

The operator Hilbert space OH embeds completely isometrically into a noncommutative L^1 -space

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Abstract

We prove that Pisier’s operator Hilbert space OH embeds completely isometrically into the predual of a QWEP von Neumann algebra, answering a longstanding question introduced by Pisier and later by Haagerup and Musat. The embedding is obtained as an ultraproduct of free probabilistic models $h_k = \Lambda^{\frac{1}{2}} c_k \Lambda^{\frac{1}{2}}$ realized in the free group factor $VN(\mathbb{F}_\infty)$, with the trace and circular family scaled according to the mass of the truncated measure $t^{-\frac{5}{3}} dt$.

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1 Introduction

The operator Hilbert space OH introduced by Pisier in the memoir [Pis96] is the unique operator space structure on the complex Hilbert space ℓ^2 which is completely isometric to its conjugate dual via the canonical identification. The question of embedding OH into the predual of a von Neumann algebra has a rich history. Junge [Jun05] constructed the first *completely isomorphic* embedding of OH into the predual of a QWEP von Neumann algebra, by means of free probability. Pisier explicitly recorded the completely isometric embedding problem as open in [Pis03, Remark following Corollary 7.16, p. 145]. He reiterated the question in the introduction of [Pis04b], whose appendix gives a simpler proof of the completely isomorphic embedding, based on complex interpolation and Shlyakhtenko’s generalized circular systems. Pisier [Pis04a] proved that OH does not even embed completely isomorphically into the predual of a *semifinite* von Neumann algebra. Junge [Jun06] showed that OH embeds completely

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isomorphically into the predual of the hyperfinite factors of type III_λ , $0 < \lambda \leq 1$, and Haagerup and Musat [HaM07, Corollary 3.8 p. 620] obtained an embedding into the predual of the hyperfinite factor of type III_1 whose cb-isomorphism constant is at most $\sqrt{2}$. The constants in the general Khintchine inequalities underlying their construction are optimal [HaM07, Theorem 3.10 p. 621]. Related questions appear in [PiS02, remark p. 208 after Corollary 3.1] and in the introduction of [JuP10]. The aim of this paper is to answer this question.

Theorem 1.1 *There exist a QWEP von Neumann algebra $\mathcal{M}$ and a completely isometric map $J: \text{OH} \rightarrow L^1(\mathcal{M})$.*

By duality, Theorem 1.1 is equivalent to the fact that the operator space OH is a complete quotient of a QWEP C^* -algebra, see Corollary 6.3. This is in sharp contrast with the operator spaces $\mathbb{R} + \mathbb{C}$ and $\mathbb{R}_n + \mathbb{C}_n$ for $n \geq 2$, which do not embed completely isometrically into non-commutative L^1 -spaces according to [Arh26], although they embed completely isomorphically [HaM07]. By Pisier's theorem [PiS04a], the von Neumann algebra $\mathcal{M}$ of Theorem 1.1 cannot be semifinite. Our construction takes $\mathcal{M} = (A^{**})^{\text{op}}$, where $A = \prod_{\mathcal{U}} \text{VN}(\mathbb{F}_\infty)$ is the C^* -algebraic (norm) ultrapower of the free group factor, see Theorem 6.2 and Remark 4.3. Corollary 6.4 also gives a complete isometry into the predual of a σ -finite factor of type III.

The QWEP property gives finite-dimensional matrix approximations: for any integer $n \geq 1$ and any $\delta > 0$, the operator space OH_n embeds into some space S_1^N with cb-distortion less than $1 + \delta$, by [EJR00, Corollary 7.2], see Corollary 6.3. Combining the complete isometry with Junge's [Jun05, Lemmas 4.4 and 4.5] gives an exact realization of completely 1-summing norms as tensor norms in a predual. Ozawa's compactness theorem [Oza00] then yields convergence of the truncated models in cb-distortion for each fixed dimension, and Proposition 6.8 identifies the joint limit of their two-factor tensor norms with $\|T\|_{\pi_1^{\text{op}}}$. For the identity on OH_n , the order $\sqrt{n(1 + \log n)}$ and the numerical comparison constants are inherited from [Jun05]. Section 7 derives the tensor distortion formula.

Approach of the paper Our proof consists of two exact computations. Let $m, n \geq 1$ be integers. For any n -tuple $x = (x_1, \dots, x_n)$ of matrices of M_m , consider the completely positive maps $\Phi_x, \Phi_x^*: M_m \rightarrow M_m$ defined by

$$(1.1) \quad \Phi_x(b) \stackrel{\text{def}}{=} \sum_{k=1}^n x_k b x_k^* \quad \text{and} \quad \Phi_x^*(b) \stackrel{\text{def}}{=} \sum_{k=1}^n x_k^* b x_k, \quad b \in M_m.$$

We first show in Theorem 3.2 that if the matrices $\Phi_x(1)$ and $\Phi_x^*(1)$ are invertible then the system of matrix equations

$$(1.2) \quad P = \Phi_x(Q^{-\frac{1}{3}}), \quad Q = \Phi_x^*(P^{-\frac{1}{3}}),$$

admits a unique solution (P, Q) in positive definite matrices and that the norm of $\sum_k x_k \otimes e_k$ in $S_1^m(\text{OH}_n)$ is equal to $\text{Tr}(Q^{\frac{2}{3}}) = \text{Tr}(P^{\frac{2}{3}})$. This turns Pisier's factorization formula [PiS98] into a characterization by a uniquely determined pair of positive matrices. The associated map $(P, Q) \mapsto (\Phi_x(Q^{-\frac{1}{3}}), \Phi_x^*(P^{-\frac{1}{3}}))$ is a $\frac{1}{3}$ -contraction for the product Thompson metric, so its iterates converge to this pair from any positive definite initial pair. The norm identity follows by constructing an optimal factorization and a norming functional, using a Perron-Frobenius argument. Thus the characterization also gives an iterative procedure for evaluating these norms, independently of the free probabilistic construction.

Next, we use a σ -system in the sense of Definition 4.1. Write $|\sigma| \stackrel{\text{def}}{=} \sigma((0, \infty))$. By Lemma 4.2, this system can be realized in a finite von Neumann algebra $\mathcal{A}$ with $\tau = |\sigma|\varphi$, where φ

is a normal faithful tracial state, and $c_k \stackrel{\text{def}}{=} |\sigma|^{\frac{1}{2}} c_k^0$, where (c_k^0) is a standard scalar-valued free circular family which is free from Λ with respect to φ . For $\mathcal{D} \stackrel{\text{def}}{=} W^*(\Lambda)$ and any $d \in \mathcal{D}$, the covariance is

$$\mathbb{E}_{\mathcal{D}}(c_k d c_l^*) = \mathbb{E}_{\mathcal{D}}(c_k^* d c_l) = \delta_{kl} \tau(d) 1.$$

We put

$$(1.3) \quad h_k \stackrel{\text{def}}{=} \Lambda^{\frac{1}{2}} c_k \Lambda^{\frac{1}{2}}.$$

Weighting on both sides leads to coupled covariance equations involving Φ_x and its adjoint Φ_x^* , which can be matched with the two equations in (1.2). With a one-sided weighting $\Lambda^{\frac{1}{2}} c_k$, the maps Φ_x and Φ_x^* enter the covariance equations with different weights, and the symmetry of (1.2) is lost. For coefficients x with $\Phi_x(1)$ and $\Phi_x^*(1)$ invertible, using the operator-valued Cauchy transform of the hermitization of $Z = \sum_k x_k \otimes h_k$, we show in Proposition 4.4 that the norm of Z in $L^1(M_m \overline{\otimes} \mathcal{A})$ is given by

$$\|Z\|_{L^1(M_m \overline{\otimes} \mathcal{A})} = \frac{1}{\pi} \int_0^\infty s^{-\frac{1}{2}} \text{Tr}(Q_s F_\sigma(Q_s)) ds,$$

where σ is the distribution of Λ with respect to τ , $F_\sigma(X) \stackrel{\text{def}}{=} \int_0^\infty t(1+tX)^{-1} d\sigma(t)$, and (P_s, Q_s) is the unique positive definite solution of $P = s^{-1} \Phi_x(F_\sigma(Q))$, $Q = s^{-1} \Phi_x^*(F_\sigma(P))$. For the formal infinite measure $\sigma = t^{-\frac{5}{3}} dt$, we have $F_\sigma(X) = \kappa X^{-\frac{1}{3}}$, where $\kappa = \frac{2\pi}{\sqrt{3}}$. If (P, Q) solves (1.2), then the corresponding formal solution is $(P_s, Q_s) = (\frac{\kappa}{s})^{\frac{3}{4}} (P, Q)$. Thus the two norm computations involve the same pair of positive matrices. Among the formal measures $t^{-1-\alpha} dt$ with $0 < \alpha < 1$, the value $\alpha = \frac{2}{3}$ is the unique one for which the integrand in the L^1 -norm formula is proportional to s^{-1} , see Remark 4.6. This accounts for the logarithmic normalization. We use the finite truncations $\sigma_\varepsilon \stackrel{\text{def}}{=} 1_{[\varepsilon, \frac{1}{\varepsilon}]}(t) t^{-\frac{5}{3}} dt$. Theorem 5.4 gives a logarithmic leading term with a bounded remainder for non-degenerate coefficients, and in particular

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\log \frac{1}{\varepsilon}} \left\| \sum_{k=1}^n x_k \otimes h_k^{(\varepsilon)} \right\|_{L^1(M_m \overline{\otimes} \mathcal{A}_\varepsilon)} = \frac{8\kappa^{\frac{3}{2}}}{3\pi} \left\| \sum_{k=1}^n x_k \otimes e_k \right\|_{S_1^m(\text{OH}_n)}.$$

An ultraproduct argument then provides the complete isometry of Theorem 1.1. Note that each truncated model lives in a finite von Neumann algebra, in accordance with Pisier's theorem: for a fixed ε , the family $(h_k^{(\varepsilon)})_{k \geq 1}$ cannot span a completely isomorphic copy of OH .

Structure of the paper Section 2 contains preliminaries on vector-valued Schatten spaces, Haagerup noncommutative L^1 -spaces, the operator Hilbert space, Thompson's part metric and operator-valued free probability. In Section 3, we describe the norm of $S_1^m(\text{OH}_n)$ by means of the system (1.2). Section 4 is devoted to the free models (1.3). In Section 5, we analyse the truncated power laws. We prove Theorem 1.1 in Section 6, where we also obtain finite-dimensional matrix approximations, convergence of the truncations in cb-distortion, and tensor norm formulas and estimates for completely 1-summing maps. Section 7 gives the tensor distortion.

2 Preliminaries

Vector-valued Schatten spaces We refer to the books [BLM04], [Efr00], [Pau02] and [Pis03] for background on operator spaces. Suppose that $1 \leq p < \infty$. For any integer $m \geq 1$,

we denote by M_m the space of complex $m \times m$ matrices, by e_{ij} its matrix units, by S_m^p the space M_m equipped with the Schatten norm $\|A\|_{S_m^p} \stackrel{\text{def}}{=} \text{Tr}(|A|^p)^{\frac{1}{p}}$ and by $\mathcal{P}_m$ the set of positive definite matrices of M_m . For any matrix $X \in M_m$, we write $X > 0$ if $X \in \mathcal{P}_m$. We denote by $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$ the smallest and the largest eigenvalues of a selfadjoint matrix X .

Let E be an operator space and $m \geq 1$ an integer. Following Pisier [Pis98], we denote by $S_m^1(E)$ the vector-valued Schatten space. We identify the elements of $S_m^1(E)$ with matrices $[u_{ij}]$ with entries in E . If $a, b \in M_m$ and $v = [v_{ij}] \in M_m(E)$, we denote by $a \cdot v \cdot b$ the matrix product. We will use the following facts.

1. By [Pis98, Theorem 1.5 p. 21], for any $u \in S_m^1(E)$ we have

$$(2.1) \quad \|u\|_{S_m^1(E)} = \inf \{ \|a\|_{S_m^2} \|v\|_{M_m(E)} \|b\|_{S_m^2} : u = a \cdot v \cdot b, a, b \in M_m, v \in M_m(E) \}.$$

2. We use the parallel duality

$$(2.2) \quad \langle [y_{ij}], [\varphi_{ij}] \rangle = \sum_{i,j=1}^m \varphi_{ij}(y_{ij}).$$

With this convention, the canonical identifications $M_m(F)^* = S_m^1(F^*)$ and $S_m^1(E)^* = M_m(E^*)$ are isometric, see [Pis98, (1.1) and (1.1)* p. 18]. This is the convention compatible with the opposite-algebra structure of Subsection 2.

3. By [Pis98, Lemma 1.7, equation (1.7)' p. 23], a linear map $u: E \rightarrow F$ between operator spaces satisfies $\|u\|_{\text{cb}, E \rightarrow F} = \sup_{m \geq 1} \|\text{Id}_{S_m^1} \otimes u\|_{S_m^1(E) \rightarrow S_m^1(F)}$. By [Pis98, Corollary 1.2 p. 19], if u is a complete isometry then $\text{Id}_{S_m^1} \otimes u$ is an isometry for any m . Consequently, a linear map $u: E \rightarrow F$ is a complete isometry if and only if $\text{Id}_{S_m^1} \otimes u: S_m^1(E) \rightarrow S_m^1(F)$ is an isometry for any integer $m \geq 1$. Indeed, if these maps are isometries then $\|u\|_{\text{cb}} \leq 1$ and the inverse $u^{-1}: u(E) \rightarrow E$ satisfies $\|u^{-1}\|_{\text{cb}} \leq 1$ since $S_m^1(u(E))$ is an isometric subspace of $S_m^1(F)$ by [Pis98, Corollary 1.2 p. 19].

Noncommutative L^1 -spaces Let $\mathcal{M}$ be a von Neumann algebra. We use the noncommutative L^1 -space $L^1(\mathcal{M})$ of Haagerup, see [Ter81] and [PiX03], which is isometric to the predual $\mathcal{M}_*$. The opposite von Neumann algebra $\mathcal{M}^{\text{op}} = \{a^{\text{op}} : a \in \mathcal{M}\}$ has the same normed involutive vector space structure as $\mathcal{M}$ and the product $a^{\text{op}}b^{\text{op}} \stackrel{\text{def}}{=} (ba)^{\text{op}}$. Following Pisier [Pis03, Chapter 7], we equip the Banach space $L^1(\mathcal{M})$ with the operator space structure of the predual $(\mathcal{M}^{\text{op}})_*$, considered as a subspace of the operator space dual $(\mathcal{M}^{\text{op}})^*$. With this convention, for any integer $m \geq 1$ the map $e_{ij} \otimes x \mapsto e_{ij} \otimes x$ induces an isometry from $S_1^m(L^1(\mathcal{M}))$ onto $L^1(M_m \overline{\otimes} \mathcal{M})$. Indeed, $(M_m \overline{\otimes} \mathcal{M})^{\text{op}} = M_m^{\text{op}} \overline{\otimes} \mathcal{M}^{\text{op}}$, and the predual of a von Neumann tensor product is the operator space projective tensor product of the preduals [Efr00, Theorem 7.2.4], see also [Xu06, Sections 1.2 and 1.3]. Since $\mathcal{M}^{\text{op}}$ is again a von Neumann algebra, an operator space embeds completely isometrically into $L^1(\mathcal{M})$ for some von Neumann algebra $\mathcal{M}$ if and only if it embeds completely isometrically into the predual of some von Neumann algebra. Moreover, for any C^* -algebra A , the dual operator space A^* is the predual of the von Neumann algebra A^{**} equipped with its predual operator space structure. Consequently, A^* is completely isometric to $L^1((A^{**})^{\text{op}})$.

If $\mathcal{M}$ is a finite von Neumann algebra equipped with a normal faithful finite trace τ , then $L^1(\mathcal{M})$ identifies with the tracial noncommutative L^1 -space $L^1(\mathcal{M}, \tau)$, whose norm is $\|h\|_1 \stackrel{\text{def}}{=} \tau(|h|)$. In this case, for any integer $m \geq 1$ the space $S_1^m(L^1(\mathcal{M}))$ identifies isometrically with $L^1(M_m \overline{\otimes} \mathcal{M}, \text{Tr} \otimes \tau)$. An element h of $L^1(\mathcal{M}, \tau)$ is identified with the normal functional $a^{\text{op}} \mapsto$

$\tau(ha)$ on $\mathcal{M}^{\text{op}}$. This identification depends on the normalization of τ , which will not be a state in our models.

In this finite tracial setting, every unital von Neumann subalgebra $\mathcal{D} \subset \mathcal{M}$ is the range of a unique normal faithful τ -preserving conditional expectation $\mathbb{E}_{\mathcal{D}}$, see [PiX03, Section 7]. Multiplying τ by a positive scalar does not change this expectation.

We will use the generalized singular numbers of Fack and Kosaki [FaK86]. Let $\mathcal{M}$ be a finite von Neumann algebra equipped with a normal faithful finite trace τ and let $X \in \mathcal{M}$. For any $t > 0$, let $\mu_t(X) \stackrel{\text{def}}{=} \inf\{\|Xe\| : e \text{ projection of } \mathcal{M}, \tau(1-e) \leq t\}$. By [FaK86, Lemma 2.5 p. 275], we have $\mu_t(aXb) \leq \|a\| \|b\| \mu_t(X)$ for any $a, b \in \mathcal{M}$, and by [FaK86, Corollary 2.8 p. 278], we have $\tau(g(|X|)) = \int_0^\infty g(\mu_t(X)) dt$ for any continuous nondecreasing function $g : [0, \infty) \rightarrow [0, \infty)$ with $g(0) = 0$.

The operator Hilbert space We denote by OH_n the n -dimensional operator Hilbert space with its canonical basis (e_k) , by (e_k^*) the dual basis of OH_n^* and by $\text{OH} = \text{OH}(\mathbb{N})$ the separable operator Hilbert space. If $E \subset \text{B}(K)$ is an operator space, its conjugate $\overline{E} = \{\overline{x} : x \in E\} \subset \text{B}(\overline{K})$ is the complex conjugate vector space with the matrix norms $\|[\overline{x}_{ij}]\|_{\text{M}_m(\overline{E})} = \|[x_{ij}]\|_{\text{M}_m(E)}$. Recall that the map $\text{OH}_n^* \rightarrow \overline{\text{OH}_n}$, $e_k^* \mapsto \overline{e_k}$ is a complete isometry [Pis96] and that any unitary of ℓ_n^2 induces a complete isometry of OH_n . If (e_k) is an orthonormal basis of OH then for any Hilbert space K and any finitely supported family (x_k) of $\text{B}(K)$ we have by [Pis96, Theorem 1.1 p. 12]

$$(2.3) \quad \left\| \sum_k x_k \otimes e_k \right\|_{\text{B}(K) \otimes_{\min} \text{OH}} = \left\| \sum_k x_k \otimes \overline{x_k} \right\|_{\text{B}(K) \otimes_{\min} \overline{\text{B}(K)}}^{\frac{1}{2}}.$$

For any integer $m \geq 1$ and any n -tuple $x = (x_1, \dots, x_n)$ of matrices of M_m , we consider the linear maps $\Phi_x, \Phi_x^* : \text{M}_m \rightarrow \text{M}_m$ defined by

$$(2.4) \quad \Phi_x(b) \stackrel{\text{def}}{=} \sum_{k=1}^n x_k b x_k^* \quad \text{and} \quad \Phi_x^*(b) \stackrel{\text{def}}{=} \sum_{k=1}^n x_k^* b x_k.$$

These maps are completely positive and Φ_x^* is the adjoint of Φ_x for the inner product $\langle a, b \rangle = \text{Tr}(b^*a)$ of the Hilbert space S_2^m . If $\Phi_x(1)$ is invertible then for any $X \in \mathcal{P}_m$ we have $\Phi_x(X) \geq \lambda_{\min}(X)\Phi_x(1)$, hence $\Phi_x(X) \in \mathcal{P}_m$. The same observation holds for Φ_x^* . We let

$$(2.5) \quad \mathcal{G}_{m,n} \stackrel{\text{def}}{=} \{x \in (\text{M}_m)^n : \Phi_x(1) \text{ and } \Phi_x^*(1) \text{ are invertible}\}.$$

This set is dense in $(\text{M}_m)^n$ since it contains all the n -tuples x such that x_1 is invertible. Finally, we let

$$(2.6) \quad \text{N}(x) \stackrel{\text{def}}{=} \left\| \sum_{k=1}^n x_k \otimes e_k \right\|_{S_m^1(\text{OH}_n)}.$$

Lemma 2.1 *Let $m, n \geq 1$ be integers and let $y = (y_1, \dots, y_n) \in (\text{M}_m)^n$. We have*

$$(2.7) \quad \left\| \sum_{k=1}^n y_k \otimes e_k \right\|_{\text{M}_m(\text{OH}_n)} = \left\| \sum_{k=1}^n y_k \otimes e_k^* \right\|_{\text{M}_m(\text{OH}_n^*)} = \|\Phi_y\|_{S_m^2 \rightarrow S_m^2}^{\frac{1}{2}}.$$

Proof : By (2.3), the first quantity is equal to $\|\sum_k y_k \otimes \overline{y_k}\|^{\frac{1}{2}}$, where the norm is computed in $M_m \otimes_{\min} \overline{M_m} = B(\mathbb{C}^m \otimes_2 \overline{\mathbb{C}^m})$. The unitary $U: \mathbb{C}^m \otimes_2 \overline{\mathbb{C}^m} \rightarrow S_2^m$, $\xi \otimes \overline{\eta} \mapsto \xi \eta^*$ satisfies $U(y_k \otimes \overline{y_k})(\xi \otimes \overline{\eta}) = y_k \xi (\eta^*)^* = y_k (\xi \eta^*) y_k^*$. Hence $U(\sum_k y_k \otimes \overline{y_k})U^* = \Phi_y$, which proves the equality of the first and the third quantities. Since OH_n^* is completely isometric to $\overline{\text{OH}_n}$ via $e_k^* \mapsto \overline{e_k}$, the second quantity is equal to $\|\sum_k \overline{y_k} \otimes e_k\|_{M_m(\text{OH}_n)} = \|\sum_k \overline{y_k} \otimes y_k\|^{\frac{1}{2}}$. The flip $\overline{\mathbb{C}^m} \otimes_2 \mathbb{C}^m \rightarrow \mathbb{C}^m \otimes_2 \overline{\mathbb{C}^m}$ is a unitary which conjugates $\sum_k \overline{y_k} \otimes y_k$ with $\sum_k y_k \otimes \overline{y_k}$. The proof is complete. $\blacksquare$

We will also use invariance under simultaneous transposition:

$$(2.8) \quad \left\| \sum_k y_k^\top \otimes e_k^* \right\|_{M_m(\text{OH}_n^*)} = \left\| \sum_k y_k \otimes e_k^* \right\|_{M_m(\text{OH}_n^*)}.$$

Indeed, $\sum_k y_k^\top \otimes \overline{y_k}^\top = (\sum_k y_k \otimes \overline{y_k})^\top$, and full transposition preserves the operator norm. The same identity holds with e_k in place of e_k^* .

We use the homogeneity of OH : every bounded linear map between operator Hilbert spaces has cb -norm equal to its operator norm, see [Pis03, Proposition 7.2]. In particular, orthogonal coordinate inclusions and projections are complete contractions. The column and row spaces are $C_n \stackrel{\text{def}}{=} B(\mathbb{C}, \mathbb{C}^n)$ and $R_n \stackrel{\text{def}}{=} B(\mathbb{C}^n, \mathbb{C})$, with their concrete operator space structures. For finite-dimensional operator spaces E and F of the same dimension, we write

$$d_{\text{cb}}(E, F) \stackrel{\text{def}}{=} \inf_S \|S\|_{\text{cb}} \|S^{-1}\|_{\text{cb}},$$

where the infimum runs over linear isomorphisms $S: E \rightarrow F$, see [Pis03]. A linear map $q: E \rightarrow F$ is a complete quotient map if, for every $m \geq 1$, the map $\text{Id}_{M_m} \otimes q$ sends the open unit ball of $M_m(E)$ onto the open unit ball of $M_m(F)$.

We will also use the following elementary Perron-Frobenius type lemma.

Lemma 2.2 *Let $\Psi: M_m \rightarrow M_m$ be a positive linear map. Suppose that there exist $X \in \mathcal{P}_m$ and $\lambda \geq 0$ such that $\Psi(X) = \lambda X$. Then the spectral radius of Ψ is equal to λ . If in addition Ψ is selfadjoint on the Hilbert space S_m^2 , then $\|\Psi\|_{S_m^2 \rightarrow S_m^2} = \lambda$.*

Proof : The real number λ is an eigenvalue of Ψ , hence the spectral radius $r(\Psi)$ satisfies $r(\Psi) \geq \lambda$. Let $Y \in M_m$ be a selfadjoint matrix. There exists $c \geq 0$ such that $-cX \leq Y \leq cX$. Since Ψ^j is positive, we obtain $-c\lambda^j X \leq \Psi^j(Y) \leq c\lambda^j X$ for any integer $j \geq 1$, hence $\|\Psi^j(Y)\| \leq c\lambda^j \|X\|$. One may take $c = \|X^{-1}\| \|Y\|$ for selfadjoint Y . Decomposing an arbitrary matrix into its real and imaginary parts therefore gives $\|\Psi^j\| \leq 2 \|X^{-1}\| \|X\| \lambda^j$. We deduce that $\sup_j \lambda^{-j} \|\Psi^j\| < \infty$ if $\lambda > 0$ and that $\Psi^j = 0$ for $j \geq 1$ if $\lambda = 0$. Hence $r(\Psi) = \lim_j \|\Psi^j\|^{\frac{1}{j}} \leq \lambda$. The last assertion follows from the equality between the norm and the spectral radius of a selfadjoint operator on a Hilbert space. $\blacksquare$

Thompson's part metric For any $X, Y \in \mathcal{P}_m$, we let

$$(2.9) \quad d(X, Y) \stackrel{\text{def}}{=} \inf \{r \geq 0 : e^{-r} X \leq Y \leq e^r X\}.$$

This is Thompson's part metric on the cone $\mathcal{P}_m$, see [Tho63]. We will use the following elementary properties. Let $X, Y \in \mathcal{P}_m$.

1. We have $d(X^{-1}, Y^{-1}) = d(X, Y)$ since the map $X \mapsto X^{-1}$ is order reversing on $\mathcal{P}_m$.

2. For any $0 \leq t \leq 1$, we have $d(X^t, Y^t) \leq td(X, Y)$. Indeed, if $e^{-r}X \leq Y \leq e^rX$ then $e^{-tr}X^t \leq Y^t \leq e^{tr}X^t$ by the Löwner-Heinz inequality. For completeness, the operator monotonicity of $X \mapsto X^t$ for $0 < t < 1$ follows from

$$X^t = \frac{\sin(\pi t)}{\pi} \int_0^\infty \lambda^{t-1} X(\lambda + X)^{-1} d\lambda,$$

since $X(\lambda + X)^{-1} = 1 - \lambda(\lambda + X)^{-1}$ is order preserving. The cases $t = 0$ and $t = 1$ are immediate.

3. If $\Psi: M_m \rightarrow M_m$ is a positive linear map with $\Psi(\mathcal{P}_m) \subset \mathcal{P}_m$ and if $c > 0$, then $d(c\Psi(X), c\Psi(Y)) \leq d(X, Y)$.
4. If $e^{-r}X \leq Y \leq e^rX$ then

$$(2.10) \quad e^{-r}\lambda_{\min}(X) \leq \lambda_{\min}(Y) \quad \text{and} \quad \lambda_{\max}(Y) \leq e^r\lambda_{\max}(X)$$

5. The metric space $(\mathcal{P}_m, d)$ is complete. Indeed, if (X_j) is a Cauchy sequence, there exists $R \geq 0$ such that $e^{-R}X_1 \leq X_j \leq e^RX_1$ for any j , and for any $r > 0$ there exists j_0 such that $e^{-r}X_j \leq X_k \leq e^rX_j$ for any $j, k \geq j_0$. Then we have $\|X_k - X_j\| \leq (e^r - 1)e^R\|X_1\|$, so (X_j) converges in norm to a matrix X satisfying $e^{-R}X_1 \leq X$, and letting $k \rightarrow \infty$ in the previous inequalities we obtain $d(X_j, X) \leq r$ for $j \geq j_0$.

For any pairs (P, Q) and (P', Q') of $\mathcal{P}_m \times \mathcal{P}_m$, we let

$$d_2((P, Q), (P', Q')) = \max \{d(P, P'), d(Q, Q')\}.$$

Then $(\mathcal{P}_m \times \mathcal{P}_m, d_2)$ is a complete metric space.

Operator-valued free probability We refer to [HiP00], [Spe98], [NSS02] and [MiS17] for background on operator-valued free probability. Let $\mathcal{A}$ be a von Neumann algebra, $\mathcal{B}$ a von Neumann subalgebra of $\mathcal{A}$ and $\mathbb{E}_{\mathcal{B}}: \mathcal{A} \rightarrow \mathcal{B}$ a normal faithful conditional expectation. The $\mathcal{B}$ -valued cumulants $(\kappa_r^{\mathcal{B}})_{r \geq 1}$ are the multilinear maps $\kappa_r^{\mathcal{B}}: \mathcal{A}^r \rightarrow \mathcal{B}$ defined by the moment-cumulant formula over noncrossing partitions [Spe98]. They are $\mathcal{B}$ -balanced, that is for any $b_0, \dots, b_r \in \mathcal{B}$ and any $a_1, \dots, a_r \in \mathcal{A}$ we have

$$\kappa_r^{\mathcal{B}}(b_0 a_1 b_1, a_2 b_2, \dots, a_r b_r) = b_0 \kappa_r^{\mathcal{B}}(a_1, b_1 a_2, \dots, b_{r-1} a_r) b_r.$$

A selfadjoint element H of $\mathcal{A}$ is called $\mathcal{B}$ -valued semicircular if $\kappa_r^{\mathcal{B}}(H b_1, \dots, H b_{r-1}, H) = 0$ for any $r \neq 2$ and any $b_1, \dots, b_{r-1} \in \mathcal{B}$. Its covariance is the completely positive map $\eta: \mathcal{B} \rightarrow \mathcal{B}$, $b \mapsto \mathbb{E}_{\mathcal{B}}(H b H) = \kappa_2^{\mathcal{B}}(H b, H)$. By the moment-cumulant formula, the $\mathcal{B}$ -valued moments $\mathbb{E}_{\mathcal{B}}(H b_1 H b_2 \dots b_{r-1} H)$ of a $\mathcal{B}$ -valued semicircular element only depend on its covariance. The operator-valued Cauchy transform $G(z) = \mathbb{E}_{\mathcal{B}}((z - H)^{-1})$ of H satisfies the equation

$$(2.11) \quad G(z) = (z - \eta(G(z)))^{-1}, \quad z \in \mathbb{C}, \quad \text{Im } z > 0.$$

Indeed, for $|z|$ large enough both sides are given by norm convergent power series in z^{-1} , and (2.11) is the functional equation relating the Cauchy transform of H to its $\mathcal{B}$ -valued R -transform $b \mapsto \eta(b)$, see [Voi95], [Spe98, Chapter 4] and [MiS17, Chapter 9]. Moreover, both sides of (2.11) are analytic functions of z on the upper half-plane. For the left-hand side this is clear. For the right-hand side, note that $\text{Im } G(z) = -(\text{Im } z) \mathbb{E}_{\mathcal{B}}((z - H)^{-1}(\bar{z} - H)^{-1}) \leq 0$. Since η is positive, we obtain $\text{Im}(z - \eta(G(z))) = (\text{Im } z)1 - \eta(\text{Im } G(z)) \geq (\text{Im } z)1$, so that $z - \eta(G(z))$ is invertible.

By analytic continuation, (2.11) holds on the whole upper half-plane, see also [HRS07, Section 2], where uniqueness of the solution with negative imaginary part is proved for an arbitrary unital C^* -algebra $\mathcal{B}$.

We will use the following consequence of the characterization of freeness with amalgamation by cumulants due to Nica, Shlyakhtenko and Speicher [NSS02, Theorem 3.6]: let $(\mathcal{A}, \varphi)$ be a tracial W^* -probability space, let $\mathcal{D} \subset \mathcal{A}$ be a von Neumann subalgebra and let $\mathcal{N} \subset \mathcal{A}$ be a von Neumann subalgebra which is free from $\mathcal{D}$ with respect to φ . Then the $\mathcal{D}$ -valued cumulants of elements of $\mathcal{N}$ with respect to the φ -preserving conditional expectation $\mathbb{E}_{\mathcal{D}}$ are given by

$$(2.12) \quad \kappa_r^{\mathcal{D}}(a_1 d_1, a_2 d_2, \dots, a_{r-1} d_{r-1}, a_r) = \kappa_r(a_1, \dots, a_r) \varphi(d_1) \cdots \varphi(d_{r-1})$$

for any $a_1, \dots, a_r \in \mathcal{N}$ and any $d_1, \dots, d_{r-1} \in \mathcal{D}$, where κ_r denotes the scalar-valued free cumulant with respect to φ .

Finally, consider the matrix amplification with conditional expectation $\text{Id}_{M_N} \otimes \mathbb{E}_{\mathcal{B}}$. For $r \geq 2$ and $A_1, \dots, A_r \in M_N(\mathcal{A})$, its cumulants satisfy

$$(2.13) \quad [\kappa_r^{M_N(\mathcal{B})}(A_1, \dots, A_r)]_{ij} = \sum_{i_1, \dots, i_{r-1}=1}^N \kappa_r^{\mathcal{B}}((A_1)_{ii_1}, \dots, (A_r)_{i_{r-1}j}).$$

For $r = 1$, the identity is $[\kappa_1^{M_N(\mathcal{B})}(A_1)]_{ij} = \mathbb{E}_{\mathcal{B}}((A_1)_{ij})$. For $\mathcal{B} = \mathbb{C}$, this is the computation of matricial cumulants of Nica, Shlyakhtenko and Speicher [NSS02b], see also [Shl96] for the semicircular case. To prove these formulas in general, expand the entries of $\mathbb{E}_{M_N(\mathcal{B})}(A_1 \cdots A_r)$ by matrix multiplication and apply the $\mathcal{B}$ -valued moment-cumulant formula to each summand. Each nesting of a cumulant corresponds to summation over the intermediate matrix indices. Thus the displayed multilinear maps satisfy the $M_N(\mathcal{B})$ -valued moment-cumulant formula, and uniqueness of cumulants proves the assertion. In particular, for $a, b \in \mathcal{A}$ and $\beta \in M_N(\mathcal{B})$, we have

$$[\kappa_2^{M_N(\mathcal{B})}((1 \otimes a)\beta, 1 \otimes b)]_{ij} = \kappa_2^{\mathcal{B}}(a\beta_{ij}, b).$$

3 The norm of $S_1^m(\text{OH}_n)$ as a fixed point

Throughout the paper, we let

$$(3.1) \quad \kappa \stackrel{\text{def}}{=} \int_0^\infty \frac{u^{-\frac{2}{3}}}{1+u} du = \frac{\pi}{\sin \frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}}.$$

Proposition 3.1 *Let $m, n \geq 1$ be integers, let $x \in \mathcal{G}_{m,n}$ and let $c > 0$. The system*

$$(3.2) \quad P = c\Phi_x(Q^{-\frac{1}{3}}), \quad Q = c\Phi_x^*(P^{-\frac{1}{3}})$$

admits a unique solution (P, Q) in $\mathcal{P}_m \times \mathcal{P}_m$. Moreover, this solution satisfies $\text{Tr}(P^{\frac{2}{3}}) = \text{Tr}(Q^{\frac{2}{3}})$.

Proof : Since $x \in \mathcal{G}_{m,n}$, the maps Φ_x and Φ_x^* send $\mathcal{P}_m$ into $\mathcal{P}_m$, see Section 2. Hence the formula $T(P, Q) = (c\Phi_x(Q^{-\frac{1}{3}}), c\Phi_x^*(P^{-\frac{1}{3}}))$ defines a map $T: \mathcal{P}_m \times \mathcal{P}_m \rightarrow \mathcal{P}_m \times \mathcal{P}_m$. For any $Q, Q' \in \mathcal{P}_m$, the properties 1, 2 and 3 below Definition 2.9 give

$$d(c\Phi_x(Q^{-\frac{1}{3}}), c\Phi_x(Q'^{-\frac{1}{3}})) \leq d(Q^{-\frac{1}{3}}, Q'^{-\frac{1}{3}}) = d(Q^{\frac{1}{3}}, Q'^{\frac{1}{3}}) \leq \frac{1}{3}d(Q, Q').$$

The same estimate holds for Φ_x^* . Hence T is a $\frac{1}{3}$ -contraction of the complete metric space $(\mathcal{P}_m \times \mathcal{P}_m, d_2)$ and the Banach fixed point theorem gives the first assertion. Now, if (P, Q) is the solution of (3.2), we have

$$\begin{aligned} \operatorname{Tr}(P^{\frac{2}{3}}) &= \operatorname{Tr}(P^{-\frac{1}{3}}P) = c \operatorname{Tr}(P^{-\frac{1}{3}}\Phi_x(Q^{-\frac{1}{3}})) \\ &= c \operatorname{Tr}(\Phi_x^*(P^{-\frac{1}{3}})Q^{-\frac{1}{3}}) = \operatorname{Tr}(QQ^{-\frac{1}{3}}) = \operatorname{Tr}(Q^{\frac{2}{3}}). \end{aligned}$$

■

The following result is the first main ingredient of the paper.

Theorem 3.2 *Let $m, n \geq 1$ be integers and let $x = (x_1, \dots, x_n) \in (\mathcal{M}_m)^n$.*

1. *Suppose that there exist $\alpha, \beta \in \mathcal{P}_m$ with $\operatorname{Tr}(\alpha^2) = \operatorname{Tr}(\beta^2) = 1$ and $T > 0$ such that*

$$(3.3) \quad \Phi_x(\beta^{-1}) = T^2 \alpha^3 \quad \text{and} \quad \Phi_x^*(\alpha^{-1}) = T^2 \beta^3.$$

Then $N(x) = T$.

2. *Suppose that $x \in \mathcal{G}_{m,n}$ and let $c > 0$. If (P, Q) is the solution of (3.2) then*

$$(3.4) \quad N(x) = c^{-\frac{1}{2}} \operatorname{Tr}(Q^{\frac{2}{3}}) = c^{-\frac{1}{2}} \operatorname{Tr}(P^{\frac{2}{3}}).$$

Proof : 1. We first show that $N(x) \leq T$. Consider the n -tuple $v = (v_1, \dots, v_n)$ defined by $v_k = \alpha^{-1}x_k\beta^{-1}$. The element $u = \sum_k x_k \otimes e_k$ of $S_1^m(\operatorname{OH}_n)$ is equal to $\alpha \cdot (\sum_k v_k \otimes e_k) \cdot \beta$. Since $\|\alpha\|_{S_2^m} = \|\beta\|_{S_2^m} = 1$, the factorization (2.1) and Lemma 2.1 give

$$N(x) \leq \left\| \sum_{k=1}^n v_k \otimes e_k \right\|_{\mathcal{M}_m(\operatorname{OH}_n)} = \|\Phi_v\|_{S_2^m \rightarrow S_2^m}^{\frac{1}{2}} = \|\Phi_v^* \Phi_v\|_{S_2^m \rightarrow S_2^m}^{\frac{1}{4}}.$$

Now, we have

$$\Phi_v(\beta) = \sum_{k=1}^n \alpha^{-1}x_k\beta^{-1}\beta\beta^{-1}x_k^*\alpha^{-1} = \alpha^{-1}\Phi_x(\beta^{-1})\alpha^{-1} \stackrel{(3.3)}{=} T^2\alpha$$

and similarly $\Phi_v^*(\alpha) = \beta^{-1}\Phi_x^*(\alpha^{-1})\beta^{-1} = T^2\beta$. Hence $\Phi_v^*\Phi_v(\beta) = T^4\beta$. The map $\Phi_v^*\Phi_v$ is positive and selfadjoint on S_2^m , and β is positive definite. By Lemma 2.2, we obtain $\|\Phi_v^*\Phi_v\|_{S_2^m \rightarrow S_2^m} = T^4$. We conclude that $N(x) \leq T$.

Now, we turn to the inequality $N(x) \geq T$. Consider the n -tuple $z = (z_1, \dots, z_n)$ defined by $z_k = T^{-1}\beta^{-1}x_k^*\alpha^{-1}$. We have

$$\Phi_z(\alpha) = T^{-2} \sum_{k=1}^n \beta^{-1}x_k^*\alpha^{-1}\alpha\alpha^{-1}x_k\beta^{-1} = T^{-2}\beta^{-1}\Phi_x^*(\alpha^{-1})\beta^{-1} \stackrel{(3.3)}{=} \beta$$

and similarly $\Phi_z^*(\beta) = T^{-2}\alpha^{-1}\Phi_x(\beta^{-1})\alpha^{-1} = \alpha$. Hence $\Phi_z^*\Phi_z(\alpha) = \alpha$ and Lemma 2.2 gives $\|\Phi_z\|_{S_2^m \rightarrow S_2^m} = 1$. By Lemma 2.1 and (2.8), the element $\varphi = \sum_k z_k^T \otimes e_k^*$ of $\mathcal{M}_m(\operatorname{OH}_n^*)$ has norm

1. Under the parallel duality, we have $\langle u, \varphi \rangle \stackrel{(2.2)}{=} \sum_{k,i,j} (x_k)_{ij} (z_k)_{ji} = \sum_k \operatorname{Tr}(x_k z_k)$. We deduce that

$$N(x) \geq |\langle u, \varphi \rangle| = \left| \sum_{k=1}^n \operatorname{Tr}(x_k z_k) \right| = T^{-1} \operatorname{Tr}(\Phi_x(\beta^{-1})\alpha^{-1}) \stackrel{(3.3)}{=} T^{-1} \operatorname{Tr}(T^2\alpha^2) = T.$$

2. Let $\nu = \text{Tr}(P^{\frac{2}{3}}) = \text{Tr}(Q^{\frac{2}{3}})$, see Proposition 3.1, and put $\alpha \stackrel{\text{def}}{=} \nu^{-\frac{1}{2}} P^{\frac{1}{3}}$ and $\beta \stackrel{\text{def}}{=} \nu^{-\frac{1}{2}} Q^{\frac{1}{3}}$. These matrices belong to $\mathcal{P}_m$ and satisfy $\text{Tr}(\alpha^2) = \text{Tr}(\beta^2) = 1$. Moreover, we have

$$\Phi_x(\beta^{-1}) = \nu^{\frac{1}{2}} \Phi_x(Q^{-\frac{1}{3}}) \stackrel{(3.2)}{=} c^{-1} \nu^{\frac{1}{2}} P = c^{-1} \nu^2 \alpha^3$$

and in the same way $\Phi_x^*(\alpha^{-1}) = c^{-1} \nu^2 \beta^3$. By the first part with $T = c^{-\frac{1}{2}} \nu$, we obtain $N(x) = c^{-\frac{1}{2}} \nu$. $\blacksquare$

Remark 3.3 1. Suppose that $\Phi_x(1) = \Phi_x^*(1) = \gamma 1$ for some $\gamma > 0$, for example if $x_1, \dots, x_n$ are unitaries, in which case $\gamma = n$. Then $\alpha = \beta = m^{-\frac{1}{2}} 1$ satisfy (3.3) with $T = m\sqrt{\gamma}$, hence $N(x) = m\sqrt{\gamma}$.

2. The proof of the first part shows that $\sum_k z_k^\top \otimes e_k^*$, where $z_k = T^{-1} \beta^{-1} x_k^* \alpha^{-1}$, is a norming functional of $\sum_k x_k \otimes e_k$ under the parallel duality.

3. Suppose that $x \in \mathcal{G}_{m,n}$. Then

$$(3.5) \quad N(x)^2 = \min \{ \text{Tr}(\alpha^{-1} \Phi_x(\beta^{-1})) : \alpha, \beta \in \mathcal{P}_m, \quad \text{Tr}(\alpha^2) = \text{Tr}(\beta^2) = 1 \}.$$

Indeed, fix such α, β and let $w \stackrel{\text{def}}{=} \sum_k w_k \otimes e_k^*$ have norm at most 1 in $M_m(\text{OH}_n^*)$. Put $w^\top \stackrel{\text{def}}{=} (w_1^\top, \dots, w_n^\top)$. Applying the Hilbert-Schmidt Cauchy-Schwarz inequality to the families $\alpha^{-\frac{1}{2}} x_k \beta^{-\frac{1}{2}}$ and $\beta^{\frac{1}{2}} w_k^\top \alpha^{\frac{1}{2}}$ gives

$$\left| \sum_k \text{Tr}(x_k w_k^\top) \right|^2 \leq \text{Tr}(\alpha^{-1} \Phi_x(\beta^{-1})) \text{Tr}(\beta \Phi_{w^\top}(\alpha)) \leq \text{Tr}(\alpha^{-1} \Phi_x(\beta^{-1})).$$

The last inequality follows from $\|\Phi_{w^\top}\|_{S_2^m \rightarrow S_2^m} \leq 1$, by (2.7) and (2.8), and from $\|\alpha\|_{S_2^m} = \|\beta\|_{S_2^m} = 1$. Taking the supremum over w under the parallel duality proves the lower bound in (3.5). The matrices furnished by the second part of Theorem 3.2 attain equality, since

$$\text{Tr}(\alpha^{-1} \Phi_x(\beta^{-1})) \stackrel{(3.3)}{=} T^2 \text{Tr}(\alpha^2) = T^2 = N(x)^2.$$

The Euler-Lagrange equations are (3.3): differentiating the objective under the two constraints gives $\Phi_x(\beta^{-1}) = 2\lambda \alpha^3$ and $\Phi_x^*(\alpha^{-1}) = 2\mu \beta^3$, and taking traces after multiplication by α^{-1} and β^{-1} shows that $2\lambda = 2\mu = T^2$ at the minimum.

For general coefficients, a minimum over positive definite matrices need not be attained. For example, if $m = 2$, $n = 1$ and $x_1 = e_{11}$, then $N(x) = 1$, whereas the objective equals $(\alpha^{-1})_{11}(\beta^{-1})_{11} > 1$ for every admissible positive definite pair. Taking $\alpha = \beta = \text{diag}(\sqrt{1 - \delta^2}, \delta)$ and letting $\delta \rightarrow 0$ shows that the infimum is 1.

4 Free models and their L^1 -norms

In this section, σ denotes a nonzero finite positive Borel measure on $(0, \infty)$ with compact support. For any $X \in M_m$ with $X \geq 0$, we let

$$(4.1) \quad F_\sigma(X) \stackrel{\text{def}}{=} \int_0^\infty t(1 + tX)^{-1} d\sigma(t).$$

For any $X \geq 0$, the matrix $F_\sigma(X)$ is positive definite, since $F_\sigma(X) \geq \int_0^\infty t(1 + t\|X\|)^{-1} d\sigma(t) \cdot 1$. For any $t > 0$, the map $X \mapsto (1 + tX)^{-1}$ is order reversing on the positive matrices. Hence F_σ is order reversing. Finally, $F_\sigma(X)$ is a continuous function of X .

Definition 4.1 A σ -system is a quadruple $(\mathcal{A}, \tau, \Lambda, (c_k)_{k \geq 1})$ where $\mathcal{A}$ is a von Neumann algebra equipped with a normal faithful finite trace τ , Λ is a positive element of $\mathcal{A}$ such that $\tau(g(\Lambda)) = \int g d\sigma$ for any bounded Borel function $g: [0, \infty) \rightarrow \mathbb{C}$, and $(c_k)_{k \geq 1}$ is a sequence of elements of $\mathcal{A}$ satisfying the following property. Let $\mathcal{D}$ be the von Neumann algebra generated by Λ and $\mathbb{E}_{\mathcal{D}}: \mathcal{A} \rightarrow \mathcal{D}$ the τ -preserving normal conditional expectation. For any integer $r \geq 1$, any $k_1, \dots, k_r \geq 1$, any $\varepsilon_1, \dots, \varepsilon_r \in \{1, *\}$ and any $d_1, \dots, d_{r-1} \in \mathcal{D}$, the $\mathcal{D}$ -valued cumulant

$$\kappa_r^{\mathcal{D}}(c_{k_1}^{\varepsilon_1} d_1, \dots, c_{k_{r-1}}^{\varepsilon_{r-1}} d_{r-1}, c_{k_r}^{\varepsilon_r})$$

vanishes unless $r = 2$, $k_1 = k_2$ and $\{\varepsilon_1, \varepsilon_2\} = \{1, *\}$, and we have $\kappa_2^{\mathcal{D}}(c_k d, c_k^*) = \kappa_2^{\mathcal{D}}(c_k^* d, c_k) = \tau(d)1$ for any $k \geq 1$ and any $d \in \mathcal{D}$. We associate to such a system the elements $h_k \stackrel{\text{def}}{=} \Lambda^{\frac{1}{2}} c_k \Lambda^{\frac{1}{2}}$ of $\mathcal{A}$.

A unital C^* -algebra B has the weak expectation property if, for a faithful unital representation $B \subset B(H)$, there is a unital completely positive map $B(H) \rightarrow B^{**}$ extending the canonical inclusion of B into B^{**} . A C^* -algebra has QWEP if it is a quotient of a C^* -algebra with the weak expectation property, see [Oza04, Definition 3.1 p. 507].

Lemma 4.2 *There exists a σ -system whose underlying von Neumann algebra has QWEP.*

Proof : Let $\hat{\sigma} = \sigma((0, \infty))^{-1} \sigma$ and let $(\mathcal{A}, \varphi)$ be the reduced free product of the tracial W^* -probability spaces $(L^\infty(\hat{\sigma}), \int \cdot d\hat{\sigma})$ and $(VN(\mathbb{F}_\infty), \tau_0)$, where τ_0 is the canonical trace. The free product state φ is a normal faithful trace on $\mathcal{A}$, see [VDN92, Section 1.6 and Chapter 2]. Let Λ be the function $t \mapsto t$ of $L^\infty(\hat{\sigma})$ and let $(c_k^0)_{k \geq 1}$ be a free family of circular elements of $VN(\mathbb{F}_\infty)$ with $\tau_0((c_k^0)^* c_k^0) = 1$. Such a family is obtained as $c_k^0 = 2^{-\frac{1}{2}}(s_{2k-1} + i s_{2k})$, where $(s_j)_{j \geq 1}$ is a free family of standard semicircular elements generating $VN(\mathbb{F}_\infty)$ [VDN92, Chapter 2]. Put $\tau \stackrel{\text{def}}{=} \sigma((0, \infty))\varphi$ and $c_k \stackrel{\text{def}}{=} \sigma((0, \infty))^{\frac{1}{2}} c_k^0$. Then $\tau(g(\Lambda)) = \int g d\sigma$. The von Neumann algebra $\mathcal{N}$ generated by the c_k^0 's is free from $\mathcal{D} = L^\infty(\hat{\sigma})$. The only nonvanishing scalar free cumulants of the family $\{c_k^0, (c_k^0)^*\}$ are $\kappa_2(c_k^0, (c_k^0)^*) = \kappa_2((c_k^0)^*, c_k^0) = 1$. Hence (2.12) shows that $(\mathcal{A}, \tau, \Lambda, (c_k))$ is a σ -system.

It remains to check the QWEP property. Abelian von Neumann algebras have the weak expectation property. The algebra $VN(\mathbb{F}_\infty)$ is the weak* closure of the increasing union of the algebras $VN(\mathbb{F}_r)$, each of which is a reduced free product of r copies of $VN(\mathbb{Z})$. The stability of QWEP under reduced free products [Jun05, Theorem 7.15 p. 279] and increasing weak* closures [Oza04, Proposition 4.1 (iii) p. 516] shows that the von Neumann algebra $VN(\mathbb{F}_\infty)$ has QWEP. Applying [Jun05, Theorem 7.15 p. 279] once more proves that the algebra $\mathcal{A}$ has QWEP. ■

Remark 4.3 Suppose that σ has no atoms. Then the algebra $\mathcal{A}$ constructed in the proof of Lemma 4.2 is isomorphic to $VN(\mathbb{F}_\infty)$. Indeed, $\hat{\sigma}$ is a nonatomic Borel probability measure on a compact interval, so $(L^\infty(\hat{\sigma}), \int \cdot d\hat{\sigma})$ is isomorphic to $(L^\infty([0, 1]), \int \cdot dt)$ [Kec95, Theorem 17.41 p. 116], hence to $(VN(\mathbb{Z}), \tau_{\mathbb{Z}})$ by Fourier transform. The reduced free product of group von Neumann algebras equipped with their canonical traces is the group von Neumann algebra of the free product group. Hence $(\mathcal{A}, \varphi)$ is isomorphic to $(VN(\mathbb{Z} * \mathbb{F}_\infty), \tau) = (VN(\mathbb{F}_\infty), \tau_0)$. This applies to the measures σ_ε of Section 5. Recall also that $VN(\mathbb{F}_\infty)^{\text{op}}$ is isomorphic to $VN(\mathbb{F}_\infty)$ via $\lambda_g^{\text{op}} \mapsto \lambda_{g^{-1}}$.

For any integers $m, n \geq 1$, any $x \in (M_m)^n$ and any σ -system, we consider the element

$$(4.2) \quad Z_x \stackrel{\text{def}}{=} \sum_{k=1}^n x_k \otimes h_k$$

of $M_m \overline{\otimes} \mathcal{A}$. We equip $M_m \overline{\otimes} \mathcal{A}$ with the normal faithful finite trace $\text{Tr} \otimes \tau$. The following result is the second main ingredient of the paper.

Proposition 4.4 *Let $m, n \geq 1$ be integers and let $x \in \mathcal{G}_{m,n}$.*

1. *For any $s > 0$, the system*

$$(4.3) \quad P = s^{-1} \Phi_x(F_\sigma(Q)), \quad Q = s^{-1} \Phi_x^*(F_\sigma(P))$$

admits a unique solution (P_s, Q_s) in $\mathcal{P}_m \times \mathcal{P}_m$.

2. *Consider a σ -system and put $Y \stackrel{\text{def}}{=} Z_x^* Z_x$. For any $s > 0$, we have*

$$(4.4) \quad (\text{Tr} \otimes \tau)(Y(s + Y)^{-1}) = \text{Tr}(Q_s F_\sigma(Q_s)).$$

3. *Consequently, we have*

$$(4.5) \quad \|Z_x\|_{L^1(M_m \overline{\otimes} \mathcal{A})} = \frac{1}{\pi} \int_0^\infty s^{-\frac{1}{2}} \text{Tr}(Q_s F_\sigma(Q_s)) ds.$$

Proof : 1. We first prove the uniqueness. Fix $s > 0$. Consider the map $R: \mathcal{P}_m \rightarrow \mathcal{P}_m$ defined by $R(P) = s^{-1} \Phi_x(F_\sigma(Q(P)))$ where $Q(P) = s^{-1} \Phi_x^*(F_\sigma(P))$. If (P, Q) is a solution of (4.3) in $\mathcal{P}_m \times \mathcal{P}_m$ then $R(P) = P$ and $Q = Q(P)$. Hence it suffices to show that R has at most one fixed point. Since F_σ is order reversing and since Φ_x and Φ_x^* are positive, the map R is order preserving. Let $\lambda > 1$ and $P \in \mathcal{P}_m$. For any $t > 0$ and any positive matrix X , the commuting positive matrices X and 1 satisfy $\lambda(1 + \lambda t X)^{-1} \geq (1 + t X)^{-1}$, hence $F_\sigma(\lambda X) \geq \lambda^{-1} F_\sigma(X)$. We deduce that $Q(\lambda P) \geq \lambda^{-1} Q(P)$ and, since F_σ is order reversing, that $F_\sigma(Q(\lambda P)) \leq F_\sigma(\lambda^{-1} Q(P))$. Moreover, for any $X \in \mathcal{P}_m$ we have

$$\lambda F_\sigma(X) - F_\sigma(\lambda^{-1} X) = \lambda(\lambda - 1) \int_0^\infty t(1 + tX)^{-1}(\lambda + tX)^{-1} d\sigma(t),$$

which is positive definite since σ is nonzero and supported in $(0, \infty)$. Since Φ_x sends $\mathcal{P}_m$ into $\mathcal{P}_m$, we obtain

$$R(\lambda P) \leq s^{-1} \Phi_x(F_\sigma(\lambda^{-1} Q(P))) = \lambda R(P) - s^{-1} \Phi_x(\lambda F_\sigma(Q(P)) - F_\sigma(\lambda^{-1} Q(P))),$$

hence

$$(4.6) \quad R(\lambda P) < \lambda R(P),$$

where $X < Y$ means that $Y - X$ is positive definite. Now, let $P, P' \in \mathcal{P}_m$ be two fixed points of R . Suppose that $P' \not\leq P$ and let $\lambda = \min\{\mu \geq 1 : P' \leq \mu P\}$. Then $\lambda > 1$ and

$$P' = R(P') \leq R(\lambda P) \stackrel{(4.6)}{<} \lambda R(P) = \lambda P.$$

Hence $\lambda P - P'$ is positive definite, so $P' \leq (\lambda - \delta)P$ for some $\delta > 0$, which contradicts the minimality of λ . Thus $P' \leq P$ and, by symmetry, $P \leq P'$. This proves the uniqueness. The existence will follow from the proof of the second part, since σ -systems exist by Lemma 4.2.

2. Let $\mathcal{D}$ be the von Neumann algebra generated by Λ , which we identify with $L^\infty(\sigma)$ in such a way that Λ corresponds to the function $t \mapsto t$ and $\tau|_{\mathcal{D}}$ to the integral with respect to σ .

Consider the von Neumann algebras $\mathcal{B} = M_{2m}(\mathcal{D}) \subset M_{2m}(\mathcal{A})$ and the conditional expectation $\mathbb{E} = \text{Id}_{M_{2m}} \otimes \mathbb{E}_{\mathcal{D}}$. Consider the hermitization of Z_x , that is the selfadjoint element

$$H \stackrel{\text{def}}{=} \begin{bmatrix} 0 & Z_x \\ Z_x^* & 0 \end{bmatrix} = \hat{\Lambda}^{\frac{1}{2}} \left(\sum_{k=1}^n (X_k \otimes 1)(1 \otimes c_k) + (X_k^* \otimes 1)(1 \otimes c_k^*) \right) \hat{\Lambda}^{\frac{1}{2}}$$

of $M_{2m}(\mathcal{A})$, where $X_k = e_{12} \otimes x_k \in M_2 \otimes M_m = M_{2m}$ and $\hat{\Lambda} = 1_{M_{2m}} \otimes \Lambda \in \mathcal{B}$. Note that the matrices $X_k \otimes 1$ and $X_k^* \otimes 1$ belong to $\mathcal{B}$. By (2.13) and Definition 4.1, the elements $1 \otimes c_k$ and $1 \otimes c_k^*$ have vanishing $\mathcal{B}$ -valued cumulants of order $r \neq 2$, and for any $\beta \in \mathcal{B}$ we have

$$\kappa_2^{\mathcal{B}}((1 \otimes c_k)\beta, 1 \otimes c_l^*) = \kappa_2^{\mathcal{B}}((1 \otimes c_k^*)\beta, 1 \otimes c_l) = \delta_{kl}(\text{Id}_{M_{2m}} \otimes \tau)(\beta) \otimes 1$$

and $\kappa_2^{\mathcal{B}}((1 \otimes c_k)\beta, 1 \otimes c_l) = \kappa_2^{\mathcal{B}}((1 \otimes c_k^*)\beta, 1 \otimes c_l^*) = 0$. Indeed, every cumulant of H expands, by multilinearity and balancing, into a finite sum of terms

$$b_0 \kappa_r^{\mathcal{B}}((1 \otimes c_{k_1}^{\varepsilon_1})b_1, \dots, (1 \otimes c_{k_r}^{\varepsilon_r})b_r), \quad b_0, \dots, b_r \in \mathcal{B}.$$

Here each ε_j is either 1 or $*$, so $c_{k_j}^{\varepsilon_j}$ denotes c_{k_j} or its adjoint. For $r \neq 2$ these terms vanish by (2.13) and Definition 4.1, the factors $\hat{\Lambda}^{\frac{1}{2}} \in \mathcal{B}$ are absorbed in the b_i 's by the balancing property. Thus H is $\mathcal{B}$ -valued semicircular. Moreover, using again the balancing property and the commutation of $\hat{\Lambda}$ with $M_{2m} \otimes 1$, its covariance is

$$\eta(\beta) = \mathbb{E}(H\beta H) = \sum_{k=1}^n (X_k \gamma_{\beta} X_k^* + X_k^* \gamma_{\beta} X_k) \otimes \Lambda,$$

where $\gamma_{\beta} = (\text{Id}_{M_{2m}} \otimes \tau)(\hat{\Lambda}^{\frac{1}{2}} \beta \hat{\Lambda}^{\frac{1}{2}}) = (\text{Id}_{M_{2m}} \otimes \tau)(\hat{\Lambda} \beta)$, since $\mathcal{D}$ is abelian. We view the elements of $\mathcal{B}$ as 2×2 block matrices $\beta = [\beta_{ij}]$ with entries in $M_m(\mathcal{D}) = L^{\infty}(\sigma, M_m)$. Since $X_k \gamma X_k^* = e_{11} \otimes x_k \gamma_{22} x_k^*$ and $X_k^* \gamma X_k = e_{22} \otimes x_k^* \gamma_{11} x_k$ for any $\gamma = [\gamma_{ij}] \in M_{2m}$, we obtain

$$(4.7) \quad \eta(\beta) = \begin{bmatrix} \eta_1(\beta_{22}) & 0 \\ 0 & \eta_2(\beta_{11}) \end{bmatrix},$$

where

$$\eta_1(b)(t) = t \Phi_x \left(\int_0^{\infty} t' b(t') d\sigma(t') \right) \quad \text{and} \quad \eta_2(b)(t) = t \Phi_x^* \left(\int_0^{\infty} t' b(t') d\sigma(t') \right)$$

for any $b \in L^{\infty}(\sigma, M_m)$ and σ -almost every t .

Fix $s > 0$ and put $z \stackrel{\text{def}}{=} i\sqrt{s}$. Since $H^2 = \text{diag}(Z_x Z_x^*, Z_x^* Z_x)$, we have $(z - H)^{-1} = (z + H)(z^2 - H^2)^{-1}$. Hence the diagonal blocks of $(z - H)^{-1}$ are $-i\sqrt{s}(s + Z_x Z_x^*)^{-1}$ and $-i\sqrt{s}(s + Z_x^* Z_x)^{-1}$. The range of η is diagonal by blocks, by (4.7). Therefore the Dyson equation (2.11) shows directly that $G(z) = \mathbb{E}((z - H)^{-1})$ is diagonal by blocks. Consequently, we have $G(z) = -i\sqrt{s} \text{diag}(W_1, W_2)$ where

$$W_1 = \mathbb{E}_{M_m(\mathcal{D})}((s + Z_x Z_x^*)^{-1}) \quad \text{and} \quad W_2 = \mathbb{E}_{M_m(\mathcal{D})}((s + Z_x^* Z_x)^{-1}).$$

These elements are positive and invertible since $(s + \|Z_x\|^2)^{-1} \leq (s + Z_x Z_x^*)^{-1}$ and $(s + \|Z_x\|^2)^{-1} \leq (s + Z_x^* Z_x)^{-1}$. By (2.11) and (4.7), we have

$$-i\sqrt{s}W_1 = (i\sqrt{s} + i\sqrt{s}\eta_1(W_2))^{-1} \quad \text{and} \quad -i\sqrt{s}W_2 = (i\sqrt{s} + i\sqrt{s}\eta_2(W_1))^{-1},$$

that is $W_1 = s^{-1}(1 + \eta_1(W_2))^{-1}$ and $W_2 = s^{-1}(1 + \eta_2(W_1))^{-1}$. Put

$$P \stackrel{\text{def}}{=} \Phi_x \left(\int_0^\infty t W_2(t) d\sigma(t) \right) \quad \text{and} \quad Q \stackrel{\text{def}}{=} \Phi_x^* \left(\int_0^\infty t W_1(t) d\sigma(t) \right).$$

By (4.7), we have $W_1(t) = s^{-1}(1 + tP)^{-1}$ and $W_2(t) = s^{-1}(1 + tQ)^{-1}$ for σ -almost every t . Substituting in the definitions of P and Q , we obtain that (P, Q) is a solution of (4.3). Moreover, we have $\int_0^\infty t W_2(t) d\sigma(t) \geq (s + \|Z_x\|^2)^{-1} \int_0^\infty t d\sigma(t) \cdot 1$, hence $P \in \mathcal{P}_m$ since $\Phi_x(1)$ is invertible, and similarly $Q \in \mathcal{P}_m$. This proves the existence in the first part. By uniqueness, we have $(P, Q) = (P_s, Q_s)$. Finally, using $(\text{Tr} \otimes \tau) \circ \mathbb{E}_{M_m(\mathcal{D})} = \text{Tr} \otimes \tau$, we obtain

$$\begin{aligned} (\text{Tr} \otimes \tau)(Y(s + Y)^{-1}) &= (\text{Tr} \otimes \tau)(1 - s(s + Y)^{-1}) = \int_0^\infty \text{Tr}(1 - sW_2(t)) d\sigma(t) \\ &= \int_0^\infty \text{Tr}(1 - (1 + tQ_s)^{-1}) d\sigma(t) = \int_0^\infty \text{Tr}(tQ_s(1 + tQ_s)^{-1}) d\sigma(t) = \text{Tr}(Q_s F_\sigma(Q_s)). \end{aligned}$$

3. For any $y \geq 0$, we have $y^{\frac{1}{2}} = \frac{1}{\pi} \int_0^\infty s^{-\frac{1}{2}} \frac{y}{s+y} ds$. By the spectral theorem and the normality of the trace, we obtain

$$\|Z_x\|_{L^1(M_m \bar{\otimes} \mathcal{A})} = (\text{Tr} \otimes \tau)(Y^{\frac{1}{2}}) = \frac{1}{\pi} \int_0^\infty s^{-\frac{1}{2}} (\text{Tr} \otimes \tau)(Y(s + Y)^{-1}) ds,$$

and (4.4) gives (4.5). ■

For any $x \in \mathcal{G}_{m,n}$ and any $s > 0$, we let

$$(4.8) \quad f_{\sigma,x}(s) \stackrel{\text{def}}{=} \text{Tr}(Q_s F_\sigma(Q_s)),$$

where (P_s, Q_s) is the solution of (4.3). By Proposition 4.4, this quantity does not depend on the choice of the σ -system.

Lemma 4.5 *Let σ' be a nonzero finite positive Borel measure on $(0, \infty)$ with compact support and let $S \subset (0, \infty)$ be a Borel subset with $\sigma'(S) > 0$. Put $\sigma \stackrel{\text{def}}{=} 1_S \sigma'$. For any integers $m, n \geq 1$, any $x \in \mathcal{G}_{m,n}$ and any $s > 0$, we have $f_{\sigma,x}(s) \leq f_{\sigma',x}(s)$.*

Proof : Let $(\mathcal{A}, \tau, \Lambda', (c'_k))$ be a σ' -system, see Lemma 4.2, let $\mathcal{D}'$ be the von Neumann algebra generated by Λ' and let $p = 1_S(\Lambda') \in \mathcal{D}'$. We first show that $(p\mathcal{A}p, \tau|_{p\mathcal{A}p}, p\Lambda', (pc'_k p))$ is a σ -system. The restriction of τ is a normal faithful finite trace on $p\mathcal{A}p$ and the functional calculus of $p\Lambda'$ in $p\mathcal{A}p$ satisfies $g(p\Lambda') = (g1_S)(\Lambda')$, hence $\tau(g(p\Lambda')) = \int g1_S d\sigma' = \int g d\sigma$. The von Neumann algebra generated by $p\Lambda'$ in $p\mathcal{A}p$ is $p\mathcal{D}'$, and the restriction of $\mathbb{E}_{\mathcal{D}'}$ to $p\mathcal{A}p$ is the τ -preserving conditional expectation onto $p\mathcal{D}'$, since $p \in \mathcal{D}'$. Write $\mathbb{E}_p = \mathbb{E}_{\mathcal{D}'}|_{p\mathcal{A}p}$. For $y_1, \dots, y_r \in p\mathcal{A}p$, the balancing relations give

$$\kappa_r^{\mathcal{D}'}(y_1, \dots, y_r) = p\kappa_r^{\mathcal{D}'}(y_1, \dots, y_r)p \in p\mathcal{D}'.$$

The restrictions of these cumulants satisfy the moment-cumulant formula for $\mathbb{E}_p$, since every intermediate value lies in $p\mathcal{D}'$. By uniqueness, they are the $p\mathcal{D}'$ -valued cumulants of the corner. Consequently, for $r \geq 2$, $a_1, \dots, a_r \in \{c'_k, c'^*_k : k \geq 1\}$ and $d_1, \dots, d_{r-1} \in p\mathcal{D}'$, we have

$$\kappa_r^{p\mathcal{D}'}(pa_1 p d_1, \dots, pa_r p) = p\kappa_r^{\mathcal{D}'}(a_1 d_1, \dots, a_r)p.$$

Indeed, the intermediate projections are absorbed by the balancing relations and $pd_j = d_jp = d_j$. The first cumulants vanish by compression of the centering identity. All cumulants of order at least 3 vanish, and the only nonzero second cumulants are

$$\begin{aligned}\kappa_2^{p\mathcal{D}'}(pc'_kpd, pc'_l{}^*p) &= \delta_{kl}\tau(d)p, \\ \kappa_2^{p\mathcal{D}'}(pc'_k{}^*pd, pc'_l p) &= \delta_{kl}\tau(d)p.\end{aligned}$$

Thus $(pc'_k p)$ has the covariance required by Definition 4.1 with the restricted trace $\tau|_{p\mathcal{A}p}$. This proves that we have a σ -system, whose associated elements are $h_k = (p\Lambda')^{\frac{1}{2}}pc'_k p(p\Lambda')^{\frac{1}{2}} = ph'_k p$.

Let $Z' \stackrel{\text{def}}{=} \sum_k x_k \otimes h'_k$ and $Z \stackrel{\text{def}}{=} (1 \otimes p)Z'(1 \otimes p) = \sum_k x_k \otimes h_k$. For any $s > 0$, the function $g_s(u) = \frac{u^2}{s+u^2}$ is continuous, nondecreasing and satisfies $g_s(0) = 0$. Since $g_s(0) = 0$, the element $g_s(|Z|)$ is the same, whether it is computed in $(1 \otimes p)(M_m \overline{\otimes} \mathcal{A})(1 \otimes p)$ or in $M_m \overline{\otimes} \mathcal{A}$. By Proposition 4.4 applied to both systems and by the facts recalled in Subsection 2, we obtain

$$\begin{aligned}f_{\sigma,x}(s) &= (\text{Tr} \otimes \tau)(g_s(|Z|)) = \int_0^\infty g_s(\mu_t(Z)) dt \\ &\leq \int_0^\infty g_s(\mu_t(Z')) dt = (\text{Tr} \otimes \tau)(g_s(|Z'|)) = f_{\sigma',x}(s),\end{aligned}$$

where we used the inequality $\mu_t(Z) = \mu_t((1 \otimes p)Z'(1 \otimes p)) \leq \mu_t(Z')$. ■

Remark 4.6 Let $x \in \mathcal{G}_{m,n}$ and $0 < \alpha < 1$. For the formal infinite measure $\sigma_\alpha = t^{-1-\alpha} dt$, a change of variables gives $F_{\sigma_\alpha}(X) = c_\alpha X^{\alpha-1}$ for $X > 0$, where $c_\alpha \stackrel{\text{def}}{=} \frac{\pi}{\sin(\pi\alpha)}$. The argument of Proposition 3.1, with contraction ratio $1 - \alpha$, gives a unique positive definite solution of the corresponding formal system (4.3) at $s = 1$. Homogeneity then gives

$$(P_s, Q_s) = s^{-\frac{1}{2-\alpha}}(P_1, Q_1).$$

Consequently, $\text{Tr}(Q_s F_{\sigma_\alpha}(Q_s))$ is proportional to $s^{-\frac{\alpha}{2-\alpha}}$, and substituting this profile into the integrand of (4.5) gives $s^{-\frac{1}{2} - \frac{\alpha}{2-\alpha}}$. This is proportional to s^{-1} if and only if $\alpha = \frac{2}{3}$. This explains the exponent $-\frac{5}{3}$ of the truncated measures of Section 5 and the logarithmic normalization.

5 Truncated power laws

For any $0 < \varepsilon < 1$, we consider the finite measure

$$(5.1) \quad \sigma_\varepsilon \stackrel{\text{def}}{=} 1_{[\varepsilon, \frac{1}{\varepsilon}]}(t)t^{-\frac{5}{3}} dt$$

on $(0, \infty)$ and we let $F_\varepsilon = F_{\sigma_\varepsilon}$ and $f_{\varepsilon,x} = f_{\sigma_\varepsilon,x}$, see (4.1) and (4.8). For any $q > 0$, we let

$$(5.2) \quad \omega_\varepsilon(q) \stackrel{\text{def}}{=} \frac{3}{\kappa} \varepsilon^{\frac{1}{3}} q^{\frac{1}{3}} + \frac{3}{2\kappa} \varepsilon^{\frac{2}{3}} q^{-\frac{2}{3}}.$$

Lemma 5.1 *Let $0 < \varepsilon < 1$.*

1. *For any $q > 0$, we have $(1 - \omega_\varepsilon(q))\kappa q^{-\frac{1}{3}} \leq F_\varepsilon(q) \leq \kappa q^{-\frac{1}{3}}$.*
2. *Let $0 < q_- \leq q_+$ and let $X \in \mathcal{P}_m$ whose spectrum is contained in $[q_-, q_+]$. Put $\omega \stackrel{\text{def}}{=} \frac{3}{\kappa} \varepsilon^{\frac{1}{3}} q_+^{\frac{1}{3}} + \frac{3}{2\kappa} \varepsilon^{\frac{2}{3}} q_-^{-\frac{2}{3}}$. Then $(1 - \omega)\kappa X^{-\frac{1}{3}} \leq F_\varepsilon(X) \leq \kappa X^{-\frac{1}{3}}$.*

3. We have $\sigma_\varepsilon((0, \infty)) \leq \frac{3}{2}\varepsilon^{-\frac{2}{3}}$ and $\int_0^\infty t d\sigma_\varepsilon(t) \leq 3\varepsilon^{-\frac{1}{3}}$.

Proof : 1. With the change of variables $u = tq$, we have $\int_0^\infty t^{-\frac{2}{3}}(1+tq)^{-1} dt = \kappa q^{-\frac{1}{3}}$. Hence $F_\varepsilon(q) = \int_\varepsilon^{\frac{1}{\varepsilon}} t^{-\frac{2}{3}}(1+tq)^{-1} dt \leq \kappa q^{-\frac{1}{3}}$ and

$$\kappa q^{-\frac{1}{3}} - F_\varepsilon(q) = \int_0^\varepsilon \frac{t^{-\frac{2}{3}}}{1+tq} dt + \int_{\frac{1}{\varepsilon}}^\infty \frac{t^{-\frac{2}{3}}}{1+tq} dt \leq \int_0^\varepsilon t^{-\frac{2}{3}} dt + \frac{1}{q} \int_{\frac{1}{\varepsilon}}^\infty t^{-\frac{5}{3}} dt = 3\varepsilon^{\frac{1}{3}} + \frac{3}{2q}\varepsilon^{\frac{2}{3}}.$$

Dividing by $\kappa q^{-\frac{1}{3}}$, we obtain the first point.

2. The matrices $F_\varepsilon(X)$ and $X^{-\frac{1}{3}}$ are functions of X . Hence the second point follows from the first one by considering an orthonormal basis of eigenvectors of X , since $\omega_\varepsilon(q) \leq \omega$ for any $q \in [q_-, q_+]$.

3. We have $\int_\varepsilon^{\frac{1}{\varepsilon}} t^{-\frac{5}{3}} dt \leq \frac{3}{2}\varepsilon^{-\frac{2}{3}}$ and $\int_\varepsilon^{\frac{1}{\varepsilon}} t^{-\frac{2}{3}} dt \leq 3\varepsilon^{-\frac{1}{3}}$. ■

Let $m, n \geq 1$ be integers and let $x \in \mathcal{G}_{m,n}$. We denote by (P_*, Q_*) the solution of the system (3.2) with $c = \kappa$, see Proposition 3.1, and we fix $0 < a \leq b$ such that the spectra of P_* and Q_* are contained in $[a, b]$. For any $0 < \varepsilon < 1$ and any $s > 0$, we let

$$(5.3) \quad \omega_x(\varepsilon, s) \stackrel{\text{def}}{=} \frac{3}{\kappa} (eb)^{\frac{1}{3}} \varepsilon^{\frac{1}{3}} s^{-\frac{1}{4}} + \frac{3}{2\kappa} \left(\frac{e}{a}\right)^{\frac{2}{3}} \varepsilon^{\frac{2}{3}} s^{\frac{1}{2}}.$$

Proposition 5.2 *Let $m, n \geq 1$ be integers and let $x \in \mathcal{G}_{m,n}$. For any $0 < \varepsilon < 1$ and any $s > 0$ such that $\omega = \omega_x(\varepsilon, s) \leq \frac{1}{3}$, we have*

$$(5.4) \quad (1 - \omega)^2 \kappa^{\frac{3}{2}} N(x) s^{-\frac{1}{2}} \leq f_{\varepsilon, x}(s) \leq (1 - \omega)^{-1} \kappa^{\frac{3}{2}} N(x) s^{-\frac{1}{2}}.$$

Proof : Consider the set $K = \{(\tilde{P}, \tilde{Q}) \in \mathcal{P}_m \times \mathcal{P}_m : e^{-1}P_* \leq \tilde{P} \leq eP_*, e^{-1}Q_* \leq \tilde{Q} \leq eQ_*\}$, which is a compact convex subset of $M_m \times M_m$, and the continuous map $\tilde{R}: K \rightarrow \mathcal{P}_m \times \mathcal{P}_m$ defined by

$$\tilde{R}(\tilde{P}, \tilde{Q}) = \left(s^{-\frac{1}{4}} \Phi_x(F_\varepsilon(s^{-\frac{3}{4}} \tilde{Q})), s^{-\frac{1}{4}} \Phi_x^*(F_\varepsilon(s^{-\frac{3}{4}} \tilde{P})) \right).$$

A pair $(\tilde{P}, \tilde{Q})$ is a fixed point of $\tilde{R}$ if and only if the pair $(s^{-\frac{3}{4}} \tilde{P}, s^{-\frac{3}{4}} \tilde{Q})$ is a solution of the system (4.3) for the measure σ_ε .

Let $(\tilde{P}, \tilde{Q}) \in K$. By (2.10), the spectrum of $s^{-\frac{3}{4}} \tilde{Q}$ is contained in $[s^{-\frac{3}{4}} e^{-1}a, s^{-\frac{3}{4}} eb]$. By the second point of Lemma 5.1 applied with $q_- = s^{-\frac{3}{4}} e^{-1}a$ and $q_+ = s^{-\frac{3}{4}} eb$, for which the associated number is exactly ω , and since $(s^{-\frac{3}{4}} \tilde{Q})^{-\frac{1}{3}} = s^{\frac{1}{4}} \tilde{Q}^{-\frac{1}{3}}$, we obtain

$$(5.5) \quad (1 - \omega) \kappa \Phi_x(\tilde{Q}^{-\frac{1}{3}}) \leq s^{-\frac{1}{4}} \Phi_x(F_\varepsilon(s^{-\frac{3}{4}} \tilde{Q})) \leq \kappa \Phi_x(\tilde{Q}^{-\frac{1}{3}}).$$

Moreover, by the properties 1, 2 and 3 below Definition 2.9 and by (3.2), we have

$$d(\kappa \Phi_x(\tilde{Q}^{-\frac{1}{3}}), P_*) = d(\kappa \Phi_x(\tilde{Q}^{-\frac{1}{3}}), \kappa \Phi_x(Q_*^{-\frac{1}{3}})) \leq \frac{1}{3} d(\tilde{Q}, Q_*) \leq \frac{1}{3}.$$

Since $\omega \leq \frac{1}{3}$, we have $(1 - \omega)e^{-\frac{1}{3}} \geq \frac{2}{3}e^{-\frac{1}{3}} \geq e^{-1}$. We deduce that $e^{-1}P_* \leq s^{-\frac{1}{4}} \Phi_x(F_\varepsilon(s^{-\frac{3}{4}} \tilde{Q})) \leq eP_*$. The same argument applies to the second component. Hence $\tilde{R}(K) \subset K$. By Brouwer's fixed point theorem, $\tilde{R}$ admits a fixed point in K . By the uniqueness in Proposition 4.4, we conclude that the pair $(\tilde{P}_s, \tilde{Q}_s) = (s^{\frac{3}{4}} P_s, s^{\frac{3}{4}} Q_s)$ belongs to K , where (P_s, Q_s) is the solution of (4.3) for the measure σ_ε .

Now, let $D \stackrel{\text{def}}{=} \max\{d(\tilde{P}_s, P_*), d(\tilde{Q}_s, Q_*)\} \leq 1$. Since $\tilde{P}_s = s^{-\frac{1}{4}} \Phi_x(F_\varepsilon(s^{-\frac{3}{4}} \tilde{Q}_s))$, the inequalities (5.5) give $d(\tilde{P}_s, \kappa \Phi_x(\tilde{Q}_s^{-\frac{1}{3}})) \leq \log \frac{1}{1-\omega}$. Using the triangle inequality and the previous

estimate, we obtain $d(\tilde{P}_s, P_*) \leq \log \frac{1}{1-\omega} + \frac{D}{3}$. The same inequality holds for $d(\tilde{Q}_s, Q_*)$. Hence $D \leq \log \frac{1}{1-\omega} + \frac{D}{3}$, that is

$$(5.6) \quad D \leq \frac{3}{2} \log \frac{1}{1-\omega}.$$

Finally, since $(\tilde{P}_s, \tilde{Q}_s) \in K$, the spectrum of $Q_s = s^{-\frac{3}{4}} \tilde{Q}_s$ is contained in $[s^{-\frac{3}{4}} e^{-1} a, s^{-\frac{3}{4}} e b]$, and the second point of Lemma 5.1 gives $(1-\omega) \kappa Q_s^{\frac{2}{3}} \leq Q_s F_\varepsilon(Q_s) \leq \kappa Q_s^{\frac{2}{3}}$, where these matrices commute. Moreover, by the second point below Definition 2.9 and (5.6), we have $e^{-\frac{2D}{3}} Q_*^{\frac{2}{3}} \leq \tilde{Q}_s^{\frac{2}{3}} \leq e^{\frac{2D}{3}} Q_*^{\frac{2}{3}}$ with $e^{\frac{2D}{3}} \leq (1-\omega)^{-1}$. Since $Q_s^{\frac{2}{3}} = s^{-\frac{1}{2}} \tilde{Q}_s^{\frac{2}{3}}$, we obtain

$$(1-\omega)^2 \kappa s^{-\frac{1}{2}} \text{Tr}(Q_*^{\frac{2}{3}}) \leq f_{\varepsilon,x}(s) = \text{Tr}(Q_s F_\varepsilon(Q_s)) \leq (1-\omega)^{-1} \kappa s^{-\frac{1}{2}} \text{Tr}(Q_*^{\frac{2}{3}}).$$

By Theorem 3.2 with $c = \kappa$, we have $\kappa \text{Tr}(Q_*^{\frac{2}{3}}) = \kappa^{\frac{3}{2}} N(x)$. This completes the proof. $\blacksquare$

Proposition 5.3 *Let $m, n \geq 1$ be integers and let $x \in \mathcal{G}_{m,n}$. For any $0 < \varepsilon < 1$ and any $s > 0$, we have*

$$(5.7) \quad f_{\varepsilon,x}(s) \leq \kappa^{\frac{3}{2}} N(x) s^{-\frac{1}{2}}.$$

Proof : Let $0 < \varepsilon' < \varepsilon$. We have $\sigma_\varepsilon = 1_{[\varepsilon, \frac{1}{\varepsilon}]} \sigma_{\varepsilon'}$. By Lemma 4.5, we obtain $f_{\varepsilon,x}(s) \leq f_{\varepsilon',x}(s)$. Since $\lim_{\varepsilon' \rightarrow 0} \omega_x(\varepsilon', s) = 0$, Proposition 5.2 gives $\limsup_{\varepsilon' \rightarrow 0} f_{\varepsilon',x}(s) \leq \kappa^{\frac{3}{2}} N(x) s^{-\frac{1}{2}}$. $\blacksquare$

For any $0 < \varepsilon < 1$, we fix a σ_ε -system $(\mathcal{A}_\varepsilon, \tau_\varepsilon, \Lambda_\varepsilon, (c_k^{(\varepsilon)}))$ with $\mathcal{A}_\varepsilon$ having QWEP, as provided by Lemma 4.2, and we denote by $h_k^{(\varepsilon)}$ the associated elements and by $Z_x^{(\varepsilon)} = \sum_k x_k \otimes h_k^{(\varepsilon)}$. We let

$$(5.8) \quad c_* \stackrel{\text{def}}{=} \frac{8\kappa^{\frac{3}{2}}}{3\pi}.$$

Theorem 5.4 *Let $m, n \geq 1$ be integers and let $x \in \mathcal{G}_{m,n}$. As $\varepsilon \rightarrow 0$, we have*

$$(5.9) \quad \|Z_x^{(\varepsilon)}\|_{L^1(M_m \overline{\otimes} \mathcal{A}_\varepsilon)} = c_* N(x) \log \frac{1}{\varepsilon} + O_x(1).$$

In particular,

$$(5.10) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{\log \frac{1}{\varepsilon}} \|Z_x^{(\varepsilon)}\|_{L^1(M_m \overline{\otimes} \mathcal{A}_\varepsilon)} = c_* N(x).$$

Proof : Set $K_x = \kappa^{\frac{3}{2}} N(x)$ and write

$$\omega_x(\varepsilon, s) = A_x \varepsilon^{\frac{1}{3}} s^{-\frac{1}{4}} + B_x \varepsilon^{\frac{2}{3}} s^{\frac{1}{2}},$$

where $A_x, B_x > 0$ are the constants in (5.3). Choose $a_x > 0$ sufficiently large and $b_x > 0$ sufficiently small that

$$A_x a_x^{-\frac{1}{4}} + B_x b_x^{\frac{1}{2}} \leq \frac{1}{3}.$$

For sufficiently small ε , the interval $J_\varepsilon = [a_x \varepsilon^{\frac{4}{3}}, b_x \varepsilon^{-\frac{4}{3}}]$ is nonempty and $\omega_x(\varepsilon, s) \leq \frac{1}{3}$ on J_ε . Propositions 5.2 and 5.3 give

$$0 \leq K_x s^{-\frac{1}{2}} - f_{\varepsilon,x}(s) \leq 2K_x \omega_x(\varepsilon, s) s^{-\frac{1}{2}}.$$

Moreover, we see that

$$\int_{J_\varepsilon} \omega_x(\varepsilon, s) \frac{ds}{s} \leq 4A_x a_x^{-\frac{1}{4}} + 2B_x b_x^{\frac{1}{2}}.$$

Consequently, we have

$$\frac{1}{\pi} \int_{J_\varepsilon} s^{-\frac{1}{2}} f_{\varepsilon, x}(s) ds = \frac{K_x}{\pi} \log \frac{b_x \varepsilon^{-\frac{4}{3}}}{a_x \varepsilon^{\frac{4}{3}}} + O_x(1) \stackrel{(5.8)}{=} c_* N(x) \log \frac{1}{\varepsilon} + O_x(1).$$

It remains to bound the two tails. Put $Y \stackrel{\text{def}}{=} (Z_x^{(\varepsilon)})^* Z_x^{(\varepsilon)}$. By (4.4) and Lemma 5.1, we have

$$f_{\varepsilon, x}(s) \leq (\text{Tr} \otimes \tau_\varepsilon)(1) = m \sigma_\varepsilon((0, \infty)) \leq \frac{3m}{2} \varepsilon^{-\frac{2}{3}},$$

and hence

$$\frac{1}{\pi} \int_0^{a_x \varepsilon^{\frac{4}{3}}} s^{-\frac{1}{2}} f_{\varepsilon, x}(s) ds \leq \frac{3m \sqrt{a_x}}{\pi}.$$

For the upper tail, the covariance identities in Definition 4.1 give

$$\mathbb{E}_D((c_k^{(\varepsilon)})^* \Lambda_\varepsilon c_l^{(\varepsilon)}) = \delta_{kl} \tau_\varepsilon(\Lambda_\varepsilon) 1.$$

It follows that $\tau_\varepsilon((h_k^{(\varepsilon)})^* h_l^{(\varepsilon)}) = \delta_{kl} \tau_\varepsilon(\Lambda_\varepsilon)^2$, and therefore

$$(\text{Tr} \otimes \tau_\varepsilon)(Y) = \left(\int_0^\infty t d\sigma_\varepsilon(t) \right)^2 \sum_{k=1}^n \|x_k\|_{S_2^m}^2 \leq 9\varepsilon^{-\frac{2}{3}} \sum_{k=1}^n \|x_k\|_{S_2^m}^2.$$

Since $f_{\varepsilon, x}(s) \leq s^{-1} (\text{Tr} \otimes \tau_\varepsilon)(Y)$, we obtain

$$\frac{1}{\pi} \int_{b_x \varepsilon^{-\frac{4}{3}}}^\infty s^{-\frac{1}{2}} f_{\varepsilon, x}(s) ds \leq \frac{18}{\pi \sqrt{b_x}} \sum_{k=1}^n \|x_k\|_{S_2^m}^2.$$

Both tails are nonnegative and bounded independently of ε . Combining the three intervals in (4.5) proves (5.9), and (5.10) follows. $\blacksquare$

6 Proof of the main theorem

We start with a description of the vector-valued Schatten norms of ultraproducts of maps with values in noncommutative L^1 -spaces. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. If $(\mathcal{A}_j)_{j \geq 1}$ is a sequence of von Neumann algebras, we denote by $\prod_{\mathcal{U}} \mathcal{A}_j^{\text{op}}$ the norm ultraproduct C^* -algebra, that is the quotient of $\ell^\infty(\mathcal{A}_j^{\text{op}})$ by the ideal $I_{\mathcal{U}} = \{(a_j) : \lim_{\mathcal{U}} \|a_j\| = 0\}$. For each fixed m , entrywise identification gives $M_m(\ell^\infty(\mathcal{A}_j^{\text{op}})) = \ell^\infty(M_m(\mathcal{A}_j^{\text{op}}))$. Under this identification, $M_m(I_{\mathcal{U}})$ consists exactly of the sequences whose matrix norms tend to zero along $\mathcal{U}$, since

$$\max_{a,b} \|(A_j)_{ab}\| \leq \|A_j\| \leq m \max_{a,b} \|(A_j)_{ab}\|.$$

Passing to the C^* -algebra quotients gives the isometric identification

$$M_m\left(\prod_{\mathcal{U}} \mathcal{A}_j^{\text{op}}\right) = \prod_{\mathcal{U}} M_m(\mathcal{A}_j^{\text{op}}).$$

We stress that this is the C^* -algebraic ultraproduct. The tracial ultraproduct of finite von Neumann algebras is again finite, and its predual cannot contain a completely isomorphic copy of OH by [Pis04a].

Lemma 6.1 *Let E be a finite-dimensional operator space and let $(\mathcal{A}_j)_{j \geq 1}$ be a sequence of von Neumann algebras. Let $u_j: E \rightarrow L^1(\mathcal{A}_j)$ be linear maps with $\sup_j \|u_j\|_{\text{cb}} < \infty$. Put $A \stackrel{\text{def}}{=} \prod_{\mathcal{U}} \mathcal{A}_j^{\text{op}}$. The formula*

$$(6.1) \quad u(\xi)((a_j)_{\mathcal{U}}) = \lim_{\mathcal{U}} \langle u_j(\xi), a_j \rangle, \quad \xi \in E, (a_j)_{\mathcal{U}} \in A,$$

where the brackets denote the duality between $L^1(\mathcal{A}_j) = (\mathcal{A}_j^{\text{op}})^*$ and $\mathcal{A}_j^{\text{op}}$, defines a linear map $u: E \rightarrow A^*$ such that for any integer $m \geq 1$ and any $x \in S_1^m[E]$

$$(6.2) \quad \|(\text{Id}_{S_1^m} \otimes u)(x)\|_{S_1^m(A^*)} = \lim_{\mathcal{U}} \|(\text{Id}_{S_1^m} \otimes u_j)(x)\|_{S_1^m(L^1(\mathcal{A}_j))}.$$

Proof : The limit in (6.1) exists since $|\langle u_j(\xi), a_j \rangle| \leq \sup_j \|u_j\| \|\xi\| \sup_j \|a_j\|$, and it vanishes if $\lim_{\mathcal{U}} \|a_j\| = 0$. Hence u is well-defined. Let $m \geq 1$, write $x = [x_{ab}] \in S_1^m(E)$, and let $a = (a_j)_{\mathcal{U}} \in M_m(A) = \prod_{\mathcal{U}} M_m(\mathcal{A}_j^{\text{op}})$ with $a_j = [a_{ab}^{(j)}]$. Under the parallel duality, we have

$$\langle (\text{Id}_{S_1^m} \otimes u)(x), a \rangle = \sum_{a,b} u(x_{ab})((a_{ab}^{(j)})_{\mathcal{U}}) \stackrel{(6.1)}{=} \lim_{\mathcal{U}} \sum_{a,b} u_j(x_{ab})(a_{ab}^{(j)}) = \lim_{\mathcal{U}} \langle (\text{Id}_{S_1^m} \otimes u_j)(x), a_j \rangle.$$

We have isometries $S_1^m(A^*) = M_m(A)^*$ and $S_1^m((\mathcal{A}_j^{\text{op}})^*) = M_m(\mathcal{A}_j^{\text{op}})^*$. Moreover, $S_1^m(L^1(\mathcal{A}_j))$ is an isometric subspace of $S_1^m((\mathcal{A}_j^{\text{op}})^*)$ by [Pis98, Corollary 1.2 p. 19], since $L^1(\mathcal{A}_j) = (\mathcal{A}_j^{\text{op}})^*$ is a completely isometric subspace of $(\mathcal{A}_j^{\text{op}})^*$. Taking the supremum over the elements a of the open unit ball of $M_m(A)$, which admit representatives (a_j) with $\|a_j\| < 1$ for any j , we obtain the inequality $\leq$ in (6.2). Conversely, $(\text{Id} \otimes u_j)(x)$ defines a normal functional on $M_m(\mathcal{A}_j^{\text{op}})$, so it attains its norm on the weak* compact closed unit ball. Multiplying a norming element by a scalar of modulus 1, we may choose a_j in that ball such that $\langle (\text{Id} \otimes u_j)(x), a_j \rangle = \|(\text{Id} \otimes u_j)(x)\|$. Then $a = (a_j)_{\mathcal{U}}$ belongs to the closed unit ball of $M_m(A)$ and the previous equality gives the reverse inequality. $\blacksquare$

We are now ready to prove Theorem 1.1.

Theorem 6.2 *Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and for any integer $j \geq 1$, let $\mathcal{A}_j = \mathcal{A}_{e^{-j}}$ be the finite von Neumann algebra of the $\sigma_{e^{-j}}$ -system fixed before Theorem 5.4. Put $A \stackrel{\text{def}}{=} \prod_{\mathcal{U}} \mathcal{A}_j^{\text{op}}$ and $\mathcal{M} \stackrel{\text{def}}{=} (A^{**})^{\text{op}}$. The algebras A and $\mathcal{M}$ have QWEP, and there exists a complete isometry $J: \text{OH} \rightarrow L^1(\mathcal{M})$ such that for any $k \geq 1$ and any $(a_j^{\text{op}})_{\mathcal{U}} \in A$, with $a_j \in \mathcal{A}_j$, we have*

$$(6.3) \quad \langle J(e_k), (a_j^{\text{op}})_{\mathcal{U}} \rangle = \lim_{\mathcal{U}} \frac{1}{c_{*j}} \tau_{e^{-j}}(h_k^{(e^{-j})} a_j),$$

where we identify $L^1(\mathcal{M})$ with A^* and use the underlying vector $a_j \in \mathcal{A}_j$ when writing the tracial pairing.

Proof : Let $V = \bigcup_{n \geq 1} \text{OH}_n$ be the dense subspace of finitely supported vectors of OH . For any $j \geq 1$, consider the linear map $u_j: V \rightarrow L^1(\mathcal{A}_j)$ defined by $u_j(e_k) \stackrel{\text{def}}{=} \frac{1}{c_{*j}} h_k^{(e^{-j})}$. Note that $\log \frac{1}{c_{*j}} = j$.

Let $m, n \geq 1$ be integers. For any $x \in (M_m)^n$, we let $\nu_j(x) = \|\sum_k x_k \otimes u_j(e_k)\|_{L^1(M_m \bar{\otimes} \mathcal{A}_j)} = \|(\text{Id}_{S_1^m} \otimes u_j)(\sum_k x_k \otimes e_k)\|_{S_1^m[L^1(\mathcal{A}_j)]}$. We claim that

$$(6.4) \quad \lim_{j \rightarrow \infty} \nu_j(x) = N(x) \quad \text{for any } x \in (M_m)^n.$$

By Theorem 5.4, this holds for any $x \in \mathcal{G}_{m,n}$. In particular, applying Theorem 5.4 with $m = n = 1$ and $x = (1) \in \mathcal{G}_{1,1}$, for which $N(x) = \|e_1\|_{\text{OH}_1} = 1$, we see that $\lim_j \|u_j(e_k)\|_1 = 1$ for any k , since $\|h_k^{(\varepsilon)}\|_1$ does not depend on k by Proposition 4.4. A linear map $\xi \mapsto \langle \xi, \eta \rangle h$ from OH_n into an operator space has cb-norm equal to $\|\eta\| \|h\|$, hence $C_n = \sup_j \|u_j|_{\text{OH}_n}\|_{\text{cb}} \leq \sup_j \sum_{k=1}^n \|u_j(e_k)\|_1 < \infty$. Consequently, the seminorms ν_j satisfy $|\nu_j(x) - \nu_j(y)| \leq \nu_j(x - y) \leq C_n N(x - y)$ for any $x, y \in (M_m)^n$. Let $x \in (M_m)^n$ and $\delta > 0$. Since $\mathcal{G}_{m,n}$ is dense in $(M_m)^n$ and since N is a norm on the finite-dimensional space $(M_m)^n$, there exists $y \in \mathcal{G}_{m,n}$ such that $N(x - y) \leq \delta$. Then

$$\limsup_{j \rightarrow \infty} |\nu_j(x) - N(x)| \leq \limsup_{j \rightarrow \infty} (|\nu_j(x) - \nu_j(y)| + |\nu_j(y) - N(y)| + |N(y) - N(x)|) \leq (C_n + 1)\delta.$$

This proves (6.4).

Now, define a linear map $u: V \rightarrow A^*$ by the formula (6.1), which makes sense since for any $\xi \in \text{OH}_n$ we have $\sup_j \|u_j(\xi)\| \leq C_n \|\xi\|$. For any $n \geq 1$, the restriction of u to OH_n is the map provided by Lemma 6.1 applied to the restrictions of the u_j 's. By (6.2) and (6.4), for any integers $m, n \geq 1$ and any $x \in (M_m)^n$ we have

$$\left\| (\text{Id}_{S_1^m} \otimes u) \left(\sum_{k=1}^n x_k \otimes e_k \right) \right\|_{S_1^m(A^*)} = \lim_{\mathcal{U}} \nu_j(x) = N(x) = \left\| \sum_{k=1}^n x_k \otimes e_k \right\|_{S_1^m(\text{OH}_n)}.$$

The restriction of u to OH_n is a complete isometry for any n . Since any element of $M_m(V)$ belongs to $M_m(\text{OH}_n)$ for some n , the map $u: V \rightarrow A^*$ is a complete isometry. Since the subspace V is dense in the space OH and A^* is complete, u extends to a complete isometry $J: \text{OH} \rightarrow A^*$. Finally, A^* is completely isometric to $L^1((A^{**})^{\text{op}}) = L^1(\mathcal{M})$, see Subsection 2. The formula (6.3) is (6.1).

By our choice of the systems, each algebra $\mathcal{A}_j$ has QWEP. This property passes to opposite algebras. Indeed, for any C^* -algebra B , the map $b^{\text{op}} \mapsto \bar{b}^*$ is a $*$ -isomorphism from B^{op} onto the conjugate C^* -algebra $\bar{B}$, see [Oza04, Section 6]. If $B = C/I$, where $C \subset B(H)$ has the weak expectation property witnessed by a unital completely positive map $P: B(H) \rightarrow C^{**}$, then $\bar{B} = \bar{C}/\bar{I}$, and the conjugate map $\bar{P}: B(\bar{H}) \rightarrow \bar{C}^{**}$ shows that $\bar{C}$ has the weak expectation property. The product $\prod_j \mathcal{A}_j^{\text{op}}$ has QWEP by [Oza04, Proposition 4.1 (i) p. 516], and its quotient A therefore has QWEP. By [Oza04, Proposition 4.1 (iv) p. 516], the bidual A^{**} has QWEP, and so does its opposite algebra $\mathcal{M}$. ■

By Remark 4.3, each $\mathcal{A}_j^{\text{op}}$ is isomorphic to the von Neumann algebra $\text{VN}(\mathbb{F}_\infty)$. Hence the C^* -algebra A of Theorem 6.2 is isomorphic to the C^* -algebraic ultrapower $\prod_{\mathcal{U}} \text{VN}(\mathbb{F}_\infty)$, and OH embeds completely isometrically into its dual. The QWEP property of A can also be deduced from this description.

Corollary 6.3 *1. For any integer $n \geq 1$ and any $\delta > 0$, there exist an integer $N \geq 1$ and an injective linear map $v: \text{OH}_n \rightarrow S_1^N$ such that*

$$(6.5) \quad \|v\|_{\text{cb}} \|v^{-1}\|_{\text{cb}} < 1 + \delta,$$

where v^{-1} is defined on $v(\text{OH}_n)$.

- 2. The operator space OH is completely isometric to a quotient of a QWEP C^* -algebra. More precisely, with the notations of Theorem 6.2, the map $q: A \rightarrow \text{OH}^*$, $a \mapsto \langle J(\cdot), a \rangle$ is a complete quotient map, and OH^* is completely isometric to OH .*
- 3. The von Neumann algebra $\mathcal{M}$ of Theorem 6.2 is not semifinite.*

Proof : 1. By Theorem 6.2, the algebra A has QWEP and $J: \text{OH} \rightarrow A^*$ is a complete isometry. By [EJR00, Corollary 7.2 p. 87], every finite-dimensional operator subspace of A^* embeds into some S_1^N with cb-distortion less than $1 + \delta$. Apply this result to $J(\text{OH}_n)$ and compose the resulting embedding with $J|_{\text{OH}_n}$.

2. The space OH is reflexive, and under the canonical identification $\text{OH}^{**} = \text{OH}$ the adjoint $q^*: \text{OH} \rightarrow A^*$ of q is the complete isometry J . Since A and OH^* are complete, [Efr00, Corollary 4.1.9 p. 73] shows that q is a complete quotient map. Finally, recall that OH^* is completely isometric to $\overline{\text{OH}}$ [Pis96], and that $\overline{\text{OH}}$ is completely isometric to OH via $\overline{e_k} \mapsto e_k$ by (2.3).

3. This follows from Pisier's theorem [Pis04a]. $\blacksquare$

Corollary 6.4 *There exist a σ -finite factor K of type III and a complete isometry $\text{OH} \rightarrow K_*$.*

Proof : Put $\mathcal{N} \stackrel{\text{def}}{=} \mathcal{M}^{\text{op}}$, so that Theorem 6.2 gives a complete isometry $J: \text{OH} \rightarrow \mathcal{N}_*$. Write $f_k = J(e_k)$ as

$$f_k = \omega_{k,1} - \omega_{k,2} + i(\omega_{k,3} - \omega_{k,4}),$$

where the $\omega_{k,l}$ are positive normal functionals on N . The series

$$\psi = \sum_{k \geq 1} \sum_{l=1}^4 \frac{2^{-k-l}}{1 + \|\omega_{k,l}\|} \omega_{k,l}$$

converges in $\mathcal{N}_*$ to a nonzero positive normal functional. Let p be its support projection and put $\mathcal{N}_0 \stackrel{\text{def}}{=} p\mathcal{N}p$. Each $\omega_{k,l}$ is supported by p , hence $f_k(a) = f_k(pap)$ for all $a \in \mathcal{N}$. The restriction $R: \mathcal{N}_* \rightarrow (\mathcal{N}_0)_*$ and the preadjoint $C_*: (\mathcal{N}_0)_* \rightarrow \mathcal{N}_*$ of the normal completely contractive map $C: \mathcal{N} \rightarrow \mathcal{N}_0$, $a \mapsto pap$, are complete contractions. They satisfy $RC_* = \text{Id}$ and $C_*RJ = J$, the latter first on the basis vectors and then by continuity. Consequently, the map $J_0 \stackrel{\text{def}}{=} RJ: \text{OH} \rightarrow (\mathcal{N}_0)_*$ is a complete isometry. The state $\varphi = \frac{1}{\psi(p)}\psi|_{\mathcal{N}_0}$ is normal and faithful, so $\mathcal{N}_0$ is σ -finite.

Form the von Neumann algebraic free product

$$(K, \chi) = (\mathcal{N}_0, \varphi) * (\text{VN}(\mathbb{F}_\infty), \tau_0).$$

Since $\mathcal{N}_0 \neq \mathbb{C}$ and $\text{VN}(\mathbb{F}_\infty)$ is diffuse, [Ued11, Theorem 3.4 p. 2652] shows that K is a factor. Its free product state is normal and faithful, so K is σ -finite. By [Ued11, Lemma 2.1 p. 2649], there is a normal state-preserving conditional expectation $\mathbb{E}: K \rightarrow \mathcal{N}_0$. Its preadjoint $\mathbb{E}_*: (\mathcal{N}_0)_* \rightarrow K_*$ is a complete isometry, since it is completely contractive and restriction to $\mathcal{N}_0$ is a completely contractive left inverse. Thus $\mathbb{E}_*J_0$ is the required complete isometry. Finally, K cannot be semifinite by [Pis04a], and hence it has type III. $\blacksquare$

Convergence of the finite models For $0 < \varepsilon < 1$ and a fixed integer $n \geq 1$, put

$$(6.6) \quad u_{\varepsilon,n}: \text{OH}_n \rightarrow L^1(\mathcal{A}_\varepsilon), \quad u_{\varepsilon,n}(e_k) \stackrel{\text{def}}{=} \frac{h_k^{(\varepsilon)}}{c_* \log \frac{1}{\varepsilon}} \quad \text{and} \quad E_{\varepsilon,n} \stackrel{\text{def}}{=} u_{\varepsilon,n}(\text{OH}_n).$$

The algebras $\mathcal{A}_\varepsilon$ are chosen to have QWEP, as in Theorem 6.2.

Proposition 6.5 *For any fixed n , the maps $u_{\varepsilon,n}$ are injective,*

$$(6.7) \quad \alpha_{\varepsilon,n} \stackrel{\text{def}}{=} \|u_{\varepsilon,n}\|_{\text{cb}, \text{OH}_n \rightarrow L^1(\mathcal{A}_\varepsilon)} \xrightarrow{\varepsilon \rightarrow 0} 1, \quad \text{and} \quad \beta_{\varepsilon,n} \stackrel{\text{def}}{=} \|u_{\varepsilon,n}^{-1}\|_{\text{cb}, E_{\varepsilon,n} \rightarrow \text{OH}_n} \xrightarrow{\varepsilon \rightarrow 0} 1.$$

Proof : Injectivity follows from $\tau_\varepsilon((h_k^{(\varepsilon)})^* h_l^{(\varepsilon)}) = \delta_{kl} \tau_\varepsilon(\Lambda_\varepsilon)^2$. The argument proving (6.4), applied to arbitrary truncation parameters tending to zero, gives

$$(6.8) \quad \lim_{\varepsilon \rightarrow 0} \|(\text{Id}_{S_1^m} \otimes u_{\varepsilon,n})(x)\|_{S_1^m(L^1(\mathcal{A}_\varepsilon))} = \|x\|_{S_1^m(\text{OH}_n)}$$

for every fixed m and $x \in S_1^m(\text{OH}_n)$. Indeed, Theorem 5.4 gives (6.8) for coefficients in $\mathcal{G}_{m,n}$. Moreover, the rank-one decomposition used in the proof of Theorem 6.2 gives $\|u_{\varepsilon,n}\|_{\text{cb}} \leq \sum_{k=1}^n \|u_{\varepsilon,n}(e_k)\|_1$, and the scalar case $m = n = 1$ of Theorem 5.4 shows that this bound is uniform for small ε . The density of $\mathcal{G}_{m,n}$ then removes the non-degeneracy assumption, exactly as in the proof of (6.4).

We now use a theorem of Ozawa [Oza00]: for each n , the set of n -dimensional operator spaces which are completely isometric to a subspace of the predual of a von Neumann algebra is compact for the cb-Banach-Mazur distance d_{cb} . Fix a sequence $\varepsilon_k \rightarrow 0$. After passing to a subsequence, there are an n -dimensional operator space F and isomorphisms $v_k: E_{\varepsilon_k,n} \rightarrow F$ whose cb-distortions tend to 1. Rescaling each v_k makes its cb-norm and the cb-norm of its inverse equal to the square root of that distortion, both therefore tend to 1. The maps $w_k = v_k u_{\varepsilon_k,n}$ are bounded as maps between the fixed finite-dimensional spaces OH_n and F . A further subsequence converges in operator norm to a map w . Equation (6.8) gives

$$\|(\text{Id}_{S_m^1} \otimes w)(x)\|_{S_m^1(F)} = \|x\|_{S_m^1(\text{OH}_n)}$$

for every m and x , so w is a surjective complete isometry.

For every linear map $L: \text{OH}_n \rightarrow F$, the rank-one decomposition gives the estimate $\|L\|_{\text{cb}} \leq \sum_{i=1}^n \|L(e_i)\| \leq n \|L\|$. Thus $w_k \rightarrow w$ in cb-norm. Since w is a surjective complete isometry, the Neumann series for $w^{-1}w_k$ gives $w_k^{-1} \rightarrow w^{-1}$ in cb-norm. Since $u_{\varepsilon_k,n} = v_k^{-1}w_k$ and $u_{\varepsilon_k,n}^{-1} = w_k^{-1}v_k$, both limsups in (6.7) are at most 1. The level-one convergence gives the opposite liminf inequalities. Every sequence tending to zero has such a subsequence, which proves (6.7). $\blacksquare$

Remark 6.6 Because of the compactness argument, Proposition 6.5 gives no rate in (6.7) and no bound on N in (6.5). For a fixed ε , the cb-distortion of $u_{\varepsilon,n}$ cannot be bounded uniformly in n , by Pisier's theorem [Pis04a], since $\mathcal{A}_\varepsilon$ is finite.

Completely 1-summing maps We denote by $E \hat{\otimes} F$ the operator space projective tensor product, recall that $(E \hat{\otimes} F)^* = \text{CB}(E, F^*)$ completely isometrically [Efr00, Corollary 7.1.5 p. 128]. Following [Jun05, Section 4], for an operator space Y we let

$$d_{\text{SL}^1}(Y) \stackrel{\text{def}}{=} \sup_E \inf \{ \|w\|_{\text{cb}} \|w^{-1}\|_{\text{cb}} : w: E \rightarrow w(E) \subset S_1^r, r \geq 1 \},$$

where the supremum runs over the finite-dimensional subspaces E of Y and the infimum over the injective linear maps w . If N is a QWEP von Neumann algebra, then $d_{\text{SL}^1}(N_*) = 1$ by [EJR00, Corollary 7.3 p. 87]. For a linear map $T: E \rightarrow F$ between operator spaces, its completely 1-summing norm is

$$(6.9) \quad \|T\|_{\pi_1^o, E \rightarrow F} \stackrel{\text{def}}{=} \sup_{m \geq 1} \|\text{Id}_{S_1^m} \otimes T\|_{S_1^m \otimes_{\min} E \rightarrow S_1^m(F)}.$$

Here the norms on the right are operator norms, see [Pis98] and [Jun05, Section 4]. By the injectivity of $S_1^m(F)$, if $j: F \rightarrow G$ is a complete isometry then $\pi_1^o(jT) = \pi_1^o(T)$. The next tensor identification can also be derived from the case $p = 1$ of [Yew08, Corollary 11 p. 1379]. We include a short proof based on Junge's lemmas [Jun05, Lemma 4.4 p. 250 and Lemma 4.5 p. 251].

Corollary 6.7 *Let $\mathcal{N}$ be a QWEP von Neumann algebra and let $j: \text{OH} \rightarrow \mathcal{N}_*$ be a complete isometry, where $\mathcal{N}_*$ has its predual operator space structure. Put $h_i \stackrel{\text{def}}{=} j(e_i)$. For any integer $n \geq 1$ and any linear map $T: \text{OH}_n \rightarrow \text{OH}_n$, we have*

$$(6.10) \quad \|T\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n} = \left\| \sum_{i=1}^n h_i \otimes j(T(e_i)) \right\|_{\mathcal{N}_* \hat{\otimes} \mathcal{N}_*}.$$

In particular, this applies to $\mathcal{N} = A^{**}$ and $j = J$ in Theorem 6.2, since $\mathcal{N}_* = A^* = L^1(\mathcal{M})$ completely isometrically.

Proof : Let $I_n: \text{OH}_n^* \rightarrow \text{OH}_n$ be the linear map $e_i^* \mapsto e_i$. The self-duality together with (2.3), shows that the map I_n is a complete isometry. Put $t \stackrel{\text{def}}{=} \sum_{i=1}^n e_i \otimes T(e_i)$. The map associated to this tensor is $T_t: \text{OH}_n^* \rightarrow \text{OH}_n$, $f \mapsto (f \otimes \text{Id})(t)$, and $T_t = TI_n$. Thus $\|T_t\|_{\pi_1^o, \text{OH}_n^* \rightarrow \text{OH}_n} = \|T\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n}$.

Put $z \stackrel{\text{def}}{=} (j \otimes j)(t)$. Since the von Neumann algebra $\mathcal{N}$ has QWEP and j is completely contractive, [Jun05, Lemma 4.5 p. 251], applied with both target preduals equal to $\mathcal{N}_*$, gives

$$(6.11) \quad \|z\|_{\mathcal{N}_* \hat{\otimes} \mathcal{N}_*} \leq \|T_t\|_{\pi_1^o, \text{OH}_n^* \rightarrow \text{OH}_n} = \|T\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n}.$$

For the reverse inequality, let $E_n = j(\text{OH}_n)$ and denote by $j_n: \text{OH}_n \rightarrow E_n$ the restriction of j . The map associated to z , regarded as a tensor of $E_n \otimes E_n$, is

$$S: E_n^* \rightarrow \mathcal{N}_*, \quad S(f) = (f \otimes \text{Id})(z) = jTI_n j_n^*(f).$$

The map $I_n j_n^*: E_n^* \rightarrow \text{OH}_n$ is a surjective complete isometry. The invariance of $\|\cdot\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n}$ under complete isometries of the domain and under the inclusion j therefore gives $\|S\|_{\pi_1^o, E_n^* \rightarrow \mathcal{N}_*} = \|T\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n}$. Since $d_{SL^1}(\mathcal{N}_*) = 1$, [Jun05, Lemma 4.4 p. 250], applied with $X = Y = \mathcal{N}_*$, yields

$$\|T\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n} = \|S\|_{\pi_1^o, E_n^* \rightarrow \mathcal{N}_*} \leq \|z\|_{\mathcal{N}_* \hat{\otimes} \mathcal{N}_*}.$$

Together with (6.11), this proves (6.10). ■

Tensor norms in the truncated models For $T: \text{OH}_n \rightarrow \text{OH}_n$ and $0 < \varepsilon, \eta < 1$, define

$$(6.12) \quad F_{\varepsilon, \eta}(T) = \left\| \sum_{i=1}^n u_{\varepsilon, n}(e_i) \otimes u_{\eta, n}(T(e_i)) \right\|_{L^1(\mathcal{A}_\varepsilon \overline{\otimes} \mathcal{A}_\eta)}.$$

The norm uses the product trace $\tau_\varepsilon \otimes \tau_\eta$. With our opposite-algebra convention, the canonical predual tensor product identification gives

$$L^1(\mathcal{A}_\varepsilon) \hat{\otimes} L^1(\mathcal{A}_\eta) = L^1(\mathcal{A}_\varepsilon \overline{\otimes} \mathcal{A}_\eta)$$

completely isometrically. This is [Efr00, Theorem 7.2.4 p. 136], applied to $\mathcal{A}_\varepsilon^{\text{op}}$ and $\mathcal{A}_\eta^{\text{op}}$, together with $(\mathcal{A}_\varepsilon \overline{\otimes} \mathcal{A}_\eta)^{\text{op}} \cong \mathcal{A}_\varepsilon^{\text{op}} \overline{\otimes} \mathcal{A}_\eta^{\text{op}}$, see also [Jun04, Fact 2.5 p. 990] for duals of C^* -algebras.

Proposition 6.8 *With the notation of (6.7), for any linear map $T: \text{OH}_n \rightarrow \text{OH}_n$ we have*

$$(6.13) \quad \frac{\pi_1^o(T)}{\beta_{\varepsilon, n} \beta_{\eta, n}} \leq F_{\varepsilon, \eta}(T) \leq \alpha_{\varepsilon, n} \alpha_{\eta, n} \|T\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n}.$$

Consequently, we have

$$(6.14) \quad \lim_{\substack{\varepsilon \rightarrow 0 \\ \eta \rightarrow 0}} F_{\varepsilon, \eta}(T) = \|T\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n}.$$

For fixed n , the convergence is uniform on $\{T : \|T\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n} \leq 1\}$.

Proof : Put $X \stackrel{\text{def}}{=} L^1(\mathcal{A}_\varepsilon)$, $Y \stackrel{\text{def}}{=} L^1(\mathcal{A}_\eta)$ and $z \stackrel{\text{def}}{=} \sum_i u_{\varepsilon, n}(e_i) \otimes u_{\eta, n}(T(e_i))$. Applying [Jun05, Lemma 4.5 p. 251] to the maps $\frac{1}{\alpha_{\varepsilon, n}} u_{\varepsilon, n}$ and $\frac{1}{\alpha_{\eta, n}} u_{\eta, n}$ gives the upper bound in (6.13), since $\mathcal{A}_\varepsilon^{\text{op}}$ has QWEP and the map associated to $\sum_i e_i \otimes T(e_i)$ is TI_n , with I_n as in the proof of Corollary 6.7.

For the lower bound, regard z as a tensor in $E_{\varepsilon, n} \otimes E_{\eta, n}$. Its associated map is

$$S_z : E_{\varepsilon, n}^* \rightarrow Y, \quad S_z = u_{\eta, n} T I_n u_{\varepsilon, n}^*.$$

Here $u_{\varepsilon, n}^*$ is the adjoint of $u_{\varepsilon, n} : \text{OH}_n \rightarrow E_{\varepsilon, n}$. Since the algebra $\mathcal{A}_\eta^{\text{op}}$ has QWEP, we have $d_{SL^1}(Y) = 1$, and [Jun05, Lemma 4.4 p. 250] gives $\|S_z\|_{\pi_1^o, E_{\varepsilon, n}^* \rightarrow Y} \leq \|z\|_{X \hat{\otimes} Y} = F_{\varepsilon, \eta}(T)$. The range of S_z lies in $E_{\eta, n}$, and regarding it as a map into this subspace does not change its completely 1-summing norm. Composing with $u_{\eta, n}^{-1}$ on the left and with $(u_{\varepsilon, n}^*)^{-1} I_n^{-1}$ on the right gives

$$\pi_1^o(T) \leq \beta_{\eta, n} \beta_{\varepsilon, n} \|S_z\|_{\pi_1^o, E_{\varepsilon, n}^* \rightarrow Y}.$$

This proves (6.13). Proposition 6.5 gives the limit and its stated uniformity. $\blacksquare$

Corollary 6.9 Put $d_n \stackrel{\text{def}}{=} \|\text{Id}_{\text{OH}_n}\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n}$. For each fixed n , as $\varepsilon \rightarrow 0$,

$$(6.15) \quad \left\| \sum_{i=1}^n h_i^{(\varepsilon)} \otimes h_i^{(\varepsilon)} \right\|_{L^1(\mathcal{A}_\varepsilon \bar{\otimes} \mathcal{A}_\varepsilon)} = c_*^2 \left(\log \frac{1}{\varepsilon} \right)^2 (d_n + o(1)).$$

Here $c_*^2 \stackrel{\text{def}}{=} \frac{512\pi}{27\sqrt{3}}$. For a rank-one map T , a more explicit formula holds at every pair of truncations:

$$(6.16) \quad F_{\varepsilon, \eta}(T) = t_\varepsilon t_\eta \|T\|_{S^2} \quad \text{with} \quad t_\varepsilon = \frac{\|h_1^{(\varepsilon)}\|_1}{c_* \log \frac{1}{\varepsilon}} = 1 + O((\log \frac{1}{\varepsilon})^{-1}).$$

In particular, we have $\|T\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n} = \|T\|_{S^2}$ for maps of rank one.

Proof : Equation (6.15) follows from (6.14) and the value of c_* in Theorem 5.4. For any $\xi = (\xi_i) \in \mathbb{C}^n$, the joint distribution of $(\Lambda_\varepsilon, \sum_i \xi_i c_i)$ agrees with that of $(\Lambda_\varepsilon, \|\xi\|_2 c_1)$: this follows from the circular cumulants in Definition 4.1. Hence $\|u_{\varepsilon, n}(\xi)\|_1 = t_\varepsilon \|\xi\|_2$. Write $T(\xi) = (\sum_i \bar{a}_i \xi_i) b$. Then the tensor in (6.12) is $u_{\varepsilon, n}(\bar{a}) \otimes u_{\eta, n}(b)$. The L^1 -norm of an elementary tensor is the product of the two norms, which proves (6.16). The estimate for t_ε is the scalar case of Theorem 5.4, and the last assertion follows by taking the limit. It can also be checked directly from (6.9). $\blacksquare$

Proposition 6.10 Let $T : \text{OH}_n \rightarrow \text{OH}_n$ have rank $r \geq 1$. Then

$$(6.17) \quad \|T\|_{S^2} \leq \|T\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n} \leq \min \left\{ \|T\|_{S^1}, 108 \sqrt{1 + \log r} \|T\|_{S^2} \right\}.$$

In particular, for every $n \geq 1$,

$$(6.18) \quad \max \left\{ \sqrt{n}, \frac{\sqrt{n(1 + \log n)}}{288\sqrt{2\pi}} \right\} \leq d_n \leq \min \left\{ n, 108 \sqrt{n(1 + \log n)} \right\}.$$

Proof : If $T = \varphi(\cdot)b$ has rank one, then $\text{Id}_{S_1^m} \otimes \varphi: S_1^m \otimes_{\min} \text{OH}_n \rightarrow S_1^m$ has norm at most $\|\varphi\|$ and $\|y \otimes b\|_{S_1^m(\text{OH}_n)} = \|y\|_{S_1^m} \|b\|_{\text{OH}_n}$, so that $\|T\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n} \leq \|\varphi\| \|b\|_{\text{OH}_n} = \|T\|_{S_2}$. The trace-norm upper bound follows by decomposing T into rank-one maps by means of its singular value decomposition. The logarithmic upper bound is the combination of [Jun05, Corollary 4.8 and Proposition 4.9 p. 253]. The same results give $d_n \geq (288\sqrt{2}\pi)^{-1} \sqrt{n(1 + \log n)}$ for $n \geq 7$. For $n < 7$, this bound follows already from $d_n \geq 1$. To replace n by r in the upper bound for T , use its singular value decomposition, the homogeneity of OH , and the completely contractive coordinate inclusions and projections. These show that $\|T\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n}$ equals the completely 1-summing norm of the diagonal map on OH_r formed by its nonzero singular values.

For completeness, the Hilbert-Schmidt lower bound follows from trace duality. Let

$$\gamma_\infty(V) \stackrel{\text{def}}{=} \inf\{\|A\|_{\text{cb}} \|B\|_{\text{cb}} : V = AB, B: \text{OH}_n \rightarrow M_k, A: M_k \rightarrow \text{OH}_n, k \geq 1\}.$$

We claim that

$$\gamma_\infty(V) \leq \|V\|_{S^2}.$$

Let $d = (d_1, \dots, d_n)$ with $d_k \geq 0$ and $D = \text{diag}(d_k)$. Let $y \in (M_m)^n$ be an element such that $\|\sum_k y_k \otimes e_k\|_{M_m(\text{OH}_n)} \leq 1$, that is $\|\Phi_y\|_{S_m^2 \rightarrow S_m^2} \leq 1$ by Lemma 2.1, and let $\xi \in \mathbb{C}^m$ be a unit vector. Since $\Phi_y^*(\xi\xi^*) = \sum_k (y_k^* \xi)(y_k^* \xi)^*$, we have $\sum_k \|y_k^* \xi\|^4 \leq \|\Phi_y^*(\xi\xi^*)\|_{S_m^2}^2 \leq 1$. By the Cauchy-Schwarz inequality,

$$\left\| \sum_k d_k y_k \otimes e_{1k} \right\|_{M_m(\mathbb{R}_n)}^2 = \sup_{\xi} \sum_k d_k^2 \|y_k^* \xi\|^2 \leq \|d\|_{\ell^4}^2.$$

Thus $\|D: \text{OH}_n \rightarrow \mathbb{R}_n\|_{\text{cb}} \leq \|d\|_{\ell^4}$. In the same way, using $\Phi_y(\xi\xi^*)$, we obtain the inequality $\|D: \text{OH}_n \rightarrow \mathbb{C}_n\|_{\text{cb}} \leq \|d\|_{\ell^4}$. Since $\mathbb{R}_n^* = \mathbb{C}_n$ and $\text{OH}_n^* \cong \text{OH}_n$ via $e_k^* \mapsto e_k$, taking adjoints gives $\|D: \mathbb{R}_n \rightarrow \text{OH}_n\|_{\text{cb}} \leq \|d\|_{\ell^4}$. Now write $V = U \text{diag}(s_k) W^*$ with unitaries U, W and singular values s_k . The maps $B = \text{diag}(\sqrt{s_k}) W^*: \text{OH}_n \rightarrow \mathbb{R}_n \subset M_n$ and $A = U \text{diag}(\sqrt{s_k}) P: M_n \rightarrow \text{OH}_n$, where $P(x) = e_{11}x$ is the completely contractive projection of M_n onto its first row $\mathbb{R}_n$, satisfy $V = AB$ and $\|A\|_{\text{cb}} \|B\|_{\text{cb}} \leq (\sum_k s_k^2)^{\frac{1}{2}} = \|V\|_{S^2}$, by homogeneity of OH_n . This proves the claim. By [Jun05, Lemma 4.1 p. 250], we have $|\text{Tr}(VT)| \leq \gamma_\infty(V) \|T\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n}$. Taking $V = \frac{1}{\|T\|_{S^2}} T^*$ proves the lower bound in (6.17). Applying it to the identity completes (6.18). $\blacksquare$

Remark 6.11 Equations (6.15) and (6.18) estimate the tensor norm of the identity in the bilateral free model, with the exact normalization c_*^2 . The logarithmic order and the constants 108 and $288\sqrt{2}\pi$ come from [Jun05]. The number d_n is related to projection constants: if $\lambda_{\text{cb}}(\text{OH}_n)$ denotes the infimum of $\|P\|_{\text{cb}}$ over projections from $B(H)$ onto a completely isometric copy of OH_n , then

$$\lambda_{\text{cb}}(\text{OH}_n) = \frac{n}{d_n}.$$

This follows directly from [JuX10, Lemma 5.2 p. 111], since OH_n is 1-homogeneous and isometric to a Hilbert space, see also the trace-duality argument in [Jun05, Equation (4.2) and Corollary 4.12 p. 255]. For the best constant in the two-state estimate for completely bounded maps from a C^* -algebra to OH , see [Pis12, Remark 13.12 p. 286] and [Pis13u, Remark 13.12 p. 48].

7 Tensor distortion and complementation

Let $\mathcal{N}$, j and h_i be as in Corollary 6.7, and put $j_n \stackrel{\text{def}}{=} j|_{\text{OH}_n}$. We regard the following tensor map as a map between Banach spaces. Its inverse norm below is an operator norm, not a cb-norm.

Proposition 7.1 *The injective map $i_n \stackrel{\text{def}}{=} j_n \otimes j_n: \text{OH}_n \hat{\otimes} \text{OH}_n \rightarrow \mathcal{N}_* \hat{\otimes} \mathcal{N}_*$ satisfies*

$$(7.1) \quad \|i_n\| = 1, \quad \|i_n^{-1}\| = \frac{n}{d_n} = \lambda_{\text{cb}}(\text{OH}_n).$$

Every completely bounded projection $P: \mathcal{N}_* \rightarrow j(\text{OH}_n)$ satisfies

$$(7.2) \quad \|P\|_{\text{cb}} \geq \sqrt{\frac{n}{d_n}}.$$

In particular, $j(\text{OH})$ is not completely complemented in $\mathcal{N}_*$.

Proof : Every tensor in $\text{OH}_n \otimes \text{OH}_n$ has a unique expression $t_T = \sum_{k=1}^n e_k \otimes T(e_k)$. Projective tensor duality identifies its dual with $\text{CB}(\text{OH}_n, \text{OH}_n^*)$, see [Efr00] and [Pis03]. Self-duality and homogeneity of OH_n identify this last space isometrically with $B(\ell_2^n)$. The resulting dual norm is the trace norm, so

$$(7.3) \quad \|t_T\|_{\text{OH}_n \hat{\otimes} \text{OH}_n} = \|T\|_{S^1}, \quad \|i_n(t_T)\|_{\mathcal{N}_* \hat{\otimes} \mathcal{N}_*} \stackrel{(6.10)}{=} \|T\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n}.$$

The ideal property and homogeneity imply that $\|\cdot\|_{\pi_1^o}$ is invariant under multiplication by unitaries on either side. Let $D = |T|$ and let μ be normalized Haar measure on the unitary group $\mathcal{U}(n)$. Since $\int_{\mathcal{U}(n)} U D U^* d\mu(U) = n^{-1} \text{Tr}(D) \text{Id}$, convexity gives

$$\frac{d_n}{n} \|T\|_{S^1} \leq \|T\|_{\pi_1^o, \text{OH}_n \rightarrow \text{OH}_n}.$$

Equality holds for $T = \text{Id}$. This proves the inverse norm formula. The map i_n is contractive, and equality of the norms on elementary tensors gives $\|i_n\| = 1$. The last equality in (7.1) was recalled in Remark 6.11.

If P is a projection as stated, put $r \stackrel{\text{def}}{=} j_n^{-1} P: \mathcal{N}_* \rightarrow \text{OH}_n$. Functoriality of the operator space projective tensor product gives

$$(r \otimes r) i_n = \text{Id}, \quad \|r \otimes r\| \leq \|r\|_{\text{cb}}^2 = \|P\|_{\text{cb}}^2.$$

Equation (7.1) therefore implies (7.2). If a completely bounded projection onto $j(\text{OH})$ existed, composition with its completely contractive orthogonal coordinate projections would yield projections onto $j(\text{OH}_n)$ with uniformly bounded cb-norms. This contradicts (7.2), since (6.18) implies $\frac{n}{d_n} \rightarrow \infty$. $\blacksquare$

By (6.18), the inverse norm of the projective tensor inclusion has order $\sqrt{\frac{n}{1+\log n}}$ as $n \rightarrow \infty$.

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Data availability No external dataset is required for the proofs. Numerical scripts and computational outputs were generated during the exploratory stage of this work. They are not included in this manuscript. All results are proved analytically, without using numerical estimates.

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