K g_K(k) \quad (k \in \mathcal{K}),$$

where $\|(\mu, c)\|_1 = \|\mu\|_1 + |c|$. Define $M_r = \max\{\bar{w}, \max_j r_j\}$ and the modulus of continuity of H on J ,

$$\omega_H(t) = \sup\{|H(a) - H(b)| : a, b \in J, |a - b| \leq t\}.$$

Theorem 7.2 (Quantitative endpoint convergence rates). *Suppose $\mathcal{K}$ is a polytope and $\kappa_i = 0$. For every $\lambda > 0$,*

$$Q_i(\infty) - \frac{M_r h_K D}{\lambda} \leq Q_i(\lambda) \leq Q_i(\infty) + \omega_H\left(\frac{h_X D}{\lambda}\right). \quad (7.1)$$

In particular, if H is L_H -Lipschitz on J , the upper bound simplifies to $Q_i(\infty) + L_H h_X D / \lambda$, establishing a two-sided $O(1/\lambda)$ rate of convergence to $Q_i(\infty)$.

Proof. Let (x_λ, k_λ) be a saddle pair for $F_i(\cdot, \cdot; \lambda)$. From the proof of Theorem 6.1, $g_X(x_\lambda) \leq D/\lambda$ and $g_K(k_\lambda) \leq D/\lambda$. Consequently, there exist $\bar{x} \in \mathcal{X}_i$ and $\bar{k} \in \mathcal{E}_i$ satisfying $\|x_\lambda - \bar{x}\|_\infty \leq h_X D / \lambda$ and $\|k_\lambda - \bar{k}\|_1 \leq h_K D / \lambda$. Because $p_x(k)$ is independent of c and $|r_j - H(x_j)| \leq M_r$, the map $k \mapsto p_x(k)$ is M_r -Lipschitz with respect to $\|\cdot\|_1$ uniformly in $x \in \mathcal{X}$. Thus,

$$Q_i(\lambda) \geq \max_{x \in \mathcal{X}_i} p_x(k_\lambda) \geq \max_{x \in \mathcal{X}_i} p_x(\bar{k}) - \frac{M_r h_K D}{\lambda} \geq Q_i(\infty) - \frac{M_r h_K D}{\lambda}.$$

Symmetrically, since $|p_{x_\lambda}(k) - p_{\bar{x}}(k)| \leq \omega_H(\|x_\lambda - \bar{x}\|_\infty)$ for all $k \in \mathcal{K}$, we have

$$Q_i(\lambda) \leq \min_{k \in \mathcal{E}_i} p_{x_\lambda}(k) \leq \min_{k \in \mathcal{E}_i} p_{\bar{x}}(k) + \omega_H\left(\frac{h_X D}{\lambda}\right) \leq Q_i(\infty) + \omega_H\left(\frac{h_X D}{\lambda}\right).$$

$\square$

Theorem 7.2 has two immediate implications. First, when $\mathcal{K}$ is a polytope and H is Lipschitz, the exact-response value V_0 can be computed to any prescribed tolerance $\eta > 0$ in finitely many steps: branches with $\kappa_i < 0$ are truncated via Theorem 6.1(i), branches with $\kappa_i = 0$ are truncated via (7.1) after solving the linear-convex endpoint game (6.2), and the remaining compact multiplier intervals are searched via the Lipschitz grid of Section 5. Second, the multiplier regime of V_0 governs the rate at which V_ε approaches V_0 as $\varepsilon \downarrow 0$.

Corollary 7.3 (Linear versus square-root tolerance loss). *If V_0 is attained at a finite multiplier $\lambda^* < \infty$, then for all $\varepsilon > 0$,*

$$0 \leq V_0 - V_\varepsilon \leq \lambda^* \varepsilon.$$

If $\mathcal{K}$ is a polytope and $V_0 = Q_i(\infty)$ is attained at the infinite-multiplier endpoint of branch i , let $B_i = M_r h_K D$. Then for all $\varepsilon > 0$,

$$0 \leq V_0 - V_\varepsilon \leq 2\sqrt{B_i \varepsilon}.$$

Proof. If (i, λ^*) attains V_0 , then (4.1) immediately gives $V_\varepsilon \geq Q_i(\lambda^*) - \lambda^* \varepsilon = V_0 - \lambda^* \varepsilon$. If $V_0 = Q_i(\infty)$, the lower bound in (7.1) implies $V_\varepsilon \geq Q_i(\lambda) - \lambda \varepsilon \geq V_0 - B_i/\lambda - \lambda \varepsilon$ for every $\lambda > 0$. Minimizing $B_i/\lambda + \lambda \varepsilon$ over $\lambda > 0$ by setting $\lambda = \sqrt{B_i/\varepsilon}$ yields $V_0 - V_\varepsilon \leq 2\sqrt{B_i \varepsilon}$ (and sending $\lambda \downarrow 0$ gives $V_\varepsilon = V_0$ when $B_i = 0$). $\square$

While square-root robustness losses also appear under payoff perturbations (Carroll and Meng 2016) and approximate best responses (Bacchiocchi et al. 2025), the next proposition proves that the $\sqrt{\varepsilon}$ rate is exact in a canonical two-action model and links its prefactor directly to the marginal cost of providing agent utility.

Proposition 7.4 (Sharp square-root loss in a two-action model). *Suppose $Y = \{0, 1\}$ with revenues 0 and $r > 0$, and let $\mathcal{K} = \text{co}\{(\delta_0, 0), (\delta_1, c)\}$ be the segment connecting a known null action and a known productive action with $0 < c < U$ and $A := r - H(c) > 0$. Then $V_0 = A$, this supremum is not attained, $V_\varepsilon = 0$ for $\varepsilon \geq U - c$, and for $0 < \varepsilon < U - c$,*

$$V_\varepsilon = \max \left\{ 0, \max_{\varepsilon \leq h \leq U-c} \left(1 - \frac{\varepsilon}{h} \right) [r - H(c+h)] \right\}. \quad (7.2)$$

If H is continuously differentiable in a neighborhood of c with $H'(c) > 0$, then as $\varepsilon \downarrow 0$,

$$V_0 - V_\varepsilon = 2\sqrt{AH'(c)\varepsilon} + o(\sqrt{\varepsilon}), \quad h_\varepsilon \sim \sqrt{\frac{A\varepsilon}{H'(c)}},$$

and the productive-branch dual value satisfies

$$Q(\lambda) = A - \frac{AH'(c)}{\lambda} + o(1/\lambda) \quad \text{as } \lambda \rightarrow \infty.$$

Proof. By Proposition 4.2, it suffices to set $x_0 = 0$. Any contract yielding positive payoff must set $x_1 = c + h$ with $h \in (0, U - c]$ and $r - H(c + h) > 0$. Under this contract, a mixture placing weight $t \in [0, 1]$ on the productive action yields agent score th and principal payoff $t[r - H(c + h)]$. Since $\sigma_0(x) = h$, the ε -response condition $th \geq h - \varepsilon$ restricts t to $[(1 - \varepsilon/h)_+, 1]$. Because $r - H(c + h) > 0$, the worst-case response chooses $t = (1 - \varepsilon/h)_+$, which yields (7.2). If $\varepsilon \geq U - c$, the null action is admissible for every contract supported on the productive outcome, so the optimal value is zero. When $\varepsilon = 0$, every $h > 0$ yields $r - H(c + h) < A$, whereas $h = 0$ induces a tie with the null action and yields 0; letting $h \downarrow 0$ shows $V_0 = A$ without attainment.

Let $b = H'(c) > 0$. Evaluating (7.2) at $h = \sqrt{A\varepsilon/b}$ and expanding $H(c+h) = H(c) + bh + o(h)$ gives $V_\varepsilon \geq A - 2\sqrt{Ab\varepsilon} + o(\sqrt{\varepsilon})$. Conversely, any maximizer h_ε of (7.2) must satisfy $h_\varepsilon \rightarrow 0$ as $\varepsilon \downarrow 0$, and the loss $A - (1 - \varepsilon/h)[r - H(c + h)]$ is bounded below by a positive multiple of $h + \varepsilon/h$. Thus $h_\varepsilon/\sqrt{\varepsilon}$ stays bounded and bounded away from zero, where the uniform expansion $A - V_\varepsilon = \min_h [bh + A\varepsilon/h] + o(\sqrt{\varepsilon})$ yields $h_\varepsilon \sim \sqrt{A\varepsilon/b}$ and $V_0 - V_\varepsilon = 2\sqrt{Ab\varepsilon} + o(\sqrt{\varepsilon})$.

Finally, for the productive reference branch at large λ , the primal optimum equates the payoffs against the two extreme actions: $Q(\lambda) = \lambda h = r - H(c + h)$ with $h > 0$. Since $h \mapsto \lambda h$ is strictly increasing and $h \mapsto r - H(c + h)$ is strictly decreasing, the solution is unique and satisfies $h(\lambda) = A/\lambda + o(1/\lambda)$, which gives $Q(\lambda) = A - Ab/\lambda + o(1/\lambda)$. $\square$

8 The Value of Performance Information

Suppose the principal can condition wages only on a coarsened performance statistic $T : Y \rightarrow \{1, \dots, q\}$ that partitions the outcome space into cells $C_1, \dots, C_q$. For any distribution $\mu \in \Delta_n$, let $T\mu \in \Delta_q$ denote the pushforward distribution with cell probabilities $(T\mu)_t = \mu(C_t) = \sum_{j \in C_t} \mu_j$. A T -measurable (coarse) contract assigns a common utility promise $x_j = \bar{x}_t$ to all outcomes $j \in C_t$; we denote the corresponding optimal robust value by V_ε^T .

Proposition 8.1 (Data processing and lossless pooling under wage bounds). *For every pair of distributions $(\nu, \mu) \in \Delta_n \times \Delta_n$ and every $\lambda \geq 0$,*

$$D_{f_\lambda}(\nu \parallel \mu) \geq D_{f_\lambda}(T\nu \parallel T\mu).$$

Equality holds if and only if, within each cell C_t , there exists a common transfer $w_t \in [0, \bar{w}]$ that simultaneously maximizes

$$w \mapsto \lambda(\nu_j - \mu_j)u(w) - \mu_j w \quad \text{for all } j \in C_t.$$

In a cell with $\mu(C_t) > 0$, constant likelihood ratios ν_j/μ_j on the support of μ are sufficient for equality provided that $\nu_j = 0$ whenever $\mu_j = 0$ in that cell. Cells with $\mu(C_t) = 0$ always satisfy equality. Constant likelihood ratios are not necessary: outcomes with distinct likelihood ratios can be pooled without loss whenever all of them call for the zero wage floor or all of them call for the wage cap $\bar{w}$.

Proof. Within each cell C_t , $D_{f_\lambda}(\nu \parallel \mu)$ sums the individual maxima $\max_{w_j \in [0, \bar{w}]} [\lambda(\nu_j - \mu_j)u(w_j) - \mu_j w_j]$ over $j \in C_t$, whereas $D_{f_\lambda}(T\nu \parallel T\mu)$ maximizes the summed objective $\sum_{j \in C_t} [\lambda(\nu_j - \mu_j)u(w) - \mu_j w]$ over a single transfer $w \in [0, \bar{w}]$. Since the maximum of a sum is at most the sum of the maxima, the inequality holds without any support restrictions, and equality holds if and only if the cell maximizer w_t attains the maximum for every summand $j \in C_t$ (outcomes with $\nu_j = \mu_j = 0$ vanish identically and impose no restriction). $\square$

In the coarse contracting problem, the branch dual $Q_i^T(\lambda)$ retains the fine-outcome expected revenue $\mu \cdot r$ and cost difference $\lambda(c - a_i)$, replacing only $D_{f_\lambda}(\nu_i \parallel \mu)$ with the aggregated divergence $D_{f_\lambda}(T\nu_i \parallel T\mu)$. Define the nonnegative divergence loss at action $k = (\mu, c)$ by

$$L_i(\lambda, k) = D_{f_\lambda}(\nu_i \parallel \mu) - D_{f_\lambda}(T\nu_i \parallel T\mu) \geq 0.$$

If k_F and k_T minimize the fine and coarse branch duals at (i, λ) , respectively, standard envelope comparisons give

$$L_i(\lambda, k_F) \leq Q_i(\lambda) - Q_i^T(\lambda) \leq L_i(\lambda, k_T). \quad (8.1)$$

The equality $L_i(\lambda, k_T) = 0$ implies $Q_i(\lambda) = Q_i^T(\lambda)$. Equality at the fine minimizer, $L_i(\lambda, k_F) = 0$, alone does not imply equality of the branch values, since a different action may minimize the coarse dual. Applying these comparisons at the optimizing branch-multiplier pairs gives bounds on the value of finer information.

Corollary 8.2 (Two-sided bounds on the value of finer information). *Fix $\varepsilon > 0$, and let $\theta_F = (i_F, \lambda_F)$ and $\theta_T = (i_T, \lambda_T)$ be optimal branch-multiplier pairs for V_ε and V_ε^T , respectively. If $k_T(\theta_F)$ minimizes the coarse branch dual at θ_F and $k_F(\theta_T)$ minimizes the fine branch dual at θ_T , then*

$$L_{i_T}(\lambda_T, k_F(\theta_T)) \leq V_\varepsilon - V_\varepsilon^T \leq L_{i_F}(\lambda_F, k_T(\theta_F)). \quad (8.2)$$

The same bounds hold at $\varepsilon = 0$ whenever both V_0 and V_0^T admit finite optimal multipliers.

Proof. Suboptimality of θ_T in the fine problem and θ_F in the coarse problem gives $Q_{i_T}(\lambda_T) - Q_{i_T}^T(\lambda_T) \leq V_\varepsilon - V_\varepsilon^T \leq Q_{i_F}(\lambda_F) - Q_{i_F}^T(\lambda_F)$, where the $-\lambda\varepsilon$ terms cancel pairwise. Applying (8.1) at θ_T and θ_F yields (8.2). $\square$

To quantify the loss for an approximately sufficient statistic, fix reconstruction distributions $q_1, \dots, q_q$, with q_t supported on C_t . For any output law $\mu \in \Delta_n$, define its reconstructed law $\hat{\mu} \in \Delta_n$ and the worst-case reconstruction error across the envelope by

$$\hat{\mu}_j = \mu(C_t)q_t(j) \quad (j \in C_t), \quad \delta = \sup_{(\mu, c) \in \mathcal{K}} \text{TV}(\mu, \hat{\mu}),$$

When $\delta = 0$, the conditional law of the fine outcome given $T = t$ equals q_t under every action assigning positive probability to that cell. Thus T is a sufficient statistic in the sense of Blackwell (1953).

Theorem 8.3 (Exact and approximate statistical sufficiency). *For every $\varepsilon \geq 0$,*

$$V_\varepsilon^T \geq V_{\varepsilon+2U\delta} - \bar{w}\delta. \quad (8.3)$$

In particular, if $\delta = 0$, then $V_\varepsilon^T = V_\varepsilon$ for all $\varepsilon \geq 0$, so restricting attention to T -measurable contracts is lossless. When T is binary, every coarse contract is affine in utility as a function of T ; this representation does not require attainment of the optimum.

More generally, for every $\varepsilon > 0$,

$$0 \leq V_\varepsilon - V_\varepsilon^T \leq \left(\bar{w} + \frac{2U\bar{\rho}_0}{\varepsilon} \right) \delta. \quad (8.4)$$

At $\varepsilon = 0$, if V_0 is attained at a finite multiplier $\lambda^* < \infty$, then $0 \leq V_0 - V_0^T \leq (\bar{w} + 2U\lambda^*)\delta$; if $\mathcal{K}$ is a polytope and $V_0 = Q_i(\infty)$ is attained at an infinite-multiplier endpoint with constant $B_i = M_r h_K D$, then

$$0 \leq V_0 - V_0^T \leq \bar{w}\delta + 2\sqrt{2B_i U \delta}.$$

Proof. Given any fine contract $x \in \mathcal{X}$, define the coarse contract $x^T \in \mathcal{X}$ by cell-wise averaging against the reconstruction kernel: $x_j^T = \sum_{\ell \in C_t} q_t(\ell) x_\ell$ for all $j \in C_t$. For any action $k = (\mu, c) \in \mathcal{K}$, $\mu \cdot x^T = \hat{\mu} \cdot x$, so

$$|s_{x^T}(k) - s_x(k)| = |\hat{\mu} \cdot x - \mu \cdot x| \leq U \text{TV}(\mu, \hat{\mu}) \leq U\delta.$$

Because the same bound applies to every reference action k_i , we have $|\sigma_0(x^T) - \sigma_0(x)| \leq U\delta$, which implies $\mathcal{R}_\varepsilon(x^T) \subseteq \mathcal{R}_{\varepsilon+2U\delta}(x)$. Moreover, by Jensen's inequality applied to the convex function H ,

$$\sum_{j=1}^n \mu_j H(x_j^T) \leq \sum_{j=1}^n \hat{\mu}_j H(x_j) \leq \sum_{j=1}^n \mu_j H(x_j) + \bar{w}\delta,$$

so $p_{x^T}(k) \geq p_x(k) - \bar{w}\delta$ for every $k \in \mathcal{K}$. Combining the response-set inclusion with this payoff bound gives $v_\varepsilon(x^T) \geq v_{\varepsilon+2U\delta}(x) - \bar{w}\delta$. Taking the supremum over $x \in \mathcal{X}$ proves (8.3).

When $\varepsilon > 0$, Theorem 5.1 ensures that V_ε is attained at a multiplier $\lambda_\varepsilon \leq \bar{\rho}_0/\varepsilon$, so convexity of $\varepsilon \mapsto V_\varepsilon$ yields $V_\varepsilon - V_{\varepsilon+2U\delta} \leq 2U\delta\lambda_\varepsilon \leq (2U\bar{\rho}_0/\varepsilon)\delta$, which combines with (8.3) to give (8.4). At $\varepsilon = 0$, combining (8.3) with the finite- and infinite-multiplier bounds for $V_0 - V_{2U\delta}$ in Corollary 7.3 yields the final two inequalities. $\square$

Theorem 8.3 varies the contractible signal. Technological information enters separately through the known actions and the envelope.

Proposition 8.4 (Comparative statics in technological information). *For any response tolerance $\varepsilon \geq 0$ and any fixed class of admissible contracts:*

- (i) *holding the envelope $\mathcal{K}$ fixed, enlarging the reference set $\mathcal{K}_0$ weakly increases the optimal robust value;*
- (ii) *holding the reference set $\mathcal{K}_0$ fixed, shrinking the envelope $\mathcal{K}$ (while retaining $\mathcal{K}_0 \subseteq \mathcal{K}$) weakly increases the optimal robust value.*

Proof. Adding reference actions weakly increases $\sigma_0(x)$ and thus shrinks $\mathcal{R}_\varepsilon(x)$ at every contract x . Likewise, shrinking $\mathcal{K}$ while preserving $\mathcal{K}_0$ shrinks $\mathcal{R}_\varepsilon(x)$. In both cases, minimizing $p_x(k)$ over a smaller response set weakly increases $v_\varepsilon(x)$ for every x . $\square$

9 Global Computation and Numerical Examples

9.1 Primal, dual, and interval bounds

When $\mathcal{K} = \text{co}\{k^1, \dots, k^N\}$ is a polytope, primal epigraph programs and dual convex mixtures give analytical lower and upper bounds on V_ε . At a fixed reference i and multiplier λ , a feasible

contract in the program from Section 5 gives a lower bound on $Q_i(\lambda)$. Any convex weight vector $\omega \in \Delta_N$ generates a feasible action $k(\omega) = (\mu(\omega), c(\omega)) = \sum_{\ell=1}^N \omega_\ell k^\ell \in \mathcal{K}$ and an upper bound

$$\Phi_i(\lambda, \omega) = \mu(\omega) \cdot r + \lambda[c(\omega) - a_i - \varepsilon] + D_{f_\lambda}(\nu_i \| \mu(\omega)) \geq Q_i(\lambda) - \lambda\varepsilon.$$

The closed-perspective divergence is convex in λ , including at zero-mass coordinates. Thus $\lambda \mapsto \Phi_i(\lambda, \omega)$ is convex for every fixed $\omega \in \Delta_N$, and on a compact interval $[a, b] \subset [0, \infty)$,

$$\sup_{\lambda \in [a, b]} [Q_i(\lambda) - \lambda\varepsilon] \leq \max\{\Phi_i(a, \omega), \Phi_i(b, \omega)\}. \quad (9.1)$$

We take the smallest upper bound supplied by (9.1), the Lipschitz bound from Section 5, and the cap $\nu_i \cdot r - a\varepsilon$. Adaptive interval subdivision continues until the gap between the global upper bound and the best feasible-contract lower bound is at most the prescribed tolerance.

To evaluate the guaranteed payoff $v_\varepsilon(x)$ of a candidate contract x across the entire convex hull $\mathcal{K}$, let $d_\ell = s_x(k^\ell)$, $p_\ell = p_x(k^\ell)$, and $\sigma = \sigma_0(x)$. The adversarial response problem is the linear program

$$v_\varepsilon(x) = \min_{\omega \in \Delta_N} \left\{ \sum_{\ell=1}^N \omega_\ell p_\ell : \sum_{\ell=1}^N \omega_\ell d_\ell \geq \sigma - \varepsilon \right\}.$$

By weak linear programming duality, any scalar multiplier $\gamma \geq 0$ provides a valid lower bound on the contract's worst-case performance across all mixtures in $\mathcal{K}$:

$$v_\varepsilon(x) \geq \min_{1 \leq \ell \leq N} \{p_\ell + \gamma(\sigma - \varepsilon - d_\ell)\}. \quad (9.2)$$

For exact best responses ($\varepsilon = 0$), when $\kappa_i = 0$ and $\mathcal{E}_i = \{k_i\}$, dual multipliers from the endpoint implementation problem provide an analytical tail bound that is often much sharper than worst-case Hoffman constants. Specifically, suppose nonnegative weights $(\eta_\ell)_{\ell=1}^N$ with $S = \sum_{\ell=1}^N \eta_\ell > 0$ and constants J_i^+, C_i satisfy

$$J_i^+ \geq \max_{x \in \mathcal{X}} \left[p_x(k_i) + \sum_{\ell=1}^N \eta_\ell A_i(x, k^\ell) \right], \quad C_i \geq \max_{x \in \mathcal{X}} \sum_{\ell=1}^N \eta_\ell [p_x(k^\ell) - p_x(k_i)].$$

Then for every $\lambda \geq S$, testing the feasible mixture $k(\lambda) = (1 - S/\lambda)k_i + \sum_{\ell=1}^N (\eta_\ell/\lambda)k^\ell \in \mathcal{K}$ in the branch dual yields the explicit tail bound

$$Q_i(\lambda) \leq J_i^+ + \frac{C_i}{\lambda} \quad (\lambda \geq S). \quad (9.3)$$

The experiments use Python, NumPy, SciPy, and Matplotlib. The code, synthetic inputs, computed results, and reproduction instructions are available from the corresponding author. The stopping tolerance for the global value gap is 5×10^{-5} , tightened to 10^{-6} for the exact-response comparison in Table 2. Candidate contracts are evaluated via (9.2) at tolerance $\varepsilon + 10^{-9}$. Bounds are recomputed from feasible wage vectors and simplex weights, with a concave-tangent correction in conjugate upper bounds and a numerical allowance of 2×10^{-8} in interval upper bounds. These computations use floating-point arithmetic without rigorous interval enclosures. Table endpoints are rounded outward from the stored values. Neither the numerical allowance nor decimal rounding certifies all preceding arithmetic operations.

9.2 A four-outcome example with an interior optimal multiplier

Consider four outcomes with revenues $Y = \{0, 1, 2, 4\}$, a wage cap $\bar{w} = 4$, and normalized CARA utility $u_a(w) = (1 - e^{-aw})/a$ (with $u_0(w) = w$). Table 1 specifies the five extreme actions generating the envelope $\mathcal{K}$; the reference set $\mathcal{K}_0$ consists of the Null action and the productive

Table 1: Vertices of the action envelope $\mathcal{K}$ in the four-outcome numerical example.

Action	$P(0)$	$P(1)$	$P(2)$	$P(4)$	Cost	Gross revenue
Null	1	0	0	0	0	0
Reference	.10	.15	.35	.40	.45	2.45
Unknown A , lower cost	.35	.15	.25	.25	.05	1.65
Unknown B	.20	.50	.20	.10	.10	1.30
Unknown A , upper cost	.35	.15	.25	.25	.40	1.65

Reference action. The two vertices sharing output distribution P_A generate a cost interval $[0.05, 0.40]$ for unknown action A .

At risk-aversion parameter $a = 0.5$, the productive reference action satisfies $\kappa_{\text{prod}} = 0$ and $\mathcal{E}_{\text{prod}} = \{k_{\text{prod}}\}$. Solving the endpoint implementation problem gives

$$Q_{\text{prod}}(\infty) = 2.45 - 0.7 \log(30/7) - 0.8 \log(45/8) \approx 0.04952218,$$

supported by endpoint wages $(0, 0, 2 \log(30/7), 2 \log(45/8))$ and a single active dual multiplier $\eta_{A,\text{low}} = 15$ against the lower-cost action A . Applying (9.3) with $S = 15$ and $C_{\text{prod}} = 3$ implies that for all $\lambda \geq 15$,

$$Q_{\text{prod}}(\lambda) \leq Q_{\text{prod}}(\infty) + \frac{3}{\lambda} \leq 0.24952218.$$

The null branch is bounded above by zero. At $\varepsilon = 0$, the feasible utility-affine contract $x_j = r_j/4$ has reference threshold $13/80$; substituting $\gamma = 2$ into (9.2) gives a payoff lower bound of $13/40$. This exceeds the tail upper bound above, so $\lambda \geq 15$ can be excluded for both unrestricted and utility-affine contracts.

Table 2: Global bounds on the optimal robust value at $a = 0.5$ (rounded outward). Utility-affine contracts take the form $u(w(y)) = \alpha + \beta y$.

ε	Contract family	Lower bound	Upper bound
0	All contracts	0.64923786	0.64923859
0	Utility-affine	0.62494654	0.62494723
0.05	All contracts	0.55803316	0.55806773
0.05	Utility-affine	0.53420835	0.53424019

Table 2 compares unrestricted and utility-affine value bounds. At $\varepsilon = 0$, they imply a gain of at least 0.02429 for unrestricted contracts at the reported numerical accuracy. The best computed multiplier is approximately 1.93, with recovered wages $(0, 0, 1.2289, 1.7413)$. The utility-affine solution has wages $(0, 0.3450, 0.7621, 2.0087)$. Requiring utility to scale with revenue forces payment at outcome 1, where the reference has no likelihood advantage over actions A and B , and shifts more compensation to outcome 4. Under normalized CARA utility, the interior wage rule is

$$w^*(y) = a^{-1} \log \left[\lambda^* \left(\frac{\nu_{\text{prod}}(y)}{\mu^*(y)} - 1 \right) \right],$$

truncated at 0 when the conjugate argument is at most 1, and at $\bar{w}$ when it reaches $e^{a\bar{w}}$. Figure 1 illustrates the rule at $\varepsilon = 0.05$, where the recovered adverse distribution mixes the null action with lower-cost action A .

9.3 Comparative statics in risk aversion and information

Table 3 and Figure 2 vary the normalized CARA coefficient over $a \in [0, 1.25]$ at $\varepsilon = 0.05$. Values decline with risk aversion, as Proposition 4.4 predicts. At $a = 1.25$, the utility-affine upper bound

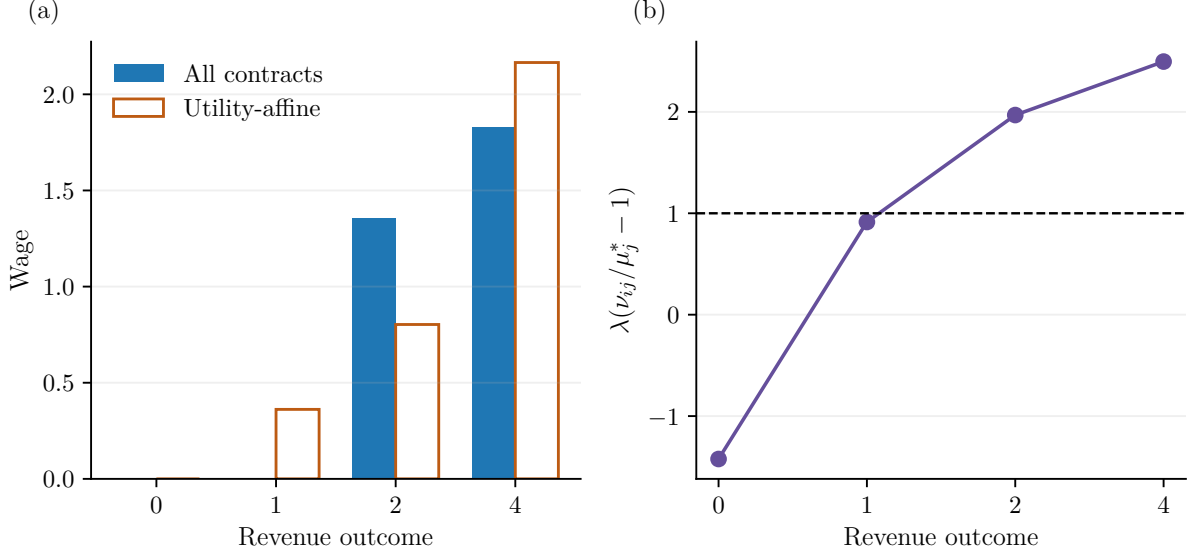


Figure 1: Recovered wages and likelihood arguments at $a = 0.5$, $\varepsilon = 0.05$. (a) Unrestricted and utility-affine wages. (b) The conjugate argument $\lambda(\nu_{ij}/\mu_j^* - 1)$ for the productive reference i . The dashed line marks the threshold of one for positive CARA wages

is 2×10^{-8} , while the unrestricted value interval is $[0.16866, 0.16871]$.

Table 3: Optimal robust value bounds across normalized CARA coefficients a at $\varepsilon = 0.05$.

a	All contracts	Utility-affine in revenue
0	[0.97145, 0.97151]	[0.79005, 0.79011]
0.1	[0.80941, 0.80945]	[0.74579, 0.74584]
0.25	[0.70024, 0.70028]	[0.67355, 0.67361]
0.5	[0.55803, 0.55807]	[0.53420, 0.53425]
0.75	[0.42612, 0.42615]	[0.36588, 0.36590]
1	[0.29794, 0.29799]	[0.16804, 0.16809]
1.25	[0.16866, 0.16871]	$[0, 2 \times 10^{-8}]$

Technological information affects the value through both the envelope and the reference set. Raising the lower bound on action A 's cost from 0 to 0.40 shrinks $\mathcal{K}$ and increases the numerical value from 0.5072 to 1.1911, as predicted by Proposition 8.4. The reported intervals coincide at the sampled lower bounds 0.30 and 0.40 (Figure 3(a)). With $\mathcal{K}$ fixed, learning that action B is available raises the value from 0.5580 to 0.7287, while learning that lower-cost action A is available raises it to 1.0447 (Table 4). Adding B to the latter reference set gives the same reported interval; any further gain is below 3.8×10^{-5} at the stated numerical accuracy.

Table 5 and Figure 3(b) vary the contractible statistic $T(Y)$. Pooling outcomes 0 and 1 gives coincident value intervals, placing the information loss below 3.5×10^{-5} at the stated numerical accuracy. Both outcomes receive zero wages in the recovered unrestricted contract, despite their distinct likelihood ratios. Pooling into the two cells $\{0, 1\}$ and $\{2, 4\}$ reduces the value to 0.5535. A bonus based only on whether the top outcome 4 occurs gives a value of 0.3436.

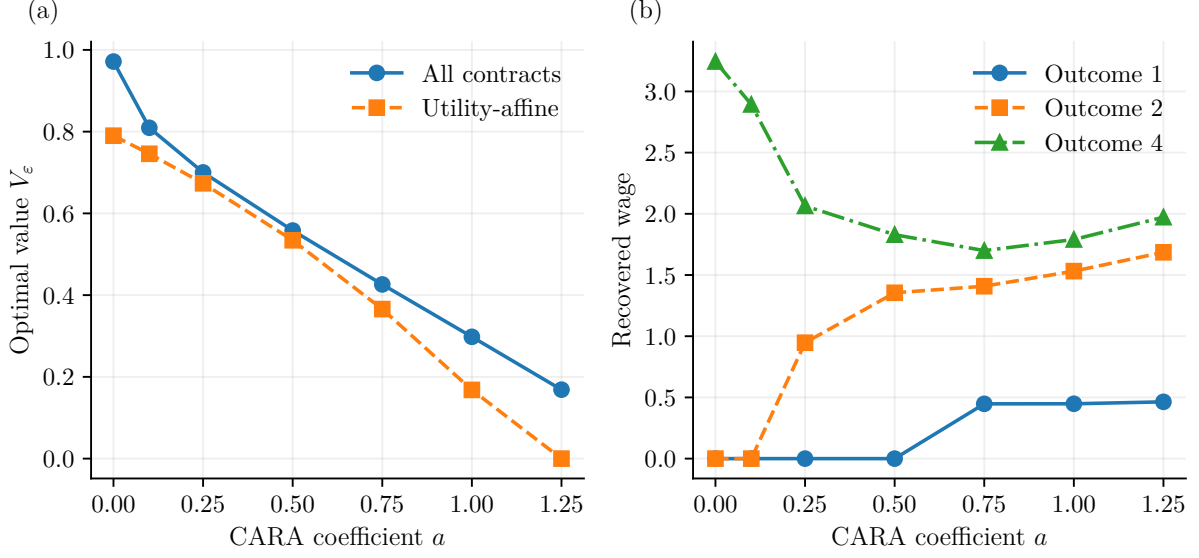


Figure 2: Normalized CARA comparisons at $\varepsilon = 0.05$. (a) Value bounds, whose widths are smaller than the markers. (b) Recovered unrestricted wages

Table 4: Effect of expanding the known reference set $\mathcal{K}_0$ at $a = 0.5$ and $\varepsilon = 0.05$.

Known reference actions	Lower bound	Upper bound
Initial references (Null, Reference)	0.55803316	0.55806773
Add action B	0.72866166	0.72868228
Add action A (lower cost)	1.04473610	1.04477369
Add both A and B	1.04473610	1.04477369

Table 5: Optimal robust value under coarsened performance signals at $a = 0.5$ and $\varepsilon = 0.05$.

Contractible partition of $\{0, 1, 2, 4\}$	Lower bound	Upper bound
Full outcome $\{\{0\}, \{1\}, \{2\}, \{4\}\}$	0.55803316	0.55806773
Pool unpaid outcomes $\{\{0, 1\}, \{2\}, \{4\}\}$	0.55803316	0.55806773
Two pooled cells $\{\{0, 1\}, \{2, 4\}\}$	0.55346923	0.55350481
Top-outcome bonus $\{\{0, 1, 2\}, \{4\}\}$	0.34362673	0.34365273

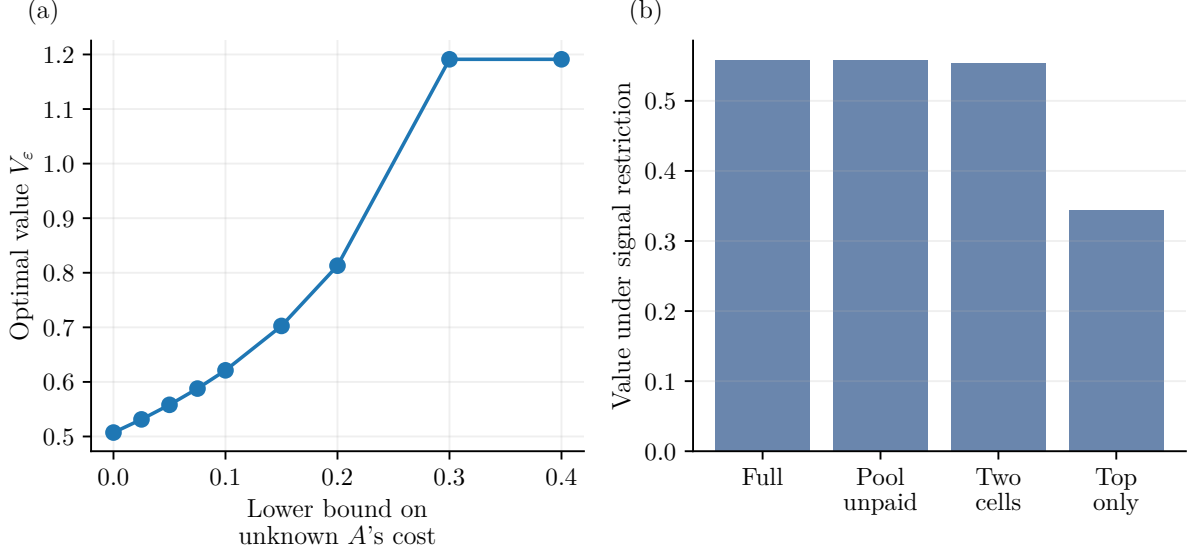


Figure 3: (a) Robust value as the lower bound on action A 's cost increases. (b) Robust value for the four contractible partitions in Table 5

9.4 Asymptotic behavior under vanishing response tolerance

For the two-action model in Proposition 7.4, set $r = 1$, $c = 0.15$, and $\bar{w} = 2$. Under square-root utility $u(w) = \sqrt{w}$, the interior optimal increment h_ε solves

$$2h^3 + (2c - \varepsilon)h^2 - \varepsilon(1 - c^2) = 0,$$

Under risk-neutral utility with a nonbinding cap, $V_\varepsilon = (\sqrt{r - c} - \sqrt{\varepsilon})_+^2$. Figure 4 evaluates these scalar formulas over $10^{-7} \leq \varepsilon \leq 0.1$. The ratio $(V_0 - V_\varepsilon)/(2\sqrt{AH'(c)\varepsilon})$ approaches one, consistent with the proved asymptotic formula.

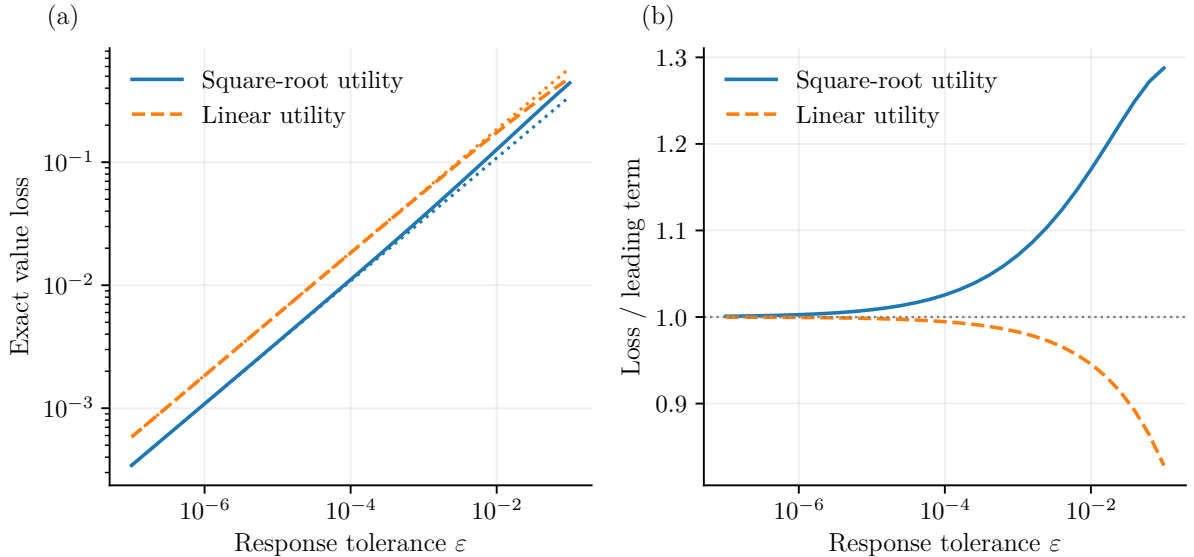


Figure 4: Two-action tolerance loss. (a) Numerical evaluation of the exact scalar value formulas, with dotted lines denoting $2\sqrt{AH'(c)\varepsilon}$. (b) Loss divided by this leading term

Figure 5(a) shows the four-outcome example as ε decreases from 0.20 to 0.005. The computed values approach $V_0 \approx 0.64924$ approximately linearly, and the recovered multipliers approach 1.93.

These observations agree with the finite-multiplier loss bound in Corollary 7.3. Panel (b) shows the risk-sharing example in Appendix A, with revenues $\{0, 1, 2\}$, $u(w) = \sqrt{w}$, and productive action $(\frac{1}{2}\delta_1 + \frac{1}{2}\delta_2, \frac{1}{2})$. Its unrestricted and utility-affine values approach the nonattained suprema $5/4$ and $11/9$, respectively. Plotting against $\sqrt{\varepsilon}$ shows the square-root behavior and the limiting gap $1/36$.

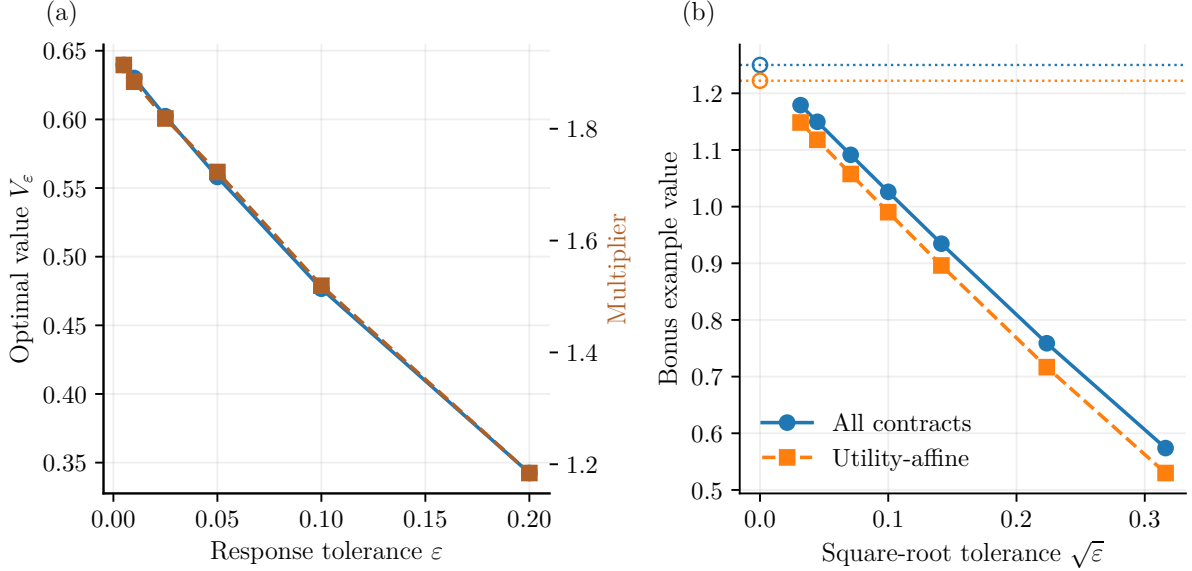


Figure 5: Computed values as $\varepsilon \downarrow 0$. (a) Four-outcome estimates: circles and the solid line use the left value axis; squares and the dashed line use the right multiplier axis. (b) Three-outcome risk-sharing values against $\sqrt{\varepsilon}$; open markers denote the nonattained zero-tolerance suprema

10 Conclusion

For bounded deterministic wages on a finite outcome space, robust contracting with a risk-averse agent admits an exact dual representation. The agent’s utility and the wage bounds determine an endogenous divergence between reference and adverse output distributions. Under strictly concave utility, dual minimizers recover finite-multiplier branch contracts through a truncated likelihood-ratio rule.

Positive response tolerance ensures existence, while the infinite-multiplier endpoint accounts for possible nonattainment at zero tolerance. A finite optimal multiplier gives a linear tolerance-loss bound; an optimal endpoint in a polyhedral envelope gives a square-root bound. The two-action example attains the square-root rate asymptotically. Coarsening contractible information aggregates the endogenous divergence. Wage bounds permit lossless pooling of distinct likelihood ratios, and approximate statistical sufficiency gives quantitative information-loss bounds.

The numerical examples distinguish the gains from better technological knowledge from those obtained through richer performance signals. They also show that requiring utility to be affine in revenue can impose a positive cost even when the unrestricted optimal wage is monotone.

A Properties and Limitations of Utility-Affine Contracts

A.1 Optimality of zero utility intercepts

If all revenues are zero, the zero contract is optimal with value zero. In this subsection assume $\max_j r_j > 0$. In the examples that follow, outcomes are labeled by their revenues, and δ_y denotes the point mass at revenue y .

Consider utility-affine contracts $x_j = \alpha + br_j \in [0, U]$, where $\min_j r_j = 0$. Subtracting a positive common constant from all coordinates preserves score differences and hence $\mathcal{R}_\varepsilon(x)$, while strictly reducing wages. Thus a contract with $b \geq 0$ and $\alpha > 0$ is weakly dominated by $\tilde{x}_j = br_j$. If $b < 0$, the zero-revenue outcome receives the highest utility payment. The null action then maximizes the score over $\mathcal{K}$ and yields a nonpositive principal payoff. Since the zero contract guarantees at least zero, optimization over the utility-affine class reduces to

$$x_j = br_j, \quad 0 \leq b \leq \frac{U}{\max_j r_j}.$$

A.2 Strict failure of utility-affine contracts under nonmonotone likelihoods

Suppose $Y = \{0, 1, 10\}$, with outcomes labeled by their revenues, transfer utility $u(w) = \sqrt{w}$, and wage cap $\bar{w} = 100$. Let the envelope $\mathcal{K}$ be the convex hull of three actions:

$$k_0 = (\delta_0, 0), \quad k_s = (\delta_1, 1/2), \quad k_r = (0.8\delta_0 + 0.2\delta_{10}, 4/5),$$

where the reference set is $\mathcal{K}_0 = \{k_0, k_s\}$. Under any utility-affine contract $x = br$ with $b \geq 0$, monetary wages are $(0, b^2, 100b^2)$, and the scores of k_0, k_s, k_r are $0, b-0.5$, and $2b-0.8$, respectively. If $b < 0.4$, then $0 > 2b-0.8$ and $0 > b-0.5$, so the null action k_0 attains the reference threshold $\sigma_0(x) = 0$ and yields zero payoff. If $b \geq 0.4$, then $2b-0.8 \geq \max\{0, b-0.5\} = \sigma_0(x)$, so the risky action k_r is an admissible exact response and yields principal payoff $0.2(10 - 100b^2) = 2 - 20b^2 \leq -1.2 < 0$. Hence no utility-affine contract can guarantee more than zero.

By contrast, consider the unrestricted monotone bonus contract $w = (0, 0.36, 0.36)$, which delivers utility vector $x = (0, 0.6, 0.6)$ and induces action scores $s_x(k_0) = 0$, $s_x(k_s) = 0.1$, and $s_x(k_r) = -0.68$. Under this contract, the safe reference action k_s is the unique exact best response across $\mathcal{K}$ and guarantees the strictly positive payoff $1 - 0.36 = 0.64$. Because the high-revenue outcome 10 is generated exclusively by the undesirable risky action k_r , tying utility promises proportionally to revenue forces the principal to reward k_r whenever she attempts to incentivize k_s .

A.3 Risk-sharing losses across multiple productive outcomes

Even when higher revenue is unambiguously associated with the productive action, utility-affine contracts can suffer a strict loss by distorting risk sharing across outcomes. Let $r = (0, 1, 2)$, $u(w) = \sqrt{w}$, $\bar{w} \geq 1$, and

$$\mathcal{K} = \text{co} \left\{ (\delta_0, 0), \left(\frac{1}{2}\delta_1 + \frac{1}{2}\delta_2, \frac{1}{2} \right) \right\},$$

where both extreme actions belong to $\mathcal{K}_0$. A utility-affine contract $x = br$ incentivizes the productive action strictly if and only if $\frac{1}{2}b + \frac{1}{2}(2b) - \frac{1}{2} > 0$, or $b > 1/3$. Its principal payoff is $\frac{3}{2} - \frac{1}{2}(b^2 + 4b^2) = \frac{3}{2} - \frac{5}{2}b^2$, which approaches the supremum $V_0^{\text{aff}} = 11/9$ as $b \downarrow 1/3$.

An unrestricted contract can instead offer equal utility $x_1 = x_2 = \bar{x} > 1/2$ across the two productive outcomes. The productive action assigns equal probabilities to these outcomes, and $H(x) = x^2$ is strictly convex. Jensen's inequality therefore implies that equalizing promised utility minimizes expected wage cost for any given incentive margin. Letting $\bar{x} \downarrow 1/2$ yields the unrestricted supremum $V_0 = \frac{3}{2} - \frac{1}{4} = 5/4$, exceeding V_0^{aff} by $1/36$.

A.4 Spanning properties of parametric utility families

Whether a restricted parametric family of contracts admits a finite-dimensional moment reduction depends on the functional form of $H \circ x$. Under constant relative risk aversion (CRRA) utility $u_\rho(w) = w^{1-\rho}/(1-\rho)$ with $\rho \in (0, 1)$ and $p = 1/(1-\rho) > 1$, we have $H(x) = p^{-p}x^p$. On $y \in [0, 1]$, the zero-intercept ray $x(y) = by$ ($b \geq 0$) induces monetary transfers $H(by) = p^{-p}b^p y^p$, whose expected cost depends on the action distribution μ solely through the single moment $\mathbb{E}_\mu[y^p]$ even

when p is not an integer. By contrast, if a positive intercept $\alpha > 0$ is varied freely in $x(y) = \alpha + by$ with noninteger p , or if utility is CARA so that $H(by) = -\log(1 - aby)/a$, repeated differentiation with respect to b at $b = 0$ generates all positive powers y^k ($k \geq 1$), precluding a finite-dimensional moment reduction on continuous outcome intervals. Our unrestricted dual representation in Section 4 operates directly on the finite outcome space and requires no finite-moment closure assumptions.

B Discontinuity at Tie-Breaking Thresholds and Grid Discretization

When the exact-response supremum V_0 is not attained, evaluating parametric contract families on coarse numerical grids can create artificial performance rankings due to tie-breaking discontinuities. Consider $Y = \{0, 1, 4\}$, with outcomes labeled by their revenues and the envelope

$$\mathcal{K} = \text{co} \{(\delta_0, 0), (\delta_1, 0.15), (0.4\delta_0 + 0.6\delta_4, 0.7)\},$$

where $\mathcal{K}_0 = \{(\delta_0, 0), (\delta_1, 0.15)\}$, $u(0) = 0$, $0.15 < U$, and $H(0.15) < 1$. Any contract achieving a positive robust payoff must give the safe reference action $(\delta_1, 0.15)$ a strictly higher score than the null action, requiring $x_1 > x_0 + 0.15 \geq 0.15$. Because the safe action itself belongs to $\mathcal{R}_0(x)$, every such contract satisfies $v_0(x) \leq 1 - H(x_1) < 1 - H(0.15)$. Conversely, setting $x = (0, 0.15 + h, 0)$ for sufficiently small $h > 0$ makes the safe action the unique best response in $\mathcal{K}$, yielding $\lim_{h \downarrow 0} v_0(x) = 1 - H(0.15)$. Thus $V_0 = 1 - H(0.15)$ is an unattained supremum, while at the boundary contract $x = (0, 0.15, 0)$ the null action ties and $v_0(x)$ drops discontinuously to 0.

Table 6 gives $V_0 = 1 - H(0.15)$ for $u(w) = \sqrt{w}$, $2\sqrt{w}$, and $1 - e^{-w}$, using $\bar{w} = 16$. This cap leaves a feasible right-hand neighborhood of the incentive threshold in each of the following contract families. Money-linear contracts $w(y) = ty$ and contracts $u(w(y)) = b\sqrt{y}$ both approach the supremum as $t \downarrow H(0.15)$ and $b \downarrow 0.15$, respectively: the third vertex remains strictly below the reference score at the threshold. The latter class is affine in utility as a function of $\sqrt{y}$, not of revenue itself. For the two square-root utilities, the classes coincide under $t = b^2$ and $t = b^2/4$, respectively. Figure 6 shows the different approximation errors produced by separate uniform grids in t and b .

Table 6: Exact unattained supremum $V_0 = 1 - H(0.15)$ across three transfer utilities.

Transfer utility $u(w)$	Exact unrestricted supremum V_0
$\sqrt{w}$	0.9775
$2\sqrt{w}$	0.994375
$1 - e^{-w}$	$1 + \log(0.85) \approx 0.83748107$

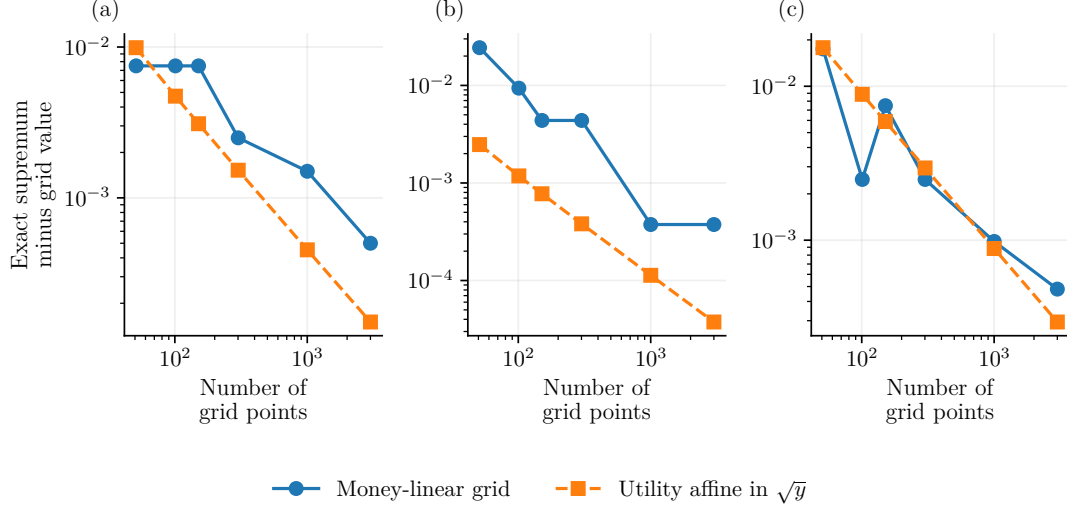


Figure 6: Grid approximation error relative to the exact supremum $1 - H(0.15)$ for money-linear contracts and contracts affine in utility as a function of $\sqrt{y}$. Panels (a), (b), and (c) use $u(w) = \sqrt{w}$, $2\sqrt{w}$, and $1 - e^{-w}$, respectively. The two grids approach the same supremum from different parameter values

Statements and Declarations

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Data and code availability. The numerical examples use synthetic data specified in the code. The source code, input specifications, computed results, and scripts for reproducing the tables and figures are available from the corresponding author.

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