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Area law for three-dimensional U(1) lattice gauge theory with the Wilson action at every coupling

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Abstract

We prove that three-dimensional U(1) lattice gauge theory with the Wilson action confines static charges at every coupling: rectangular Wilson loops obey an area law, uniformly in the finite box, with an explicit lower bound on the string tension at weak coupling. For the Villain action, confinement at every coupling is the theorem of Göpfert and Mack; for the Wilson action the question had remained open. The Wilson weight is a mixture of Villain weights, so Ginibre's inequality bounds the Wilson loop by an average of Villain loops at a coupling larger by a fixed factor, in which a sparse random set of plaquettes is frozen at angle zero; after this set is regularized, the multiscale analysis of Göpfert and Mack, made uniform in it, and a Peierls estimate give the area law. As corollaries, the area law holds at every coupling for all closed subgroups of $\mathrm{U}(n)$ containing the scalars, also with a Higgs field blind to the centre of $\mathrm{U}(n)$, and for the Georgi–Glashow model at infinite Higgs coupling.

Keywords lattice gauge theory, confinement, Wilson loop, Wilson action, area law, Coulomb gas, Debye screening, Villain action

Mathematics Subject Classification (2020) 81T25, 82B20, 60K35

1 Introduction

1.1 The model and the main result

Three-dimensional U(1) lattice gauge theory is the simplest gauge theory in which static charges are expected to be confined at every coupling. By Polyakov's mechanism [Pol77, BMK77], magnetic monopoles form a plasma that screens the magnetic field at the Debye length and squeezes the electric flux between two static charges into a tube of finite tension. Göpfert and Mack proved this for the Villain action [GM82]. In this paper we prove it for the Wilson action.

Definition 1.1 (Wilson and Villain measures and the Wilson loop)**.** A finite box $\Lambda \subset \mathbb{Z}^3$ is a cell complex with vertex set Λ, bonds (oriented edges) e and plaquettes p (free boundary conditions throughout); for $\theta \in (\mathbb{R}/2\pi\mathbb{Z})^{E(\Lambda)}$ and an oriented plaquette p, θ_p is the sum of θ_e around ∂p.

1. The *Wilson measure* at coupling $\beta > 0$ (weak coupling means large β) and the *Villain weight* at coupling $\tau > 0$ are

$$\mu^W_{\Lambda,\beta}(d\theta) = \frac{1}{Z^W_{\Lambda,\beta}} \prod_p e^{\beta \cos\theta_p} \prod_e d\theta_e, \qquad \tilde{v}_\tau(\theta) \propto \sum_{m\in\mathbb{Z}} e^{-\tau(\theta-2\pi m)^2/2}; \tag{1.1}$$

   the Villain measure $\mu^V_{\Lambda,\tau}$ is $\mu^W_{\Lambda,\beta}$ with $e^{\beta\cos\theta_p}$ replaced by $\tilde{v}_\tau(\theta_p)$.

2. For a rectangle $C = C_{R,T}$ of sides $R, T \in \mathbb{N}$ in a coordinate plane, an oriented closed lattice path, $\theta_C := \sum_{e\in C} \theta_e$, and the Wilson loop expectation is $\langle \cos\theta_C \rangle^W_{\Lambda,\beta}$.

3. The model has an *area law at coupling* β if there are $\sigma > 0$, $c < \infty$ and $R_1 < \infty$ such that $\langle \cos\theta_C \rangle^W_{\Lambda,\beta} \le \exp(-\sigma RT + c(R+T))$ for every rectangle C with $\min(R,T) \ge R_1$ and every finite box $\Lambda \supseteq C$.

This area law, uniform in the box and with an explicit perimeter term, is stronger than positivity of the iterated-limit string tension α of [GM82, (1.7)],[1] whose finite-loop estimate carries the loss term P_F of [GM82, (7.7)]. Theorem 2.1(c) shows that the infinite-volume limit exists and that the area law passes to it. By Remark 9.5, the proof also gives the area law in this form for the Villain action.

Theorem 1.2 (Main theorem)**.** *For every $\beta > 0$ there are $\sigma(\beta) > 0$ and $R_1(\beta) < \infty$ such that for every rectangle C of sides R, T with $\min(R,T) \ge R_1(\beta)$ and every finite box $\Lambda \supseteq C$,*

$$\langle \cos\theta_C \rangle^W_{\Lambda,\beta} \le \exp\bigl(-\sigma(\beta) RT + c(R+T)\bigr), \tag{1.2}$$

where c is an absolute constant. Moreover, for every $K > K_* := 377.38$ there is $\beta_0(K) < \infty$ such that, for every $\beta \ge \beta_0(K)$, $\sigma(\beta)$ and $R_1(\beta)$ can be chosen, depending on K, so that (1.2) *holds and*

$$\sigma(\beta) \ge 1.35 \cdot 10^{-4}\, c_g c_L\, (K\beta)^{-1/2} e^{-2.49436\, K\beta}, \tag{1.3}$$

where c_g, c_L are the fixed regularization constants of Section 8.2; for $\beta < \beta_0(K)$, (1.2) *then holds with $\sigma(\beta)$ and $R_1(\beta)$ equal to the values so chosen at $\beta_0(K)$.*

The first assertion follows from the second with one fixed $K > K_*$. In the first assertion the absoluteness of c is no restriction: if (1.2) holds with a constant $c' = c'(\beta)$, then it holds with $c = 0$ and with $\sigma(\beta)/2$ and $\max(R_1(\beta), 4c'/\sigma(\beta))$ in place of $\sigma(\beta)$ and $R_1(\beta)$, since $c'(R+T) \le 2c' \max(R,T) \le \frac{1}{2}\sigma(\beta)RT$ when $\min(R,T) \ge 4c'/\sigma(\beta)$. The content of the first assertion is the uniformity in Λ; the absolute c matters together with (1.3). With $c_g = 1/4$ and $c_L = 1.6\cdot 10^{-19}$ (Section S1.1.1), (1.3) reads $\sigma(\beta) \ge 5.4\cdot 10^{-24} (K\beta)^{-1/2} e^{-2.49436 K\beta}$ (Section S1.1.3).

Corollary 1.3 (Groups containing the scalars, with or without Higgs fields blind to the centre of $\mathrm{U}(n)$)**.** *Let $G \subseteq U(n)$ be a closed subgroup with $\{zI : |z| = 1\} \subseteq G$, in $d = 3$ with the Wilson action $\beta \operatorname{Re}\operatorname{Tr} U_p$ (free boundary conditions), with or without a Higgs field coupled through any action that depends on the links only through $\rho(U_e)$, ρ a representation of G trivial on the scalars. At every $\beta > 0$ and all Higgs couplings, for every rectangle C with $\min(R,T) \ge R_1(2n\beta)$ and every finite box $\Lambda \supseteq C$, $\bigl|\langle \operatorname{Tr}\prod_{e\in C} U_e \rangle_{G,\beta}\bigr| \le 2n \exp\bigl(-\sigma(2n\beta)RT + c(R+T)\bigr)$; in particular for U(2) with an adjoint Higgs field.*

[1] Equation, section and page numbers of [GM82] and [GM81a] refer to the preprints [GM81c] (DESY 81-036) and [GM81b] (DESY 81-014), equation numbers of [Frö79] to its preprint IHES/P/79/02, and lemma numbers of [Gut80] to its preprint CLNS 79/439.

Corollary 1.4 (Georgi–Glashow model at infinite Higgs coupling)**.** *The SU(2) lattice gauge theory with the Wilson action $\beta \cdot \frac{1}{2} \operatorname{Tr} U_p$ and a unit adjoint Higgs field at hopping coupling $\kappa = \infty$, defined in each finite box as the limit $\kappa \to \infty$, is the U(1) Wilson model at coupling β, and it obeys the area law* (1.2) *at every $\beta > 0$.*

1.2 Background

The Villain action. Göpfert and Mack [GM82] made Polyakov's mechanism rigorous for the Villain action. By duality the model is a Coulomb gas of monopoles, which they control by an iterated Mayer expansion down to the Debye scale [GM81a] and a single-scale expansion built on the analysis of Debye screening of Brydges and Federbush [BF80, Bry78]. Their Theorem 1 is the large-β bound $\alpha \geq c\, m_D \beta^{-1}$ [GM82, (1.9)], with m_D the Debye mass, and all couplings follow [GM82, Corollary 2] by the 3-dimensional version of Guth's inequality [Gut80, Lemma 4.1]. The expansions use the Gaussian form of the dual weight: in [GM82, Appendix A] the Villain action is chosen explicitly, in place of their Wilson action (A.4).

The Wilson action. In three dimensions the rigorous results are the area law at strong coupling [OS78], for all compact groups, and logarithmic confinement bounds at all couplings, for U(1) [GJ77] and for closed $G \subseteq U(n)$ containing the scalars [Cha26]. Fröhlich's comparison [Frö79] bounds the Wilson loop of such a group by a U(1) loop with the Wilson action $A' = -\beta' \sum_p \cos(\theta_{\partial p})$ of [Frö79, (6)], at coupling $2n\beta$ [Frö79, (23), (24)]. Thus, through this comparison, an area law at every coupling for these groups requires the Wilson case of U(1), not the Villain case. This case is sometimes cited as known [CNS25, For26, CPS25], as is a mass gap for all β [Nis25], but no proof for the Wilson action has been given: the expansions of [GM82] do not apply to it as they stand, and whether the Villain result can be combined with Fröhlich's comparison to give an area law for U(n) with the Wilson action was left open in [Cha26, v2].

Related work. The mixture of Lemma 2.2 is used to prove lower bounds for the XY model [AHPS21, Lemma 9.6] and the annealed Villain model [DvEG26]. Here we use it for an upper bound. The same Bessel dual weights and their Gaussian mixture enter recent work on the critical planar XY model [Ope26, §2.1]. In four dimensions a Coulomb phase is proved for the Villain [Gut80, FS82] and the Wilson action [Kin86].

1.3 Strategy

The proof has four steps.

(i) Comparison (Section 2). Lemma 2.2 expresses $e^{\beta \cos\theta}/I_0(\beta)$ as a mixture of Villain weights $\tilde{v}_{1/u}$, by a theorem of Hartman and Watson [HW74], and Lemma 2.3 shows, by Ginibre's inequality, that Villain loops are nondecreasing in each coupling. We raise the couplings below $\tau = K\beta$ to τ and the others to ∞, and Lemmas 2.4 and 2.6 allow us to dominate the set of the latter by a Bernoulli set of density $\delta = e^{-c_\delta(K)\beta}$, where $c_\delta(K)/K \to \infty$ as $K \to \infty$. In this way Theorem 2.5 bounds the Wilson loop at β by the average over a Bernoulli set D of the Villain loop at τ with D frozen. Proposition 2.9 places this Villain loop in one box Λ° that depends neither on D nor on Λ (Lemma 3.3), where frozen plaquettes are insulating walls (Proposition 2.10).

(ii) Regularization (Sections 3 and 4). A deterministic construction maps each D to a hole set and frozen sheets. It freezes more plaquettes, which only increases the loop. It works on the blocks of a loop-adapted coarse grid, and a coarse site touched by D is a mark.

The hole set of its first phase is the smallest minimizer of a cut functional that penalizes leaving blocks crowded with marks unfilled. Thus each cost bound is a comparison with an explicit competitor. A local seal adds thin pieces of blocks to the hole set, and a second phase enforces non-fragmentation, which is not local. The output, a slit domain $G'(D)$, has uniform isoperimetry, cells of controlled size and shape with a Poincaré inequality, and non-fragmentation, with constants bounded independently of β, K, C, D (Theorems 3.20, 3.21).

(iii) Deterministic area law (Sections 6 and 7, Appendices A to D). On each output $G'(D)$ we run the analysis of [GM82, GM81a, BF80] with uniform constants, grading the outer partition functions by the periods of their walls (the bonds where the integer field of the expansion jumps, net of the loop current) around the handles of the insulating set (Definition 7.25). We adapt the steps we need from these sources (Part S4 proves them in full), and Appendix A proves a tree-graph counting step of [Bry78] for which the deduction given there is incomplete. Theorem 7.57 states that on each output the loop decays exponentially in the min-cut area $A_{G'}(C)$, in which frozen plaquettes are free.

(iv) Min-cut lower tail (Section 5). A Peierls estimate shows that $A_{G'}(C) \geq (1-\eta_T)RT$ outside an event of probability at most $2^{4|C|+5}e^{-RT}$, with $\eta_T \to 0$ as $K \to \infty$ (Theorem 5.1). We impose the condition $K > K_*$ for this estimate alone (Lemma 2.6 and the remark after it).

In Section 8 we combine the four steps into the following chain: for $\beta \geq \beta_0(K)$, with $D \sim \mathrm{Ber}(\delta)$ on the interior plaquettes of Λ°,

$$\begin{aligned}
\langle \cos\theta_C \rangle^W_{\Lambda,\beta} &\leq \mathbb{E}_D\, g^{\mathrm{fix}}_{\Lambda^\circ,\tau}(C;D) && (1.4)\\
&\leq \mathbb{E}_D\, W_{G'(D)}(C) && (1.5)\\
&\leq \mathbb{E}_D \exp\bigl(-\sigma_V A_{G'(D)}(C) + \sigma_V \eta_S RT + c_N(R+T)\bigr) && (1.6)\\
&= \exp\bigl(-\sigma_V RT + \sigma_V \eta_S RT + c_N(R+T)\bigr)\, \mathbb{E}_D \exp\bigl(\sigma_V (RT - A_{G'(D)}(C))\bigr) && (1.7)\\
&\leq \exp\bigl(-\sigma_V(1-\eta_T-\eta_S)RT + c_N(R+T) + 4|C| + 4\bigr), && (1.8)
\end{aligned}$$

where $g^{\mathrm{fix}}_{\Lambda^\circ,\tau}(C;D)$ is the Villain loop on Λ° with D and the boundary plaquettes frozen, $W_{G'(D)}(C)$ is the Villain loop on the output, and σ_V, η_S and c_N are the string tension, the deficit and the perimeter constant of Theorem 7.57. Theorem 2.1, by which the Wilson loop is nondecreasing in β, extends the resulting bound to all β (Remark 8.1).

Two natural routes fail. First, the Wilson dual weight $I_n(\beta)/I_0(\beta)$, of order $e^{-|n|\log|n|(1+o(1))}$, is dominated by no Villain dual weight $e^{-n^2/(2t)}$. Second, on the carpet graphs of [CG25], where the Villain action is a limit of Wilson actions, the Wilson model is the limit with the auxiliary plaquettes frozen, and hence Ginibre's inequality bounds the Wilson loop from below, not from above. Appendix E shows that the first obstruction persists on every finite box (Theorem E.2): no comparison that holds for all charges at once, such as those obtained from correlation inequalities, bounds Wilson loops by Villain loops with bounded couplings, even with a perimeter factor (Corollary E.4). Theorem 2.5 is such a comparison, since it holds for every integer 1-cycle, and this is why it keeps a random set of frozen plaquettes (Remark E.7). Appendix E also shows that Wilson loops are not monotone in the character coefficients of the plaquette weights (Theorem E.8), so the Villain monotonicity of Lemma 2.3(b) is not an instance of a general monotonicity in these coefficients (Remark E.9).

How the paper is organized. Sections 2 and 8 use one fact about the construction, that its output is a deterministic function of D whose set $\mathfrak{F}$ of frozen plaquettes contains D (Theorem 3.20); Sections 6 and 7 can be read independently of the construction, given $A_{G'}(C)$ and

the items listed after Theorem 7.57; Section 5 relies on the construction of Section 3 and on Theorem 3.21; and Appendix E uses only Section 2, and no other part of the paper uses it. Part S1 of the supplementary material gives the values of the constants with their arithmetic, and Part S2 completes individual proofs that the main text sketches. Part S4, a separate file, proves in full, in the order of the main text, the parts of the argument that the main text takes from the literature with changes or presents in compressed form. It uses the sources only as a guide, and its introduction lists which of its sections completes which part of the main text.

Theorem 1.2, Corollaries 1.3 and 1.4, Fröhlich's comparison (Proposition 9.1) and all results of Appendix E are formalized in Lean 4 with Mathlib and checked by the Lean kernel [Tom26]. The formal theorems carry no hypotheses beyond those of the statements and use only Lean's three standard axioms, so the steps that their proofs take from the literature are formalized as well.

1.4 Conventions

Couplings. We write β for the Wilson coupling, $\tau = K\beta$ for the Villain coupling of the comparison, and $\beta_{\mathrm{GM}} = 4\pi^2\tau$ for the coupling in the normalization of [GM82], which we use in Sections 6 and 7 and in Appendices A to D.

Constants and units. Section 8.2 fixes the constants in order. Those fixed before K do not depend on β, τ, K, C or D, and c, C with or without indices denote such constants, except where a dependence is shown or stated. This is the case for $c_\delta(K)$, which also depends on β, for c'_g and c_u, which depend on β through the size and the rounding of L_c, and for c_{dep}, which grows like $\log\tau$ (Proposition 6.17). The letter C also denotes the loop. One site is one coarse cell of the loop-adapted grid in Sections 3 to 5, unless "fine" is written, and the lattice spacing is one in Sections 6 and 7 (Section 2.5).

2 From the Wilson action to a diluted Villain model

In this section we bound the Wilson loop by an average of Villain loops over a Bernoulli set of frozen plaquettes, and we show that frozen plaquettes are insulating walls.

Complexes and fields. Let X be a finite 2-dimensional cell complex with oriented edges $E(X)$, oriented plaquettes $P(X)$ and incidence numbers $[p : e]$. For $\theta \in \mathbb{T}^{E(X)}$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, we set $\theta_p := \sum_e [p : e]\theta_e$, and for $n \in \mathbb{Z}^{P(X)}$, $(\partial n)_e := \sum_p n_p[p : e]$. An *integer* 1*-cycle* is a $C \in \mathbb{Z}^{E(X)}$ with zero boundary, $\theta_C := \sum_e C_e\theta_e$, and $Z_2(X) := \ker\partial$. The Villain weight of Definition 1.1, normalized by $\int \tilde{v}_t\, d\theta/2\pi = 1$, is $\tilde{v}_t(\theta) = \sum_{n\in\mathbb{Z}} e^{-n^2/(2t)}e^{in\theta} = (2\pi t)^{1/2}\sum_{m\in\mathbb{Z}} e^{-t(\theta-2\pi m)^2/2} > 0$ by Poisson summation, and $\tilde{v}_t \to 1$ uniformly as $t \to 0$. Let $\mu^V_{X,t} \propto \prod_p \tilde{v}_{t_p}(\theta_p)\prod_e d\theta_e$ for $t \in (0,\infty)^{P(X)}$, and $\mu^W_{X,\beta} \propto \prod_p e^{\beta\cos\theta_p}\prod_e d\theta_e$. For $\tau > 0$, $D \subseteq P(X)$ and $s > 0$ let $t_p^{\tau,D,s}$ be τ off D and s on D. Then the *frozen expectation* $g_{X,\tau}(C; D) := \lim_{s\to\infty}\langle\cos\theta_C\rangle^V_{X,t^{\tau,D,s}}$ exists by Lemma 2.3, and the plaquettes of D are *frozen*.

Ginibre's inequality. Let $\mu \propto \exp(\sum_A J_A\cos(A\cdot\phi))\, d\phi$ on a torus $\mathbb{T}^m$, where A runs over a finite family of integer vectors and $J_A \geq 0$. Then for all integer vectors B, B'

$$\langle\cos(B\cdot\phi)\rangle_\mu \geq 0, \qquad \langle\cos(B\cdot\phi)\cos(B'\cdot\phi)\rangle_\mu - \langle\cos(B\cdot\phi)\rangle_\mu\langle\cos(B'\cdot\phi)\rangle_\mu \geq 0, \tag{2.1}$$

by [Gin70, Propositions 2 and 3] applied to the torus of [Gin70, Example 4], with S the set of functions $\cos(m\cdot\phi)$. In particular, $\partial\langle\cos(B\cdot\phi)\rangle_\mu/\partial J_A \geq 0$ for all integer vectors B.

2.1 Monotonicity

Theorem 2.1 (Monotonicity for the Wilson action)**.** *Let C be an integer 1-cycle supported in a box $\Lambda \subset \mathbb{Z}^3$, in the U(1) Wilson model with free boundary conditions. (a) $\langle e^{i\theta_C}\rangle^W_{\Lambda,\beta} = \langle\cos\theta_C\rangle^W_{\Lambda,\beta} \in [0,1]$. (b) $\langle\cos\theta_C\rangle^W_{\Lambda,\beta}$ is nondecreasing in β, and nondecreasing in Λ under inclusion. (c) Along every increasing sequence of boxes Λ_k exhausting $\mathbb{Z}^3$ the limit $\langle\cos\theta_C\rangle^W_\beta := \lim_k\langle\cos\theta_C\rangle^W_{\Lambda_k,\beta}$ exists, equals the supremum of $\langle\cos\theta_C\rangle^W_{\Lambda,\beta}$ over all boxes Λ containing the support of C, and is nondecreasing in β.*

Proof. (a) and the monotonicity in β are (2.1) with $J_p = \beta$, together with the substitution $\theta \mapsto -\theta$. For the volume monotonicity, let $\Lambda \subseteq \Lambda'$ be boxes. The expectation on Λ' with coupling $s \in [0,\beta]$ on the plaquettes not in Λ is nondecreasing in s and equals the one on Λ at $s = 0$, where the edges outside Λ interact with nothing. (c) follows, since each box lies in Λ_k for large k.

2.2 The Hartman–Watson mixture and the annealed coupling field

Lemma 2.2 (Hartman–Watson mixture)**.** *For every $\beta > 0$ there is a unique probability measure ω_β on $(0,\infty)$ such that, for all $\nu \geq 0$ and $\theta \in \mathbb{T}$,*

$$\int_{(0,\infty)} e^{-\nu^2 u/2}\,\omega_\beta(du) = \frac{I_\nu(\beta)}{I_0(\beta)}, \qquad \frac{e^{\beta\cos\theta}}{I_0(\beta)} = \int_{(0,\infty)} \tilde v_{1/u}(\theta)\,\omega_\beta(du), \tag{2.2}$$

and $\omega_\beta((0,\varepsilon)) \leq e^{\nu^2\varepsilon/2} I_\nu(\beta)/I_0(\beta)$ for all $\varepsilon > 0$ and $\nu \geq 0$.

Proof. By [HW74, Theorem 1.3], $I_\nu(\beta)/I_0(\beta) = \int_0^\infty e^{-\nu^2 t}\,W(dt,\beta)$ $(0 \leq \nu < \infty)$ with a distribution function $W(\cdot,\beta)$ on $[0,\infty)$. We let ω_β be its image under $u = 2t$, which has no atom at 0 because $I_\nu(\beta) \leq (\beta/2)^\nu e^{\beta^2/4}/\Gamma(\nu+1) \to 0$ as $\nu \to \infty$. The second identity follows from the first by Fourier inversion and Fubini, as $\sum_n I_{|n|}(\beta)/I_0(\beta) = e^\beta/I_0(\beta) < \infty$ (for the XY model it is [AHPS21, Lemma 9.6]). The tail bound is the Chernoff bound $\mathbf{1}_{u<\varepsilon} \leq e^{\nu^2(\varepsilon-u)/2}$. It is unique, since a finite measure on $[0,\infty)$ is determined by its Laplace transform on $[0,\infty)$, which the first identity of (2.2) gives at $\nu^2/2$ for every real $\nu \geq 0$.

Lemma 2.3. *Let X be a finite 2-complex and C an integer 1-cycle. (a) For $t \in (0,\infty)^{P(X)}$, $\langle\sin\theta_C\rangle^V_{X,t} = 0$ and $\langle\cos\theta_C\rangle^V_{X,t} \geq 0$. (b) $\langle\cos\theta_C\rangle^V_{X,t}$ is nondecreasing in each t_p. (c) (a) and (b) hold when some plaquettes are removed (weight 1, the limit $t_p \to 0$); a removed plaquette gives a value no larger than any $t_p > 0$. (d) For fixed couplings off $D \subseteq P(X)$, the limit as the couplings on D tend to ∞ (jointly, or all equal to $s \to \infty$; by monotonicity the two limits agree) exists, lies in $[0,1]$, and is nondecreasing in each remaining coupling and in D; in particular $g_{X,\tau}(C;D) \in [0,1]$ exists and is nondecreasing in D (for inclusion) and in τ.*

Proof. On a cube of $\mathbb{Z}^3$ with couplings in $(0,\infty)$, (b) is proved in [CG25, Corollary 1.6], and also in [Gut80, Lemma C.1] for plaquette-dependent couplings. We need the lemma for boxes and adapt these results to them as follows. Removal, $t_p \to 0$, is dominated convergence in (2.6), whose terms decrease as $t_p \downarrow 0$ to those of the complex without p, and (c) follows. A box

is a cube with the outside plaquettes removed, and an integer 1-cycle C has $\theta_C = \theta_\ell$ for one closed path ℓ. Freezing, $t_p \to \infty$, is a monotone limit in $[0,1]$. This yields (d), as $t^{\tau,D,s} \le t^{\tau,D',s}$ for $D \subseteq D'$, $s \ge \tau$. For (a) we use $\theta \mapsto -\theta$ and the fact that the value with all plaquettes removed is $\int \cos\theta_C \, d\theta/(2\pi)^{|E|} \ge 0$. This proves the lemma on boxes. In Section S2.1 we prove the lemma on all finite 2-complexes, by verifying Ginibre's hypothesis for the Villain weights directly.

The annealed coupling field. Fix X and $\beta > 0$. Let $Z(t) := \int \prod_p \tilde{v}_{t_p}(\theta_p)\, d\theta/(2\pi)^{|E(X)|} > 0$ for $t \in (0,\infty)^{P(X)}$, and let μ_{ann} be the probability measure on $(0,\infty)^{P(X)}$ with density proportional to $Z(1/u)$, $1/u := (1/u_p)_p$, with respect to $\omega_\beta^{\otimes P(X)}$. The normalization is finite and positive by (2.2) and Fubini. We also put $\vartheta_J(u) := \sum_{n\in\mathbb{Z}} e^{-n^2 u/2}$ and $\delta_\beta(\varepsilon) := \int_{(0,\varepsilon)} \vartheta_J(u)\, \omega_\beta(du)$.

Lemma 2.4 (Annealed coupling field)**.** *Let X be a finite 2-complex, $\beta > 0$ and $\varepsilon > 0$. (i) For every bounded measurable F on $\mathbb{T}^{E(X)}$, $\langle F\rangle^W_{X,\beta} = \int \langle F\rangle^V_{X,1/u}\, \mu_{\mathrm{ann}}(du)$. (ii) (Tilt bound.) For every $p \in P(X)$ and every $u_{-p} := (u_q)_{q\neq p}$, $\mu_{\mathrm{ann}}(u_p < \varepsilon \mid u_{-p}) \le \delta_\beta(\varepsilon)$. (iii) Under μ_{ann} the random set $D_\varepsilon(u) := \{p : u_p < \varepsilon\}$ is stochastically dominated by $\mathrm{Ber}(\delta')^{\otimes P(X)}$ for every $\delta' \in [\min(\delta_\beta(\varepsilon),1),1]$: $\int F(D_\varepsilon(u))\, \mu_{\mathrm{ann}}(du) \le \mathbb{E}_{D\sim \mathrm{Ber}(\delta')^{\otimes P(X)}} F(D)$ for every nondecreasing function F on the subsets of $P(X)$. (iv) $\delta_\beta(\varepsilon) \le \inf_{\nu\ge 0} e^{\nu^2\varepsilon/2} \sum_{n\in\mathbb{Z}} I_{\sqrt{n^2+\nu^2}}(\beta)/I_0(\beta)$.*

Proof. (i) We multiply the second identity of (2.2) over p, integrate against F and against 1, and use Fubini.

(ii) The tilt is bounded because, as a function of one coupling, Z is a combination of the functions $e^{-n^2 u_p/2}$ with nonnegative coefficients, each at most the constant coefficient. We fix p and u_{-p}. As μ_{ann} has density proportional to $Z(1/u)$ with respect to a product measure, the conditional law of u_p given u_{-p} has density proportional to $u_p \mapsto Z(1/u)$ with respect to ω_β. Expanding $\tilde{v}_{1/u_p}(\theta_p) = \sum_n e^{-n^2 u_p/2} e^{in\theta_p}$, an absolutely convergent series, we obtain $Z(1/u) = \sum_{n\in\mathbb{Z}} e^{-n^2 u_p/2} A_n$, $A_n := \int e^{in\theta_p} \prod_{q\neq p} \tilde{v}_{1/u_q}(\theta_q)\, d\theta/(2\pi)^{|E|}$, with $A_0 > 0$ independent of u_p. Here $A_n/A_0 = \langle e^{in\theta_p}\rangle$ in the system with p removed. Since $n\theta_p = \theta_{C'}$ for the integer 1-cycle $C' = n\,\partial p$, Lemma 2.3(a),(c) shows that A_n is real and $0 \le A_n \le A_0$. Hence $1 \le Z(1/u)/A_0 \le \vartheta_J(u_p)$, where the lower bound comes from the term $n = 0$, and $\mu_{\mathrm{ann}}(u_p < \varepsilon \mid u_{-p}) = \int_{(0,\varepsilon)} (Z/A_0)\, d\omega_\beta \big/ \int_{(0,\infty)} (Z/A_0)\, d\omega_\beta \le \int_{(0,\varepsilon)} \vartheta_J\, d\omega_\beta = \delta_\beta(\varepsilon)$.

(iii) Let $Y_p := \mathbf{1}[u_p \ge \varepsilon]$ and fix an order $s_1, s_2, \dots$ of $P(X)$. Since $Y_{s_1}, \dots, Y_{s_j}$ are functions of $u_{-s_{j+1}}$, (ii) and the tower property give $\mathbb{P}(Y_{s_{j+1}} = 1 \mid Y_{s_1}, \dots, Y_{s_j}) \ge 1 - \min(\delta_\beta(\varepsilon), 1)$. By [LSS97, Lemma 1.1] this yields the domination for $\delta' = \min(\delta_\beta(\varepsilon),1)$, and a monotone coupling shows that $\mathbb{E}F(D)$ is nondecreasing in δ'.

(iv) is the Chernoff bound $\mathbf{1}_{u<\varepsilon} \le e^{\nu^2(\varepsilon-u)/2}$ integrated against $\vartheta_J(u)\, \omega_\beta(du)$, with (2.2).

Theorem 2.5 (Comparison with diluted Villain loops)**.** *For every finite 2-complex X, integer 1-cycle C, $\beta > 0$, $\varepsilon > 0$ and $\delta' \in [\min(\delta_\beta(\varepsilon),1),1]$,*

$$0 \le \langle \cos\theta_C\rangle^W_{X,\beta} \le \mathbb{E}_{D\sim \mathrm{Ber}(\delta')^{\otimes P(X)}}\, g_{X,1/\varepsilon}(C;D). \tag{2.3}$$

Proof. Lemma 2.4(i) represents the Wilson model as a Villain model with random couplings $1/u_p$, and by monotonicity we round all couplings up, those at most $1/\varepsilon$ to $1/\varepsilon$ and the others to ∞. Positivity is a case of (2.1). For fixed u and $D = D_\varepsilon(u)$ we have $1/u_p \le 1/\varepsilon$ off D, and hence Lemma 2.3(b),(d) give $\langle \cos\theta_C\rangle^V_{X,1/u} \le g_{X,1/\varepsilon}(C; D_\varepsilon(u))$. We integrate this against

$\mu_{\mathrm{ann}}(du)$, use Lemma 2.4(i) once more, and apply Lemma 2.4(iii) to the function $g_{X,1/\varepsilon}(C;\cdot)$, which is nondecreasing by Lemma 2.3(d).

2.3 The density of frozen plaquettes

From now on we take $\varepsilon = 1/(K\beta)$, so that the Villain coupling is $\tau = K\beta$, and set $\delta := \delta_\beta(1/(K\beta))$ and $c_\delta(K) := -(\log\delta)/\beta$, a function of K and β, although the notation shows only K. Moreover $\delta > 0$: $\vartheta_J \geq 1$ gives $\delta \geq \omega_\beta((0,1/(K\beta)))$, and $\omega_\beta((0,\varepsilon)) = 0$ would give $I_\nu(\beta)/I_0(\beta) \leq e^{-\nu^2\varepsilon/2}$ for all $\nu \geq 0$ by (2.2), against $I_\nu(\beta) \geq (\beta/2)^\nu/\Gamma(\nu+1)$. For $K > e/2$ let $a_* = a_*(K) > e/2$ be the solution of $a_* = K\log(2a_*)$ (unique on $(e/2,\infty)$, where $a \mapsto a/\log 2a$ increases), and put $c_\delta^{\mathrm{sp}}(K) := a_*\big(\frac{1}{2}\log 2a_* - 1\big) + 1$.

Lemma 2.6. *Let $K > e/2$ and $\beta \geq \pi^{-2}$. Then*

$$\log\delta \leq -\beta\Big(c_\delta^{\mathrm{sp}}(K) - \frac{1}{4a_*}\Big) + \log\big((2a_*\beta+5)\pi\sqrt{\beta}\big) + \frac{1}{2}, \tag{2.4}$$

so that $c_\delta(K) \geq c_\delta^{\mathrm{sp}}(K) - 1/(4a_*) - O_K(\log\beta/\beta)$. The function $K \mapsto c_\delta^{\mathrm{sp}}(K)/(2K)$ is increasing on $(e/2,\infty)$; it equals 14.9662 at $K = 377.3635$, and it is at least 14.9663 for $K \geq K_* = 377.38$. Consequently, for every $K > K_*$ there are $\epsilon_K > 0$ and $\beta_1(K)$ such that

$$\frac{c_\delta(K)}{2K} \geq 14.9662 + \epsilon_K \qquad (\beta \geq \beta_1(K)). \tag{2.5}$$

The number 14.9662 is $6\pi^2 v_{\mathrm{Cb}}(0) = 14.96613052\ldots$ rounded up. Here $v_{\mathrm{Cb}}(0) = (-\Delta_{\mathbb{Z}^3})^{-1}(0,0) = 0.2527310\ldots$ is half of Watson's integral [Wat39] in the normalization of [Zuc11], whose closed form is [Zuc11, (3.6)] (Section S1.1.1), and the Debye exponent $\pi^2 v_{\mathrm{Cb}}(0) = 2.4943550871\ldots$ gives $\log(1/\tilde{m}) = \pi^2 v_{\mathrm{Cb}}(0)\,\tau + O(\log\tau)$ (Definition 3.1). Thus, for $K > K_*$, (2.5) says $\delta^{1/2} \leq \tilde{m}^6 e^{-\epsilon_K\tau + O(\log\tau)}$. In the Peierls count of Lemma 5.12 this means that the factor $\delta^{1/2}$ that each mark costs (Definition 3.6 and Lemma 5.7) is smaller than the reciprocal of the entropy $(\tilde{m}^{-1})^{6+o(1)}$ of a root and a part. For the first assertion of Theorem 1.2 the value of K_* is not needed: with $t := \log 2a_*(K)$ we have $K = e^t/(2t)$ and $c_\delta^{\mathrm{sp}}(K)/(2K) = \frac{1}{4}t(t-2) + te^{-t}$, which tends to ∞ with K (as $t > \log 2K$), so every sufficiently large K satisfies (2.5).

Proof. Heuristically, the heavy dual tail of the Wilson weight comes from ω_β near $u = 0$. Applying Lemma 2.4(iv) at its saddle point $\nu = a_*\beta$, with the series of I_μ, log-convexity of Γ, Stirling's inequality and $I_0(\beta) \geq e^{\beta-1/2}/(\pi\sqrt{\beta})$ for $\beta \geq \pi^{-2}$, we obtain (2.4) (Section S1.1.2 gives the computation, the monotonicity and the two values). For $K > K_*$, $\epsilon_K := \frac{1}{2}\big(c_\delta^{\mathrm{sp}}(K)/(2K) - 14.9662\big) \geq 9\cdot 10^{-5}$, since $c_\delta^{\mathrm{sp}}(K)/(2K)$ is increasing and, by the computation in Section S1.1.2, equals 14.96639 at K_*. For these K the value 14.9663 of the statement gives $\epsilon_K \geq 5\cdot 10^{-5}$, and as $1/(8Ka_*) < 10^{-7}$ for $K \geq K_*$, either lower bound on ϵ_K shows that (2.4) implies (2.5) once the $O_K(\log\beta/\beta)$ term is below $K\epsilon_K$.

2.4 The dual representation and the frozen limit

Let Λ' be a box, regarded as a closed cubical complex with cubes Λ'_3, plaquettes Λ'_2 and boundary plaquettes $\partial_2\Lambda'$, and let $\partial : \mathbb{Z}^{\Lambda'_3} \to \mathbb{Z}^{\Lambda'_2}$ be the boundary map, cubes oriented outward. Since the complex is contractible, ∂ is injective, $Z_2(\Lambda') = \partial\mathbb{Z}^{\Lambda'_3}$, and each integer 1-cycle C in Λ' bounds an integer 2-chain n_0.

Lemma 2.7 (Dual representation)**.** *Let $t \in (0,\infty)^{\Lambda'_2}$, let C be an integer 1-cycle in Λ', and let n_0 be any integer 2-chain of Λ' with $\partial n_0 = C$. Then*

$$\langle \cos\theta_C \rangle^V_{\Lambda',t} = \frac{\sum_{k\in\mathbb{Z}^{\Lambda'_3}} \prod_p e^{-(n_0+\partial k)_p^2/(2t_p)}}{\sum_{k\in\mathbb{Z}^{\Lambda'_3}} \prod_p e^{-(\partial k)_p^2/(2t_p)}}. \tag{2.6}$$

Proof. By the expansion $\prod_p \tilde{v}_{t_p}(\theta_p) = \sum_{n\in\mathbb{Z}^{\Lambda'_2}} \prod_p e^{-n_p^2/(2t_p)} e^{i(\partial n)\cdot\theta}$, which converges absolutely and has even coefficients, $\langle e^{i\theta_C}\rangle = \langle \cos\theta_C\rangle$ is the ratio of the sums over $\partial n = C$ and over $\partial n = 0$. Since $Z_2(\Lambda') = \partial\mathbb{Z}^{\Lambda'_3}$ and ∂ is injective, the chains $n = n_0 + \partial k$, $k \in \mathbb{Z}^{\Lambda'_3}$, parametrize the first set bijectively for every choice of n_0 (a point that matters when frozen plaquettes surround an edge of C).

Dually, cubes are sites, an interior plaquette is a bond between its two cubes, and a boundary plaquette is a bond to an exterior ground site, where $k = 0$. We now fix $\tau > 0$ and a set $F \subseteq \Lambda'_2$ of plaquettes to be frozen, and let t_p^s be τ for $p \notin F$ and s for $p \in F$. The *unfrozen graph* has the cubes and the ground as vertices and the plaquettes not in F as bonds, and $\mathcal{K}_F$ denotes the set of its components not containing the ground. A cube with all faces in F is such a component. Fix $c_{\mathfrak{K}} \in \mathfrak{K}$ and put $W := \{\sum_{\mathfrak{K}\in\mathcal{K}_F} a_{\mathfrak{K}} \mathbf{1}_{\mathfrak{K}} : a \in \mathbb{Z}^{\mathcal{K}_F}\}$ and $W' := \{k \in \mathbb{Z}^{\Lambda'_3} : k(c_{\mathfrak{K}}) = 0 \text{ for all } \mathfrak{K} \in \mathcal{K}_F\}$. Then $\mathbb{Z}^{\Lambda'_3} = W \oplus W'$, and $(\partial w)_p = 0$ for $w \in W$, $p \notin F$.

Lemma 2.8. *In this setting, for every integer 1-cycle C in Λ' and every integer 2-chain n_0 of Λ' with $\partial n_0 = C$,*

$$\lim_{s\to\infty} \langle \cos\theta_C \rangle^V_{\Lambda',t^s} = \frac{\sum_{k\in W'} \exp\bigl(-\frac{1}{2\tau}\sum_{p\notin F}(n_0+\partial k)_p^2\bigr)}{\sum_{k\in W'} \exp\bigl(-\frac{1}{2\tau}\sum_{p\notin F}(\partial k)_p^2\bigr)}. \tag{2.7}$$

The quadratic form $k \mapsto \sum_{p\notin F}(\partial k)_p^2$ is positive definite on $W' \otimes \mathbb{R}$, so both sums converge; the summands are invariant under W, so the sums run over $\mathbb{Z}^{\Lambda'_3}/W$ (k modulo one constant on each component in $\mathcal{K}_F$) and do not depend on the choice of W'. In particular every cube all of whose faces are frozen drops out.

Proof. Freezing deletes dual bonds; the only issue is the soft directions W, one zero mode for each component in $\mathcal{K}_F$, isolated cubes included, such as the cubes outside Λ' in Proposition 2.9. Along these directions only the frozen weights act, at strength $1/s$. Poisson summation over $w \in W$, where $k = w' + w$ and the quadratic form of this summation is positive definite because ∂ is injective, gives a prefactor that cancels in the ratio times a factor in $[0,2]$ tending to 1. A w' with $\mathbf{1}_{F^c}\partial w' = 0$ is constant on components and vanishes on that of the ground, which proves the positive definiteness stated in the lemma. Dominated convergence then yields (2.7) (see Section S2.1 for the details of this proof).

2.5 One box with insulating walls

For a box Λ', $D \subseteq \Lambda'_2 \setminus \partial_2\Lambda'$ and $\tau > 0$, $g^{\mathrm{fix}}_{\Lambda',\tau}(C;D) := g_{\Lambda',\tau}(C; D \cup \partial_2\Lambda')$ is the Villain expectation on Λ' with the plaquettes of D and the boundary plaquettes frozen. We write $C \subset \Lambda' \setminus \partial\Lambda'$ if all edges of C lie in Λ' and none lies in the boundary of Λ'.

Proposition 2.9. *Let C be an integer 1-cycle, Λ' a box with $C \subset \Lambda' \setminus \partial\Lambda'$, $K, \beta > 0$, $\tau = K\beta$, δ as in Section 2.3 and $\delta' := \min(\delta, 1)$. For every finite box $\Lambda \supseteq C$, $\langle \cos\theta_C \rangle^W_{\Lambda,\beta} \le \mathbb{E}_{D \sim \mathrm{Ber}(\delta')^{\otimes(\Lambda'_2 \setminus \partial_2\Lambda')}} g^{\mathrm{fix}}_{\Lambda',\tau}(C; D)$.*

Proof. We use volume monotonicity and then freeze everything outside Λ'. Let $\Lambda'' \supseteq \Lambda \cup \Lambda'$ be a box and $F_{\mathrm{out}} := \Lambda''_2 \setminus (\Lambda'_2 \setminus \partial_2\Lambda')$. Then Theorem 2.1(b), Theorem 2.5 on $X = \Lambda''$ with $\varepsilon = 1/(K\beta)$, and Lemma 2.3(d) give $\langle \cos\theta_C \rangle^W_{\Lambda,\beta} \le \mathbb{E}_{D''} g_{\Lambda'',\tau}(C; D \cup F_{\mathrm{out}})$, where $D'' \sim \mathrm{Ber}(\delta')^{\otimes \Lambda''_2}$ and $D := D'' \cap (\Lambda'_2 \setminus \partial_2\Lambda')$. By Lemma 2.8 on Λ'' and on Λ' (n_0 in Λ'), the right side is $g^{\mathrm{fix}}_{\Lambda',\tau}(C; D)$, as the unfrozen plaquettes are the same, the cubes outside Λ' are isolated, and the ground is not reached.

Slit graphs. Let Λ' be a box, $\mathfrak{F} \subseteq \Lambda'_2$ with $\partial_2\Lambda' \subseteq \mathfrak{F}$, and $\mathbb{X} \subseteq \Lambda'_3$ a set of cubes all of whose faces lie in $\mathfrak{F}$. The *slit graph* $G' = (V, E)$ has $V := \Lambda'_3 \setminus \mathbb{X}$ and $E := E(V) \setminus S$, with $E(V)$ the nearest-neighbour pairs in V and S those whose common plaquette lies in $\mathfrak{F}$. Conversely, each triple $(\Lambda', \mathbb{X}, S)$, S a set of bonds with both ends in $\Lambda'_3 \setminus \mathbb{X}$, arises in this way, with $\mathfrak{F} := F(\mathbb{X}) \cup F_S \cup \partial_2\Lambda'$, where $F(\mathbb{X})$ is the set of faces of the cubes of $\mathbb{X}$ and F_S is the set of plaquettes dual to S. The phrase "every slit graph" means every such triple. As E omits the bonds of S, a slit graph is in general not an induced subgraph of $\mathbb{Z}^3$. Its components are denoted $\mathfrak{K}$ (an isolated vertex is a component), its vertices have degree at most 6, and $d_{G'}(x, y) \ge \|x - y\|_1$, with $d_{G'} = \infty$ across components. We orient a bond $b = (x, y) \in E$ and its plaquette p_b compatibly, with p_b oriented as in ∂x. For an integer 2-chain Ξ of Λ' with $\partial\Xi = C$ let $\xi_b \in \mathbb{Z}$ be the coefficient of p_b in Ξ, which for the flat rectangle spanned by C is the signed crossing number, in $\{0, \pm 1\}$, and for $k \in \mathbb{Z}$ put

$$Z_{G'}(k, \Xi) := \sum_{[m]} \exp\Big(-\frac{1}{2\tau} \sum_{b=(x,y)\in E} \big(m(y) - m(x) - k\xi_b\big)^2\Big), \qquad W_{G'}(C) := \frac{Z_{G'}(1, \Xi)}{Z_{G'}(0, \Xi)}, \quad (2.8)$$

where $[m]$ runs over the classes of $m \in \mathbb{Z}^V$ modulo functions constant on each component.

Coarse and fine. In Sections 3 to 5 the hole set $\mathbb{X}$ is a set of coarse sites and the frozen set S is a set of coarse bonds, as in Definition 3.7, with the coarse slit graph $(\Lambda^\circ \setminus \mathbb{X}, E(\Lambda^\circ \setminus \mathbb{X}) \setminus S)$. In view of Definition 3.4, a coarse face carries between r_0^2 and $\bar{r}_0^2$ fine plaquettes. On the fine lattice the hole cubes are the fine cubes in sites of $\mathbb{X}$, $F(\mathbb{X})$ is the set of their faces (Definition 3.6), F_S is the set of fine plaquettes in the coarse faces dual to the bonds of S, and

$$\mathfrak{F} := F(\mathbb{X}) \cup F_S \cup \partial_2\Lambda^\circ. \quad (2.9)$$

For the output $(\mathbb{X}, S) = (\mathbb{X}(D), S(D))$ of Theorem 3.20 the fine slit graph of this $\mathfrak{F}$ is the *output* $G' = G'(D)$. Its vertices are the fine cubes of Λ° outside the holes, and a bond at a vertex is missing from E exactly when it leads into a hole, is frozen (dual to F_S), or leaves Λ°.

Proposition 2.10 (Frozen plaquettes as insulating walls)**.** *Let Λ', $\mathfrak{F}$, $\mathbb{X}$ and G' be as above, $\tau > 0$, and C an integer 1-cycle in Λ'. For every $D \subseteq \mathfrak{F} \setminus \partial_2\Lambda'$, $g^{\mathrm{fix}}_{\Lambda',\tau}(C; D) \le g^{\mathrm{fix}}_{\Lambda',\tau}(C; \mathfrak{F} \setminus \partial_2\Lambda') = W_{G'}(C)$, and $W_{G'}(C)$ depends neither on the choice of Ξ nor on the choice of $\mathbb{X}$.*

Proof. By Lemma 2.3(d), freezing raises the loop, and the inequality follows. Since freezing also deletes a dual bond, holes, sheets, cuts and box walls are admissible. For the identity we

apply Lemma 2.8 with $F = \mathfrak{F}$, $n_0 = \Xi$. Here no component reaches the ground, cubes with all faces in $\mathfrak{F}$ (those of $\mathbb{X}$ among them) are isolated and drop out, and $(\Xi + \partial k)_{p_b} = \xi_b + k(x) - k(y)$ gives $Z_{G'}(1, \Xi)/Z_{G'}(0, \Xi)$ with $m = k|_V$. A change $\Xi' = \Xi + \partial U$ is absorbed by $m \mapsto m - U|_V$. On outputs, $D \subseteq \mathfrak{F}(D)$ for all D by Theorem 3.20, and hence the inequality applies to every D.

Definition 2.11 (Min-cut area)**.** For $\mathfrak{F}$ as in (2.9), the *min-cut area* of C in G' is $A_{G'}(C) := \min\{\#(\operatorname{supp}\Sigma \setminus \mathfrak{F}) : \Sigma \text{ an integer 2-chain of } \Lambda^\circ \text{ with } \partial\Sigma = C\}$.

3 The regularization: construction and main properties

In this section we define a deterministic map $D \mapsto G'(D)$ from sets D of frozen plaquettes to slit domains and state its main properties, Theorems 3.20 (terminal state) and 3.21 (history), which we prove in Section 4; the details of the proof of Lemma 3.9, including the lattice of minimizers, are in Section S4.1. Throughout Sections 3 to 5 we fix $K > K_*$, $\beta \geq \beta_0(K)$, $\tau = K\beta$ and the constants fixed before K, for which we refer to Section 8 and, for their values, to Section S1. The loop $C = \partial\Xi$ bounds the rectangle $\Xi := [r_1, r_1 + R] \times [t_1, t_1 + T] \times \{z_0\}$ ($r_1, t_1, z_0 \in \mathbb{Z}$, fine units), with $\min(R, T) \geq R_1$. We treat other planes by permuting the axes. All statements hold for arbitrary $D \subseteq \Lambda_2^\circ \setminus \partial_2\Lambda^\circ$. Units are coarse (one site is one coarse cell of the grid) and distances are coarse ℓ^∞ distances unless "fine" or "Euclidean" is written. We write $Q_\infty(u, r) := \{x : |x - u|_\infty \leq r\}$.

Before the formal definitions we outline the design of the construction. The Phase-1 hole set is the smallest minimizer of a cut functional, given in Definition 3.11, and in Section 4.3 we pay its cost by comparison with competitors that replace a region, at a bounded cost per mark. Smallest minimizers are canonical, so the definition of Phase 1 involves no tie-breaking rule. Fills add whole blocks and are triggered by a count of marks. Penalizing defective interfaces or weak links instead fails by cascades, since freezing the bonds of one defect can create defects of the same kind in other blocks, cells being defined through windows, whereas filling blocks changes no bond inside other blocks. We enforce (NF_*) by a rule of Phase 2, the cut rule, and not by a term of the functional, since the deletion competitor used in that comparison can create a fragmented split. The local seal of Definition 3.16 fills the thin pieces of blocks that minimality does not exclude.

3.1 Scales, the box, the grid

Definition 3.1 (Scales and units)**.** Let $\tilde{m} \in [m_D/2, 2m_D]$ be the screening mass of Section 6 (fine units). The scales are: the fine side $r_0 \geq 1$ of a coarse cell, which is at most $\bar{r}_0 := 2r_0 - 1$; the block scale L_c, the largest even integer $\leq c_L \tilde{m}^{-1}/r_0$ (coarse), and $L := r_0 L_c$ (fine); $\rho_\lambda := L_c/12$ and the hull penalty $\lambda := 1/\rho_\lambda = 12/L_c$; $V_* := (1 - \eta_w)^3 \rho_\lambda^3/2$ with $\eta_w := 1/10$; the window bound $k_P := 304 L_c$; the removal scale $\ell := \ell_D/r_0$ with $\ell_D := L\lceil \tilde{m}^{-1}/L \rceil$; and $R_1 := (2r_\square + 5)L$ (fine) with $r_\square := (k_P + 3)/L_c \leq 305$.

The lower bounds on τ used here are $L_c \geq 4$ and $\tilde{m}L \geq c_L/2$ (Section S1.2). Since $c_L \leq 1/308$, we have

$$\ell \geq k_P + 4L_c = 308\, L_c, \qquad \tilde{m}L \leq c_L \leq \tfrac{1}{2}. \tag{3.1}$$

Definition 3.2 (The box Λ°)**.** Put $m := \max(R,T)$, $M_\Lambda := \max\big(3m,\ r_0(\ell+3)\big)$ (fine), M'_Λ the even integer in $\{M_\Lambda, M_\Lambda+1\}$, and $o = (o_1,o_2,o_3) := (r_1 + R/2,\ t_1 + T/2,\ z_0)$, the centre of Ξ. Then $\Lambda^\circ := \prod_{\mu=1}^3 [a_\mu, b_\mu]$ with $a_\mu := \lfloor o_\mu \rfloor - M'_\Lambda/2$ and $b_\mu := a_\mu + M'_\Lambda$, a cube whose faces are lattice planes; Λ°_2 is its set of plaquettes and $\partial_2\Lambda^\circ$ the set of those contained in $\partial\Lambda^\circ$.

Lemma 3.3. *(a) Λ° and its grid $\mathfrak{G}(C)$ depend only on C and the constants. (b) Consecutive special planes are at fine distance $\geq L$. (c) Every coarse side of Λ° is $\geq \ell \geq k_P + 4L_c$, and Λ° is a union of blocks and of removal cells. (d) The coarse sides n_μ of Λ° satisfy $\max_\mu n_\mu \leq 2\min_\mu n_\mu$. (e) Λ° contains the fine ℓ^∞-neighbourhood of radius m of Ξ, so $\operatorname{dist}(C, \partial\Lambda^\circ) \geq \max(R,T)$, $C \subset \Lambda^\circ \setminus \partial\Lambda^\circ$, and the fine side of Λ° is $\geq 3\max(R,T)$.*

Proof. Parts (a) to (d) are proved in Section S2.2, and for (e) we use $M'_\Lambda \geq 3m$ and $o_\mu - \frac{1}{2} \leq \lfloor o_\mu \rfloor \leq o_\mu$. These inequalities give $r_1 - a_1 \geq (M'_\Lambda - R)/2 \geq m$, and the integer $b_1 - (r_1 + R)$ is at least $o_1 - \frac{1}{2} + M'_\Lambda/2 - r_1 - R = (M'_\Lambda - R - 1)/2 \geq m - \frac{1}{2}$, hence at least m. Direction 2 is the same with T, and $z_0 - a_3 = b_3 - z_0 = M'_\Lambda/2 \geq m$.

Definition 3.4 (The loop-adapted grid $\mathfrak{G}(C)$)**.** The grid $\mathfrak{G}(C)$, constructed in Section S2.2, is a function of C and the constants. Its *special planes* are the planes of C and the faces of Λ°. Its *sites* are boxes of fine sides in $[r_0, \bar{r}_0]$ forming, combinatorially, a box of $\mathbb{Z}^3$, with nearest-neighbour bonds E. The coarse face dual to a bond is the common face of its two sites. *Blocks* are the boxes between consecutive *block planes* of the three directions. A *block face* F' is the common face of two adjacent blocks, and $|F'|$ is the number of bonds across it. The *block distance* of two blocks is the ℓ^∞ distance of their indices. For a *removal cell* Δ, $|\partial\Delta|$ is the number of bonds of $\mathbb{Z}^3$ with exactly one end in Δ, bonds leaving Λ° included. The *window* $\hat{Q}(Q)$ of a block Q is the union of the blocks at block distance at most 75 from Q, and $\mathcal{Q}$ is the set of windows. For a block face F' between blocks Q, Q', in the plane P', the *tube zone* $Z(F')$ is the set of sites of $Q \cup Q'$ in the $L_c/2$ layers nearest to P' on each side, and the *tube* τ_b of a bond b across F' is the segment of its line formed by its L_c sites in $Z(F')$. The grid has the following properties. (G1) Consecutive parallel block planes are at coarse distance in $[L_c, 2L_c - 1]$. Blocks have coarse sides in $[L_c, 2L_c - 1]$, fine sides in $[L, 2L - 1]$, volume in $[L_c^3, 8L_c^3)$, and $|F'| \in [L_c^2, (2L_c - 1)^2]$. (G2) Every block is a box with six faces, each a block face or contained in $\partial\Lambda^\circ$. A line in direction μ crosses only block planes perpendicular to e_μ. Every special plane is a block plane, and Λ° and every removal cell are unions of blocks. (G3) Removal cells tile Λ°, have coarse sides in $[\ell, 2\ell + 2L_c - 3] \subseteq [\ell, 3\ell)$ and fine sides at most $3\ell_D$ (a block of coarse side $a \geq L_c \geq 4$ has fine side at most $r_0 a + r_0 - 1$, and $\ell \geq 308L_c$; Section S2.2), and $|\partial\Delta| \in [6\ell^2, 54\ell^2)$. (G4) Every window $\hat{Q}(Q)$ is a box of coarse sides less than k_P that contains $Q_\infty(u, 75L_c) \cap \Lambda^\circ$ for every site $u \in Q$, and $Q \cup Q' \subseteq \hat{Q}(Q)$ for every block Q' adjacent to Q. Every site lies in at most 151^3 windows. (G5) $Z(F') \subseteq Q \cup Q'$, tube zones of distinct parallel block faces are disjoint, and a tube crosses no block plane other than P'.

We prove (G1) to (G5) in Section S2.2 and list the sizes in Table S1.2.

Lemma 3.5 (Alignment)**.** *By (G2) the planes of C are block planes; so, with $P := \{x_3 = z_0\}$: (A1) every plaquette of Ξ lies in the block plane P, and no bond with both ends in one block crosses Ξ; (A2) every block face lies in Ξ or contains no plaquette of Ξ; (A3) every block lies on one side of P, and C lies on lines of block planes. Every plaquette of Ξ is interior to Λ°.*

Proof. By the construction of $\mathfrak{G}(C)$ (Section S2.2, Steps 1–3) the special planes are block planes, and in each direction the block intervals subdivide each special interval. (A1) A bond crossing Ξ joins two fine cubes on opposite sides of the block plane P, which lie in different blocks. (A2) A block face in a plane $x_\mu = \text{const}$ with $\mu \neq 3$, or in a plane $x_3 \neq z_0$, contains no plaquette of Ξ. A block face in P is $I^1 \times I^2 \times \{z_0\}$ with block intervals I^1, I^2; since r_1, $r_1 + R$, t_1, $t_1 + T$ are block planes, either $I^1 \times I^2 \subseteq [r_1, r_1 + R] \times [t_1, t_1 + T]$, and the face lies in Ξ, or the interiors are disjoint. (A3) A block lies between consecutive block planes and P is one; each side of C lies on the intersection of P with one of the block planes $x_1 \in \{r_1, r_1 + R\}$, $x_2 \in \{t_1, t_1 + T\}$. The last claim is Lemma 3.3(e).

Definition 3.6 (Marks)**.** For a set Y of sites, $F(Y)$ is the set of fine plaquettes that are faces of fine cubes contained in sites of Y. A site is *marked* iff one of its fine cubes has a face in D. The set of marked sites is $\mathbb{M} = \mathbb{M}(D)$, and $N(P) := |P|$ for $P \subseteq \mathbb{M}$. Then $D \subseteq F(\mathbb{M})$, and a plaquette of D marks at most two sites, since it is a face of at most two fine cubes.

3.2 States, functionals and steps

For a set Y of sites, we write $E(Y)$ for the set of bonds with both ends in Y, ∂Y for the set of bonds of $E = E(\Lambda^\circ)$ with exactly one end in Y, and $V(A)$ for the set of endpoints of a bond set A. A bond *touches* Y if it has an end in Y. Bonds leaving Λ° are not in E, that is, the faces of the box are free.

Definition 3.7 (States)**.** A *state* is a triple $\sigma = (\mathbb{X}, \mathbb{U}, S)$: the *holes* $\mathbb{X} \subseteq \Lambda^\circ$, the *filled blocks* $\mathbb{U} \subseteq \mathbb{X}$, a union of blocks, and the *frozen bonds* $S \subseteq E(\Lambda^\circ \setminus \mathbb{X})$, each crossing a block face and each of one *kind*: $S = S_1 \sqcup S_{\text{cut}}$, seal bonds and cut bonds (Definitions 3.16 and 3.18). A block is *surviving* if it is not contained in $\mathbb{U}$. A frozen bond that $\mathbb{X}$ grows to touch leaves S. Such a bond is *converted*, and from then on it is an ordinary hole face. The *slit graph* is $G'(\sigma) := \big(\Lambda^\circ \setminus \mathbb{X},\ E(\Lambda^\circ \setminus \mathbb{X}) \setminus S\big)$, and $\mathcal{R}(\sigma) := S \cup \{b \in E : b \text{ touches } \mathbb{X}\}$ is the set of bonds missing from it. Along a history, that is, a finite sequence of states produced by the steps below (Definition 4.1), the *base* is $B_t := \mathbb{M} \cup \mathbb{U}_t$, and the *ever-frozen set* $\bar{S}_t$ is the set of bonds frozen at some time $\le t$ that do not touch B_t, converted bonds included, each keeping its kind.

Definition 3.8 (Hull functional)**.** Let $S \subseteq E$ carry weights $w_b \in [0, 2]$. For $X' \subseteq \Lambda^\circ$,

$$F^S_\lambda(X') := |\partial X'| + \sum_{b \in S \cap E(\Lambda^\circ \setminus X')} w_b + \lambda |X'|. \tag{3.2}$$

Every frozen bond has the *hull weight* $w_b := 2 - \eta_w = \frac{19}{10}$.

Lemma 3.9. *Let $S \subseteq E$ carry weights $w_b \in [0, 2]$. (a) F^S_λ is the sum of the cut function of a graph on Λ° with nonnegative capacities and of a modular function; in particular it is submodular. (b) For every $Y \subseteq \Lambda^\circ$, the minimizers of F^S_λ over $\{X' : Y \subseteq X' \subseteq \Lambda^\circ\}$ are closed under union and intersection; the smallest, $H^S_\lambda(Y)$, is the source side of a minimum s–t cut in a network with nonnegative capacities. Bonds of S that touch Y may be added to S or removed from it, with any weights, without changing $H^S_\lambda(Y)$. (c) H^S_λ is extensive and idempotent: $Y \subseteq H^S_\lambda(Y) = H^S_\lambda(H^S_\lambda(Y))$. (d) Let S, S' carry weights that agree on $S \cap S'$. If $Y \subseteq Y'$ and $\mathcal{R}(Y, S) \subseteq \mathcal{R}(Y', S')$, where $\mathcal{R}(Y, S) := S \cup \{b : b \text{ touches } Y\}$, then $H^S_\lambda(Y) \subseteq H^{S'}_\lambda(Y')$. In particular H^S_λ is a closure operator for fixed S and monotone in S.*

Proof. For (a), we write a frozen bond of weight $w \le 2$ as a modular term plus a cut term of capacity $\frac{2-w}{2} \ge 0$. Indeed, with $c_b(X') := \mathbf{1}[\text{exactly one end of } b = \{u, v\} \text{ in } X']$, the three cases give $w_b \mathbf{1}[u, v \notin X'] + c_b(X') = w_b - \frac{w_b}{2}(\mathbf{1}[u \in X'] + \mathbf{1}[v \in X']) + \frac{2-w_b}{2} c_b(X')$ for $b \in S$, hence

$$F_\lambda^S(X') = \sum_{b \in E} \kappa_b \, c_b(X') + \sum_{x \in X'} \mu(x) + \sum_{b \in S} w_b, \tag{3.3}$$

with $\kappa_b := 1$ for $b \notin S$, $\kappa_b := (2 - w_b)/2$ for $b \in S$, and $\mu(x) := \lambda - \frac{1}{2} \sum_{b \in S,\, b \ni x} w_b$. This proves (a) and explains why frozen bonds weigh at most 2: weights above 2 would give negative capacities. Parts (b) and (c) are the lattice and closure properties of the minimizers of a submodular function on $\{X' \supseteq Y\}$. The network has capacities κ_b on the bonds, $\mu(x)^{\pm}$ to the sink or from the source, and ∞ from the source to Y. For the last assertion of (b) we observe that, on $\{X' \supseteq Y\}$, a bond of S touching Y contributes 1 or 0 whatever its weight.

(d) By (b) we may add the bonds of $S \setminus S'$, which touch Y', to S' with their S-weights, so that $S \subseteq S'$. Then $g := F_\lambda^{S'} - F_\lambda^S = \sum_{b \in S' \setminus S} w_b \mathbf{1}[\text{no end of } b \text{ in } X']$ is non-increasing. With $M := H_\lambda^S(Y)$ and $M' := H_\lambda^{S'}(Y')$, submodularity gives $F_\lambda^S(M \cap M') + F_\lambda^{S'}(M \cup M') \le F_\lambda^S(M) + F_\lambda^S(M') + g(M \cup M') \le F_\lambda^S(M) + F_\lambda^{S'}(M')$. Each term on the left is at least the corresponding one on the right. Hence $M \cap M' \supseteq Y$ is a minimizer, and $M \subseteq M'$.

Definition 3.10 (Hull step)**.** In a history, the *hull step* at time t replaces the hole set by $H_\lambda^{\bar{S}_t}(B_t)$, the hull from the base computed with the ever-frozen set and the hull weight, and converts the frozen bonds that touch it. A time t (or its state) is *hull-closed* if $\mathbb{X}_t = H_\lambda^{\bar{S}_t}(B_t)$.

Definition 3.11 (Forced blocks and the regularization functional)**.** For a block Q let $3Q$ be the union of the at most 27 blocks at block distance at most 1 from Q, and $N(3Q) := |\mathbb{M} \cap 3Q|$. The block Q is *forced* if $N(3Q) \ge \vartheta L_c^2$, where $\vartheta := 3 \cdot 10^{-5}$ is the *forcing threshold*, and $\mathfrak{B}_F = \mathfrak{B}_F(D)$ is the set of forced blocks. Put $\lambda_U := 2c_s/\ell$ and $\mu_{\mathrm{B}} := 25$. For a set $\bar{S}$ of bonds that cross block faces, each with the hull weight, and for pairs $(\mathbb{X}, \mathbb{U})$ with $\mathbb{M} \subseteq \mathbb{X} \subseteq \Lambda^\circ$ and $\mathbb{U} \subseteq \mathbb{X}$ a union of blocks,

$$\mathcal{E}^{\bar{S}}(\mathbb{X}, \mathbb{U}) := F_\lambda^{\bar{S}}(\mathbb{X}) - (\lambda - \lambda_U)|\mathbb{U}| + \mu_{\mathrm{B}} L_c^2 \, \#\{Q \in \mathfrak{B}_F : \; Q \not\subseteq \mathbb{U}\}. \tag{3.4}$$

Filling a block Q costs at most $|\partial Q| + \lambda_U |Q| < 24.6 L_c^2 < \mu_{\mathrm{B}} L_c^2$, filling the forced blocks costs at most $27 \cdot 24.6/\vartheta$ per mark, and filling a removal cell Δ costs at most w_* (we give the arithmetic behind these bounds in Section S1.2). In Lemma 4.9 we deduce from the first bound that forced blocks are filled. The last bound implies that a minimizer fills each cell whose insulation exceeds w_*, as Theorem 4.17 shows.

Lemma 3.12. *Let $\bar{S}$ be as in Definition 3.11. (a) $\mathcal{E}^{\bar{S}}$ is, up to a constant, the cut function of an s–t network with nonnegative capacities; its minimizers are closed under componentwise union and intersection, and the smallest one, $(\mathbb{X}^*, \mathbb{U}^*)$, is the source side of a minimum cut, a deterministic function of $\mathbb{M}$ and $\bar{S}$. (b) $\mathbb{X}^* = H_\lambda^{\bar{S}}(\mathbb{M} \cup \mathbb{U}^*)$. (c) The smallest minimizer is componentwise nondecreasing in $\bar{S}$.*

Proof. (a) We add to the network of Lemma 3.9(b) for $F_\lambda^{\bar{S}}$ (with ∞ from the source to the marks) a node u_Q per block Q, with capacity ∞ from u_Q to each site of Q and $(\lambda - \lambda_U)|Q| + \mu_{\mathrm{B}} L_c^2 \mathbf{1}[Q \in \mathfrak{B}_F]$ from the source to u_Q. Putting u_Q on the source side means $Q \subseteq \mathbb{U}$. A finite cut has $\mathbb{M} \cup \mathbb{U} \subseteq \mathbb{X}$, and since the blocks tile Λ° by (G2), its capacity is $\mathcal{E}^{\bar{S}}(\mathbb{X}, \mathbb{U}) + (\lambda -$

$\lambda_U)|\Lambda^\circ|$ up to the constant of (3.3). The capacities are nonnegative, as $\lambda_U < \lambda$. Hence the minimum cuts form a lattice, and the smallest source side is the set reachable from the source in the residual network of a maximum flow. (b) For fixed $\mathbb{U}^*$, $\mathcal{E}^{\bar S}(\cdot, \mathbb{U}^*) - F_\lambda^{\bar S}$ is constant on $\{\mathbb{X} \supseteq \mathbb{M} \cup \mathbb{U}^*\}$, and a smaller minimizer there would give a smaller joint minimizer. (c) Let $\bar S \subseteq \bar S'$, and m, m' the smallest minimizers of $\mathcal{E} := \mathcal{E}^{\bar S}$ and $\mathcal{E}' := \mathcal{E}^{\bar S'} = \mathcal{E} + g$, $g(\mathbb{X}, \mathbb{U}) := \sum_{b \in \bar S' \setminus \bar S} w_b \mathbf{1}[\text{no end of } b \text{ in } \mathbb{X}]$, which does not increase when the pair grows. Minimality of m' for $\mathcal{E}'$ and $g(m \cup m') \le g(m')$ give $\mathcal{E}(m') \le \mathcal{E}(m \cup m')$. Combined with submodularity, this yields $\mathcal{E}(m \cap m') \le \mathcal{E}(m)$. Thus $m \cap m'$ is a minimizer of $\mathcal{E}$, and $m \subseteq m'$.

For a state $(\mathbb{X}, \mathbb{U}, S)$ and a bond set A, we define the *two-sided insulation* by $\mathcal{A}_2(A) := \#\{b \in A :\ b \text{ has exactly one end in } \mathbb{X}\} + 2\,\#(S \cap A)$, in which a frozen bond counts 2 whatever its weight.

Definition 3.13 (Removal)**.** Let σ be a hull-closed state with base B. For a removal cell $\Delta \not\subseteq B$ put $w(\Delta) := \mathcal{A}_2(E(\Delta))$, with $E(\Delta)$ the set of bonds with both ends in Δ and S the current frozen set. The *removal step* removes, all at once, every removal cell $\Delta \not\subseteq B$ with $w(\Delta) > w_* := (1 + c_s)|\partial\Delta|$, where $c_s \in [3.3 \cdot 10^{-7}, 10]$ is the removal slack: $\mathbb{X} := \mathbb{X} \cup \Delta$, $\mathbb{U} := \mathbb{U} \cup \Delta$, and the frozen bonds touching Δ are converted. A hull step follows every removal step.

Definition 3.14 (Cells, interfaces)**.** Let σ be a state, $G' = G'(\sigma)$, and for $\hat Q \in \mathcal{Q}$ let $G' \cap \hat Q$ be the subgraph of G' induced on $\hat Q \setminus \mathbb{X}$, with components $\hat{\mathfrak{K}}$. For a block $Q \not\subseteq \mathbb{X}$, the *cells* of Q are the sets $\Omega_{Q,\hat{\mathfrak{K}}} := Q \cap \hat{\mathfrak{K}}$ for the components $\hat{\mathfrak{K}}$ of $G' \cap \hat Q(Q)$ that meet Q. For a block face F' between blocks $Q, Q' \not\subseteq \mathbb{X}$, an *interface* I is a nonempty set consisting of all G'-bonds across F' that join a given cell of Q to a given cell of Q', and $F(I) := F'$. Frozen bonds and hole faces belong to no interface. A bond $b \in I$ is *good* if its tube τ_b is a G'-path (its sites lie in $\Lambda^\circ \setminus \mathbb{X}$ and its bonds are G'-bonds), and $\mathrm{good}(I)$ is the number of good bonds of I.

Definition 3.15 (Weak links)**.** Let $\hat Q \in \mathcal{Q}$, $\hat{\mathfrak{K}}$ a component of $G' \cap \hat Q$, and $f_{G'}(A, A')$ the number of G'-bonds between disjoint site sets A, A'. For $\chi' > 0$, $A \subseteq \hat{\mathfrak{K}}$ is a *χ'-weak link* of $\hat{\mathfrak{K}}$ if $0 < |A| \le |\hat{\mathfrak{K}}|/2$ and $f_{G'}(A, \hat{\mathfrak{K}} \setminus A) < \chi'|A|/k_P$. The *weak-link constants* are $\chi := 2.0 \cdot 10^{-8}$ and $\chi_{\mathrm{N3}} := \chi/3$.

Weak links are excluded by Theorem 4.16(c).

Definition 3.16 (Auxiliary material and the seal)**.** Let $\sigma = (\mathbb{X}, \mathbb{U}, S)$ be a hull-closed state with ever-frozen set $\bar S$, let j be a surviving block, and put $V_j := j \setminus \mathbb{X}$. The *auxiliary functional* F_j^{aux} is $F_\lambda^{\bar S \cup \partial j}$, every bond of ∂j counted as frozen with the hull weight, on the sets Y with $\mathbb{X} \subseteq Y \subseteq \mathbb{X} \cup V_j$. Its minimizers are closed under union and intersection (Lemma 3.9(a)). The smallest one is M_j, and $A_j := M_j \setminus \mathbb{X}$ is the *auxiliary material* of j. The *seal step* freezes every G'-bond that crosses a face between two surviving blocks and has an end in $\bigcup_j A_j$. These bonds form S_1 and carry the hull weight. A hull step follows every seal step.

The set M_j is obtained from one maximum flow inside j. By Lemma 4.13, a block with a thin piece, attached to the rest of the block by few bonds however large its contacts across the faces of j, has auxiliary material, and Lemma 4.14 shows that after the seal the next hull step fills the auxiliary material.

3.3 Phase 1 and Phase 2

Definition 3.17 (Phase 1 and the terminal properties)**.** *Phase 1* starts from the state $(\mathbb{M}, \emptyset, \emptyset)$ with $\bar{S} := \emptyset$ and repeats the following *outer step* until an outer step changes neither $\mathbb{U}$ nor $\bar{S}$, that is, until both loops are at rest: (1) $(\mathbb{X}, \mathbb{U}) :=$ the smallest minimizer of $\mathcal{E}^{\bar{S}}$ (Lemma 3.12); (2) with $\mathbb{U}$ fixed, as long as some surviving block has auxiliary material, perform a seal step, add its bonds to $\bar{S}$ (in Phase 1, $\bar{S}$ is not pruned: it keeps the seal bonds that touch the base, unlike $\bar{S}_t$ of Definition 3.7; the hulls are the same, as the two sets differ only by bonds touching B_t, but step (1) and Theorem 4.17 use this unpruned set), and perform the hull step $\mathbb{X} := H_\lambda^{\bar{S}}(\mathbb{M} \cup \mathbb{U})$. Its final state $(\mathbb{X}^1, \mathbb{U}^1, S^1)$, with $S^1 := \bar{S} \cap E(\Lambda^\circ \setminus \mathbb{X}^1)$, base $B^1 := \mathbb{M} \cup \mathbb{U}^1$ and ever-frozen set $\bar{S}^1 := \bar{S}$, is the *Phase-1 output*.

The *terminal properties* of a hull-closed state with base B, frozen set S and ever-frozen set $\bar{S}$ are: (S-hull) $\mathbb{X} = H_\lambda^{\bar{S}}(B) = H_\lambda^{S}(\mathbb{X})$; (S-rem) no removal cell $\Delta \not\subseteq B$ has $w(\Delta) > w_*$; (S-face) every interface I has $\mathrm{good}(I) \geq c_g|F(I)|$; (S-plug) no component of any $G' \cap \hat{Q}$, $\hat{Q} \in \mathcal{Q}$, has a χ_{N3}-weak link.

We show in Theorem 4.17 that Phase 1 terminates and that its output is the smallest minimizer of $\mathcal{E}^{\bar{S}^1}$, has no auxiliary material and no frozen bond, and has the four terminal properties. As $\bar{S}$ does not change in the last outer step, the output is the pair returned by step (1) of that outer step, and step (2) finds no auxiliary material.

Definition 3.18 (Scale cost, fragmented splits, the cut rule, Phase 2)**.** Let σ be a state with slit graph G' and cells as in Definition 3.14. Every site of $\Lambda^\circ \setminus \mathbb{X}$ lies in exactly one cell, and every cell in one component of G'. A *cell set* is a union U of cells. Its cut $\partial_{G'}U$ is the set of G'-bonds with exactly one end in U, and $|\partial_{G'}U|$ is its number of (coarse) bonds. U is *c-connected* if it is connected in the graph on U whose edges are the G'-bonds inside U and the pairs of sites in a common cell. The distance of two interfaces is the least Euclidean distance (fine units) between the coarse faces dual to their bonds. For a finite set $\mathcal{B}$ of interfaces and $s > 0$, $\varphi_s(\mathcal{B})$ is the number of classes of the equivalence relation generated by "distance $\leq s$". The *scale cost* is $J(U) := \sum_{j \geq 0:\ 2^j L \geq R_{\mathrm{NF}}} (\varphi_{2^j L}(\partial_{G'}U) - 1)$, where $R_{\mathrm{NF}} := L\, e^{c_3^{\mathrm{NF}} F_{\mathrm{unit}}}$ (fine), $F_{\mathrm{unit}} := \sigma_F L^2$ with σ_F the wall tension (7.15), and $\theta_{\mathrm{NF}}, c_3^{\mathrm{NF}} > 0$ are the cut-rule constants. A pair $(\mathfrak{K}, U)$ is *fragmented* if $\mathfrak{K}$ is a component of G', $U \subsetneq \mathfrak{K}$ is a nonempty c-connected cell set, $\mathfrak{K} \setminus U$ is c-connected, and $J(U) > \theta_{\mathrm{NF}} F_{\mathrm{unit}} |\partial_{G'}U| / L_c^2$.

The cut rule acts at a hull-closed state at which no removal cell is over threshold and no surviving block has auxiliary material: a firing freezes $\partial_{G'}U$ for the first fragmented pair $(\mathfrak{K}, U)$ in a fixed total order of the sets $U \subseteq \Lambda^\circ$ that depends only on C and the constants (for instance: order the coarse sites lexicographically by their coordinates, and let $U \prec U'$ iff the least site of $U \triangle U'$ lies in U; any such order works, since the proofs below use only that a fragmented pair is cut, never which one; see Section S4.1); these bonds form S_{cut}, have the hull weight and count 2 in $\mathcal{A}_2$ and in $w(\Delta)$; nothing is filled.

Phase 2 starts from the Phase-1 output, with base B^1 and ever-frozen set $\bar{S}^1$ minus the bonds touching B^1, the ever-frozen set of Definition 3.7 (the unpruned $\bar{S}^1$ is used only in Theorem 4.17 and Section 4.3), and repeats the following pass until a full pass changes nothing: (1) a hull step; (2) a removal step, then a hull step; (3) a seal step, then a hull step; (4) if (1) to (3) changed nothing, one firing of the cut rule, then a hull step. Its terminal state $(\mathbb{X}, \mathbb{U}, S)$ is the output of the construction.

The scale cost is well defined, since $\partial_{G'}U$ is a union of whole interfaces (Lemma 4.27(a)).

The trigger of the cut rule uses the coarse cut size, and the non-fragmentation condition that the rule enforces implies the same condition with the fine cut size. Since a coarse face carries at least r_0^2 fine plaquettes, we have $|\partial_{G'}U|_{\mathrm{fine}}/L^2 \geq |\partial_{G'}U|/L_c^2$, and hence a split fragmented with the fine cut size is fragmented.

Proposition 3.19. *Phase 2 terminates. Its terminal state has (S-hull), (S-rem), (S-face), (S-plug) on the global slit graph and (S-NF): for every component $\mathfrak{K}$ of the final G' and every nonempty c-connected cell set $U \subsetneq \mathfrak{K}$ with $\mathfrak{K} \setminus U$ c-connected,*

$$J(U) \leq \theta_{\mathrm{NF}}\, F_{\mathrm{unit}}\, |\partial_{G'}U|/L_c^2. \tag{NF$_*$}$$

If the Euclidean diameter of Λ° (fine units) is at most R_{NF}, then Phase 2 makes no firing and the output is the Phase-1 output.

3.4 The output and the regularization theorem

The output $(\mathbb{X}, \mathbb{U}, S)$, with base $B = \mathbb{M} \cup \mathbb{U}$ and $S = S_1 \sqcup S_{\mathrm{cut}} \subseteq E(\Lambda^\circ \setminus \mathbb{X})$, defines the slit domain $G' = G'(D) := G'(\mathbb{X}, S)$, the free set $\mathfrak{F}$ of (2.9) and the min-cut area $A_{G'}(C)$ of Definition 2.11. The fine slit graph is that of Section 2.5, and cells and interfaces are those of Definition 3.14. The *cell graph* of a component $\mathfrak{K}$ of G' has the cells in $\mathfrak{K}$ as vertices and the interfaces as edges. The weight of a cell Ω is $|\Omega|_{\mathrm{fine}}$, its number of fine cubes. The $\mathcal{U}$-cells are the parts of a partition of each cell graph given by Lemma 4.21 with $V = V_{\mathcal{U}} := \tilde{m}^{-3}$, one fixed choice being made for each G'. A component of weight $< V_{\mathcal{U}}$ is a single $\mathcal{U}$-cell.

Theorem 3.20 (Regularization theorem: terminal form)**.** *For every $D \subseteq \Lambda_2^\circ \setminus \partial_2\Lambda^\circ$:*

*(R1) (*Termination, determinism.*) Phase 1 and Phase 2 terminate; the output $(\mathbb{X}, S)$ is a deterministic function of D; and $\mathbb{M} \subseteq \mathbb{X}$, so that $D \subseteq F(\mathbb{M}) \subseteq F(\mathbb{X}) \subseteq \mathfrak{F}$.*

The output satisfies the hull identity and the terminal properties, read on its slit graph: (N1) (S-hull), (N2) (S-face), (N3) (S-plug) and (N4) (S-rem) of Definition 3.17, and (N5) (S-NF) of Proposition 3.19. Consequently the following hold, with constants that depend only on the constants fixed before K and on none of β, τ, K, C, D, except c_g' and c_u, which lie between two such constants.

*(R2) (*Geometry.*) For every $W \subseteq \Lambda^\circ \setminus \mathbb{X}$, $|\partial_{\Lambda^\circ}W| \leq a_\eta|\partial_{G'}W| + b_\eta\lambda|W|$ with $(a_\eta, b_\eta) = (19/9, 10/9)$, where $\partial_{\Lambda^\circ}W$ is the set of bonds of E with exactly one end in W and $\partial_{G'}W$ the set of G'-bonds with exactly one end in W; with $c_I := 2^{2/3}$, $|\partial_{G'}W| \geq (9c_I/38)|W|^{2/3}$ whenever $|W| \leq V_*$. No component of G' has at most V_* sites.*

*(R5) (*Cell volume.*) Every cell satisfies $c_1 L_c^3 \leq |\Omega| < 8L_c^3$, with $c_1 := \min\bigl(c_g/2,\ ((1 - \eta_w)c_I/24)^3\bigr)$.*

*(R6) (*Poincaré inequality on cells.*) For every cell Ω of a block Q there is a connected set $\mathfrak{K}^\flat(\Omega)$ with $\Omega \subseteq \mathfrak{K}^\flat(\Omega) \subseteq \hat{Q}(Q)$ such that every $A \subseteq \mathfrak{K}^\flat(\Omega)$ with $0 < |A| \leq |\mathfrak{K}^\flat(\Omega)|/2$ has $f_{G'}(A, \mathfrak{K}^\flat(\Omega) \setminus A) \geq \chi|A|/(3k_P)$ (expansion form), and, for every $f : \Lambda^\circ \setminus \mathbb{X} \to \mathbb{R}$,*

$$\sum_{x \in \Omega} |f(x) - \bar{f}_\Omega|^2 \leq \frac{108}{\chi^2}\, k_P^2 \sum_{\{x,y\} \in E_{G'}(\mathfrak{K}^\flat(\Omega))} (f(x) - f(y))^2,$$

where $\bar f_\Omega$ is the average of f over Ω and $E_{G'}(\cdot)$ the set of G'-bonds with both ends in the set; each site lies in $\mathfrak{K}^\flat(\Omega)$, and each G'-bond lies in $E_{G'}(\mathfrak{K}^\flat(\Omega))$, for at most $N_{\rm ov} \le 27(k_P/L_c)^3$ cells Ω (for sites: a site lies in at most 151^3 windows, and the sets $\mathfrak{K}^\flat(\Omega)$ of the cells of one block are disjoint).

*(R7) (*Cells and $\mathcal{U}$-cells.*) Every cell lies in one block and in one component of G'; every interface lies in one block face, and that face lies in Ξ or contains no plaquette of Ξ. Every $\mathcal{U}$-cell is a union of cells, connected in the cell graph, of weight in $[V_{\mathcal{U}}, (2+6/c_g)V_{\mathcal{U}})$, or a whole component of G' of weight in $(c_u V_{\mathcal{U}}, V_{\mathcal{U}})$, with $c_u := r_0^3 V_* \tilde m^3 = ((1-\eta_w)c_I \tilde m L/24)^3$; and every $\mathcal{U}$-cell has Euclidean diameter at most $C_{\mathcal{U}}\tilde m^{-1}$.*

*(R8) (*Interfaces.*) Every interface I has $\mathrm{good}(I) \ge c_g|F(I)| \ge c_g \max(|I|, L_c^2)$, good bonds on one line are at distance $\ge L_c$, and the tube of a good bond of I lies in $Q \cup Q' \subseteq \hat Q(Q) \cap \hat Q(Q')$, with one half in each cell of I. A block face carries at most $1/c_g$ interfaces, and a cell has at most $6/c_g$ neighbouring cells. In fine units, $\mathrm{good}_{\rm fine}(I) \ge c_g' \max(|I|_{\rm fine}, L^2)$ with $c_g' = c_g(1+1/L_c)^{-2}$.*

*(R9) (*Density bound.*) Every removal cell Δ satisfies $\mathcal{A}_2(E(\Delta)) \le w_*$ and, counting every bond of E with an end in Δ, $\mathcal{A}_2(E(\Delta) \cup (\partial\Delta \cap E)) \le w_* + 2|\partial\Delta| = (3+c_s)|\partial\Delta|$.*

*(R10) (*Non-fragmentation.*) (NF_*) holds in the split form (S-NF) of Proposition 3.19.*

Item (R1) and the terminal properties (N1) to (N5) come from the construction, and in Section 4 we derive them from Propositions 4.3(b), (d) and 3.19. The items (R2) and (R5) to (R10), also proved in Section 4, then follow from the terminal state alone, through (N1) to (N5), (R10) being (N5). About cells and interfaces the construction gives more than (R5) to (R8): by Theorem 4.16 there is one cell per surviving block, of volume above $(1-3\vartheta)L_c^3$, and one interface per block face.

Below we write $P^{+R} := \{x : \mathrm{dist}(x,P) \le R\}$ and $r' := 2R_* + 2$, where R_* is the clearing-out radius of Theorem 4.23. We let $\mathcal{P}$ be the partition of $\mathbb{M}$ into the components of the graph joining two marks at distance at most r'. The constants $C'_\star$, $C_{\rm free}$ are those of Proposition 4.24.

Theorem 3.21 (Regularization theorem: history form)**.** *(R3) (*Block-local hull.*) At every hull-closed time of the full history (Definition 4.1), in particular at the output, for every block q, $|(\mathbb{X} \setminus B) \cap q| < 2L_c m_q$ unless it vanishes, where m_q is the number of marks in q. So hull material lies only in blocks containing a mark: frozen bonds, filled blocks, removal cells and box faces support no hull material, alone or jointly.*

*(a) (*Area per part.*) The Phase-1 output $(\mathbb{X}^1, \mathbb{U}^1, S^1)$ has $S^1 = \emptyset$. For $P \in \mathcal{P}$ put $Y_P := \mathbb{X}^1 \cap P^{+R_*}$. Then $\mathbb{X}^1 = \bigsqcup_{P\in\mathcal{P}} Y_P$, no bond joins Y_P to $Y_{P'}$ for $P \ne P'$, every block of $\mathbb{U}^1$ lies in one Y_P, and*

$$|\partial Y_P| + \lambda|Y_P \setminus \mathbb{U}^1| + \lambda_U|Y_P \cap \mathbb{U}^1| \le C'_\star N(P), \qquad 2|\partial Y_P| + |Y_P|/\ell \le C_{\rm free} N(P).$$

*(b) (*Uniform reach.*) Every site of $\mathbb{X}^1$ lies within distance R_* of $\mathbb{M}$, uniformly in $|\mathbb{M}|$, Λ°, C and D; so $Y_P \subseteq P^{+R_*}$.*

The constants $C'_\star$, $C_{\rm free}$ and the ratio $R_*/(\ell\log\ell)$ are bounded independently of β, τ, K, C and D.

We prove (R3) in Corollary 4.7, (a) in Proposition 4.24 and (b) in Theorem 4.23.

4 The regularization: proofs

In this section we prove Theorems 3.20 and 3.21 and Proposition 3.19. Several steps below are abbreviated, for instance in the proofs of Proposition 4.3, of Theorem 4.5(iii), (iv) and of Lemma 4.19. We first fix the class of histories to which the arguments apply.

Definition 4.1 (Admissible history)**.** An *admissible history* is a finite sequence of states $\sigma_t = (\mathbb{X}_t, \mathbb{U}_t, S_t)$, $0 \le t \le T$, such that (H1) $\sigma_0 = (\mathbb{M}, \emptyset, \emptyset)$ and the first step is a hull step; (H2) every other step is a hull step or, at a hull-closed state, a *fill*, which adds a union $\mathbb{U}'$ of blocks to $\mathbb{U}$ and to $\mathbb{X}$ and converts the frozen bonds touching $\mathbb{U}'$, or a *freeze*, which adds to S a nonempty set of G'-bonds crossing block faces, each with its kind; (H3) every fill and every freeze is followed by a hull step. The *full history* is Phase 1 followed by Phase 2.

Phase 1 is admissible: its first outer step is a hull step, the fill of $\mathbb{U}^*$ and a hull step, every later outer step that changes $\mathbb{U}$ is a fill and a hull step (Lemmas 3.12(b), (c) and 4.2), and a seal step is a freeze. The full history is admissible as well, since its removal steps are fills and its seal steps and cut firings are freezes.

4.1 The hull

We first show in Lemma 4.2 that the hull steps of an admissible history only enlarge the hole set, and then derive the geometric consequences of the minimality of the hull.

Lemma 4.2. *Let (σ_t) satisfy (H1) to (H3) of Definition 4.1. At every hull-closed time t, $\mathbb{X}_t = H_\lambda^{\bar S_t}(B_t) = H_\lambda^{S_t}(\mathbb{X}_t)$ and $S_t \subseteq \bar S_t \subseteq E(\Lambda^\circ \setminus B_t)$, and every bond of $\bar S_t \setminus S_t$ touches $\mathbb{X}_t$. Every hull step $t \to t+1$ satisfies $\mathbb{X}_{t+1} = H_\lambda^{S_t}(\mathbb{X}_t) \supseteq \mathbb{X}_t$.*

Proof. We use that H is a closure operator, jointly monotone in the base and in the set of missing bonds, as stated in Lemma 3.9(d). We argue by induction over the steps. By (H1) to (H3), each time is 0, or hull-closed, or follows a fill or a freeze at a hull-closed time t and is followed by a hull step. A hull step at a hull-closed time changes nothing, and the first hull step produces $H_\lambda^{\emptyset}(B_0) \supseteq B_0 = \mathbb{X}_0$. A freeze at t of a set A of G'-bonds, which do not touch $\mathbb{X}_t \supseteq B_t$, keeps B and $\mathbb{X}$, while $\bar S_{t+1} = \bar S_t \cup A$. Thus $\mathbb{X}_t = H_\lambda^{\bar S_t}(B_t) \subseteq H_\lambda^{\bar S_{t+1}}(B_t) = \mathbb{X}_{t+2}$. A fill of $\mathbb{U}'$ gives $B_{t+1} = B_t \cup \mathbb{U}'$ and drops from $\bar S$ only bonds touching B_{t+1}. It follows that $\mathcal{R}(B_t, \bar S_t) \subseteq \mathcal{R}(B_{t+1}, \bar S_{t+1})$ and $\mathbb{X}_t \subseteq \mathbb{X}_{t+2}$. In this case, moreover, $\mathbb{U}' \subseteq B_{t+1} \subseteq \mathbb{X}_{t+2}$. In both cases $B_{t+1} \subseteq \mathbb{X}_{t+1} \subseteq \mathbb{X}_{t+2}$, and by monotonicity and idempotence $H_\lambda^{\bar S_{t+1}}(\mathbb{X}_{t+1}) = \mathbb{X}_{t+2}$.

We now pass from $\bar S$ to S. Since a bond of $S_{t'}$ does not touch $\mathbb{X}_{t'} \supseteq B_{t'}$, we have $S_{t'} \subseteq \bar S_{t'}$. A bond of $\bar S_{t'} \setminus S_{t'}$ was converted, so it touches an earlier hole set and hence $\mathbb{X}_{t'}$, and by Lemma 3.9(b) it may be dropped: $H_\lambda^{\bar S_{t'}}(\mathbb{X}_{t'}) = H_\lambda^{S_{t'}}(\mathbb{X}_{t'})$. With $t' = t+1$ this is the claim on hull steps. At a hull-closed time, with $B_t \subseteq \mathbb{X}_t = H_\lambda^{\bar S_t}(B_t)$ and idempotence, the same identity implies $\mathbb{X}_t = H_\lambda^{S_t}(\mathbb{X}_t)$. The inclusion $\bar S_t \subseteq E(\Lambda^\circ \setminus B_t)$ holds by definition.

Proof of (R1) and (N1) to (N5). These properties follow from Propositions 4.3(b), (d) and 3.19.

Proposition 4.3. *(a) Order states by $\sigma \le \sigma'$ iff $\mathbb{X} \subseteq \mathbb{X}'$, $\mathbb{U} \subseteq \mathbb{U}'$ and $\mathcal{R}(\sigma) \subseteq \mathcal{R}(\sigma')$. Under (H1) to (H3), every step that changes the state strictly increases it; in particular (H4)*

$\mathbb{X}_t \subseteq \mathbb{X}_{t+1}$ *and* $\mathcal{R}(\sigma_t) \subseteq \mathcal{R}(\sigma_{t+1})$ *for every* t*, so that* G' *only shrinks, and a bond is frozen at most once along an admissible history. (b) Phase 1 and Phase 2 terminate, and their outputs are deterministic functions of* D*. (c) The terminal state of Phase 2 satisfies (S-hull), (S-rem) and (S-NF), and no surviving block has auxiliary material there. (d)* $\mathbb{M} \subseteq B_t \subseteq \mathbb{X}_t$ *at every time; hence, for the output,* $D \subseteq F(\mathbb{M}) \subseteq F(\mathbb{X}) \subseteq \mathfrak{F}$*.*

Sketch. Hull steps do not shrink $\mathbb{X}$ by Lemma 4.2, a fill adds blocks to $\mathbb{U}$, a freeze adds G'-bonds to S, and converted bonds stay missing. This proves (a). Phase 1 terminates by Theorem 4.17, and Phase 2 terminates because each effective step strictly increases the state in a finite box. Smallest minimizers are canonical and the cut rule uses a fixed order, so the outputs are deterministic. Since a terminal pass of Phase 2 changes nothing, Lemmas 4.2 and 4.14(a) imply (c). Part (d) holds because the base stays in $\mathbb{X}$.

Below, we denote by $f(A, A')$ the number of bonds of E between disjoint site sets A, A', whereas $f_{G'}$ of Definition 3.15 counts G'-bonds. Similarly, $s(A, A')$ denotes the number of bonds of S between A and A', and $s(E(A))$ the number of those with both ends in A. We put $N := |\mathbb{M}|$. A state is *closed* if $\mathbb{X} = H^S_\lambda(\mathbb{X})$ for the hull weight. By Lemma 4.2, each hull-closed state of an admissible history is closed, and so is the output.

Proof of (R2). We apply Lemma 4.4 and Theorem 4.5(i), (ii) at the output, which is closed by (N1).

Lemma 4.4. *Let* $(\mathbb{X}, \mathbb{U}, S)$ *be closed,* $V := \Lambda^\circ \setminus \mathbb{X}$ *and* $W \subseteq V$*. Then*

$$f(W, \mathbb{X}) + \sum_{b \in S,\ |b \cap W| = 1} (w_b - 1) + \sum_{b \in S,\ b \subseteq W} w_b \ \le\ |\partial_{G'} W| + \lambda |W|. \tag{4.1}$$

Consequently, with $(a_\eta, b_\eta) = (1 + \frac{1}{1-\eta_w}, \frac{1}{1-\eta_w}) = (\frac{19}{9}, \frac{10}{9})$*,* $|\partial_{\Lambda^\circ} W| \le a_\eta |\partial_{G'} W| + b_\eta \lambda |W|$*, and* $N_2(W) := f(W, \mathbb{X}) + s(W, V \setminus W) + 2s(E(W)) \le (|\partial_{G'} W| + \lambda |W|)/(1 - \eta_w)$*.*

Proof. Filling W does not lower the functional: $F^S_\lambda(\mathbb{X} \cup W) \ge F^S_\lambda(\mathbb{X})$. Since bonds of S do not touch $\mathbb{X}$, passing from $\mathbb{X}$ to $\mathbb{X} \cup W$ changes a bond between W and $\mathbb{X}$ from 1 to 0, a G'-bond between W and $V \setminus W$ from 0 to 1, a bond of S between W and $V \setminus W$ from w_b to 1 and a bond of S inside W from w_b to 0, and adds $\lambda |W|$. This is (4.1). The hull weight has $w_b - 1 = 1 - \eta_w$ and $w_b \ge 2(1 - \eta_w)$, and the bound on $N_2(W)$ follows. For the remaining bound, note that $|\partial_{\Lambda^\circ} W| = f(W, \mathbb{X}) + s(W, V \setminus W) + |\partial_{G'} W| \le |\partial_{G'} W| + N_2(W)$.

Theorem 4.5 (Geometry of the slit domain)**.** *Let* $(\mathbb{X}, \mathbb{U}, S)$ *be closed and* $c_I := 2^{2/3}$*. For* $W \subseteq \Lambda^\circ \setminus \mathbb{X}$*: (i)* $|\partial_{G'} W| \ge (9c_I/38)|W|^{2/3}$ *whenever* $|W| \le V_* = (1 - \eta_w)^3 \rho_\lambda^3/2$*; (ii) no component of* G' *has at most* V_* *sites; (iii)* $|B_{G'}(x, r)| \ge \min\{(c_I r/(48 a_\eta))^3, V_*\}$ *for all* $x \in \Lambda^\circ \setminus \mathbb{X}$ *and integers* $r \ge 0$*; (iv)* $\lambda_1^D(\Omega) \ge (c_I^2/(48 a_\eta^2))|\Omega|^{-2/3}$ *for* $\Omega \subseteq \Lambda^\circ \setminus \mathbb{X}$ *with* $|\Omega| \le V_*$*.*

Proof. (i) In a box $\prod_{i=1}^3 \{1, \dots, n_i\}$, a line parallel to e_i that meets W carries a bond of the box with exactly one end in W unless its n_i sites lie in W, which happens for at most $|W|/n_i$ lines, and the lines meeting W number at least $3|W|^{2/3}$ by the discrete Loomis–Whitney inequality [LW49, Theorem 2] and the arithmetic-geometric mean inequality. In this box, such bonds therefore number at least $3|W|^{2/3} - |W| \sum_i n_i^{-1} \ge c_I |W|^{2/3}$ if $|W|^{1/3} \le 0.47 \min_i n_i$, since $1 - 2^{2/3}/3 = 0.4708\ldots$. Lemma 4.4 controls the boundary in the box up to $b_\eta \lambda |W|$,

which is small below $V_* = (c_I\rho_\lambda/(2b_\eta))^3$, an identity that follows from $(c_I/2)^3 = \frac{1}{2}$. Since $|W|^{1/3} \le V_*^{1/3} < \rho_\lambda = L_c/12$ and each side of Λ° is at least $\ell \ge 308L_c$ by Lemma 3.3(c), this and Lemma 4.4 give $c_I|W|^{2/3} \le |\partial_{\Lambda^\circ} W| \le a_\eta|\partial_{G'}W| + b_\eta\lambda|W|^{1/3}|W|^{2/3} \le a_\eta|\partial_{G'}W| + \frac{c_I}{2}|W|^{2/3}$. We conclude that $|\partial_{G'}W| \ge (c_I/(2a_\eta))|W|^{2/3} = (9c_I/38)|W|^{2/3}$.

(ii) A component $\mathfrak{K}$ of G' has $\partial_{G'}\mathfrak{K} = \emptyset$, so (i) excludes $0 < |\mathfrak{K}| \le V_*$.

(iii) (Sketch.) We put $v(r) := |B_{G'}(x, r)|$. Each G'-bond leaving $B_{G'}(x, r)$ ends at distance $r + 1$. Thus $|\partial_{G'}B_{G'}(x, r)| \le 6(v(r + 1) - v(r))$, and by (i), $v(r + 1) \ge v(r) + \alpha v(r)^{2/3}$ with $\alpha := c_I/(12a_\eta) \le \frac{1}{4}$ while $v(r) \le V_*$. Consequently, $v(r + 1)^{1/3} > v(r)^{1/3} + \alpha/4$, and $v(r)^{1/3} \ge 1 + \alpha r/4 \ge c_I r/(48a_\eta)$ until v exceeds V_*.

(iv) By (i), each nonempty $A \subseteq \Omega$ has $|\partial_{G'}A| \ge h|A|$ with $h := (9c_I/38)|\Omega|^{-1/3}$, and the Cheeger argument in a graph of maximal degree 6, based on the coarea formula for f^2 and the Cauchy–Schwarz inequality, yields $\lambda_1^D(\Omega) \ge h^2/12 \ge (c_I^2/(48a_\eta^2))|\Omega|^{-2/3}$.

Segment accounting. In Lemma 4.6 and Corollary 4.7 we assume that $B = \mathbb{M}' \cup U$ with $\mathbb{M}'$ a set of sites and U a union of blocks and that $S \subseteq E(\Lambda^\circ \setminus B)$ carries weights $w_b \in [0, 2]$, and we let $\mathbb{X} := H_\lambda^S(B)$ and $W \subseteq \mathbb{X} \setminus B$. By Lemma 4.2, this setting holds at a hull-closed time of an admissible history, with $\mathbb{M}' = \mathbb{M}$, $U = \mathbb{U}_t$ and $S = \bar{S}_t$ carrying the hull weight. A *segment* R of W is a maximal set of consecutive sites of W on an axis-parallel line, and $m(R) := |R|$. It has $m(R) - 1$ *interior bonds* and two *ends*, the positions just beyond its extremities. An end is an $\mathbb{X}$-*end* if it is a bond to $\mathbb{X} \setminus W$, a *frozen end* if it is a bond of S to $\Lambda^\circ \setminus \mathbb{X}$, an *open end* if it is a bond of $E \setminus S$ to $\Lambda^\circ \setminus \mathbb{X}$, and a *box end* if it is on $\partial\Lambda^\circ$. With $x(R)$ the number of $\mathbb{X}$-ends and $o(R)$ the number of open ends, we put $c_W(R) := x(R) + \sum_{\text{frozen ends } b}(w_b - 1) + \sum_{\text{interior } b \in S} w_b - o(R) - \lambda m(R)/3$. A *grid bond* has its ends in different blocks. Such a bond lies on a block plane. The *grid-type terms* of R are its $\mathbb{X}$-ends, frozen ends and interior frozen bonds that are grid bonds, and its box ends. The number $x_{\mathrm{ng}}(R)$ counts the $\mathbb{X}$-ends, frozen ends and interior frozen bonds that are not grid bonds. Similarly, S^{n} is the set of bonds of S that are not grid bonds, and $f_{\mathrm{ng}}(A, A')$ counts the bonds between A and A' that are not grid bonds. Frozen bonds cross block faces and are therefore grid bonds, so $S^{\mathrm{n}} = \emptyset$ at each hull-closed time of an admissible history.

Lemma 4.6. *Let $\lambda \ge 12/L_c$. Every segment R of W satisfies $c_W(R) \le 2x_{\mathrm{ng}}(R) - 2m(R)/L_c$.*

Proof. Insulation on block planes, which are at least L_c apart, contributes at most 2 per plane met, while the volume term is $\lambda m/3 \ge 4m/L_c$. Indeed, since $o = 2 - x - e_{\mathrm{fr}} - e_b$, where e_{fr} is the number of frozen ends and e_b that of box ends, we have

$$c_W(R) = 2x + \sum_{\text{frozen ends}} w_b + \sum_{\text{interior frozen}} w_b + e_b - 2 - \lambda m/3. \tag{4.2}$$

The terms that are not grid bonds contribute at most $2x_{\mathrm{ng}}(R)$, since $w_b \le 2$. Each grid-type term has coefficient at most 2 and sits at its own bond position on a block plane perpendicular to the line. The $m + 1$ bond positions of R lie in an interval of length m, which meets at most $m/L_c + 1$ block planes by (G1). Therefore $c_W(R) \le 2x_{\mathrm{ng}}(R) + 2(m/L_c + 1) - 2 - 4m/L_c$.

Corollary 4.7 (Block-local hull)**.** *For every union N of blocks, $W := (\mathbb{X} \setminus B) \cap N$ is empty or satisfies $|W| < \frac{L_c}{3}\big(f_{\mathrm{ng}}(W, B) + \sum_{v \in W} \deg_{S^{\mathrm{n}}}(v)\big)$, and every bond counted in $f_{\mathrm{ng}}(W, B)$ ends in $\mathbb{M}' \setminus U$. Consequently, for every block q, $|(\mathbb{X} \setminus B) \cap q| < (L_c/3)(6m_q + s_q)$ unless*

it vanishes, with $m_q := |\mathbb{M}' \cap q|$ and s_q the number of endpoints in q of bonds of S^{n}, and $\mathbb{X} \subseteq B \cup \bigcup\{q :\ q \cap (\mathbb{M}' \cup V(S^{\mathrm{n}})) \neq \emptyset\}$. In other words, grid bonds of S of any weight in $[0,2]$, filled blocks and box faces support no hull material, alone or jointly. At a hull-closed time of an admissible history, m_q is the number of marks in q and $S^{\mathrm{n}} = \emptyset$.

Proof. Let $W \subseteq \mathbb{X} \setminus B$ be nonempty. Then the set $\mathbb{X} \setminus W \supseteq B$ is strictly smaller than the smallest minimizer $\mathbb{X}$. Since bonds of S do not touch B and each site of W lies in three segments, passing from $\mathbb{X}$ to $\mathbb{X} \setminus W$ changes the contribution of an $\mathbb{X}$-end by $+1$, that of a frozen end by $w_b - 1$, that of an open end by -1 and that of an interior frozen bond by $+w_b$, and it changes the volume term by $-\lambda|W|$, so that

$$F^S_\lambda(\mathbb{X} \setminus W) - F^S_\lambda(\mathbb{X}) = \sum_R c_W(R) > 0, \tag{4.3}$$

over the segments of W in the three directions. For $W = (\mathbb{X} \setminus B) \cap N$, an $\mathbb{X}$-end that is not a grid bond has both ends in one block $q \subseteq N$, so it ends in $\mathbb{X} \cap N \setminus W \subseteq B$, but not in U, since otherwise $q \subseteq U$. It is not frozen either, and it is therefore counted in $f_{\mathrm{ng}}(W, B)$. A frozen end that is not a grid bond is counted once in the degree sum, and an interior bond of S^{n} is counted twice. Combining (4.3), Lemma 4.6 and $\sum_R m(R) = 3|W|$, we obtain $0 < \sum_R c_W(R) \leq 2\sum_R x_{\mathrm{ng}}(R) - 6|W|/L_c \leq 2(f_{\mathrm{ng}}(W, B) + \sum_{v \in W} \deg_{S^{\mathrm{n}}}(v)) - 6|W|/L_c$. For $N = q$, $f_{\mathrm{ng}}(W, B) \leq 6m_q$ and the degree sum is at most s_q, so the bound for a single block holds. At a hull-closed time, such as the output, the last assertion of Corollary 4.7 applies, and the bound for a single block then gives (R3).

For a hull-closed time t and a block face F' between surviving blocks Q, Q', in the plane P', we put $W_t := (\mathbb{X}_t \setminus B_t) \cap (Q \cup Q')$ and $\mathrm{supp}_t(F') := f_{\mathrm{ng}}(W_t, B_t)$. For a line $\ell \perp F'$, b_ℓ denotes its bond across F'. As $B_t \cap (Q \cup Q')$ consists of marks, $\mathrm{supp}_t(F') \leq 6(m_Q + m_{Q'})$.

Lemma 4.8. *At every hull-closed time t of an admissible history and for every such face F': (a) the lines $\ell \perp F'$ such that $b_\ell \notin \bar{S}_t$ and $\ell \cap Z(F')$ contains a site of W_t and no mark, number fewer than $2\,\mathrm{supp}_t(F')$ unless there is none; (b) without the condition on b_ℓ, these lines number fewer than $2\,\mathrm{supp}_t(F')/\eta_w$ unless there is none. A frozen bond lies on the tube of a line $\ell \perp F'$ only if it is b_ℓ, that is, frozen bonds block no tube of another line.*

Proof. A line counted in (a) or (b) carries a segment that is either long, and pays through Lemma 4.6, or short, and then has no grid-type term except b_ℓ and has deficit at least η_w. We put $d(R) := 2x_{\mathrm{ng}}(R) - c_W(R)$ for the segments of W_t. By Lemma 4.6, $d(R) \geq 2m(R)/L_c \geq 0$, and (4.3) implies $\sum_R d(R) < 2\sum_R x_{\mathrm{ng}}(R) \leq 2\,\mathrm{supp}_t(F')$, as in the proof of Corollary 4.7. For a counted line ℓ let R_ℓ be the segment of W_t along ℓ through a site of $\ell \cap Z(F') \cap W_t$. Distinct lines give distinct segments. If $m(R_\ell) \geq L_c/2$, then $d(R_\ell) \geq 1$. Otherwise all bond positions of R_ℓ are at distance $< L_c$ from P', so by (G1) the only possible grid-type term is b_ℓ, which is not an $\mathbb{X}$-end, since its other end would lie in $\mathbb{X}_t \cap (Q \cup Q') \setminus W_t \subseteq B_t$, a mark of $\ell \cap Z(F')$. Thus, in this case, $d(R_\ell) = 2 + \lambda m/3 - \beta_\ell$, with $\beta_\ell = 2 - \eta_w$ if $b_\ell \in \bar{S}_t$ and b_ℓ is a bond position of R_ℓ, and $\beta_\ell = 0$ otherwise. It follows that $d(R_\ell) \geq 1$ for each line counted in (a) and $d(R_\ell) \geq \eta_w$ for each line counted in (b). Summing these bounds, we obtain (a) and (b). For the last claim, recall that a frozen bond is a grid bond, and use (G5): a tube consists of b_ℓ and of bonds interior to Q or to Q', which are not grid bonds.

4.2 Cells, giants and the seal

In this subsection we control the cells by sparseness. Since forced blocks are filled (Lemma 4.9), a surviving block has fewer than ϑL_c^2 marks near it, its hole sites lie on few lines by Corollary 4.7, and one component of its free sites, the giant, contains all but a small volume. The seal excludes thin sets (Lemma 4.13 and Theorem 4.16), at a cost of at most 726 frozen bonds per mark (Proposition 4.15).

Lemma 4.9. *Let $\bar{S}$ be as in Definition 3.11. The smallest minimizer $(\mathbb{X}^*, \mathbb{U}^*)$ of $\mathcal{E}^{\bar{S}}$ has $\bigcup \mathfrak{B}_F \subseteq \mathbb{U}^*$. Hence at every hull-closed time of the full history other than the first every forced block lies in $\mathbb{U}$, and every surviving block q has $m_q \le N(3q) < \vartheta L_c^2$.*

Proof. Suppose that $Q \in \mathfrak{B}_F$ and $Q \not\subseteq \mathbb{U}^*$. Then $Q \cap \mathbb{U}^* = \emptyset$, and we compare with $(\mathbb{X}^* \cup Q, \mathbb{U}^* \cup Q)$. Only the at most $|\partial Q|$ bonds leaving Q gain weight, at most 1 each, the volume terms change by $\lambda_U|Q| - \lambda|Q \cap \mathbb{X}^*| \le \lambda_U|Q|$, and the penalty drops by $\mu_{\mathrm{B}} L_c^2 > |\partial Q| + \lambda_U|Q|$ (see Section S1.2). This is a contradiction. For the second assertion we use that $\mathbb{U}$ never decreases along the full history, by Proposition 4.3(a), and that the second hull-closed time follows the first outer step of Phase 1. If that step fills nothing, then $\mathfrak{B}_F = \emptyset$.

Lemma 4.10. *At every hull-closed time of an admissible history, every surviving block q satisfies: (a) $|\mathbb{X} \cap q| < 3L_c m_q$ unless $\mathbb{X} \cap q = \emptyset$; (b) unless there is none, fewer than $9m_q$ maximal runs of $\mathbb{X}$ in the three directions lie inside q and are joined by both end bonds to sites of $V_q := q \setminus \mathbb{X}$.*

Proof. (a) The set $\mathbb{X} \cap q$ consists of the m_q marks of q and, by Corollary 4.7, fewer than $2L_c m_q$ sites of hull material. (b) At most $3m_q$ runs contain a mark. A run without marks is a segment R of $W := (\mathbb{X} \setminus B) \cap q$ with two open ends inside q, since in-block bonds are not frozen, and with no grid-type term. For such a segment, $d(R) := 2x_{\mathrm{ng}}(R) - c_W(R) = 2 + \lambda m(R)/3$ by (4.2). As in Lemma 4.8, we have $d \ge 0$ for each segment of W and $\sum_R d(R) < 2f_{\mathrm{ng}}(W, B) \le 12m_q$. Therefore fewer than $6m_q$ runs have no mark, which proves (b).

Lemma 4.11. *(a) Let Q be a box with sides L_1, L_2, L_3, $Z \subseteq \mathsf{Q}$, $\mathsf{Q} \setminus Z = A \sqcup B$, and n_μ the number of lines of Q in direction μ meeting both A and B. Then $\min(|A|, |B|) \le 4|Z| + 6.4 \max_\mu(n_\mu L_\mu)$. (b) Let R be a rectangle of $\mathbb{Z}^2$ with sides $s_1, s_2 \ge s$, and $\sigma \subseteq \mathsf{R}$ with $|\sigma| \le s^2/4$. At least $2|\sigma|^{1/2}$ bonds of R have exactly one end in σ.*

Proof. (a) Let a_μ be the number of lines in direction μ meeting A but not B. Then $|A| \le L_\mu(a_\mu + n_\mu)$, and the lines meeting B are among the other $|\mathsf{Q}|/L_\mu - a_\mu$. By the discrete Loomis–Whitney inequality [LW49, Theorem 2], it follows that $|B|^2 \le \prod_\mu(|\mathsf{Q}|/L_\mu - a_\mu) \le \prod_\mu(|\mathsf{Q}| - |A| + n_\mu L_\mu)/L_\mu$. With $\alpha := |A|/|\mathsf{Q}|$, $\beta := 1-\alpha$, $\zeta := |Z|/|\mathsf{Q}|$ and $\nu := \max_\mu n_\mu L_\mu/|\mathsf{Q}|$ this reads $(\beta - \zeta)^2 \le (\beta + \nu)^3$. If $\nu > 0.1$, then $\min(|A|, |B|) \le |\mathsf{Q}|/2 < 6.4\nu|\mathsf{Q}|$. If $\nu \le 0.1$, then $(\beta+\nu)^{3/2} \le \beta^{3/2} + 1.6\nu$, so $\beta(1 - \sqrt{\beta}) \le \zeta + 1.6\nu$. Since $1 - \sqrt{\beta} \ge (1-\beta)/2$, in this case $\alpha(1-\alpha) \le 2\zeta + 3.2\nu$, and $\min(|A|, |B|) \le 2|\mathsf{Q}|\alpha(1-\alpha) \le 4|Z| + 6.4\nu|\mathsf{Q}|$.

(b) Each row and each column that meets σ without lying in it carries such a bond, a different one for each. If no row or column lies in σ, the r rows and c columns meeting σ give $r + c \ge 2(rc)^{1/2} \ge 2|\sigma|^{1/2}$ bonds. If a row lies in σ (the case of a column is symmetric), all columns meet σ, and if moreover a column lies in σ, at most $|\sigma|/s_2 + |\sigma|/s_1 \le s/2$ rows and columns lie in σ, so that either way there are at least $s \ge 2|\sigma|^{1/2}$ bonds.

Proposition 4.12 (The giant)**.** *At every hull-closed time of the full history other than the first, every surviving block q has $|V_q| > (1-3\vartheta)L_c^3$, and the graph G'_q of the G'-bonds with both ends in q has a unique component C_q, the* giant*, with $|V_q \setminus C_q| < 128 L_c m_q$ unless $V_q = C_q$.*

Proof. Since bonds inside a block are never frozen, on a line of q that meets two disjoint unions A, B of components of G'_q, two consecutive sites of V_q in A and in B are separated by a run of Lemma 4.10(b), a different run for each line. If $A \sqcup B = V_q$, Lemma 4.11(a) with $Z = \mathbb{X} \cap q$, $|Z| < 3L_c m_q$ and $L_\mu < 2L_c$ yields $\min(|A|,|B|) < 12 L_c m_q + 6.4 \cdot 9 m_q \cdot 2L_c < 128 L_c m_q$. By Lemma 4.9, $m_q < \vartheta L_c^2$, so $|V_q| > (1-3\vartheta)L_c^3 > 3 \cdot 128 L_c m_q$. If each component had fewer than $|V_q|/3$ sites, a union A of components with $|V_q|/3 \le |A| < 2|V_q|/3$ would violate the bound. The largest component C_q therefore has at least $|V_q|/3$ sites, and the bound with $A = C_q$ implies $|V_q \setminus C_q| < 128 L_c m_q$.

A *thin set* of a surviving block j is a nonempty $A \subseteq V_j$ with $|A| < \varepsilon_{\mathrm{th}} L_c^3 := 0.008 L_c^3$ and $\mathrm{cut}_j(A) < |A|/L_c$, $\mathrm{cut}_j(A)$ being the number of G'-bonds inside j between A and $V_j \setminus A$.

Lemma 4.13 (Thin sets)**.** *At every hull-closed time of the full history other than the first, every thin set A of a surviving block j satisfies $F_j^{\mathrm{aux}}(\mathbb{X} \cup A) < F_j^{\mathrm{aux}}(\mathbb{X})$. Hence a surviving block without auxiliary material has no thin set, and $V_j = C_j$ is connected in G'_j.*

Proof. We show that in each planar slice a thin set is enclosed by in-block insulation and that, by planar isoperimetry, this insulation exceeds the volume penalty. We fix a direction ν. A line of j in direction ν that meets A either meets $V_j \setminus A$, and then carries a bond counted in $\mathrm{cut}_j(A)$ or a run of Lemma 4.10(b), or lies in $A \cup (\mathbb{X} \cap j)$, which happens for at most $(|A| + 3L_c m_j)/L_c$ lines. A slice σ of A perpendicular to ν meets each line once. Therefore, with $m_j < \vartheta L_c^2$ from Lemma 4.9, $|\sigma| \le a := \mathrm{cut}_j(A) + 9 m_j + (|A| + 3L_c m_j)/L_c < (2\varepsilon_{\mathrm{th}} + 12\vartheta) L_c^2 \le 0.0164\, L_c^2 \le L_c^2/4$. The bonds of the cross-section of j (sides at least L_c) with exactly one end in σ end in $\mathbb{X}$ or in $V_j \setminus A$, and there are at least $2|\sigma|^{1/2} \ge 2|\sigma|/a^{1/2}$ of them, where the first bound is Lemma 4.11(b). Summing over the slices, we obtain $f_j(A, \mathbb{X}) + \mathrm{cut}_j(A) \ge 2|A|/a^{1/2} \ge 15.6|A|/L_c$, with $f_j(A, \mathbb{X})$ the number of bonds inside j from A to $\mathbb{X}$.

Passing from $\mathbb{X}$ to $\mathbb{X} \cup A$ changes F_j^{aux} by $-f_j(A, \mathbb{X})$ on these hole faces, by at most 0 on the bonds of ∂j at A (from w_b to 1, or from 1 to 0), by $+\mathrm{cut}_j(A)$ on the in-block bonds to $V_j \setminus A$, and by $+\lambda|A|$. In total the change is at most $-15.6|A|/L_c + 2\,\mathrm{cut}_j(A) + 12|A|/L_c < 0$. If j has no auxiliary material, $\mathbb{X}$ minimizes F_j^{aux}, so j has no thin set. Moreover, a component of G'_j other than C_j has $\mathrm{cut}_j = 0$ and, by Proposition 4.12, fewer than $128 L_c m_j < \varepsilon_{\mathrm{th}} L_c^3$ sites, so it would be thin.

Lemma 4.14 (The seal)**.** *Let t be a hull-closed time of an admissible history and j a surviving block with $A_j \neq \emptyset$. (a) Some G'-bond that crosses a face between j and a surviving block has an end in A_j. (b) If a seal step at t is followed by a hull step with the same $\mathbb{U}$, then $A_j \subseteq \mathbb{X}_{t+2}$.*

Proof. On the sets between $\mathbb{X} := \mathbb{X}_t$ and $\mathbb{X} \cup A_j$, the functionals $F := F_\lambda^{\bar{S}_t}$ and F_j^{aux} differ by a constant unless a G'-bond leaves A_j across a face of j. Indeed, since a block adjacent to j that is not surviving lies in $\mathbb{X}$, the other bonds of ∂j at A_j touch $\mathbb{X}$ or are frozen. Those at $V_j \setminus A_j$ keep their values. (a) Otherwise $F(\mathbb{X} \cup A_j) - F(\mathbb{X}) = F_j^{\mathrm{aux}}(M_j) - F_j^{\mathrm{aux}}(\mathbb{X}) < 0$, since $M_j \neq \mathbb{X}$ is the smallest minimizer. This contradicts $\mathbb{X} = H_\lambda^{\bar{S}_t}(B_t)$, as $\mathbb{X} \cup A_j \supseteq B_t$.

(b) Let $X' := \mathbb{X}_{t+2} \supseteq \mathbb{X}$ (Lemma 4.2), $F' := F_\lambda^{\bar{S}_{t+2}}$, $A' := A_j \setminus X'$ and $C := \mathbb{X} \cup (A_j \cap X') \subseteq X'$. On the sets between C and $M_j = C \cup A'$ only the sites of A' vary. The bonds inside j

at these sites count alike in F' and F_j^{aux}, and after the seal each bond of ∂j at them is frozen or touches $\mathbb{X}$, in both. Consequently, $F' - F_j^{\mathrm{aux}}$ is constant on these sets. If $A' \neq \emptyset$, then $C \subsetneq M_j$, and submodularity of F' implies $F'(X' \cup A') - F'(X') \leq F'(C \cup A') - F'(C) = F_j^{\mathrm{aux}}(M_j) - F_j^{\mathrm{aux}}(C) < 0$, contradicting the minimality of $X' = H_\lambda^{\bar{S}_{t+2}}(B_t)$.

Proposition 4.15. *Along the full history, a block face F' between blocks Q, Q' carries at most $121(m_Q + m_{Q'})$ bonds frozen by seal steps, converted ones included. Hence seal steps freeze at most $C_{\mathrm{seal}}N$ bonds, all on faces F' with $m_Q + m_{Q'} > 0$, where $C_{\mathrm{seal}} := 6 \cdot 121 = 726$.*

Proof. Each seal bond on F' has an end in auxiliary material, which the next hull step fills while Q and Q' still survive. This follows from Lemma 4.14(b), since in Phase 1 $\mathbb{U}$ is fixed during the inner loop and in Phase 2 the hull step follows the seal before any removal. Consequently, at the last hull-closed time T at which Q and Q' both survive, each seal bond on F' has an end in W_T in the layer next to F', hence in $Z(F')$, and distinct bonds lie on distinct lines $\ell \perp F'$. At most $m_Q + m_{Q'}$ of these lines carry a mark in $Z(F')$, and fewer than $2\,\mathrm{supp}_T(F')/\eta_w \leq 120(m_Q + m_{Q'})$ of the others meet W_T in $Z(F')$, the first inequality being Lemma 4.8(b). Since a block has six faces, summing over all block faces yields the bound on the total number of seal bonds.

Theorem 4.16 (Cells at rest)**.** *Let t be a hull-closed time of the full history other than the first at which no surviving block has auxiliary material. Then: (a) every surviving block q carries exactly one cell, $\Omega_q = V_q$, connected in q, with $(1 - 3\vartheta)L_c^3 < |\Omega_q| < 8L_c^3$, and every other block lies in $\mathbb{X}$; (b) every block face F' between surviving blocks contains no G'-bond, or carries exactly one interface I; in the latter case F' contains no cut bond and $\mathrm{good}(I) \geq (1 - 134\vartheta)|F'|$; in particular (S-face) holds; (c) every component $\mathfrak{K}$ of $G' \cap \hat{Q}$, $\hat{Q} \in \mathcal{Q}$, satisfies $f_{G'}(A, \mathfrak{K} \setminus A) \geq h_\star |A|$ for $0 < |A| \leq |\mathfrak{K}|/2$, with $h_\star := 2.2 \cdot 10^{-11}/L_c$, and (S-plug) holds.*

Proof. (a) By Lemma 4.13, $V_q = C_q$ is connected in q, and therefore lies in one component of $G' \cap \hat{Q}(q)$ and is the only cell of q. The volume bounds follow from Proposition 4.12 and (G1). Finally, a block that is not surviving lies in $\mathbb{U} \subseteq \mathbb{X}$, which is the last assertion of (a).

(b) By (a), the G'-bonds across F', if any, join the cell of Q to the cell of Q' and form one interface. Cuts fire only at such times, where by (a) a cut, which is a union of whole interfaces by Lemma 4.27(a), freezes all G'-bonds of each face its bonds cross. As G' only shrinks, a face with a G'-bond carries no cut bond. A line $\ell \perp F'$ is good unless $\ell \cap Z(F')$ contains a mark, or b_ℓ is a seal bond, or else $b_\ell \notin \bar{S}_t$ and $\ell \cap Z(F')$ contains a site of W_t. Indeed, on each remaining line the tube has no hole site, and its bonds are G'-bonds, since bonds inside a block are never frozen. The first case holds for at most $m_Q + m_{Q'}$ lines and, by Proposition 4.15, the second holds for at most $121(m_Q + m_{Q'})$ lines. The third holds for fewer than $2\,\mathrm{supp}_t(F') \leq 12(m_Q + m_{Q'})$ lines, where the first inequality comes from Lemma 4.8(a). With $m_Q + m_{Q'} \leq N(3Q) < \vartheta L_c^2 \leq \vartheta |F'|$, we obtain $\mathrm{good}(I) \geq (1 - 134\vartheta)|F'| \geq c_g|F'|$.

(c) We show that expansion can fail only through a thin set, which Lemma 4.13 excludes. Let $A \subseteq \mathfrak{K}$, $0 < |A| \leq |\mathfrak{K}|/2$, and $\mathrm{cut} := f_{G'}(A, \mathfrak{K} \setminus A)$. Since windows are unions of blocks, by (a) $\mathfrak{K}$ is a union of sets V_j of surviving blocks $j \subseteq \hat{Q}$, connected through faces that contain G'-bonds. We put $A_j := A \cap V_j$ and $B_j := V_j \setminus A_j$. A line of j meeting A_j and B_j carries a bond counted in cut or a run of Lemma 4.10(b). If $\min(|A_j|, |B_j|) \geq \varepsilon_{\mathrm{th}} L_c^3$ for some j, then $\varepsilon_{\mathrm{th}} L_c^3 \leq 12 L_c m_j + 12.8 L_c(\mathrm{cut} + 9m_j)$ by Lemma 4.11(a), so $\mathrm{cut} \geq (\varepsilon_{\mathrm{th}} - 127.2\vartheta)L_c^2/12.8 \geq \varepsilon_{\mathrm{th}} L_c^2/25.6$. Otherwise each block of $\mathfrak{K}$ is either *in* ($|B_j| < \varepsilon_{\mathrm{th}} L_c^3$) or *out* ($|A_j| < \varepsilon_{\mathrm{th}} L_c^3$), but not both, since $|V_j| > 2\varepsilon_{\mathrm{th}} L_c^3$. If an in-block and an out-block share a face F' that contains

G'-bonds, we follow each line $\perp F'$ from the face into either block until it reaches the majority side of that block through a bond counted in cut or a run, or stays in the minority side and $\mathbb{X}$ for the whole block. In this case the minority side of each block meets the face layer in at most $\mathrm{cut} + 9\vartheta L_c^2 + (\varepsilon_{\mathrm{th}} + 3\vartheta)L_c^2$ sites, all other bonds among the G'-bonds across F', of which there are at least $(1 - 134\vartheta)L_c^2$ by (b), join A to $\mathfrak{K} \setminus A$, and $3\,\mathrm{cut} \geq (1 - 158\vartheta - 2\varepsilon_{\mathrm{th}})L_c^2$. Otherwise all blocks are in, which is impossible since then $|A| \geq \sum_j (|V_j| - \varepsilon_{\mathrm{th}} L_c^3) > |\mathfrak{K}|/2$, or all are out. In the latter case each nonempty A_j has fewer than $\varepsilon_{\mathrm{th}} L_c^3$ sites and is not thin, so $\mathrm{cut} \geq \sum_j \mathrm{cut}_j(A_j) \geq |A|/L_c$. In all cases $\mathrm{cut} \geq \min(3.1 \cdot 10^{-4} L_c^2, |A|/L_c) \geq h_\star |A|$, where we use that $|A| < k_P^3/2$ by (G4), together with the arithmetic of Section S1.2. As $3k_P h_\star \geq \chi$, we have $\chi_{\mathrm{N3}}/k_P = \chi/(3k_P) \leq h_\star$, and (S-plug) holds.

Theorem 4.17 (The Phase-1 output)**.** *Phase 1 terminates. Its output* $(\mathbb{X}^1, \mathbb{U}^1, S^1)$ *is the smallest minimizer of* $\mathcal{E}^{\bar{S}^1}$*, no surviving block has auxiliary material there, and: (i) every bond of* $\bar{S}^1$ *touches* $\mathbb{X}^1$*, so* $S^1 = \emptyset$*, and crosses a face between two blocks that together contain a mark; (ii) the four terminal properties (S-hull), (S-rem), (S-face), (S-plug) hold.*

Proof. By Lemma 3.12(c), each outer step returns $\mathbb{U}^* \supseteq \mathbb{U}$, since $\bar{S}$ only grows, and by Lemma 4.14(a) each inner round freezes at least one bond. Each outer step except the last enlarges $\mathbb{U}$ or $\bar{S}$, which only grow in a finite box. Phase 1 thus terminates. In the last outer step the inner loop performs no seal step, so $\bar{S} = \bar{S}^1$ throughout that step. Thus step (1) of the last outer step returns the smallest minimizer of $\mathcal{E}^{\bar{S}^1}$, with hole set $H_\lambda^{\bar{S}^1}(\mathbb{M} \cup \mathbb{U}^1)$ by Lemma 3.12(b). This pair is the output, and step (2) finds no auxiliary material at it.

(i) Each bond of $\bar{S}^1$ has an end in auxiliary material, which the next hull step fills by Lemma 4.14(b). For the faces crossed by these bonds we use Proposition 4.15.

(ii) Property (S-hull) follows from Lemma 4.2, and (S-face), (S-plug) follow from Theorem 4.16. For (S-rem), let $\Delta \not\subseteq B^1$ be a removal cell with $\Delta \not\subseteq \mathbb{X}^1$ (otherwise $w(\Delta) = 0$). We compare with $(\mathbb{X}^1 \cup \Delta, \mathbb{U}^1 \cup \Delta)$, admissible since Δ is a union of blocks. The bonds of $E(\Delta)$ with one end in $\mathbb{X}^1$ lose their weight 1, since by (i) no bond is frozen. At most $|\partial\Delta|$ bonds gain at most 1, the volume terms grow by at most $\lambda_U|\Delta|$, and the penalty does not grow. Therefore $w(\Delta) \leq |\partial\Delta| + \lambda_U|\Delta| \leq (1 + c_s)|\partial\Delta| = w_*$, since $|\Delta|/|\partial\Delta| < \ell/2$ for a box with sides in $[\ell, 3\ell)$ (see Section S1.2).

Lemma 4.18. *Let the state satisfy (S-plug),* $\hat{Q} \in \mathcal{Q}$*, and* $\mathfrak{K}$ *a component of* $G' \cap \hat{Q}$*. Then* $f_{G'}(A, \mathfrak{K} \setminus A) \geq \chi|A|/(3k_P)$ *for* $A \subseteq \mathfrak{K}$ *with* $0 < |A| \leq |\mathfrak{K}|/2$*, and* $\sum_{x \in \mathfrak{K}} |f(x) - \bar{f}_{\mathfrak{K}}|^2 \leq (108/\chi^2) k_P^2 \sum_{\{x,y\} \in E_{G'}(\mathfrak{K})} (f(x) - f(y))^2$ *for every* f*.*

Proof. The first claim is (S-plug), since $\chi_{\mathrm{N3}}/k_P = \chi/(3k_P)$. The Poincaré inequality follows from the Cheeger argument and a median. Let μ be a median of f on $\mathfrak{K}$. Since each of $(f - \mu)_\pm$ is supported on at most $|\mathfrak{K}|/2$ sites, the argument of Theorem 4.5(iv) in the graph $(\mathfrak{K}, E_{G'}(\mathfrak{K}))$, with $h := \chi/(3k_P)$, bounds $\sum (f - \mu)_\pm^2$ by $12h^{-2} = 108k_P^2/\chi^2$ times the Dirichlet form of $(f - \mu)_\pm$. We add the two bounds, using $(a - b)^2 \geq (a_+ - b_+)^2 + (a_- - b_-)^2$, and replace μ by the mean.

Lemma 4.19 (Good bonds)**.** *At every state satisfying (S-face), (R8) holds, a fine bond across* $F(I)$ *being good if the segment of* L *fine sites of its line centred at it is a path of the fine slit graph.*

Sketch. The first claim follows from (S-face) and (G1), and the second holds because bonds across block faces lie on block planes, at least L_c apart. A good tube is a G'-path in $Z(F') \subseteq Q \cup Q' \subseteq \hat{Q}(Q) \cap \hat{Q}(Q')$, by (G5) and (G4). Thus its half in Q lies in the component of $G' \cap \hat{Q}(Q)$ of its end, the cell of I in Q. The interfaces of one face are disjoint with at least $c_g|F'|$ bonds each, and a block has six faces. Together, these facts bound the number of interfaces of a face and the number of neighbouring cells of a cell. In fine units, a block face meets at most one exceptional layer on each side, so $|F'|_{\text{fine}} < r_0^2|F'|(1+1/L_c)^2$, while each coarse good bond carries at least r_0^2 fine good bonds.

Theorem 4.20 (Cells)**.** *At every closed state with (S-face) and (S-plug), such as the output, (R5), (R6) and the cell part of (R7) hold, with $\mathfrak{K}^{\flat}(\Omega) := \hat{\mathfrak{K}}$ for the cell $\Omega = Q \cap \hat{\mathfrak{K}}$ of a block Q.*

Proof. For (R5), $|\Omega| \le |Q| < 8L_c^3$ by (G1). An interface on a face F' of Q provides, by Lemma 4.19, at least $c_g L_c^2$ good half-tubes of $L_c/2$ sites in Ω, on distinct lines. Without an interface no G'-bond leaves Ω, since inside Q a G'-bond joins two sites of one cell and across a face it belongs to an interface. In that case Ω is therefore a union of components of G', each with more than $V_* = ((1-\eta_w)c_I/24)^3 L_c^3$ sites by Theorem 4.5(ii). For (R6), we apply Lemma 4.18 in $\hat{\mathfrak{K}} \supseteq \Omega$ and use that the mean minimizes the sum of squares. The overlap bound holds because a bond lies in at most $151^3 \le 27(k_P/L_c)^3$ windows (G4) and the components $\hat{\mathfrak{K}}$ of distinct cells of one block are distinct. For (R7), we use that $\hat{\mathfrak{K}}$ is connected in G', together with Lemma 3.5(A2).

Lemma 4.21 ($\mathcal{U}$-cells)**.** *Let $\mathcal{T}$ be a connected graph of maximal degree at most D, with vertex weights $w(v) \in (0, V)$ and total weight at least V. Then the vertex set of $\mathcal{T}$ splits into parts, each connected in $\mathcal{T}$, of weight in $[V, (2+D)V)$. Applied to the cell graphs of the output with $D = 6/c_g$, cell weights $|\Omega|_{\text{fine}}$ and $V = V_{\mathcal{U}} = \tilde{m}^{-3}$, it gives the $\mathcal{U}$-cell part of (R7), with $C_{\mathcal{U}}$ of Section S1.2, provided τ is so large that $\tilde{m}L \ge c_L/2$.*

Proof. We cut a rooted spanning tree from the leaves upward, cutting off a vertex with its uncut descendants once their weight reaches V. A cut part weighs less than $w(v) + DV < (1+D)V$, and the light remainder at the root joins an adjacent cut part. In the application, the cell graph is connected, since G'-bonds between cells are interface bonds, and Lemma 4.19 bounds its degree by $6/c_g$. A cell weighs less than $8L^3 \le \tilde{m}^{-3}$ by (3.1). By Theorem 4.5(ii), a light component has more than V_* sites, and thus weight more than $r_0^3 V_* = c_u V_{\mathcal{U}}$. Finally, a $\mathcal{U}$-cell holds fewer than $(2+6/c_g)\tilde{m}^{-3}/(c_1 L^3)$ connected cells by (R5), in blocks of Euclidean diameter less than $2\sqrt{3}L$.

4.3 Payment and reach of the Phase-1 output

In this subsection we bound the cost of the Phase-1 output by means of the three comparisons of Lemma 4.22. By filling everything near a point we bound the cost of a ball; by deleting a region without marks we bound its cost by the bonds of a sphere around it; and with a good sphere, chosen by averaging over the radii, the cost decays geometrically away from the marks. Consequently, no hole lies far from a mark, as Theorem 4.23 states, and in Proposition 4.24 we pay for the pieces near the clusters of marks by comparison with the cheapest state that covers the marks and fills the forced blocks of the cluster.

Throughout, $(\mathbb{X}^1, \mathbb{U}^1)$ is the Phase-1 output and $\mathcal{E} := \mathcal{E}^{\bar{S}^1}$. By Theorem 4.17, $(\mathbb{X}^1, \mathbb{U}^1)$ minimizes $\mathcal{E}$, and each bond of $\bar{S}^1$ touches $\mathbb{X}^1$ and has both ends within distance $4L_c$ of a

mark. Thus, at $(\mathbb{X}^1, \mathbb{U}^1)$, the frozen bonds count as ordinary bonds, and the penalty vanishes by Lemma 4.9. For a set A of sites we put $\mathcal{E}_A := \#\{b \in \partial\mathbb{X}^1 : b \subseteq A\} + \lambda|(\mathbb{X}^1 \setminus \mathbb{U}^1) \cap A| + \lambda_U|\mathbb{U}^1 \cap A|$, let $B_r(x) := Q_\infty(x, r) \cap \Lambda^\circ$, and for a fixed site x let $f(r) := \mathcal{E}_{B_r(x)}$, nondecreasing in r. Each competitor $(\mathbb{X}', \mathbb{U}')$ below has $\mathbb{M} \subseteq \mathbb{X}'$ and $\mathbb{U}' \subseteq \mathbb{X}'$ a union of blocks, and so satisfies $\mathcal{E}(\mathbb{X}', \mathbb{U}') \geq \mathcal{E}(\mathbb{X}^1, \mathbb{U}^1)$.

Lemma 4.22. *(a) For every site x and every $r \geq 0$, $f(r) \leq g(r) := 6(2r+4L_c)^2 + \lambda_U(2r+4L_c)^3$. (b) Let B be a set of sites with $\operatorname{dist}(B, \mathbb{M}) \geq 6L_c$, and $\hat{B}$ the union of the blocks meeting B. Then $\mathcal{E}_B \leq \operatorname{sl}(B) := f(B, \mathbb{X}^1 \setminus B) + \lambda|\mathbb{U}^1 \cap (\hat{B} \setminus B)|$. (c) Put $w := \lceil 120\ell/c_s \rceil$ and $w' := w + 2L_c$. If $\operatorname{dist}(B_{r+w'}(x), \mathbb{M}) \geq 6L_c$, then $f(r) \leq f(r + w')/5$.*

Proof. (a) Let $\hat{B}$ be the union of the blocks meeting $B_r(x)$, a box of sides less than $2r + 4L_c$. We compare with $(\mathbb{X}^1 \cup \hat{B}, \mathbb{U}^1 \cup \hat{B})$. In this competitor the bonds of $\partial\mathbb{X}^1$ with both ends in $\hat{B}$ disappear, at most $|\partial\hat{B}| \leq 6(2r + 4L_c)^2$ bonds appear, the frozen bonds still touch the hole set, each site of $\hat{B}$ ends with the weight λ_U, and the penalty does not grow. It follows that $f(r) \leq \mathcal{E}_{\hat{B}} \leq |\partial\hat{B}| + \lambda_U|\hat{B}|$.

(b) A forced block has a mark within distance $4L_c$ of each of its sites. Consequently, no block meeting B is forced, and no bond of $\bar{S}^1$ has an end in B. We compare with $(\mathbb{X}^1 \setminus B, \mathbb{U}')$, where $\mathbb{U}'$ is the union of the blocks of $\mathbb{U}^1$ that do not meet B. This competitor contains the marks and all forced blocks. In passing to it, the bonds of $\partial\mathbb{X}^1$ with both ends in B disappear, at most $f(B, \mathbb{X}^1 \setminus B)$ bonds appear, the frozen bonds still touch the hole set, the sites of $B \cap \mathbb{X}^1$ lose their weights, and those of $\mathbb{U}^1 \cap (\hat{B} \setminus B)$ change from λ_U to λ.

(c) We put $A := B_{r+w'}(x) \setminus B_r(x)$ and apply (b) to $B_s := B_s(x)$ for $r \leq s < r + w$. Since a site of $\mathbb{X}^1$ at distance exactly $s + 1$ from x has at most six bonds into B_s, we have $\sum_s f(B_s, \mathbb{X}^1 \setminus B_s) \leq 6|\mathbb{X}^1 \cap A|$. A site of $\mathbb{U}^1$ outside B_s lies in a block meeting B_s for fewer than $2L_c$ values of s, and therefore $\sum_s \lambda|\mathbb{U}^1 \cap (\hat{B}_s \setminus B_s)| \leq 2\lambda L_c|\mathbb{U}^1 \cap A| = 24|\mathbb{U}^1 \cap A|$. Combining the two bounds, we obtain $\sum_s \operatorname{sl}(B_s) \leq 30|\mathbb{X}^1 \cap A| \leq 30(\lambda^{-1} + \lambda_U^{-1})\mathcal{E}_A \leq (30\ell/c_s)\mathcal{E}_A$, using $\lambda^{-1} + \lambda_U^{-1} = L_c/12 + \ell/(2c_s) \leq \ell/c_s$. Thus some s has $\operatorname{sl}(B_s) \leq \mathcal{E}_A/4$. The bonds and sites counted in $\mathcal{E}_A$ are not counted in $f(r)$. Therefore $\mathcal{E}_A \leq f(r + w') - f(r)$, and $f(r) \leq f(s) \leq \operatorname{sl}(B_s) \leq (f(r + w') - f(r))/4$.

Theorem 4.23. *Let J be the least integer with $5^J > g((J+1)w')$, and $R_* := (J+1)w' + 6L_c$. Every site of $\mathbb{X}^1$ lies within distance R_* of a mark, and $R_* = O((\ell/c_s)\log(\ell/c_s))$, uniformly in $|\mathbb{M}|$, Λ°, C and D.*

Proof. Let $\operatorname{dist}(x, \mathbb{M}) > R_*$. Then $\operatorname{dist}(B_{(J+1)w'}(x), \mathbb{M}) > 6L_c$, and applying Lemma 4.22(c) J times from $r = w'$, together with Lemma 4.22(a), we obtain $f(w') \leq 5^{-J} f((J + 1)w') \leq 5^{-J} g((J + 1)w') < 1$. So no bond of $\partial\mathbb{X}^1$ has both ends in the box $B_{w'}(x)$, which therefore lies in $\mathbb{X}^1$ or misses it. In the first case $f(w') \geq \lambda_U|B_{w'}(x)| > 1$. Therefore $x \notin \mathbb{X}^1$. With the arithmetic of Section S1.3, this proves Theorem 3.21(b).

Proposition 4.24. *Theorem 3.21(a) holds with $C'_\star := C_1 + 2C_{\text{seal}}$, $C_1 := 7 + 27 \cdot 24.6/\vartheta$, and $C_{\text{free}} := C'_\star(2 + 1/(2c_s) + 1/3696)$.*

Proof. By Theorem 4.17(i), $S^1 = \emptyset$. We compare with the competitor that replaces Y_P by the marks and the forced blocks of P. The sets Y_P cover $\mathbb{X}^1$ by Theorem 4.23. Marks of distinct parts are more than $2R_* + 2$ apart, and hence no bond joins P^{+R_*} to P'^{+R_*}. Moreover, a block of $\mathbb{U}^1$ is connected and lies in $\mathbb{X}^1$. The marks of a set $3Q$ are within distance $6L_c \leq r'$

of each other, and hence those of a forced block lie in one part. The union $U_F(P)$ of the forced blocks with marks in P lies in Y_P, by Lemma 4.9 and because a forced block lies within $4L_c$ of its marks. It has at most $27N(P)/(\vartheta L_c^2)$ blocks, since a mark lies in at most 27 sets $3Q$. The competitor $\mathbb{X}' := (\mathbb{X}^1 \setminus Y_P) \cup P \cup U_F(P)$, $\mathbb{U}' := (\mathbb{U}^1 \setminus Y_P) \cup U_F(P)$ contains the marks, $\mathbb{U}'$ is a union of blocks containing all forced blocks, and the change is confined to Y_P, which no bond joins to the rest of $\mathbb{X}^1$. A bond of $\bar{S}^1$ that loses contact with the hole set touches Y_P, so the marks of its two blocks lie within $R_* + 4L_c < r'$ of P and therefore in P. There are at most $C_{\mathrm{seal}}N(P)$ such bonds by Proposition 4.15, each costing at most 2. Since $|\partial Q| + \lambda_U|Q| < 24.6L_c^2$ and $\lambda = 12/L_c \le 1$ ($L_c \ge 35$ by item (e) of Section S1.1.4), minimality implies $|\partial Y_P| + \lambda|Y_P \setminus \mathbb{U}^1| + \lambda_U|Y_P \cap \mathbb{U}^1| \le (6+\lambda)N(P) + 24.6 \cdot 27N(P)/\vartheta + 2C_{\mathrm{seal}}N(P) \le C'_\star N(P)$. The second inequality follows from $|Y_P|/\ell \le (\lambda\ell)^{-1}\lambda|Y_P \setminus \mathbb{U}^1| + (\lambda_U\ell)^{-1}\lambda_U|Y_P \cap \mathbb{U}^1|$, with $(\lambda\ell)^{-1} \le 1/3696$ and $(\lambda_U\ell)^{-1} = 1/(2c_s)$.

4.4 Phase 2: insulation, density bound and cuts

In this subsection we first bound the insulation in Phase 2, in Lemma 4.25, and then prove the density bound (R9), Lemma 4.27 and Proposition 3.19. Each hole site added in Phase 2 outside the removed cells is hull material, whose removal does not lower the hull functional, while the Phase-1 insulation $\mathrm{ins}^1 := \partial\mathbb{X}^1$ is fixed in advance and bounded in Theorem 3.21(a). Throughout, R_t is the union of the cells removed in Phase 2 before time t. A cell Δ removed in Phase 2 is removed at a hull-closed time t_Δ, at which $w(\Delta)$ is evaluated. We denote by s_Δ the number of bonds of $E(\Delta)$ frozen in Phase 2 before t_Δ, converted ones included, with no end in $\mathbb{X}^1$.

Lemma 4.25. *(a) Let t be a hull-closed time of Phase 2, N' a box, $W' := (\mathbb{X}_t \setminus (\mathbb{X}^1 \cup R_t)) \cap N'$, $\mathrm{Ins}_{N'}(Y) := \#\{b \in E(N') : \text{exactly one end in } Y\} + 2\#\{b \in \bar{S}_t \cap E(N') : \text{no end in } Y\}$, and $\mathrm{cut}_{N'}$ the number of bonds across $\partial N'$ with inner end in W' whose outer end lies in $\mathbb{X}_t$ or which are frozen. Then*

$$\mathcal{A}_2(E(N'); t) \le \mathrm{Ins}_{N'}(\mathbb{X}^1 \cup R_t) + \mathrm{cut}_{N'}. \tag{4.4}$$

(b) Every cell Δ removed in Phase 2 satisfies $w(\Delta) \le |\mathrm{ins}^1 \cap E(\Delta)| + 2s_\Delta + |\partial\Delta|$, hence $c_s|\partial\Delta| < |\mathrm{ins}^1 \cap E(\Delta)| + 2s_\Delta$.

Proof. (a) We put $\mathbb{X} := \mathbb{X}_t$ and $V := \Lambda^\circ \setminus \mathbb{X}$. Since $B_t = \mathbb{M} \cup \mathbb{U}^1 \cup R_t \subseteq \mathbb{X}^1 \cup R_t \subseteq \mathbb{X}$ by Proposition 4.3(a), W' misses B_t and $\mathrm{Ins}_{N'}(\mathbb{X} \setminus W') = \mathrm{Ins}_{N'}(\mathbb{X}^1 \cup R_t)$, while $\mathrm{Ins}_{N'}(\mathbb{X}) = \mathcal{A}_2(E(N'); t)$, since by Lemma 4.2 the bonds of $\bar{S}_t$ with no end in $\mathbb{X}$ are those of S_t. This proves (a) if $W' = \emptyset$. Otherwise $F_\lambda^{\bar{S}_t}(\mathbb{X} \setminus W') > F_\lambda^{\bar{S}_t}(\mathbb{X})$, because $\mathbb{X} = H_\lambda^{\bar{S}_t}(B_t)$. Written bond by bond as in (4.3), this inequality reads $f_{E \setminus \bar{S}_t}(W', V) + \lambda|W'| < f(W', \mathbb{X} \setminus W') + \sum_{b \in \bar{S}_t \text{ between } W', V}(w_b - 1) + \sum_{b \in \bar{S}_t, b \subseteq W'} w_b$, with $f_{E \setminus \bar{S}_t}$ counting unfrozen bonds. When we pass from $\mathrm{Ins}_{N'}(\mathbb{X} \setminus W')$ to $\mathrm{Ins}_{N'}(\mathbb{X})$, the contribution of a bond of $E(N')$ between W' and $\mathbb{X} \setminus W'$ changes by -1, that of a bond of $\bar{S}_t$ inside W' by -2, that of a bond between W' and V by $+1$ if unfrozen and by -1 if frozen, and that of every other bond by 0. Inserting the inequality, we see that the bonds between W' and $\mathbb{X} \setminus W'$ inside N' cancel, frozen bonds inside N' with an end in W' get the coefficient $w_b - 2 \le 0$, and the bonds counted in $\mathrm{cut}_{N'}$ remain, each with coefficient at most 1. We conclude that $\mathcal{A}_2(E(N'); t) < \mathrm{Ins}_{N'}(\mathbb{X}^1 \cup R_t) + \mathrm{cut}_{N'} - \lambda|W'|$.

(b) We take $N' := \Delta$ and $t := t_\Delta$. Since $\Delta \cap R_t = \emptyset$ and, by Theorem 4.17, each bond of $\bar{S}^1$ touches $\mathbb{X}^1$, $\mathrm{Ins}_\Delta(\mathbb{X}^1 \cup R_t) = |\partial\mathbb{X}^1 \cap E(\Delta)| + 2\#\{b \in \bar{S}_t \cap E(\Delta) : \text{no end in } \mathbb{X}^1\} \le |\mathrm{ins}^1 \cap E(\Delta)| + 2s_\Delta$, and $\mathrm{cut}_\Delta \le |\partial\Delta|$. The last claim follows from the trigger.

Proposition 4.26 (Density bound)**.** *At every state satisfying (S-rem), in particular at the output, (R9) holds, and* $(3+c_s)|\partial\Delta| < 54(3+c_s)\ell^2$.

Proof. Since a frozen bond is a missing pair of neighbours counted at both ends, the density count of Section 6 is two-sided, namely the hull's own count $\mathcal{A}_2$. If $\Delta \not\subseteq B$, then $\mathcal{A}_2(E(\Delta)) = w(\Delta) \le w_*$ by (S-rem). If $\Delta \subseteq B \subseteq \mathbb{X}$, this count is 0. A bond of $\partial\Delta \cap E$ weighs at most 2. This yields the bound in which these bonds are counted as well, and the last assertion holds as $|\partial\Delta| < 54\ell^2$ by (G3). For the constant c_0 of the conversion into fine plaquettes we refer to Section S1.2.

We recall that Phase 2 fills only by removal and freezes by the seal and the cut rule, and that the cut rule fires only at hull-closed states without auxiliary material, where Theorem 4.16 holds.

Lemma 4.27. *Let the firings $j = 1, 2, \dots$ of the cut rule have times t_j, components $\mathfrak{K}_j$ and sides U_j, $\mathfrak{K}_j \setminus U_j$. (a) Every bond of $\partial_{G'}U$ joins two different cells, so it crosses a block face; $\partial_{G'}U$ is a union of whole interfaces. (b) After the cut j no G'-bond joins U_j to $\mathfrak{K}_j \setminus U_j$; every later component of G' lies in U_j, in $\mathfrak{K}_j \setminus U_j$, or outside $\mathfrak{K}_j$. The sides of the cuts form a laminar family. (c) $|\partial_{G'}U_j| \ge c_g L_c^2$ for every cut. (d) Cut bonds support no hull material, alone or with seal bonds, filled blocks and box faces, and cut bonds block no tube of another interface. (e) If $J(U) > 0$, then $\partial_{G'}U$ contains two interfaces at distance $> R_{\mathrm{NF}}$.*

Proof. (a) A G'-bond with both ends in a block Q lies in $G' \cap \hat{Q}(Q)$, and hence its ends lie in one cell. Therefore a bond of $\partial_{G'}U$ joins cells of different blocks, and all bonds across F' between a cell of U and a cell outside U lie in $\partial_{G'}U$. (b) The cut consists of all G'-bonds between U_j and $\mathfrak{K}_j \setminus U_j$, where $\mathfrak{K}_j$ is a component of G' at time t_j, and G' only shrinks by (H4) of Proposition 4.3. Consequently, each later component lies in U_j, in $\mathfrak{K}_j \setminus U_j$, or outside $\mathfrak{K}_j$. If j_a fires before j_b, the sides of j_b lie in $\mathfrak{K}_{j_b}$, a component at time t_{j_b}, so each of them lies in a side of j_a or misses both. As the two sides of one cut are disjoint, any two sides are disjoint or nested. (c) By (a) the cut contains a whole interface I, and (S-face) holds at t_j by Theorem 4.16(b). Thus $|\partial_{G'}U_j| \ge \mathrm{good}(I) \ge c_g L_c^2$, where the second inequality comes from Lemma 4.19. (d) Cut bonds are grid bonds with the hull weight, and they therefore enter Corollary 4.7 and Lemma 4.8 as grid-type terms, as seal bonds do. A tube crosses no other block plane, by (G5), and this proves the last assertion of (d). (e) If $J(U) > 0$, then $\varphi_s(\partial_{G'}U) \ge 2$ for some $s = 2^j L \ge R_{\mathrm{NF}}$, and interfaces in different s-clusters are more than s apart.

Proof of Proposition 3.19. Termination is Proposition 4.3(b). At the terminal state a full pass changes nothing. Thus the state is hull-closed, which by Lemma 4.2 implies (S-hull), and no removal cell is over threshold, which is (S-rem). For the same reason, no surviving block has auxiliary material, by Lemma 4.14(a), so (S-face) and (S-plug) hold by Theorem 4.16. Finally, the cut rule has no candidate at the terminal state, which is (S-NF) and contains the form used in Section 7.8.

We now assume that the Euclidean diameter of Λ° is at most R_{NF}. Then no pair is fragmented, since the right side of (NF_*) is positive and, by Lemma 4.27(e), $J(U) > 0$ would give two interfaces more than R_{NF} apart. Since, moreover, by Theorem 4.17 the Phase-1 output is hull-closed, satisfies (S-rem) and has no auxiliary material, Phase 2 changes nothing in this case.

5 The min-cut lower tail

In this section we show that the frozen set of the construction of Section 3 lowers the min-cut area of Definition 2.11 by at most $\eta_T RT$, except with probability at most $2^{4|C|+5}e^{-RT}$. The proof has two ingredients. Section 5.2 shows that, as Λ° is acyclic, a minimizer can be taken constant on each piece of the hole set. In Sections 5.3 to 5.7 we then prove that the frozen plaquettes which a part of the marks attaches are paid for by the marks of that part.

The units, Ξ, Λ_2°, $\partial_2\Lambda^\circ$, $F(Y)$, F_S, $\mathfrak{F}$, $G'(D)$, $\mathbb{X}$ and $S = S_1 \sqcup S_{\mathrm{cut}}$ are those of Sections 2.5, 3 and 3.4. A coarse face carries at most $\bar r_0^2$ fine plaquettes, $\bar r_0 = 2r_0 - 1$, and Ξ is a union of coarse faces, since the plane of C is a special plane of $\mathfrak{G}(C)$. We write $\mathbb{P}_D := \mathrm{Ber}(\delta)^{\otimes(\Lambda_2^\circ \setminus \partial_2\Lambda^\circ)}$ for the law of D and $\mathbb{E}_D$ for its expectation.

5.1 Statement

Theorem 5.1 (Min-cut lower tail)**.** *Fix $K > K_*$ and $\beta \ge \beta_0(K)$. For every rectangle C with $\min(R,T) \ge R_1(\beta)$ and every $\sigma \in (0,1]$,*

$$\mathbb{P}_D\big(A_{G'(D)}(C) \le (1-\eta_T)RT\big) \le 2^{4|C|+5}e^{-RT}, \tag{5.1}$$
$$\mathbb{E}_D \exp\big(\sigma(RT - A_{G'(D)}(C))\big) \le \exp\big(\sigma\eta_T RT + 4|C| + 4\big),$$

where

$$\eta_T := \frac{3c_T''\bar r_0^2 C_3}{|\log \varepsilon''|}, \tag{5.2}$$

with ε'' of (5.8) *and the constants c_T'' and C_3 of Section S1.4. Moreover, there is $\tau_0(K)$ such that for $\tau \ge \tau_0(K)$ we have $\eta_T \le 6c_T''\bar r_0^2 C_3/(\gamma_T(K)\tau)$, with $\gamma_T(K)$ of Lemma 5.12, and $\eta_T = O\big(\bar r_0^2(C'_\star + C_{\mathrm{resp}})(1 + 1/c_s)/(\tau \log^2 K)\big)$ with an absolute implied constant, uniformly in $K > K_*$; so $\eta_T \to 0$ as $K \to \infty$, and as $\beta \to \infty$ at fixed $K > K_*$.*

The conditions that define $\beta_0(K)$ in item (a) of Section S1.1.4 are lower bounds on τ, while the constants c_T'', C_3, C_{free}, $C'_\star$, C_{resp} are free of β, τ, K, C, D. The condition $\tau \ge \tau_0(K)$ is used only for the order of η_T, not for (5.1).

5.2 The deterministic side

In this subsection we let $(\mathbb{X}, S)$ be any state with $S \subseteq E(\Lambda^\circ \setminus \mathbb{X})$ and compute $A_{G'(D)}(C)$ with $\mathfrak{F} = F(\mathbb{X}) \cup F_S \cup \partial_2\Lambda^\circ$. Then $F_S \cap F(\mathbb{X}) = \emptyset$.

Box topology. Let B be a box of $\mathbb{Z}^3$ with its cubical complex. (i) Over $\mathbb{Z}$ and over $\mathbb{Z}_2$ this complex is the tensor product of the complexes of three intervals, each chain-homotopy equivalent to the coefficients in degree 0; hence every k-cycle with $k \ge 1$ is a boundary. (ii) ∂ is injective on 3-chains: in a nonzero 3-chain, a cube that is extreme in the lexicographic order has a face shared with no other cube of the chain, and the boundary has a nonzero coefficient there. (iii) $H_1(\partial B; \mathbb{Z}) = 0$: if z is an integer 1-cycle on the boundary bonds, then $z = \partial c$ for a 2-chain c of B by (i); the part of c on the interior plaquettes is a closed 1-cochain on the dual box, hence the coboundary of a 0-cochain f on the cubes, and $c - \partial f$ is a 2-chain of boundary plaquettes with boundary z. (iv) By (i) over $\mathbb{Z}_2$, the boundaries of the unit squares generate the $\mathbb{Z}_2$ cycle space of the box graph.

Lemma 5.2. *Every integer 2-chain Σ in Λ° with $\partial\Sigma = C$ satisfies $\#(\operatorname{supp}\Sigma \setminus \partial_2\Lambda^\circ) \geq RT$. Moreover $A_{G'(D)}(C) \leq RT$.*

Proof. There are RT disjoint closed dual paths that link C, namely nested closed paths of fine cubes in the planes perpendicular to Ξ through its plaquette rows. They lie inside Λ° off $\partial_2\Lambda^\circ$ by Lemma 3.3(e), and each of them crosses Ξ once (Section S2.3). As Λ° is acyclic (Box topology (i)), $\Sigma - \Xi$ is a boundary. Each path therefore meets $\operatorname{supp}\Sigma$, with intersection number ± 1, and different paths meet it in different plaquettes. The second claim holds because $A_{G'(D)}(C) \leq |\operatorname{supp}\Xi| = RT$.

A *piece partition* is a partition $\mathcal{Y}$ of the sites of $\mathbb{X}$ into pieces, which need not be connected. For a site set Y, let ∂Y be the set of coarse bonds with exactly one end in Y, $\operatorname{int} F(Y)$ the set of fine plaquettes whose two fine cubes lie in sites of Y, and $\partial^f Y$ the set of those with one fine cube in a site of Y and the other in a site of $\Lambda^\circ \setminus Y$. Then

$$F(Y) = \operatorname{int} F(Y) \sqcup \partial^f Y \sqcup \big(F(Y) \cap \partial_2\Lambda^\circ\big), \qquad |\partial^f Y| \leq \bar{r}_0^2 |\partial Y|. \tag{5.3}$$

Indeed, a plaquette of $F(Y)$ lies in $\partial_2\Lambda^\circ$ iff it is a face of only one fine cube of Λ°, and otherwise its second cube lies in a site of Y or of $\Lambda^\circ \setminus Y$; a plaquette of $\partial^f Y$ lies in the common coarse face of two adjacent sites $y \in Y$, $y' \notin Y$, which is dual to the bond $\{y, y'\} \in \partial Y$ and carries at most $\bar{r}_0^2$ fine plaquettes.

Lemma 5.3. *There is a minimizer $\tilde{\Sigma}$ of Definition 2.11 ($\partial\tilde{\Sigma} = C$, $\#(\operatorname{supp}\tilde{\Sigma} \setminus \mathfrak{F}) = A_{G'(D)}(C)$), determined by $(\mathbb{X}, S)$, such that for every Y of every piece partition*

$$\operatorname{supp}\tilde{\Sigma} \cap F(Y) \subseteq \partial^f Y \cup \big(\operatorname{supp}\Xi \cap \operatorname{int} F(Y)\big) \cup \partial_2\Lambda^\circ. \tag{5.4}$$

Proof. By acyclicity a minimizer is the flat rectangle plus a boundary, and we may change the boundary freely on hole cubes, where all plaquettes are free. Let Σ be the first minimizer in a fixed order and $\Sigma - \Xi = \partial k$, k an integer function on the fine cubes of Λ° (as in Lemma 5.2; k is unique by Box topology (ii)). We define $\tilde{k} := k$ on the fine cubes of sites of $\Lambda^\circ \setminus \mathbb{X}$, $\tilde{k} := 0$ on those of sites of $\mathbb{X}$, and $\tilde{\Sigma} := \Xi + \partial\tilde{k}$. Then $\partial\tilde{\Sigma} = C$, and $\tilde{\Sigma}$, Σ differ only on faces of fine cubes of sites of $\mathbb{X}$, which lie in $F(\mathbb{X}) \subseteq \mathfrak{F}$. It follows that $\#(\operatorname{supp}\tilde{\Sigma} \setminus \mathfrak{F}) = A_{G'(D)}(C)$. For $p \in \operatorname{int} F(Y)$ both cubes of p carry $\tilde{k} = 0$, and therefore $\tilde{\Sigma}_p = \Xi_p$. Together with (5.3), this is (5.4), and $\tilde{\Sigma}$ does not involve $\mathcal{Y}$.

Lemma 5.4 (C-rooted minimizer)**.** *There is a minimizer Σ°, determined by $(\mathbb{X}, S)$ alone, with the following properties, in which $\operatorname{supp}\Sigma^\circ$ denotes its support in $\Lambda_2^\circ \setminus \partial_2\Lambda^\circ$ and $a_\Sigma := |\operatorname{supp}\Sigma^\circ|$. We have $\operatorname{supp}\Sigma^\circ \subseteq \operatorname{supp}\tilde{\Sigma}$, and every component of $\operatorname{supp}\Sigma^\circ$ for edge adjacency contains a plaquette incident to an edge of C. Moreover, (5.4) holds for Σ° for every piece partition, and $a_\Sigma \geq RT$.*

Proof. In a fixed order, we replace each component of $\operatorname{supp}\tilde{\Sigma} \setminus \partial_2\Lambda^\circ$ with no plaquette at an edge of C by a free 2-chain on $\partial_2\Lambda^\circ$ with the same boundary, which exists because that boundary lies on $\partial\Lambda^\circ$ (Box topology (iii)) (the plaquettes of $\operatorname{supp}\tilde{\Sigma}$ at an edge off $C \cup \partial\Lambda^\circ$ are pairwise adjacent). Then (5.4) passes to subsets, and $a_\Sigma \geq RT$ follows from Lemma 5.2. Since $(\mathbb{X}, S)$ is a function of D by Theorem 3.20(R1), so is Σ°.

Proposition 5.5. *For a piece partition $\mathcal{Y}$, call $Y \in \mathcal{Y}$* touched *if* $\operatorname{supp}\Sigma^\circ \cap F(Y) \neq \emptyset$*, and $b \in S$* touched *if* $\operatorname{supp}\Sigma^\circ$ *meets the coarse face dual to b. Then*

$$a_\Sigma - A_{G'(D)}(C) = \#(\operatorname{supp}\Sigma^\circ \cap \mathfrak{F}) \le \bar{r}_0^2 \Big[\sum_{Y \text{ touched}} \big(2|\partial Y| + |Y|/\ell\big) + \#\{b \in S \text{ touched}\} \Big]. \quad (5.5)$$

In particular, if $\eta \in (0,1]$ and $A_{G'(D)}(C) \le (1-\eta)RT$, then the right side of (5.5) *is at least $a_\Sigma - (1-\eta)RT \ge \eta RT$.*

Proof. The identity holds as $\partial_2\Lambda^\circ \subseteq \mathfrak{F}$ and Σ° is a minimizer, and $a_\Sigma \ge RT$ by Lemma 5.4. Off $\partial_2\Lambda^\circ$ the free set is $F(\mathbb{X}) \sqcup F_S$, and we bound the contributions of the two sets in turn. Each touched bond of S contributes at most $\bar{r}_0^2$ plaquettes. A face of Ξ carrying a plaquette of $\operatorname{int} F(Y)$ has both its sites in Y. Its coarse column perpendicular to the plane of C, which is disjoint from those of the other faces, therefore either lies in Y or contains a bond of ∂Y, and in the first case it has $n_3 \ge \ell$ sites, in view of Lemma 3.3(c). Consequently, $|\operatorname{supp}\Xi \cap \operatorname{int} F(Y)| \le \bar{r}_0^2(|\partial Y| + |Y|/\ell)$, and (5.4) and (5.3) show that a touched Y contributes at most $\bar{r}_0^2(2|\partial Y| + |Y|/\ell)$.

5.3 The Phase-1 Peierls estimate

In this subsection we work with the Phase-1 output $(\mathbb{X}^1, \mathbb{U}^1)$, which has no frozen bond, take $\mathcal{P}$, $r' = 2R_* + 2$ and the pieces Y_P, $P \in \mathcal{P}$, from Theorem 3.21, and compute $\mathfrak{F}^1$, $A^1 := A_{G'(D)}(C)$ and Σ° for $(\mathbb{X}^1, \emptyset)$. For a plaquette set Ξ', let $Z(\Xi')$ be the set of sites with a fine cube having a face in Ξ'. Then $|Z(\Xi')| \le 2|\Xi'|$, as a plaquette is a face of at most two fine cubes. We also write $Z(\Sigma^\circ) := Z(\operatorname{supp}\Sigma^\circ)$. A part P is *touched* if Y_P is.

Lemma 5.6. $\#(\operatorname{supp}\Sigma^\circ \cap \mathfrak{F}^1) \le \bar{r}_0^2 \sum_{P \text{ touched}}(2|\partial Y_P| + |Y_P|/\ell) \le \bar{r}_0^2 C_{\text{free}} \sum_{P \text{ touched}} N(P)$*, and every touched part satisfies* $\operatorname{dist}(P, Z(\Sigma^\circ)) \le R_*$.

Proof. The two inequalities follow from Proposition 5.5, applied to $\mathcal{Y}^1 := \{Y_P\}$, and from Theorem 3.21(a). For the last assertion, note that a touched plaquette is a face of a fine cube of a site of Y_P. Thus $Z(\Sigma^\circ)$ meets $Y_P \subseteq P^{+R_*}$ (Theorem 3.21(b)).

Lemma 5.7 (Marks)**.** *Put $q_m := 3(\bar{r}_0 + 1)^3 \delta^{1/2} \le 24 r_0^3 \delta^{1/2}$. For every finite set W of sites, $\mathbb{P}_D(W \subseteq \mathbb{M}) \le q_m^{|W|}$.*

Proof. A plaquette marks at most two sites, and a site has at most $3(\bar{r}_0 + 1)\bar{r}_0^2 \le 3(\bar{r}_0 + 1)^3$ plaquettes on its fine cubes. If $W \subseteq \mathbb{M}$, a choice for each $x \in W$ of such a plaquette in D takes at least $|W|/2$ distinct values, and there are at most $(3(\bar{r}_0 + 1)^3)^{|W|}$ choices.

Lemma 5.8 (Animals)**.** *In a graph of maximal degree $D_{\max}$, the number of connected vertex sets of size n containing a given vertex is at most $(eD_{\max})^{n-1}$. The number of plaquette sets $\Xi' \subseteq \Lambda_2^\circ$ with $|\Xi'| = a$ each of whose components (for edge adjacency) contains a plaquette incident to an edge of C is at most $2^{4|C|}(2\kappa_{\text{pl}})^a$, with $\kappa_{\text{pl}} := 12e$.*

Proof. Section S2.3 gives both counts in more detail. By [GK04, Lemma 2.1 and the sentence after it, p. 141] the first number is at most $\binom{Dn}{n-1}/n \le (eD)^{n-1}$, where $D := D_{\max}$ denotes the maximal degree of the graph. For the second, we choose the first plaquette at an edge of C of each component, the sizes and the components. There are $2^{4|C|}$ choices for the first plaquettes and 2^a for the sizes, and the components are counted by the first count with $D_{\max} = 12$.

We put $\varepsilon_c := 8e\,(2R_*+1)^3(2r'+1)^3 q_m$.

Proposition 5.9. *Assume $\varepsilon_c \le \frac12$. For every finite set Z of sites and every $u_0 > 0$, $\mathbb{P}_D\big(\sum_{P\in\mathcal{P}:\ \mathrm{dist}(P,Z)\le R_*} N(P) \ge u_0\big) \le 2^{|Z|+1}\varepsilon_c^{u_0}$.*

Proof. Since the partition is random, we take the union bound over all candidate families of mark sets, not over the partition, and we use only that parts are pairwise disjoint subsets of $\mathbb{M}$, r'-connected by definition (the graph on P joining sites at distance at most r' is connected, see Theorem 3.21). Thus the event lies in the union, over $u \ge u_0$ and all families $(P_1,\dots,P_m)$ of pairwise disjoint r'-connected site sets with $\mathrm{dist}(P_i,Z)\le R_*$ and $\sum_i |P_i| = u$, of the deterministic events $\{P_1\cup\dots\cup P_m \subseteq \mathbb{M}\}$, each of which has probability at most q_m^u by Lemma 5.7. A family is fixed by the multiset of roots $z_i \in Z$ within R_* of P_i, the sizes, marks $y_i \in P_i$ within R_* of z_i, and the sets $P_i \ni y_i$. These sets are connected in a graph of degree less than $(2r'+1)^3$, and Lemma 5.8 bounds their number. There are $2^{|Z|+m}$ values for the multiset of roots, 2^u for the sizes and $(2R_*+1)^3$ for each mark. Summing over $m \le u$, we bound the probability by $2^{|Z|}\sum_{u\ge u_0}\varepsilon_c^u \le 2^{|Z|+1}\varepsilon_c^{u_0}$.

Lemma 5.10. *Let $k \ge 1$ and $j \in \{0,1,2,3\}$ be integers, put $c_T(k) := 1 + \log(2^{k+1}\kappa_{\mathrm{pl}})$ and $c_T := c_T(2) = 2 + \log 96$, and let $\eta \in (0,1]$, $\varepsilon \in (0,\frac12]$ and $F_* \ge 1$. Let E be an event with $E \subseteq \bigcup_{\Xi'} E_{\Xi'}$, the union over the plaquette sets Ξ' of Lemma 5.8 with $|\Xi'| \ge RT$, where $\mathbb{P}_D(E_{\Xi'}) \le 2^{k|\Xi'|+j}\varepsilon^{(|\Xi'|-(1-\eta)RT)/F_*}$. If $\eta|\log\varepsilon|/F_* \ge c_T(k)$, then $\mathbb{P}_D(E) \le 2^{4|C|+j+2}e^{-RT}$, and every X with $0 \le X \le RT$ and $X < \eta RT$ off E satisfies $\mathbb{E}_D e^{\sigma X} \le e^{\sigma\eta RT+4|C|+4}$ for every $\sigma \in (0,1]$.*

Proof. With $\lambda := \log(2^{k+1}\kappa_{\mathrm{pl}})$ and $\omega := |\log\varepsilon|/F_*$, Lemma 5.8 bounds $\mathbb{P}_D(E)$ by $2^{4|C|+j}$ times a geometric series in $a \ge RT$ of ratio $e^{-(\omega-\lambda)} \le e^{-1}$, and the hypothesis reads $\eta\omega \ge 1+\lambda$. The moment bound then follows from $0 \le X \le RT$, as we show in Section S2.3.

Proposition 5.11 (Phase-1 output)**.** *Assume $\varepsilon_c \le \frac12$ and $\eta_T^{(1)} := c_T\bar r_0^2 C_{\mathrm{free}}/|\log\varepsilon_c| \le 1$. Then for every rectangle C with $\min(R,T) \ge R_1(\beta)$ and every $\sigma \in (0,1]$, the two bounds of* (5.1) *hold with A^1, $\eta_T^{(1)}$ and $2^{4|C|+3}$ in place of $A_{G'(D)}(C)$, η_T and $2^{4|C|+5}$.*

Proof. We argue as in the proof of Theorem 5.1, without block families. On $E := \{A^1 \le (1-\eta_T^{(1)})RT\}$, Proposition 5.5 with $\mathcal{Y}^1$ and Lemma 5.6 give $\sum N(P) \ge u_0(|\Xi'|) := (|\Xi'| - (1-\eta_T^{(1)})RT)/(\bar r_0^2 C_{\mathrm{free}})$ over the parts within R_* of $Z(\Xi')$, where $\Xi' := \mathrm{supp}\,\Sigma^\circ$ is a function of D by Lemma 5.4. For each deterministic Ξ', Proposition 5.9 bounds the probability of this event by $2^{2|\Xi'|+1}\varepsilon_c^{u_0(|\Xi'|)}$. Lemma 5.10, applied with $k=2$, $j=1$, $\varepsilon=\varepsilon_c$, $F_* = \bar r_0^2 C_{\mathrm{free}}$ and $X := RT - A^1 \in [0,RT]$ (by Lemma 5.2), yields both bounds.

Lemma 5.12 (Threshold K_*)**.** *For $K > K_* = 377.38$ there is a β-independent $\gamma_T(K) > 0$ with*

$$|\log\varepsilon_c| \ge \tau\Big(\frac{c_\delta(K)}{2K} - 14.9662\Big) - O(\log\tau + \log r_0) \ge \gamma_T(K)\,\tau - O(\log\tau+\log r_0), \tag{5.6}$$

and (5.6) *holds with ε'' of* (5.8) *in place of ε_c.*

Proof. One mark costs $\delta^{1/2} = e^{-\tau c_\delta(K)/(2K)}$, and the root and part entropies of Proposition 5.9 cost $e^{6\log R_* + O(\log r_0)} \le e^{14.9662\,\tau + O(\log\tau+\log r_0)}$, since $6\pi^2 v_{\mathrm{Cb}}(0) < 14.9662$. For the second inequality we use the lower bound on $c_\delta(K)$ of Lemma 2.6 rather than its asymptotic form, and we take $\gamma_T(K) := (c_\delta^{\mathrm{sp}}(K) - 1/(4a_*))/(2K) - 14.9662$, which vanishes at $K = 377.3635$ by the computation in Section S1.4.

5.4 Phase 2: separators and leaves

In Sections 5.4 to 5.6 we bound the Phase-2 cut bonds by the insulation around them. Since a bond can lie next to many cuts, the main tool is the Leaf Lemma below: n cuts at one place and scale have $n+1$ pairwise disjoint sides, each of which crosses an annulus around that place and carries insulation of its own there.

Setting. The cuts j, their times t_j, components $\mathfrak{K}_j$ and sides U_j are those of Section 4.4. We put $U'_j := \mathfrak{K}_j \setminus U_j$, $\mathrm{cut}_j := \partial_{G'_{t_j}} U_j$ and $A_j := |\mathrm{cut}_j|$. At a cut time the state is hull-closed without auxiliary material, so (S-face) and (S-plug) hold (Theorem 4.16) and each cell at time t_j lies in one block and has at least $c_1 L_c^3$ sites (Theorem 4.20). A *c-path* is a sequence of sites, consecutive ones joined by a G'-bond or lying in a common cell. A step of a c-path moves less than $2L_c$ in each coordinate. The *box graph* is Λ° with all coarse nearest-neighbour bonds (holes and frozen bonds ignored), with box-components and box paths, and $\partial(Y, W)$ denotes the set of box bonds from Y to W. Bonds are *square-adjacent* if they are edges of a common unit square. The distance of two bonds is the least ℓ^∞ distance of their ends, and a family of bond sets is *k_P-linked* if the graph joining members with bonds at distance at most k_P is connected.

Definition 5.13 (Separators)**.** Fix a cut j and write $U := U_j$, $U' := U'_j$. Let U^α be the box-components of U, and $\mathcal{W}(\alpha)$ the set of box-components W of $\Lambda^\circ \setminus U^\alpha$ with $W \cap U' \neq \emptyset$. Put $\mathfrak{S}_j^* := \bigcup_\alpha \bigcup_{W \in \mathcal{W}(\alpha)} \partial(U^\alpha, W)$, and call each $\partial(U^\alpha, W)$ a *piece* of $\mathfrak{S}_j^*$. Define $\mathfrak{S}_j^{**}$ and its pieces in the same way with U and U' exchanged, and put $\mathfrak{S}_j := \mathfrak{S}_j^* \cup \mathfrak{S}_j^{**}$.

Lemma 5.14 (Separators)**.** *For every cut j: (a) every bond of $\mathfrak{S}_j^*$ joins a site of U to a site outside U and is a cut bond of j or insulating at time t_j (a bond of $\partial\mathbb{X}_{t_j}$ or of S_{t_j}); the same holds for $\mathfrak{S}_j^{**}$ with U'; (b) every piece is square-connected; (c) every piece contains a cut bond of j, distinct pieces of $\mathfrak{S}_j^*$ are disjoint, and in particular $\mathfrak{S}_j^*$ has at most A_j pieces; the same holds for $\mathfrak{S}_j^{**}$; (d) every cut bond of j lies in a piece of $\mathfrak{S}_j^*$ and in a piece of $\mathfrak{S}_j^{**}$; (e) every box path from U to U' contains a bond of $\mathfrak{S}_j^*$ and a bond of $\mathfrak{S}_j^{**}$; (f) the pieces of $\mathfrak{S}_j$ are k_P-linked.*

Proof. Items (a) and (c) to (e) are elementary and are proved in Section S2.3. The separators are defined in the box graph, and the argument stays in the box graph except for (a) and the paths of (c) and (f).

(b) The set $\Pi := \partial(U^\alpha, W)$ is a minimal set of box bonds separating a site $y \in U^\alpha$ from a site $x \in W$. It separates, because each box neighbour of W outside W lies in U^α. It is minimal, because for $e = (u, v) \in \Pi$, $u \in U^\alpha$, $v \in W$, a box path inside U^α from y to u, then e, then a box path inside W from v to x, meets Π only in e. The unit squares generate the cycle space of the box graph over $\mathbb{Z}_2$ (Box topology (iv), Section 5.2). We use Timár's lemma only in the box graph, since in the slit graph the unit squares need not generate the cycle space. By [Tim13, Lemma 1], for a minimal separating edge set Π and a generating set $\mathcal{C}$ of the cycle space, each partition (Π_1, Π_2) of Π admits a cycle of $\mathcal{C}$ meeting both Π_1 and Π_2. Taking for $\mathcal{C}$ the unit squares, we obtain the square-connectedness of Π.

(f) A cell is $\Omega = Q \cap \hat{\mathfrak{K}}$ with $\hat{\mathfrak{K}}$ a component of $G'_{t_j} \cap \hat{Q}(Q)$. In particular, two sites of one cell are joined by a G'-path inside $\hat{Q}(Q)$, which by (G4) is a box of sides less than k_P, and this path lies in $\mathfrak{K}_j$ if the cell does. Bonds with all ends in $\hat{Q}(Q)$ are at distance at most $k_P - 1$. A G'_{t_j}-path in $\mathfrak{K}_j$ enters or leaves U^α only through cut bonds, and such a bond with outer end v lies in the piece $\partial(U^\alpha, W)$, where $W \ni v$ is a box-component of $\Lambda^\circ \setminus U^\alpha$. Here $v \in \mathfrak{K}_j \setminus U = U'$, as a site of U box-adjacent to U^α lies in U^α.

We first link the pieces with the same α and different W. A G'-bond inside U' joins box-adjacent sites outside U^α and therefore stays in one box-component of $\Lambda^\circ \setminus U^\alpha$. For $W_1 \neq W_2$ in $\mathcal{W}(\alpha)$ we take a c-path in U' from $W_1 \cap U'$ to $W_2 \cap U'$, which exists because U' is c-connected. Along it the box-component of $\Lambda^\circ \setminus U^\alpha$ changes only at same-cell steps, between two sites of a cell $\Omega' \subseteq U'$ lying in components $W \neq \tilde{W}$. The G'-path inside $\hat{Q}$ joining them is a box path from W to $\tilde{W}$ and hence meets U^α. Its first entry crosses a cut bond of $\partial(U^\alpha, W)$ and its last exit one of $\partial(U^\alpha, \tilde{W})$, since the path stays in W before the entry and in $\tilde{W}$ after the exit. As both bonds lie in $\hat{Q}$, these two pieces contain bonds at distance at most k_P. Following the c-path, we conclude that all pieces $\partial(U^\alpha, W)$, $W \in \mathcal{W}(\alpha)$, are linked.

We now turn to pieces with different α. A G'-bond inside U stays in one U^α. Thus, along a c-path in U, the box-component changes only at same-cell steps between sites of a cell $\Omega \subseteq U$ lying in $U^{\alpha_1} \neq U^{\alpha_2}$. The G'-path inside $\hat{Q}$ joining them leaves U^{α_1} and enters U^{α_2} through cut bonds of pieces of α_1 and of α_2, both in $\hat{Q}$. Each α has a piece, since the box-component of $\Lambda^\circ \setminus U^\alpha$ containing a site of U' lies in $\mathcal{W}(\alpha)$. Combining this with the first case, we find that all pieces of $\mathfrak{S}_j^*$ are linked. The same holds for the pieces of $\mathfrak{S}_j^{**}$, with U and U' exchanged, and by (d) a cut bond lies in a piece of each. Hence all pieces of $\mathfrak{S}_j$ are linked.

For a site x and $r > 0$, $B_\infty(x, r)$ is the set of bonds with both ends within ℓ^∞ distance r of x. Put $c_{\mathrm{sh}} := c_1/216$, and let $\mathrm{side}_{\min}$ be the least coarse side of Λ°.

Lemma 5.15 (Admissible cuts)**.** *For a site x of Λ° and $27L_c \leq \rho \leq \mathrm{side}_{\min}/24$, a cut j is* admissible at (x, ρ) *if (i) a cut bond of j has both ends within distance $\rho/2+2$ of x; (ii) U_j and U_j' (at time t_j) each contain a site farther than 9ρ from x; (iii) $|\mathrm{cut}_j \cap B_\infty(x, 9\rho)| \leq c_{\mathrm{sh}}\rho L_c/2$. If j is admissible at (x, ρ), then $B_\infty(x, 9\rho)$ contains at least $c_{\mathrm{sh}}\rho L_c$ bonds of each of $\mathfrak{S}_j^*$ and $\mathfrak{S}_j^{**}$.*

Proof. Both sides of the cut cross about $\rho/(9L_c)$ disjoint shells around x, and in each shell the separator, which by Lemma 5.14(e) contains all box bonds between the sides it separates, is large by a box Cheeger inequality. We carry this out briefly here, and in full in Section S2.3. In a box $B = [n_1] \times [n_2] \times [n_3] \subset \mathbb{Z}^3$ (shell boxes need not be cubes), with $n := \max_i n_i$, each $A \subseteq B$ with $|A| \leq |B|/2$ has at least $|A|/(2n)$ bonds to $B \setminus A$. To see this, we route all ordered pairs along coordinate-ordered paths. At most $n|B|$ of them use a given bond, and at least $|A|\,|B|/2$ join A to $B \setminus A$. The *shell boxes* form a tensor-product partition of Λ° with sides in $[6L_c, 12L_c)$, and the shell Sh_i is the union of the boxes at index distance $3i$, $3i+1$ or $3i+2$ from the box of x. A c-path moves less than $2L_c$ per step and cells lie in one block. It follows that each of the at least $\rho/(9L_c)$ disjoint, face-connected shells Sh_i with $\rho/(36L_c) + 1 \leq i \leq \rho/(4L_c) - 1$ lies within 9ρ of x and contains a whole cell of U_j and one of U_j', by (i) and (ii). In each shell, the Cheeger inequality in one shell box or in the union of two adjacent ones gives $(c_1/24)L_c^2$ bonds of $\mathfrak{S}_j^*$ and as many of $\mathfrak{S}_j^{**}$. Summing over the shells, we obtain $(\rho/(9L_c))(c_1/24)L_c^2 = c_{\mathrm{sh}}\rho L_c$ bonds of each.

Lemma 5.16 (Leaf Lemma)**.** *Let $n \geq 1$ and let $j_1, \ldots, j_n$ be distinct cuts admissible at (x, ρ). Let I be the set of bonds $b \in B_\infty(x, 9\rho)$ such that for some i, $b \in \mathfrak{S}_{j_i}$, b is not a cut bond of j_i, and b is insulating at time t_{j_i}. Then $|I| \geq (n+1)\, c_{\mathrm{sh}}\rho L_c/4$.*

Proof. The $2n$ sides U_{j_i}, U'_{j_i} (at the respective times) are pairwise disjoint or nested (Lemma 4.27(b)) and distinct, since the component of a later cut lies in one side of an earlier cut or outside its component. Hence the minimal sides, the *leaves*, are pairwise disjoint. We

show that there are at least $n+1$ of them by adding the cuts in firing order. When a cut j is added, its two sides are leaves, since an earlier side contains $\mathfrak{K}_j$ or is disjoint from it. An earlier leaf stops being one only if it contains $\mathfrak{K}_j$, while the earlier sides containing $\mathfrak{K}_j$ form a chain, with at most one minimal element. Thus each cut adds two leaves and removes at most one. For a leaf $L = U_j$ (resp. U'_j), Lemma 5.15 gives at least $c_{\rm sh}\rho L_c$ bonds of $\mathfrak{S}^*_j$ (resp. $\mathfrak{S}^{**}_j$) in $B_\infty(x, 9\rho)$, of which at most $c_{\rm sh}\rho L_c/2$ are cut bonds of j by Lemma 5.15(iii), while Lemma 5.14(a) shows that the others are insulating at t_j and have an end in L, so that they lie in I. Since a bond has two ends and the leaves are disjoint, a bond of I is counted for at most two leaves.

5.5 Phase 2: charges and domination

We use that a cut fires only on a fragmented split, whose scale cost exceeds its area. We spread the area of the cut over its thin clusters and, using the Leaf Lemma, charge each thin cluster to insulation of its own, at a rate ε_D that is doubly exponentially small.

Scales. For the scales $s = 2^j L \geq R_{\rm NF}$ of Definition 3.18, which are lattice lengths, we put $\rho(s) := s/(128\bar{r}_0)$, a length in coarse units. Points of two sites at coarse distance d are at lattice Euclidean distance at most $\sqrt{3}\,\bar{r}_0(d+1)$. Consequently, (U1) two cut bonds whose ends are pairwise within coarse distance d lie in one s-cluster if $\sqrt{3}\,\bar{r}_0(d+1) \leq s$, and (U2) cut bonds of different s-clusters have all their ends at coarse distance greater than $s/(\sqrt{3}\bar{r}_0) - 1 > 73\rho(s) - 1$. The *charge boxes* at scale s partition the coarse lattice into cubes Q of side $\lceil \rho(s) \rceil$ with centres x_Q.

Thin clusters, good scales. Fix a cut j and a scale s. An s-cluster c of cut_j is *thin* if it contains a bond with an end x_c such that $|c \cap B_\infty(x_c, 10\rho(s))| \leq c_{\rm sh}\rho(s)L_c/2$, and *fat* otherwise. For a thin cluster we fix such a bond and end x_c. The scale s is *good* for j if $\varphi_s \geq 2$ and at least half of the s-clusters are thin. Let $\varphi_s^{\rm thin}$ be the number of thin s-clusters.

Lemma 5.17 (Charges)**.** *Assume $R_{\rm NF} \geq \max\bigl(3456\bar{r}_0 L_c,\ 2048\bar{r}_0 L_c/(c_{\rm sh}\theta_{\rm NF}F_{\rm unit})\bigr)$. For every cut j: (a) $J'_j := \sum_{s\ \rm good}(\varphi_s - 1) \geq J(U_j)/2$; (b) with $q_j(s,c) := A_j(\varphi_s - 1)/(J'_j\varphi_s^{\rm thin})$ (s good, c a thin s-cluster), $\sum_{s\ \rm good}\sum_{c\ \rm thin} q_j(s,c) = A_j$ and $q_j(s,c) < 4L_c^2/(\theta_{\rm NF}F_{\rm unit})$; (c) for good s and thin c, with $Q \ni x_c$ the charge box, j is admissible at $(x_Q, \rho(s))$ and $x_{c'} \notin Q$ for every other thin s-cluster c'.*

Proof. (a) A fat cluster has more than $c_{\rm sh}\rho L_c/2$ bonds, $\rho = \rho(s)$, and clusters are disjoint subsets of cut_j. Consequently, at a scale with $\varphi_s \geq 2$ that is not good, $\varphi_s - 1 < 2\varphi_s^{\rm fat} < 4A_j/(c_{\rm sh}\rho L_c) = 512\bar{r}_0 A_j/(c_{\rm sh} s L_c)$. Summing over dyadic $s \geq R_{\rm NF}$ gives less than $1024\bar{r}_0 A_j/(c_{\rm sh}R_{\rm NF}L_c) \leq J(U_j)/2$, by the assumption on $R_{\rm NF}$ and the inequality $J(U_j) > \theta_{\rm NF}F_{\rm unit}A_j/L_c^2$, which holds because the split is fragmented.

(b) We have $\sum_c q_j(s,c) = A_j(\varphi_s - 1)/J'_j$, and these sums add up to A_j over good s. At a good scale $\varphi_s - 1 < \varphi_s \leq 2\varphi_s^{\rm thin}$, so $q_j(s,c) < 2A_j/J'_j \leq 4A_j/J(U_j) < 4L_c^2/(\theta_{\rm NF}F_{\rm unit})$.

(c) Take for x_Q a central site of $Q \cap \Lambda^\circ$ (nonempty, as it contains x_c; for instance the centre of Q moved coordinatewise into Λ°), so that $x_Q \in Q$ and $|x_c - x_Q|_\infty \leq \lceil\rho\rceil/2 \leq \rho/2 + 1$, $\rho = \rho(s)$. The numerical inequalities used in (c), and the range $27L_c \leq \rho \leq \mathrm{side}_{\min}/24$ of Lemma 5.15, are established in Section S1.4. (i) The chosen bond of c has both ends within $\rho/2 + 2$ of x_Q. (ii) Since $\varphi_s \geq 2$, there is a cut bond in another s-cluster, with one end in U_j and one in U'_j. By (U2) its ends are farther than 9ρ from x_Q. (iii) $B_\infty(x_Q, 9\rho) \subseteq B_\infty(x_c, 10\rho)$, and by (U1) all cut bonds of j in $B_\infty(x_c, 10\rho)$ lie in c, which is thin. Finally, if $x_c, x_{c'} \in Q$, the two chosen bonds lie in one cluster by (U1).

Classes. For a bond set Y, $\mathfrak{B}(Y)$ is the set of blocks containing an end of a bond of Y. The *classes* $\mathcal{J}$ are the connected components of the graph on cuts with $j \sim j'$ if $\mathfrak{B}(\mathfrak{S}_j) \cap \mathfrak{B}(\mathfrak{S}_{j'}) \neq \emptyset$. We put $\mathfrak{X}(\mathcal{J}) := \bigcup_{j\in\mathcal{J}} \mathfrak{S}_j$, $\mathfrak{B}(\mathcal{J}) := \mathfrak{B}(\mathfrak{X}(\mathcal{J}))$ and $A(\mathcal{J}) := \sum_{j\in\mathcal{J}} A_j$. Let S_{cut} be the set of all bonds frozen by the cut rule and $S_1^{(2)}$ the set of bonds frozen by seal steps in Phase 2, and put $\mathrm{ins}^1 := \partial\mathbb{X}^1$. The *payable bonds* form $B_{\mathrm{pay}} := \mathrm{ins}^1 \cup S_{\mathrm{cut}} \cup S_1^{(2)} \cup \partial\mathbb{X}_{\mathrm{final}} \cup \bigcup_\Delta \partial\Delta$, the last union over the cells removed in Phase 2. We denote by m_q the number of marks of a block q, and by $h(\mathfrak{B})$ the number of sites in the blocks of $\mathfrak{B}$ that are hull material at some time of Phase 2. By Proposition 4.3 a bond is frozen at most once, so each cut bond belongs to exactly one cut.

Lemma 5.18 (Classes)**.** *(a) Distinct classes have disjoint $\mathfrak{X}$'s and disjoint $\mathfrak{B}$'s. (b) The cut bonds of every $j \in \mathcal{J}$ lie in $\mathfrak{X}(\mathcal{J})$, and every cut bond lying in $\mathfrak{X}(\mathcal{J})$ belongs to a cut of $\mathcal{J}$. Moreover, $|\mathfrak{X}(\mathcal{J}) \cap S_{\mathrm{cut}}| = A(\mathcal{J})$. (c) $\mathfrak{B}(\mathcal{J})$ is a union of at most $2A(\mathcal{J})$ sets, each $*$-connected in the block lattice (26-adjacency), which are linked at block distance at most $k_b := \lceil k_P/L_c \rceil + 2$. (d) Every bond of $\mathfrak{X}(\mathcal{J}) \setminus B_{\mathrm{pay}}$ is, at some time of Phase 2, a hole face of a site $y \notin \mathbb{X}^1$ that is hull material at that time and lies in a block of $\mathfrak{B}(\mathcal{J})$. So $|\mathfrak{X}(\mathcal{J}) \setminus B_{\mathrm{pay}}| \le 6\, h(\mathfrak{B}(\mathcal{J})) \le 12 L_c \sum_{q\in\mathfrak{B}(\mathcal{J})} m_q$.*

Proof. (a) A common bond or block relates two cuts.

(b) The first claim follows from Lemma 5.14(d). For the second, a cut bond of j'' in $\mathfrak{S}_j$ lies in $\mathfrak{S}_{j''}$, so $j'' \sim j$. The equality in (b) follows from the two claims.

(c) In view of Lemma 5.14(c), the set $\mathfrak{S}_j$ has at most $2A_j$ pieces, and Lemma 5.14(b), (f) shows that they are square-connected, hence with $*$-connected blocks, and k_P-linked. Bonds at distance $\le k_P$ have blocks at block distance $\le k_P/L_c + 1 \le k_b$, and related cuts share a block, which gives the linking asserted in (c).

(d) Let $b \in \mathfrak{S}_j \setminus B_{\mathrm{pay}}$, $j \in \mathcal{J}$. Cut bonds and bonds frozen at t_j lie in $S_{\mathrm{cut}} \cup S_1^{(2)} \subseteq B_{\mathrm{pay}}$, since by Theorem 4.17 the bonds frozen in Phase 1 are converted. It follows by Lemma 5.14(a) that $b = (y,u)$ is a hole face at t_j, with $y \in \mathbb{X}_{t_j}$ and $u \in \mathfrak{K}_j \subseteq G'_{t_j}$. If $y \in \mathbb{X}^1$, then $b \in \partial\mathbb{X}^1 = \mathrm{ins}^1$, as $u \in G'_{t_j} \subseteq G'_{\mathbb{X}^1}$, where the inclusion holds because G' only shrinks. If y lies in a cell Δ removed in Phase 2 before t_j, then $b \in \partial\Delta$, as $u \notin \Delta \subseteq \mathbb{X}_{t_j}$. Otherwise $y \notin \mathbb{X}^1 \cup R_{t_j} \supseteq \mathbb{M} \cup \mathbb{U}^1 \cup R_{t_j} = B_{t_j}$, where R_t is as in Section 4.4 and we use that Phase 2 fills only by removal. Hence, in this case, y is hull material at t_j, also if it lies in a cell removed after t_j. Its block contains an end of b and hence lies in $\mathfrak{B}(\mathcal{J})$. Since a site carries six bonds, this gives the first inequality in (d). For the last bound, let $q \not\subseteq \mathbb{U}^1$ be a block. If its removal cell is never removed, the hull material $(\mathbb{X}_t \setminus B_t) \cap q = (\mathbb{X}_t \cap q) \setminus \mathbb{M}$ only grows in t. If the cell is removed at time t_Δ, all hull material of q before t_Δ is hull material just before t_Δ, because removal cells are unions of blocks by Lemma 3.3(c). Corollary 4.7 at the final time, or just before the removal, gives fewer than $2L_c m_q$ sites per block.

We call a hole face of (d) *transient* if its other end is absorbed later. There can be L_c times as many transient faces as local sources. We absorb them into ε_D and attach them to the contour only through blocks.

Proposition 5.19. *Under the assumption on R_{NF} of Lemma 5.17, for every set $\mathcal{J}$ of cuts, $A(\mathcal{J}) \le \varepsilon_D |\mathfrak{X}(\mathcal{J})|$ with $\varepsilon_D := 3.2768 \cdot 10^7 \bar{r}_0 L_c/(c_{\mathrm{sh}}\theta_{\mathrm{NF}} F_{\mathrm{unit}} R_{\mathrm{NF}})$, and if $\mathcal{J}$ is a class and $\varepsilon_D \le \frac{1}{2}$,*

$$A(\mathcal{J}) \le 2\varepsilon_D \Big(|\mathfrak{X}(\mathcal{J}) \cap B_{\mathrm{pay}} \setminus S_{\mathrm{cut}}| + 12 L_c \sum_{q\in\mathfrak{B}(\mathcal{J})} m_q \Big). \tag{5.7}$$

Proof. We assign the charge $q_j(s,c)$ of Lemma 5.17 to the pair (Q,s) with $Q \ni x_c$. Fix (Q,s) and let $j_1,\dots,j_n$ be the cuts of $\mathcal{J}$ with a charge at (Q,s). By Lemma 5.17(b), (c), each of them has exactly one charge there, of size less than $4L_c^2/(\theta_{\mathrm{NF}}F_{\mathrm{unit}})$, and is admissible at $(x_Q,\rho(s))$. The Leaf Lemma gives $I_Q \subseteq B_\infty(x_Q,9\rho(s)) \cap \mathfrak{X}(\mathcal{J})$ with $|I_Q| \geq (n+1)c_{\mathrm{sh}}\rho(s)L_c/4$. Hence the charge at (Q,s) is at most $4nL_c^2/(\theta_{\mathrm{NF}}F_{\mathrm{unit}}) \leq 16L_c\,|\mathfrak{X}(\mathcal{J}) \cap B_\infty(x_Q,9\rho(s))|/(c_{\mathrm{sh}}\theta_{\mathrm{NF}}F_{\mathrm{unit}}\rho(s))$. A bond lies in $B_\infty(x_Q,9\rho(s))$ for at most 20^3 charge boxes Q of one scale, and $\sum_s 1/\rho(s) \leq 256\bar{r}_0/R_{\mathrm{NF}}$ over dyadic $s \geq R_{\mathrm{NF}}$. Summing over (Q,s), we obtain $A(\mathcal{J}) \leq 20^3 \cdot 16 \cdot 256\,\bar{r}_0 L_c|\mathfrak{X}(\mathcal{J})|/(c_{\mathrm{sh}}\theta_{\mathrm{NF}}F_{\mathrm{unit}}R_{\mathrm{NF}}) = \varepsilon_D|\mathfrak{X}(\mathcal{J})|$. For (5.7), we write $\mathfrak{X}(\mathcal{J})$ as the disjoint union of $\mathfrak{X}\cap S_{\mathrm{cut}}$, $\mathfrak{X}\cap B_{\mathrm{pay}}\setminus S_{\mathrm{cut}}$ and $\mathfrak{X}\setminus B_{\mathrm{pay}}$. The first has $A(\mathcal{J})$ elements and the last is bounded by Lemma 5.18(b), (d). Thus $A \leq \varepsilon_D(A+R)$ with R the bracket of (5.7), and $A \leq \varepsilon_D R/(1-\varepsilon_D) \leq 2\varepsilon_D R$.

5.6 Phase 2: responses

Cuts change G' only on their cut bonds, which cross block faces and carry the hull weight $2-\eta_w$. The hull, removal and seal steps therefore respond to cuts as they respond to marks, and we bound the responses locally by the marks, the cut bonds and the bonds of $\mathrm{ins}^1 = \partial\mathbb{X}^1$. Since each frozen bond is a grid bond and supports no hull material (Corollary 4.7), a response creates no new source, so the bound on the responses need not be iterated. Section S1.4 collects the constants of this subsection and the next, with the arithmetic behind them. Let $m(Z)$ be the number of marks in Z, and let $n_{\mathrm{cut}}(Z)$ and $n_{\mathrm{seal}}(Z)$ be the numbers of cut bonds and of bonds frozen by seal steps in Phase 2 (converted ones included) with both ends in Z. For a removal cell Δ_0, $\mathcal{N}(\Delta_0)$ is the set of the at most 27 removal cells within index distance 1 of Δ_0, with union $U(\Delta_0)$, and $\mathcal{N}_2(\Delta_0)$ is the set of the at most 125 removal cells within index distance 2 of Δ_0, with union $U_2(\Delta_0)$.

Response units. These are (1) the bonds of $S_1^{(2)}$, (2) the bonds of $\partial\mathbb{X}_{\mathrm{final}} \setminus \mathrm{ins}^1$, that is, the hole faces created in Phase 2 and present at the end, and (3) the pairs (Δ,b) with Δ a cell removed in Phase 2 and $b \in \partial\Delta$, and transient hole faces are not units. Each unit is placed at its bond, and units(Z) denotes the number of units with an end in Z. We assign each unit to the removal cell of the first of its ends in Λ°, in a fixed order (a bond of a unit of type (3) may leave Λ°). Each bond of $B_{\mathrm{pay}} \setminus S_{\mathrm{cut}}$ lies in ins^1 or is the bond of a unit.

Lemma 5.20 (Units)**.** *For every removal cell Δ_0,* $\mathrm{units}(\Delta_0) \leq C_a\,m(U_2(\Delta_0)) + C_b\,n_{\mathrm{cut}}(U_2(\Delta_0)) + C_c\,|\mathrm{ins}^1 \cap E(U_2(\Delta_0))|$, *with* $C_a := 4362 + 5808/c_s$, $C_b := 2 + 4/c_s$ *and* $C_c := 1 + 2/c_s$.

Proof. We bound the units near Δ_0 by the seal count or, through Lemma 4.25, by comparison with the Phase-1 output. Let $U := U(\Delta_0)$, $m := m(U_2(\Delta_0))$, $n := n_{\mathrm{cut}}(U)$ and $i := |\mathrm{ins}^1 \cap E(U)|$. A seal bond with an end in a block q lies on one of the six faces of q, and by Proposition 4.15 the face shared with q' carries at most $121(m_q + m_{q'})$ of them. Summing over the blocks of Δ_0, resp. of U, we find that at most $1452\,m$ units of type (1) have an end in Δ_0 and that $n_{\mathrm{seal}}(U) \leq 1452\,m$. A unit (Δ,b) with an end in Δ_0 has $\Delta \in \mathcal{N}(\Delta_0)$, and by Lemma 4.25(b) the units of type (3) of the cells of $\mathcal{N}(\Delta_0)$ number less than $u_3 := c_s^{-1}(i + 2n_{\mathrm{seal}}(U) + 2n)$. We turn to the units of type (2). For $1 \leq r \leq L_c$ the boxes $N_r := \{x : \mathrm{dist}(x,\Delta_0) \leq r\}$ lie in U and have disjoint outer layers. A final hole face with an end in Δ_0 lies in $E(N_r)$ and has exactly one end in $\mathbb{X}_\infty$, so it is counted in $\mathcal{A}_2(E(N_r);\infty)$, which (4.4) bounds by $\mathrm{Ins}_{N_r}(\mathbb{X}^1 \cup R_\infty) + \mathrm{cut}_{N_r}$. A bond of $E(N_r)$ with exactly one end in $\mathbb{X}^1 \cup R_\infty$ is

a bond of ins^1 or the bond of a unit of type (3) of a cell of $\mathcal{N}(\Delta_0)$, and a bond of $\bar{S}_\infty \cap E(N_r)$ with no end in $\mathbb{X}^1$ was frozen in Phase 2. Hence $\mathrm{Ins}_{N_r}(\mathbb{X}^1 \cup R_\infty) \le i + u_3 + 2n_{\mathrm{seal}}(U) + 2n$. The sites counted in cut_{N_r} are hull material in the outer layer of N_r, each with at most three bonds leaving N_r, and the blocks of U hold fewer than $2L_c m$ sites of hull material (Corollary 4.7). Consequently, $\mathrm{cut}_{N_r} \le 6m$ for some r. Adding these bounds, we obtain $\mathrm{units}(\Delta_0) \le 1452m + 2u_3 + i + 2n_{\mathrm{seal}}(U) + 2n + 6m$. Inserting $n_{\mathrm{seal}}(U) \le 1452\,m$ here and in the bound on the units of type (3) gives the claim.

Lemma 5.21. *There are weights $\varpi(u,z) \ge 0$, for u a response unit and z a* source *(a mark, a cut bond or a bond of* ins^1*), with $\varpi(u,z) = 0$ if $\mathrm{dist}(u,z) > R_{\mathrm{resp}} := 10\ell$ and $\sum_z \varpi(u,z) = 1$ for every u, the weights being functions of D, such that every mark receives $\sum_u \varpi(u,z) \le 125C_a$, every cut bond at most $125C_b$, and every bond of* ins^1 *at most $125C_c$.*

Proof. Since Lemma 5.20 bounds the units of a cell by the sources around it, we spread each unit over those sources. For a unit u assigned to Δ_0 put $\varpi(u,z) := c_z / D(\Delta_0)$ for the sources z of $U_2(\Delta_0)$, that is, the marks in $U_2(\Delta_0)$, the cut bonds and the bonds of ins^1 in $E(U_2(\Delta_0))$. Here $c_z = C_a$, C_b or C_c according to the kind of z, and $D(\Delta_0)$ is the right side of Lemma 5.20. Then $\sum_z \varpi(u,z) = 1$, and $D(\Delta_0) \ge \mathrm{units}(\Delta_0) > 0$ if a unit is assigned to Δ_0. A source lies in $U_2(\Delta)$ for at most 125 cells Δ and therefore receives at most $125c_z$. A unit assigned to Δ_0 has an end in Δ_0, a source of $U_2(\Delta_0)$ lies in a cell at index distance at most 2, and cells have sides less than 3ℓ. Hence $\mathrm{dist}(u,z) \le 9\ell + 1 \le R_{\mathrm{resp}}$.

The constant $C_{\mathrm{resp}} := 375C_a + 125C_cC'_\star$ is fixed before K. A part P owns the bonds of ∂Y_P, which by Theorem 3.21(a) are at most $C'_\star N(P)$ bonds of ins^1. The marks and the ins^1 bonds of P therefore receive at most $(125C_a + 125C_cC'_\star)N(P) \le C_{\mathrm{resp}}N(P)$, and a cut bond receives at most $125C_b \le C_{\mathrm{resp}}$.

5.7 Counting and the proof of Theorem 5.1

In this subsection $(\mathbb{X}, S)$ is the Phase-2 output, with $\mathfrak{F}$ and the minimizer Σ° of Lemma 5.4 computed for it, and we write $\Xi' := \operatorname{supp}\Sigma^\circ$ and $a := a_\Sigma = |\Xi'| \ge RT$. Let $R' := R_* + 2R_{\mathrm{resp}}$ and

$$\varepsilon'' := 8e\,(2R' + 4L_c + 1)^3\,(2r' + 1)^3\,q_m. \tag{5.8}$$

The hole set only grows, as Proposition 4.3 shows, and hence $\mathbb{X}^1 \subseteq \mathbb{X}$.

As pieces we take the sets Y_P, $P \in \mathcal{P}$, of Theorem 3.21(a), the sets $Y_\Delta := \Delta \setminus \mathbb{X}^1$ for the cells Δ removed in Phase 2, which lie in $\mathbb{X}$, and the sets $Y_{\Delta'} := (\mathbb{X} \setminus \mathbb{X}^1) \cap \Delta'$ for the removal cells Δ' not removed in Phase 2. These sets partition $\mathbb{X}$. A piece is touched if Ξ' meets its $F(\cdot)$, and a part P is touched if Y_P is. The cell of a touched Y_Δ or $Y_{\Delta'}$ is a *touched cell*. Let u_{touch} be the number of response units that have an end in a touched cell or are of type (1) with a coarse face containing a plaquette of Ξ' (each unit counted once), c_{touch} the number of bonds of $S \cap S_{\mathrm{cut}}$ whose coarse face contains a plaquette of Ξ', and i_{touch} the number of bonds of ins^1 with an end in a touched cell.

Proposition 5.22. $\#(\Xi' \cap \mathfrak{F}) \le \bar{r}_0^2\big[C_{\mathrm{free}} \sum_{P\ \mathrm{touched}} N(P) + 20(u_{\mathrm{touch}} + c_{\mathrm{touch}} + i_{\mathrm{touch}}) + (26L_c/\ell) \sum_{\Delta'\ \mathrm{touched}} m(\Delta')\big]$*, the last sum over the touched cells Δ'.*

Proof. We apply Proposition 5.5 to the pieces defined above. The removal cells cut the Phase-2 hole material into pieces of diameter less than 9ℓ, each entering with its volume divided by

ℓ, and we charge the boundary of each piece to Phase-1 insulation, to response units and to its volume. A touched part costs at most $C_{\mathrm{free}}N(P)$, as in Lemma 5.6. A touched Y_Δ costs $2|\partial Y_\Delta| + |Y_\Delta|/\ell$, with $|Y_\Delta|/\ell < 27\ell^2 \le 4.5|\partial\Delta|$. A bond of ∂Y_Δ either lies in $\partial\Delta$, in which case it is the bond of a unit (Δ, b) of type (3), or joins $\mathbb{X}^1 \cap \Delta$ to $\Delta \setminus \mathbb{X}^1$, in which case it is a bond of ins^1 with both ends in Δ.

A touched $Y := Y_{\Delta'}$ costs $2|\partial Y| + |Y|/\ell$. A bond (y, v) of ∂Y, $y \in Y$, is of one of the following four types. (a) $v \in G'$, and then the bond is a final hole face with no end in $\mathbb{X}^1$, the bond of a unit of type (2). (b) $v \in \mathbb{X}^1$, so the bond lies in ins^1. (c) v lies in a cell Δ removed in Phase 2, so $(\Delta, (y, v))$ is a unit of type (3). (d) $v \in (\mathbb{X} \setminus \mathbb{X}^1) \cap \Delta''$ for another cell Δ'' not removed in Phase 2. For a bond of type (d), we walk from y along its coordinate line into Δ'. If the walk leaves Y inside Δ', it leaves through a bond from Y to $\Delta' \setminus Y \subseteq G' \cup \mathbb{X}^1$, of type (a) or (b) with both ends in Δ', and distinct walks end at distinct bonds. Otherwise the walk crosses Δ' inside Y, through at least ℓ sites, and we charge $1/\ell$ to each of them. Since a site lies on three lines, each with at most two type-(d) bonds at its ends, it receives at most $6/\ell$. Hence $|\partial Y|$ is at most the number of bonds of types (a) to (c) with an end in Y, plus those of types (a), (b) inside Δ', plus $6|Y|/\ell$. Phase 2 fills only by removal, so Y is hull material and Corollary 4.7 at the final time gives $|Y| < 2L_c m(\Delta')$.

Since the Phase-1 output has no frozen bond, a touched bond of S lies in $S_1^{(2)}$ or in $S \cap S_{\mathrm{cut}}$. In the first case it is a unit of type (1), counted in u_{touch}, and in the second it is counted in c_{touch}. Each unit or bond of ins^1 charged above has an end in the touched cell charging it, so it is counted in u_{touch} or i_{touch}, and the coefficients add up to at most 20.

The attached family. Let $\mathcal{T}$ be the set of classes with a cut bond counted in c_{touch}, and $\mathcal{F}$ the smallest set of classes containing $\mathcal{T}$ with the closure property: if $\varpi(u, z) > 0$ for a cut bond z of a class $\mathcal{J}'$, where (i) u is a unit counted in u_{touch}, or (ii) the bond of the unit u lies in $\mathfrak{X}(\mathcal{J})$ for some $\mathcal{J} \in \mathcal{F}$, then $\mathcal{J}' \in \mathcal{F}$. We put $\mathfrak{B}_\mathcal{F} := \bigsqcup_{\mathcal{J}\in\mathcal{F}} \mathfrak{B}(\mathcal{J})$ and $A_\mathcal{F} := \sum_{\mathcal{J}\in\mathcal{F}} A(\mathcal{J})$. Let $Z_\mathcal{F}$ be the set of sites of the blocks of $\mathfrak{B}_\mathcal{F}^{+1}$ (the blocks at block distance at most 1 from $\mathfrak{B}_\mathcal{F}$) and $\mathcal{Q}_{\mathrm{att}}$ the set of parts P with $\mathrm{dist}(P, Z(\Xi') \cup Z_\mathcal{F}) \le R'$, and put $N_\mathcal{Q} := \sum_{P\in\mathcal{Q}_{\mathrm{att}}} N(P)$. In Lemma 5.23 and Proposition 5.24 we assume the condition

$$8\varepsilon_D(1 + 8L_c)(C'_\star + 3C_{\mathrm{resp}} + 6)(48C_{\mathrm{resp}} + 2) \le 1, \tag{5.9}$$

which holds for $\beta \ge \beta_0(K)$, as ε_D is doubly exponentially small.

Lemma 5.23 (Attached family)**.** *(a) the units counted in u_{touch} and the units whose bond lies in $\bigcup_\mathcal{F} \mathfrak{X}(\mathcal{J})$ number at most $C_{\mathrm{resp}}(N_\mathcal{Q} + A_\mathcal{F})$ each, so at most $2C_{\mathrm{resp}}(N_\mathcal{Q} + A_\mathcal{F})$ together; (b) $A_\mathcal{F} \le 8\varepsilon_D(1 + 8L_c)(C'_\star + 3C_{\mathrm{resp}} + 6)\, N_\mathcal{Q} \le N_\mathcal{Q}/(48C_{\mathrm{resp}} + 2)$; (c) $c_{\mathrm{touch}} \le A_\mathcal{F}$, $i_{\mathrm{touch}} \le C'_\star N_\mathcal{Q}$ and $|\mathfrak{B}_\mathcal{F}| \le C_\mathfrak{B} N_\mathcal{Q}$ ($C_\mathfrak{B}$ of Section S1.4); (d) $\mathfrak{B}_\mathcal{F}$ has $m \le 2A_\mathcal{F}$ $*$-components.*

Proof. (a) A unit of the first family has an end within 3ℓ of $Z(\Xi')$, since a touched cell meets $Z(\Xi')$ and has sides less than 3ℓ, and since a unit of type (1) whose face meets Ξ' has both ends in $Z(\Xi')$. A unit of the second family has an end in $Z_\mathcal{F}$. Thus, by Lemma 5.21, a source z with $\varpi(u, z) > 0$ lies within $R_{\mathrm{resp}} + 3\ell + 1$ of $Z(\Xi') \cup Z_\mathcal{F}$. If such a source is a mark, it is a mark of a part of $\mathcal{Q}_{\mathrm{att}}$. If it is a cut bond, it belongs to a class of $\mathcal{F}$ by the closure property, and there are at most $A_\mathcal{F}$ such bonds, by Lemma 5.18(b). If it is a bond of $\mathrm{ins}^1 = \bigsqcup_P \partial Y_P$, its owner P has a site of $Y_P \subseteq P^{+R_*}$ within $R_{\mathrm{resp}} + 3\ell + 2$ of $Z(\Xi') \cup Z_\mathcal{F}$ and so lies in $\mathcal{Q}_{\mathrm{att}}$. As each unit has total weight at least 1, the loads stated after Lemma 5.21 imply that each family has at most $C_{\mathrm{resp}}N_\mathcal{Q} + 125C_b A_\mathcal{F} \le C_{\mathrm{resp}}(N_\mathcal{Q} + A_\mathcal{F})$ elements.

(b) We sum (5.7) over $\mathcal{J} \in \mathcal{F}$. The bonds of $\mathfrak{X}(\mathcal{J}) \cap B_{\mathrm{pay}} \setminus S_{\mathrm{cut}}$ are bonds of ins^1 or of units. A bond of ins^1 lies in ∂Y_P for one part P, which has an end of it in $Y_P \subseteq P^{+R_*}$, so P lies within $R_* + 1$ of $Z_{\mathcal{F}}$ and belongs to $\mathcal{Q}_{\mathrm{att}}$. A part owns at most $C'_\star N(P)$ such bonds, by Theorem 3.21(a). The units are bounded by (a), and the marks in blocks of $\mathfrak{B}_{\mathcal{F}}$ belong to parts of $\mathcal{Q}_{\mathrm{att}}$. Therefore

$$A_{\mathcal{F}} \le 2\varepsilon_D \big[C'_\star N_{\mathcal{Q}} + 2C_{\mathrm{resp}}(N_{\mathcal{Q}} + A_{\mathcal{F}}) + 12L_c N_{\mathcal{Q}}\big].$$

The coefficient of $A_{\mathcal{F}}$ on the right is $4\varepsilon_D C_{\mathrm{resp}} \le \frac{1}{2}$ by (5.9). Hence $A_{\mathcal{F}} \le 4\varepsilon_D(C'_\star + 2C_{\mathrm{resp}} + 12L_c)N_{\mathcal{Q}}$, which gives the first inequality. The second inequality is (5.9).

(c) Lemma 5.18(b) implies that a touched cut bond belongs to a cut of a class in $\mathcal{T}$, and thus $c_{\mathrm{touch}} \le A_{\mathcal{F}}$. A bond counted in i_{touch} has an end within 3ℓ of $Z(\Xi')$, so its owner lies within $R_* + 3\ell + 1 \le R'$ of $Z(\Xi')$, in $\mathcal{Q}_{\mathrm{att}}$, and Theorem 3.21(a) gives $i_{\mathrm{touch}} \le C'_\star N_{\mathcal{Q}}$. For $|\mathfrak{B}_{\mathcal{F}}|$, we charge each block of $\mathfrak{B}(\mathcal{J})$ to a bond of $\mathfrak{X}(\mathcal{J})$ with an end in it if that bond lies in B_{pay}, with at most two blocks per bond. Otherwise, by Lemma 5.18(d), the block is that of a site y of hull material or one of the six face neighbours of the block of that site, and we charge it to the first mark of the block of y, which exists by Corollary 4.7. In this way each mark receives at most seven blocks. Counting as in (a) and (b) gives $|\mathfrak{B}_{\mathcal{F}}| \le C_{\mathfrak{B}} N_{\mathcal{Q}}$.

(d) By Lemma 5.18(c), $\mathfrak{B}(\mathcal{J})$ is a union of at most $2A(\mathcal{J})$ $*$-connected sets.

Proposition 5.24. *$\#(\Xi' \cap \mathfrak{F}) \le \bar{r}_0^2 C_3 N_{\mathcal{Q}}$, $|\mathfrak{B}_{\mathcal{F}}| \le C_{\mathfrak{B}} N_{\mathcal{Q}}$ and $m \le M_m N_{\mathcal{Q}}$, with C_3, $C_{\mathfrak{B}}$ free of β, τ, K, C, D and $M_m = O(\varepsilon_D L_c C_{\mathrm{resp}})$ (Section S1.4).*

Proof. We insert Lemma 5.23 into Proposition 5.22. Touched parts lie in $\mathcal{Q}_{\mathrm{att}}$ by Lemma 5.6. By Lemma 5.23(c) and the proof of (a), $u_{\mathrm{touch}} + c_{\mathrm{touch}} + i_{\mathrm{touch}} \le (C_{\mathrm{resp}} + C'_\star)N_{\mathcal{Q}} + (C_{\mathrm{resp}} + 1)A_{\mathcal{F}}$. The marks of the touched cells belong to parts of $\mathcal{Q}_{\mathrm{att}}$, so $\sum_{\Delta'} m(\Delta') \le N_{\mathcal{Q}}$, and $26L_c/\ell \le 26/308$. Together with $A_{\mathcal{F}} \le N_{\mathcal{Q}}/(48C_{\mathrm{resp}} + 2)$ from Lemma 5.23(b), these estimates give the first bound, and the other two follow from Lemma 5.23(c), (b), (d).

Lemma 5.25. *Assume $\varepsilon'' \le \frac{1}{2}$. For a fixed set Z of sites, a fixed set $\mathfrak{B}'$ of blocks and $u_0 > 0$, $\mathbb{P}_D\big(\sum_{P \in \mathcal{P}:\ \mathrm{dist}(P, Z \cup \bigcup \mathfrak{B}') \le R'} N(P) \ge u_0\big) \le 2^{|Z| + |\mathfrak{B}'| + 1} (\varepsilon'')^{u_0}$.*

Proof. We argue as in the proof of Proposition 5.9, with roots in $Z \cup \mathfrak{B}'$, which give at most $2^{|Z| + |\mathfrak{B}'| + m}$ multisets of roots, and with the mark y_i within R' of its root. As block sides are less than $2L_c$, there are $(2R' + 4L_c + 1)^3$ choices for each mark.

For Lemma 5.26 we set $d_0 := \lceil R'/L_c \rceil + 2$. The block distance is the ℓ^∞ distance of block indices, and $q(x)$ is the block of a site x. Since block sides are at least L_c, sites at distance r lie in blocks at block distance at most $r/L_c + 1$. A block set $\mathfrak{B}'$ is *anchored to* a plaquette set Ξ' if the graph on $\{o\} \cup \{*\text{-components of } \mathfrak{B}'\}$, in which o is joined to a component C when the component C has a block within block distance d_0 of $q(x)$ for some $x \in Z(\Xi')$, and two components are joined when they have blocks within block distance d_0, is connected.

Lemma 5.26. *(a) $\mathfrak{B}_{\mathcal{F}}$ is anchored to $\Xi' = \operatorname{supp} \Sigma^\circ$. (b) For every plaquette set Ξ' with $|\Xi'| = a$, the number of block sets anchored to Ξ' with b blocks and m $*$-components is at most $2^{2a+2b+m}(2d_0 + 1)^{3m}(26e)^b$.*

Proof. (a) We build $\mathcal{F}$ by the closure, starting from $\mathcal{F}_0 := \mathcal{T}$ and letting $\mathcal{F}_{t+1}$ be $\mathcal{F}_t$ with the classes forced by $\mathcal{F}_t$. A class of $\mathcal{T}$ has a touched cut bond, whose ends lie in $Z(\Xi')$ and

whose blocks lie in its $\mathfrak{B}$. A class $\mathcal{J}' \in \mathcal{F}_{t+1} \setminus \mathcal{F}_t$ has a cut bond z with $\varpi(u,z) > 0$, hence $\operatorname{dist}(u,z) \le R_{\mathrm{resp}}$, for a unit u of type (i) or (ii) relative to $\mathcal{F}_t$. In case (i), u has an end within 3ℓ of $Z(\Xi')$, so the block of an end of z is within block distance $(R_{\mathrm{resp}} + 3\ell + 1)/L_c + 1 \le d_0$ of $q(x)$ for some $x \in Z(\Xi')$. In case (ii), the block of an end of z is within block distance $(R_{\mathrm{resp}} + 1)/L_c + 2 \le d_0$ of $\mathfrak{B}(\mathcal{J})$, $\mathcal{J} \in \mathcal{F}_t$, as u has an end in a block of $\mathfrak{B}(\mathcal{J})$. In both cases this block lies in $\mathfrak{B}(\mathcal{J}')$, as $z \in \mathfrak{X}(\mathcal{J}')$, and $\mathfrak{B}(\mathcal{J}')$ is a union of $*$-connected sets linked at block distance $k_b \le d_0$ by Lemma 5.18(c), each of these sets lying in one component of $\mathfrak{B}_{\mathcal{F}}$. By induction on t, each component meeting $\mathfrak{B}(\mathcal{J})$, $\mathcal{J} \in \mathcal{F}_t$, is connected to o, and each component of $\mathfrak{B}_{\mathcal{F}}$ meets some $\mathfrak{B}(\mathcal{J})$.

(b) Along a breadth-first spanning tree from o, built with fixed orders, the block set is fixed by the following data, which Section S2.3 lists in more detail:

- the component sizes, with 2^b choices;
- the anchors of the children of o in $Z(\Xi')$, where $|Z(\Xi')| \le 2a$, and of each component among its blocks, with 2^{2a+b+m} choices in all;
- the root positions relative to the anchors, with $(2d_0+1)^{3m}$ choices;
- the components, for which Lemma 5.8 with $D_{\max} = 26$ gives $(26e)^b$ choices.

Proof of Theorem 5.1. The conditions of item (a) of Section S1.1.4 include (5.9) and

$$2^{29} \cdot 26e \cdot (\varepsilon'')^{1/(3C_{\mathfrak{B}})} \le \tfrac{1}{2}, \qquad 2(2d_0+1)^3 (\varepsilon'')^{1/(3M_m)} \le \tfrac{1}{2}. \tag{5.10}$$

On $E := \{A_{G'(D)}(C) \le (1-\eta_T)RT\}$, as $a = A_{G'(D)}(C) + \#(\Xi' \cap \mathfrak{F})$, Proposition 5.24 gives $N_{\mathcal{Q}} \ge \frac{1}{3}(u_1(a) + b/C_{\mathfrak{B}} + m/M_m)$ with $b := |\mathfrak{B}_{\mathcal{F}}|$ and $u_1(a) := (a - (1-\eta_T)RT)/(\bar{r}_0^2 C_3)$.

We first reduce the estimate to deterministic pairs. The minimizer Σ° and the block family $\mathfrak{B}_{\mathcal{F}}$ are functions of D, defined through fixed orders. Indeed, this holds for the output $(\mathbb{X}, S)$ by Theorem 3.20(R1), for the minimizer by Lemma 5.4, for the classes, for the weights ϖ by Lemma 5.21, and for the smallest closed family $\mathcal{F}$. Thus, on E, the pair $(\operatorname{supp}\Sigma^\circ, \mathfrak{B}_{\mathcal{F}})$ takes one deterministic value $(\Xi', \mathfrak{B}')$, with Ξ' a plaquette set as in Lemma 5.8 and with $\mathfrak{B}'$ anchored to Ξ' by Lemma 5.26(a). Hence $E \subseteq \bigcup_{(\Xi',\mathfrak{B}')} E(\Xi', \mathfrak{B}')$, the union over all such deterministic pairs, where $E(\Xi', \mathfrak{B}')$ is the event that the parts P within R' of $Z(\Xi') \cup \bigcup \mathfrak{B}'^{+1}$ have $\sum N(P) \ge \frac{1}{3}(u_1(|\Xi'|) + |\mathfrak{B}'|/C_{\mathfrak{B}} + m(\mathfrak{B}')/M_m)$. In $E(\Xi', \mathfrak{B}')$ the random partition appears only as a family of disjoint r'-connected mark sets, by Theorem 3.21, and Lemma 5.25 sums over all such families.

It remains to sum over these pairs. Lemma 5.25, applied with $Z = Z(\Xi')$, where $|Z| \le 2a$, and with the block set $\mathfrak{B}'^{+1}$ of at most $27b$ blocks, gives $\mathbb{P}_D(E(\Xi', \mathfrak{B}')) \le 2^{2a+27b+1} (\varepsilon'')^{(u_1 + b/C_{\mathfrak{B}} + m/M_m)/3}$. Summing over the anchored $\mathfrak{B}'$ with Lemma 5.26(b), we obtain the factors $(2^{29} \cdot 26e\,(\varepsilon'')^{1/(3C_{\mathfrak{B}})})^b$ and $(2(2d_0+1)^3(\varepsilon'')^{1/(3M_m)})^m$, each at most 2^{-b}, resp. 2^{-m}, by (5.10). The union over $\mathfrak{B}'$ therefore has probability at most $2^{4a+3}(\varepsilon'')^{u_1(a)/3}$. Finally, Lemma 5.10 with $k = 4$, $j = 3$, $\varepsilon = \varepsilon''$, $F_* = 3\bar{r}_0^2 C_3$ and $X := RT - A_{G'(D)}(C)$, for which $\eta_T |\log \varepsilon''|/F_* = c''_T \ge c_T(4)$ and, by Lemma 5.2, $0 \le X \le RT$, gives both bounds of (5.1).

The order of η_T. By Lemma 5.12 with ε'', $|\log \varepsilon''| \ge \gamma_T(K)\tau/2$ for $\tau \ge \tau_0(K)$, where $\tau_0(K)$ is chosen so that the error term of (5.6) is at most $\gamma_T(K)\tau/2$ (it grows logarithmically in τ). This gives the first bound, and the second follows from $C_3 = O((C'_\star + C_{\mathrm{resp}})(1 + 1/c_s))$ (Section S1.4) and $\gamma_T(K) \ge 1.4 \cdot 10^{-6} \log^2 K$ for $K > K_*$ (Section S4.2).

6 Stage 1: the Mayer expansion to the Debye scale on slit domains

In this section we prove [GM82, Prop. 3], with the iterated Mayer expansion of [GM81a], for the gas whose kernel is the Neumann Green function of a slit graph (Theorem 6.5), and we derive heat-kernel, covariance and depletion bounds on outputs. Since the Neumann field of a distant source is enhanced near a free edge of a frozen sheet, the only pointwise bounds on a Neumann kernel or its gradient that Sections 6 and 7 use near such an edge are the diagonal lower bounds of Lemma 6.1 and heat-kernel decay from the isoperimetry (R2), in the form of Theorem 6.8 and Lemma 6.14. The remaining inputs are the exact L^1 sums of Lemma 6.2 and, in Lemma 7.32, the free-space solid angle.

Conventions. The lattice is the fine lattice (spacing 1), $\beta_{\mathrm{GM}} = 4\pi^2\tau$, and distances are Euclidean unless $d_{G'}$ is written. Slit graphs $G' = (V, E)$, $V = \Lambda' \setminus \mathbb{X}$, $E = E(V) \setminus S$, their components $\mathfrak{K}$, the bound $d_{G'}(x, y) \geq \|x - y\|_1$ and the outputs $G' = G'(D)$ are those of §2.5. We define $(-\Delta_{G'} f)(y) := \sum_{y' \sim_{G'} y} (f(y) - f(y'))$ over the G'-neighbours of y. For $m > 0$, $v^{(m)}_{G'} := (-\Delta_{G'} + m^2)^{-1}$, $v_{G'} := v^{(M_Y)}_{G'}$, $v := v^{(M_Y)}_{\mathbb{Z}^3}$, $v_{\Lambda'}$ is $v^{(M_Y)}$ of the box Λ' with all its bonds, and v_{Cb} is the Coulomb kernel of $\mathbb{Z}^3$. On $\mathbb{R}^V_0 := \{\varphi : \sum_{x \in \mathfrak{K}} \varphi(x) = 0 \text{ for all } \mathfrak{K}\}$, $G_N := (-\Delta_{G'})^+$ is the inverse of $-\Delta_{G'}$. The *image energies* are $\Phi(x) = \Phi_{G'}(x) := \frac{1}{2}\beta_{\mathrm{GM}}(v_{G'}(x, x) - v(0))$ and $I_{G'}(x) := v_{G'}(x, x) - v_{\Lambda'}(x, x)$, and the *depleted set* is $\mathfrak{D} := \{x \in V : \Phi_{G'}(x) > \theta_0\}$, $\theta_0 = 0.01$. For a matrix K on V and $r \geq 0$, $\|K\|_r$ is the larger of $\sup_x \sum_y |K(x, y)| e^{r\, d_{G'}(x,y)}$ and the same expression with x and y exchanged. This norm is submultiplicative, and $\|K\|_{2\to 2} \leq \|K\|_0 \leq \|K\|_r$ for all such K and r. The activity of a unit monopole is $z := e^{-\beta_{\mathrm{GM}} v_{\mathrm{Cb}}(0)/2}$, and $m_D^2 := 2\beta_{\mathrm{GM}} z$ [GM82, (1.8a)]. Hence $m_D = (8\pi^2\tau)^{1/2} e^{-\pi^2 v_{\mathrm{Cb}}(0)\tau}$. Numerical values are given in §S1.5. The mass $\tilde{m} \in [m_D/2, 2m_D]$ and the Pauli–Villars mass $M_Y = \tilde{m}/\lambda_{\mathrm{PV}}$ are fixed in (6.11). Until then M_Y is any mass in $I_Y := [m_D/(2\lambda_{\mathrm{PV}}), 2m_D/\lambda_{\mathrm{PV}}]$, and all statements hold uniformly in that range. Besides the loop, the letter C denotes, depending on the context, the parameter $C \in (0, 1)$ of Theorem 6.5 (the notation of [GM82]), a generic constant, such as those of (S5), (6.7) and Proposition 6.6, and, in Theorem 6.11 and Appendix C, the covariance $(C_0^{-1} + \nu)^{-1}$.

6.1 Kernel facts on every slit graph

Lemma 6.1 (Thomson monotonicity)**.** *Let $H \subseteq H'$ be graphs with $V(H) \subseteq V(H')$, $E(H) \subseteq E(H')$ and $V(H)$ finite; $H' = \mathbb{Z}^3$ is allowed. Let f be supported on $V(H)$. (i) For every $m > 0$, $\big(f, v^{(m)}_H f\big) \geq \big(f, v^{(m)}_{H'} f\big)$. (ii) If f has zero sum on every component of H, then it has zero sum on every component of H', and $\big(f, (-\Delta_H)^+ f\big) \geq \big(f, (-\Delta_{H'})^+ f\big)$; for $H' = \mathbb{Z}^3$ the right side is $(f, v_{\mathrm{Cb}} f)$. (iii) For every slit graph $G' \subseteq \Lambda'$ and $x \in V$: $v^{(m)}_{G'}(x, x) \geq v^{(m)}_{\mathbb{Z}^3}(0)$ and $v_{G'}(x, x) \geq v_{\Lambda'}(x, x)$. Hence $\Phi_{G'} \geq 0$ and $I_{G'} \geq 0$.*

Proof. In the variational formula $(f, v^{(m)}_H f) = \sup_\varphi [2(f, \varphi) - \sum_{\{y,y'\} \in E(H)} (\varphi(y) - \varphi(y'))^2 - m^2 \sum_{y \in V(H)} \varphi(y)^2]$, restricting φ from $V(H')$ to $V(H)$ keeps (f, φ) and does not raise the energy, which gives (i). In (ii), f is orthogonal to the kernels of both Laplacians, and the formula holds at $m = 0$; for $H' = \mathbb{Z}^3$ let $m \downarrow 0$ in (i): the left side tends to $(f, (-\Delta_H)^+ f)$, by the spectral theorem on $f \perp \ker(-\Delta_H)$, and the right side increases to $(f, v_{\mathrm{Cb}} f)$, by monotone convergence in the Fourier formula. We obtain (iii) by taking $f = \delta_x$ in (i).

Lemma 6.2. *Let G' be a slit graph and $0 < m < m'$. Then: (i) $v^{(m)}_{G'}$ is symmetric and entrywise nonnegative, and $v^{(m)}_{G'}(x,y) = 0$ when x and y lie in different components; (ii) $\sum_{y\in V} v^{(m)}_{G'}(x,y) = m^{-2}$ for every $x \in V$; (iii) $v^{(m)}_{G'} - v^{(m')}_{G'} = (m'^2 - m^2)\, v^{(m)}_{G'} v^{(m')}_{G'}$, and this matrix is entrywise nonnegative and positive semidefinite, with row sums $m^{-2} - m'^{-2}$; (iv) $\operatorname{spec}(-\Delta_{G'}) \subseteq [0, 12]$.*

Proof. Since $\deg_{G'} \le 6$, the matrix $P := I - (-\Delta_{G'})/6$ is symmetric, nonnegative and stochastic, with no entry across components, and the series $v^{(m)}_{G'} = (6+m^2)^{-1} \sum_{n\ge 0} (6/(6+m^2))^n P^n$ gives (i) and (ii). Part (iii) is the resolvent identity, and (iv) is Gershgorin's theorem.

Proposition 6.3. *Let G' be a slit graph, $\beta_{\mathrm{GM}} > 0$, $k \in \mathbb{R}$ and $f : V \to \mathbb{R}$. Put $\tilde Z(k,f) := \sum_{[n]} \exp\bigl(-\frac{1}{2\beta_{\mathrm{GM}}} \sum_{b\in E} (\nabla_b (n - kf))^2\bigr)$, where n runs over $(2\pi\mathbb{Z})^V$ modulo the addition of a constant in $2\pi\mathbb{Z}$ on each component. Let $u := G_N - v_{G'}$ on $\mathbb{R}^V_0$ and let $d\mu_u$ be the centred Gaussian measure on $\mathbb{R}^V_0$ with covariance u. Then: (a) $u = M_Y^2(-\Delta_{G'})^+(-\Delta_{G'} + M_Y^2)^{-1}$ on $\mathbb{R}^V_0$, and u is positive definite there. (b) $\tilde Z(k,f) = \mathcal{N}_0 \sum_q e^{ik(q,f)} e^{-\beta_{\mathrm{GM}}(q, G_N q)/2}$, the sum over $q \in \mathbb{Z}^V$ with zero sum on every component, where $\mathcal{N}_0 = \prod_{\mathfrak{K}} \mathcal{N}_{\mathfrak{K}} > 0$ depends only on G' and β_{GM}. An isolated vertex carries only $q = 0$ and has $\mathcal{N}_{\mathfrak{K}} = 1$. (c) With $\theta := \sum_{\mathfrak{K}} \theta_{\mathfrak{K}} 1_{\mathfrak{K}}$,*

$$\sum_{q \text{ neutral}} e^{ik(q,f)} e^{-\beta_{\mathrm{GM}}(q,G_N q)/2} = \int_{[0,2\pi)^{\mathcal{K}}} \prod_{\mathfrak{K}} \frac{d\theta_{\mathfrak{K}}}{2\pi} \int d\mu_u(\varphi)\, Z^Y\bigl(\varphi + k\beta_{\mathrm{GM}}^{-1/2} f + \beta_{\mathrm{GM}}^{-1/2}\theta\bigr), \quad (6.1)$$

where $Z^Y(\psi) := \sum_{q\in\mathbb{Z}^V} e^{i\beta_{\mathrm{GM}}^{1/2}(q,\psi)} e^{-\beta_{\mathrm{GM}}(q, v_{G'} q)/2}$ is the Yukawa gas on G', with no neutrality constraint. (d) $Z^Y(\psi) > 0$ for real ψ, and $Z^Y = \prod_{\mathfrak{K}} Z^Y_{\mathfrak{K}}(\psi|_{\mathfrak{K}})$. Consequently the coefficients ρ_s of the expansion of $\ln Z^Y$ in Theorem 6.5 vanish unless all their points lie in one component.

Proof. On each component $\mathfrak{K}$ we carry out the argument of [GM82, (2.4) to (2.12)], which is written there for a measure on $\mathbb{Z}^3$ with charges confined to Λ and no zero mode (change (c) of Table 6.1). We root $\mathfrak{K}$ at a point $x_0 \in \mathfrak{K}$. Poisson summation over $(2\pi\mathbb{Z})^{|\mathfrak{K}|-1}$ then gives the neutral charges $\hat q = (q, -\sum q)$, with $\langle q, A^{-1} q\rangle = \langle \hat q, G_N \hat q\rangle$ for the restriction A of $-\Delta_{\mathfrak{K}}$ to $\mathfrak{K}\setminus\{x_0\}$. One variable $\theta_{\mathfrak{K}}$ for each component imposes neutrality on every component, for any number of components of any sizes. Finally, $G_N = v_{G'} + u$, and (a) follows from $(-\Delta_{G'} + M_Y^2)^{-1} 1_{\mathfrak{K}} = M_Y^{-2} 1_{\mathfrak{K}}$, which holds by Lemma 6.2(ii). The positivity $Z^Y > 0$ is cited in [GM82, p. 21]. We obtain it by Poisson summation over $\mathbb{Z}^V$, which writes $Z^Y(\psi)$ as a sum of positive Gaussians in $n - \beta_{\mathrm{GM}}^{1/2}\psi$, $n \in (2\pi\mathbb{Z})^V$, with covariance $\beta_{\mathrm{GM}} v_{G'}$. The factorization and (d) hold because $v_{G'}$ and the potentials $v^\ell_{G'}$, defined as in [GM81a, (2.5), p. 6, and (4.1), p. 17], vanish across components by Lemma 6.2(i), and [GM82, §3] builds ρ_s from these potentials through the tree formula [GM81a, (3.7)].

Proposition 6.4. *Let $M_1 \ge M_Y$ be the mass of [GM82, (3.20)], and on a slit graph G' let $v^1_{G'} := v^{(M_1)}_{G'}$, with the coincident-site rule [GM81a, (5.1a)], and $v^2_{G'} := v_{G'} - v^1_{G'}$. Then the stability bounds [GM81a, (4.2)] hold on G' with $\gamma_1 = \frac12 v^1_{\mathbb{Z}^3}(0)$, with the same $\varepsilon_1 > 0$ as on $\mathbb{Z}^3$ [GM81a, Prop. 8], and with $\varepsilon_2 = \delta_2 = 0$. Moreover $v^\ell_{G'}(x,y) \ge 0$, and*

$$\tilde v^1 := \sup_y \sum_x v^1_{G'}(x,y) = M_1^{-2}, \qquad \tilde v^2 := \sup_y \sum_x v^2_{G'}(x,y) = M_Y^{-2} - M_1^{-2} \quad (6.2)$$

exactly, as on $\mathbb{Z}^3$ [GM81a, (5.3)].

Proof. By Lemma 6.1 each Yukawa energy on G' is at least its $\mathbb{Z}^3$ value for the same configuration. Hence the margins of [GM81a, Prop. 8] transfer. At level one, with the rule (5.1a) and $m(x) := \sum_i m_i \delta_{x_i x}$, $E_{G'} = \frac{1}{2}(m, v^1_{G'} m) + \frac{1}{2}\sum_{x_i = x_j,\ \operatorname{sign} m_i = -\operatorname{sign} m_j} |m_i m_j|\, v^1_{G'}(x_i, x_i) \geq E_{\mathbb{Z}^3}$ by Lemma 6.1(i), (iii), and $E_{\mathbb{Z}^3} \geq \gamma_1 + \varepsilon_1(-1 + \sum_i m_i^2)$, coincident positions included. At level two, $\frac{1}{2}(m, v^2_{G'} m) \geq 0$ by Lemma 6.2(iii), that is (4.2) with $\varepsilon_2 = \delta_2 = 0$. Positivity and (6.2) follow from Lemma 6.2(i) to (iii).

6.2 The Stage 1 theorem

Theorem 6.5 (Stage 1 on slit domains)**.** *Fix $\kappa_S, \mu < \infty$ and $C, \delta_{\mathrm{GM}} \in (0,1)$. There are $\lambda_0 = \lambda_0(\kappa_S, \mu, C, \delta_{\mathrm{GM}}) > 0$ and, for $\lambda_{\mathrm{PV}} \leq \lambda_0$, $\beta^0_{\mathrm{GM}} = \beta^0_{\mathrm{GM}}(\lambda_{\mathrm{PV}}, \kappa_S, \mu, C, \delta_{\mathrm{GM}})$ such that for $\beta_{\mathrm{GM}} \geq \beta^0_{\mathrm{GM}}$ and every $M_Y \in I_Y$ the following hold on every slit graph G', with constants independent of G' and M_Y. Write $\xi = (m, x)$ with $m \in \mathbb{Z} \setminus \{0\}$ and $x \in V$, $-\xi := (-m, x)$, and $\int d\xi := \sum_m \sum_{x \in V}$.*

(S1) Expansion. *For real ψ on V, $Z^Y(\psi) = \exp(-V_{\mathrm{eff}}(\psi))$ with*

$$-V_{\mathrm{eff}}(\psi) = \rho_0 + \sum_{s \geq 1} \frac{1}{s!} \int d\xi_1 \cdots d\xi_s\, \rho_s(\xi_1, \dots, \xi_s)\, \varepsilon(\xi_1) \cdots \varepsilon(\xi_s), \tag{6.3}$$

where $\varepsilon(\xi) := e^{i m \beta_{\mathrm{GM}}^{1/2} \psi(x)} - 1$, the coefficients $\rho_s = \rho_s^{G'}$ are real and symmetric, ρ_0 is a constant, and the series converges absolutely, uniformly in real ψ. With Proposition 6.3, for every k and f,

$$\tilde{Z}(k, f) = \mathcal{N}_0 \int \prod_{\mathfrak{K}} \frac{d\theta_{\mathfrak{K}}}{2\pi} \int d\mu_u(\varphi)\, \exp\bigl[-V_{\mathrm{eff}}\bigl(\varphi + k\beta_{\mathrm{GM}}^{-1/2} f + \beta_{\mathrm{GM}}^{-1/2}\theta\bigr)\bigr]. \tag{6.4}$$

(S2) Coefficient bounds. *For $s \geq 1$ and every $a_1 \subseteq V$,*

$$\int_{x_1 \in a_1} d\xi_1 \int d\xi_2 \cdots \int d\xi_s\, |\rho_s(\xi_1, \dots, \xi_s)|\, e^{2\kappa_S(-1 + \sum_i |m_i|)}\, e^{(1-\delta_{\mathrm{GM}}) A\, L_{G'}(x_1, \dots, x_s)} \leq s!\, |a_1|\, \frac{m_D^2}{\beta_{\mathrm{GM}}}\, e^{-\mu(s-1)} (1 - C)^{-s-10} \tag{6.5}$$

with $A = 0$, where $L_{G'}(a_1, \dots, a_n)$ is the tree length of [GM82, (1.11a), (1.11b)] computed with the distance $3^{-1/2} d_{G'}$ and auxiliary points in V, $L_{G'}(x_1, \dots, x_s) := L_{G'}(\{x_1\}, \dots, \{x_s\})$, and $0 \cdot \infty := 0$. Moreover, for all $a_1, \dots, a_s \subseteq V$

$$\int_{x_1 \in a_1} d\xi_1 \cdots \int_{x_s \in a_s} d\xi_s\, |\rho_s(\xi_1, \dots, \xi_s)|\, e^{2\kappa_S(-1 + \sum_i |m_i|)} \leq s!\, |a_1|\, \frac{m_D^2}{\beta_{\mathrm{GM}}}\, e^{-(1-\delta_{\mathrm{GM}}) A\, L_{G'}(a_1, \dots, a_s) - \mu(s-1)} (1 - C)^{-s-10}. \tag{6.6}$$

(S2$^+$) Charge moments. *If $\mu \geq \ln 3$, then for every $a_1 \subseteq V$*

$$\sum_{s \geq 1} \frac{2^s}{s!} \int_{x_1 \in a_1} d\xi_1 \int d\xi_2 \cdots \int d\xi_s\, |\rho_s(\xi_1, \dots, \xi_s)|\, e^{2\kappa_S \sum_i |m_i|} \leq |a_1|\, \frac{m_D^2}{\beta_{\mathrm{GM}}}\, e^{2\kappa_S + \mu} (1 - C)^{-9};$$

in particular $\sum_{s \geq 1} \frac{2^s}{s!} \int |\rho_s| (\sum_i |m_i|)^n \leq n! (2\kappa_S)^{-n} |V| \frac{m_D^2}{\beta_{\mathrm{GM}}} e^{2\kappa_S + \mu} (1 - C)^{-9}$ for every $n \geq 0$.

(S3) Symmetries. $\rho_s(-\xi_1,\dots,-\xi_s)=\rho_s(\xi_1,\dots,\xi_s)$*; in particular* $\sum_m m\,\rho_1(m,x)=0$ *at every* $x\in V$*. Moreover* $V_{\rm eff}(\psi+2\pi\beta_{\rm GM}^{-1/2}I)=V_{\rm eff}(\psi)$ *for every* $I\in\mathbb{Z}^V$*.*

(S4) One-body term. *The bounds of Proposition 6.6 hold.*

(S5) Dipole kernel. $\nu(x_1,x_2):=\beta_{\rm GM}\sum_{m_1,m_2} m_1m_2\,\rho_2((m_1,x_1),(m_2,x_2))$ *is real and symmetric, and* $\|\nu\|_{2\to2}\le\|\nu\|_0\le Ce^{-\mu}m_D^2$*. If* $Ce^{-\mu}\le\frac18$*, the right side is at most* $\frac12\tilde m^2$*.*

With M_Y *fixed in* (6.11)*, on every output (S2) holds also with* $A=\sqrt3\,\kappa_Y M_Y$ *(Proposition 6.10), and, if* $\delta_{\rm GM}\le\frac12$*, (S5) in the form* $\|\nu\|_{r_{\rm cov}}\le Ce^{-\mu}m_D^2$ *(*$r_{\rm cov}$ *of Theorem 6.11).*

Proof of Theorem 6.5. By Lemma 6.1, removing bonds raises all quadratic energies, and by Lemma 6.2 it conserves the mass of the walk. The stability margins of [GM81a] and the pinned L^1 norms of the links are therefore the same as on $\mathbb{Z}^3$, without image factors. Hence the proof of [GM82, Prop. 3], together with the results of [GM81a] that it uses, runs on G' with the changes (a) to (h) of Table 6.1, and its other steps carry over without change, for the same reason. Numbers refer to the DESY preprints identified in the footnote in §1.1.

	other steps
[GM81a]	§2, (2.5) to (2.25), Prop. 2, (2.26) (its finite pieces are left arbitrary until the estimates are made, p. 6), Prop. 3, (2.30), App. A (only (2.27) is used), Prop. 4, (3.7), App. B, (B.4) (stated on p. 32 for any real symmetric pair array $w(ij)$, and used here for the $v^\ell_{G'}$, $\ell\ge1$), (3.8) to (3.12), (4.3) (with (4.2) for all subconfigurations, by Proposition 6.4), Lemma 5, (4.8), App. C (combinatorial), Prop. 6, Cor. 7, Lemma 9 (through the pinned norms of change (a) only)
[GM82]	(3.1) to (3.18) (for real ψ, (3.14) uses only $\lvert1+\varepsilon(\xi)\rvert=1$, p. 25), (4.1) to (4.11), Lemma 6 (any metric), (4.14) to (4.22), Prop. 7, Cor. 8 (through $\|v^\ell\|_A$ only), (4.25) (γ_1 is the constant of $\mathbb{Z}^3$), Prop. 9, (4.26) to (4.30), (4.32) (through the preceding steps, with $L_{G'}$ and the rate $(1-\delta_{\rm GM})A$)

We read the statement of [GM82, Prop. 3] with two corrections: $\varepsilon(\xi)=-1+\exp[im\beta_{\rm GM}^{1/2}\varphi(x)]$, as in its (3.4), and $2\beta_{\rm GM}m_D^{-2}\rho_1(\pm1,x)\to1$ at its end. The constants depend only on $(\lambda_{\rm PV},\kappa_S,\mu,C,\delta_{\rm GM})$ and, on outputs, on those of Theorem 6.8.

A complete proof is given in Section S4.3. There the tree formula [GM81a, Prop. 4, (3.7), App. B] and the tree estimate [GM81a, Lemma 5, App. C] are replaced by the forest formula of Brydges, Kennedy, Abdesselam and Rivasseau [BK87, AR95], its decoupling property and Cayley's formula; the recursive bound obtained there is slightly stronger than [GM82, (4.16)], so that the analogues of [GM82, Cor. 8, Prop. 9] follow by their printed arguments.

(a) *Pinned norms.* Where [GM81a, p. 18] and [GM82, p. 32] use translation invariance, the former after a preparatory majorization by functions of differences of the arguments, we take instead, on G', the norms in the max form [GM81a, (4.5b)], $\|\sigma_\alpha\|(m_\alpha):=\max_{i\in\alpha}\sup_{x\in V}\sum_{x_\alpha\in V^n:\,x_i=x}|\sigma_\alpha(\xi_\alpha)|$, with the weights of [GM82, (4.12)] inserted in the same way, and $\tilde v^\ell:=\sup_y\sum_x|v^\ell_{G'}(x,y)|$. We sum the positions as in the trimming of the tree in [GM81a, p. 18], after re-rooting the tree at the vertex containing the pinned constituent, in order of decreasing tree distance from it. Each summed vertex is then a leaf attached by one link (j,k), and $\sum_{x_j}|v^\ell_{G'}(x_j,x_k)|\le\tilde v^\ell$ uniformly in x_k. The s-integrals and tree sums are bounded by [GM81a, Lemma 5] independently of the positions. Hence [GM81a, (4.7), (4.10), (4.13), (4.16), (4.17)] hold without change, and $\int d\xi_\alpha|\sigma_\alpha|\le|V|\,\|\sigma_\alpha\|$.

(f) *Link norms.* For $A\ge0$ we put $\|v^\ell_{G'}\|_A:=\sup_y\sum_x|v^\ell_{G'}(x,y)|\,e^{A\|x-y\|_{G'}}$. At $A=0$ the values are (6.2), on all slit graphs. On an output, Proposition 6.10 gives $\|v_{G'}\|_A,\|v^2_{G'}\|_A\le3M_Y^{-2}$ and $\|v^1_{G'}\|_A\le3M_1^{-2}$ for $A\le\sqrt3\,\kappa_YM_Y$, since $\|x-y\|_{G'}=3^{-1/2}d_{G'}(x,y)$, the rate of

Table 6.1: Changes to [GM82, Prop. 3] and the results of [GM81a] it uses, on a slit graph G'.

change	steps	use of $\mathbb{Z}^3 \to$ on G'
(a) pinning	[GM81a, (4.5a), (4.6), pp. 18 to 20]; [GM82, (4.12), (4.13), p. 32]	translation invariance $\to$ pinned norms in max form, re-rooted sums
(b) stability	[GM81a, Prop. 8, (5.1a) to (5.13), App. D]; [GM82, Lemma 5, (3.25)]	Fourier analysis $\to$ Proposition 6.4: same γ_1, ε_1; $\varepsilon_2 = \delta_2 = 0$
(c) Gaussian	[GM82, (2.4) to (2.12), pp. 20, 21]	measure on $\mathbb{Z}^3$, no zero mode, $Z_\Lambda(\phi) > 0$ cited $\to$ Proposition 6.3: one zero mode per component, $Z^Y > 0$ proved
(d) resolvents	[GM82, (3.19) to (3.22)]	resolvents of $\mathbb{Z}^3 \to$ of G'; rule (3.19) (= [GM81a, (5.1a)]), same M_1 (3.20)
(e) metric	[GM82, (3.23), (3.24)]	$3^{-1/2}\|x-y\|_1 \to \|x-y\|_{G'} := 3^{-1/2} d_{G'}(x,y)$, in $\|\cdot\|_A$ and $L_{G'}$
(f) link norms	[GM82, Lemma 5, (3.26); (4.23), (4.24), p. 34]	momentum space $\to$ (6.2), Proposition 6.10; Debye condition $\lambda_{\mathrm{PV}}^2 \le C^3(1-C)^8 e^{-2\kappa_S-\mu}/24$
(g) σ^R	[GM82, (4.33), (4.34), p. 36]	$v(0) \to v_{G'}(x,x)$ in $\sigma^R = \exp[-\beta_{\mathrm{GM}} m^2 v(0)/2]$
(h) tree length	[GM82, (4.30) to (4.32), pp. 35, 36; (4.12), p. 32]	$L(a_1,\dots,a_s)$ on the right of (4.31) $\to$ $L_{G'}(x_1,\dots,x_s)$ inside the sum over all positions

(6.13) is nondecreasing in the mass, and $M_Y \le M_1 \le m_W$. These bounds replace [GM82, (3.26)] in [GM82, (4.23)] and in [GM82, (4.24)]. In [GM82, (4.23)], $3M_1^{-2}$ is a power of β_{GM}, absorbed by $e^{\beta_{\mathrm{GM}}\varepsilon_1}$. In [GM82, (4.24)], $3M_Y^{-2}$ replaces $2(\delta M)^{-2}$ at $A = (1-\delta)M$, so the factor $\frac{1}{4}\delta^2$ of the Debye condition [GM82, p. 34] becomes $\frac{1}{6}$. The condition then holds on I_Y once $\lambda_{\mathrm{PV}}^2 \le C^3(1-C)^8 e^{-2\kappa_S-\mu}/24$ (§S1.5). This threshold does not depend on G', and all other thresholds of the proof are those of $\mathbb{Z}^3$.

(g) *The leading one-body term.* For a single constituent the recursion [GM82, (3.12)] gives $\sigma^{\ell+1} = \sigma^\ell \exp[-\beta_{\mathrm{GM}} m^2 v_{G'}^\ell(x,x)/2]$, hence $\sigma^R(m,x) = \exp[-\beta_{\mathrm{GM}} m^2 v_{G'}(x,x)/2]$ in place of [GM82, (4.33)]. The argument on [GM82, p. 36], through the pinned norm of [GM82, Prop. 9], bounds the remainder in [GM82, (4.34)] by $e^{-\mu}(1-e^{-\mu})^{-2} m_D^2 \beta_{\mathrm{GM}}^{-1}(1-C)^{-9}$, uniformly in x. With the weights $e^{2\kappa_S(|m|-1)}$ the same argument gives this bound up to a constant factor, since we use the pinned norm with weight $e^{(2\kappa_S+\mu)\sum_j |m_j|}$ and $\sum_{j\ge 2} |m_j| \ge t-1$. Thus, with $A = 0$ and uniformly in x and G',

$$\rho_1(m,x) = e^{-\beta_{\mathrm{GM}} m^2 v_{G'}(x,x)/2} + R(m,x), \qquad \sum_m |R(m,x)|\, e^{2\kappa_S(|m|-1)} \le C e^{-\mu} m_D^2/\beta_{\mathrm{GM}}. \quad (6.7)$$

It replaces the final assertion of [GM82, Prop. 3], whose error at fixed μ is of order $e^{-\mu}$.

(h) *The tree length inside the sum.* We state the bound (S2) with the tree length of the points inside the integral, which is what the chain [GM82, (4.30) to (4.32)] gives. By [GM82, (4.30)], $|\rho_s(\xi_1,\dots,\xi_s)| \le s! \sum_{t\ge s} \binom{t}{s} \sum_{[\alpha]:\, C(\alpha)=\{1,\dots,t\}} \int d\xi_{s+1}\cdots d\xi_t\, |\sigma_\alpha^R(\xi_\alpha)|$. Applied to the sets $\{x_i\}$, [GM82, (4.9)] gives $L_{G'}(x_1,\dots,x_s) \le \tilde{L}(x_1,\dots,x_t)$ for all $t \ge s$ and $x_{s+1},\dots,x_t \in V$, where $\tilde{L}$ is the least length of a tree on exactly the vertices $x_1,\dots,x_t$ [GM82, (4.8)], computed with $3^{-1/2} d_{G'}$. In [GM82, (4.31)] the restriction $x_i \in a_i$ ($i \ge 2$) enters only through $L(a_1,\dots,a_s) \le \tilde{L}(x_1,\dots,x_t)$, since the pinned norm [GM82, (4.12)], taken in the max form of change (a), sums all positions with the weight $e^{A\tilde{L}(x_\alpha)}$. Hence (4.31) holds with $x_2,\dots,x_s$ summed over V and the factor $e^{(1-\delta_{\mathrm{GM}})AL_{G'}(x_1,\dots,x_s)}$ on the left in place of $e^{-(1-\delta_{\mathrm{GM}})AL(a_1,\dots,a_s)}$ on the right. Combining this with (4.32), [GM82, Prop. 9] and the sum over $x_1 \in a_1$, we obtain (6.5). The bound (6.6), which has the form of [GM82, (1.18)], follows from it, since $L_{G'}(a_1,\dots,a_s) \le L_{G'}(x_1,\dots,x_s)$ whenever $x_i \in a_i$.

With these changes the proof of [GM82, Prop. 3] gives (S1) and (S2). Since the Debye condition is the only threshold that involves M_Y, they hold uniformly in $M_Y \in I_Y$. By Proposition 6.3(d), $\rho_s = 0$ in (S2) unless $x_1, \dots, x_s$ lie in one component. Together with Proposition 6.3, (S1) gives (6.4). For (S3), the activities are equal, the potentials depend on the charges through $m_i m_j$ and the rule (3.19), both invariant under $m \mapsto -m$, and $\varepsilon(\xi)$ is $2\pi\beta_{\mathrm{GM}}^{-1/2}$-periodic in $\psi(x)$. (S4) is Proposition 6.6. For (S5), ν is real and symmetric because ρ_2 is, and it vanishes across components by Proposition 6.3(d). Since $|m_1 m_2| \le C e^{2\kappa_S(|m_1|+|m_2|-1)}$, (6.6) at $s=2$ with $a_1 = \{x_1\}$ and any $a_2 \subseteq V$ gives $\sum_{x_2 \in a_2} |\nu(x_1,x_2)| \le C e^{-\mu} m_D^2 e^{-(1-\delta_{\mathrm{GM}}) A\, L_{G'}(\{x_1\}, a_2)}$. With $A = 0$ and $a_2 = V$ this is $\|\nu\|_0 \le C e^{-\mu} m_D^2$, at most $\frac{1}{8} m_D^2 \le \frac{1}{2}\tilde{m}^2$ once $C e^{-\mu} \le \frac{1}{8}$. On outputs we take $A = \sqrt{3}\,\kappa_Y M_Y$ and, for an integer $k \ge 0$, $a_2 = \{x_2 : d_{G'}(x_1,x_2) \ge k\}$, whose tree length with $\{x_1\}$ is at least $3^{-1/2}k$. With these choices and $\gamma_\nu := (1-\delta_{\mathrm{GM}})\kappa_Y M_Y$, the tail sums $T_k := \sum_{x_2 :\, d_{G'}(x_1,x_2) \ge k} |\nu(x_1,x_2)|$ satisfy $T_k \le C e^{-\mu} m_D^2 e^{-\gamma_\nu k}$. Summation by parts, $e^{rd} = 1 + \sum_{1 \le k \le d} (e^{rk} - e^{r(k-1)})$, gives for $0 \le r < \gamma_\nu$

$$\begin{aligned} \sum_{x_2} |\nu(x_1,x_2)|\, e^{r\, d_{G'}(x_1,x_2)} &= T_0 + \sum_{k \ge 1} (e^{rk} - e^{r(k-1)}) T_k \\ &\le \Big(1 + \frac{1-e^{-r}}{e^{\gamma_\nu - r} - 1}\Big) C e^{-\mu} m_D^2 \le \Big(1 + \frac{r}{\gamma_\nu - r}\Big) C e^{-\mu} m_D^2, \end{aligned}$$

which is at most $2Ce^{-\mu}m_D^2$ for $r \le \gamma_\nu/2$. For $\delta_{\mathrm{GM}} \le \frac{1}{2}$ we have $\gamma_\nu \ge \frac{1}{2}\kappa_Y M_Y \ge 2\kappa_0 \tilde{m} \ge 2r_{\mathrm{cov}}$, where the middle inequality follows from the last assertion of Proposition 6.10, and hence $\|\nu\|_{r_{\mathrm{cov}}} \le 2Ce^{-\mu}m_D^2$. This is (S5) on outputs, with the constant C doubled. Finally, ($\mathrm{S2}^+$) follows from the pinned-norm bound of [GM82, Prop. 9] at $\kappa_S + \frac{1}{2}\ln 3$, since $\sum_{s=1}^{t} \binom{t}{s} 2^s \le 3^t \le \prod_{i \le t} 3^{|m_i|}$, and the moments from $(\sum_i |m_i|)^n \le n!(2\kappa_S)^{-n} e^{2\kappa_S \sum_i |m_i|}$ (Section S4.3).

Proposition 6.6 (Additive one-body bound)**.** *Let $\rho^{\mathrm{ref}}(m) := \lim_{\Lambda \uparrow \mathbb{Z}^3} \rho_1^{\Lambda}(m,0)$, the one-point coefficient of the Yukawa gas on $\mathbb{Z}^3$: $\rho_1^{\Lambda}(m,x)$ is the series of [GM82, Prop. 3] over the configurations in Λ, whose terms do not depend on Λ, so the limit exists by dominated convergence, does not depend on the base point, and satisfies* (6.7) *and* (6.10) *with the same constants (Section S4.3). Let $\Phi = \Phi_{G'}$. Uniformly in $x \in V$ and G',*

$$\big|\rho_1(\pm 1, x) - \rho^{\mathrm{ref}}(1) e^{-\Phi(x)}\big| \le C e^{-\mu} \rho^{\mathrm{ref}}(1), \tag{6.8}$$

$$\rho_1(\pm 1, x) \le \rho^{\mathrm{ref}}(1)\big(1 + C e^{-\mu}\big), \tag{6.9}$$

$$\sum_{|m| \ge 2} |\rho_1(m,x)|\, m^2 e^{2\kappa_S(|m|-1)} \le C m_D^2/\beta_{\mathrm{GM}}. \tag{6.10}$$

Consequently $|\rho_1(\pm 1, x) - \rho^{\mathrm{ref}}(1)| \le \rho^{\mathrm{ref}}(1)\big(1 - e^{-\Phi(x)} + Ce^{-\mu}\big)$.

Proof. Subtracting from (6.7) its form on $\mathbb{Z}^3$, $\rho^{\mathrm{ref}}(1) = e^{-\beta_{\mathrm{GM}} v(0)/2} + R^{\infty}$, $|R^{\infty}| \le Ce^{-\mu} m_D^2/\beta_{\mathrm{GM}}$, gives (6.8) and (6.9). Here we use $e^{-\beta_{\mathrm{GM}} v_{G'}(x,x)/2} = e^{-\beta_{\mathrm{GM}} v(0)/2} e^{-\Phi(x)}$: the coefficient $\rho^{\mathrm{ref}}(1)$ carries the same self-energy factor as ρ_1 up to the image term $e^{-\Phi}$, so the subtraction involves no error of order $\beta_{\mathrm{GM}} M_Y$. We also use the bound $\Phi \ge 0$ of Lemma 6.1 and $m_D^2/\beta_{\mathrm{GM}} \le 2\rho^{\mathrm{ref}}(1)(1 - 2Ce^{-\mu})^{-1}$, which holds because $e^{-\beta_{\mathrm{GM}} v(0)/2} \ge z$. In (6.10) the leading terms are at most $e^{-2\beta_{\mathrm{GM}} v(0)} \le e^{-\mu} m_D^2/\beta_{\mathrm{GM}}$, and in the remainders m^2 is absorbed by the μ-part of the weight $e^{(2\kappa_S + \mu)\sum_j |m_j|}$ of change (g), since $m^2 e^{-\mu|m|} \le 4e^{-2\mu}$ for $|m| \ge 2$ and $\mu \ge 1$, so that the Debye condition of change (f) suffices at the same κ_S.

Remark 6.7. We state Proposition 6.6 in additive form only, because the multiplicative form $\rho_1(\pm 1, x) = \rho^{\mathrm{ref}}(1)e^{-\Phi(x)}(1 + \mathcal{O}(e^{-\mu}))$ fails at a first-layer corner site of a box. There a tight neutral pair, which feels only its dipole image, makes the one-site density exponentially larger in β_{GM}.

The masses $\tilde{m}$ and M_Y. For $M \in I_Y$ let ρ_M^{ref} be the one-point coefficient of the Yukawa gas on $\mathbb{Z}^3$ with kernel $(-\Delta_{\mathbb{Z}^3} + M^2)^{-1}$, and $\tilde{m}(M)^2 := \beta_{\mathrm{GM}} \sum_m m^2 \rho_M^{\mathrm{ref}}(m)$. By (6.7) and (6.10) on $\mathbb{Z}^3$, in which the terms with $|m| \geq 2$ carry the weight $e^{2\kappa_S(|m|-1)} \geq e^{2\kappa_S}$, we have $\tilde{m}(M)^2 = 2\beta_{\mathrm{GM}} e^{-\beta_{\mathrm{GM}} v_{\mathbb{Z}^3}^{(M)}(0)/2} + \mathcal{O}(e^{-\mu} + e^{-2\kappa_S}) m_D^2 = m_D^2(1 + \mathcal{O}(\beta_{\mathrm{GM}} M)) + \mathcal{O}(e^{-\mu} + e^{-2\kappa_S}) m_D^2$, as $v_{\mathbb{Z}^3}^{(M)}(0) = v_{\mathrm{Cb}}(0) - \mathcal{O}(M)$. Since $\beta_{\mathrm{GM}} m_D \to 0$, $\tilde{m}(M) \in [m_D/2, 2m_D]$ once μ and κ_S exceed a threshold that depends on C through the factor $(1 - C)^{-9}$ in the remainder of (6.7) [GM82, p. 36], and $\beta_{\mathrm{GM}} \geq \beta_{\mathrm{GM}}^0$. Since M_1 and the rule (3.19) do not depend on M, the mass M appears only in $v_M^2 = v^{(M)} - v^{(M_1)}$, so each term of the expansion of ρ_M^{ref} is continuous in M. The t-th term is $\mathcal{O}(te^{-\mu(t-1)})$ uniformly in $M \in I_Y$ [GM82, p. 36]. With $M_- := m_D/(2\lambda_{\mathrm{PV}})$, we have $0 \leq v_M^2 \leq v_{M_-}^2$ entrywise by the resolvent identity, so at fixed t the terms are dominated by those with $v_{M_-}^2$ in place of v_M^2, whose bound in [GM82] needs exactly the Debye condition of change (f) at $M = M_-$. By dominated convergence, $M \mapsto \tilde{m}(M)/\lambda_{\mathrm{PV}}$ is a continuous map of I_Y into itself. Its fixed points form a nonempty closed set. We let M_Y be the smallest fixed point and put

$$\tilde{m} := \tilde{m}(M_Y) = \lambda_{\mathrm{PV}} M_Y \in [m_D/2, 2m_D], \qquad \rho^{\mathrm{ref}} := \rho_{M_Y}^{\mathrm{ref}}, \qquad \tilde{m}^2 = \beta_{\mathrm{GM}} \sum_m m^2 \rho^{\mathrm{ref}}(m). \tag{6.11}$$

The inverse mass $\tilde{m}^{-1}$, which like ρ^{ref} is determined by the gas on $\mathbb{Z}^3$ alone, before any construction, is the unit of length of Sections 3 to 5, and ρ^{ref} is the reference density of Proposition 6.6 and of Section 7.

6.3 Heat kernel and covariance on outputs of the construction

We prove the statements of this subsection in Appendix D, from (R2) and, for Proposition 6.9, from (R5) and (R6), without any Poincaré inequality or volume-growth hypothesis. Full proofs of Proposition 6.10 and Theorem 6.11 are given in Section S2.4. The *coarse slit graph* of an output is $\Gamma = (V_c, E_c)$, $V_c := \Lambda^\circ \setminus \mathbb{X}$, $E_c := E(V_c) \setminus S$, the graph G' of Sections 3 to 5 (§2.5). For $W \subseteq V_c$, $\partial_\Gamma W$ is the set of bonds of E_c with exactly one end in W. A *coarse cell* is a site of the grid of Definition 3.4, a box of fine sides in $[r_0, \bar{r}_0]$, not a cell of Definition 3.14. In fine units we put

$$V_*^f := r_0^3 V_*/2, \qquad R_{\mathrm{HK}} := (1 - \eta_w) c_I c_\rho L/3, \qquad K_0 := (1 - \eta_w) c_I c_\rho c_L/6 \leq \tilde{m} R_{\mathrm{HK}} \leq 2K_0, \tag{6.12}$$

with $c_I = 2^{2/3}$, $c_\rho = 1/12$, $V_* = (1 - \eta_w)^3 \rho_\lambda^3/2$ (coarse units). The bounds on $\tilde{m} R_{\mathrm{HK}}$ follow from Definition 3.1 (§S1.5). Let $p_t(x, y)$ be the kernel of $e^{t\Delta_{G'}}$, the transition probability of the walk X_t that jumps across each bond of G' at rate 1, and let $\tau(x, \rho) := \inf\{t : d_{G'}(x, X_t) > \rho\}$.

Theorem 6.8 (Heat kernel at scale R_{HK})**.** *Let G' be an output, and let C_H (a function of c_F of Proposition 6.9) and $\varepsilon_1^{\mathrm{HK}}$ be the absolute constants of §S1.5. There are $C_{\mathrm{mo}}, c_{\mathrm{ex}}, C_{\mathrm{gs}}, c_{\mathrm{gs}}, C_M > 0$, depending only on C_H, such that for $\beta_{\mathrm{GM}} \geq \beta_{\mathrm{GM}}^0$, $x, y \in V$ and $d := d_{G'}(x, y)$: (HK0) $p_t(x, y) \leq \exp(-\alpha d + 6(\cosh \alpha - 1)t)$ for all $\alpha, t \geq 0$; in particular $p_t(x, y) \leq e^{-d/17}$ for*

$t \le d$; *(HK1)* $p_t(x,y) \le \max\bigl(C_H t^{-3/2}, 4/V_*^f\bigr) \le C_H \max\bigl(t^{-3/2}, R_{\mathrm{HK}}^{-3}\bigr)$ *for all* $t > 0$; *(HK1′)* $\mathbb{E}_x d_{G'}(x, X_t) \le C_{\mathrm{mo}} t^{1/2}$ *for* $0 \le t \le R_{\mathrm{HK}}^2$; *(HK2)* $\mathbb{P}_x(\tau(x,\rho) < t) \le \sqrt{2}\exp(-c_{\mathrm{ex}}\min(\rho^2/t, \rho))$ *for* $\rho \ge 1$ *and* $0 < t \le \varepsilon_1^{\mathrm{HK}} R_{\mathrm{HK}}^2$; *(HK3)* $p_t(x,y) \le C_{\mathrm{gs}} t^{-3/2}\exp(-c_{\mathrm{gs}}\min(d^2/t, d))$ *for* $1 \le t \le 2\varepsilon_1^{\mathrm{HK}} R_{\mathrm{HK}}^2$; *(HK4)* $\mathbb{E}_x e^{\theta d_{G'}(x,X_s)} \le 1 + C_M \theta s^{1/2}$ *for* $0 < s \le \varepsilon_1^{\mathrm{HK}} R_{\mathrm{HK}}^2$ *and* $0 \le \theta \le \min(s^{-1/2}, c_{\mathrm{ex}}/2)$.

Proposition 6.9 (Coarse-to-fine transfer)**.** *Let G' be an output, read on the fine lattice. (i)* Isoperimetry. $|\partial_{G'} A| \ge c_F |A|^{2/3}$ *for every $A \subseteq V$ with $|A| \le V_*^f$, where the absolute constant c_F (§S1.5) does not depend on r_0. (ii)* Faber–Krahn, components. *For $\Omega \subseteq V$ with $|\Omega| \le V_*^f$, $\lambda_1^D(\Omega) \ge (c_F^2/12)|\Omega|^{-2/3}$, where $\lambda_1^D(\Omega)$ is the smallest eigenvalue of $-\Delta_{G'}$ on functions vanishing off Ω; and every component of G' has more than $2V_*^f$ vertices. (iii)* Cheeger transfer. *If a set K of coarse cells satisfies $|\partial_K W| \ge h_c \min(|W|, |K \setminus W|)$ for every $W \subseteq K$, then its blow-up K^f satisfies $|\partial_{K^f} A| \ge r_0^{-1}\min\{\frac{1}{2}, h_c/C_{\mathrm{bl}}\}\min(|A|, |K^f \setminus A|)$ for every $A \subseteq K^f$, with the absolute constant C_{bl} of §S1.5. (iv)* Cell Poincaré inequality. *For every fine cell Ω of a block Q there is a connected set $\mathfrak{K}^\flat$ with $\Omega \subseteq \mathfrak{K}^\flat \subseteq V \cap \hat{Q}(Q)$ such that $\sum_{x\in\Omega} |\varphi(x) - \bar{\varphi}_\Omega|^2 \le C_W^f k_\square^2 \sum_{b \in E(\mathfrak{K}^\flat)} (\nabla_b \varphi)^2$ for every $\varphi : V \to \mathbb{R}$, with $C_W^f := 108 C_{\mathrm{bl}}^2/\chi^2$ and $k_\square := r_0(k_P + 3)$; every fine site and every fine bond lies in at most $N_{\mathrm{ov}} \le 27(k_P/L_c)^3$ of the sets $\mathfrak{K}^\flat$, resp. $E(\mathfrak{K}^\flat)$; and $|\Omega| \ge c_1 L^3$. (v)* Diameter. *Under (L8-2) of §7.6, every graph $(\mathfrak{K}^\flat, E(\mathfrak{K}^\flat))$ is connected, of diameter $d_\flat \le 2 + 0.469\,\beta_{\mathrm{GM}}/\tilde{m} = 2 + 0.235 L'$ in its own graph distance; in particular every $\mathfrak{K}^\flat$ has $d_{G'}$-diameter less than L'.*

Proposition 6.10. *Let G' be an output, $\kappa_W := \min(1, 1/(2C_M))$ and $m_W := \min(1, c_{\mathrm{ex}}/2)$. For every mass $0 < M \le m_W$,*

$$\sup_x \sum_y v_{G'}^{(M)}(x,y) e^{\theta_M d_{G'}(x,y)} \le 3M^{-2}, \qquad \theta_M := \kappa_W M \min\bigl(1, (\varepsilon_1^{\mathrm{HK}})^{1/2} M R_{\mathrm{HK}}\bigr). \tag{6.13}$$

In particular, if $M_Y \le m_W$,

$$\sup_x \sum_y v_{G'}(x,y) e^{\kappa_Y M_Y d_{G'}(x,y)} \le 3M_Y^{-2}, \qquad \kappa_Y := \kappa_W \min\bigl(1, (\varepsilon_1^{\mathrm{HK}})^{1/2} K_0/\lambda_{\mathrm{PV}}\bigr), \tag{6.14}$$

and the same bound holds for $v_{G'}^2$ (since $0 \le v_{G'}^2 \le v_{G'}$ entrywise) and for $v_{G'}^1$ at mass $M_1 \le m_W$. Moreover $\kappa_Y M_Y \ge 4\kappa_0 \tilde{m}$, where $\kappa_0 := \kappa_W (\varepsilon_1^{\mathrm{HK}})^{1/2} K_0$.

We use only one property of the constant κ_W: since C_M depends only on the absolute constant C_H (Theorem 6.8), κ_W does not depend on τ, K, C or D.

Theorem 6.11 (Covariance bounds)**.** *Let G' be an output and $C_0 := (M_Y^{-2}\Delta_{G'}^2 - \Delta_{G'} + \tilde{m}^2)^{-1}$ on $\mathbb{R}^V$. Assume $\lambda_{\mathrm{PV}} \le \frac{1}{4}$, $M_Y \le m_W$ and $\tilde{m} \le \min(1/68, m_W/2)$ (all true for $\beta_{\mathrm{GM}} \ge \beta_{\mathrm{GM}}^0$). Put $\kappa_0' := \min(\kappa_0/2, (2c_{\mathrm{gs}})^{1/2})$ and $r_{\mathrm{cov}} := \min(\kappa_0, \kappa_0')\tilde{m}$, and let K_1, K_1', K_1'' and $K_3 \le 3.5$ be the constants of §S1.5, which depend only on $(c_F, c_\rho, c_L, \lambda_{\mathrm{PV}})$. Then for all $x, y \in V$: (H1)* $C_0(x,x) \le K_1 M_Y$ *(the zero modes included); (H1′)* $0 \le C_0(x,y) \le K_1' M_Y e^{-\kappa_0' \tilde{m} d_{G'}(x,y)}$; *(H2)* $\|C_0\|_{\kappa_0\tilde{m}} \le K_3 \tilde{m}^{-2}$; *($H_C$) if ν is symmetric with $\|\nu\|_{r_{\mathrm{cov}}} \le \varepsilon_\nu \tilde{m}^2$ and $2K_3\varepsilon_\nu \le 1$, then $C := (C_0^{-1} + \nu)^{-1}$ exists and is positive definite, $C(x,x) \le 2K_1 M_Y$, $|C(x,y)| \le 2K_1'' M_Y e^{-r_{\mathrm{cov}} d_{G'}(x,y)}$ and $\|C\|_{r_{\mathrm{cov}}} \le 2K_3\tilde{m}^{-2}$; the same bounds hold for every matrix C_s with $|C_s(x,y)| \le |C(x,y)|$ entrywise and $C_s(x,x) \le C(x,x)$, such as the interpolated covariances of Appendix C.*

6.4 Depletion

In this subsection we bound the number of depleted sites per cell. The face kernel of Theorem 6.15 contains all collective effects of the walls. By Chebyshev's inequality, (R9) bounds the number of depleted sites; sparse insulation costs a factor of order $\log\tau$ in this bound, and the estimates of Section 7 allow such a factor. Throughout, $\Lambda' = \Lambda^\circ$, $M := M_Y$, and $u_x := v_{\Lambda'}(\cdot, x)$ is the kernel of the box without holes. The *missing pairs* are the (y, y') with $y \in V$, $|y - y'| = 1$, and $y' \in \mathbb{X}$ (the set $\mathfrak{M}_{\mathbb{X}}$) or $\{y, y'\} \in S$ (the set $\mathfrak{M}_S$). Their union is $\mathfrak{M} := \mathfrak{M}_{\mathbb{X}} \cup \mathfrak{M}_S$. A frozen bond gives one pair at each end (frozen bonds are *two-sided*), and $N_2(U) := \#\{(y, y') \in \mathfrak{M} : y \in U\}$. For $x, y \in V$ we put $g_x(y) := \sum_{y' : (y,y') \in \mathfrak{M}} (u_x(y) - u_x(y'))$.

Lemma 6.12. *For every slit graph $G' \subseteq \Lambda'$ and $x \in V$: (a) for $y \in V$,*

$$\big((-\Delta_{G'} + M^2)u_x|_V\big)(y) = \delta_x(y) - g_x(y);$$

(b) $I_{G'}(x) = \sum_{(y,y') \in \mathfrak{M}} v_{G'}(x, y)\big[u_x(y) - u_x(y')\big]$.

Proof. For (a), at $y \in V$ the operator $-\Delta_{\Lambda'} + M^2$ is $-\Delta_{G'} + M^2$ plus the terms of the pairs of $\mathfrak{M}$ at y, and $(-\Delta_{\Lambda'} + M^2)u_x = \delta_x$. For (b) we apply $v_{G'}$ to (a) at x.

Lemma 6.13. *There is an absolute constant A_0^{AD} such that on every output, for every z and every $r \geq 1$ (fine units), $N_2(V \cap B(z, r)) \leq A_0^{\mathrm{AD}} r^2 + A_0^{\mathrm{AD}} r^3/(r_0\rho_\lambda)$.*

Proof. At the final state Lemma 4.4 applies to all sets $W \subseteq V_c$ of coarse cells, with $\partial_\Gamma W$ as the coarse cut $\partial_{G'} W$. In (4.1) a frozen bond with one end in W carries $w_b - 1 = 1 - \eta_w$ and one inside W carries $w_b \geq 2(1 - \eta_w)$, where $w_b = 2 - \eta_w$ is the hull weight. Hence the coarse count $N_2^c(W)$ of missing pairs with first entry in W is at most $(|\partial_\Gamma W| + \lambda|W|)/(1 - \eta_w)$. Here a frozen bond inside W gives two missing pairs with first entry in W, one for each orientation, and its weight pays for both; this reduces the density bound to the minimality of the hull on a ball. We first consider $r \geq r_0$, and let W be the set of coarse cells of V_c that meet $B(z, r)$. Then $|W| \leq C(r/r_0)^3$, and each bond of $\partial_\Gamma W$ joins a coarse cell meeting $B(z, r)$ to a coarse cell of V_c that does not, so $|\partial_\Gamma W| \leq C(r/r_0)^2$. Since a coarse missing pair carries fewer than $4r_0^2$ fine ones, $N_2(V \cap B(z, r)) \leq 4r_0^2 \cdot \frac{10}{9}\big(C(r/r_0)^2 + C(r/r_0)^3/\rho_\lambda\big)$. For $1 \leq r < r_0$, the fine missing pairs with first entry in $B(z, r)$ cross at most six coarse grid planes, each in at most Cr^2 bonds, counted with both orientations.

Lemma 6.14. *On every output, for all $x, y \in V$, $v_{G'}(x, y) \leq C_{\mathrm{Gr}}\big[(1 + |x - y|)^{-1} + \eta_G\big]$ with $\eta_G := \tilde{m}(K_0^{-1} + \lambda_{\mathrm{PV}}^2 K_0^{-3})$ and C_{Gr} depending only on the constants of Theorem 6.8; and $v_{G'}(x, y) = 0$ when x and y lie in different components.*

Proof. We split the time integral in $v_{G'} = \int_0^\infty e^{-M^2 t} p_t \, dt$, with $R := R_{\mathrm{HK}}$. On $[0, 1]$ we use $p_t \leq 1$ and (HK0). On $[1, 2\varepsilon_1^{\mathrm{HK}} R^2]$, (HK3), split at $t = d_{G'}(x, y)$, gives $C/(1 + |x - y|)$. Beyond this interval, (HK1) gives $C(R^{-1} + M^{-2}R^{-3}) \leq C\eta_G$, since $R \geq K_0/\tilde{m}$ and $M = \tilde{m}/\lambda_{\mathrm{PV}}$. Across components $v_{G'} = 0$ by Lemma 6.2(i). The zero mode of a component $\mathfrak{K}$ contributes $M^{-2}/|\mathfrak{K}| \leq 1/(2M^2 V_*^f)$, by Proposition 6.9(ii), and this lies inside $M^{-2}R^{-3}$.

Theorem 6.15 (Face-kernel bound)**.** *On every output, for every $x \in V$, with $r(y) := |x - y|$, $I_{G'}(x) \leq C_F S_3(x) + C_F \eta_G S_2(x)$, where $S_3(x) := \sum_{(y,y') \in \mathfrak{M}} (1 + r(y))^{-3} e^{-M_Y r(y)/2}$, $S_2(x) := \sum_{(y,y') \in \mathfrak{M}} (1 + r(y))^{-2} e^{-M_Y r(y)/2}$, $C_F := C_{\mathrm{Gr}} C_L'$ with C_L' the constant of Lemma D.2, and $S_2(x) \leq C_S \log(2/M_Y)$ with $C_S := C A_0^{\mathrm{AD}}(1 + \lambda_{\mathrm{PV}}/(c_\rho c_L))$ (§S1.5).*

Proof. We take absolute values in Lemma 6.12(b), so no cancellation across a sheet is used and the Neumann kernel appears only through its values. For each pair, $0 \le v_{G'}(x,y) \le C_{\mathrm{Gr}}[(1+r(y))^{-1}+\eta_G]$ by Lemma 6.14, and $|u_x(y)-u_x(y')| \le C_L'(1+r(y))^{-2}e^{-M_Y r(y)/2}$ by (L2) of Lemma D.2. That lemma applies with $M=M_Y$, because $|y-y'|=1$, the sides of Λ° are, by Lemma 3.3(c), at least $\ell_D \ge \frac{1}{2}\tilde{m}^{-1} \ge M_Y^{-1}$, and $M_Y \le \frac{1}{5}$ for $\beta_{\mathrm{GM}} \ge \beta_{\mathrm{GM}}^0$. A frozen bond appears twice, each time with its own first entry. For the bound on S_2 we sum Lemma 6.13 over dyadic shells $2^k \le r < 2^{k+1}$. Each shell inside $r_0\rho_\lambda$ gives at most $8A_0^{\mathrm{AD}}$, and up to $r \le M_Y^{-1}$ these contributions yield the logarithm. For the shells beyond we use $e^{-M_Y r/2}$, and the growth term gives at most $CA_0^{\mathrm{AD}}/(M_Y r_0\rho_\lambda) \le 2CA_0^{\mathrm{AD}}\lambda_{\mathrm{PV}}/(c_\rho c_L)$ (§S1.5(f)).

We put $\varepsilon_0 := \theta_0/(2\pi^2\tau)$, so that $\Phi(x) > \theta_0$ if and only if $v_{G'}(x,x) - v(0) > \varepsilon_0$, and $\varepsilon_F := \varepsilon_0/(4C_F)$. With the constants L_g, C_R of §S1.5, let $R_{\mathrm{dep}} := \lceil 32A_0^{\mathrm{AD}}L_g/\varepsilon_F\rceil = \lceil C_R\tau/\theta_0\rceil$, and let (F) be the condition $C_F\eta_G C_S\log(2/M_Y) \le \varepsilon_0/4$. The constant $c_0 := z\tau\,\bar{r}_0^2(w_* + 2|\partial\Delta|)$, maximized over removal cells Δ, comes from the two-sided bound (R9), proved in Proposition 4.26. It satisfies $c_0 < 25(3+c_s)$ (§S1.2).

Theorem 6.16 (Depletion per removal cell)**.** *Let G' be an output and assume $R_{\mathrm{dep}} \le r_0\rho_\lambda$, $R_{\mathrm{dep}} \le \ell_D$ and (F). Then for every removal cell Q, $\#(\mathfrak{D}\cap Q) \le C(z\theta_0)^{-1}[c_0 + C_{\mathrm{box}} + c_0^{\mathrm{sp}}\log(C'\tau/\theta_0)]$, where $c_0^{\mathrm{sp}} \le c_0$ is the part of the insulation lying in sparse blocks (defined in the proof), C_{box} is the box-face term, and C, C' are absolute up to C_R and C_F.*

Proof. (i) *Reduction to S_3.* Since $v_{G'}(x,x) - v(0) = I_{G'}(x) + [v_{\Lambda'}(x,x) - v(0)]$, we have $\mathfrak{D} \subseteq \{I_{G'} > \varepsilon_0/2\} \cup \{\Phi_{\Lambda'} > \theta_0/2\}$, where $\Phi_{\Lambda'}(x) := \frac{1}{2}\beta_{\mathrm{GM}}(v_{\Lambda'}(x,x) - v(0))$. On the first set Theorem 6.15 and (F) give $S_3(x) > \varepsilon_F$.

(ii) *Far insulation.* The pairs at distance more than $R := R_{\mathrm{dep}}$ from x contribute to $S_3(x)$ at most $\sum_{k\ge 0}(2^kR)^{-3}N_2(V\cap B(x,2^{k+1}R))e^{-M_Y 2^k R/2}$. By Lemma 6.13, with $R \le r_0\rho_\lambda$ and $M_Y r_0\rho_\lambda \ge c_\rho c_L/(2\lambda_{\mathrm{PV}})$, this is at most $16A_0^{\mathrm{AD}}L_g/R \le \varepsilon_F/2$ (§S1.5(g)). There the shells with $2^{k+1}R \le r_0\rho_\lambda$ give $16A_0^{\mathrm{AD}}/R$, and the others give a logarithm. Hence each $x \in \mathfrak{D}$ outside the box set has $S_3^{\mathrm{near}}(x) > \varepsilon_F/2$, where S_3^{near} counts the pairs with first entry within R of x.

(iii) *Blocks.* We cover Q by grid cubes q of side R. Let $3q$ be the block of $3\times 3\times 3$ such cubes centred at q, put $N_{3q} := N_2(V\cap 3q)$, and call q dense if $N_{3q} \ge R^2$, sparse otherwise. A dense q has $\#(\mathfrak{D}\cap q) \le R^3 \le RN_{3q}$. For a sparse q, Chebyshev's inequality gives $\#(\mathfrak{D}\cap q) \le (2/\varepsilon_F)\sum_{x\in q} S_3^{\mathrm{near}}(x) \le (2/\varepsilon_F)N_{3q}(4\pi\log(4R) + C_3)$ plus the box set. Each pair lies in at most 27 blocks $3q$, and the pairs counted for $q\cap Q \ne \emptyset$ lie within $2R \le 2r_0\rho_\lambda < \ell_D$ of Q, hence in the 27 removal cells Q' around Q, as the removal cells form a product grid with sides at least ℓ_D. Each of these cells has $N_2(Q') \le 4r_0^2(w_* + 2|\partial\Delta|) \le 4c_0/(z\tau)$ by (R9), since a coarse pair carries fewer than $4r_0^2$ fine ones and removed cells contain no site of V. Hence $A_{\mathrm{dense}} + A_{\mathrm{sparse}} \le 4\cdot 27^2 c_0/(z\tau)$, where $A_\bullet := \sum_q {}_\bullet\, N_{3q}$. We put $c_0^{\mathrm{sp}} := z\tau A_{\mathrm{sparse}}/(4\cdot 27^2) \le c_0$ and substitute $R \le 2C_R\tau/\theta_0$ and $2/\varepsilon_F = 16\pi^2 C_F\tau/\theta_0$.

(iv) *Box faces.* By the image representation of the box kernel (proof of Lemma D.2) and (L1), $v_{\Lambda'}(x,x) - v(0) \le C\sum_F d_F(x)^{-1} + 3C_{\mathrm{im}}/\ell_D$, the sum over the faces F of Λ', $d_F(x)$ the layer distance of x from F, and the last term from the images other than the 26 nearest ones, which have a coordinate at distance at least $2n_\mu \ge 2\ell_D$ from x (n_μ the sides of Λ'). As $R_{\mathrm{dep}} \le \ell_D$ and $C_R \ge 24\pi^2 C_{\mathrm{im}}$ (§S1.5), $3C_{\mathrm{im}}/\ell_D \le \theta_0/(2\beta_{\mathrm{GM}})$. Hence the set $\{\Phi_{\Lambda'} > \theta_0/2\}$ lies within distance $C\tau/\theta_0$ of $\partial\Lambda'$ and meets a removal cell, of side at most $3\ell_D$, in at most $C\ell_D^2\tau/\theta_0 \le C_{\mathrm{box}}/(z\theta_0)$ sites, since $\ell_D^2\tau \le C\tau/\tilde{m}^2 \le C/z$.

Proposition 6.17. *Under the hypotheses of Theorem 6.16, for every $\mathcal{U}$-cell $\Delta_{\mathcal{U}}$ of the output, $(\tilde{m}^2/\beta_{\mathrm{GM}})\,\#(\mathfrak{D}\cap\Delta_{\mathcal{U}}) \le c_{\mathrm{dep}} := A_0^D + A_1^D \log(C'\tau/\theta_0)$, with $A_0^D := CN_{\mathrm{cov}}(c_0 + C_{\mathrm{box}})/\theta_0$, $A_1^D := CN_{\mathrm{cov}}c_0^{\mathrm{sp}}/\theta_0$ and $N_{\mathrm{cov}} \le (C_{\mathcal{U}}\tilde{m}^{-1}/\ell_D + 6)^3$.*

Proof. Steps (i) to (iv) of the proof of Theorem 6.16 apply to any region U in place of Q, the pairs being counted in the union of the blocks $3q$ over the cubes q meeting U. By (R7) the cell $\Delta_{\mathcal{U}}$ has diameter at most $C_{\mathcal{U}}\tilde{m}^{-1}$. Hence that union lies in a cube of side $C_{\mathcal{U}}\tilde{m}^{-1} + 4R$, which meets at most N_{cov} removal cells, each with sides $\ge \ell_D \ge R$ and with $N_2 \le 4c_0/(z\tau)$. Finally $\tilde{m}^2/\beta_{\mathrm{GM}} \le 4m_D^2/\beta_{\mathrm{GM}} = 8z$. Since a $\mathcal{U}$-cell lies in one component and depletion is counted site by site, the bound does not depend on the number of components.

Lemma 6.18. *Let $\theta_0 = 0.01$ and*

$$C_*^{\mathrm{dep}} := \max\{17.8\,C_R/(c_\rho c_L),\ 3.6\cdot 10^3 C_F C_S(K_0^{-1} + \lambda_{\mathrm{PV}}^2 K_0^{-3})\}.$$

If $\tau \ge 2$, $2.494\,\tau \ge \ln(C_*^{\mathrm{dep}}/\theta_0) + 2.5\ln\tau$ and $m_D/2 \le \tilde{m} \le 2m_D$, then $R_{\mathrm{dep}} \le r_0\rho_\lambda$, $R_{\mathrm{dep}} \le \ell_D$ and (F) hold.

Proof. By hypothesis, $C_*^{\mathrm{dep}}\tau^{5/2}e^{-2.494\tau} \le \theta_0$. With $\tilde{m} \le 2m_D$, $r_0\rho_\lambda = c_\rho L \ge c_\rho c_L/(2\tilde{m})$ and $\log(2/M_Y) \le \ln(1/m_D) \le 2.4944\,\tau$, the three conclusions follow by arithmetic (§S1.5).

The value $\theta_0 = 0.01$ is dictated by Section 7, which needs $\varepsilon_\nu + \theta' \le 1/7$ with $\theta' \ge 1 - e^{-\theta_0}$. The condition of Lemma 6.18 is a lower bound on τ.

7 Stage 2: the single-scale expansion and the deterministic area law

In this section we prove Theorem 7.57: for each output $G' = G'(D)$ of the construction and each rectangle C with $\min(R,T) \ge R_1(\beta)$, $W_{G'}(C) \le \exp(-\sigma_V[A_{G'}(C) - \eta_S RT] + c_N(R+T))$. We carry out the single-scale analysis of [GM82, Section 6, Appendices C to E], with the lemmas of [BF80, Section 9] that it uses, on G'. The structural changes are the following. First, the outer partition functions are graded by the class of their walls (their periods around the handles of the insulating set), and the Kirkwood–Salsburg system of [BF80, Appendix 4] is solved for all classes at once in a weighted ℓ^1 algebra. Second, the handles enter the bounds in a single place: the area weight of a class comes from one bound at fixed class (Proposition 7.53). In Subsections 7.1 to 7.5 we derive the class-graded polymer expansion (Corollary 7.28), and in Subsections 7.6 to 7.9 we bound it. Propositions 7.3, 7.4 and 7.7 are proved in full with Proposition 7.22, in Section S4.5.

7.0 Standing conventions

We use the following conventions in Section 7 and Appendices A to C. The *cells* of Definition 3.14 may be disconnected. Interfaces, walls and tubes are defined on cells, and polymers, seeds and the Kirkwood–Salsburg system on the $\mathcal{U}$-cells of Theorem 3.20 (R7). By $|X|$ we mean the number of $\mathcal{U}$-cells of X, not a Debye volume, and the constants of Appendix B are per $\mathcal{U}$-cell. We work in fine lattice units, $a = 1$. Sums over sites and bonds are plain sums, each bond counted once. The collar scale L' is fixed by $\tilde{m}L' = K_{\mathrm{col}}\beta_{\mathrm{GM}}$, $K_{\mathrm{col}} = 2$, where $\beta_{\mathrm{GM}} = 4\pi^2\tau$. The min-cut area $A_{G'}(C)$ of Definition 2.11 (a support count) and the forced wall

content $A(p)$ of Lemma 7.43 (an ℓ^1 quantity) are related by Theorem 7.14 and Lemma 7.55. Constants C, c are absolute or depend only on the constants fixed before K (Section 8), and change from line to line. We write $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ for the ℓ^2 inner product and norm on V or E, and $\|\varphi\|_Y^2 := \sum_{x\in Y}\varphi(x)^2$.

7.1 Representation

Throughout, $G' = (V, E)$ is the fine slit graph of Subsection 2.5 for the set (2.9), with set of components $\mathrm{Comp}(G') \ni \mathfrak{K}$. From Subsection 7.2 on, G' will be an output of the construction.

Definition 7.1 (Operators and the loop on G')**.** (a) For an oriented bond $b = (x, y) \in E$ and $\varphi : V \to \mathbb{R}$, $\nabla\varphi(b) := \varphi(y) - \varphi(x)$, and $\Delta := -\nabla^*\nabla$ is the Neumann Laplacian of G'. (b) On $\mathbb{R}^V$, $u^{-1} := M_Y^{-2}(-\Delta)(-\Delta + M_Y^2)$ and $C_0 := (u^{-1} + \tilde{m}^2)^{-1} = F(-\Delta)$, $F(t) := (M_Y^{-2}t^2 + t + \tilde{m}^2)^{-1}$. (c) For an integer 2-chain Ξ in Λ° with $\partial\Xi = C$ and $b = (x, y) \in E$, ξ_b is the coefficient in Ξ of the plaquette p_b dual to b, oriented as ∂c_y (c_y the cube of y), and $j(b) := 2\pi\xi_b$ (the opposite orientation of Subsection 2.5 amounts to $\Xi \mapsto -\Xi$ and leaves (2.8) unchanged, $m \mapsto -m$). With $n = 2\pi m$, (2.8) reads $Z(k, \Xi) := Z_{G'}(k, \Xi) = \sum_{[n]} \exp(-\frac{1}{2\beta_{\mathrm{GM}}}\sum_{b\in E}(\nabla n(b) - kj(b))^2)$, $[n]$ running over $(2\pi\mathbb{Z})^V$ modulo functions constant on each component; $W_{G'}(C) = Z(1, \Xi)/Z(0, \Xi)$ does not depend on Ξ (Proposition 2.10). (d) The Neumann solid angle $f = f_{G'}$ solves $-\Delta f = \nabla^* j$; $R_f := \nabla f - j$ and $E_{G'} := \|R_f\|^2$.

The operator C_0 is positive definite, since $u^{-1} \geq 0$. As C_0 and u^{-1} are both functions of $-\Delta$ and $\Delta\mathbf{1}_{\mathfrak{K}} = 0$, we have

$$C_0\mathbf{1}_{\mathfrak{K}} = \tilde{m}^{-2}\mathbf{1}_{\mathfrak{K}} \quad (\mathfrak{K} \in \mathrm{Comp}(G')), \qquad \tilde{m}^2 C_0 - 1 = -C_0 u^{-1} = -u^{-1}C_0. \tag{7.1}$$

Since both ends of each bond lie in one component, $\nabla^* j \perp \ker\Delta$. Hence f exists, unique up to one constant per component, and R_f, $E_{G'}$ do not depend on the choice.

Lemma 7.2. *Let Π be the orthogonal projection of $\ell^2(E)$ onto $\operatorname{ran}\nabla$. Then $\nabla f = \Pi j$ and $R_f = -(1-\Pi)j$; hence $\nabla^* R_f = 0$, $\langle\nabla\varphi, R_f\rangle = 0$ for every $\varphi : V \to \mathbb{R}$, $\langle j, R_f\rangle = -E_{G'}$, and $E_{G'} = \min_\varphi \|\nabla\varphi - j\|^2$. For every $n : V \to \mathbb{R}$ and $k \in \mathbb{Z}$, $\sum_E(\nabla n - kj)^2 = \sum_E(\nabla(n - kf))^2 + k^2 E_{G'}$, so $Z(k, \Xi) = e^{-k^2 E_{G'}/2\beta_{\mathrm{GM}}}\,\tilde{Z}(k, \Xi)$ with $\tilde{Z}(k, \Xi) := \sum_{[n]} \exp\bigl(-\frac{1}{2\beta_{\mathrm{GM}}}\sum_E(\nabla(n - kf))^2\bigr)$.*

Proof. The solid angle is the gradient part of the loop current, since f solves the Euler–Lagrange equation $\nabla^*(\nabla\varphi - j) = 0$ of $\|\nabla\varphi - j\|^2$. Hence $\nabla f = \Pi j$ and $R_f \in \operatorname{ran}(1-\Pi) = \ker\nabla^*$, and the identities follow. For the last one we write $\nabla n - kj = \nabla(n - kf) + kR_f$ and use $R_f \perp \operatorname{ran}\nabla$.

The input from Stage 1. We use (S1) to (S5) of Theorem 6.5, with $\xi = (m, x)$ and $\int d\xi$ as there, u and $d\mu_u$ of Proposition 6.3 (on $\mathbb{R}_0^V$, where u inverts u^{-1}), and $P := 2\pi\beta_{\mathrm{GM}}^{-1/2}$. By (6.4) with $\theta_{\mathfrak{K}} = \beta_{\mathrm{GM}}^{1/2} c_{\mathfrak{K}}$ there is $\mathcal{N}_0 > 0$, independent of k and Ξ, such that

$$\tilde{Z}(k, \Xi) = \mathcal{N}_0 \int_{[0,P)^{\mathrm{Comp}(G')}} \prod_{\mathfrak{K}} \frac{dc_{\mathfrak{K}}}{P} \int d\mu_u(\varphi)\, e^{-V_{\mathrm{eff}}(\varphi + c \mid k\Xi)}, \qquad c := \sum_{\mathfrak{K}} c_{\mathfrak{K}}\mathbf{1}_{\mathfrak{K}}, \tag{7.2}$$

$$-V_{\mathrm{eff}}(\varphi \mid k\Xi) = \rho_0 + \sum_{s\geq 1} \frac{1}{s!} \int d\xi_1 \cdots d\xi_s\, \rho_s(\xi_1, \ldots, \xi_s)\, \varepsilon_k(\xi_1) \cdots \varepsilon_k(\xi_s), \tag{7.3}$$

with $\varepsilon_k(\xi) := e^{im(\beta_{\mathrm{GM}}^{1/2}\varphi(x)+kf(x))} - 1$ and the real coefficients $\rho_s = \rho_s^{G'}$ of (S1), and the series converges absolutely, uniformly in real φ. By (S3), $V_{\mathrm{eff}}(\varphi + PI \mid k\Xi) = V_{\mathrm{eff}}(\varphi \mid k\Xi)$ for $I \in \mathbb{Z}^V$. On outputs the dipole kernel ν of (S5) has $\|\nu\|_{r_{\mathrm{cov}}} \le \varepsilon_\nu \tilde{m}^2$, with $2K_3\varepsilon_\nu \le 1$. The space $\mathbb{R}_0^V$, the norms $\|\cdot\|_r$, the image energy $\Phi = \Phi_{G'} \ge 0$ and the depleted set $\mathfrak{D}$ are those of the conventions of Section 6. The quantities ρ^{ref}, $\tilde{m}$ and $M_Y = \tilde{m}/\lambda_{\mathrm{PV}}$ are as in (6.11), and (S4′) denotes Proposition 6.6 together with (6.10) for ρ^{ref} in place of $\rho_1(\cdot, x)$ (proved in Section S4.3).

Proposition 7.3 ([GM82, (6.1) to (6.4)] on G')**.** *Assume (S1) to (S3). (a) $Z(k,\Xi) = e^{-k^2 E_{G'}/2\beta_{\mathrm{GM}}}\tilde{Z}(k,\Xi)$, with $\tilde{Z}$ given by (7.2): one zero-mode variable $c_{\mathfrak{K}} \in [0,P)$ per component of G'. (b) The integrand of (7.2) is P-periodic in each $c_{\mathfrak{K}}$; hence (a) does not change when f is changed by a constant on each component. (c) ρ_0 cancels in $W_{G'}(C)$.*

Lemma 7.2 and (7.2) give part (a) as in [GM82, (6.1) to (6.4)], except that there is one zero mode per component (GM's Gaussian measure is defined on $(a\mathbb{Z})^3$ and has none), the loop entering only through the phases ε_k, and that $e^{-k^2 E_{G'}/2\beta_{\mathrm{GM}}}$ replaces $e^{-k^2(J,v_{Cb}J)/2\beta}$. Part (b) is (S3) with $I = \mathbf{1}_{\mathfrak{K}}$. Part (c) holds as in [GM82, (6.5)], because ρ_0 gives the same factor in $Z(1,\Xi)$ and $Z(0,\Xi)$. By (c), we set $\rho_0 = 0$ from now on.

Cells, cell averages and block spins. We write $-V_{\mathrm{eff}}(\varphi \mid k\Xi) = \int d\xi\, \rho_1(\xi)\varepsilon_k(\xi) + E'(\varphi)$, where E' collects the terms $s \ge 2$ of (7.3). The cells $\Omega = \Omega_{Q,\hat{\mathfrak{K}}}$ of Definition 3.14 partition V, each in one component. For a cell Ω and $x \in \Omega$ let $A(x) = A_\Omega$ and $\bar{f}(x) = \bar{f}_\Omega$ be the averages of φ and f over Ω, $\delta := \varphi - A$, $\delta f := f - \bar{f}$ and $\tilde{\delta} := \delta + k\beta_{\mathrm{GM}}^{-1/2}\delta f$. Then $\sum_{x\in\Omega}\tilde{\delta}(x) = 0$ for each cell, and $\beta_{\mathrm{GM}}^{1/2}\varphi + kf = (\beta_{\mathrm{GM}}^{1/2}A + k\bar{f}) + \beta_{\mathrm{GM}}^{1/2}\tilde{\delta}$. The block spins are the elements of $\mathcal{H} := \{h : V \to P\mathbb{Z} \text{ constant on every cell}\}$.

Proposition 7.4. *Assume (S3) and $\sum_m m^2|\rho^{\mathrm{ref}}(m)| < \infty$. (a) For every real φ on V, $e^{\int d\xi\,\rho_1(\xi)\varepsilon_k(\xi)} = \Theta(\varphi)\, e^{G(\varphi)}$ with $\Theta(\varphi) := \sum_{h\in\mathcal{H}} e^{-\frac{1}{2}\tilde{m}^2\|\varphi+k\beta_{\mathrm{GM}}^{-1/2}f-h\|^2}$, $G = G_1 + G_2$, $e^{G_1} = \prod_\Omega r_\Omega(A_\Omega)$, $G_2 = G_{2,1} + G_{2,2} + G_{2,3}$,*

$$r_\Omega(A) := \frac{\exp\{|\Omega|\sum_m \rho^{\mathrm{ref}}(m)\,[e^{im(\beta_{\mathrm{GM}}^{1/2}A + k\bar{f}_\Omega)} - 1]\}}{\sum_{n\in\mathbb{Z}} \exp\{-\frac{1}{2}\tilde{m}^2|\Omega|\,(A + k\beta_{\mathrm{GM}}^{-1/2}\bar{f}_\Omega - nP)^2\}}, \tag{7.4}$$

$$\begin{aligned}
G_{2,1} &:= \int d\xi\, \rho^{\mathrm{ref}}(m)\,[e^{im\beta_{\mathrm{GM}}^{1/2}\tilde{\delta}(x)} - 1 - im\beta_{\mathrm{GM}}^{1/2}\tilde{\delta}(x) + \tfrac{1}{2}m^2\beta_{\mathrm{GM}}\tilde{\delta}(x)^2],\\
G_{2,2} &:= \int d\xi\, \rho^{\mathrm{ref}}(m)\,[e^{im(\beta_{\mathrm{GM}}^{1/2}A(x) + k\bar{f}(x))} - 1]\,[e^{im\beta_{\mathrm{GM}}^{1/2}\tilde{\delta}(x)} - 1 - im\beta_{\mathrm{GM}}^{1/2}\tilde{\delta}(x)],\\
G_{2,3} &:= \int d\xi\, \big(\rho_1(m,x) - \rho^{\mathrm{ref}}(m)\big)\,[e^{im(\beta_{\mathrm{GM}}^{1/2}\varphi(x) + kf(x))} - 1].
\end{aligned} \tag{7.5}$$

Θ, r_Ω, G_1 and the $G_{2,i}$ are real, $\Theta > 0$, $r_\Omega > 0$, and e^G does not depend on h. The identity holds for every partition of V in place of the cells. (b) Under (S1) to (S3),

$$\tilde{Z}(k,\Xi) = \mathcal{N}_0 \sum_{h\in\mathcal{H}} \int \prod_{\mathfrak{K}} \frac{dc_{\mathfrak{K}}}{P} \int d\mu_u(\varphi)\, e^{-\frac{1}{2}\tilde{m}^2\|\varphi+c+k\beta_{\mathrm{GM}}^{-1/2}f-h\|^2}\, e^{G(\varphi+c)}\, e^{E'(\varphi+c)}. \tag{7.6}$$

We obtain part (a) as in [GM82, (6.6), (6.8) to (6.11)] and part (b) as in [GM82, (6.12)], with cells in place of blocks, $|\Omega|$ in place of L^3 and $\rho^{\mathrm{ref}}(m)$ in place of $\rho_1(m,0)$ in [GM82, (6.7), (6.9), (6.10)], since on G' the one-body coefficient depends on the site. The difference is the term $G_{2,3}$. Also, (6.12) needs a sign correction. In (b) we move the sum over h out of the integrals by Tonelli's theorem, as the moduli of the h-terms sum to $e^{\int \rho_1 \varepsilon_k}|e^{E'}|$.

The deficit term. Since $e^{im\beta_{\mathrm{GM}}^{1/2}h}=1$ for $h\in\mathcal{H}$, (S3) for ρ_1 and ρ^{ref} gives, for each $h\in\mathcal{H}$ and with $u_h:=\beta_{\mathrm{GM}}^{1/2}(\varphi+k\beta_{\mathrm{GM}}^{-1/2}f-h)$, $G_{2,3}=\sum_m\sum_{x\in V}(\rho_1(m,x)-\rho^{\mathrm{ref}}(m))(\cos(m\,u_h(x))-1)$, and $G_{2,3}(Y)$ is the same sum over $x\in Y$. For a union Y of $\mathcal{U}$-cells, $N_{\mathfrak{D}}(Y)$ denotes the number of its $\mathcal{U}$-cells that meet the depleted set $\mathfrak{D}=\{\Phi>\theta_0\}$, $\theta_0=0.01$.

Proposition 7.5. *Assume (S3) and (S4′), and put*

$$\theta' := 1-e^{-\theta_0}+C\big(e^{-\mu}+e^{-2\kappa_S}\big). \tag{7.7}$$

Uniformly in real φ, in $h\in\mathcal{H}$ and in G': (a) $|G_{2,3}(V\setminus\mathfrak{D})|\le\theta'\cdot\frac12\tilde m^2\|\varphi+k\beta_{\mathrm{GM}}^{-1/2}f-h\|^2_{V\setminus\mathfrak{D}}$; (b) for every $Y\subseteq V$, $-C(e^{-\mu}+e^{-2\kappa_S})\frac{\tilde m^2}{\beta_{\mathrm{GM}}}\#(\mathfrak{D}\cap Y)\le G_{2,3}(\mathfrak{D}\cap Y)\le C\frac{\tilde m^2}{\beta_{\mathrm{GM}}}\#(\mathfrak{D}\cap Y)$; (c) if G' is an output of the construction, then $e^{-Cc_{\mathrm{dep}}N_{\mathfrak{D}}(Y)}\le e^{G_{2,3}(\mathfrak{D}\cap Y)}\le e^{Cc_{\mathrm{dep}}N_{\mathfrak{D}}(Y)}$ for every union Y of $\mathcal{U}$-cells, with c_{dep} the constant of Proposition 6.17.

Sketch. Off $\mathfrak{D}$, $\Phi\le\theta_0$ and (S4′) give $\sum_m|\rho_1(m,x)-\rho^{\mathrm{ref}}(m)|m^2\le\theta'\tilde m^2/\beta_{\mathrm{GM}}$ (Section S1.6), and $|\cos(mu)-1|\le\frac12 m^2u^2$ gives (a). On $\mathfrak{D}$ the deviation is at most the density, and its negative part is at most $Ce^{-\mu}\rho^{\mathrm{ref}}(1)$ plus the tails, as $e^{-\Phi}\le1$. This proves (b). Part (c) follows from (b) and Proposition 6.17, which gives $(\tilde m^2/\beta_{\mathrm{GM}})\#(\mathfrak{D}\cap\Delta_{\mathcal{U}})\le c_{\mathrm{dep}}$ for each $\mathcal{U}$-cell $\Delta_{\mathcal{U}}$.

In Lemma 7.29 we will move the quadratic part of $G_{2,3}(V\setminus\mathfrak{D})$ into the covariance. Throughout Section 7 we assume the *standing hypothesis*

$$\varepsilon_\nu+\theta'\le\tfrac17, \tag{7.8}$$

which, by Lemma 6.18 and Section 8, holds for $\theta_0=0.01$, μ and κ_S large, and $\beta_{\mathrm{GM}}\ge\beta^0_{\mathrm{GM}}$.

Walls. For $h\in\mathcal{H}$ we put $n:=(\beta_{\mathrm{GM}}^{1/2}/2\pi)h\in\mathbb{Z}^V$, define the *wall field* $w:=\nabla h-k\beta_{\mathrm{GM}}^{-1/2}j$ on E, and put $N:=(\beta_{\mathrm{GM}}^{1/2}/2\pi)w$, that is, $N(b)=n(y)-n(x)-k\xi_b\in\mathbb{Z}$ for $b=(x,y)$. The wall bonds are $\Sigma_E:=\{b\in E:N(b)\neq0\}$, the wall sites Σ_V are their endpoints, and we put $N^\vee:=\sum_{b\in E}N(b)p_b$, with p_b oriented as in Definition 7.1 (c). A wall is where the cell-constant field fails to absorb the jump of the loop current, which is constant on interfaces because they are aligned with Ξ.

Lemma 7.6 (Walls)**.** *Let $h\in\mathcal{H}$ and $k\in\mathbb{Z}$. (a) $N(b)\neq0$ only on interface bonds, and N is constant on each interface I; we write N_I. (b) Frozen bonds, bonds into $\mathbb{X}$ and bonds leaving Λ° are not in E; they carry no wall. (c) $\partial N^\vee=-kC+\partial Y$ for an integer 2-chain Y supported in $\mathfrak{F}=F(\mathbb{X})\cup F_S\cup\partial_2\Lambda^\circ$. (d) For every cycle γ of G', $\sum_{b\in\gamma}N(b)=-k\,\mathrm{lk}(\gamma,C)$. Moreover $N(h)=N(h')$ if and only if $h-h'$ is constant on every component of G'; so N determines the class $[h]$ of h modulo $\mathcal{C}_P:=\{\sum_{\mathfrak{K}}Pn_{\mathfrak{K}}\mathbf{1}_{\mathfrak{K}}:n_{\mathfrak{K}}\in\mathbb{Z}\}$. (e)* (Relatively closed walls.) *For an integer field $M:E\to\mathbb{Z}$ with $M^\vee:=\sum_{b\in E}M(b)p_b$, the following are equivalent: (i) M is relatively closed: $\partial M^\vee=\partial Y$ for an integer 2-chain Y supported in $\mathfrak{F}$; (ii) M has zero periods: $\sum_{b\in\gamma}M(b)=0$ for every cycle γ of G'; (iii) $M=\nabla n'$ for some $n':V\to\mathbb{Z}$.*

Proof. (a) We have $\nabla h = 0$ on bonds inside a cell, which do not cross Ξ by Lemma 3.5 (A1). Other bonds join two blocks, and by Theorem 3.20 (R7) an interface lies in one block face, which lies in Ξ or misses it. (b) is the definition of E. (c) Extend n by 0 to $\mathbb{X}$ and let $U := \sum_{x\in V} n(x)c_x$. A plaquette of a cube of V not dual to a bond of E is a face of a hole cube, a frozen face or in $\partial_2\Lambda^\circ$, so $\partial U = \sum_{b\in E} \nabla n(b) p_b + Y_1$ with $\operatorname{supp} Y_1 \subseteq \mathfrak{F}$. Likewise, $\sum_{b\in E} \xi_b p_b = \Xi - Y_2$, with Y_2 the part of Ξ in $\mathfrak{F}$, since Ξ is interior to Λ° by Lemma 3.5 and $F(\mathbb{X})$ contains all faces of hole cubes. So $N^\vee = \partial U - Y_1 - k\Xi + kY_2$ and $\partial N^\vee = -kC + \partial(kY_2 - Y_1)$. Thus the wall spans C relative to the insulating set and may end on faces of hole cubes, on frozen faces and on $\partial\Lambda^\circ$. (d) $\sum_{b\in\gamma} \nabla n(b) = 0$, and $\sum_{b\in\gamma} \xi_b$ counts the crossings of γ with Ξ, that is, $\mathrm{lk}(\gamma, C)$. If $N(h) = N(h')$, then $\nabla(n - n') = 0$. (e) Since the box is acyclic, the relative cycle $M^\vee - Y$ is a boundary ∂W, which a cycle of G' crosses as often inwards as outwards. As plaquettes dual to bonds of E are not in $\mathfrak{F}$, this proves (i)⇒(ii). Integrating along paths, we obtain (ii)⇔(iii), and the computation of (c) with $k = 0$ gives (iii)⇒(i).

Handles. In $\mathbb{Z}^3$ the period condition of Lemma 7.6 (d) is local, and by Lemma 7.6 (e) walls with zero periods are removed by $h \mapsto h - Pn'$. In G' a cycle γ around a handle of the insulating set with $\mathrm{lk}(\gamma, C) = 0$ may be crossed once each, with opposite signs, by two separated wall pieces. We therefore write the shift in the source form of Proposition 7.8 and, in Definition 7.25, grade the outer partition functions by the class of their walls.

Proposition 7.7. *Assume (S1) to (S3) and (S5). For each $h \in \mathcal{H}$ let $g = g_h : V \to \mathbb{R}$ be* equivariant*: $g_{h+c} = g_h + c$ for $c \in \mathcal{C}_P$. Put $H := h - k\beta_{\mathrm{GM}}^{-1/2} f$, $C^{-1} := C_0^{-1} + \nu$, $E(\psi) := E'(\psi+g) + \frac{1}{2}\langle\psi, \nu\psi\rangle$, $g_c := \tilde{m}^2 C_0 H$, $F_1 := \frac{1}{2}\tilde{m}^2\|g - H\|^2 + \frac{1}{2}\langle g, u^{-1}g\rangle$ and $F_2 := \langle\psi, C_0^{-1}(g - g_c)\rangle$. Then $C^{-1} \geq \frac{1}{2}C_0^{-1} > 0$, and with $d\mu_C$ the centred Gaussian measure on $\mathbb{R}^V$ with covariance C,*

$$\tilde{Z}(k, \Xi) = \sum_{[h]\in\mathcal{H}/\mathcal{C}_P} \mathcal{N} \int d\mu_C(\psi)\, e^{E(\psi)}\, e^{G(\psi+g)}\, e^{-F_1}\, e^{-F_2}, \tag{7.9}$$

where $\mathcal{N} > 0$ depends on G', M_Y, $\tilde{m}$ and ν, and not on k, Ξ or h. Each term of (7.9) *does not depend on the representative of $[h]$.*

The proof follows [GM82, (6.13) to (6.20)], with $\chi_\Lambda \equiv 1$ on V and E the E of [GM82, (6.14c)], except that the sum over h is unfolded over $\mathcal{H}/\mathcal{C}_P$ and the mass acts at every site. The integrand of (7.6) is invariant under $(\varphi + c, h) \mapsto (\varphi + c + c', h + c')$, $c' \in \mathcal{C}_P$, since each cell lies in one component, r_Ω is P-periodic, and the phases of E' shift by multiples of 2π. Hence the sum over c' turns the zero-mode integrals into Lebesgue measure on $\mathbb{R}^{\mathrm{Comp}(G')}$. As $u^{-1}\mathbf{1}_{\mathfrak{K}} = 0$, the measure becomes $e^{-\frac{1}{2}\langle\hat{\varphi}, u^{-1}\hat{\varphi}\rangle} d\hat{\varphi}$ on $\mathbb{R}^V$, and $C_0 = (u^{-1} + \tilde{m}^2)^{-1}$ is one operator on $\mathbb{R}^V$, whereas in [GM82, (6.15)] the mass term is $\tilde{m}^2\chi_\Lambda$. By (S5), $C^{-1} \geq \frac{1}{2}C_0^{-1}$, and equivariance with (7.1) gives the independence of the representative.

7.2 The shift and the localized F_1

From now on G' is an output of the construction and $\beta_{\mathrm{GM}} \geq \beta_{\mathrm{GM}}^0$, so that $L' \geq 16$ and $L/2 < L'$. We write $d = d_{G'}$ for the graph distance of G' ($d(x, y) = \infty$ for sites in different components) and $N_r(A) := \{x \in V : d(x, A) \leq r\}$. By Theorem 3.20 (R7), each cell lies in one $\mathcal{U}$-cell and each $\mathcal{U}$-cell lies in one component. A good tube of an interface bond b (Subsection 7.3) is a straight path of G' of L sites with middle bond b.

The wall neighbourhood. The set $\hat\Sigma$ is the union of the $\mathcal{U}$-cells Δ with $d(\Delta,\Sigma_V) < L'$, as on [GM82, p. 47] with $\mathcal{U}$-cells for unit cubes and d for GM's distance $3^{-1/2}\|x-y\|_1$ of [GM82, (1.11a), (3.23)]. The *wall components* S_β are the unions of the $\mathcal{U}$-cells of the connected components of the graph on the $\mathcal{U}$-cells in $\hat\Sigma$, two being adjacent when a bond of G' joins them. We put $\Sigma_\beta := \Sigma_V \cap S_\beta$ and $w_\beta := w\mathbf{1}_{E(S_\beta)}$, where $E(S_\beta)$ is the set of bonds of E with both ends in S_β. The *wall sources* and the *standard cutoff* are $s_\beta := M_Y^{-2}(-\Delta + M_Y^2)\nabla^* w_\beta$ and $\chi_\beta := \zeta(d(\cdot,\Sigma_\beta))$, with $\zeta = 1$ on $[0, L'/2]$, $\zeta(t) = 3 - 4t/L'$ on $[L'/2, 3L'/4]$ and $\zeta = 0$ on $[3L'/4, \infty)$. The transition set A_β is the set of endpoints of bonds $\{x,y\} \in E$ with $\chi_\beta(x) \neq \chi_\beta(y)$. The shift, which near each wall component subtracts the response $C_0 s_\beta$ to the wall source of that component, is

$$g := H - \sum_\beta \chi_\beta C_0 s_\beta, \qquad D := g - H = \sum_\beta D_\beta, \quad D_\beta := -\chi_\beta C_0 s_\beta. \tag{7.10}$$

Proposition 7.8. *For every $h \in \mathcal{H}$ and $k \in \{0,1\}$: (a) every wall bond lies in $E(S_\beta)$ for exactly one β, so $w = \sum_\beta w_\beta$ with disjoint supports; $\{x : d(x,\Sigma_\beta) < L'\} \subseteq S_\beta$; every interface carrying a wall lies in one $E(S_\beta)$, and the good tubes of its bonds lie in S_β; (b) $\chi_\beta = 0$ off S_β, $\chi_\beta = 1$ on Σ_β, $d(A_\beta,\Sigma_\beta) \geq L'/2 - 1 \geq 7$, $N_1(A_\beta) \subseteq S_\beta$, and χ_β is $(4/L')$-Lipschitz for d; (c) $u^{-1}H = \sum_\beta s_\beta$ and $\operatorname{supp} s_\beta \subseteq N_1(\Sigma_\beta)$; (d) $g = H$ off $\bigcup_\beta \operatorname{supp}\chi_\beta$, in particular $g + k\beta_{\mathrm{GM}}^{-1/2} f = h$ on $V \setminus \hat\Sigma$; on S_β, $g = H - \chi_\beta C_0 s_\beta$; and g is equivariant; (e) $F_2' := C_0^{-1}(g - g_c)$ satisfies*

$$F_2' = \sum_\beta F_{2,\beta}', \qquad F_{2,\beta}' := -[C_0^{-1}, \chi_\beta]\, C_0 s_\beta, \qquad \operatorname{supp} F_{2,\beta}' \subseteq N_1(A_\beta) \subseteq S_\beta, \tag{7.11}$$

$C_0^{-1} D_\beta + s_\beta = F_{2,\beta}'$ (as $\chi_\beta = 1$ on $\operatorname{supp} s_\beta$), and on $N_1(A_\beta)$ one has $s_\beta = 0$ and $F_2' = -C_0^{-1}\chi_\beta C_0 s_\beta$; (f) given S_β, the functions s_β, $D|_{S_\beta}$ and $F_{2,\beta}'$ depend on h only through $N|_{E(S_\beta)}$.

Proof. Parts (a), (b), (d) and (f) are support checks. A geodesic of length less than L' to Σ_β stays in $\hat\Sigma$, hence in one wall component, and good tubes lie within $L/2 < L'$ of their middle bond. The function ζ is $(4/L')$-Lipschitz, which gives the Lipschitz bound of (b). Since $A_\beta \subseteq \{L'/2 - 1 < d(\cdot,\Sigma_\beta) \leq 3L'/4 + 1\}$ and $L' \geq 16$, we have $N_1(A_\beta) \subseteq \{d(\cdot,\Sigma_\beta) < L'\} \subseteq S_\beta$. Finally, $\nabla c = 0$ for $c \in \mathcal{C}_P$, and $w_\beta = 2\pi\beta_{\mathrm{GM}}^{-1/2} N$ on $E(S_\beta)$, whose wall bonds have endpoint set Σ_β.

(c) Inserting $-\Delta f = \nabla^* j$, we find $-\Delta H = \nabla^*\nabla h - k\beta_{\mathrm{GM}}^{-1/2}\nabla^* j = \nabla^* w$, so $u^{-1}H = M_Y^{-2}(-\Delta + M_Y^2)\nabla^* w = \sum_\beta s_\beta$ by (a). That is, on each bond with $N = 0$ the jumps of h across Ξ cancel j, and hence $u^{-1}H$ is a pure wall source. Moreover, $\nabla^* w_\beta$ is supported in Σ_β, and $-\Delta + M_Y^2$ enlarges supports by at most one step.

(e) By (7.1), $C_0^{-1} g_c = \tilde m^2 H$ and $C_0^{-1} H - \tilde m^2 H = u^{-1} H$, and by (c) we obtain $F_2' = \sum_\beta (s_\beta - C_0^{-1}\chi_\beta C_0 s_\beta)$. A site $x \in N_1(\Sigma_\beta) \setminus \Sigma_\beta$ is adjacent to some $z \in \Sigma_\beta$, and $\chi_\beta(x) = 1$, since otherwise $z \in A_\beta$, against (b). Thus $\chi_\beta = 1$ on $N_1(\Sigma_\beta) \supseteq \operatorname{supp} s_\beta$, and hence $s_\beta - C_0^{-1}\chi_\beta C_0 s_\beta = -[C_0^{-1}, \chi_\beta] C_0 s_\beta$. The kernel $C_0^{-1}(x,y) = (M_Y^{-2}\Delta^2 - \Delta + \tilde m^2)(x,y)$ vanishes unless $d(x,y) \leq 2$. Hence $([C_0^{-1}, \chi_\beta]v)(x) = \sum_y C_0^{-1}(x,y)(\chi_\beta(y) - \chi_\beta(x))v(y)$ vanishes unless χ_β changes along a path of at most two bonds from x, which puts x in $N_1(A_\beta) \subseteq S_\beta$. Since $d(A_\beta,\Sigma_\beta) \geq 7 \geq 3$, the sets $N_1(A_\beta)$ and $N_1(\Sigma_\beta)$ are disjoint, so $s_\beta = 0$ on $N_1(A_\beta)$, and there $F_2' = -C_0^{-1}\chi_\beta C_0 s_\beta$.

The construction above uses no extension of $h|_{S_\beta}$ to V. A construction through an extension h^e_β of $h|_{S_\beta}$ that is constant on the components of $V \setminus S_\beta$, as in [GM82, (6.24)], is not available on G': such an extension exists only if w_β has zero periods on all cycles of G', and a wall spanning the central hole of a solid torus of $\mathbb{X}$ has period ± 1 on a meridian of the torus. When an extension exists, (7.10) agrees with [GM82, (6.24), (6.25)] up to the choice of cutoff and C_0 for C_{00}.

Proposition 7.9. *For every $h \in \mathcal{H}$, $k \in \mathbb{Z}$ and $D : V \to \mathbb{R}$, with $g := H + D$,*

$$2F_1 = \tilde{m}^2\|D\|^2 + \|w + \nabla D\|^2 + M_Y^{-2}\|\Delta D - \nabla^* w\|^2 - k^2\beta_{\mathrm{GM}}^{-1}E_{G'}. \tag{7.12}$$

Proof. As $u^{-1} = -\Delta + M_Y^{-2}\Delta^2$ and $-\Delta = \nabla^*\nabla$, we have $\langle g, u^{-1}g\rangle = \|\nabla g\|^2 + M_Y^{-2}\|\Delta g\|^2$. Here the gradient of g differs from the wall field plus ∇D by a multiple of the Biot–Savart field, which is orthogonal to gradients and has inner product $-E_{G'}$ with j. Now $\nabla g = \nabla h - k\beta_{\mathrm{GM}}^{-1/2}(j + R_f) + \nabla D = (w + \nabla D) - k\beta_{\mathrm{GM}}^{-1/2}R_f$ and, by Lemma 7.2, $\langle w + \nabla D, R_f\rangle = -k\beta_{\mathrm{GM}}^{-1/2}\langle j, R_f\rangle = k\beta_{\mathrm{GM}}^{-1/2}E_{G'}$, so $\|\nabla g\|^2 = \|w + \nabla D\|^2 - k^2\beta_{\mathrm{GM}}^{-1}E_{G'}$, and the perimeter energy separates exactly. Next $\Delta g = \Delta h + k\beta_{\mathrm{GM}}^{-1/2}\nabla^* j + \Delta D = \Delta D - \nabla^* w$. Adding $\tilde{m}^2\|g - H\|^2 = \tilde{m}^2\|D\|^2$ gives (7.12).

Corollary 7.10 (Localization)**.** *Let D be the shift* (7.10)*. For each wall component put*

$$2F_1(S_\beta) := \tilde{m}^2\|D_\beta\|^2 + \|w_\beta + \nabla D_\beta\|^2 + M_Y^{-2}\|\Delta D_\beta - \nabla^* w_\beta\|^2, \tag{7.13}$$

and for every union Y of wall components and of $\mathcal{U}$-cells outside $\hat{\Sigma}$, $F_1^{\mathrm{loc}}(Y) := \sum_{\beta : S_\beta \subseteq Y} F_1(S_\beta)$. Then: (a) $F_1 = F_1^{\mathrm{loc}}(V) - \frac{1}{2}k^2\beta_{\mathrm{GM}}^{-1}E_{G'}$; (b) $F_1(S_\beta) \geq 0$, and given S_β it is a function of $N|_{E(S_\beta)}$; if $E(S_\beta)$ contains no Ξ-bond, it does not depend on (k, Ξ); (c) for every such Y,

$$F_1^{\mathrm{loc}}(Y) \geq \tfrac{1}{2}\tilde{m}^2 \sum_{x \in Y}(g - H)^2 + \tfrac{1}{2} \sum_{b \in E(Y)} \left|\nabla g + k\beta_{\mathrm{GM}}^{-1/2}R_f\right|^2; \tag{7.14}$$

(d) $Z(k, \Xi)$ is the right side of (7.9) *with e^{-F_1} replaced by $e^{-F_1^{\mathrm{loc}}(V)}$, and the prefactor $e^{-k^2E_{G'}/2\beta_{\mathrm{GM}}}$ of Lemma 7.2 cancels.*

Proof. Each term of (7.12) but the last is a sum of squares of functions supported in the wall components: ∇D_β is supported on $E(S_\beta)$ and ΔD_β on S_β, since a bond leaving S_β from $\operatorname{supp}\chi_\beta$ would end in $A_\beta \subseteq S_\beta$. This gives (a). Part (b) is Proposition 7.8 (f), and off Ξ-bonds $N = \nabla n$ does not involve k. For (c) we drop the terms with M_Y^{-2}, as $\nabla g + k\beta_{\mathrm{GM}}^{-1/2}R_f = w + \nabla D$. Part (d) follows from Lemma 7.2, Proposition 7.7 and (a), and since the prefactor cancels, $E_{G'}$ does not enter the perimeter constant.

7.3 The wall tension and the spanning area

Tubes on the fine lattice. Let $b = (x, y)$ be a bond of an interface $I = I(\Omega, \Omega')$, oriented from Ω to Ω', with $y = x + \sigma e_\mu$, $\sigma = \pm 1$. We put $\tau_b^- := \{x - i\sigma e_\mu : 0 \leq i < L/2\}$, $\tau_b^+ := \{y + i\sigma e_\mu : 0 \leq i < L/2\}$ and $\tau_b := \tau_b^- \cup \tau_b^+$, a segment of L sites (L is even). The bond b is *good* if $\tau_b \subseteq V$ and the $L - 1$ consecutive bonds of τ_b lie in E. We write $\mathrm{good}(I)$ for the number of good bonds of I. On the fine lattice the face rule gives, by Theorem 3.20 (R8) and Lemma 4.19, $\mathrm{good}(I) \geq c'_g \max(|I|, L^2)$ for each interface, with $c'_g := c_g(1 + 1/L_c)^{-2} \geq \frac{9}{16}c_g$.

Lemma 7.11. *Let $z_1, \dots, z_L$ be a path, let $\tilde H = c_-$ on $\{z_1, \dots, z_{L/2}\}$ and $\tilde H = c_+$ on $\{z_{L/2+1}, \dots, z_L\}$, $J := c_+ - c_-$, and let $m\,(L/2) \le 1/\sqrt{2}$. Then for every g, $\frac12 m^2 \sum_{i\le L}(g(z_i) - \tilde H(z_i))^2 + \frac12 \sum_{i<L}(g(z_{i+1}) - g(z_i))^2 \ge \frac16\, m^2\,(L/2)\, J^2$.*

Sketch. With $\bar g_\pm$ the averages of g on the two halves, we write $J = (c_+ - \bar g_+) + (\bar g_+ - \bar g_-) + (\bar g_- - c_-)$ and apply Cauchy–Schwarz to each term. Heuristically, the mass term supplies the lower bound if g does not follow the jump, and the gradient term supplies it if g does.

Lemma 7.12 (Tube geometry)**.** *(a) If $b \in I(\Omega, \Omega')$ is good, then $\tau_b^- \subseteq \Omega$ and $\tau_b^+ \subseteq \Omega'$. (b) Good tubes on a common lattice line are site-disjoint. Hence every site lies in at most 3 good tubes and every bond in at most one. (c) No bond of τ_b other than b crosses Ξ, and ξ, ∇h, N are constant on every interface.*

Sketch. A good tube is a straight path of G' crossing one block face at its middle bond, and by (G1) of Definition 3.4 blocks are at least L long. The chain Ξ lies in grid planes by Lemma 3.5 (A1), and for (c) we use Lemma 7.6 (a).

Theorem 7.13 (Wall tension)**.** *Let $h \in \mathcal{H}$, $k \in \{0, 1\}$, and let Y be a union of wall components and of $\mathcal{U}$-cells outside $\hat\Sigma$. With $J_I := 2\pi\beta_{\mathrm{GM}}^{-1/2} N_I$,*

$$F_1^{\mathrm{loc}}(Y) \ge \frac{1}{18}\,\tilde m^2\,\frac{L}{2}\sum_{I\subseteq Y} \mathrm{good}(I)\, J_I^2 \ge 2\sigma_F \sum_{b\in E(Y)} N(b)^2, \qquad \sigma_F := \frac{\pi^2 c_g'}{18}\,\frac{\tilde m^2 L}{\beta_{\mathrm{GM}}}, \tag{7.15}$$

where $I \subseteq Y$ means that both cells of I lie in Y. We put $F_{\mathrm{unit}} := \sigma_F L^2$.

Proof. For a good bond b of an interface $I \subseteq Y$, Lemma 7.12 (a), (c) shows that $\tau_b \subseteq Y$ and, with $\vartheta_b := D + J_b \mathbf{1}_{\tau_b^+}$, that $w + \nabla D = \nabla\vartheta_b$ on the bonds of τ_b and $g - H = \vartheta_b - J_b\mathbf{1}_{\tau_b^+}$ on τ_b. Hence, as $\tilde m L/2 \le c_L/2 \le 1/\sqrt{2}$, Lemma 7.11 with $m = \tilde m$, $c_- = 0$ and $c_+ = J_b$ bounds the part of the right side of (7.14) on τ_b below by $\frac16 \tilde m^2 \frac{L}{2} J_b^2$. Summing over good bonds, we count each site at most three times and each bond at most once, by Lemma 7.12 (b). The second inequality follows from $\sum_{E(Y)} N^2 = \sum_{I\subseteq Y} |I| N_I^2$ and $\mathrm{good}(I) \ge c_g'|I|$. The constant $\frac{1}{18}$ is the $\frac16$ of Lemma 7.11 divided by the overlap 3 of good tubes, and the remaining factor 2 in front of σ_F is used in Subsection 7.8. By comparison, [GM82, Lemma 12, (C.9)] gives $\frac{1}{60}$ together with a perimeter term, which the bound of the theorem does not contain.

Theorem 7.14 (Walls spanning the loop)**.** *Let $k = 1$, let $n : V \to \mathbb{Z}$ be arbitrary, and $N(b) := \nabla n(b) - \xi_b$ on E. Then $\sum_{b\in E} N(b)^2 \ge \#\{b \in E : N(b) \ne 0\} \ge A_{G'}(C)$.*

Proof. With U as in the proof of Lemma 7.6 (c), we correct Ξ by the boundary of the integer field and obtain the integer 2-chain $\Sigma := \Xi - \partial U$ in Λ°, with $\partial\Sigma = C$. The coefficient of p_b in Σ is $\xi_b - \nabla n(b) = -N(b)$ for $b \in E$, and all other plaquettes of Λ° are faces of hole cubes, frozen faces or faces in $\partial_2\Lambda^\circ$, hence lie in $\mathfrak{F}$. So the support of this chain outside the free set $\mathfrak{F}$ is exactly the set of wall bonds, $\operatorname{supp}\Sigma \setminus \mathfrak{F} = \{p_b : b \in E,\ N(b) \ne 0\}$, and Definition 2.11, applied to this 2-chain spanning C, gives $A_{G'}(C) \le \#\{N \ne 0\} \le \sum_E N^2$, since N is integer valued.

7.4 Smallness of the fluctuation term F_2

Throughout this subsection we assume $\lambda_{\mathrm{PV}} \le \frac{1}{4}$ and $\tilde{m} \le M_Y \le 1$, which hold for $\beta_{\mathrm{GM}} \ge \beta_{\mathrm{GM}}^0$, and put $R_s := (-\Delta + s)^{-1}$ for $s > 0$ and

$$r_\pm := \tfrac{1}{2} M_Y^2 \big(1 \pm (1 - 4\lambda_{\mathrm{PV}}^2)^{1/2}\big), \qquad \mathcal{G} := C_0 M_Y^{-2}(-\Delta + M_Y^2). \tag{7.16}$$

As $r_+ + r_- = M_Y^2$ and $r_+ r_- = \tilde{m}^2 M_Y^2$, partial fractions in $t = -\Delta$ give $C_0 = \varpi(R_{r_-} - R_{r_+})$ and $\mathcal{G} = c_+ R_{r_-} - c_- R_{r_+}$, with $\varpi := M_Y^2/(r_+ - r_-)$ and $c_\pm := r_\pm/(r_+ - r_-)$. Bounds on these quantities, the derivation of the numerical constants of Lemma 7.15 and the arithmetic in the proofs of Theorem 7.16 and Corollary 7.18 are given in Section S1.6. For X a union of wall components and of $\mathcal{U}$-cells outside $\hat{\Sigma}$ we put $F_2(X) := \langle \mathbf{1}_X \psi, F_2' \rangle = \sum_{\beta : S_\beta \subseteq X} \langle \psi, F_{2,\beta}' \rangle$, where the last equality holds by Proposition 7.8 (e).

Lemma 7.15. *(a)* (Combes–Thomas.) *Let $\rho : V \to \mathbb{R}$ satisfy $|\rho(x) - \rho(y)| \le \alpha$ on every bond of G', and let $\kappa_{\mathrm{CT}} := s - 6(\cosh\alpha - 1) > 0$. Then $\|e^{\rho} R_s e^{-\rho}\| \le \kappa_{\mathrm{CT}}^{-1}$. (b) Let $B \subseteq E$ have endpoint set V_B, and let $A \subseteq V$ with $d(A, V_B) \ge r \ge 0$. Then $\|\mathbf{1}_A \mathcal{G} \nabla^* \mathbf{1}_B\| \le 8.71\, \tilde{m}^{-2} e^{-\tilde{m} r/3}$. (c) If $\chi : V \to [0,1]$ is Λ_χ-Lipschitz for d, then $\|[C_0^{-1}, \chi]\| \le \Lambda_\chi(144 M_Y^{-2} + 6) \le 150\, \Lambda_\chi M_Y^{-2}$.*

Proof. All three are operator norms from the Schur test and the degree bound 6, so missing bonds enter only through the degree. (a) The bound follows by the method of [CT73, Theorem 1, p. 257], in a lattice form not stated there. With $e^{\rho}(-\Delta + s)e^{-\rho} = -\Delta + s + \mathfrak{B}$, pairing the two orientations of each bond gives $\mathrm{Re}\langle f, \mathfrak{B} f\rangle = -\sum_{\text{bonds}} 2(\cosh t - 1)\mathrm{Re}(\bar{f}(x) f(y))$, $t := \rho(x) - \rho(y)$, which is at least $-6(\cosh\alpha - 1)\|f\|^2$. As $-\Delta \ge 0$, it follows that $\mathrm{Re}\langle f, (-\Delta + s + \mathfrak{B}) f\rangle \ge \kappa_{\mathrm{CT}} \|f\|^2$. The antisymmetric part of $\mathfrak{B}$ drops out, so ρ may have kinks, such as those of d at slit edges and hole corners, and the second-order form matters: in (b), at $s = r_- \le 1.08\, \tilde{m}^2$ and $\alpha = \tilde{m}/3$, the first-order quantity $s - 6(e^\alpha - 1)$ is negative. (b) We apply (a) with $\rho := \frac{\tilde{m}}{3} \min\{d(\cdot, V_B), r\}$ to the two resolvents of $\mathcal{G}$, and the Schur test to $e^{\rho} \nabla^* e^{-\rho}$. (c) We use $[C_0^{-1}, \chi] = M_Y^{-2}(\Delta[\Delta, \chi] + [\Delta, \chi]\Delta) - [\Delta, \chi]$.

Theorem 7.16 (Control of F_2 by F_1; [GM82, Lemma 14] on G')**.** *For every $h \in \mathcal{H}$, $k \in \{0, 1\}$ and every union X of wall components and of $\mathcal{U}$-cells outside $\hat{\Sigma}$, $\sum_{x \in X} F_2'(x)^2 = \sum_{\beta : S_\beta \subseteq X} \|F_{2,\beta}'\|^2 \le \tilde{m}^2\, \varepsilon(L')\, F_1^{\mathrm{loc}}(X)$, where*

$$\varepsilon(L') := C_\varepsilon\, c_g^{-1}\, \lambda_{\mathrm{PV}}^4\, \tilde{m}^{-9}\, (\tilde{m}L/2)^{-1} (\tilde{m}L')^{-2} e^{-\tilde{m}L'/3}, \tag{7.17}$$

with the absolute constant C_ε of Section S1.6.

Proof. By Proposition 7.8 (e), $F_{2,\beta}' = -[C_0^{-1}, \chi_\beta] \mathbf{1}_{A'} \mathcal{G} \nabla^* \mathbf{1}_B w_\beta$ with $A' := N_1(A_\beta)$, $B := \operatorname{supp} w_\beta$, $d(A', V_B) \ge L'/2 - 2$ and $\Lambda_\chi = 4/L'$. The commutator is localized where χ_β changes, at distance at least $L'/2 - 1$ from the walls, and at that distance the Combes–Thomas decay dominates the polynomial loss, because $\tilde{m}L' \propto \beta_{\mathrm{GM}}$. We apply Lemma 7.15 (b), (c), and bound $\sum_\beta \|w_\beta\|^2$ by Theorem 7.13 with $\mathrm{good}(I) \ge c_g' |I|$.

This proof follows [GM82, Appendix D, (D.3) to (D.16)], with the following changes.

(1) The operator norms of Lemma 7.15 replace GM's pointwise bounds [GM82, (D.10)], which come from [GM82, Lemma B.3] and fail within distance 2 of a missing bond. This loses a factor $M_Y^{-4} \tilde{m}^{-4}$.

(2) The source form of Proposition 7.8 (e) replaces [GM82, (D.3) to (D.5)], since h^e_β need not exist on G'.

(3) Theorem 7.13 replaces [GM82, Lemma 12, (C.21)], so the bound has no perimeter term P_F and is uniform in Λ°, C, Ξ, the hole set $\mathbb{X}$, S, h, k and X.

Corollary 7.17 (Gaussian F_2 factors)**.** *Let C' be positive semidefinite on $\mathbb{R}^V$ with $\|C'\| \le t_C \tilde m^{-2}$, and let $d\mu_{C'}$ be the centred Gaussian measure with covariance C'. For every $p \in \mathbb{R}$ and every X as in Theorem 7.16,*

$$\int d\mu_{C'}(\psi)\, e^{-pF_2(X)} = \exp(\tfrac12 p^2 \langle \mathbf{1}_X F_2', C' \mathbf{1}_X F_2'\rangle) \le \exp(\tfrac12 p^2 t_C\, \varepsilon(L')\, F_1^{\mathrm{loc}}(X)).$$

Proof. Since $F_2(X) = \langle \psi, \mathbf{1}_X F_2'\rangle$ is linear in ψ, its Laplace transform is Gaussian. The bound then follows from $\langle F, C'F\rangle \le \|C'\|\|F\|^2$ and Theorem 7.16. We will apply the corollary with the C' of Lemma 7.29, for which $C' \le \alpha_D C_0 \le \frac76 \tilde m^{-2}$, so that $t_C = \frac76 < 2$, and with block-diagonal restrictions $\sum_j \mathbf{1}_{Y_j} C_0 \mathbf{1}_{Y_j}$, for which $t_C = 1$.

Corollary 7.18 (Collar)**.** *Let p_2, p_3, p', q' and c_λ be the Hölder exponents and the constant of Subsection 7.6, and $\varepsilon_0 := \min\{(1 - 1/p_3 - c_\lambda)/(2p_2),\, 2(p'/q')^2\}$ (values in Section S1.6). There is β^0_{GM} such that $\varepsilon(L') \le \varepsilon_0$ for $\beta_{\mathrm{GM}} \ge \beta^0_{\mathrm{GM}}$.*

Proof. We have $\tilde m L' = 2\beta_{\mathrm{GM}}$, while for $\varepsilon(L') \le \varepsilon_0$ we need only $\tilde m L' \ge 27 \log(1/\tilde m) + O(1)$ (Section S1.6), with $\log(1/\tilde m) \le 2.4944\tau$ and $K_{\mathrm{col}} = 2 > 27 \cdot 2.4944/(4\pi^2) = 1.706$. A collar with $\tilde m L'$ fixed, as in [GM82, p. 47], does not suffice on G'.

Proposition 7.19. *Let $D_E := C_{\mathcal U} \tilde m^{-1}$ bound the Euclidean diameter of $\mathcal U$-cells (Theorem 3.20 (R7)) and $\eta_{\mathrm{col}} := (2\sqrt{2}L + D_E + 3)/L'$. For every wall component S_β, with $n_I(\beta)$ the number of interfaces of S_β carrying a wall: (a) $F_1(S_\beta) \ge \frac{\pi^2 c'_g}{9} \frac{\tilde m^2 L^3}{\beta_{\mathrm{GM}}} n_I(\beta) = 2F_{\mathrm{unit}}\, n_I(\beta)$; (b) S_β has at most $n_I(\beta) \frac{4\pi}{3c_u} (\tilde m L')^3 (1 + \eta_{\mathrm{col}})^3$ $\mathcal U$-cells; (c) for $\beta_{\mathrm{GM}} \ge \beta^0_{\mathrm{GM}}$, every $\mathcal U$-cell of $\hat\Sigma$ receives a share at least $\Phi_{\mathcal U}$ of F_1^{loc}, that is, $F_1(S_\beta) \ge \Phi_{\mathcal U} \cdot \#\{\mathcal U\text{-cells of } S_\beta\}$, where the constant $\Phi_{\mathcal U}$ (Section S1.6) satisfies $\Phi_{\mathcal U} \to \infty$ as $\beta_{\mathrm{GM}} \to \infty$.*

Sketch. (a) is Theorem 7.13 with $Y = S_\beta$, $J_I^2 \ge 4\pi^2/\beta_{\mathrm{GM}}$ and $\mathrm{good}(I) \ge c'_g L^2$. (b) Each $\mathcal U$-cell of S_β lies within d-distance L' of a wall interface, and by (R7) this interface lies in one block face and $\mathcal U$-cells are disjoint with at least $c_u \tilde m^{-3}$ sites. (c) We divide (a) by (b). Then $\Phi_{\mathcal U} \to \infty$, because the energy is exponential in β_{GM}, whereas the collar is polynomial in $\tilde m L' \propto \beta_{\mathrm{GM}}$.

7.5 The class-graded expansion

By Corollary 7.10 (d), $Z(k, \Xi) = \sum_{[h]} \mathcal N \int d\mu_C\, e^E Q_h$ with $Q_h := e^{G(\psi+g)} e^{-F_1^{\mathrm{loc}}(V)} e^{-F_2}$. We expand the Gaussian integral at fixed h and then resum the walls, graded by their class.

Definition 7.20 (Closures, wall polymers, seeds)**.** Fix $h \in \mathcal H$. (a) The closure $\mathrm{cl}(S)$ of a union S of $\mathcal U$-cells is S together with the $\mathcal U$-cells joined to S by a bond of G'. Closures of wall components are *linked* if they share a $\mathcal U$-cell; a *wall polymer* $\mathcal S_\gamma(h)$ is the union of a maximal linked family of closures. (b) The wall polymers and the $\mathcal U$-cells contained in no wall polymer form a partition $\tilde{\mathcal Y}(h)$ of V, whose members are the *elements*. A set is *admissible* (for h) if it is a union of elements; $\mathrm{sat}_h(S)$ is the union of the elements that meet

S. (c) X_1 is the union of the $\mathcal{U}$-cells that contain an endpoint of a bond of G' crossing Ξ, and Δ_1 is a fixed $\mathcal{U}$-cell. The *seeds* are $Y_1(h) := \mathrm{sat}_h(X_1)$ in the expansion of $Z(1,\Xi)$ and $Y_1(h) := \mathrm{sat}_h(\Delta_1)$ in that of $Z(0,\Xi)$. (d) $\mathfrak{M}(h) := \{X \supseteq Y_1(h) : X \text{ admissible}\}$. (e) For admissible Y, $G(Y) := \sum_{\Omega\subseteq Y} \ln r_\Omega(A_\Omega) + \sum_i G_{2,i}(Y)$ (the site sums of (7.5) restricted to Y) and $Q_h(Y) := e^{G(Y)} e^{-F_1^{\mathrm{loc}}(Y)} e^{-F_2(Y)}$ ($F_2(Y)$ as in Subsection 7.4); $E(Y)$ and $E(Y,s)$ are defined in Appendix C.

An admissible set is a union of wall components and of $\mathcal{U}$-cells outside $\hat\Sigma$, as a $\mathcal{U}$-cell of $\hat\Sigma$ joined by a bond to S_β belongs to S_β. Hence $F_1^{\mathrm{loc}}(Y)$ and $F_2(Y)$ are defined. An element that meets X_1 lies in each admissible $X \supseteq X_1$, since the elements partition V and admissible sets are unions of elements. Therefore $\mathfrak{M}(h) = \{X \supseteq X_1 : X \text{ admissible}\}$, as in the rule of [GM82, p. 50], and with Δ_1 for $k = 0$, as on [GM82, p. 51].

Lemma 7.21. *Let Y, Y' be disjoint admissible sets. Then $Q_h(Y \cup Y') = Q_h(Y)Q_h(Y')$; in particular $Q_h = Q_h(X)Q_h(V \setminus X)$ for admissible X. Moreover $Q_h(Y)$, and every term of $E(Y,s)$, are functions of $\psi|_Y$ and $N|_{E(Y)}$. The same holds with $G^\natural$ of Lemma 7.29 in place of G.*

Sketch. We follow [GM82, (E.4)], with admissible sets in place of unions of unit cubes. The action is a sum of cell terms ($\ln r_\Omega$), site terms ($G_{2,i}$, the terms of $G^\natural$ and of E) and wall-component terms ($F_1(S_\beta)$, $F'_{2,\beta}$), and an admissible set contains each cell and wall component entirely or not at all. As $g + k\beta_{\mathrm{GM}}^{-1/2} f = h + D$ with h cell-constant and $P\mathbb{Z}$-valued, each phase is $e^{im\beta_{\mathrm{GM}}^{1/2}(\psi+D)}$, and r_Ω is P-periodic. By Proposition 7.8 (f) and Corollary 7.10 (b), $D|_Y$, $F_1^{\mathrm{loc}}(Y)$ and $F_2(Y)$ depend on h only through $N|_{E(Y)}$.

Since F_1^{loc} does not split inside a wall polymer, this factorization holds for admissible sets only, so we saturate the seed, as X_1 itself need not be admissible.

Proposition 7.22 (Cluster expansion at fixed h; [GM82, Lemma E.1, Proposition E.2] on G')**.** *Assume (S1) to (S5) and the hypotheses of Lemma C.2, and let $\mathbf{C} \in \{C, C'\}$ (C' of Lemma 7.29). Let $U = V$, or $U = V \setminus X_0$ for a union X_0 of $\mathcal{U}$-cells; let N be a wall field on $E(U)$ (see the paragraph on wall fields before Lemma 7.24, where $E(Y,s)$ and Q_N are defined for wall fields) whose wall polymers have their closures in U (for $U = V$, $N = N(h)$ with any $h \in \mathcal{H}$), with D, S_β, $F_1(S_\beta)$, $F'_{2,\beta}$ and the elements defined from N (Proposition 7.8 (f), Corollary 7.10 (b)); and let $Y_1 \subseteq U$ be nonempty and admissible for N on U (it may depend on N and need not be connected). With $\mathbf{C}_U := \mathbf{1}_U \mathbf{C} \mathbf{1}_U$ on $\mathbb{R}^U$,*

$$\int d\mu_{\mathbf{C}_U}(\psi)\, e^{E(U)} Q_N(U) = \sum_X \mathcal{K}_N(X) \int d\mu_{\mathbf{C}_U}(\psi)\, e^{E(U\setminus X)} Q_N(U \setminus X), \tag{7.18}$$

where X runs over the admissible subsets of U containing Y_1, $\mathcal{K}_N(X) := \sum_{\bar y:\, X(\bar y)=X} \mathcal{K}_{\bar y}$ over the sequences $\bar y = (Y_1, \dots, Y_n)$ of pairwise disjoint nonempty admissible sets with union $X(\bar y) = X$, and

$$\mathcal{K}_{\bar y} := \int_{[0,1]^{n-1}} ds \int d\mu_s^{\bar y}(\psi)\, e^{E^{\bar y}(X(\bar y), s)}\, \kappa(\bar y, s)\, Q_N(X(\bar y)), \tag{7.19}$$

with $d\mu_s^{\bar y}$ (covariance (C.1) at $s_n = 0$), $E^{\bar y}$ and the operator $\kappa(\bar y, s)$ of Appendix C, built from $\mathbf{C}_U$. Every sequence has length at most $e_N + 1$, e_N the number of elements of U not contained

in Y_1; all sums are finite, all integrals converge absolutely, and $\int d\mu_{\mathbf{C}_U} e^{E(Y)} Q_N(Y) \in (0,\infty)$ for every admissible $Y \subseteq U$. $\mathcal{K}_{\bar y}$ depends only on $\bar y$, the $X(\bar y) \times X(\bar y)$ block of $\mathbf{C}$ and $N|_{E(X(\bar y))}$, not on U.

We give the proof in Appendix C.

Admissibility in terms of walls. For a set W of wall bonds with endpoint set $V(W)$, $\hat\Sigma(W)$ is the union of the $\mathcal{U}$-cells Δ with $d(\Delta, V(W)) < L'$. For a union X of $\mathcal{U}$-cells, ∂X is the set of bonds of G' joining X to $X^c := V \setminus X$, and we put $W_X := \Sigma_E \cap E(X)$ and $W_{X^c} := \Sigma_E \cap E(X^c)$. We consider the conditions (in) $\mathrm{cl}(\hat\Sigma(W_X)) \subseteq X$, (out) $\mathrm{cl}(\hat\Sigma(W_{X^c})) \subseteq X^c$, and ($\partial$) $N = 0$ on ∂X.

Lemma 7.23. *Let $k \in \{0,1\}$, $h \in \mathcal{H}$, and let X be a union of $\mathcal{U}$-cells with $X \supseteq X_1$ if $k = 1$ and $X \supseteq \Delta_1$ if $k = 0$. Then $X \in \mathfrak{M}(h)$ if and only if (in), (out) and (∂) hold. In that case the elements of $\tilde{\mathcal{Y}}(h)$ inside X are determined by X and $N|_{E(X)}$, and those inside X^c by X and $N|_{E(X^c)}$.*

Sketch. By Proposition 7.8 (a), each wall component, with its closure, belongs to the side that contains its walls. No element straddles ∂X when ∂X carries no wall, since under (∂) we have $\hat\Sigma = \hat\Sigma(W_X) \cup \hat\Sigma(W_{X^c})$, no $\mathcal{U}$-cell of one being adjacent to one of the other by (in) and (out), and the wall components inside X are the components of $\hat\Sigma(W_X)$, with $W_X = \operatorname{supp} N|_{E(X)}$.

At $k = 1$, since both ends of each Ξ-bond lie in $X_1 \subseteq X$, the set ∂X contains no Ξ-bond, and (∂) is equivalent to $h(x) = h(y)$ for all $\{x,y\} \in \partial X$, as at $k = 0$. By Lemmas 7.21 and 7.23, for admissible $X \supseteq X_1$ the activity $\mathcal{K}_h(X)$ with first set $\mathrm{sat}_h(X_1)$ depends on h only through $N|_{E(X)}$.

Wall fields. A *wall field* on a union U of $\mathcal{U}$-cells is a function $N : E(U) \to \mathbb{Z}$, extended by 0 to E, that vanishes on the bonds inside cells and is constant on each interface. By Lemma 7.6 (a), the restriction $N(h)|_{E(U)}$ is a wall field for $h \in \mathcal{H}$ and $k \in \{0,1\}$. The objects S_β, χ_β, s_β, D, $F_1(S_\beta)$, F_1^{loc}, $F'_{2,\beta}$ and $F_2(Y)$ of Subsection 7.2 are functions of $w := 2\pi\beta_{\mathrm{GM}}^{-1/2} N$ alone, as Proposition 7.8 (f) and Corollary 7.10 (b) show. For a wall field N we define $G(Y)$, $G^\natural(Y)$ (Lemma 7.29), $E(Y)$ and $E(Y,s)$ by their formulas with $\psi + g + k\beta_{\mathrm{GM}}^{-1/2} f$ replaced by $\psi + D$ in all phases and all cell averages, and put $Q_N(Y) := e^{G(Y)} e^{-F_1^{\mathrm{loc}}(Y)} e^{-F_2(Y)}$. Under this replacement, $A_\Omega + k\beta_{\mathrm{GM}}^{-1/2} \bar f_\Omega$ becomes $\mathrm{avg}_\Omega(\psi + D)$, and $\tilde\delta$ becomes $(\psi + D) - \mathrm{avg}_\Omega(\psi + D)$.

Lemma 7.24 (Wall form)**.** *(a) If U is admissible for $h \in \mathcal{H}$ ($U = V$, or $U = X^c$ with $X \in \mathfrak{M}(h)$), $k \in \{0,1\}$ and $N = N(h)|_{E(U)}$, then the elements of N are the elements of h inside U, $Q_N(Y) = Q_h(Y)$, and $E(Y,s)$ is the same in both definitions, for every admissible $Y \subseteq U$. (b) Definition 7.20 applies, and Lemma 7.21, Proposition 7.22 and Lemma 7.23 hold, for every wall field N on $E(U)$ whose wall polymers have their closures in U, with V replaced by U and without the condition on the seed in Lemma 7.23; Theorem 7.13 holds for every wall field; and Proposition 7.5 holds with $\|\varphi + k\beta_{\mathrm{GM}}^{-1/2} f - h\|$ replaced by $\|\psi + D\|$.*

Sketch. (a) As in the proof of Lemma 7.21, h drops out. (b) Since the action consists of cell, site and wall-component terms, these proofs use h only through (a) and the walls. In Theorem 7.13, (7.14) becomes (7.13) without its last term, and $w = 0$ inside cells. The proof

of Proposition 7.5 is pointwise in $\beta_{\mathrm{GM}}^{1/2}(\psi + D)(x)$, and Appendix C uses Lemma C.1 and the Poincaré inequality on cells, pointwise in (ψ, D) (details in Section S2.5).

Definition 7.25 (Classes and outer partition functions)**.** (a) The *m3-graph* has the sites of V as vertices, joined by the bonds of E and by the pairs of sites in a common cell; a closed walk γ in it has the length $|\gamma|$, the sum of the $d_{G'}$-distances of its steps. The *class* of a function N on E (extended by 0 to the same-cell pairs) is $[N] : \gamma \mapsto \sum_{e\in\gamma} N(e)$ (oriented sum) on closed m3-walks. (b) A wall field N is *long* if $[N](\gamma) = 0$ for every closed m3-walk γ with $|\gamma| < 2L'$. $\mathcal{N}^{\mathrm{long}}(U)$ is the set of long wall fields on $E(U)$, and $\Gamma_{\mathrm{long}} := \{[N] : N \in \mathcal{N}^{\mathrm{long}}(V)\}$; it is a group. (c) For a union X of $\mathcal{U}$-cells and $p \in \Gamma_{\mathrm{long}}$,

$$\mathcal{Z}_p(X) := \sum_N \mathcal{N} \int d\mu_C(\psi)\, e^{E(X^c)}\, Q_N(X^c), \tag{7.20}$$

the sum over the $N \in \mathcal{N}^{\mathrm{long}}(X^c)$ that satisfy (out) and have $[N] = p$, with the integrand of Lemma 7.24; $\mathcal{Z}(X) := (\mathcal{Z}_p(X))_{p\in\Gamma_{\mathrm{long}}}$, $\mathcal{Z}_p(X) := 0$ for $p \notin \Gamma_{\mathrm{long}}$, $Z_p := \mathcal{Z}_p(\emptyset)$ and $Z := Z_0$. (d) For $k \in \{0, 1\}$ and a union X of $\mathcal{U}$-cells, with $X \supseteq X_1$ if $k = 1$, an *inside field* at coupling k is a wall field N_X on $E(X)$ that satisfies (in) and for which $[(N_X + k\xi)\mathbf{1}_{E(X)}]$ vanishes on the closed m3-walks in X; its *outer class* is $p_k(N_X) := -[(N_X + k\xi)\mathbf{1}_{E(X)}]$.

Integrating along m3-paths, we see that, for N vanishing on the bonds inside cells (for instance a wall field), $[N] = 0$ if and only if $N = \nabla n$ with $n : V \to \mathbb{Z}$ constant on each cell (on the cycles of G' alone n need not be constant on cells, which may be disconnected). With Lemmas 7.6 (d) and 7.24 (a), this characterization gives $Z = Z(0, \Xi)$: the configurations of Z_0 are the walls of the $[h] \in \mathcal{H}/\mathcal{C}_P$ at $k = 0$. A long wall field has period 0 around each plaquette of G', which is a closed walk of length 4. For $X \supseteq X_1$, ξ vanishes off $E(X)$, so by Lemma 7.6 (d) an inside field at $k = 1$ has the periods $-\operatorname{lk}(\cdot, C)$ on the cycles of G' in X, and $p_1(N_X)(\gamma) = -\operatorname{lk}(\gamma, C) - \sum_{e\in\gamma\cap E(X)} N_X(e)$ on the cycles of G'. In general, such a field is not long.

Lemma 7.26 (Long fields)**.** *(a) A closed m3-walk γ with $|\gamma| < 2L'$ that contains a wall bond of a wall component S_β has all its sites in S_β. Hence, if $N \in \mathcal{N}^{\mathrm{long}}(U)$ has its wall polymers' closures in U and $Y \subseteq U$ is a union of elements of N, then $N\mathbf{1}_{E(Y)}$ and $N\mathbf{1}_{E(U\setminus Y)}$ are long, and their classes add up to $[N]$. (b) Let $h \in \mathcal{H}$, $k = 1$ and $X \in \mathfrak{M}(h)$. Then $N(h)|_{E(X)}$ is an inside field at $k = 1$, $N(h)|_{E(X^c)}$ is long and satisfies (out), and $[N(h)|_{E(X^c)}] = p_1(N(h)|_{E(X)}) \in \Gamma_{\mathrm{long}}$.*

Proof. A closed walk of length less than $2L'$ stays at $d_{G'}$-distance less than L' from each of its sites. (a) So all sites of γ lie in S_β (Proposition 7.8 (a)), all wall bonds of γ lie in $E(S_\beta)$, and S_β lies in one element. Thus the period of $N\mathbf{1}_{E(Y)}$ on γ is 0 or that of N, and N vanishes on the bonds between Y and $U \setminus Y$ by Lemma 7.23. (b) We put $N := N(h) = \nabla n - \xi$, $N_X := N|_{E(X)}$ and $N' := N|_{E(X^c)}$. By Lemma 7.23, N_X satisfies (in), N' satisfies (out), and $N = 0$ on ∂X. Since ξ vanishes off $E(X)$, we have $N_X + \xi\mathbf{1}_{E(X)} = \nabla n$ on $E(X)$ and $[N'] = -[N_X + \xi\mathbf{1}_{E(X)}] = p_1(N_X)$, as $[N + \xi] = 0$. A short closed walk through a site of X carries no wall of N', by (out), and on one inside X^c the period of $N' = \nabla n$ is 0. Hence N' is long.

For unions $Z \subseteq Y$ of $\mathcal{U}$-cells and a wall field N_Y on $E(Y)$ of which Z is a union of elements, we write $\mathcal{K}_{N_Y}^{(Z)}(Y)$ for the activity of Proposition 7.22 at N_Y with first set Z, and $\tilde{\mathcal{K}}(\Delta)$ for

that of a wall-free $\mathcal{U}$-cell Δ ($n = 1$). The finiteness of $\sum_{N_Y} |\mathcal{K}^{(Z)}_{N_Y}(Y)|$ and of $\sup_p \mathcal{Z}_p(X)$ used below is Proposition 7.47, proved without Theorem 7.27 for the activities and outer partition functions built with $(C', G^\natural, \mathfrak{c}\mathcal{N})$ (Lemma 7.29 (f)), the only ones used from Lemma 7.29 on.

Theorem 7.27 (Resummation)**.** *Assume the hypotheses of Proposition 7.22, and that the sums below converge absolutely; for the objects built with $(C', G^\natural, \mathfrak{c}\mathcal{N})$ of Lemma 7.29 (f) they do, by Proposition 7.47. (a) For $X \supseteq X_1$ and an inside field N_X at $k = 1$, let $\mathcal{K}_{N_X}(X) := \mathcal{K}^{(\mathrm{sat}(X_1))}_{N_X}(X)$. Then*

$$Z(1, \Xi) = \sum_{X \supseteq X_1} \sum_{N_X} \mathcal{K}_{N_X}(X)\, \mathcal{Z}_{p_1(N_X)}(X),$$

the inner sum over the inside fields at $k = 1$. (b) (Kirkwood–Salsburg identity.) *Let $X_0 \subsetneq V$ be a union of $\mathcal{U}$-cells and $\Delta \subseteq X_0^c$ a $\mathcal{U}$-cell. For every $p \in \Gamma_{\mathrm{long}}$,*

$$\mathcal{Z}_p(X_0) = \sum_{Y \ni \Delta} \sum_{N_Y} \mathcal{K}^{(\mathrm{sat}(\Delta))}_{N_Y}(Y)\, \mathcal{Z}_{p-[N_Y]}(X_0 \cup Y), \tag{7.21}$$

the sums over the unions $Y \subseteq X_0^c$ of $\mathcal{U}$-cells containing Δ and the $N_Y \in \mathcal{N}^{\mathrm{long}}(Y)$ satisfying (in), with first set $\mathrm{sat}_{N_Y}(\Delta)$. The term $Y = \Delta$, $N_Y = 0$ is $\tilde{\mathcal{K}}(\Delta)\mathcal{Z}_p(X_0 \cup \Delta)$. (c) (Prescribed walls.) *Let R be a family of disjoint unions $\mathcal{S}$ of $\mathcal{U}$-cells, each with a long wall field $N_{\mathcal{S}}$ on $E(\mathcal{S})$ of which it is a wall polymer, $\sigma := (\bigcup R, \sum_{\mathcal{S}} N_{\mathcal{S}})$, and for $r \in \Gamma_{\mathrm{long}}$ let $Z_r(R)$ be the part of Z_r from the configurations N in which every $\mathcal{S} \in R$, with $N|_{E(\mathcal{S})} = N_{\mathcal{S}}$, is an element. Then $Z_r(R) = \sum_{Y \supseteq \bigcup R} \sum_{N_Y} \mathcal{K}^{(\bigcup R)}_{N_Y}(Y)\, \mathcal{Z}_{r-[N_Y]}(Y)$, the sum over the $N_Y \in \mathcal{N}^{\mathrm{long}}(Y)$ satisfying (in) with $N_Y|_{E(\bigcup R)} = \sum_{\mathcal{S}} N_{\mathcal{S}}$ of which $\bigcup R$ is a union of elements.*

Proof. (a) By Corollary 7.10 (d) and Proposition 7.22 with $U = V$ and first set $\mathrm{sat}_h(X_1)$, $Z(1, \Xi) = \sum_{[h]} \mathcal{N} \sum_{X \in \mathfrak{M}(h)} \mathcal{K}_h(X) \int d\mu_C\, e^{E(X^c)} Q_h(X^c)$. Fix $X \supseteq X_1$. The map $[h] \mapsto (N(h)|_{E(X)}, N(h)|_{E(X^c)})$ is a bijection from $\{[h] : X \in \mathfrak{M}(h)\}$ onto the pairs (N_X, N') of an inside field at $k = 1$ and a configuration N' of $\mathcal{Z}_{p_1(N_X)}(X)$. It maps into this set by Lemma 7.26 (b), and it is injective by Lemma 7.6 (d). It is onto, since $N := N_X + N'$ has $[N + \xi] = [(N_X + \xi)\mathbf{1}_{E(X)}] + [N'] = 0$, hence $N = N(Pn)$ with n cell-constant, and (in), (out), (∂) give $X \in \mathfrak{M}(Pn)$ (Lemma 7.23). The activity depends only on N_X (Lemmas 7.21, 7.23). Since both ends of each Ξ-bond lie in $X_1 \subseteq X$, the outer integral depends only on N' and is the summand of (7.20) (Lemma 7.24 (a)). The outer integrals are ≥ 0 (Lemma 7.42), and the absolute convergence allows the sums to be exchanged. Only inside fields with $p_1(N_X) \in \Gamma_{\mathrm{long}}$ occur, by Lemma 7.26 (b), and for them $\sum_{N_X} |\mathcal{K}_{N_X}(X)| < \infty$ (Proposition 7.47, for the objects built with $(C', G^\natural, \mathfrak{c}\mathcal{N})$). (b), (c) In (b) we fix a configuration N of $\mathcal{Z}_p(X_0)$ and expand on $U = X_0^c$ with first set $\mathrm{sat}_N(\Delta)$ by Proposition 7.22 in wall form (Lemma 7.24 (b)). In (c) we fix a configuration N of $Z_r(R)$ and expand in the same way on $U = V$ with first set $\bigcup R$, possibly disconnected. The index sets agree: if each $\mathcal{S} \in R$ is an element of N, then $\bigcup R$ is admissible; conversely, every site within distance L' of a wall of $N_{\mathcal{S}}$ lies in a wall component of $N_{\mathcal{S}}$, hence in $\mathcal{S}$ (Proposition 7.8 (a)), so the wall components of $\sum_{\mathcal{S}} N_{\mathcal{S}}$ are those of the $N_{\mathcal{S}}$, closures from different $\mathcal{S}$ are disjoint, and the elements of N_Y in $\bigcup R$ are the $\mathcal{S}$ (Lemma 7.23). By Lemma 7.23 in wall form and Lemma 7.26 (a), the map $N \mapsto (Y, N\mathbf{1}_{E(Y)}, N\mathbf{1}_{E(U \setminus Y)})$ is a bijection onto the triples of Y, N_Y as stated and a configuration of $\mathcal{Z}_{p-[N_Y]}(X_0 \cup Y)$ in (b), and of $\mathcal{Z}_{r-[N_Y]}(Y)$ in (c). The summand factorizes (Lemma 7.21), and the absolute convergence allows the sums to be exchanged. For $Y = \Delta$ and $N_Y = 0$, the $\mathcal{U}$-cell Δ is a wall-free element.

At fixed walls the expansion factorizes over the elements, admissibility splits into conditions on the two sides (Lemma 7.23), and the class of the walls is additive. This class is therefore all that the inside constrains on the outer side. In (b) the removal of Y moves the class by $[N_Y]$, and each pair (X,p), $X\neq\emptyset$, enters the identity for $(X\setminus\Delta,p)$ through the wall-free term $Y=\Delta$, the form of [BF80, (A4.3), (A4.5)]. The class cannot be dropped: removing the polymer of one of the two walls in the paragraph headed *Handles* (Section 7.1) leaves outer walls of nonzero class, which are not the walls of a height on the complement.

Corollary 7.28 (Loop expansion)**.** *Under the hypotheses of Theorem 7.27, in the finite box,*

$$W_{G'}(C)=\sum_{X\supseteq X_1}\sum_{N_X}\mathcal{K}_{N_X}(X)\,\frac{\mathcal{Z}_{p_1(N_X)}(X)}{Z},\tag{7.22}$$

the inner sum over the inside fields at $k=1$, and $\mathcal{Z}_{p_1(N_X)}(X)\geq 0$.

Proof. We divide Theorem 7.27 (a) by $Z=Z(0,\Xi)>0$ and use Lemma 7.42. Here $\mathcal{Z}_p(X)/Z$ is a ratio of two absolutely convergent sums in the finite box, which Theorem 7.49 only bounds.

7.6 Bounds on the activities

In this subsection we bound the activities of Section 7.5 on G' by [BF80, Section 9] and [GM82, Section 7]. On G' the argument differs from these references in the following points. There is a one-body deficit, and Lemma 7.29 moves its quadratic part into the covariance. Lemmas 7.32 and 7.33 avoid both an extension of the wall field across Ξ and a Poincaré inequality across a slit. Finally, the covariance rate is small and independent of τ (Proposition 7.36). Lemma 7.29 is proved in full in Section S4.6, which uses Section S4.4 for its one-body bounds and for Theorem 7.27 and Corollary 7.28 with $(C',G^\natural,\mathfrak{c}\mathcal{N})$. Lemmas 7.37 and 7.44 and Propositions 7.46, 7.47 and 7.52 are proved in full in Section S4.8.

Throughout, X is a polymer (a union of elements of $\tilde{Y}(h)$ in the sense of Definition 7.20), $|X|$ is its number of $\mathcal{U}$-cells, $\vartheta:=\psi+D$, where $D:=g-H$ is the field in the phases, and $\tilde{\delta}:=P_X\vartheta$, where $P_X\vartheta:=\vartheta-\operatorname{avg}_\Omega\vartheta$ on each cell $\Omega\subseteq X$ and $P_X\vartheta:=0$ off X. We put $C_u:=2+6/c_g$, so that $\#\Delta<C_u\tilde{m}^{-3}$ for each $\mathcal{U}$-cell Δ by (R7), and $\|A\|_r:=\sup_x\sum_y|A(x,y)|e^{r\,d_{G'}(x,y)}$ for a matrix A on V and $r\geq 0$. Constants C,C_n depend only on those of Theorem 6.5 and Proposition 6.6.

The deficit decomposition. By Proposition 7.4 the inhomogeneity is confined to the term $G_{2,3}$ of (7.5), which by (S3) is $\sum_{m\neq0}\sum_{x\in V}(\rho_1(m,x)-\rho^{\mathrm{ref}}(m))(\cos(m\beta_{\mathrm{GM}}^{1/2}\vartheta(x))-1)$. Let $G_{2,3}^{\mathfrak{D}}$ be its part with $x\in\mathfrak{D}$, and put

$$\mu_{\mathfrak{D}}(x):=\beta_{\mathrm{GM}}\sum_{m\neq0}m^2\big(\rho_1(m,x)-\rho^{\mathrm{ref}}(m)\big)\,1_{V\setminus\mathfrak{D}}(x),\tag{7.23}$$

$$G_{2,3}^{\natural}:=\sum_{m\neq0}\sum_{x\in V\setminus\mathfrak{D}}\big(\rho_1(m,x)-\rho^{\mathrm{ref}}(m)\big)\big(\cos(m\beta_{\mathrm{GM}}^{1/2}\vartheta(x))-1+\tfrac12 m^2\beta_{\mathrm{GM}}\psi(x)^2\big).$$

Then, identically in ψ,

$$G_{2,3}=G_{2,3}^{\mathfrak{D}}-\tfrac12\langle\psi,\mu_{\mathfrak{D}}\psi\rangle+G_{2,3}^{\natural}.\tag{7.24}$$

We write $G^{\mathfrak{D}}_{2,3}(Y)$, $G^{\natural}_{2,3}(Y)$ for the sums over $x \in Y$. Finally, let $G^{\natural} := G_1 + G_{2,1} + G_{2,2} + G^{\mathfrak{D}}_{2,3} + G^{\natural}_{2,3}$ and $C' := (C_0^{-1} + \nu + \mu_{\mathfrak{D}})^{-1}$, where the operator $\mu_{\mathfrak{D}}$ depends only on G' (through ρ_1). At a site with $D(x) = 0$ we have $\partial_{\psi(x)} G_{2,3} = -\mu_{\mathfrak{D}}(x)\psi(x) + O(\theta' \tilde{m}^2 \beta_{\mathrm{GM}} |\psi(x)|^3)$, with $\theta' \geq 1 - e^{-\theta_0}$ fixed, that is, with no power of β_{GM}, whereas [BF80, (9.813) to (9.819)] need $\beta_{\mathrm{GM}}^{1/2}$ per derivative of a one-body coefficient of order $\tilde{m}^2/\beta_{\mathrm{GM}}$. For this reason we move the quadratic part into the covariance, as ν is, and $G^{\natural}_{2,3}$ is of fourth order (type vi of [BF80, (9.811)]).

Lemma 7.29. *Assume the standing hypothesis* (7.8), $\varepsilon_\nu + \theta' \leq 1/7$, *with* ε_ν *of Section 7.1 (the output form of (S5)) and* θ' *from* (7.7). *Then, uniformly in* ψ, h, k, Ξ *and* G', *with* $C_{\mathfrak{D}}$ *the constant* C *of Proposition 7.5(b): (a)* $|\mu_{\mathfrak{D}}(x)| \leq \theta' \tilde{m}^2$ *for every* x; *for every union* Y *of* $\mathcal{U}$*-cells,* $-C_{\mathfrak{D}}(e^{-\mu} + e^{-2\kappa_S})\, c_{\mathrm{dep}} N_{\mathfrak{D}}(Y) \leq G^{\mathfrak{D}}_{2,3}(Y) \leq C_{\mathfrak{D}} c_{\mathrm{dep}} N_{\mathfrak{D}}(Y)$ *and*

$$|G^{\natural}_{2,3}(Y)| \leq \tfrac{1}{2}\theta' \tilde{m}^2 \textstyle\sum_{x \in Y \setminus \mathfrak{D}} \big(\vartheta(x)^2 + \psi(x)^2\big).$$

(b) At a site $x \notin \mathfrak{D}$ *with* $D(x) = 0$, *the* n*-th derivative in* $\psi(x)$ *of the* x*-term of* $G^{\natural}_{2,3}$ *is at most* $C_n \theta' \tilde{m}^2 \beta_{\mathrm{GM}} |\psi(x)|^{4-n}$ *for* $0 \leq n \leq 4$ *and at most* $C_n \theta' \tilde{m}^2 \beta_{\mathrm{GM}}^{n/2-1}$ *for* $n \geq 4$. *At* $x \in \mathfrak{D}$, *the* n*-th derivative* ($n \geq 1$) *of the* x*-term of* $G^{\mathfrak{D}}_{2,3}$ *is at most* $C_n(\tilde{m}^2/\beta_{\mathrm{GM}})\beta_{\mathrm{GM}}^{n/2}$. *(c)* C' *exists, is positive definite, and* $C' \leq \alpha_D C_0$ *as forms, with* $\alpha_D := (1 - \varepsilon_\nu - \theta')^{-1} \leq 7/6$. *(d) The bounds of Theorem 6.11 hold for* C' *and for every interpolated covariance* C'_s *of* (C.2): $C'_s(x,x) = C'(x,x) \leq \alpha_D K_1 M_Y$, $|C'_s(x,y)| \leq 2K_1'' M_Y e^{-r_{\mathrm{cov}} d_{G'}(x,y)}$, $\|C'_s\|_{r_{\mathrm{cov}}} \leq 2K_3 \tilde{m}^{-2}$ *and* $\|C'_s\|_{2\to 2} \leq \alpha_D \tilde{m}^{-2}$. *(e) As measures on* $\mathbb{R}^V$,

$$e^{G(\psi+g)}\, d\mu_C(\psi) = \mathfrak{c}\, e^{G^{\natural}}\, d\mu_{C'}(\psi), \qquad \mathfrak{c} := \Big(\frac{\det C'}{\det C}\Big)^{1/2}, \tag{7.25}$$

and $\mathfrak{c}$ *does not depend on* h, k *or* Ξ. *(f) The representation* (7.9), *Corollary 7.10 (d) and Proposition 7.22 hold with* $(C, G, \mathcal{N})$ *replaced by* $(C', G^{\natural}, \mathfrak{c}\mathcal{N})$; *Theorem 7.27 and Corollary 7.28 hold for the activities and outer partition functions built with* $(C', G^{\natural}, \mathfrak{c}\mathcal{N})$, *with the same left sides* $Z(1, \Xi)$, $Z_r(R)$ *and* $W_{G'}(C)$; *and* $W_{G'}(C)$, *every* Z_p *and every* $Z_r(R)$ *are unchanged.*

Proof. (a) Arguing as in the proof of Proposition 7.5(a), with m^2 replaced by $|m|^n$, we obtain, for $x \notin \mathfrak{D}$ and $n \geq 2$,

$$\beta_{\mathrm{GM}} \sum_{m \neq 0} |m|^n \, |\rho_1(m, x) - \rho^{\mathrm{ref}}(m)| \leq C_n \theta' \tilde{m}^2, \tag{7.26}$$

with $C_2 = 1$ once C in (7.7) is large, and at each x the left side is at most $C_n \tilde{m}^2$. For $n = 2$ this bounds $\mu_{\mathfrak{D}}$. In the same case, together with $|\cos y - 1 + \frac{1}{2}z^2| \leq \frac{1}{2}y^2 + \frac{1}{2}z^2$, applied with $y = m\beta_{\mathrm{GM}}^{1/2}\vartheta(x)$ and $z = m\beta_{\mathrm{GM}}^{1/2}\psi(x)$, it also bounds $G^{\natural}_{2,3}$. The bound on $G^{\mathfrak{D}}_{2,3}(Y)$ follows from Proposition 7.5(b) together with Proposition 6.17.

(b) If $D(x) = 0$, then $\vartheta(x) = \psi(x)$ and $f(\psi) := \cos(m\beta_{\mathrm{GM}}^{1/2}\psi) - 1 + \frac{1}{2}m^2\beta_{\mathrm{GM}}\psi^2$ vanishes to fourth order with $|f^{(n)}| \leq |m|^n \beta_{\mathrm{GM}}^{n/2}$. At $x \in \mathfrak{D}$, $|\partial_\psi^n \cos(m\beta_{\mathrm{GM}}^{1/2}\vartheta)| \leq |m|^n \beta_{\mathrm{GM}}^{n/2}$. We then sum with (7.26).

(c) The Schur test gives $\|\nu\|_{2\to 2} \leq \|\nu\|_{r_{\mathrm{cov}}} \leq \varepsilon_\nu \tilde{m}^2$, part (a) gives $\|\mu_{\mathfrak{D}}\|_{2\to 2} \leq \theta' \tilde{m}^2$, and $C_0^{-1} = u^{-1} + \tilde{m}^2 \geq \tilde{m}^2$ by Definition 7.1. Hence $C_0^{-1} + \nu + \mu_{\mathfrak{D}} \geq (1 - \varepsilon_\nu - \theta')C_0^{-1} \geq \frac{6}{7}C_0^{-1} > 0$, and (c) follows, since inversion reverses the order of positive definite forms.

(d) Theorem 6.11 applies to $C' = (C_0^{-1} + \nu')^{-1}$, $\nu' := \nu + \mu_{\mathfrak{D}}$, as $2K_3\|\nu'\|_{r_{\mathrm{cov}}} \le 2K_3(\varepsilon_\nu + \theta')\tilde m^2 \le \tilde m^2$, where we used $K_3 \le 3.5$. The diagonal bound follows from (c). Block compressions keep the diagonal and positivity and increase no other bound, which extends these bounds to the interpolated covariances.

(e) By (7.24), $e^{G(\psi+g)} = e^{G^\natural} e^{-\frac12\langle\psi,\mu_{\mathfrak{D}}\psi\rangle}$. Since $C^{-1} = C_0^{-1} + \nu$ by Proposition 7.7, we have $e^{-\frac12\langle\psi,\mu_{\mathfrak{D}}\psi\rangle} d\mu_C(\psi) = (\det 2\pi C)^{-1/2} e^{-\frac12\langle\psi,C'^{-1}\psi\rangle} d\psi = \mathfrak{c}\, d\mu_{C'}(\psi)$. The operators C_0, ν, $\mu_{\mathfrak{D}}$ depend only on G', M_Y and $\tilde m$, so the factor $\mathfrak{c}$ does not depend on h, k or Ξ, and since (7.25) is an identity of measures, it also preserves absolute integrability.

(f) We insert (7.25) in Proposition 7.7. Those proofs use only factorization over cells and sites, the margin of Lemma C.2 ($q_M \le 1/6$ for C') and the quadratic bound on $|E(X,s)|$, and, as $G - G^\natural = -\frac12\langle\psi,\mu_{\mathfrak{D}}\psi\rangle$ does not depend on the wall field, (7.25) applies to each configuration of $Z(k,\Xi)$, Z_p and $Z_r(R)$, so these are unchanged; the outer partition functions with $X \neq \emptyset$ are not the same for the two choices, and from here on only those built with $(C', G^\natural, \mathfrak{c}\mathcal{N})$ are used.

Like the $\mathcal{Z}_p(X)$, the activities are now built with $(C', G^\natural, \mathfrak{c}\mathcal{N})$. For a polymer X, $\mathcal{N}(X)$ is the set of wall configurations $N_X := N|_{E(X)}$ of cell-constant labels h_X with values in $P\mathbb{Z}$ ($P = 2\pi\beta_{\mathrm{GM}}^{-1/2}$) whose walls lie in $E(X)$. At $k = 0$ the statements of this subsection and Lemma 7.48 hold for the long wall fields of Definition 7.25, by Proposition 7.46. We write $n_I(X)$ for the number of interfaces with both cells in X, and put $A_w := 2F_{\mathrm{unit}}$.

Lemma 7.30 (Wall entropy)**.** *(a) $F_1^{\mathrm{loc}}(Y) \ge A_w \sum_{I\subseteq Y} N_I^2$ for every union Y of elements of $\tilde Y(h)$, where N_I is the constant value of N on the interface I (Lemma 7.6(a)). (b) For every polymer X and every $c_4 > 0$ with $e^{-c_4A_w} \le \frac13$, $\sum_{N_X\in\mathcal{N}(X)} e^{-c_4F_1^{\mathrm{loc}}(X)} \le (1 + 3e^{-c_4A_w})^{n_I(X)} \le e^{c_4'|X|}$ with $c_4' := 9C_u e^{-c_4A_w}/(c_g c_1(\tilde m L)^3)$.*

Proof. We adapt the proof of [BF80, Lemmas 9.2, 9.3] to G' as follows. (1) The face sum of [BF80, (9.117)] becomes the interface sum. Indeed, walls are supported on interfaces by Lemma 7.6(a), and by Theorem 7.13 with $\mathrm{good}(I) \ge c_g' L^2$, which holds by (R8) in fine units and Lemma 4.19, an interface $I \subseteq Y$ gives at least $\frac{1}{18}\tilde m^2 \frac{L}{2} c_g' L^2 \cdot 4\pi^2 N_I^2/\beta_{\mathrm{GM}} = 2F_{\mathrm{unit}}N_I^2$. (2) The sum over h becomes the injection $N_X \mapsto (N_I)_I \in \mathbb{Z}^{n_I(X)}$, each interface giving $\sum_n e^{-c_4A_w n^2} \le 1+3e^{-c_4A_w}$. (3) The cube count becomes $n_I(X) \le 3C_u|X|/(c_g c_1(\tilde m L)^3)$, since a cell has at least c_1L^3 sites by (R5) and at most $6/c_g$ neighbours by (R8), and a $\mathcal{U}$-cell has fewer than $C_u\tilde m^{-3}$ sites. With $\log(1+3t) \le 3t$ this gives the second inequality of (b), and by Corollary 7.10(d) no perimeter term arises. For the value $c_4 = c'/4$ of Remark S1.1, c_4' is doubly exponentially small in τ, as $A_w \ge (\pi^2 c_g'/72)c_L^3(\tilde m\beta_{\mathrm{GM}})^{-1}$ by $\tilde m L \ge c_L/2$.

In Debye units, for the Peierls factor r_Ω of (7.4), we put $v_\Omega := \tilde m^3|\Omega|$, $\hat\rho(m) := \tilde m^{-3}\rho^{\mathrm{ref}}(m)$, $b_D := \beta_{\mathrm{GM}}\tilde m$, $\hat P := 2\pi b_D^{-1/2}$, and

$$\hat R_v(x) := \frac{\exp\{v\sum_m \hat\rho(m)\,[e^{imb_D^{1/2}x} - 1]\}}{\sum_{n\in\mathbb{Z}} \exp\{-\frac12 v(x - n\hat P)^2\}}, \tag{7.27}$$

so that $r_\Omega(A) = \hat R_{v_\Omega}(\tilde m^{-1/2}(A + k\beta_{\mathrm{GM}}^{-1/2}\bar f_\Omega))$. By (R5) in fine units and $c_L/2 \le \tilde m L \le c_L$, we have $v_\Omega \in [v_-, v_+] := [c_1c_L^3/8,\, 8c_L^3]$. Let $\eta_* := \sup_{0<s\le\pi}[1 - 2b_D\sum_m \hat\rho(m)(1 - \cos ms)/s^2] \le \bar\eta := 1 - \frac{4}{\pi^2}\big(1 - C_{\hat\rho}e^{-2\kappa_S}\big) + C_{\hat\rho}e^{-2\kappa_S} \le 1 - \frac{4}{\pi^2} + 2C_{\hat\rho}e^{-2\kappa_S}$, where $C_{\hat\rho} :=$

$4C \geq b_D \sum_{|m|\geq 2} |\hat\rho(m)|\, m^2 e^{2\kappa_S(|m|-1)}$ with C of (6.10) ($\tilde m \geq m_D/2$); unlike η_*, the bound $\bar\eta$ does not depend on b_D.

Lemma 7.31. *Fix $\eta_H \in (\bar\eta, 1)$. There are $b_0 > 0$ and c_7, c_7', c_7'', depending only on $v_\pm$, η_H, $C_{\hat\rho}$ and κ_S, such that for $b_D \leq b_0$, $v \in [v_-, v_+]$, all real x and all integers $N \geq 0$,*

$$|\partial_x^N \hat R_v(x)| \leq c_7\, (c_7' b_D^{1/6})^N e^{c_7'' N \log N} e^{\frac12 \eta_H v x^2}.$$

Consequently, uniformly in h, with $\partial A_\Omega/\partial\psi(x) = 1/|\Omega|$ ($x \in \Omega$),

$$|\partial_A^N r_\Omega(A_\Omega)| \leq c_7\, (c_7' b_D^{1/6} \tilde m^{-1/2})^N e^{c_7'' N \log N} \exp\Big(\tfrac12 \eta_H \tilde m^2 \sum_{x\in\Omega} \vartheta(x)^2\Big).$$

The first bound is proved as [BF80, Lemma 9.7, (9.51)], with $\tilde l_D = 1$, $L^3 \to v$, $\beta \to b_D$, $\rho_i \to \hat\rho(m)$, $\tau \to \hat P$, except for the following points. (1) There are infinitely many charges. The Taylor expansions of the factors I, II of [BF80, (9.53)] converge for $|\mathrm{Im}\, x| \leq b_D^{-1/6}$ by the tails of (S4′), as $e^{|m| b_D^{1/3}} e^{-2\kappa_S|m|}$ is summable. (2) The constants are uniform in $v \in [v_-, v_+] \subset (0,\infty)$, not for a fixed volume L^3. (3) We need an explicit exponent for $\alpha_D p' \eta_H < 1$ in (L8-1). The inequalities $1-\cos s \geq 2s^2/\pi^2$ on $[0,\pi]$ and $2\beta_{\mathrm{GM}} \rho^{\mathrm{ref}}(1) \geq (1 - C_{\hat\rho} e^{-2\kappa_S})\tilde m^2$ bound η_*, and each $\eta_H \in (\bar\eta, 1)$ is admissible in [BF80, (9.59)]. The second bound follows from the $\hat P$-periodicity, from $A_\Omega + k\beta_{\mathrm{GM}}^{-1/2} \bar f_\Omega = h_\Omega + \mathrm{avg}_\Omega\, \vartheta$ with $h_\Omega \in P\mathbb{Z}$, and from $|\Omega|(\mathrm{avg}_\Omega\, \vartheta)^2 \leq \sum_{x\in\Omega} \vartheta(x)^2$. With $\gamma := \eta_H \tilde m^2$, the second bound is the analogue on G' of [GM82, (7.9), (7.10)]: each derivative of G_1 contributes a factor $b_D^{1/6}$, times $e^{\frac12 \gamma \sum_X \vartheta^2}$.

Hölder exponents and the hypotheses (L8-1) to (L8-4). We fix $\eta_H \in (\bar\eta, 1)$, Hölder exponents $p_1, \dots, p_4 > 1$ with $\sum_i 1/p_i = 1$ and p_1 even, and $p' > p_3$, q' with $1/p' + 1/q' = 1/p_3$, at the values given in Section S1.7.1. Let C_W^f, $k_\square = r_\square L$, N_{ov} and $\mathfrak{K}_\Omega^\flat \supseteq \Omega$, with sites in the box $\hat Q(Q)$ of fine side at most $k_\square$, be those of the cell Poincaré inequality of Proposition 6.9(iv), and let $C_P' := C_W^f r_\square^2 N_{\mathrm{ov}}$, so that $C_P' L^2 = C_W^f k_\square^2 N_{\mathrm{ov}}$. Hypothesis (L8-1) is the standing hypothesis (7.8) together with $\hat\eta := \alpha_D p' \eta_H < 1$, which implies $\alpha_D p_3 \eta_H < 1$ (in [BF80, (9.638)] the constant $\hat\eta$ is denoted $\hat\beta < 1$). (L8-2) requires $C_P'(\tilde m L)^2 \leq \eta_H/8$, so that $\theta_P := (4\hat\eta/\eta_H) C_P'(\tilde m L)^2 \leq \frac12$ and $\nu_0 := \max(\hat\eta, \theta_P) < 1$. (L8-3) requires $\varepsilon(L') \leq 2(p'/q')^2$, with $\varepsilon(L')$ of Theorem 7.16, and (L8-4) states that each $\mathfrak{K}_\Omega^\flat$ has $d_{G'}$-diameter less than L'. At the values fixed above, (L8-1) holds. Hypothesis (L8-2) holds by the choice of c_L, since $\tilde m L \leq c_L$, (L8-3) holds by Corollary 7.18, and (L8-4) follows from (L8-2) by Proposition 6.9(v). A cell $\Omega \subseteq Q$ is *near* if the ℓ^∞-distance from the sites of $\hat Q(Q)$ to C is at most $\frac12$, and *far* otherwise. For a union Y of elements, $E(\mathcal{N}(Y))$ is the union of the $E(\mathfrak{K}_\Omega^\flat)$ over the near cells $\Omega \subseteq Y$.

The free-space solid angle. Let $\Omega_\Xi(y) \in (-2\pi, 2\pi)$ be the signed solid angle of the flat rectangle Ξ seen from a point y off its plane, increasing by 4π when y crosses Ξ in the direction in which $\xi_b = +1$. For a site x of G' (a dual site of $\mathbb{Z}^3$, never in the plane of Ξ) we put $f^C(x) := \frac12 \Omega_\Xi(x)$ and, for each dual bond $b = (x, x')$ of $\mathbb{Z}^3$, $R^C(b) := f^C(x') - f^C(x) - j(b)$, where $j(b) = 2\pi\xi_b$, which on the bonds of G' is the jump field of Ξ (Section 7.1). The field f^C depends only on C, not on G', unlike the Neumann solid angle $f_{G'}$ of Lemma 7.2, and R^C differs from R_f. Let $S(D) := \sum_{b:\, d(b,C)\leq D} |R^C(b)|^2$, the sum running over dual bonds of $\mathbb{Z}^3$, and $D_{\mathcal{N}} := \sqrt{3}\, k_\square + 1$.

Lemma 7.32. *(a) Assume (L8-4). Let Y be a union of elements of $\tilde{Y}(h)$ and $\Omega \subseteq Y$ a cell. For every $\vartheta \in \mathbb{R}^V$, every function a on the sites of $\mathfrak{K}^\flat_\Omega$ and every $\varepsilon \in (0,1]$,*

$$\|P_\Omega \vartheta\|^2 \le (1+\varepsilon) C_W^f k_\square^2 \|\nabla\vartheta + w_Y\|^2_{E(\mathfrak{K}^\flat_\Omega)} + (1+1/\varepsilon) k^2 \beta_{\mathrm{GM}}^{-1} \delta_\Omega(a)^2,$$
$$\delta_\Omega(a) := (C_W^f)^{1/2} k_\square \|\nabla a - j\|_{E(\mathfrak{K}^\flat_\Omega)} + \|P_\Omega a\|_\Omega,$$

where $w_Y := \sum_{S_\beta \subseteq Y} w_\beta$. If Ω is far, there is an a with $\delta_\Omega(a) = 0$. (b) For every dual bond b of $\mathbb{Z}^3$, $d(b,C) \ge \frac12$ and $|R^C(b)| \le 2\pi/d(b,C)$; at every dual site, $|f^C| < \pi$. (c) On every output G', $\sum_{\Omega \text{ near}} \|R^C\|^2_{E(\mathfrak{K}^\flat_\Omega)} \le N_{\mathrm{ov}} \|R^C\|^2_{E(\mathcal{N}(V))} \le N_{\mathrm{ov}} S(D_\mathcal{N})$, and if $|C| \ge 8k_\square$, then

$$S(D_\mathcal{N}) \le C_{\mathrm{BS}} |C| \,(\log k_\square + C_{\mathrm{sh}}), \qquad C_{\mathrm{BS}} := 24\pi^3, \quad C_{\mathrm{sh}} := 92.$$

Proof. Heuristically, on a box crossed by C the trivial primitive leaves a flat sheet of jumps, of energy $\asymp k_\square^2$, and with the solid angle only the Biot–Savart residual remains, of energy $\asymp k_\square \log k_\square$ per unit length of C.

(a) We first show that $w = w_Y$ on $E(\mathfrak{K}^\flat_\Omega)$. Let $b' \in E(\mathfrak{K}^\flat_\Omega)$ be a wall bond. It belongs to one component $w_{\beta'}$ and has a wall site z as an endpoint. Fix $x \in \Omega$. Then $d_{G'}(x,z) < L'$ by (L8-4). Hence each $\mathcal{U}$-cell met by a G'-path of length $d_{G'}(x,z)$ from x to z lies in $\hat{\Sigma}$, consecutive ones being joined by a bond, and by Proposition 7.8 the $\mathcal{U}$-cell of x lies in $S_{\beta'}$. This $\mathcal{U}$-cell also lies in the element of Y containing Ω, and each S_β lies in one element, in view of Definition 7.20. Hence $S_{\beta'} \subseteq Y$. Other bonds carry $w = 0 = w_Y$. Next we define $a_\Omega := h - k\beta_{\mathrm{GM}}^{-1/2} a$ on the sites of $\mathfrak{K}^\flat_\Omega$. Here, at $k = 0$, for long wall fields, the label is replaced by the primitive of Lemma 7.45 (Proposition 7.46), and at $k = 1$, for the inside fields of outer class in Γ_{long}, by the primitive of Lemma 7.45b (b). As h is constant on cells, $P_\Omega h = 0$ and $P_\Omega \vartheta = P_\Omega(\vartheta + a_\Omega) + k\beta_{\mathrm{GM}}^{-1/2} P_\Omega a$. On $E(\mathfrak{K}^\flat_\Omega)$, since $w = \nabla h - k\beta_{\mathrm{GM}}^{-1/2} j$, we have $\nabla(\vartheta + a_\Omega) = \nabla\vartheta + w_Y + k\beta_{\mathrm{GM}}^{-1/2}(j - \nabla a)$. The Poincaré inequality of Proposition 6.9(iv) for $\vartheta + a_\Omega$ on $\mathfrak{K}^\flat_\Omega$ and the triangle inequality give $\|P_\Omega\vartheta\| \le (C_W^f)^{1/2} k_\square \|\nabla\vartheta + w_Y\|_{E(\mathfrak{K}^\flat_\Omega)} + k\beta_{\mathrm{GM}}^{-1/2}\delta_\Omega(a)$. We then square, using $(x+y)^2 \le (1+\varepsilon)x^2 + (1+1/\varepsilon)y^2$.

Now let $\Omega \subseteq Q$ be far, and let Π be the plane of Ξ. A dual bond crossing Π crosses it at the centre of a plaquette, and $j(b) \ne 0$ iff that plaquette lies in Ξ. The plaquettes crossed by bonds of $\hat{Q}(Q)$ form a rectangle. If this rectangle contained plaquettes in and outside Ξ, it would contain two sharing an edge $e \in \partial\Xi = C$ at ℓ^∞-distance $\frac12$ from a site of $\hat{Q}(Q)$, which is excluded for a far cell. Hence $j = 2\pi\sigma\nabla 1_{\Pi^+}$ on $E(\hat{Q}(Q))$ for some $\sigma \in \{-1,0,1\}$, where Π^+ is the set of sites on one side of Π. With this σ, the function $a := 2\pi\sigma 1_{\Pi^+}$ has $\nabla a = j$ on $E(\mathfrak{K}^\flat_\Omega)$ and is constant on Ω, as Q lies on one side of Π by (A3) of Lemma 3.5. Thus $\delta_\Omega(a) = 0$.

(b) Primal edges are at distance at least $\frac12$ from dual bonds, which gives the first assertion. Off C, $\nabla\Omega_\Xi$ extends continuously across Ξ as the Biot–Savart field of C. Hence $|R^C(b)| \le \frac12 \max_{y \in b} J(y)$, where $J(y) := \oint_C |dl|\,|y-l|^{-2}$. Let $y = y' + h\mathbf{e}$ with $y' \in \Pi$ and $h \ge 0$. The length of $\{l \in C : |y'-l| \le t\}$ vanishes for $t < d(y',C)$ and is at most $2\pi t$, since this set lies in the boundary of a convex subset of a disc of radius t. Integration by parts and the substitution $t = h\tan\varphi$ give $J(y) \le 4\pi/d(y,C)$. Finally, $|\Omega_\Xi| < 2\pi$, since the radial projection of the rectangle from a point off Π lies in an open hemisphere.

(c) For a near cell $\Omega \subseteq Q$, the bonds of $\mathfrak{K}^\flat_\Omega$ are dual bonds of $\mathbb{Z}^3$ joining sites of $\hat{Q}(Q)$, a box of diameter at most $\sqrt{3}\,k_\square$ within Euclidean distance $\frac{\sqrt{3}}{2}$ of C, so $d(b,C) \le D_\mathcal{N}$. Each

$|R^C(b)|^2$ is a fixed nonnegative number attached to a dual bond of $\mathbb{Z}^3$, independent of G', and passing from $\mathbb{Z}^3$ to G' can only delete terms of free-space ℓ^2 sums, since freezing a plaquette deletes a bond and a hole deletes sites and bonds. The bounds are therefore free-space bounds, independent of the Neumann field and its enhancement at the slit edges, and with the overlap N_{ov} these facts give the first two inequalities. For the last one, we use the number $\mathcal{N}(r)$ of dual bonds with $d(b,C)\le r$, which vanishes for $r<\frac12$. The midpoints of these bonds lie within $r+\frac12$ of C and form three translates of $\mathbb{Z}^3$. Hence, with $r_*:=\frac12+\frac{\sqrt3}{2}$, $\mathcal{N}(r)\le 3|\{y: d(y,C)\le r+r_*\}|\le 3\pi|C|(r+r_*)^2+16\pi(r+r_*)^3$. By (b) and summation by parts, $\sum_{d(b,C)\le D} d(b,C)^{-2}=\mathcal{N}(D)D^{-2}+2\int_{1/2}^{D}\mathcal{N}(r)r^{-3}dr$. The term $3\pi|C|r^2$ gives $6\pi|C|\log D+\mathcal{O}(|C|)$, and the other terms give $\mathcal{O}(|C|+D)$. The factor $(2\pi)^2$ of (b) gives $C_{\mathrm{BS}}=(2\pi)^2\cdot 6\pi=24\pi^3$. At $D=D_{\mathcal{N}}\le 2k_\square$ and $|C|\ge 8k_\square$ the remaining constants add up to $C_{\mathrm{sh}}=92$ (Section S1.7.2).

For a polymer X put

$$B_X(\vartheta):=\tfrac12\tilde m^2\sum_{x\in X}\vartheta(x)^2+\frac{2\tilde m^2}{\eta_H}\sum_{x\in X}(P_X\vartheta)(x)^2,\qquad T_3:=\Big[\int d\mu_{C'_s}(\psi)\,e^{p_3\eta_H B_X(\psi+D)}\Big]^{1/p_3}.$$

With $\gamma=\eta_H\tilde m^2$ we have $\frac12 p_3\gamma\sum_X\vartheta^2+2p_3\tilde m^2\sum_X\tilde\delta^2=p_3\eta_H B_X(\vartheta)$. Hence T_3 is the third factor of the Hölder split (Proposition 7.35), and it is the analogue on G' of [BF80, (9.64)]. The near-cell factor is

$$\mathfrak{n}'(Y):=2k^2\beta_{\mathrm{GM}}^{-1}\Big[\theta_P\|R^C\|^2_{E(\mathcal{N}(Y))}+\frac{4\hat\eta\tilde m^2}{\eta_H}\sum_{\Omega\subseteq Y\ \mathrm{near}}\|P_\Omega f^C\|^2_\Omega\Big].$$

We also use its cellwise version $\mathfrak{n}''(Y):=2k^2\beta_{\mathrm{GM}}^{-1}\sum_{\Omega\subseteq Y\ \mathrm{near}}\big[\frac{\theta_P}{N_{\mathrm{ov}}}\|R^C\|^2_{E(\mathfrak{K}^\flat_\Omega)}+\frac{4\hat\eta\tilde m^2}{\eta_H}\|P_\Omega f^C\|^2_\Omega\big]$, which is additive over disjoint unions of cells and satisfies $\mathfrak{n}''\le\mathfrak{n}'$, as each bond of $E(\mathcal{N}(Y))$ lies in at most N_{ov} of the sets $E(\mathfrak{K}^\flat_\Omega)$ (Theorem 3.20 (R6)).

Lemma 7.33. *Assume (L8-1) to (L8-4), and let c_8 and c_{near} be the constants of Section S1.7.3. For every h, every polymer X and every s,*

$$T_3\le\exp\Big\{\frac{c_8|X|+\mathfrak{n}'(X)}{p'}+\frac{F_1^{\mathrm{loc}}(X)}{p_3}\Big\},$$

and $\mathfrak{n}'=0$ for $k=0$. If $\min(R,T)\ge R_1(\beta)=(2r_\square+5)L$, then $\mathfrak{n}'(V)\le c_{\mathrm{near}}|C|$, and uniformly for $\beta_{\mathrm{GM}}\ge\beta^0_{\mathrm{GM}}$,

$$c_{\mathrm{near}}\le 0.1264\,\theta_P C_{\mathrm{BS}}+\frac{2\theta_P C_{\mathrm{BS}}C_{\mathrm{sh}}+1024\pi^2\hat\eta(2r_\square+5)^2c_L^2/(\eta_H c_1)}{\beta_{\mathrm{GM}}}.$$

Proof. Step 1 (local untranslation identity). We absorb the shift D by a translation, which turns $\frac12\langle\psi,C_0^{-1}\psi\rangle+F_2+F_1^{\mathrm{loc}}$ into one quadratic form of $\psi+D$. In local form, this is the step of [BF80, p. 226] in which (7.8) to (7.10) are applied backwards. Let Y be a union of elements, let D_Y, w_Y, s_Y be the sums of D_β, w_β, s_β over $S_\beta\subseteq Y$, and let $\mathcal{Q}_Y(\vartheta):=\tilde m^2\|\vartheta\|^2+\|\nabla\vartheta+w_Y\|^2+M_Y^{-2}\|\Delta\vartheta-\nabla^*w_Y\|^2$. Then for all $\psi\in\mathbb{R}^V$

$$\tfrac12\langle\psi,C_0^{-1}\psi\rangle+F_2(Y)+F_1^{\mathrm{loc}}(Y)=\tfrac12\mathcal{Q}_Y(\psi+D_Y).\tag{7.28}$$

Indeed, since $C_0^{-1} = \tilde m^2 - \Delta + M_Y^{-2}\Delta^2$ and $-\Delta = \nabla^*\nabla$, the right side is $\frac12\langle\psi, C_0^{-1}\psi\rangle + \langle\psi, C_0^{-1}D_Y + s_Y\rangle + \frac12\mathcal{Q}_Y(D_Y)$ with $s_Y = M_Y^{-2}(-\Delta + M_Y^2)\nabla^* w_Y$. By Proposition 7.8, $D_\beta = -\chi_\beta C_0 s_\beta$ and $(1-\chi_\beta)s_\beta = 0$, so $C_0^{-1}D_\beta + s_\beta = -[C_0^{-1}, \chi_\beta]C_0 s_\beta = F'_{2,\beta}$, supported in S_β. Therefore the cross term is $F_2(Y)$. Since the functions D_β, ∇D_β, w_β, ΔD_β, $\nabla^* w_\beta$ are supported in S_β or $E(S_\beta)$, terms of different β have disjoint supports, and $\frac12\mathcal{Q}_Y(D_Y) = \sum_{S_\beta\subseteq Y} F_1(S_\beta) = F_1^{\mathrm{loc}}(Y)$ by Proposition 7.9 and Corollary 7.10.

Step 2 (the Gaussian integral at fixed Y). We show

$$J_Y := \int d\mu_{C_0}(\psi)\exp\bigl\{\hat\eta B_Y(\psi + D) - F_1^{\mathrm{loc}}(Y) - F_2(Y)\bigr\} \le \exp\{c_8|Y| + \mathfrak{n}''(Y)\}. \tag{7.29}$$

Each S_β lies in Y or is disjoint from it, so $D = D_Y$ on Y. Let $N := \hat\eta\tilde m^2 1_Y + (4\hat\eta\tilde m^2/\eta_H)P_Y$, so that $\hat\eta B_Y(\vartheta) = \frac12\langle\vartheta, N\vartheta\rangle$, since $P_Y = 1_Y P_Y 1_Y$ is an orthogonal projection. By (7.28) and $\vartheta = \psi + D_Y$, J_Y is the ratio of the integrals over $\mathbb{R}^V$ of $\exp\{-\frac12\mathcal{Q}_Y(\vartheta) + \frac12\langle\vartheta, N\vartheta\rangle\}$ and of $e^{-\frac12\langle\psi, C_0^{-1}\psi\rangle}$, with quadratic part $-\frac12\langle\vartheta, (C_0^{-1} - N)\vartheta\rangle$. For positivity we use [BF80, (9.637), (9.640) to (9.642)], with the cell Poincaré inequality summed with overlap N_{ov}, that is, $\|P_Y\vartheta\|^2 \le C'_P L^2\|\nabla\vartheta\|^2$, in place of the Neumann Laplacian of the cubes in (9.642). We obtain $\langle\vartheta, N\vartheta\rangle \le \hat\eta\tilde m^2\|\vartheta\|^2 + \theta_P\|\nabla\vartheta\|^2 \le \nu_0\langle\vartheta, C_0^{-1}\vartheta\rangle$, the last step because $C_0^{-1} \ge \tilde m^2 - \Delta$. Thus $K := C_0^{1/2}NC_0^{1/2}$ satisfies $0 \le K \le \nu_0 < 1$, and hence $J_Y = \det(1-K)^{-1/2}\exp\{\sup_\vartheta[-\frac12\mathcal{Q}_Y(\vartheta) + \frac12\langle\vartheta, N\vartheta\rangle]\}$.

We first bound the supremum by Lemma 7.32, and only near cells leave a remainder. For a far cell we apply Lemma 7.32(a) with $\delta_\Omega(a) = 0$ and $\varepsilon \to 0$, and for a near cell with $\varepsilon = 1$, $a = f^C$ and $\delta_\Omega(f^C)^2 \le 2C_W^f k_\square^2\|R^C\|^2_{E(\mathfrak{K}_\Omega^\flat)} + 2\|P_\Omega f^C\|_\Omega^2$. Summing over the cells of Y with overlap N_{ov} and multiplying by $2\hat\eta\tilde m^2/\eta_H$ gives, as $(2\hat\eta\tilde m^2/\eta_H)\cdot 2C_W^f k_\square^2 N_{\mathrm{ov}} = \theta_P$, $\frac12\langle\vartheta, N\vartheta\rangle \le \frac12\hat\eta\tilde m^2\|\vartheta\|_Y^2 + \theta_P\|\nabla\vartheta + w_Y\|^2 + \mathfrak{n}''(Y)$, the near-cell remainders adding up exactly to $\mathfrak{n}''(Y)$. Since $\mathcal{Q}_Y(\vartheta) \ge \tilde m^2\|\vartheta\|^2 + \|\nabla\vartheta + w_Y\|^2$, the exponent is at most $-\frac12(1-\hat\eta)\tilde m^2\|\vartheta\|^2 - (\frac12 - \theta_P)\|\nabla\vartheta + w_Y\|^2 + \mathfrak{n}''(Y) \le \mathfrak{n}''(Y)$.

For the determinant, we have $-\frac12\log\det(1-K) \le \operatorname{tr}K/(2(1-\nu_0))$ [BF80, Appendix 3], and $\operatorname{tr}K = \operatorname{tr}(NC_0) \le \hat\eta(1 + 4/\eta_H)\tilde m^2 K_1 M_Y \#Y \le 2(1-\nu_0)c_8|Y|$ by $P_Y \le 1_Y$, Theorem 6.11, $M_Y = \tilde m/\lambda_{\mathrm{PV}}$ and $\#Y < C_u\tilde m^{-3}|Y|$.

Step 3 (assembly). We follow [BF80, (9.65) to (9.626)], adapted to G' as follows. (1) We apply the convexity step [BF80, (9.67) to (9.611)], fact S3 of Section S1.6, to the block compressions $C'_\epsilon = \sum_J 1_{Y_J}C'1_{Y_J}$ of (C.2), $C'_s = \sum_\epsilon \lambda_\epsilon(s)C'_\epsilon$. The factors in this step are finite, because for $\frac12\langle\vartheta, N_1\vartheta\rangle = p_3\eta_H B_X(\vartheta)$ we have $\alpha_D N_1 \le N$ (as $p_3 < p'$), N commutes with each 1_{Y_J}, and $C'_\epsilon \le \alpha_D C_{0\epsilon}$, so that $\|N_1^{1/2}C'_\epsilon N_1^{1/2}\| \le \nu_0 < 1$ by the positivity step. (2) In the monotonicity step (9.612) to (9.617) we use Lemma 7.29(c), $C'_\epsilon \le \alpha_D C_{0\epsilon}$ with $C_{0\epsilon} := \sum_J 1_{Y_J}C_0 1_{Y_J}$, instead of (9.618) to (9.622). (3) After Hölder's inequality (9.625) with exponents p'/p_3, q'/p_3, both integrals factor over the sets Y_J, unions of elements (9.626), and the first factor is at most $\exp\{(c_8|X| + \mathfrak{n}''(X))/p'\} \le \exp\{(c_8|X| + \mathfrak{n}'(X))/p'\}$ by (7.29), the additivity of $\mathfrak{n}''$ and $\mathfrak{n}'' \le \mathfrak{n}'$. (4) Corollary 7.17 with $t_C = 1$ and (L8-3) replaces (9.632): $\bigl(\int d\mu_{C_0} e^{(q'/p')F_2(Y_j)}\bigr)^{1/q'} e^{F_1^{\mathrm{loc}}(Y_j)/p'} \le e^{F_1^{\mathrm{loc}}(Y_j)/p_3}$, as $1/p_3 - 1/p' = 1/q'$. For $k = 0$, $\mathfrak{n}' = 0$, since $\mathfrak{n}'$ carries the factor k^2.

Step 4 (the perimeter bound). Lemma 7.32(c) and $k \le 1$ show that the first part of $\mathfrak{n}'(V)$ is at most $2\beta_{\mathrm{GM}}^{-1}\theta_P C_{\mathrm{BS}}|C|(\log k_\square + C_{\mathrm{sh}})$, as $|C| = 2(R+T) \ge 4R_1(\beta) \ge 8k_\square$. By Lemma 7.32(b), $\|P_\Omega f^C\|_\Omega^2 \le \pi^2|\Omega| \le 8\pi^2 L^3$. It remains to count the near cells. A near cell

$\Omega \subseteq Q$ lies in a block $Q \subseteq \hat{Q}(Q)$ all of whose sites are within ℓ^∞-distance $k_\square + \frac{1}{2}$ of C. The ℓ^∞-neighbourhood of width s of C has volume at most $(2s)^2(|C| + 8s)$, and for $s = k_\square + 2$ and $L \geq 4$, $2s \leq (2r_\square + 5)L$ and $8s \leq (8r_\square + 20)L$. Blocks are disjoint with at least L^3 sites, and by (R5) a block contains fewer than $8/c_1$ cells. Hence, as $|C| \geq 4R_1(\beta) = (8r_\square + 20)L$, $N_{\mathrm{near}} \leq \frac{8}{c_1}(2r_\square + 5)^2(|C| + (8r_\square + 20)L)/L \leq \frac{16}{c_1}(2r_\square + 5)^2|C|/L$, and the second part of $\mathfrak{n}'(V)$ is at most $2\beta_{\mathrm{GM}}^{-1}\frac{4\hat{\eta}\tilde{m}^2}{\eta_H}8\pi^2L^3N_{\mathrm{near}} \leq \frac{1024\pi^2\hat{\eta}}{\eta_H c_1}(2r_\square+5)^2c_L^2\frac{|C|}{\beta_{\mathrm{GM}}}$, using $\tilde{m}L \leq c_L$. Adding the two parts gives $\mathfrak{n}'(V) \leq c_{\mathrm{near}}|C|$. The bound on c_{near}, uniform in $\beta_{\mathrm{GM}} \geq \beta_{\mathrm{GM}}^0$, then follows (Section S1.7.3) from $\log k_\square \leq \log(r_\square c_L) + \log(1/\tilde{m})$ and the value of $\tilde{m}$.

By Proposition 7.7, $E(\psi) = E'(\psi + g) + \frac{1}{2}\langle\psi, \nu\psi\rangle$, where E' collects the Mayer terms of order $s \geq 2$ of Theorem 6.5. We write $E(X)$ for the part of E localized in X, as in Section 7.5. Both $G_{2,1}$ and $G_{2,2}$ are real, by (S3). Let $c_\lambda := \frac{1}{4}(1 - 1/p_3)$, $c'_\lambda := C(e^{-\mu} + \varepsilon_\nu)$, $c_E := c'_\lambda + \theta'$, $G_2^\natural := G_{2,1} + G_{2,2} + G_{2,3}^{\mathfrak{D}} + G_{2,3}^\natural$, and assume

$$2\alpha_D p_4 c_E \leq \tfrac{1}{2}, \qquad \frac{p_4\alpha_D c_E^2}{1 - 2\alpha_D p_4 c_E} + c_E \leq c_\lambda. \tag{7.30}$$

For the values of Section S1.7.4, (7.30) follows from $c_E \leq 1/70$, which is the case for $\theta_0 = 0.01$ and μ, κ_S, β_{GM} large.

Lemma 7.34. *Assume $\varepsilon_\nu + \theta' \leq 1/7$ and (7.30). For every h, every polymer X and every s,*

$$T_4 := \Big[\int d\mu_{C'_s}\big|e^{E(X)}e^{G_2^\natural(X)}e^{-2\tilde{m}^2\sum_X\tilde{\delta}^2}\big|^{p_4}\Big]^{1/p_4} \leq \exp\big\{c_9|X| + C_{\mathfrak{D}}c_{\mathrm{dep}}N_{\mathfrak{D}}(X) + c_\lambda F_1^{\mathrm{loc}}(X)\big\},$$

with $c_9 := 2\alpha_D c_E K_1 C_u/\lambda_{\mathrm{PV}}$.

Proof. The proof of [BF80, Lemma 9.9, (9.72) to (9.718)] carries over to G' with the following changes. (1) In (9.72), the bracket of $G_{2,1}$ lies in $[0, \frac{1}{2}y^2]$, where $y = m\beta_{\mathrm{GM}}^{1/2}\tilde{\delta}$, and that of $G_{2,2}$ is at most y^2, so by (S4′) $G_{2,1}(X) + G_{2,2}(X) \leq \frac{3}{2}(1 + Ce^{-2\kappa_S})\tilde{m}^2\sum_X\tilde{\delta}^2 \leq 2\tilde{m}^2\sum_X\tilde{\delta}^2$. (2) By Lemma 7.29(a), $e^{G_{2,3}^{\mathfrak{D}}(X)} \leq e^{C_{\mathfrak{D}}c_{\mathrm{dep}}N_{\mathfrak{D}}(X)}$ and $|G_{2,3}^\natural(X)| \leq \frac{1}{2}\theta'\tilde{m}^2\sum_X(\vartheta^2 + \psi^2)$. Thus c'_λ of (9.712) becomes c_E. (3) On G' the bounds (9.73) to (9.712), $|E(X)| \leq \frac{1}{2}c'_\lambda\tilde{m}^2\sum_X(\vartheta^2 + \psi^2)$, remain valid. Indeed, $|\varepsilon_k| \leq |m|\beta_{\mathrm{GM}}^{1/2}|\vartheta|$ as in (9.78), since the phase at $\psi + g$ is $m\beta_{\mathrm{GM}}^{1/2}(\vartheta + h)$ with $h \in P\mathbb{Z}$, and (9.711) follows from (S2) of Theorem 6.5, which is uniform in G', by the Schur test [BF80, Lemma 9.6] together with the bound $\|\nu\| \leq \varepsilon_\nu\tilde{m}^2$ of (S5). (4) In (9.716) to (9.718), the bound $C'_s \leq \alpha_D\tilde{m}^{-2}$ of Lemma 7.29(d) takes the place of the lower bound on C_s^{-1}. With it, $K_4 := 2p_4c_E\tilde{m}^2C_s'^{1/2}1_XC_s'^{1/2}$ has $\|K_4\| \leq 2\alpha_D p_4 c_E \leq \frac{1}{2}$, and $-\frac{1}{2}\log\det(1 - K_4) \leq \operatorname{tr} K_4 \leq p_4c_9|X|$, the last inequality by Theorem 6.11. Condition (7.30) is (9.718), and the bound $\frac{1}{2}\tilde{m}^2\|1_XD\|^2 \leq F_1^{\mathrm{loc}}(X)$ of Corollary 7.10 replaces $\frac{1}{2}\int f^2 \leq F_1$.

Let κ' be a term of the expansion of an activity, with the parameters of [GM82, (7.3) to (7.5)] referring to $\mathcal{U}$-cells: as in Lemma A.6, the cubes $(a_1, \dots, a_t)$ of a Mayer factor are replaced by its number t of points and its set of $\mathcal{U}$-cells. We put $\mathcal{P}'_G := e^{\mathfrak{n}'(V)/p'}$ and

$$T_1 := \Big[\int d\mu_{C'_s}\prod_i|\psi(x_i)|^{p_1}\Big]^{1/p_1}, \qquad T_2 := \Big[\int d\mu_{C'_s}e^{-p_2F_2(X)}\Big]^{1/p_2}.$$

Proposition 7.35 (Hölder split)**.** *Assume (L8-1) to (L8-4),* (7.30) *and*

$$p_2\varepsilon(L') \le \tfrac{1}{2}(1 - 1/p_3 - c_\lambda). \tag{7.31}$$

Fix h, X, s and a term κ'. (a) There is a kernel $k((x_i)) \ge 0$ such that

$$\int d\mu_{C'_s} \big| e^{E(X,s)} \kappa' e^{G^\natural(X)} e^{-F_1^{\mathrm{loc}}(X)} e^{-F_2(X)} \big|_0 \\ \le \int_{x_i \in X} k((x_i)) \int d\mu_{C'_s} e^{E} e^{G_2^\natural} e^{-F_2} \prod_i |\psi(x_i)|\, e^{\frac{1}{2}\gamma \sum_X \vartheta^2} e^{-F_1^{\mathrm{loc}}(X)}.$$

(b) The right side of (a) is at most $\int k((x_i))\, T_1 T_2\, e^{-F_1^{\mathrm{loc}}(X)}\, T_3 T_4$, with T_3 as in Lemma 7.33 and T_4 as in Lemma 7.34. (c) $T_1 \le c_T^{\sum_j n_j} \prod_j n_j!$, where n_j is the number of points x_i in the j-th $\mathcal{U}$-cell and c_T depends only on p_1, r_C, $\alpha_D K_1/\lambda_{\mathrm{PV}}$, $K_1''/\lambda_{\mathrm{PV}}$, c_u and $C_{\mathcal{U}}$; and $T_2 \le e^{p_2\varepsilon(L')F_1^{\mathrm{loc}}(X)}$. (d) We have

$$T_2\, e^{-F_1^{\mathrm{loc}}(X)}\, T_3\, T_4 \le \mathcal{P}'_G\, e^{c_{13}|X| + C_{\mathfrak{D}} c_{\mathrm{dep}} N_{\mathfrak{D}}(X)} e^{-c' F_1^{\mathrm{loc}}(X)},$$

with $c_{13} := c_8/p' + c_9$ and $c' := 1 - 1/p_3 - c_\lambda - p_2\varepsilon(L') \ge c'_0 := \frac{1}{2}(1 - 1/p_3 - c_\lambda)$; at the values of Section S1.7.4, $c'_0 = 1/16$. For $k = 0$, $\mathcal{P}'_G = 1$.

Proof. (a) is the analogue on G' of [GM82, (7.9)], that is, of [BF80, (9.22), (9.23), (9.81) to (9.823)], with the differentiated factors of Appendix B, the rates of Proposition 7.36 and Lemma 7.31 for the derivatives of G_1. By (S3), E and $G_2^\natural$ are real for real ψ. *(b)* follows as in [GM82, (7.11)], that is, [BF80, (9.25)], since $(e^{\frac{1}{2}\gamma\sum_X \vartheta^2} e^{2\tilde{m}^2 \sum_X \tilde{\delta}^2})^{p_3} = e^{p_3 \eta_H B_X(\vartheta)}$. *(c)* For T_1 we use [BF80, (9.31) to (9.33)], with Lemma 7.29(d) in place of the Neumann series of (9.32) and (R7) for the number of $\mathcal{U}$-cells within a given distance. The bound on T_2 follows from Corollary 7.17 with $t_C = \alpha_D \le 2$. *(d)* This is [GM82, Lemma 13] on G'. By (c) and Lemmas 7.33, 7.34, the exponents of $F_1^{\mathrm{loc}}(X)$ add up to $-1 + p_2\varepsilon(L') + 1/p_3 + c_\lambda = -c'$, and (7.31) gives $c' \ge c'_0$. Finally, $e^{\mathfrak{n}'(X)/p'} \le \mathcal{P}'_G$, as $\mathfrak{n}'$ is monotone in X.

Remark. In these bounds the perimeter is carried by the factor $\mathcal{P}'_G$ alone. Far cells contribute nothing, and $E_{G'}$ of Corollary 7.10 (d) drops out of the bounds, since the factor $e^{-k^2 E_{G'}/2\beta_{\mathrm{GM}}}$ of Proposition 7.3 cancels against $F_1^{\mathrm{loc}}(V) = F_1 + \frac{1}{2}k^2\beta_{\mathrm{GM}}^{-1} E_{G'}$ and every later use of F_1^{loc} is a lower bound. Hence Theorem 7.57 contains no factor $\log(1 + \min(R, T))$, and $\mathcal{P}'_G \le e^{2c_{\mathrm{near}}(R+T)/p'}$.

We measure distances between $\mathcal{U}$-cells in Debye units, $\mathrm{dist}(A, B) := \tilde{m}\, d_{G'}(A, B)$. The covariance rate in these units, $r_C := r_{\mathrm{cov}}/\tilde{m}$, does not depend on τ, by Theorem 6.11. The rate conditions [BF80, (9.123), (9.124)] require a rate of at least $(1 - \delta_1)/\tilde{l}_D \ge \frac{1}{2}$, whereas on G' the rate r_C is small. In (9.95) we take the exponent with the minus sign that (9.124) has.

Proposition 7.36. *Let $0 < \delta_2 < \bar{\delta} < r_C$ and put $c'_{\mathrm{rate}} := \frac{1}{2}(r_C - \bar{\delta})$. Every step of [BF80, Section 9] that uses the rate conditions [BF80, (9.123), (9.124)], the covariance bounds [BF80, (9.32), (9.34)] or [BF80, (9.95)] holds on G' with the bounds of Lemma 7.29(d) for C'_s, with (S2) of Theorem 6.5 and with any τ-independent margin, in particular c'_{rate}, in place of a lower bound on the rates. The pinning factor of [BF80, Lemmas 9.10, 9.11] is used with the margin $c_p := (r_C - \bar{\delta})/18$.*

Proof. The argument uses the rate conditions through margins, and the changes are as follows.

(i) *Tree factor* $e^{\bar\delta d}$, $d = \sum_i \operatorname{dist}(\Delta_i', \Delta_i'')$. Link i decays at rate r_C by Lemma 7.29(d), and pinning brings a factor $e^{9c_p d}$. The rest $r_C - \bar\delta - 9c_p = c'_{\mathrm{rate}}$ is left for the sum over the positions of Δ_i'', as $d_{G'}$ dominates the Euclidean distance and by (R7) at most $\frac{4\pi}{3}(r + 2C_{\mathcal U})^3/c_u$ $\mathcal U$-cells lie within Debye distance r of a given one.

(ii) *Mayer factors.* The factors $e^{\bar\delta L_0}$ and e^{rO_0} of [GM82, (7.3), (7.4)] are absorbed by the tree decay of (S2) at the Stage 1 rate $\sqrt{3}\kappa_Y M_Y \geq 4\sqrt{3}\, r_{\mathrm{cov}} > \bar\delta\tilde m$ of Proposition 6.10 and by $e^{-\mu(s-1)}$ of (S2). The details are in the *Tree weights* step of the proof of Proposition B.7.

(iii) *Wick's theorem.* The rate $\frac12$ is needed only to obtain (9.32) by a Neumann series, whereas Lemma 7.29(d) gives the pointwise bound directly. The contraction count holds for all rates, with constants depending on r_C, $K_1''/\lambda_{\mathrm{PV}}$, c_u, $C_{\mathcal U}$.

(iv) *Clustering.* Since each application has $W \subseteq Z$ or $W \supseteq Z$, we do not use the factor $e^{-(1-2\delta_1)\operatorname{dist}(X_1, W)}$ of [BF80, (9.122)].

(v) We work in the finite box, and the only role of [BF80, (A4.25), (A4.26)] is in the infinite-volume limit of [BF80, Lemma A4.1].

Seed configurations and the seeded activity. A pair (U, k) is *admissible* if $U = V$ and $k \in \{0, 1\}$, or $U = X_0^c := V \setminus X_0$ with $X_0 \subsetneq V$ a union of $\mathcal U$-cells and $k = 0$. A *seed configuration* in (U, k) is a pair $\sigma = (Z, N_Z)$ with $Z \subseteq U$ a nonempty, possibly disconnected union of $\mathcal U$-cells, $Z \supseteq X_1$ if $k = 1$, and N_Z a wall field on $E(Z)$ whose wall polymers have their closures in Z (so that Z is a union of elements of N_Z); for label configurations, $N_Z \in \mathcal N(Z)$. For $X \supseteq Z$, $\mathcal N_\sigma(X)$ is the set of $N_X \in \mathcal N(X)$ with $N_X|_{E(Z)} = N_Z$ for which Z is a union of elements of N_X. The *seeded activity* $\mathcal K^{(Z)}_{N_X}(X)$ is the activity of Proposition 7.22 at N_X with first set $Y_1 = Z$, built with $(C', G^\natural)$: $\mathcal K^{(Z)}_{N_X}(X) = \sum_{\bar y} \int_{[0,1]^{n-1}} ds \int d\mu_{C'_s}(\psi)\, e^{E(X,s)} \kappa(\bar y, s)\, Q^\natural_{N_X}(X)$, the sum over $\bar y = (Z, Y_2, \dots, Y_n)$ with the Y_j ($j \geq 2$) disjoint unions of elements and $\bigcup_j Y_j = X$, and $Q^\natural_{N_X}(X) := e^{G^\natural(X)} e^{-F_1^{\mathrm{loc}}(X)} e^{-F_2(X)}$. We write $X = Z \sqcup X_\Sigma \sqcup X_o$, where X_Σ consists of the non-seed $\mathcal U$-cells in wall polymers and X_o of the other non-seed $\mathcal U$-cells (the *lone* cells). A cell is of *class B* if it is one of the $\mathcal U$-cells of $\bigcup_\beta S_\beta$, and of *class A* otherwise. Since $\operatorname{supp} D \cup \operatorname{supp} F_2' \subseteq \bigcup_\beta S_\beta$ by Proposition 7.8 (b), (e), we have $D = F_2' = 0$ on every class-A cell. As in [BF80, p. 214], $|\cdot|_0$ is the absolute value taken inside the sums and integrals that result when all differentiations in κ' are performed.

Lemma 7.37. *Let (U, k) be admissible, $\sigma = (Z, N_Z)$ a seed configuration in (U, k), $c_1' > 0$, c_4' as in Lemma 7.30 with $c_4 = c_1'$, $e^{a''} \geq 4C_\varphi$, where $C_\varphi < \infty$ is the link sum of Lemma A.6, and $c_2^L := \log(2eD_{\mathcal U}) + 4 + c_4' + a'' + \alpha$. Then*

$$\sum_{\substack{X \supseteq Z \\ X \subseteq U}} \sum_{N_X \in \mathcal N_\sigma(X)} |\mathcal K^{(Z)}_{N_X}(X)|\, e^{\alpha|X|} \leq \sup_{(\cdot)} e^{c_1' F_1^{\mathrm{loc}}(X, N_X) + c_2^L |X| + \delta_2 d} \int d\mu_{C'_s} \big| e^{E(X,s)} \kappa' Q^\natural_{N_X}(X) \big|_0,$$

where the supremum runs over n, T, $\bar y$, the elementary terms, $s \in [0,1]^{n-1}$ and $N_X \in \mathcal N_\sigma(X)$; κ' is the elementary term without its tree weight $q(T, s)$ (Lemma A.1) and with every Mayer factor $\mathcal E$ replaced by $\sum_{\eta^A} e^{c_{10} L_{\eta^A}} e^{c_{10}'' t} \mathcal E_{\eta^A}$ ([BF80, (9.18), (9.18a), (9.19)]); and $d := \sum_i \operatorname{dist}(\Delta_i', \Delta_i'')$, the sum over the links i with a covariance line (types (2) to (5), Lemma A.1). The constant c_2^L depends only on $D_{\mathcal U}$, c_4', C_φ, α, not on L', and it is bounded uniformly in β_{GM} and τ (c_4' depends on them through A_w, but it is doubly exponentially small

by Lemma 7.30). The bound also holds for the sum restricted by $X \cap W \neq \emptyset$ if $W \subseteq Z$ or $W \supseteq Z$.

This lemma is the version on G' of [BF80, Lemmas 9.1, 9.4], in the distance-zero form, for seeds that may depend on the configuration, be disconnected, omit X_1 when $k = 0$, and lie in X_0^c, and we prove it in Proposition A.7. For the last assertion of the lemma, note that every term of the sum then satisfies the restriction. In the supremum we take together the elementary terms with the same link data in the sense of Lemma A.6, as in the definition of κ' above and in Proposition A.7; that is, κ' contains, inside $|\cdot|_0$, the sums over the assignments of the points of each $\mathcal{E}$-factor to the cells of its set of cells, over their positions and over their charges (and, for $t = 2$, over the forms).

Lemma 7.38. *Insert Proposition 7.35 and the Wick bound into the right side of Lemma 7.37. In the resulting bound, the factor $e^{c_2'|X|}$ of [BF80, (9.823)], with $e^{c_2^L|X|}$ and $e^{c_{13}|X|}$, is at most $\prod_{\Delta\subseteq X} e^{c_o} \cdot \prod_{\Delta \text{ class } B} e^{c_B(b_D)} \cdot \prod_{\Delta\cap\mathfrak{D}\neq\emptyset} e^{C_{\mathfrak{D}}c_{\mathrm{dep}}}$, the products running over $\mathcal{U}$-cells, and the near-cell factor is pulled out globally as $\mathcal{P}'_G$. Here c_o does not depend on L' and is bounded uniformly in b_D and τ, and, with the leftover pinning exponent $q_1 = 2$ of Lemma B.9, $c_B(b_D) \leq 2(C_{\mathcal{E}}^{1/2} + C_G^{1/2})b_D^{-1/2}$, where $C_{\mathcal{E}}$ and C_G, the constants of Proposition B.10, are independent of L'. Every class-B cell lies in X_Σ or in the seed, and every factor on a class-A cell carries in addition a nonnegative power of b_D. The resulting bound depends on the scale L' only through the set of cells in X_Σ, through $\varepsilon(L')$, which is decreasing, and through the monotone hypotheses (L8-3), (L8-4) and* (7.31).

Proposition 7.39. *Insert Proposition 7.35 and the Wick bound into Lemma 7.37 with $c_1' = c'/4$, and distribute F_1^{loc} by the fractions of Remark S1.1. Then for $\beta_{\mathrm{GM}} \geq \beta_{\mathrm{GM}}^0$, in the resulting bound, every non-seed $\mathcal{U}$-cell Δ of X carries a factor at most ε_Δ, where*

$$\varepsilon_\Delta := K_{\mathrm{cell}}\, e^{\alpha + C_{\mathfrak{D}}c_{\mathrm{dep}}} (b_D/\lambda_{\mathrm{PV}})^{1/6} \quad (\Delta \subseteq X_o),$$
$$\varepsilon_\Delta := K_{\mathrm{cell}}\, e^{\alpha + C_{\mathfrak{D}}c_{\mathrm{dep}} + c_B(b_D) - (c'/8)\Phi_{\mathcal{U}}/(1+D_{\mathcal{U}})} \quad (\Delta \subseteq X_\Sigma),$$

and every seed cell carries at most $K_{\mathrm{cell}} e^{\alpha + C_{\mathfrak{D}}c_{\mathrm{dep}}}$, times $e^{c_B(b_D)}$ if it is of class B. Here $K_{\mathrm{cell}} := K_B e^{(c_2^L + c_{13})C_u} \max(1, \lambda_{\mathrm{PV}}^{1/6})$ with K_B independent of b_D, L' and τ, $\Phi_{\mathcal{U}}$ is the share of Proposition 7.19, and the exponent $p_0 = 1/6$ is attained.

Lemma 7.38 and Proposition 7.39 are a quantitative version on G' of [BF80, Section 9.8, (9.81) to (9.824)] and are proved in Appendix B. With $p_0 := 1/6$ we put

$$c_{16}^D := 2K_{\mathrm{cell}}\, e^{\alpha + C_{\mathfrak{D}}c_{\mathrm{dep}}} \max\Big\{(b_D/\lambda_{\mathrm{PV}})^{p_0},\ e^{c_B(b_D) - (c'/8)\Phi_{\mathcal{U}}/(1+D_{\mathcal{U}})}\Big\}. \tag{7.32}$$

Theorem 7.40 (Removal of the logarithm)**.** *Fix α. There is τ_0 such that $c_{16}^D \leq 1$ for $\tau \geq \tau_0$, and $c_{16}^D \to 0$ as $\tau \to \infty$. Consequently the depletion factor $e^{C_{\mathfrak{D}}c_{\mathrm{dep}}N_{\mathfrak{D}}(X)}$, and with it the $\log\tau$ in c_{dep}, is absorbed by the per-cell smallness of Proposition 7.39.*

Proof. The first branch of (7.32) is at most 1 if $\alpha \leq p_0 \log(\lambda_{\mathrm{PV}}/b_D) - C_{\mathfrak{D}}c_{\mathrm{dep}} - \log(2K_{\mathrm{cell}})$. This holds for $\tau \geq \tau_0$, because the per-cell gains are exponentially small in τ and the depletion exponent is $O(\log\tau)$: we have $\log(1/b_D) \geq \pi^2 v_{\mathrm{Cb}}(0)\tau - \frac{3}{2}\log\tau - C$ as $\tilde{m} \leq 2m_D$, $c_{\mathrm{dep}} = O(\log\tau)$ by Proposition 6.17, and $\log K_{\mathrm{cell}}$ is affine in α. In the second branch, Proposition 7.19 with $\tilde{m}L \geq c_L/2$ and $\tilde{m}L' = 2\beta_{\mathrm{GM}}$ gives $(c'/8)\Phi_{\mathcal{U}}/(1+D_{\mathcal{U}}) \geq c\, b_D^{-1}\beta_{\mathrm{GM}}^{-3}$, which dominates the bound $c_B(b_D) \leq C b_D^{-1/2}$ of Lemma 7.38. A depleted cell that is lone or in X_Σ has its factor $e^{C_{\mathfrak{D}}c_{\mathrm{dep}}}$ inside $\varepsilon_\Delta \leq \frac{1}{2}$ (Section S1.7.5).

Let $\tilde{\mathcal{K}}(\Delta)$ be the activity of the single element Δ at $k=0$ (first set Δ, $n=1$), with no wall on Δ.

Lemma 7.41. *For $\tau \geq \tau_0$ and every wall-free $\mathcal{U}$-cell Δ, $\tilde{\mathcal{K}}(\Delta) \geq \frac{1}{2}$.*

Proof. A single $\mathcal{U}$-cell that is an element lies in no S_β, so $D=0$ on Δ and $F_1^{\mathrm{loc}}(\Delta) = F_2(\Delta) = 0$. The label is constant on Δ and, at $k=0$, contributes only a multiple of 2π to each phase. Hence $\vartheta = \psi$ on Δ, $\tilde{\mathcal{K}}(\Delta) = \int d\mu_{C'_\Delta} e^{E(\Delta)+G^\natural(\Delta)}$ with C'_Δ the $\Delta \times \Delta$ block of C', and the exponent is real by (S3). Positivity at $k=0$ allows us to use Jensen's inequality, which gives

$$\tilde{\mathcal{K}}(\Delta) \geq \exp\Big(-\mathbb{E}|E(\Delta)| - \mathbb{E}|G_1(\Delta)| - \mathbb{E}|G_{2,1}(\Delta) + G_{2,2}(\Delta)| - \mathbb{E}\big(G^{\mathfrak{D}}_{2,3}(\Delta)\big)_- - \mathbb{E}|G^\natural_{2,3}(\Delta)|\Big),$$

expectations under $d\mu_{C'_\Delta}$. A supremum bound does not suffice here, since $\sup_\psi(-G^{\mathfrak{D}}_{2,3}(\Delta))$ is only bounded by $C(e^{-\mu}+e^{-2\kappa_S})c_{\mathrm{dep}}$, which grows like $\log\tau$. Let $x_b := \alpha_D K_1 b_D/\lambda_{\mathrm{PV}}$. Then $\beta_{\mathrm{GM}} C'(x,x) \leq x_b$ by Lemma 7.29(d) and $M_Y = \tilde{m}/\lambda_{\mathrm{PV}}$. Moreover, $\#\Delta < C_u \tilde{m}^{-3}$, and Δ contains at most C_u/v_- cells. As we will show below, each expectation is at most $C(b_D^{1/2} + c_{\mathrm{dep}} b_D)$, with C depending only on the constants fixed before τ_0. Since $c_{\mathrm{dep}} b_D = O(\log\tau) e^{-2.494\tau} \to 0$, the sum is then at most $\ln 2$ for $\tau \geq \tau_0$.

We begin with $G^{\mathfrak{D}}_{2,3}$. At $x \in \mathfrak{D}$ the x-term can be negative only because of the excess $\rho_1(m,x) > \rho^{\mathrm{ref}}(m)$, which is at most $Ce^{-\mu}\rho^{\mathrm{ref}}(1)$ for $m = \pm 1$ by Proposition 6.6 and $e^{-\Phi} \leq 1$, and is bounded by the $|m| \geq 2$ moment bound otherwise. With $1 - \cos u \leq \frac{1}{2}u^2$, $\mathbb{E}\psi(x)^2 = C'(x,x)$ and Proposition 6.17, $\mathbb{E}(G^{\mathfrak{D}}_{2,3}(\Delta))_- \leq C(e^{-\mu}+e^{-2\kappa_S})\tilde{m}^2\, \alpha_D K_1 M_Y\, \#(\mathfrak{D} \cap \Delta) \leq C(e^{-\mu}+e^{-2\kappa_S})\alpha_D K_1 c_{\mathrm{dep}} b_D/\lambda_{\mathrm{PV}}$.

The other four terms are moment bounds with $\beta_{\mathrm{GM}} C'(x,x) \leq x_b$. First, $\mathbb{E}|E(\Delta)| \leq C\#\Delta(m_D^2/\beta_{\mathrm{GM}}) x_b^{3/2} \leq C b_D^{1/2}$, as each term of $E(\Delta)$ has at least three factors ε or $\tilde{\varepsilon} := \varepsilon - im\beta_{\mathrm{GM}}^{1/2}\psi$ and the weight $e^{2\kappa_S(\sum|m_i|-1)}$ of (S2) absorbs the charges. Second, $\mathbb{E}|G_1(\Delta)| \leq C b_D$, as $\log \hat{R}_v(x) = O(v b_D x^4 + e^{-\pi^2 v/b_D})$ for $|x| \leq b_D^{-1/2}$ by Taylor's formula and the periodization, with a Gaussian tail beyond. Third, $\mathbb{E}|G_{2,1}+G_{2,2}| \leq C(b_D + b_D^{1/2})$, as the brackets are of fourth and third order and $\operatorname{Var}\tilde{\delta}(x) \leq 2\alpha_D(K_1/\lambda_{\mathrm{PV}} + 1/v_-)\tilde{m}$. Finally, $\mathbb{E}|G^\natural_{2,3}(\Delta)| \leq 3C_4\theta' C_u (\alpha_D K_1)^2 b_D/\lambda_{\mathrm{PV}}^2$ by Lemma 7.29(b) with $n=0$ and $\mathbb{E}\psi^4 = 3C'(x,x)^2$.

The bound $\tilde{\mathcal{K}}(\Delta) \geq \frac{1}{2}$ replaces [BF80, (A4.16) to (A4.21)], which use translation invariance, whereas here the positivity at $k=0$ of Lemma 7.42 and the diagonal bound of Theorem 6.11 suffice.

7.7 The Kirkwood–Salsburg system in the class algebra

In this subsection we solve the Kirkwood–Salsburg system of Theorem 7.27 (b) by the argument of [BF80, Appendix 4] in a weighted ℓ^1 algebra over Γ_{long}, with the area weight of the classes (Theorem 7.49). The area decay of $\mathcal{Z}_p(X)/Z$ will then follow from that of Z_p/Z, established in Subsection 7.8. Activities and outer partition functions are built with the triple $(C', G^\natural, \mathfrak{cN})$ of Lemma 7.29 (f).

Lemma 7.42 (Positivity)**.** *At $k=0$ and for real ψ, the functions E, G and $G^\natural$ of Lemma 7.24 are real, and $e^{E(Y)} Q^\natural_N(Y) > 0$ for every wall field N and every union Y of its elements. Hence $\mathcal{Z}_p(X) \geq 0$ for every X and p, $Z > 0$, and every activity with $n = 1$ is positive.*

Proof. By (S3) the terms of m and $-m$ in the charge sums of E, G_1, $G_{2,1}$, $G_{2,2}$ are complex conjugate, and $G^{\mathfrak{D}}_{2,3}$, $G^{\natural}_{2,3}$, F_1^{loc}, F_2 are real. Hence $e^{E(Y)}Q^{\natural}_N(Y)$ is the exponential of a real number. The outer partition functions $\mathcal{Z}_p(X)$ and the activities with $n = 1$ are integrals of such factors, with $\mathfrak{c}\mathcal{N} > 0$, and Z contains $N = 0$.

Lemma 7.43. *For an additive function p on the cycle space of G' let $\mathrm{A}(p)$ be the minimum of $\sum_{e\in E}|N(e)|$ over the $N \in \mathbb{Z}^E$ with $[N](\gamma) = p(\gamma)$ for every cycle γ of G', and $\mathrm{A}(p) := \infty$ if no such N exists; for $p \in \Gamma_{\mathrm{long}}$, $\mathrm{A}(p)$ is that of the restriction of p to the cycles of G'. Then $\mathrm{A}(0) = 0$, $\mathrm{A}(-p) = \mathrm{A}(p)$, $\mathrm{A}(p+q) \le \mathrm{A}(p) + \mathrm{A}(q)$, and $\mathrm{A}([N]) \le \sum_E |N|$ for every $N \in \mathbb{Z}^E$; in particular $\mathrm{A} < \infty$ on Γ_{long}. Fix $\varepsilon_E \in (0, c_0'/8]$ and put $\sigma' := \varepsilon_E\sigma_F$ and $\omega(p) := e^{\sigma'\mathrm{A}(p)}$. Then $\omega(0) = 1 \le \omega$ and $\omega(p+q) \le \omega(p)\,\omega(q)$ on Γ_{long}, so $\mathcal{A}_\omega := \{a : \Gamma_{\mathrm{long}} \to \mathbb{C} : \|a\|_\omega := \sum_p |a(p)|\,\omega(p) < \infty\}$, with $(a*b)(p) := \sum_q a(q)\,b(p-q)$, is a commutative Banach algebra with unit δ_0, and $\|a*b\|_\omega \le \|a\|_\omega\|b\|_\omega$.*

Proof. Sums and negatives of admissible cochains are admissible, N is admissible for $[N]$, and $\sum_p |(a*b)(p)|\,\omega(p) \le \sum_{p,q} |a(q)|\,\omega(q)\,|b(p-q)|\,\omega(p-q)$.

Lemma 7.44. *Let Ω be a cell of the output. For all sites $x, y \in \Omega$, $d_{G'}(x,y) \le D_{\mathrm{cell}} := 7r_0(2t_1+1)$, where $t_1 := 1 + \lceil (12k_P/\chi_{\mathrm{N3}})\ln k_P^3\rceil$ (k_P in coarse units). If $c_L \le 1.99\cdot 10^{-4}\chi_{\mathrm{N3}}$, then $D_{\mathrm{cell}} < L'$ for $\beta_{\mathrm{GM}} \ge \beta^0_{\mathrm{GM}}$.*

Sketch. By (N3) of Theorem 3.20, there is no χ_{N3}-weak link. Hence, in the component $\mathfrak{K}$ of $G' \cap \hat{Q}(Q)$ with $\Omega = Q \cap \mathfrak{K}$ as in Definition 3.14, where $|\mathfrak{K}| \le k_P^3$, balls with $|B_t| \le |\mathfrak{K}|/2$ grow by the factor $1 + \chi_{\mathrm{N3}}/(6k_P)$ per step. Balls of radius t_1 about x, y therefore meet. A coarse path of length ℓ gives a G'-path of at most $7r_0(\ell+1)$ fine steps. With $L_c \ge 35$ and $c_L \le 1.99\cdot 10^{-4}\chi_{\mathrm{N3}}$, we obtain $D_{\mathrm{cell}} < L'$ (Section S1.8).

Lemma 7.45. *Let N be a long wall field, $w := 2\pi\beta_{\mathrm{GM}}^{-1/2}N$, and let Ω be a cell with the set $\mathfrak{K}^\flat_\Omega$ of Proposition 6.9 (iv). Then there is $h' : \mathfrak{K}^\flat_\Omega \to P\mathbb{Z}$, constant on Ω, with $\nabla h' = w$ on $E(\mathfrak{K}^\flat_\Omega)$.*

Proof. The graph $(\mathfrak{K}^\flat_\Omega, E(\mathfrak{K}^\flat_\Omega))$ is connected, of diameter $d_\flat \le 2 + 0.469\,\beta_{\mathrm{GM}}/\tilde{m} = 2 + 0.235L'$ (Proposition 6.9 (iv), (v)). We define $h'(x)$ as the sum of w along the path from $x_0 \in \Omega$ to x in a breadth-first spanning tree. Then h' is $P\mathbb{Z}$-valued. Such a primitive can fail only on closed walks of about twice the diameter of $\mathfrak{K}^\flat_\Omega$, which is less than L'. A bond of $E(\mathfrak{K}^\flat_\Omega)$ off the tree closes a walk of length at most $2d_\flat + 1 < 2L'$ with two tree paths, on which N has period 0. Hence $\nabla h' = w$ on $E(\mathfrak{K}^\flat_\Omega)$. For $x \in \Omega$, the tree path from x_0 to x and the same-cell step back form a closed m3-walk of length at most $d_\flat + D_{\mathrm{cell}} \le 2 + 0.235L' + L + 0.964L' < 2L'$, by Lemma 7.44 and Section S1.8, using $L' \ge 16$ and $L \le L'/(4\beta_{\mathrm{GM}})$. Therefore $h'(x) = h'(x_0)$ for each such x.

Definition 7.45a (Primitive-admissible wall fields)**.** Let $k \in \{0,1\}$, X a union of $\mathcal{U}$-cells ($X \supseteq X_1$ if $k = 1$), and N a wall field on $E(X)$ whose wall polymers have their closures in X. Then N is *primitive-admissible at coupling k* if for every union $Y \subseteq X$ of elements of N and every cell $\Omega \subseteq Y$ there is $h' : \mathfrak{K}^\flat_\Omega \to P\mathbb{Z}$, constant on Ω, with $\nabla h' = P(N\mathbf{1}_{E(Y)} + k\xi)$ on $E(\mathfrak{K}^\flat_\Omega)$.

Lemma 7.45b (Primitives for long and inside fields)**.** *(a) Every long wall field on $E(X)$ whose wall polymers have their closures in X is primitive-admissible at $k = 0$. (b) If $X \supseteq X_1$ and N_X is an inside field at $k = 1$ with $p_1(N_X) \in \Gamma_{\mathrm{long}}$, then N_X is primitive-admissible at $k = 1$.*

Proof. (a) For Y a union of elements, $N\mathbf{1}_{E(Y)}$ is long (Lemma 7.26 (a)); apply Lemma 7.45. (b) Let $Y \subseteq X$ be a union of elements, $N_Y := N_X\mathbf{1}_{E(Y)}$ and $\Omega \subseteq Y$ a cell. Each wall of N_X on $E(\mathfrak{K}^\flat_\Omega)$ lies in Y (first step of the proof of Lemma 7.32 (a)), and ξ vanishes off $E(X)$. Hence $N_Y + \xi = (N_X + \xi)\mathbf{1}_{E(X)}$ on $E(\mathfrak{K}^\flat_\Omega)$, and its period on a closed m3-walk γ in $\mathfrak{K}^\flat_\Omega$ is $-p_1(N_X)(\gamma)$, which is 0 if $|\gamma| < 2L'$. The breadth-first-tree argument of Lemma 7.45, whose closed walks (fundamental cycles of length at most $2d_\flat + 1$, same-cell closures of length at most $d_\flat + D_{\mathrm{cell}}$) are shorter than $2L'$, gives h' with values in $P\mathbb{Z}$.

Proposition 7.46 (Long wall fields)**.** *At $k = 0$, the bounds of Subsection 7.6 and of Appendices A and B, and Lemma 7.48, hold with $\mathcal{N}(X)$ and $\mathcal{N}_\sigma(X)$ replaced by $\mathcal{N}^{\mathrm{long}}(X)$ and $\mathcal{N}^{\mathrm{long}}_\sigma(X) := \{N_X \in \mathcal{N}^{\mathrm{long}}(X) : N_X|_{E(Z)} = N_Z,\ Z \text{ a union of elements of } N_X\}$, for seed configurations $\sigma = (Z, N_Z)$ with $N_Z \in \mathcal{N}^{\mathrm{long}}(Z)$, with the integrand of Lemma 7.24 and the same constants. In Lemma 7.32 (a) the label is replaced by the primitive h' of Lemma 7.45. The same holds at $k = 1$ for the inside fields N_X with $p_1(N_X) \in \Gamma_{\mathrm{long}}$, with the primitive of Lemma 7.45b (b) and $\mathcal{P}' = \mathcal{P}'_G$; in general, the bounds hold for every class of primitive-admissible wall fields (Definition 7.45a).*

Sketch. For the sums over configurations, in Lemma 7.30 (b) and in steps (2), (3) of the proof of Proposition A.8, the injection $N_X \mapsto (N_I)_{I \subseteq X}$ into integer interface values is all that is needed, and it is valid for all wall fields. At a fixed configuration the label occurs in four places: the phases (Lemma 7.24), $w = 0$ inside cells, the walls in (7.28) and, in Lemma 7.32 (a), a primitive of w_Y on $E(\mathfrak{K}^\flat_\Omega)$ constant on Ω. In the last case $N\mathbf{1}_{E(Y)}$ is long by Lemma 7.26 (a), and Lemma 7.45 gives such a primitive (details in Section S2.6). Only the existence of the primitive is used, so the argument applies to every primitive-admissible field.

A fixed fraction of the F_1^{loc} of each configuration accounts for its walls, which carry at least the area of their class.

Proposition 7.47 (Absolute convergence)**.** *There is $c < \infty$, depending on τ and the fixed constants only, such that $\|\mathcal{Z}(X)\|_\omega \le \mathfrak{c}\mathcal{N}\, e^{(c+c'_4)|X^c|}$ for every union X of $\mathcal{U}$-cells, with c'_4 from Lemma 7.30 at $c_4 = c'/2$. Moreover $\sum_Y \sum_{N_Y} |\mathcal{K}^{(Z)}_{N_Y}(Y)|\, \omega([N_Y])\, e^{\alpha|Y|} < \infty$ for the seeds Z of Theorem 7.27 (b), (c), and $\sum_{N_X} |\mathcal{K}_{N_X}(X)| < \infty$ in Theorem 7.27 (a), the sum over the inside fields N_X at $k = 1$ with $p_1(N_X) \in \Gamma_{\mathrm{long}}$. Hence the sums of Theorem 7.27 converge absolutely.*

Proof. By Hölder's inequality as in Proposition 7.35 (b), (d) at $k = 0$, with Lemma 7.31 at $N = 0$ and Proposition 7.46, the term of N in $\mathcal{Z}_p(X)$ is at most $\mathfrak{c}\mathcal{N} e^{c|X^c|} e^{-c' F_1^{\mathrm{loc}}(X^c, N)}$. Moreover, $\omega([N]) \le e^{\sigma' \sum |N|} \le e^{(\varepsilon_E/2) F_1^{\mathrm{loc}}(X^c, N)}$, the first inequality by Lemma 7.43 and the second by Theorem 7.13 and $|N| \le N^2$. We also have $\varepsilon_E \le c'$, and by Lemma 7.30 (b) the sum of $e^{-(c'/2) F_1^{\mathrm{loc}}(X^c, N)}$ over the wall fields on $E(X^c)$ is at most $e^{c'_4 |X^c|}$. We bound the seeded sums by Lemma 7.48 (a), (c′), with $\varepsilon = \varepsilon_E/2$ at $k = 0$ and $\varepsilon = 0$ at $k = 1$. At $k = 1$ the inside fields with $p_1(N_X) \notin \Gamma_{\mathrm{long}}$ contribute 0 to Theorem 7.27 (a), as $\mathcal{Z}_p(X) = 0$ for $p \notin \Gamma_{\mathrm{long}}$ by Definition 7.25(c). For the inside fields at $k = 1$ with $p_1(N_X) \in \Gamma_{\mathrm{long}}$, Lemma 7.48 (a) applies by Lemma 7.45b (b) and Proposition 7.46.

Constants of the seeded sums. The constant C_9 bounds the number of interfaces within Euclidean distance r of a given interface or site by $C_9(r/L + 1)^3$. The Euclidean link range is $g := 2L' + 4D_E + 2$ (D_E of Proposition 7.19). We put $c_2 := c_S + 2 = O(\log \tau)$ and

$c_{16} := c_{16}^D + \eta_{\text{seed}} \to 0$ as $\tau \to \infty$, where c_{16}^D is defined in (7.32) and c_S, η_{seed} in Section S1.8. Let $\mathcal{P}' := \mathcal{P}'_G$ if $k = 1$ and $\mathcal{P}' := 1$ if $k = 0$. For a seed configuration $\sigma = (Z, N_Z)$, Z^Σ is the union of its wall polymers and $Z^o := Z \setminus Z^\Sigma$. For an anchor $S_0 \subseteq U$, that is, a nonempty union of $\mathcal{U}$-cells with $S_0 \supseteq X_1$ if $k = 1$, and for $N_X \in \mathcal{N}(X)$, $\mathrm{sat}_{N_X}(S_0)$ is the union of the elements of N_X meeting S_0.

Lemma 7.48 (Seeded sums at $k = 0$ and $k = 1$)**.** *Let (U, k) be admissible, $\beta_{\mathrm{GM}} \geq \beta_{\mathrm{GM}}^0$, and $\varepsilon \in [0, c'/2]$. (a)* (Variable first set.) *For every anchor $S_0 \subseteq U$,*

$$\sum_{S_0 \subseteq X \subseteq U} \sum_{N_X \in \mathcal{N}(X)} \bigl|\mathcal{K}_{N_X}^{(\mathrm{sat}_{N_X}(S_0))}(X)\bigr|\, e^{\alpha|X| + \varepsilon F_1^{\mathrm{loc}}(X, N_X)} \leq \mathcal{P}'\, e^{c_2|S_0|}.$$

(b) (Strict containment.) *The sum in (a) without the term $X = S_0$, $N_X = 0$ is at most $c_{16}\,\mathcal{P}'\, e^{c_2|S_0|}$. (c′)* (Prescribed seed walls, $k = 0$.) *For every seed configuration $\sigma = (Z, N_Z)$ in $(U, 0)$ with $Z^o = \emptyset$, every $\varepsilon \in [0, c'/8]$ and $c_w := c'/16$,*

$$\sum_{Z \subseteq Y \subseteq U} \sum_{N_Y \in \mathcal{N}_\sigma(Y)} |\mathcal{K}_{N_Y}^{(Z)}(Y)|\, e^{\alpha|Y| + \varepsilon F_1^{\mathrm{loc}}(Y, N_Y)}\, e^{c_w F_1^{\mathrm{loc}}(Z, N_Z)} \leq \prod_{\mathcal{S}} e^{-(c'/2) F_1^{\mathrm{loc}}(\mathcal{S})},$$

the product over the wall polymers $\mathcal{S}$ of N_Z.

We prove Lemma 7.48 in Appendix A. Proposition A.8 treats a fixed seed configuration, and for (a), (b) the entropy of the first set is controlled by its walls.

Theorem 7.49 (The Kirkwood–Salsburg system in $\mathcal{A}_\omega$; [BF80, Lemma A4.1] on G')**.** *Fix a linear order on the $\mathcal{U}$-cells; for $X \neq \emptyset$ let $\ell(X)$ be its last $\mathcal{U}$-cell and $X^* := X \setminus \ell(X)$. For a $\mathcal{U}$-cell Δ and a union $Y \ni \Delta$ of $\mathcal{U}$-cells let $\mathcal{K}'_\Delta(Y) \in \mathcal{A}_\omega$ be the sequence $q \mapsto \sum \mathcal{K}_{N_Y}^{(\mathrm{sat}(\Delta))}(Y)$, the sum over the N_Y of (7.21) with $[N_Y] = q$ and $(Y, N_Y) \neq (\Delta, 0)$. There is exactly one family $(r(X))_X$ in $\mathcal{A}_\omega$ with $r(\emptyset) = \delta_0$ and $\max_{X \neq \emptyset} e^{-\alpha|X|}\|r(X)\|_\omega < \infty$ such that, for every $X \neq \emptyset$,*

$$r(X^*) = \tilde{\mathcal{K}}(\ell(X))\, r(X) + \sum_{Y \ni \ell(X),\ Y \cap X^* = \emptyset} \mathcal{K}'_{\ell(X)}(Y) * r(X^* \cup Y), \tag{7.33}$$

and it satisfies, with $\alpha = \ln 8$: (a) $\|r(X)\|_\omega \leq \frac{8}{31} e^{\alpha|X|}$ for every $X \neq \emptyset$; (b) $\mathcal{Z}(X) = r(X) * \mathcal{Z}(\emptyset)$ for every X, that is, $\mathcal{Z}_p(X) = \sum_q r(X)(q)\, Z_{p-q}$.

Proof. By Theorem 7.27 (b) the system has the form of [BF80, (A4.5)] with values in $\mathcal{A}_\omega$, and we run the contraction of [BF80, (A4.6) to (A4.13)] with $\|\cdot\|_\omega$ in place of $|\cdot|$. The normalization $r(\emptyset) = \delta_0$ replaces the division by Z, which is not available for the sequence $\mathcal{Z}(\emptyset)$, and (b) follows from uniqueness in the finite box.

We need two bounds. (K1) $\tilde{\mathcal{K}}(\Delta) \geq \frac{1}{2}$, by Lemma 7.41, and of the two bounds this is the one that rests on positivity, at $k = 0$ in the proof of Lemma 7.41. (K2) $\sum_{Y \ni \Delta} \|\mathcal{K}'_\Delta(Y)\|_\omega e^{\alpha|Y|} \leq \frac{1}{8}$. By Lemma 7.43 and Theorem 7.13, $\omega([N_Y]) \leq e^{\sigma' \sum |N_Y|} \leq e^{(\varepsilon_E/2) F_1^{\mathrm{loc}}(Y, N_Y)}$. Hence (K2) follows from Lemma 7.48 (b) with $S_0 = \Delta$, $k = 0$ and $\varepsilon = \varepsilon_E/2$, over long fields by Proposition 7.46, and from the bound $c_{16} e^{c_2} \leq \frac{1}{8}$ of Lemma S1.2.

Step 1 (the map). Let $\mathfrak{B}$ be the Banach space of families $s = (s(X))_{X \neq \emptyset}$ in $\mathcal{A}_\omega$ with $|||s||| := \max_{X \neq \emptyset} e^{-\alpha|X|}\|s(X)\|_\omega$, put $s(\emptyset) := \delta_0$, and let $(Ts)(X)$ be $\tilde{\mathcal{K}}(\ell(X))^{-1}$ times $s(X^*)$ minus the sum in (7.33). A family satisfies (7.33) if and only if $Ts = s$.

Step 2 (contraction). By $\|a*b\|_\omega \le \|a\|_\omega \|b\|_\omega$, $|X^* \cup Y| = |X| - 1 + |Y|$, $\|s(\emptyset)\|_\omega = 1$, (K1) and (K2), we obtain $e^{-\alpha|X|}\|(Ts)(X)\|_\omega \le 2e^{-\alpha}[\max(1, |||s|||) + |||s|||/8] = \frac{1}{4}\max(1, |||s|||) + |||s|||/32$, and in the same way $|||Ts - Ts'||| \le 2e^{-\alpha}(1 + \frac{1}{8})|||s - s'||| = \frac{9}{32}|||s - s'|||$, since $(s - s')(\emptyset) = 0$. Let r be the fixed point of T.

Step 3 ((a)). If $|||r||| \ge 1$, Step 2 gives $|||r||| \le \frac{9}{32}|||r|||$, which is impossible. Therefore $|||r||| \le \frac{1}{4} + |||r|||/32$, that is, $|||r||| \le \frac{8}{31}$.

Step 4 ((b)). We set $s(X) := r(X) * \mathcal{Z}(\emptyset) - \mathcal{Z}(X)$, which has $s(\emptyset) = 0$. By associativity and commutativity, $X \mapsto r(X) * \mathcal{Z}(\emptyset)$ satisfies (7.33), and so does $X \mapsto \mathcal{Z}(X)$, by (7.21) with $X_0 = X^*$ and $\Delta = \ell(X)$. Consequently s satisfies the linear part of (7.33) with $s(\emptyset) = 0$. Moreover, $|||s||| < \infty$, since there are finitely many X and $\|\mathcal{Z}(X)\|_\omega < \infty$ by Proposition 7.47. As in Step 2, $|||s||| \le \frac{9}{32}|||s|||$, and hence $s = 0$.

The bounds in the proof above are uniform in G', Λ° and C. We obtain the invertibility of $\mathcal{Z}(\emptyset)$ in $\mathcal{A}_\omega$ as a consequence of Theorem 7.49 (b): $r(V) * \mathcal{Z}(\emptyset) = \mathcal{Z}(V) = \mathfrak{c}\mathcal{N}\,\delta_0$.

7.8 The area cost of frozen flux: group charge and the count

In this subsection we prove that the partition function at fixed class r carries the area weight of r (Proposition 7.53). As announced at the start of Section 7, the handles of the insulating set enter the bounds only here; so does (NF_*), which holds by (R10) of Theorem 3.20. In a configuration of class r, a group of wall polymers realizing r is re-expanded as a seed at $X = \emptyset$, and its entropy is controlled by Proposition 7.52.

Polymers, realizations and the reference realization. Fix $r \in \Gamma_{\mathrm{long}} \setminus \{0\}$ with $Z_r \neq 0$. For a configuration N of Z_r (Definition 7.25), $\mathcal{P}(N)$ is the set of its wall polymers, and we put $N_{\mathcal{S}} := N\mathbf{1}_{E(\mathcal{S})}$ for $\mathcal{S} \in \mathcal{P}(N)$ and $N_R := \sum_{\mathcal{S}\in R} N_{\mathcal{S}}$, $q_R := [N_R]$, $|R| := \|N_R\|_1$ for $R \subseteq \mathcal{P}(N)$. By Proposition 7.8 (a), each wall bond lies in exactly one wall polymer. Hence $\sum_{\mathcal{S}\in\mathcal{P}(N)} N_{\mathcal{S}} = N$ and $q_{\mathcal{P}(N)} = r$, and $q_R \in \Gamma_{\mathrm{long}}$ by Lemma 7.26 (a). A family R *realizes* r if $q_R = r$, and is *irreducible* if no nonempty $T \subseteq R$ has $q_T = 0$. An inclusion-minimal realizing subfamily of $\mathcal{P}(N)$ is irreducible: if $\emptyset \neq T \subseteq R$ had $q_T = 0$, then $T \neq R$ as $r \neq 0$, and $R \setminus T$ would realize r. Let $\mathfrak{S}(r)$ be the set of families R, with their wall configurations, that occur as irreducible realizations of r in some configuration of Z_r. We fix $R_0 \in \mathfrak{S}(r)$ of least area by a deterministic choice, so that $|R| \ge |R_0| \ge \mathrm{A}(r)$ for all $R \in \mathfrak{S}(r)$, the last inequality by Lemma 7.43. A polymer is determined by its wall interfaces and their values, since closures and links are functions of the walls. Thus R is determined by N_R: the wall components of N_R are those of the members of R, closures from different members are not linked (they are distinct wall polymers of one configuration), so the wall polymers of N_R are exactly the members of R. We also set $\nu(R) := \sum_{I\subseteq \operatorname{supp} N_R} N_I^2$, $c_w := c'/16$, $\lambda_E := 2c_w\sigma_F = \frac{c'}{8}\sigma_F$ and $\Theta_4(r) := \sum_{R\in\mathfrak{S}(r)} e^{-\lambda_E|R|}\prod_{\mathcal{S}\in R} e^{-(c'/2)F_1^{\mathrm{loc}}(\mathcal{S})}$. The notion of c-connectedness is that of Definition 3.18, and distances between bond sets are Euclidean.

Lemma 7.50. *Let $R \in \mathfrak{S}(r)$. Then $N_R - N_{R_0} = \sum_k \varepsilon_k \nabla 1_{P_k}$ with $\varepsilon_k \in \{-1, 1\}$, where (i) each P_k is a nonempty c-connected cell set with c-connected complement in its component $\mathfrak{K}$ of G'; (ii) $\partial_{G'} P_k \subseteq \operatorname{supp} N_R \cup \operatorname{supp} N_{R_0}$; (iii) $\sum_k |\partial_{G'} P_k| = \|N_R - N_{R_0}\|_1 \le |R| + |R_0| \le 2|R|$; and (iv) every P_k is joined to $\operatorname{supp} N_{R_0}$ by a chain $P_k = P_{k_0}, \dots, P_{k_j}$ with $\operatorname{dist}(\partial P_{k_{i-1}}, \partial P_{k_i}) \le g$ and $\operatorname{dist}(\partial P_{k_j}, \operatorname{supp} N_{R_0}) \le g$.*

Proof. (1) Since $[N_R - N_{R_0}] = q_R - q_{R_0} = 0$, Definition 7.25 gives $N_R - N_{R_0} = \nabla f$ with $f : V \to \mathbb{Z}$ constant on each cell.

(2) By the coarea formula for the level sets of f, $\nabla f = \sum_t \pm \nabla 1_{U_t}$ with $\sum_t |\partial U_t| = \|\nabla f\|_1$. We split each U_t into its c-components W and write $\nabla 1_W = -\sum_i \nabla 1_{C_i}$ over the c-components C_i of $\mathfrak{K} \setminus W$, where $\mathfrak{K}$ is the component of W. Since each C_i is c-connected with c-connected complement and the cuts partition, (i) to (iii) hold.

(3) For (iv) we use irreducibility, which forbids groups of pieces far from R_0, since these would be sub-families of R of class zero. We group the pieces into clusters by the relation "cuts within distance g". Let $\mathcal{V}$ be a cluster no cut of which is within g of $\operatorname{supp} N_{R_0}$, and let $B := \bigcup_{k \in \mathcal{V}} \partial P_k$. Within distance g of B, $N_{R_0} = 0$ and the other pieces contribute nothing. It follows that $N_R = N_{\mathcal{V}} := \sum_{k \in \mathcal{V}} \varepsilon_k \nabla 1_{P_k}$ in this region, and each wall bond of N_R within distance g of B lies in B. Two wall bonds of one polymer are joined by a chain of wall bonds of that polymer with consecutive gaps at most g, since walls of one S_β have gaps less than $2L' + 2D_E$ and linked closures add at most $2D_E + 2$. Hence the polymers of R meeting B form a sub-family T with $N_T = N_R 1_B = N_{\mathcal{V}}$, the gradient of a cell-constant function, and therefore $q_T = 0$. By irreducibility $T = \emptyset$, so that $N_{\mathcal{V}} = 0$. By (iii) the terms of $N_R - N_{R_0}$ do not cancel, and hence neither do those of the sub-sum: $\|N_{\mathcal{V}}\|_1 = \sum_{k \in \mathcal{V}} |\partial P_k|$. So $N_{\mathcal{V}} = 0$ forces $\partial P_k = \emptyset$ for every $k \in \mathcal{V}$. This is impossible, since by (i) each P_k is a nonempty proper cell set of its component $\mathfrak{K}$. Therefore no such cluster exists: every cluster has a cut within g of $\operatorname{supp} N_{R_0}$, and chaining inside the cluster gives (iv).

For a finite set B of interfaces and $s > 0$, the s-clusters of B and their number $\varphi_s(B)$ are those of Definition 3.18 (for a cut, of its set of interfaces), and $\mathrm{MST}(B)$ denotes a minimal spanning tree of the complete graph on B with edge lengths ℓ_e the distances of interfaces. Let $j(\ell) := \#\{j \geq 0 : 2^j L < \ell\}$.

Lemma 7.51 (Kruskal's identity)**.** *For every $s > 0$, $\#\{e \in \mathrm{MST}(B) : \ell_e > s\} = \varphi_s(B) - 1$, and $\sum_{e \in \mathrm{MST}(B)} j(\ell_e) = \sum_{j \geq 0} (\varphi_{2^j L}(B) - 1)$.*

Proof. The edges of $\mathrm{MST}(B)$ of length $\leq s$ form a spanning forest of the threshold graph $\{\ell \leq s\}$, since Kruskal's algorithm processes them first. Thus they number $|B| - \varphi_s(B)$. For the second identity we sum $1[\ell_e > 2^j L]$ over $j \geq 0$ and e.

Let $\kappa_{\mathrm{NF}} := 4(c_g + 4)/c_g^2$, and let R_{NF}, θ_{NF}, c_3^{NF}, J be as in Definition 3.18. Applying (R10) in the form of Proposition 3.19 to the pieces P_k of Lemma 7.50, we obtain

$$J(P_k) \leq \theta_{\mathrm{NF}} F_{\mathrm{unit}} |\partial_{G'} P_k| / L^2, \tag{7.34}$$

with the cut measured in fine bonds, of which a coarse face carries at least r_0^2. Here J depends only on the cut.

Proposition 7.52 (The count)**.** *For $\beta_{\mathrm{GM}} \geq \beta_{\mathrm{GM}}^0$, every $r \in \Gamma_{\mathrm{long}} \setminus \{0\}$ and every $\nu \geq 1$, $\#\{R \in \mathfrak{S}(r) : \nu(R) = \nu\} \leq \exp((c'/2) F_{\mathrm{unit}}\, \nu)$. Consequently $\Theta_4(r) \leq \frac{1}{4}$.*

Proof. By Kruskal's identity, specifying the positions of the cut patches costs, per dyadic scale, the number of clusters minus one. Below R_{NF} this is at most $\log_2(R_{\mathrm{NF}}/L)$ per interface, within the factor $e^{(c'/2)F_{\mathrm{unit}}}$, and above R_{NF} it is J, bounded by the cut (7.34). We first bound the count and then prove the second assertion, $\Theta_4 \leq \frac{1}{4}$.

Fix R with $\nu(R)=\nu$. Let S, S_0 be the interface sets of $\operatorname{supp} N_R$ and $\operatorname{supp} N_{R_0}$, put $n:=|S|\le\nu$, $n_0:=|S_0|$ and $T:=S\cup S_0$, and let I_k be the interface set of ∂P_k. An interface has between $c_g'L^2$ and $4L^2$ bonds by (R8) and since block sides are less than $2L$, and $|R_0|\le|R|\le 4L^2\nu$. Hence $n_0\le(4/c_g')\nu$, and by Lemma 7.50(iii) $\sum_k|\partial P_k|\le 8L^2\nu$, $\sum_k|I_k|\le(8/c_g')\nu\le\kappa_{\mathrm{NF}}\nu$. The last inequality follows from $8/c_g'\le 128/(9c_g)\le 4(c_g+4)/c_g^2$, which holds since $c_g'\ge\frac{9}{16}c_g$ by Section S1.1.4(e) and $c_g\le\frac14$. Since a bond where $N_R\ne N_{R_0}$ lies on some ∂P_k, we have $S\subseteq S_0\cup\bigcup_k I_k\subseteq T$.

Step 1 (a spanning forest). For each k let t_k be the least j for which a chain $P_k=P_{k_0},\dots,P_{k_j}$ as in Lemma 7.50(iv) exists. Let H be the graph on T whose edges are the edges of the $\mathrm{MST}(I_k)$ and, for each k, one parent edge at distance at most $2g$: from an interface of I_k to one of S_0 if $t_k=0$, and otherwise to one of $I_{k'}$, where $P_{k'}$ is the second piece of a chain of length t_k, so that $t_{k'}=t_k-1$. Here we use that a bond and its dual coarse face are within distance $\sqrt3\,r_0\le g/2$. Each node of $T\setminus S_0$ lies in some I_k, and t decreases along parent edges. Hence the parent edges lead to S_0. Consequently H contains a spanning forest F of T with roots in S_0, as one sees by adding a virtual node joined to all of S_0 and removing it from a spanning tree. Let $B:=\sum_{e\in F}j(\ell_e)\le\sum_{e\in H}j(\ell_e)$.

Step 2 (encoding). We record the roots (at most 2^{n_0} choices), the shape of the plane forest on $|T|\le n+n_0$ nodes (at most 4^{n+n_0}), and for each child the number j_e and the child among the at most $C_9(2^{j_e}+1)^3\le 8C_9\,8^{j_e}$ interfaces within distance $2^{j_e}L$ of its parent. Sequences of $|F|$ nonnegative integers with sum at most $B_{\max}$ (the bound of Step 3, a function of ν only) number at most $2^{B_{\max}+|F|}$. We then record $S\subseteq T$ (at most 2^{n+n_0}) and the values N_I with $\sum N_I^2=\nu$ (at most 4^ν, counting signs and compositions of $\sum|N_I|\le\nu$). Since R is determined by N_R, we obtain $\ln\#\{R:\nu(R)=\nu\}\le C(n+n_0+\nu)+B_{\max}\ln 16$ with $C:=\ln(64C_9)+2$.

Step 3 (the bound on B). By Lemma 7.51, $\sum_{e\in\mathrm{MST}(I_k)}j(\ell_e)=\sum_{j\ge0}(\varphi_{2^jL}(I_k)-1)$. The scales with $2^jL<R_{\mathrm{NF}}$ number at most $j_*:=c_3^{\mathrm{NF}}F_{\mathrm{unit}}/\ln 2+1$, each contributing at most $|I_k|-1$, while the scales with $2^jL\ge R_{\mathrm{NF}}$ contribute exactly $J(P_k)$, by Definition 3.18. A parent edge has $j\le j_g:=\log_2(2g/L)+1=O(\ln\beta_{\mathrm{GM}})$, and there are at most $\sum_k|I_k|\le\kappa_{\mathrm{NF}}\nu$ pieces. With (7.34),

$$B\le(j_*+j_g)\kappa_{\mathrm{NF}}\nu+\theta_{\mathrm{NF}}F_{\mathrm{unit}}\sum_k|\partial P_k|/L^2\le\Big(\frac{c_3^{\mathrm{NF}}\kappa_{\mathrm{NF}}}{\ln 2}+8\theta_{\mathrm{NF}}\Big)F_{\mathrm{unit}}\,\nu+(1+j_g)\kappa_{\mathrm{NF}}\nu=:B_{\max}.$$

Step 4. By Definition 3.18, $4c_3^{\mathrm{NF}}\kappa_{\mathrm{NF}}=c_0'/4$ and $8\theta_{\mathrm{NF}}\ln 16\le c_0'/8$. Hence $\ln\#\le\frac38c_0'F_{\mathrm{unit}}\nu+O((1+\ln\beta_{\mathrm{GM}})\nu)\le\frac{c'}{2}F_{\mathrm{unit}}\nu$ for $\beta_{\mathrm{GM}}\ge\beta_{\mathrm{GM}}^0$, as F_{unit} is exponentially large by Section S1.8.

It remains to prove $\Theta_4\le\frac14$, which rests on the factor 2 of Lemma 7.30(a), that is, on the constant $1/18$ of Theorem 7.13, set against the bound of the count. By Lemma 7.30(a), $F_1^{\mathrm{loc}}(\mathcal S)\ge 2F_{\mathrm{unit}}\,\nu(\mathcal S)$ for each wall polymer $\mathcal S$. Hence the summand of Θ_4 is at most $e^{-c'F_{\mathrm{unit}}\nu(R)}$, and $\Theta_4(r)\le\sum_{\nu\ge1}e^{(c'/2)F_{\mathrm{unit}}\nu}e^{-c'F_{\mathrm{unit}}\nu}\le 2e^{-(c'/2)F_{\mathrm{unit}}}\le\frac14$.

Proposition 7.53 (Area cost of frozen flux)**.** *For every $r\in\Gamma_{\mathrm{long}}$, $Z_r\le e^{-\sigma'\mathrm{A}(r)}Z$.*

Proof. We extract an irreducible realization from each configuration and re-expand it as a seed, whose own F_1^{loc} covers its cells and shape and leaves the fraction $c'/2$ over. The outer factors are bounded by Theorem 7.49 and by the quantity being bounded, which closes the argument. We put $C_{\mathrm{FF}}:=\sup_{s\in\Gamma_{\mathrm{long}}}\omega(s)Z_s/Z$. Then $1\le C_{\mathrm{FF}}<\infty$, by taking $s=0$ and by Proposition 7.47. Let $r\ne0$ with $Z_r\ne0$.

(1) *Union bound.* For each configuration we choose one irreducible realization $R(N) \subseteq \mathcal{P}(N)$ by a fixed rule. All terms are nonnegative by Lemma 7.42. Hence $Z_r \le \sum_{R \in \mathfrak{S}(r)} Z_r(R)$, with $Z_r(R)$ as in Theorem 7.27 (c). Dropping the condition $R = R(N)$ only enlarges the right side.

(2) *Re-expansion.* By Theorems 7.27 (c) and 7.49 (b), with $\sigma := (\bigcup R, N_R)$, $Z_r(R) \le \sum_{Y \supseteq \bigcup R} \sum_{N_Y \in \mathcal{N}_\sigma^{\log}(Y)} |\mathcal{K}_{N_Y}^{(\bigcup R)}(Y)| \sum_q |r(Y)(q)|\, Z_{r-[N_Y]-q}$.

(3) *Weights.* By Lemma 7.43, $\omega(r) \le \omega([N_Y])\,\omega(q)\,\omega(r - [N_Y] - q)$, and $\omega([N_Y]) \le e^{(\varepsilon_E/2)F_1^{\mathrm{loc}}(Y)}$ by Theorem 7.13. The definition of C_{FF} and Theorem 7.49 (a), which applies as $Y \supseteq \bigcup R \neq \emptyset$, give $\sum_q |r(Y)(q)|\,\omega(q)\,\omega(r - [N_Y] - q) Z_{r-[N_Y]-q} \le \frac{8}{31} e^{\alpha|Y|} C_{\mathrm{FF}} Z$.

(4) *Activity.* So that the sum takes the form $\Theta_4(r)$ of Proposition 7.52, we insert $1 \le e^{c_w F_1^{\mathrm{loc}}(\bigcup R)} e^{-\lambda_E |R|}$, which is valid since $F_1^{\mathrm{loc}}(\bigcup R) \ge 2\sigma_F |R|$ and $2c_w\sigma_F = \lambda_E$. The final estimate of this step holds without this insertion as well, since the proof of $\Theta_4 \le \frac{1}{4}$ bounds $e^{-\lambda_E|R|}$ by 1 and Lemma 7.48 (c′) without the factor $e^{c_w F_1^{\mathrm{loc}}(Z, N_Z)} \ge 1$ follows from (c′). Lemma 7.48 (c′) over long fields, which holds by Proposition 7.46, applied with the seed configuration σ, all of whose $\mathcal{U}$-cells lie in wall polymers, and with $\varepsilon = \varepsilon_E/2 \le c'/8$, bounds the sum over Y and N_Y, including $e^{\alpha|Y|} e^{(\varepsilon_E/2)F_1^{\mathrm{loc}}(Y)} e^{c_w F_1^{\mathrm{loc}}(\bigcup R)}$, by $\prod_{\mathcal{S} \in R} e^{-(c'/2)F_1^{\mathrm{loc}}(\mathcal{S})}$. Collecting (1) to (4), we obtain $\omega(r) Z_r/Z \le \frac{8}{31} C_{\mathrm{FF}} \Theta_4(r) \le \frac{2}{31} C_{\mathrm{FF}}$ by Proposition 7.52.

(5) *The supremum.* With $s = 0$, we obtain $C_{\mathrm{FF}} \le \max(1, \frac{2}{31} C_{\mathrm{FF}})$. Since $C_{\mathrm{FF}} < \infty$, this gives $C_{\mathrm{FF}} = 1$.

Remark. The position entropy is part of the count of $\mathfrak{S}(r)$, and so the bound just proved carries no position factor, such as a sum over a cycle detecting r or $|\gamma|$, even for a long handle with a tapered thin neck, since its cuts of one class are compared with R_0 by Lemma 7.50.

Lemma 7.54. *Let $\alpha = \ln 8$, $\varepsilon_E \in (0, c_0'/8]$, $\sigma' = \varepsilon_E \sigma_F$ and $\beta_{\mathrm{GM}} \ge \beta_{\mathrm{GM}}^0$. For every nonempty union X of $\mathcal{U}$-cells and every $p \in \Gamma_{\log}$, $0 \le \mathcal{Z}_p(X)/Z \le \frac{8}{31} e^{\alpha|X|} e^{-\sigma' \mathrm{A}(p)}$. The bound is uniform in G', Λ° and C.*

Proof. By Theorem 7.49 (b) and Lemma 7.43, we have $\omega(p)\mathcal{Z}_p(X)/Z \le \sum_q |r(X)(q)|\,\omega(q) \cdot \omega(p-q) Z_{p-q}/Z$, and this is at most $\|r(X)\|_\omega \le \frac{8}{31} e^{\alpha|X|}$ by Proposition 7.53 and Theorem 7.49 (a). Nonnegativity follows from Lemma 7.42.

7.9 The deterministic area law

We now prove the deterministic area law, Theorem 7.57.

Lemma 7.55. *Let $X \supseteq X_1$ be a union of $\mathcal{U}$-cells, N_X an inside field at $k = 1$ on X, and $p := p_1(N_X)$ its outer class (Definition 7.25). Then $\sum_{e \in E(X)} |N_X(e)| + \mathrm{A}(p) \ge A_{G'}(C)$.*

Proof. Since linking numbers are additive, the inside walls and a cheapest cochain with the forced periods form a wall field at $k = 1$, and by Theorem 7.14 this field spans C modulo the frozen complex. Let $N' \in \mathbb{Z}^E$ realize $\mathrm{A}(p)$, which is finite since $-\xi - N_X 1_{E(X)}$ has periods p. By definition of p_1, $N_X 1_{E(X)}$ has periods $-\mathrm{lk}(\cdot, C) - p$, and hence $M := N_X 1_{E(X)} + N'$ has periods $-\mathrm{lk}(\cdot, C)$. Since ξ has periods $\mathrm{lk}(\cdot, C)$ by Lemma 7.6(d), we get $M + \xi = \nabla m$ for some $m : V \to \mathbb{Z}$, and $M = \nabla m - \xi$ is the $k = 1$ wall field of m. By Theorem 7.14, $\sum_E |M| \ge \#\{M \neq 0\} \ge A_{G'}(C)$, and $\sum_E |M| \le \sum_{E(X)} |N_X| + \sum_E |N'| = \sum_{E(X)} |N_X| + \mathrm{A}(p)$.

Lemma 7.56. *With $C_{\mathcal{U}}$ and c_u as in (R7) and $C_X := 3C_{\mathcal{U}}/c_u$, $|X_1| \le C_X(\tilde{m}^2RT + 3C_{\mathcal{U}}\tilde{m}(R+T) + 9C_{\mathcal{U}}^2)$.*

Proof. The $\mathcal{U}$-cells of X_1 meet bonds crossing Ξ, by Definition 7.20. Thus they lie within Euclidean distance $D_E + 2$ of Ξ, where $D_E = C_{\mathcal{U}}\tilde{m}^{-1} \ge 4$, in a set of volume at most $3D_E(R + 3D_E)(T + 3D_E)$. Each of these cells has at least $c_u\tilde{m}^{-3}$ sites by (R7).

For a rectangle C of sides R, T we put $R_1(\beta) := (2r_\square + 5)L$, where L is the block scale of Definition 3.1 at $\tau = K\beta$.

Theorem 7.57 (Deterministic area law)**.** *Fix the constants of Section S1.1.1 of the supplement. There is β^0_{GM} such that for $\beta_{\mathrm{GM}} \ge \beta^0_{\mathrm{GM}}$, for every output $G' = G'(D)$ of the construction, for every D, and for every rectangle C with $\min(R,T) \ge R_1(\beta) = (2r_\square + 5)L$,*

$$W_{G'}(C) \le \exp\bigl(-\sigma_V[A_{G'}(C) - \eta_S RT] + c_N(R+T)\bigr),$$

with

$$\sigma_V = \varepsilon_E\sigma_F = \frac{c'_g\,\tilde{m}L}{9216}\,\frac{\tilde{m}}{\tau} \quad (\varepsilon_E = \tfrac{1}{128}), \qquad \eta_S = \frac{18c_2C_X}{\varepsilon_E\pi^2 c'_g\,\tilde{m}L}\,\tilde{m}\beta_{\mathrm{GM}} = O(\tilde{m}\beta_{\mathrm{GM}}\log\tau),$$

and c_N as in Remark 7.58. The degenerate output $\mathbb{X} = \Lambda^\circ$ ($G' = \emptyset$, $W_{G'}(C) = 1$, $A_{G'}(C) = 0$) is included.

Section S1.8 gives the values of σ_V and η_S. The geometric input of the proof consists of (R2), (R5) to (R10), Theorems 6.5 and 6.8 to 6.11, Proposition 6.17, the frame, the coarse alignment of the holes and frozen bonds, Lemma 4.4, in Lemma 6.13, and (N3) of Theorem 3.20, in Lemma 7.44. The frame, which does not depend on D, is made up of the box (Lemma 3.3), the grid with its windows and removal cells (Definition 3.4) and the alignment (A1) to (A3) (Lemma 3.5). The holes and frozen bonds are unions of coarse sites and of whole coarse faces (Section 2.5).

Proof. We bound the outer factors of the loop expansion by Lemma 7.54. If no bond of G' crosses Ξ (for instance if $G' = \emptyset$), then $X_1 = \emptyset$ and the bound holds trivially. Indeed, in this case $j = 0$ on E, which gives $Z(1,\Xi) = Z(0,\Xi)$ and $W_{G'}(C) = 1$. Moreover, by the proof of Lemma 7.6(c), each plaquette of Ξ is then a face of a hole cube or a frozen face, hence lies in $\mathfrak{F}$, and Definition 2.11 with $\Sigma = \Xi$ gives $A_{G'}(C) = 0$. Finally, $\sigma_V\eta_S RT + c_N(R+T) \ge 0$. Otherwise $X_1 \ne \emptyset$, and

$$\begin{aligned} W_{G'}(C) &= \sum_{X \supseteq X_1}\sum_{N_X} \mathcal{K}_{N_X}(X)\,\frac{\mathcal{Z}_{p_1(N_X)}(X)}{Z} \le \frac{8}{31}\sum_{X\supseteq X_1}\sum_{N_X} |\mathcal{K}_{N_X}(X)|\, e^{\alpha|X|} e^{-\sigma' \mathrm{A}(p_1(N_X))} \qquad (7.35)\\ &\le \frac{8}{31}\, e^{-\varepsilon_E\sigma_F A_{G'}(C)} \sum_{X\supseteq X_1}\sum_{N_X} |\mathcal{K}_{N_X}(X)|\, e^{\alpha|X| + \varepsilon_E F_1^{\mathrm{loc}}(X,N_X)} \le \frac{8}{31}\,\mathcal{P}'_G\, e^{c_2|X_1|} e^{-\sigma_V A_{G'}(C)} \\ &\le \exp\Bigl(-\sigma_V A_{G'}(C) + \sigma_V\eta_S RT + \Bigl[\frac{2c_{\mathrm{near}}}{p'} + 3c_2C_XC_{\mathcal{U}}\tilde{m} + \frac{9c_2C_XC_{\mathcal{U}}^2}{R_1(\beta)}\Bigr](R+T)\Bigr), \end{aligned}$$

and the bracket, $b(\beta)$, is at most c_N. The equality follows from Corollary 7.28 together with Lemma 7.29(f). Its terms with $p_1(N_X) \notin \Gamma_{\mathrm{long}}$ vanish by Definition 7.25(c). Consequently, in

(7.35), each sum over N_X runs over the inside fields at $k=1$ with $p_1(N_X)\in\Gamma_{\mathrm{long}}$. Lemma 7.54 gives the first inequality. The second uses Lemma 7.55, $\sigma'=\varepsilon_E\sigma_F$ and $F_1^{\mathrm{loc}}(X,N_X)\geq 2\sigma_F\sum_{E(X)}|N_X|$, which holds by Theorem 7.13 since X is a union of elements. The third follows from Lemma 7.48(a) with $k=1$, $U=V$, $S_0=X_1$, $\varepsilon=\varepsilon_E\leq c'/2$, applied to these fields. These fields are primitive-admissible at $k=1$ (Lemma 7.45b (b)), so Lemma 7.48(a) applies to them by Proposition 7.46. The last inequality uses $\log(8/31)<0$, Lemma 7.56 with $c_2C_X\tilde{m}^2=\sigma_V\eta_S$ and $R+T\geq 2R_1(\beta)$, and $\mathcal{P}'_G=e^{\mathfrak{n}'(V)/p'}\leq e^{2c_{\mathrm{near}}(R+T)/p'}$ (Lemma 7.33).

Remark 7.58 (The perimeter constant). The proof above obtains $\varepsilon_E\sigma_F A_{G'}(C)$ from a fraction ε_E of the wall energy of the seeded polymer, with the area weight, by Lemma 7.55; the seed contributes $e^{c_2|X_1|}$, whose bulk part gives η_S and whose remainder is a perimeter term, and the near cells give the perimeter factor $\mathcal{P}'_G$. Let $c_N:=\sup_{\beta_{\mathrm{GM}}\geq\beta^0_{\mathrm{GM}}}b(\beta)$, where $b(\beta)$ is the bracket in (7.35) (see also the remark after the proof of Proposition 7.35) and $p'=1.3$. It is finite and depends only on the constants fixed before K, since c_{near} is bounded uniformly for $\beta_{\mathrm{GM}}\geq\beta^0_{\mathrm{GM}}$ by Lemma 7.33 together with $\log(1/\tilde{m})/\beta_{\mathrm{GM}}<0.0632$, and the last two terms tend to 0 (Section S1.8). Since β and K appear in Sections 3, 6 and 7 only in the combination $\tau=K\beta$, that is, in $\beta_{\mathrm{GM}}=4\pi^2\tau$, and every condition on β_{GM} used there is a lower bound in terms of items 0 to 14 of Section S1.1.1, it follows that Theorem 7.57, its threshold β^0_{GM} and c_N use items 0 to 14, not K and $\beta_0(K)$ (items 15 and 16). Section 8 needs this, since there $c=c_N+10$ must be absolute.

8 Proof of the main theorem

8.1 The chain

The deterministic area law holds for each frozen set D, and averaged over D it gives $e^{-\sigma_V(1-\eta_S)RT+c_N(R+T)}\cdot\mathbb{E}_De^{\sigma_V(RT-A_{G'}(C))}$. The second factor is the exponential moment of the area deficit $RT-A_{G'}(C)$, which is exactly what the min-cut lower tail bounds, and the two deficits η_S and η_T add. We fix $K>K_*$ and $\beta\geq\beta_0(K)$ as in Section 8.2. With $\tau=K\beta$, the conditions defining $\beta_0(K)$ include $\beta\geq\pi^{-2}$, $\delta<1$ and the condition (2.5) of Lemma 2.6, that is $\beta\geq\beta_1(K)$, as well as the hypotheses of Theorems 5.1 and 7.57 for β and $\beta_{\mathrm{GM}}=4\pi^2\tau$. They also include $\eta_T+\eta_S\leq\frac{1}{2}$ and $\sigma_V\leq 1$, which are true for large τ, since $\eta_T\to 0$ by Theorem 5.1, $\eta_S=O(\tilde{m}\beta_{\mathrm{GM}}\log\tau)\to 0$ by Theorem 7.57, and $\sigma_V=(c'_g\tilde{m}L/9216)\tilde{m}/\tau\to 0$. Let C be a rectangle with $\min(R,T)\geq R_1(\beta)=(2r_\square+5)L$ (Definition 3.1), $\Lambda^\circ=\Lambda^\circ(C)$ the box of Definition 3.2, and $\Lambda\supseteq C$ any finite box.

Proof of Theorem 1.2. We prove the chain (1.4) to (1.8), where $\mathbb{E}_D$ is the expectation over $D\sim\mathrm{Ber}(\delta)^{\otimes(\Lambda_2^\circ\setminus\partial_2\Lambda^\circ)}$.

Step (1.4). Lemma 3.3(e) gives $\mathrm{dist}(C,\partial\Lambda^\circ)\geq\max(R,T)\geq 1$. Hence Proposition 2.9 applies with $\Lambda'=\Lambda^\circ$ and $\delta'=\delta$. By Lemma 3.3(a), the box Λ° is determined by C and the constants, so it is the same for all Λ and D and is fixed before D is drawn.

Step (1.5). Theorem 3.20 shows that for each $D\subseteq\Lambda_2^\circ\setminus\partial_2\Lambda^\circ$ the output $(\mathbb{X}(D),S(D))$ is a deterministic function of D with $D\subseteq\mathfrak{F}(D)$. Hence Proposition 2.10 applies to each D.

Step (1.6). Theorem 7.57 applies to each output $G'(D)$ and each rectangle with $\min(R,T)\geq R_1(\beta)$, pointwise inside $\mathbb{E}_D$. We need it for every D, because $\mathrm{Ber}(\delta)$ charges each subset of $\Lambda_2^\circ\setminus\partial_2\Lambda^\circ$ and the construction is deterministic. Its functional $A_{G'(D)}(C)$ is the min-cut area of Definition 2.11 for the same output on the same box Λ°.

Step (1.7) is the identity $-\sigma_V A = -\sigma_V RT + \sigma_V(RT - A)$.

Step (1.8). Theorem 5.1 at $\sigma = \sigma_V \in (0,1]$, applied to the same random set D, the box Λ°, the output $G'(D)$ and the functional $A_{G'(D)}(C)$ of Steps (1.5) and (1.6), bounds the expectation in (1.7) by $\exp(\sigma_V \eta_T RT + 4|C| + 4)$.

Constants. Since $|C| = 2(R+T)$ and $R+T \geq 2$, we have $4|C|+4 \leq 10(R+T)$, and (1.8) gives (1.2) with $\sigma(\beta) := (1-\eta_T-\eta_S)\sigma_V$, $R_1(\beta) = (2r_\square + 5)L$, both depending on K through $\tau = K\beta$, and $c := c_N + 10$. This constant is absolute in the sense of Theorem 1.2, since c_N, a supremum over $\beta_{\mathrm{GM}} \geq \beta^0_{\mathrm{GM}}$, depends only on constants fixed before K (Remark 7.58). Using $\eta_T + \eta_S \leq \frac{1}{2}$ and the bound $c'_g \tilde{m} L \geq \frac{9}{16} c_g c_L$ of Section S1.1.3, we obtain

$$\sigma(\beta) \geq \frac{c_g c_L\, \tilde{m}}{2^{15}\tau} \geq \frac{(8\pi^2)^{1/2}}{2^{16}}\, c_g c_L\, \tau^{-1/2} e^{-\pi^2 v_{\mathrm{Cb}}(0)\tau} \geq 1.35 \cdot 10^{-4}\, c_g c_L\, \tau^{-1/2} e^{-2.49436\,\tau}, \qquad (8.1)$$

by $\tilde{m} \geq m_D/2$ and the formula $m_D = (8\pi^2\tau)^{1/2} e^{-\pi^2 v_{\mathrm{Cb}}(0)\tau}$ of Section 6, with the Debye exponent $\pi^2 v_{\mathrm{Cb}}(0) = 2.4943550871\ldots$ rounded up and the prefactor rounded down (see Section S1.1.3). This proves the second assertion of Theorem 1.2 for $\beta \geq \beta_0(K)$. Remark 8.1 gives the case $\beta < \beta_0(K)$ at the same K, and the first assertion follows with one fixed $K > K_*$, for instance $K = 378$ (any sufficiently large K also works, by the remark after Lemma 2.6).

Remark 8.1 (Small β and the infinite-volume limit). For fixed $K > K_*$ and $\beta < \beta_0(K)$, Theorem 2.1(b), applied pointwise in Λ and C, gives Theorem 1.2 at β with the three constants $\sigma(\beta_0(K))$, c, $R_1(\beta_0(K))$. We use monotonicity in β on the Wilson side only: raising β raises τ but lowers δ, and these two changes push the right side of Theorem 2.5 in opposite directions. The bound depends on Λ only through Proposition 2.9, on D only inside $\mathbb{E}_D$ and on C only through R, T, and by Theorem 2.1(c) it holds for the infinite-volume limit $\langle \cos\theta_C \rangle^W_\beta$, which is nondecreasing in β. Hence the string tension $\sigma_\infty(\beta) := \liminf_{\min(R,T)\to\infty} -\frac{1}{RT} \log \langle \cos\theta_{C_{R,T}} \rangle^W_\beta \in [0,\infty]$, where $C_{R,T}$ is a coordinate rectangle of sides R, T, is nonincreasing in β and at least $\sigma(\beta)$.

Remark 8.2 (Loops below the non-fragmentation scale). If $\operatorname{diam}\Lambda^\circ \leq R_{\mathrm{NF}}$, for instance if both $3\sqrt{3}\max(R,T) + \sqrt{3}$ and $\sqrt{3}(r_0(\ell+3)+1)$ are at most R_{NF}, then Phase 2 makes no firing by Proposition 3.19, and Proposition 5.11, with the deficit $\eta_T^{(1)}$, replaces Theorem 5.1 in Step (1.8). The string tension, by contrast, uses Theorem 5.1 in full.

8.2 The order of the constants

We fix the constants once, in the order of the list in Section S1.1.1, which also gives their values and the statements that use each of them. The absolute constants come first (item 0), among them $\pi^2 v_{\mathrm{Cb}}(0)$, the threshold 14.9662 of Lemma 2.6 and the constants quoted from [GM82, GM81a, BF80] that do not depend on the parameters of Theorem 6.5; those that do are fixed with these parameters. Then come r_0, c_g and the other constants of the construction that precede the scales (items 1 to 10); c_L and the scales L_c, L, k_P, ℓ, which depend on β only through $\tilde{m}$ (item 11); the constants of Stages 1 and 2, among them the parameters C, δ_{GM}, μ and κ_S of Theorem 6.5, and the number $c'_0 = \frac{1}{16}$ (item 12); λ_{PV}, and with it M_Y and $\tilde{m} = \lambda_{\mathrm{PV}} M_Y$ (item 13); the cut-rule constants θ_{NF} and c_3^{NF} of Definition 3.18 (item 14); $K > K_*$ (item 15); and $\beta_0(K)$, defined by the conditions listed in Section S1.1.4 (item 16).

Four placements are forced. First, k_P/L_c, ϑ and $\varepsilon_{\mathrm{th}}$ precede χ, because the expansion constant of Theorem 4.16(c), which bounds χ, is computed from them. Second, χ precedes

c_L, because the condition $C'_P c_L^2 \le (1-4/\pi^2)/8$ that fixes c_L has $C'_P \le 2916 r_\square^5 C_{\rm bl}^2/\chi^2$. Since $1-4/\pi^2 < \eta_H$ and $\tilde m L \le c_L$, the condition that fixes c_L also gives the condition $C'_P(\tilde m L)^2 \le \eta_H/8$ of Stage 2. Third, C, $\delta_{\rm GM}$, c_{10}, c''_{10}, μ and κ_S precede $\lambda_{\rm PV}$, because the Debye condition of Theorem 6.5 gives $\lambda_0 = \lambda_0(\kappa_S, \mu, C, \delta_{\rm GM})$ and each earlier constant uses $\lambda_{\rm PV}$ only through $\lambda_{\rm PV} \le \frac14$. Fourth, $\theta_{\rm NF}$ and $c_3^{\rm NF}$ are built from the pure number c'_0, and no earlier constant depends on them.

Each condition defining $\beta_0(K)$ is, at fixed K, a lower bound on $\tau = K\beta$ that persists for larger τ. Those of Section 2 are $\beta \ge \pi^{-2}$, $\beta \ge \beta_1(K)$ and $\delta < 1$, which (2.4) gives for all large β. In all the others the left side is a fixed function of τ (built from $\tilde m$, τ, $\log\tau$ and the doubly exponential scale $R_{\rm NF}$) tending monotonically to its limit. The value $\beta_0(K)$ is the largest of these thresholds divided by K. It is very large ($|\log\varepsilon''| \gtrsim 10^{14}$ in Theorem 5.1), and the constants are far from sharp, but the proof uses only that those fixed before K depend on τ at most through $\tilde m$, the scales of item 11 and $c_{\rm dep}$ (Proposition 6.17).

9 Corollaries

In this section we prove the two corollaries of the introduction. The central phases are described by a U(1) Wilson model with configuration-dependent couplings at most $n\beta$ (Section 1.2), also in the presence of a Higgs term invariant under the centre. Since the plaquettes of a finite 2-complex are attached along closed edge paths, U_p (the ordered product of the links around p) and θ_p are defined. We write Tr for the trace of $n \times n$ matrices.

Proposition 9.1 (Fröhlich's comparison)**.** *Let $G \subseteq U(n)$ be a closed subgroup with $\{zI : |z| = 1\} \subseteq G$, X a finite 2-complex, and $\mu_{G,\beta}(dU) \propto \prod_p \exp(\beta \operatorname{Re}\operatorname{Tr} U_p) \prod_e dU_e$, dU_e Haar measure on G (free boundary conditions). For every closed edge path ℓ in X,*

$$\Big|\Big\langle \operatorname{Tr}\prod_{e\in\ell} U_e \Big\rangle_{G,\beta}\Big| \le 2n \,\langle \cos\theta_\ell\rangle^W_{X,2n\beta}, \tag{9.1}$$

where the right side is the U(1) Wilson model on the same complex.

Proof. Let ν be the law on $U(1)^{E(X)} \times G^{E(X)}$ with density proportional to $\prod_p \exp(\beta \operatorname{Re}\operatorname{Tr}(\xi_p U_p))$, where $\xi_e = e^{i\phi_e}$ and $\xi_p = e^{i\phi_p}$ is the product of the $\xi_e^{\pm1}$ around p. Since $\xi_e I \in G$ is central, its image under $(\xi, U) \mapsto \xi U$ is $\mu_{G,\beta}$, and $\operatorname{Tr}\prod_{e\in\ell}(\xi_e U_e) = e^{i\phi_\ell}\operatorname{Tr}\prod_{e\in\ell} U_e$. Given U, ξ has the law $\langle\cdot\rangle_U$ with density proportional to $\prod_p \exp(\beta|\operatorname{Tr} U_p|\cos(\phi_p - \psi_p))$, where $\psi_p := -\arg\operatorname{Tr} U_p$. Hence $|\langle \operatorname{Tr}\prod_{e\in\ell} U_e\rangle_{G,\beta}| \le n \sup_U \sup_\alpha \langle\cos(\phi_\ell - \alpha)\rangle_U$. We bound $\langle\cos(\phi_\ell-\alpha)\rangle_U$ by $\langle\cos\theta_\ell\rangle^W_{X,n\beta}$ using [MMSP78, Proposition 1 and the remark after its proof, p. 21], with couplings $n\beta \ge \beta|\operatorname{Tr} U_p|$ on the plaquette vectors, where terms with the same vector up to sign are combined. The remark states this bound (p. 21) for every α, as a particular case of Proposition 1, which gives it only for α equal to the phase of the interaction; for each α it follows from the proof of Proposition 1, since for the edge vector M of ℓ and two copies ϕ, ϕ' of the angles $\cos(M\cdot\phi) - \cos(M\cdot\phi' - \alpha) = 2\sin(M\cdot\Phi - \frac{\alpha}{2})\sin(M\cdot\Phi' - \frac{\alpha}{2})$, where $\Phi = \frac12(\phi'+\phi)$ and $\Phi' = \frac12(\phi'-\phi)$. Consequently, the left side of (9.1) is at most $n\langle\cos\theta_\ell\rangle^W_{X,n\beta}$, which is at most the right side, since U(1) Wilson loops are nonnegative and nondecreasing in the coupling by Ginibre's inequality, on all finite complexes, as in Theorem 2.1(a), (b). The constants $2n$ and $2n\beta$ in (9.1) are those of [Frö79, (23), (24)].

Corollary 9.2 (Corollary 1.3)**.** *Let G be as in Proposition 9.1, with the Wilson action $\beta \operatorname{Re}\operatorname{Tr} U_p$ on finite boxes of $\mathbb{Z}^3$, ρ a representation of G with $\rho(zI) = 1$ for $|z| = 1$, (Φ, ρ_0) a probability space of Higgs field values, and $h \geq 0$ a measurable function of $((\rho(U_e))_e, (\varphi_x)_x)$, the Higgs weight, for any values of the Higgs couplings, with $0 < \int h < \infty$ ($h \equiv 1$ is the model without Higgs field). On a finite box Λ take the measure with density proportional to $h((\rho(U_e))_e, (\varphi_x)_x)$ times $\prod_p \exp(\beta \operatorname{Re}\operatorname{Tr} U_p)$ with respect to $\prod_e dU_e \prod_x \rho_0(d\varphi_x)$. Then* (9.1) *holds for every closed path ℓ, and the bound of Corollary 1.3 for every rectangle C with $\min(R, T) \geq R_1(2n\beta)$ and every finite box $\Lambda \supseteq C$, with σ, R_1, c of Theorem 1.2. In particular the string tension of G at coupling β is at least $\sigma(2n\beta) > 0$, also for $U(2)$ with an adjoint Higgs field, at all gauge and Higgs couplings.*

Proof. Since $\rho(\xi_e U_e) = \rho(\xi_e I)\rho(U_e) = \rho(U_e)$, the proof of Proposition 9.1 applies to the joint law with the extra factor $h((\rho(U_e))_e, \varphi) \prod_x \rho_0(d\varphi_x)$, which cancels from the conditional law of ξ. We then apply Theorem 1.2 at $2n\beta$. The case of $U(2)$ with an adjoint Higgs field is included, since $\operatorname{Ad}(zU) = \operatorname{Ad}(U)$. ([Frö79, Corollary 2] states confinement of static quarks for all $U(n)$ in three dimensions from the comparison and the results cited there; the area law is new.)

The Georgi–Glashow model. On a finite box Λ we take links $U_e \in SU(2)$, $U_{-e} = U_e^{-1}$, a unit adjoint Higgs field $\mathsf{n}_x \in S^2$ (the London limit), the adjoint representation $R : SU(2) \to SO(3)$, $\beta, \kappa > 0$, and

$$\mu_{\Lambda;\beta,\kappa} \propto \exp\Bigl(\beta \sum_p \tfrac{1}{2} \operatorname{Tr} U_p + \kappa \sum_{e=(x,y)} \mathsf{n}_x \cdot R(U_e)\mathsf{n}_y\Bigr) \prod_e dU_e \prod_x d\mathsf{n}_x \tag{9.2}$$

(dU_e Haar, $d\mathsf{n}_x$ uniform on S^2). The weight in (9.2) is invariant under $U_e \mapsto g_x U_e g_y^{-1}$, $\mathsf{n}_x \mapsto R(g_x)\mathsf{n}_x$ ($g \in SU(2)^\Lambda$), and $f(U, \mathsf{n})$ is *gauge invariant* if it is. Write $D(\theta)_e := \operatorname{diag}(e^{i\theta_e}, e^{-i\theta_e})$ and e_3 for the constant field $\mathsf{n}_x = e_3$.

Proposition 9.3 (Georgi–Glashow model at infinite Higgs coupling)**.** *For every finite box Λ, every $\beta > 0$ and every bounded continuous gauge invariant f, the limit $\lim_{\kappa\to\infty} \langle f \rangle_{\Lambda;\beta,\kappa}$ equals $\langle f(D(\theta), e_3) \rangle^W_{\Lambda,\beta}$, the expectation in the U(1) Wilson model at coupling β. In particular, for every closed path C, $\langle \frac{1}{2} \operatorname{Tr} \prod_{e\in C} U_e \rangle_{\Lambda;\beta,\kappa} \to \langle \cos\theta_C \rangle^W_{\Lambda,\beta}$.*

Proof. We pass to the unitary gauge. Since $R(SU(2))$ is transitive on S^2, gauge invariance gives $\iint F\, d\mathsf{n}\, dU = \int F(U, e_3)\, dU$ for $F = fw$ and $F = w$, where w is the weight of (9.2). With U_e of diagonal entries a_e, $\bar{a}_e$ and off-diagonal entries b_e, $-\bar{b}_e$, the hopping factor is $\prod_e e^{\kappa - 2\kappa|b_e|^2}$, Haar measure is b_e uniform on the unit disc times $\arg a_e = \theta_e$ uniform, and as $\kappa \to \infty$ the weights $e^{-2\kappa|b_e|^2}$ concentrate at $b = 0$, where $U_e = D(\theta)_e$ and $\frac{1}{2} \operatorname{Tr} U_p = \cos\theta_p$. The limit is taken in each fixed box, in the London limit $|\mathsf{n}_x| = 1$, and for $\frac{1}{2}\operatorname{Tr}$ (the full trace gives $2\cos\theta_C$).

Corollary 9.4 (Corollary 1.4)**.** *At $\kappa = \infty$, for every $\beta > 0$, every rectangle C with $\min(R, T) \geq R_1(\beta)$ and every finite box $\Lambda \supseteq C$, the limit of $\langle \frac{1}{2} \operatorname{Tr} \prod_{e\in C} U_e \rangle_{\Lambda;\beta,\kappa}$ as $\kappa \to \infty$ has absolute value $\langle \cos\theta_C \rangle^W_{\Lambda,\beta}$, which is at most $\exp(-\sigma(\beta)RT + c(R + T))$.*

Proof. This follows from Proposition 9.3, Theorem 2.1(a) and Theorem 1.2.

Remark 9.5 (Villain action and other models). The theorem of [GM82] holds in the form of Definition 1.1: for $\tau \geq K\beta_0(K)$ and $\beta := \tau/K$, Lemma 2.3(c),(d) as in Proposition 2.9 give $\langle \cos\theta_C \rangle^V_{\Lambda,\tau} \leq g^{\mathrm{fix}}_{\Lambda^\circ,\tau}(C;\emptyset) \leq \mathbb{E}_D g^{\mathrm{fix}}_{\Lambda^\circ,\tau}(C;D)$, Steps (1.5) to (1.8) give the bound of Theorem 1.2 at β, and Lemma 2.3(b), pointwise in Λ, gives smaller τ. The proof uses $d = 3$ throughout. The Georgi–Glashow model at finite Higgs and weak gauge coupling is not covered: the centre $\mathbb{Z}_2$ of $SU(2)$ contains no U(1), and [Frö79, Theorem 1] leads to the pure $\mathbb{Z}_2$ gauge theory, which confines only at strong coupling.

A The activities in tree form: proofs of Lemmas 7.37 and 7.48

In this appendix we prove Lemmas 7.37 and 7.48, carrying [BF80, §9.1] with [Bry78, §5] over to G'. The arguments are given here in condensed form, and Section S1.6 lists the structural facts about the expansion that the arguments use.[2] Proposition A.4 counts the sets with the tree weight $w(T)$ kept outermost, which cancels the moments $d_j!$ that the count produces.

Conventions. We assume that (U,k) is admissible and that $\sigma = (Z, N_Z)$ is a seed configuration in (U,k) (Section 7.6). By Proposition 7.46, at $k = 0$ the configuration sets may be those of long wall fields. We write $|Y|$ for the number of $\mathcal{U}$-cells of Y and set $\mathrm{dist}(A,B) := \tilde{m}\, d_{G'}(A,B)$. The $\mathcal{U}$-cell graph joins two $\mathcal{U}$-cells when a bond of G' joins them, and $D_{\mathcal{U}}$ bounds its degree; by (R7) we may take $D_{\mathcal{U}} := \frac{4\pi}{3}(2C_{\mathcal{U}}+2)^3/c_u$, a constant, since the $\mathcal{U}$-cells joined to a given one lie in the ball of radius $2C_{\mathcal{U}}\tilde{m}^{-1}+2$ about one of its sites and each has more than $c_u\tilde{m}^{-3}$ sites ($\tilde{m} \leq 1$).

The quantity $\mathcal{K}^{(Z)}_{N_X}(X)$ is the sum over $\bar{y} = (Y_1,\dots,Y_n)$, $Y_1 = Z$, of the terms of Proposition 7.22. For $2 \leq j \leq n$, *link* j is the operator $\kappa(j-1)[Y_j]$ of (C.6), which creates Y_j, differentiates s_{j-1}, and is a sum of elementary terms of the five types listed after (C.5). We also write $X_j := Y_1 \cup \dots \cup Y_j$. A *tree* is a map $T : \{2,\dots,n\} \to \mathbb{N}$ with $T(j) < j$. We put $d_i(T) := \#\{j : T(j) = i\}$. Thus $d_n(T) = 0$ and $\sum_i d_i(T) = n-1$. The weight of [BF80, (9.115)] is

$$q(T,s) := \prod_{j=2}^{n} \prod_{k=T(j)}^{j-2} s_k, \qquad w(T) := \int_{[0,1]^{n-1}} q(T,s)\, ds \tag{A.1}$$

since $\frac{d}{ds_{j-1}}(s_{T(j)} \cdots s_{j-1}) = s_{T(j)} \cdots s_{j-2}$. The variable s_{n-1} does not occur.

Lemma A.1 (Tree form)**.** *Let $X \supseteq Z$, $N_X \in \mathcal{N}_\sigma(X)$ and $\bar{y} = (Z, Y_2, \dots, Y_n)$, $n \geq 2$. Expand $\kappa(\bar{y},s)Q^\natural_{N_X}(X)$ into elementary terms $\theta = (\theta_2,\dots,\theta_n)$, θ_j a term of link j, and assign to θ a tree $T = T(\theta)$ and $\mathcal{U}$-cells $\Delta'_j \subseteq Y_{T(j)}$, $\Delta''_j \subseteq Y_j$: if θ_j is of type (2) to (5), it contains $\partial_{s_{j-1}}\mathbf{C}^{\bar{y}}(x,y)$ with $x \in Y_j$, $y \in Y_a$, $a < j$, and $T(j) := a$, Δ''_j, Δ'_j the $\mathcal{U}$-cells of x, y; if $\theta_j = (\partial_{s_{j-1}} w_s(\tau))e_\tau$ is of type (1), $T(j) := \min\{i : \mathrm{supp}\,\tau \cap Y_i \neq \emptyset\}$ and Δ'_j, Δ''_j are the first $\mathcal{U}$-cells, in a fixed order, of $\mathrm{supp}\,\tau$ in $Y_{T(j)}$ and in Y_j. Then*

$$(\theta\text{-term}) = q(T,s)\, R_\theta(s), \tag{A.2}$$

where $R_\theta(s)$ depends on s only through the weights $w_s(\tau') \in [0,1]$ of E-terms, through the covariance $\mathbf{C}^{\bar{y}}_s$ and through $e^{E(X,s)}$. Each θ receives exactly one tree.

[2] Of these facts, S6 is used in the proof of Lemma A.1, S5 in the paragraph on link data and S13 in item (4) of the proof of Proposition A.7.

Proof. We apply the tree decomposition [Bry78, (5.1), (5.2)] to the expansion of Proposition 7.22, with the following modifications. (1) We interpolate the covariance as in [GM82, App. E, (E.3)]: for $x \in Y_j$, $y \in Y_a$, $\partial_{s_{j-1}} \mathbf{C}^{\bar y}(x,y) = s_a \cdots s_{j-2}\, \mathbf{C}(x,y)$. (2) For the weights $w_s(\tau)$ we use the rule of [GM82, p. 89], under which $i \in I$ if $1 \le i \le n-1$ and τ has a cube $a_\alpha \subset Y_j$ for some $j > i$ and a cube $a_\beta \subset \bigcup_{\ell \le i} Y_\ell$. Under this rule the expansion is an identity, by the proof of Proposition 7.22, and $\partial_{s_{j-1}} w_s(\tau) = \prod_{k=T(j)}^{j-2} s_k$, as among $k \le j-1$ the set $\operatorname{supp}\tau$ meets both X_k and its complement exactly for $k \ge T(j)$. The rule stated in [BF80, p. 212] has Y_{i+1} in place of $\bigcup_{j>i} Y_j$. Under that rule a term meeting Y_1 and Y_3 but not Y_2 carries no factor s_1, and no tree weight dominates it. Finally, link j differentiates s_{j-1} and no other s_k, so each s_k is differentiated once, in link $k+1$.

As the bounds of Section 7.6 and Appendix B on $R_\theta(s)$ are uniform in s (fact S1 of Section S1.6), the weights lie in $[0,1]$ and $\int q(T,s)\,ds = w(T)$, grouping the elementary terms by their link data $\lambda = (\lambda_j)_{j\ge 2}$ (defined below; they determine the tree $T = T_\lambda$) gives

$$|\mathcal{K}^{(Z)}_{N_X}(X)| \le \sum_{\bar y} \sum_{\lambda} w(T_\lambda) \sup_s \sum_{\theta:\, \lambda(\theta)=\lambda} \Big| \int d\mu_s\, e^{E(X,s)} R_\theta(s)\, Q^\natural_{N_X}(X) \Big|_0, \tag{A.3}$$

with the supremum outside the sum over the terms with one link datum, which Proposition B.7 bounds together.

Lemma A.2. *For $n \ge 2$ and $y_1, \dots, y_n \ge 0$,*

$$\sum_T \int_{[0,1]^{n-1}} q(T,s) \prod_{j=2}^{n} y_{T(j)}\, ds \le e^{y_1 + \cdots + y_{n-1}}. \tag{A.4}$$

In particular $\sum_T w(T) \le e^{n-1}$.

Proof. This is [Bry78, (5.5)] and [BF77, Lemma 5.5], for real weights. The sum over T is $\prod_{m=1}^{n-1}(y_m + P_{m-1}(s))$ with $P_m := s_m(y_m + P_{m-1})$, $P_0 := 0$, and we integrate over $s_{n-2}, \dots, s_1$ in turn using $\int_0^1 x\, e^{sx}\, ds \le e^x$.

Lemma A.3. *For $n \ge 2$, $y_1 \ge 0$ and $b_2, \dots, b_{n-1} \in [0,1)$,*

$$\sum_T w(T)\, y_1^{d_1(T)} \prod_{i=2}^{n-1} d_i(T)!\, b_i^{d_i(T)} \le e^{y_1} \prod_{i=2}^{n-1} (1-b_i)^{-1}. \tag{A.5}$$

Proof. We use the identity $d!\, b^d = \int_0^\infty (bt)^d e^{-t}\, dt$ for each $i = 2, \dots, n-1$. Since $\prod_j y_{T(j)} = \prod_i y_i^{d_i(T)}$ and $d_n(T) = 0$, Tonelli's theorem and (A.4) with $y_i := b_i t_i$ ($2 \le i \le n-1$) bound the left side by $\int_{\mathbb{R}_+^{n-2}} e^{y_1 + \sum_i b_i t_i} e^{-\sum_i t_i}\, dt = e^{y_1} \prod_{i=2}^{n-1} (1-b_i)^{-1}$.

For a finite set A of $\mathcal{U}$-cells, $\mathfrak{Y}(A)$ is the set of nonempty unions Y of $\mathcal{U}$-cells each of whose components in the $\mathcal{U}$-cell graph meets A. For $a = \log(2eD_{\mathcal{U}}) + a' + a''$ with $a' \ge 4$, $a'' \ge 0$, and all integers $d \ge 0$,

$$\textstyle\sum_{Y \in \mathfrak{Y}(A)} |Y|^d\, e^{-a|Y|} \le e^{-a''}\, 2^{|A|}\, d!\, 2^{-d}\,: \tag{A.6}$$

if W contains each $\mathcal{U}$-cell independently with probability $p \in [0,1]$, then $p^{|Y|}(1-p)^{|\partial Y|}$, ∂Y the set of cells outside Y adjacent to Y, is the probability that $W \supseteq Y$ and $W \cap \partial Y = \emptyset$; for each W

at most $2^{|A|}$ sets $Y \in \mathfrak{Y}(A)$ have this property, since such a Y is the union of the components of W that meet $A \cap Y$. Hence $\sum_{Y \in \mathfrak{Y}(A)} p^{|Y|}(1-p)^{|\partial Y|} \le 2^{|A|}$, and with $|\partial Y| \le D_{\mathcal{U}}|Y|$ and $p = 1/(D_{\mathcal{U}}+1)$, at most $2^{|A|}(e(D_{\mathcal{U}}+1))^N \le 2^{|A|}(2eD_{\mathcal{U}})^N$ sets $Y \in \mathfrak{Y}(A)$ have $|Y| = N$. We conclude with $N^d e^{-a'N/2} \le d!\,(2/a')^d$.

Link data. The *link datum* λ_j of an elementary term θ_j consists of its type, of the number of points and the set of $\mathcal{U}$-cells containing them for each of its E-terms (at most two), and of the cells Δ'_j, Δ''_j of Lemma A.1. The set $A(\lambda_j)$ consists of Δ''_j and the $\mathcal{U}$-cells of its E-terms, and $\varphi \ge 0$ is a weight on link data with

$$C_\varphi := \sup_{\Delta_0} \sum_{\lambda:\, \Delta' = \Delta_0} \varphi(\lambda)\, 2^{|A(\lambda)|} < \infty. \tag{A.7}$$

Each Y_j ($j \ge 2$) of a term with $\theta_j \ne 0$ lies in $\mathfrak{Y}(A(\lambda_j))$. Indeed, each element is connected in the $\mathcal{U}$-cell graph, since a wall component is a component of $\hat{\Sigma}$ and, by Definition 7.20, a closure adds bond neighbours and linked closures share a $\mathcal{U}$-cell. Each element of Y_j also meets $S(\theta_j) \setminus X_{j-1}$, because link j creates Y_j as the union of the elements that meet this set. This set lies in the cells of $A(\lambda_j)$, as the point y of a covariance line lies in X_{j-1}.

Proposition A.4. *Let a be as in* (A.6)*, with $2C_\varphi e^{-a''} \le \frac{1}{2}$. For every seed configuration $\sigma = (Z, N_Z)$,*

$$\sum_{n \ge 1} \sum_{T} \sum_{(Y_j, \lambda_j)_{j=2}^{n}} w(T) \prod_{j=2}^{n} \varphi(\lambda_j)\, e^{-a|Y_j|} \le 2e^{|Z|}, \tag{A.8}$$

where $Y_1 := Z$, the sum runs over all trees T, all link data with $\Delta'_j \subseteq Y_{T(j)}$, and all $Y_j \in \mathfrak{Y}(A(\lambda_j))$ (the Y_j may overlap, and $Y_j \subseteq U$ need not hold), and the term $n = 1$ equals 1.

Proof. We keep the tree outermost with its weight; each set is rooted at its own link, and each link in an earlier set. The roots in earlier sets produce the moments of Lemma A.3. We fix $n \ge 2$ and T and sum in the order $(Y_n, \lambda_n), \dots, (Y_2, \lambda_2)$, innermost first, and inside stage j first over Y_j, then over λ_j with Δ'_j fixed, then over $\Delta'_j \subseteq Y_{T(j)}$. In this order every constraint ties a variable only to variables summed further out.

We claim that at stage j the summand depends on Y_j only through $|Y_j|^{d_j(T)} e^{-a|Y_j|}$ and the constraint $Y_j \in \mathfrak{Y}(A(\lambda_j))$. At stage j, (A.6) gives $\sum_{Y_j \in \mathfrak{Y}(A(\lambda_j))} |Y_j|^{d_j} e^{-a|Y_j|} \le e^{-a''} 2^{|A(\lambda_j)|} d_j!\, 2^{-d_j}$. The sum over λ_j with Δ'_j fixed then gives at most $C_\varphi e^{-a''} d_j!\, 2^{-d_j}$ by (A.7), and the remaining sum contributes $\sum_{\Delta'_j \subseteq Y_{T(j)}} 1 = |Y_{T(j)}|$, a factor that is handled at stage $T(j) < j$. This preserves the claim for the stages that follow.

After stage 2 the factor $|Z|^{d_1(T)}$ remains, and the sum over $(Y_j, \lambda_j)_{j \ge 2}$ is at most $(C_\varphi e^{-a''})^{n-1} |Z|^{d_1(T)} \prod_{j=2}^{n-1} d_j(T)!\, 2^{-d_j(T)}$. Multiplying by $w(T)$ and summing over T, we obtain the bound $(C_\varphi e^{-a''})^{n-1} e^{|Z|} 2^{n-2}$ from Lemma A.3 with $y_1 = |Z|$ and $b_i = \frac{1}{2}$. Finally $1 + e^{|Z|} \sum_{n \ge 2} (C_\varphi e^{-a''})^{n-1} 2^{n-2} \le 1 + \frac{1}{2} e^{|Z|} \le 2e^{|Z|}$.

Remark A.5 (The printed deduction). In [Bry78, p. 329] the bound corresponding to Proposition A.4, (5.4), is deduced from (5.5), $\sum_T \int ds\, q(T, s, \bar{y}) \le e^{|X|}$ (trees at fixed sets), and (5.6) (sets at fixed tree, with the supremum over trees outside). Writing the summand of (5.4) as $f(T, \bar{y}) g(T, \bar{y})$ with $f := \int ds\, q(T, s, \bar{y}) e^{-|X|}$, the deduction uses

$$\sum_{T, \bar{y}} f(T, \bar{y})\, g(T, \bar{y}) \le \Big(\sup_{\bar{y}} \sum_{T} f(T, \bar{y}) \Big) \Big(\sup_{T} \sum_{\bar{y}} g(T, \bar{y}) \Big),$$

which fails for general nonnegative f, g: for $T, \bar{y}$ ranging over $\{1, \dots, N\}$ and $f = g = \delta_{T,\bar{y}}$ the left side is N and the right side 1. The conclusion holds with a different per-cell constant, by Proposition A.4, or by keeping (5.6) and using $\sum_{T,\bar{y}} fg \le (\sum_T \sup_{\bar{y}} f e^{-\epsilon|X|}) \sup_T \sum_{\bar{y}} g e^{\epsilon|X|}$, whose first factor is at most $((1+\epsilon)/\epsilon)^{n-1}$ by Lemma A.3 and whose second is (5.6) with c_5 raised by 2ϵ.

In [BF80, p. 215] the joint choice for 5Σ and 6Σ cites "(5.6) of [1] or Lemma 5.5 of [2]". The inequality used there is [Bry78, (5.5)], which [BF77, Lemma 5.5] states at fixed weights, and the conclusion of [BF80, Lemma 9.1] holds. In [BF80, pp. 213–217] the sums are listed in the inverse order in which they are performed, so the constraint of 3Σ, that the set contain one of the cubes 'mentioned in 10Σ', refers to an inner sum; in the count of Proposition A.4 each constraint links a variable only to variables outer to it.

For a link datum λ, $O_0(\lambda)$ is the number of points of its E-terms and $L_0(\lambda)$ the sum over its E-terms of half the dist-length of a shortest tree spanning their $\mathcal{U}$-cells ($L_0 = O_0 = 0$ for type (2)). An E-term with cells in two components of G' has coefficient 0 by Proposition 6.3(d), and is absent. We put

$$\varphi(\lambda) := e^{-\delta_2 \mathbf{1}_{\text{line}}(\lambda) \operatorname{dist}(\Delta', \Delta'')} e^{-c_{10} L_0(\lambda)} e^{-c''_{10} O_0(\lambda)}, \qquad c_\gamma := \sup_{\Delta_0} \sum_{\Delta} e^{-\gamma \operatorname{dist}(\Delta_0, \Delta)} \quad (\gamma > 0); \tag{A.9}$$

where $\mathbf{1}_{\text{line}}(\lambda) = 1$ if λ is of a type with a covariance line (types (2) to (5), Lemma A.1) and $\mathbf{1}_{\text{line}}(\lambda) = 0$ for type (1); $c_\gamma < \infty$, since by (R7) at most $\frac{4\pi}{3}(r + 2C_{\mathcal{U}} + 1)^3 / c_u$ $\mathcal{U}$-cells lie within distance r of a given one.

Lemma A.6 (Link sum)**.** *Let $c_{10}, \delta_2 > 0$ and $e^{c''_{10}} \ge 16e \max(1, c_{c_{10}/2})$. Then $C_\varphi \le 1 + 8c_{\delta_2} < \infty$. Moreover, for every E-term $\mathcal{E}$ of λ, with $\mathfrak{a}$ the set of $\mathcal{U}$-cells of its points, and every augmented tree η^A of [BF80, Appendix 1] whose marked points lie in the cells of $\mathfrak{a}$, one in each, the contribution of $\mathcal{E}$ to $L_0(\lambda)$ is at most L_{η^A}; so $e^{c_{10} L_0(\lambda)} e^{c''_{10} O_0(\lambda)}$ is dominated, inside $|\cdot|_0$, by replacing each E-term $\mathcal{E}$ by $\sum_{\eta^A} e^{c_{10} L_{\eta^A}} e^{c''_{10} t} \mathcal{E}_{\eta^A}$, for any decomposition $\mathcal{E} = \sum_{\eta^A} \mathcal{E}_{\eta^A}$ ([BF80, (9.18), (9.18a)]). Which point of $\mathcal{E}$ lies in which cell of $\mathfrak{a}$, the positions and the charges are not part of λ; they are summed inside $|\cdot|_0$.*

Proof. The argument is that of 8Σ and 10Σ of [BF80, p. 217], with the choice $B_8 B_{10} = e^{c''_{10} O_0} e^{c_{10} L_0}$, together with (9.18) and the lengths [BF80, (A1.3)], adapted as follows.

(1) Positions are sets of $\mathcal{U}$-cells. We count a set of k cells once for its $(k-1)!$ orderings, and the tree sum gives $e^k c_{c_{10}/2}^{k-1}$ (by Cayley's formula, k^{k-2} labelled trees on k vertices, and $k^{k-2}/(k-1)! \le e^k$). The terms with the same link datum differ in the assignment of the points of each E-term to the cells of its set, in their positions and in their charges, and for $t = 2$ also in the form. Proposition B.7 bounds their sum through the bound (M4) of Lemma B.1(d), which holds for the sum over all assignments. This is needed because a set of k cells carries $k!\, S(t,k)$ assignments of t points, where S denotes the Stirling numbers of the second kind, and $t!$ assignments if $k = t$.

(2) The factor $2^{|A(\lambda)|}$ costs 4^k per E-term with a marked cell. This raises by $\log 2$ the constant c''_{10}; [BF80, p. 217] gives no value for it. The cell Δ'' costs $2c_{\delta_2}$ in each of the four types with a covariance line. Hence $C_\varphi \le 1 + 8c_{\delta_2}$.

(3) The length L_{η^A} is an infimum over the positions of the marked points in the cells of $\mathfrak{a}$ and of the auxiliary points, measured with dist, and a tree on given points has at least half the length of a shortest tree spanning them, which dominates one spanning their cells.

Hence $L_0 \le L_{\eta^A}$, and $e^{c_{10}L_0}|\mathcal{E}| \le \sum_{\eta^A} e^{c_{10}L_{\eta^A}}|\mathcal{E}_{\eta^A}|$ as $\mathcal{E} = \sum_{\eta^A} \mathcal{E}_{\eta^A}$. Under the conditions $c_{10} \le \frac{7}{9} r_C$ and $\mu \ge 1 + 2c''_{10}$, whose constants are fixed in the order given in Section S1.8, the proof of Proposition B.7 absorbs the factors $e^{c_{10}L_{\eta^A}} e^{c''_{10}t}$ of κ' when it bounds the weights of the augmented trees.

In the count of [GM82, (7.3) to (7.5)] the Mayer factors $E(a_1, \dots, a_t)$ keep the ordered tuple of cubes $(a_1, \dots, a_t)$ among the summed parameters, with a weight exponential in the least tree length $L(a_1, \dots, a_t)$ of [GM82, (1.11b)]. This count is that of [BF80, 10Σ]. There, on [BF80, p. 214], the positions of the cubes are summed at a fixed augmented tree η^A, with the constant weights b_{η^A} of [BF80, (A1.4)]. With the least tree length in place of these weights, the counting step needs a coefficient bound for the sum over all positions of the points with their tree length inside the sum. The derivation of [GM82, (4.31)] provides it through the pinned norm [GM82, (4.12)]. Theorem 6.5 states (S2) in this form, and the count above uses it as explained in Lemma A.6 and its proof.

Proposition A.7 (Lemma 7.37 on G')**.** *Lemma 7.37 holds, with C_φ the constant of Lemma A.6. The same proof gives the bound with $e^{\alpha|X|}$ replaced on the left by $e^{\alpha|X| + \varepsilon F_1^{\mathrm{loc}}(X, N_X)}$, $\varepsilon \ge 0$, and $e^{c'_1 F_1^{\mathrm{loc}}}$ replaced on the right by $e^{(c'_1+\varepsilon)F_1^{\mathrm{loc}}}$.*

Proof. We follow the proof of [BF80, Lemma 9.1] by the device of [BF80, (9.12)], the bound $\sum_v |f(v)| \le (\sum_v B(v)^{-1}) \sup_v |B(v)f(v)|$ over the data $v = (n, T, \bar{y}, (\lambda_j)_{j\ge 2}, N_X)$ of (A.3), with $B(v) := e^{a|X\setminus Z|} e^{c'_1 F_1^{\mathrm{loc}}(X,N_X)} \prod_j \varphi(\lambda_j)^{-1}$, $a := \log(2eD_{\mathcal{U}}) + 4 + c'_4 + a''$, $e^{a''} \ge 4C_\varphi$. Terms with the same link data differ only inside $|\cdot|_0$, namely in the maps l of Lemma B.2 and in the data of the E-terms listed in the proof of Lemma A.6, and their sum is bounded as there. We change the argument at the following points.

(1) For the count of the sets we use Proposition A.4 with $a' = 4$, after summing N_X innermost by Lemma 7.30(b) and keeping only $Y_j \in \mathfrak{Y}(A(\lambda_j))$. Thus $\sum_v w(T)B(v)^{-1} \le e^{(1+c'_4+\log 2)|Z|} \le e^{a|Z|}$, the supremum carries $e^{c_2^L|X|}$, $c_2^L := a + \alpha$, and Lemma A.6 turns $\prod_j e^{c_{10}L_0(\lambda_j) + c''_{10}O_0(\lambda_j)}$ and R_θ into κ'. (2) The seed is fixed and enters through $|Z|$ and $\mathcal{N}_\sigma(X)$ alone, not through its connectivity. Seeds depending on the configuration are reduced to fixed ones in Proposition A.8. (3) No step refers to the set X_1 of Definition 7.20(c). (4) For $U = X_0^c$ the activity is independent of the region, since it depends only on the sets of the sequence, the block of the covariance on their union and the configuration there. (5) We use only the distance-zero form ($W \subseteq Z$ or $W \supseteq Z$, where the restriction $X \cap W \ne \emptyset$ is vacuous), since on G' a $\mathcal{U}$-cell can have $d_{G'}$-diameter of order L' and the clustering form [BF80, (9.122)] would therefore cost $e^{O(\tilde{m}L')} = e^{O(\tau)}$ per non-seed cell. As $F_1^{\mathrm{loc}}(X, N_X)$ depends neither on ψ nor on s, we may move $e^{\varepsilon F_1^{\mathrm{loc}}}$ into the supremum.

For an anchor S_0, $\Sigma_{\mathrm{sat}}(S_0)$ is the set of seed configurations $\sigma = (Z, N_Z)$ with $Z \supseteq S_0$ each element of which meets S_0. We write $F_1^{\mathrm{loc}}(\mathcal{S}, N_{\mathcal{S}})$ for the sum of $F_1(S_\beta)$ over the wall components of a wall polymer $\mathcal{S}$, D_I for the largest number of interfaces within Euclidean distance g of a given interface or site, and $(\beta 1)$, $(\beta 3)$, $(\beta 4)$ for the conditions (S1.5) on β^0_{GM} of Section S1.8.

Proposition A.8 (Core estimate)**.** *Let (U, k) be admissible, $\beta_{\mathrm{GM}} \ge \beta^0_{\mathrm{GM}}$ and $\varepsilon \in [0, c'/2]$.*

For every seed configuration $\sigma = (Z, N_Z)$ in (U, k),

$$\sum_{\substack{X \supseteq Z \\ X \subseteq U}} \sum_{N_X \in \mathcal{N}_\sigma(X)} |\mathcal{K}^{(Z)}_{N_X}(X)|\, e^{\alpha|X| + \varepsilon F_1^{\mathrm{loc}}(X, N_X)} \le \mathcal{P}'\, e^{c_S |Z^o|}\, e^{-(11c'/16 - \varepsilon) F_1^{\mathrm{loc}}(Z, N_Z)}, \tag{A.10}$$

and the sum restricted to $X \neq Z$ is at most c_{16}^D times the right side. Consequently Lemma 7.48 holds.

Proof. The proof combines the per-cell bound of [BF80, §9.8], applied after Lemma 7.37 with the factor $e^{-F_1^{\mathrm{loc}}}$ divided region by region, with a split of the sum by the first set.

(1) *Booking.* Proposition A.7, with $c_1' = c'/4$ and the extraction $e^{\varepsilon F_1^{\mathrm{loc}}}$, together with Proposition 7.35, Lemma 7.38 and Proposition 7.39, leaves $e^{-(3c'/4 - \varepsilon) F_1^{\mathrm{loc}}(X)}$, whose part on Z is constant on $\mathcal{N}_\sigma(X)$. We divide this factor between the two regions as in Remark S1.1 of the supplement. On $X \setminus Z$ we use $c'/16$ for $|D|$, $|F_2'|$ and $c'/8$ for the wall polymers, and on Z we use $c'/32$ for $|D|$, $|F_2'|$ and $c'/32$ for the cells of Z^Σ under $(\beta 1)$. Then each non-seed cell carries at most $\varepsilon_\Delta \le \frac{1}{2} c_{16}^D \le \frac{1}{2}$ by $(\beta 4)$, lone seed cells carry at most e^{c_S}, and the fraction $c' - c'/4 - \varepsilon - c'/32 - c'/32 = 11c'/16 - \varepsilon$ remains on the seed. This gives (A.10). Terms with $X \neq Z$ contain a non-seed cell and gain $\frac{1}{2} c_{16}^D$.

(2) *Splitting.* By Lemma 7.23, the map $(X, N_X) \mapsto (\sigma(N_X), X, N_X)$, $\sigma(N_X) := (Z, N_X|_{E(Z)})$ with $Z := \mathrm{sat}_{N_X}(S_0)$, is a bijection onto the triples with $\sigma \in \Sigma_{\mathrm{sat}}(S_0)$, $X \supseteq Z$, $N_X \in \mathcal{N}_\sigma(X)$, preserving the summands.

(3) *Entropy of the first set.* Section S2.7 gives this step in detail. As $11c'/16 - \varepsilon \ge c'/8$, the left side of Lemma 7.48(a) is at most $\mathcal{P}' e^{c_S |S_0|} \sum_\sigma \prod_{\mathcal{S}} e^{-(c'/8) F_1^{\mathrm{loc}}(\mathcal{S}, N_{\mathcal{S}})}$. We assign the wall polymers of σ to their first cells in S_0 and count them by their wall interfaces and values with Lemmas 5.8 and 7.30(a). Under $(\beta 3)$ this gives the factor $1 + \eta_{\mathrm{seed}}$ per cell of S_0. From this we obtain (a), then (b), and (c′) with $\mathcal{P}' = 1$, $Z^o = \emptyset$.

B Per-cell bounds: proofs of Lemma 7.38 and Proposition 7.39

In this appendix we prove Lemma 7.38 and Proposition 7.39, the bookkeeping of [BF80, §9.8, (9.81) to (9.824)] on G', again in condensed form. In Debye units ($b_D := \beta_{\mathrm{GM}} \tilde{m}$) each field derivative carries $b_D^{1/2}$ and each coefficient $1/b_D$. A lone cell is of class A, and by the minimality of [GM82, (E.11)] it contains a link end or a Mayer cube, which gives it net weight $\ge \frac{1}{6}$. The cells of X_Σ are controlled by F_1^{loc}, and the factorials by pinning. Since $\mathcal{O}((1+r)^3)$ cells lie within distance r of a cell, a sum of N distances grows like $N^{4/3}$, which grows faster than $\log((N!)^{q_{\mathrm{pin}}})$.

Setting. Cells are $\mathcal{U}$-cells, grid cells are denoted Ω, and $\mathrm{vol}_D(\Delta) := \tilde{m}^3 \#\Delta \in [c_u, C_u]$. Classes A, B and $X = Z \sqcup X_\Sigma \sqcup X_o$ are those of Section 7.6 (seed cells in wall polymers count with X_Σ). On a class-A cell $\vartheta = \psi$ modulo $P\mathbb{Z}$ in each phase. In the seeded activity of Section 7.6, $X_i := Y_1 \cup \cdots \cup Y_i$ and $\kappa(\bar{y}, s) = \kappa(n-1) \cdots \kappa(1)$ as in [GM82, (E.9), (E.10)] and Appendix C. When its brackets are expanded, $\kappa(i)$ is of one of five *kinds*: K0 $= \partial_{s_i} E^{(i)}$, K1 $= \dot{C} \partial_x \partial_y$, K2 $= \dot{C} \partial_x (\partial_y E)$, K3 $= \dot{C} (\partial_x E) \partial_y$ and K4 $= \dot{C} (\partial_x E)(\partial_y E)$. Here $\partial_x := \partial / \partial \psi(x)$, $x \in Y_{i+1}$ and $y \in X_i$. An *$\mathcal{E}$-factor* is one Mayer term of $E(X, s)$ of the forms [GM82, (E.5a) to (E.5d)], with its coefficient ρ_t and its t factors. Derivatives act on the factors to the right of the differentiation, as in [GM82, (E.13) to (E.16)], which is the rule of [BF80, p. 229].

The ψ-dependent *atoms* of $Q^\natural(X)$ are: (a1) r_Ω of (7.4), for each grid cell $\Omega \subseteq X$; (a2) $e^{G_2(\Omega)}$, $G_2(\Omega) := G_{2,1}(\Omega)+G_{2,2}(\Omega)$; (a3) $e^{g^{\mathfrak{D}}_z}$, $z \in X \cap \mathfrak{D}$; (a4) $e^{g^\natural_z}$, $z \in X \setminus \mathfrak{D}$; (a5) $e^{-F_2(X)} = e^{-\langle 1_X \psi, F_2' \rangle}$. Here $g^{\mathfrak{D}}_z$, $g^\natural_z$ are the z-terms of $G^{\mathfrak{D}}_{2,3}$, $G^\natural_{2,3}$ and, pairing m with $-m$ by (S3),

$$G_{2,1}(\Omega) = \sum_{z\in\Omega}\sum_m \rho^{\mathrm{ref}}(m) f_2(m\beta_{\mathrm{GM}}^{1/2}\tilde\delta(z)), \qquad G_{2,2}(\Omega) = \sum_{z\in\Omega}\sum_m \rho^{\mathrm{ref}}(m) v_m(\Omega) f_1(m\beta_{\mathrm{GM}}^{1/2}\tilde\delta(z)), \tag{B.1}$$

with $f_1(y) := e^{iy} - 1 - iy$, $f_2(y) := \cos y - 1 + \frac{1}{2}y^2$ and $v_m(\Omega) := e^{im\beta_{\mathrm{GM}}^{1/2}\mathrm{avg}_\Omega \vartheta} - 1$. Thus $G_{2,1}$ is of fourth order in $\tilde\delta$. The factors of an $\mathcal{E}$-factor are $\varepsilon(\xi) = e^{im\beta_{\mathrm{GM}}^{1/2}(\vartheta(x)+h(x))} - 1$ in (E.5a), $\tilde\varepsilon(\xi) := \varepsilon(\xi) - im\beta_{\mathrm{GM}}^{1/2}\psi(x)$ in (E.5b) to (E.5d), and the *ψ-factor* $im\beta_{\mathrm{GM}}^{1/2}\psi(x)$ in (E.5c), (E.5d).

Lemma B.1 (Debye units)**.** *Put $d\hat x := \tilde m^3 \times$ counting measure on V, $\hat\psi := \tilde m^{-1/2}\psi$, $\delta/\delta\hat\psi(\hat x) := \tilde m^{-5/2}\partial/\partial\psi(x)$, $\hat C(\hat x,\hat y) := \tilde m^{-1}C'_s(x,y)$ (likewise for $\partial_{s_i}C'_s$), and $\hat c := \tilde m^{-3}c$ for a coefficient $c(x)$ per site. Then (a) $\sum_{x,y} K(x,y)\partial_{\psi(x)}\partial_{\psi(y)} = \iint d\hat x\, d\hat y\, \tilde m^{-1}K(x,y)\frac{\delta}{\delta\hat\psi(\hat x)}\frac{\delta}{\delta\hat\psi(\hat y)}$; (b) if $F(\psi) = \sum_z c(z) f(\beta_{\mathrm{GM}}^{1/2}\psi(z))$, then $\delta F/\delta\hat\psi(\hat x) = \hat c(\hat x)\, b_D^{1/2} f'(b_D^{1/2}\hat\psi(\hat x))$: each derivative of a function of $\beta_{\mathrm{GM}}^{1/2}\psi$ produces exactly $b_D^{1/2}$; (c) $|\hat C(\hat x,\hat y)| \le (2K_1''/\lambda_{\mathrm{PV}})e^{-r_{\mathrm{cov}}d(x,y)}$, $\hat C(\hat x,\hat x) \le \alpha_D K_1/\lambda_{\mathrm{PV}}$ and $\sup_{\hat x}\int d\hat y\, |\hat C(\hat x,\hat y)|e^{r_{\mathrm{cov}}d(x,y)} \le 2K_3$; (d) for a cell Δ, $k \ge 2$ and $C_k \le Ck!(2\kappa_S)^{-k}$: (M1) $\int_\Delta d\hat x \sum_m |\hat\rho^{\mathrm{ref}}(m)||m|^k \le C_k \mathrm{vol}_D(\Delta)/b_D$; (M2) $\int_{\mathfrak{D}\cap\Delta} d\hat x \sum_m |\hat\rho_1 - \hat\rho^{\mathrm{ref}}||m|^k \le C_k c_{\mathrm{dep}}$; (M3) $\int_{\Delta\setminus\mathfrak{D}} d\hat x \sum_m |\hat\rho_1 - \hat\rho^{\mathrm{ref}}||m|^k \le C_k\theta'\mathrm{vol}_D(\Delta)/b_D$; (M4) for $t \ge 2$, a set $\mathfrak{a}$ of cells and a cell $a_* \in \mathfrak{a}$,*

$$\frac{1}{t!}\int_{\Xi_t(\mathfrak{a})} d\hat x_1 \cdots d\hat x_t \sum_{m_1,\dots,m_t} |\hat\rho_t|\, e^{2\kappa_S\sum_i |m_i|} \le t\, K_{M4} C_{M4}^t \,\frac{4\,\mathrm{vol}_D(a_*)}{b_D}\, e^{-\mu(t-1)} e^{-(1-\delta_{\mathrm{GM}})(A/\tilde m)\hat L(\mathfrak{a})},$$

and hence, for all integers $k_1,\dots,k_t \ge 0$, the left side with $\prod_i |m_i|^{k_i}$ in place of $e^{2\kappa_S\sum_i|m_i|}$ is at most $\prod_i k_i!(2\kappa_S)^{-k_i}$ times the right side. Here $\Xi_t(\mathfrak{a})$ is the set of $(x_1,\dots,x_t) \in V^t$ such that $\mathfrak{a}$ is the set of the cells that contain some x_i, A is the rate of (S2), $\hat L(\mathfrak{a}) := \tilde m L_{G'}(\mathfrak{a})$ is the tree length of (S2) of the cells of $\mathfrak{a}$, each listed once, in Debye units, $C_{M4} := (1-C)^{-1}$ per point and $K_{M4} := e^{2\kappa_S}(1-C)^{-10}$, where C is the parameter of (S2), fixed in $(0,\frac14]$ (item 12 of the list in Section S1.1.1). The left side is the sum over all assignments of the t points to the cells of $\mathfrak{a}$ that use every cell; it contains $\int_{a_1}\cdots\int_{a_t}$ for each t-tuple of cells with $\{a_1,\dots,a_t\} = \mathfrak{a}$.

Proof. Parts (a) and (b) follow from the change of variables, since $\tilde m^{-5/2}\beta_{\mathrm{GM}}^{1/2} = \tilde m^{-3}b_D^{1/2}$. Part (c) is Theorem 6.11 for $C' \le \alpha_D C_0$, which holds by Lemma 7.29, with $M_Y/\tilde m = 1/\lambda_{\mathrm{PV}}$ and entrywise domination of C'_s. Bound (M1) follows from $\beta_{\mathrm{GM}}\sum_m m^2\rho^{\mathrm{ref}}(m) = \tilde m^2$ and (S4′). Bounds (M2) and (M3) follow from Propositions 6.17, 6.6 and (7.26). We derive (M4) from (S2). As $e^{2\kappa_S\sum_i|m_i|} = e^{2\kappa_S}e^{2\kappa_S(\sum_i|m_i|-1)}$, the left side is $e^{2\kappa_S}\frac{1}{t!}\sum_{x\in\Xi_t(\mathfrak{a})}\sum_m |\rho_t| e^{2\kappa_S(-1+\sum_i|m_i|)}$, since in Debye units $\int d\hat x\,|\hat\rho| = \sum_x|\rho|$. Each $x \in \Xi_t(\mathfrak{a})$ has a coordinate in a_*. As ρ_t, the weight and $\Xi_t(\mathfrak{a})$ are symmetric, the sum is at most t times its part with $x_1 \in a_*$. For $x \in \Xi_t(\mathfrak{a})$, $L_{G'}(x_1,\dots,x_t) \ge L_{G'}(\mathfrak{a})$. Indeed, in a tree for $x_1,\dots,x_t$ we keep one marked vertex in each cell of $\mathfrak{a}$ and declare the others auxiliary. Hence the part is at most $e^{-(1-\delta_{\mathrm{GM}})AL_{G'}(\mathfrak{a})}$ times the left side of (6.5) with $s = t$ and $a_1 = a_*$, and (S2), $(1-C)^{-t-10} = (1-C)^{-10}C_{M4}^t$ and $|a_*|m_D^2/\beta_{\mathrm{GM}} \le 4\,\mathrm{vol}_D(a_*)/b_D$, which holds as $m_D \le 2\tilde m$, give (M4). Here the sum over the $|\mathfrak{a}|!\,S(t,|\mathfrak{a}|)$ assignments (S as in the proof of Lemma A.6) is a part of the sum over all ordered tuples of points in (S2), and the decay

is read off the weight on that part, so the sum over these assignments is bounded by (M4) without a factorial. The form with $\prod_i |m_i|^{k_i}$ follows from $|m|^k \le k!(2\kappa_S)^{-k} e^{2\kappa_S|m|}$.

Thus the expansion is as in [BF80, §9.8], with $\tilde{l}_D = 1$, $\beta \to b_D$, on a fine lattice with a regularized covariance, and infinitely many charges only produce the $k!$ in (d). A factor f has *weight* w if $|f| \le K b_D^w \cdot$ (field monomial) $\cdot$ (numerical factors) for all positions in their cells and all fields, with K uniform in b_D, L' and τ. Here a field monomial is a product of values $|\hat{\psi}(\hat{x})|$ and averages $\mathrm{avg}_\Omega|\hat{\psi}|$, paid by the Wick bound T_1 (the Wick bound gives a constant c_T, independent of b_D, L', τ, per field factor, together with the factorials $\prod_j n_j!$ counted in Lemma B.9). A numerical factor is $|\delta\hat{D}(\hat{x})| := |\hat{D}(\hat{x}) - \mathrm{avg}_\Omega \hat{D}|$ or $|\hat{F}_2'(\hat{x})|$, with $\hat{D} := \tilde{m}^{-1/2} D$, $\hat{F}_2' := \tilde{m}^{-5/2} F_2'$. The weight of a term is the sum of those of its factors.

Lemma B.2. *Expand every derivative of $\kappa(\bar{y}, s)$ by the Leibniz rule, as in [BF80, (9.84), (9.85)]. (a) $\kappa(\bar{y}, s) Q^\natural(X)$ is a finite sum of terms, each labelled by the kinds, the $\mathcal{E}$-factors, and a map l that sends each derivative to one atom or one factor of an $\mathcal{E}$-factor which depends on the derivative's point and lies to its right; the term is $[\prod_i \partial_{s_i} C'(x_i, y_i|s)] \prod_a \partial^{l^{-1}(a)} a \prod_e [$coefficient of $e \cdot \prod_{\phi \in e} \partial^{l^{-1}(\phi)} \phi]$. (b) The multiple derivatives are of the types: i) $\partial^S r_\Omega$; ii) from an observable (absent here); iii) $\partial^S e^{-F_2} = (-1)^{|S|} \prod_{j \in S} F_2'(x_j) e^{-F_2}$; iv) $\partial^S \varepsilon$, $\partial^S \tilde{\varepsilon}$, $\partial^S (im\beta_{\mathrm{GM}}^{1/2}\psi)$; v), vi) the Faà di Bruno blocks $\rho^{\mathrm{ref}}(m)\partial^B[v_m f_1]$ and $\rho^{\mathrm{ref}}(m)\partial^B[f_2]$ of $\partial^S e^{G_2(\Omega)}$; vii), viii) the Faà di Bruno blocks of $\partial^S e^{g_z^{\mathfrak{D}}}$ and $\partial^S e^{g_z^\natural}$. (c) With n_α link ends, m_α factors of $\mathcal{E}$-factors and w_α ($= 0$ here) factors of an observable in the cell Δ_α, the number of maps l is at most $\prod_\alpha (m_\alpha + w_\alpha + 5)^{n_\alpha}$, and for $c_p > 0$, $n \ge 1$ and integers $m, w \ge 0$, $(m+w+5)^n \le C_r^{n+w} n!\, e^{c_p m}$ with $C_r := 4e^5 \max(1, 1/c_p)$.*

Proof. The lemma extends [BF80, (9.84) to (9.87), Lemma 9.10] by two new types, vii) and viii). A bare derivative at $x \in \Delta_\alpha$ can hit $r_{\Omega(x)}$, $e^{G_2(\Omega(x))}$, $e^{g_x^{\mathfrak{D}}}$ or $e^{g_x^\natural}$, e^{-F_2}, or one of the $m_\alpha + w_\alpha$ factors in Δ_α. The count therefore has $+5$ in place of the $+3$ of [BF80, (9.86)]. The undifferentiated exponentials recombine into $e^{G^\natural(X)}$ and e^{-F_2}, which stay in T_4 and T_2 of Proposition 7.35. We bound each r_Ω by Lemma 7.31, whose factor is included in T_3. The last inequality follows from $(m + w + 5)^n \le 2^n (m^n + (w+5)^n)$, $m^n \le n! c_p^{-n} e^{c_p m}$ and $(w+5)^n \le n! e^{w+5}$. The factor $n!$ is controlled by the pinning of Lemma B.4. Within a fixed l the further expansions, namely Faà di Bruno, the Leibniz rule inside $v_m f_1$, the two parts of $\partial\tilde{\delta}(z)/\partial\psi(x)$ and the charge sums of Lemma B.1(d), cost at most $C^{n_\alpha}(n_\alpha!)^2$ per cell, and $e^{c_p \sum_\alpha m_\alpha} = \prod_e e^{c_p t_e}$ is a constant per $\mathcal{E}$-factor.

Lemma B.3 (Witness)**.** *(a)* Link localization. *Let $1 \le i \le n-1$ and let $Y \in \tilde{Y}(h)$ with $Y \subseteq Y_{i+1}$. Then Y contains the point x_i of the bare derivative of $\kappa(i)$ (kinds K1, K2), or a cube of an $\mathcal{E}$-factor of $\kappa(i)$. (b) Every lone cell is of class A, and $F_2' = 0$ on it. (c) Let $\Delta \in X_o$ with $\Delta \subseteq Y_{i+1}$, $i \ge 1$. Then (W1) the bare derivative of $\kappa(i)$ sits at $x_i \in \Delta$ and hits an atom of kind (a1) to (a4) of Δ, or (W2) an $\mathcal{E}$-factor of $\kappa(i)$ has a cube equal to Δ.*

Proof. (a) By [GM82, (E.11)], Y_{i+1} is the smallest union of elements of $\tilde{Y}(h)$ containing $\mathcal{O} - X_i$, where $\mathcal{O}$, the support of the term of $\kappa(i)$, consists of the cubes of its $\mathcal{E}$-factors and the cube of a bare x_i if there is one. An element $Y \subseteq Y_{i+1}$ meeting no point of $\mathcal{O} - X_i$ would make $Y_{i+1} \setminus Y$ a smaller such union. As cells are $\mathcal{U}$-cells, "meets" is "contains". (b) We have $\mathrm{supp}\, D_\beta$, $\mathrm{supp}\, F_{2,\beta}' \subseteq S_\beta$ by Proposition 7.8, and a lone cell lies in no wall polymer. (c) By (a), Δ contains x_i, or it contains a cube of an $\mathcal{E}$-factor of $\kappa(i)$, which is case (W2). In the first case

the bare derivative hits a factor to its right that depends on $\psi(x_i)$. An atom (a1) to (a4) at x_i gives (W1). The atom (a5) and the derivatives of e^{-F_2} give $F_2'(x_i) = 0$ by (b). A factor at x_i of the $\mathcal{E}$-factor of the y-bracket of $\kappa(i)$ (kind K2) gives (W2). All other factors are functions of ψ on X_i, which is disjoint from Y_{i+1}.

For each $\Delta \in X_o$ we fix one witness, (W1) if available and otherwise an $\mathcal{E}$-factor e as in (W2). Let $W(e)$ denote the set of lone cells witnessed by e. Each lone cell lies in exactly one Y_{i+1}, and by (W2) of Lemma B.3(c) a witness e of a lone cell $\Delta \subseteq Y_{i+1}$ is an $\mathcal{E}$-factor of $\kappa(i)$. It follows that all cells of $W(e)$ lie in $Y_{i(e)+1}$, where $\kappa(i(e))$ is the link of e. Each $\kappa(i)$ has at most two $\mathcal{E}$-factors.

The cells of G' have polynomial growth: $\#\{\Delta : \mathrm{dist}(\Delta_0, \Delta) \le r\} \le K_g(1+r)^3$ in Debye units, for the Euclidean distance between cells and $K_g := (4\pi/3)(1+2C_{\mathcal{U}})^3/c_u$, by (R7), $\mathrm{vol}_D \ge c_u$ and $\tilde{m} \le 1$ (the sites of the cells lie in a ball of radius $r + 2C_{\mathcal{U}} + \sqrt{3}\tilde{m}/2 \le (1+r)(1+2C_{\mathcal{U}})$). Links have one end in Y_{i+1} and one in X_i, and the sets Y_{i+1} are pairwise disjoint.

Lemma B.4. *(a) Let Δ_0 be a cell and $\Delta_1, \dots, \Delta_N$ cells, each repeated at most twice, with $r_k := \mathrm{dist}(\Delta_0, \Delta_k)$ (Euclidean, in Debye units). For $q \ge 1$ and $c_p > 0$, $(N!)^q \le c_N^N \exp(c_p \sum_k r_k)$ with $c_N := e^{c_p} \max(1, 3q/(c_p a_g))^{3q}$, $a_g := \frac{3}{4}(2K_g)^{-1/3}$. (b) Let $d := \sum_i \mathrm{dist}(\Delta_i', \Delta_i'')$ be the total length of the links, $\hat{L}^E(e)$ the Euclidean Steiner length, in Debye units, of the cells of the $\mathcal{E}$-factor e (the least Steiner length of points chosen in its cells), and $\hat{L}_0 := \sum_e \hat{L}^E(e)$. Pin each $\mathcal{E}$-factor e at a set $\Pi(e)$ of at most four of its cells (chosen in Rule D, before Theorem B.6), and let $N_\alpha := \#\{e : \Delta_\alpha \in \Pi(e)\}$, in either class. Then, with $c = c(q, c_p, K_g, C_{\mathcal{U}})$, $\prod_\alpha (n_\alpha!)^q \le c^{\sum n_\alpha} e^{c_p d}$ and $\prod_\alpha (N_\alpha!)^q \le c^{\#\mathcal{E}} e^{4c_p(\hat{L}_0 + 2d)}$.*

Proof. For (a) we argue as in [BF80, Lemma 9.11, (9.820)], with the growth $K_g(1+r)^3$ in place of the lattice count of [BF80, Appendix 2]. In increasing order the distances satisfy $r_{(k)} \ge (k/(2K_g))^{1/3} - 1$. Summing these bounds, we obtain $\sum_k r_k \ge a_g N^{4/3} - N$. Moreover, $qN \log N - c_p a_g N^{4/3} \le 3qN \log^+(3q/(c_p a_g))$. (b) We first treat the derivatives. Since Δ_α lies in exactly one Y_{i+1}, at most one x-end lies in Δ_α. All other ends in Δ_α are y-ends y_j of links j with $\Delta_\alpha \subseteq X_j$, whose partners x_j lie in Y_{j+1}, disjoint for distinct j. We apply (a) to the y-ends, with $n! \le 2^n (n-1)!$. For the $\mathcal{E}$-factors, let e be pinned at Δ_α, with link $i = i(e)$. If $\Delta_\alpha \subseteq Y_{i+1}$, there are at most two such e. Otherwise $\Delta_\alpha \subseteq X_i$, and e reaches Y_{i+1}. For K0 this holds since $\partial_{s_i} E^{(i)}$ keeps only Mayer terms with cubes in X_i and in Y_{i+1}. The x-side $\mathcal{E}$-factor of K3, K4 does so through a factor at x_i, and the y-side one of K2, K4 through a factor at y_i, which is joined to x_i by link i. Thus $\mathrm{dist}(\Delta_\alpha, Y_{i+1}) \le \hat{L}^E(e) + 2C_{\mathcal{U}} + d_i$, as $\hat{L}^E(e)$ is a Steiner length over points of the cubes of e and cells have Euclidean diameter at most $C_{\mathcal{U}}$ in Debye units. The partner is the cell of Y_{i+1} nearest to Δ_α. Since distinct links give disjoint Y's, a partner is used only by $\mathcal{E}$-factors of a single link, hence at most twice (see the paragraph after Lemma B.3). We apply (a). Each e appears at most $|\Pi(e)| \le 4$ times and each link at most twice, and the factor $e^{8c_p C_{\mathcal{U}}}$ is absorbed into c.

With c_p of Proposition 7.36 and $r_C = r_{\mathrm{cov}}/\tilde{m}$, the link decay $e^{-r_C \mathrm{dist}(\Delta_i', \Delta_i'')}$ of Lemma B.1(c) pays the factor $e^{9c_p d}$ and leaves $e^{-(\bar{\delta} + (r_C - \bar{\delta})/2)\mathrm{dist}}$. In the proof of Proposition B.7, the Mayer decay of (M4) absorbs the factor $e^{4c_p \hat{L}_0}$ together with the factors $e^{c_{10} L_{\eta^A}}$ of κ'. This uses $\hat{L} \ge 3^{-1/2} \hat{L}^E$ and, on outputs, $(1 - \delta_{\mathrm{GM}})A/\sqrt{3} \ge \frac{1}{2}\kappa_Y M_Y \ge 2r_C \tilde{m} > 4c_p \tilde{m}$ for $\delta_{\mathrm{GM}} \le \frac{1}{2}$ (Proposition 6.10). Finally, $c^{\#\mathcal{E}} \le c^{2|X|}$ is a constant per cell, as a link has at most two $\mathcal{E}$-factors and there are $n - 1 < |X|$ links (each Y_{i+1} contains a $\mathcal{U}$-cell).

factor ([BF80])	k	bound	weight
v) $v_m(\Omega)$, any class (9.814)	≥ 1; 0	$(\lvert m\rvert b_D^{1/2}/v_-)^k$; 2	$k/2$; 0
$v_m(\Omega)$, class A	0	$\lvert m\rvert b_D^{1/2}\lvert\hat\psi_\Omega\rvert$	1/2
vi) $f_1(mb_D^{1/2}\hat\delta)$ (9.815)	0; 1; ≥ 2	$\frac12 m^2 b_D\lvert\hat\delta\rvert^2$; $m^2 b_D\lvert\hat\delta\rvert$; $(\lvert m\rvert b_D^{1/2})^k$	1; 1; $k/2$
f_1, bounded form	1	$2\lvert m\rvert b_D^{1/2}$	1/2
vi) $f_2(mb_D^{1/2}\hat\delta)$	≤ 4; ≥ 4	$m^4 b_D^2\lvert\hat\delta\rvert^{4-k}$; $(\lvert m\rvert b_D^{1/2})^k$	2; $k/2$
f_2, bounded form	2; 3	$2m^2 b_D$; $\lvert m\rvert^3 b_D^{3/2}$	1; 3/2
iv) $\tilde\varepsilon$, class A (9.816)	0; 1; ≥ 2	$\frac12 m^2 b_D\lvert\hat\psi\rvert^2$; $m^2 b_D\lvert\hat\psi\rvert$; $(\lvert m\rvert b_D^{1/2})^k$	1; 1; $k/2$
$\tilde\varepsilon$, class B (9.818)	0; 1	$2+\lvert m\rvert b_D^{1/2}\lvert\hat\psi\rvert$; $2\lvert m\rvert b_D^{1/2}$	0; 1/2
ψ-factor, any class (9.816)	0; 1	$\lvert m\rvert b_D^{1/2}\lvert\hat\psi\rvert$; $\lvert m\rvert b_D^{1/2}$ (zero for $k\geq 2$)	1/2; 1/2
ε, any class (9.817)	≥ 1	$(\lvert m\rvert b_D^{1/2})^k$	$k/2$
ε undistinguished; distinguished, class A (9.819)	0	2; $\lvert m\rvert b_D^{1/2}\lvert\hat\psi\rvert$	0; 1/2
vii) $\cos(mb_D^{1/2}\hat\vartheta)-1$	≥ 1	$(\lvert m\rvert b_D^{1/2})^k$	$k/2$
viii) $f_2(mb_D^{1/2}\hat\psi)$, class A	≤ 4; ≥ 4	$m^4 b_D^2\lvert\hat\psi\rvert^{4-k}$; $(\lvert m\rvert b_D^{1/2})^k$	2; $k/2$
viii), class B	1; 2; ≥ 3	$\lvert m\rvert b_D^{1/2}+m^2 b_D\lvert\hat\psi\rvert$; $2m^2 b_D$; $(\lvert m\rvert b_D^{1/2})^k$	$\frac12$; 1; $k/2$
i) r_Ω (Lemma 7.31)	≥ 0	$c_7(c_7' b_D^{1/6}/v_-)^k e^{c_7'' k\log k} e^{\frac12\eta_H\tilde m^2\sum_\Omega\vartheta^2}$	$k/6$
iii) $\hat F_2'$, class B only		numerical factor	0

Table B.1: Bounds per type (Lemma B.5(a)).

In [BF80, (9.821)], N_α counts distinguished factors ε in class-A cubes, but the proof in [BF80, Appendix 2] does not use the class, and (b) pins $\mathcal{E}$-factors at cells of either class, as required by Proposition B.10. As in [BF80, (9.822)], we multiply each term by $1 = f_q\cdot\prod_\alpha(n_\alpha!)^q(N_\alpha!)^q$, $f_q := \prod_\alpha(n_\alpha!)^{-q}(N_\alpha!)^{-q}$, and bound the second factor by (b).

We state the bounds below in the variables of Lemma B.1, at fixed positions, for k derivatives at points of the factor's cell (for site factors, at the site). Here $\hat\delta := \tilde m^{-1/2}\tilde\delta$, $\hat\psi_\Omega := \mathrm{avg}_\Omega\hat\psi$, and $v_\Omega\in[v_-,v_+]$ is the Debye volume of a grid cell. The chain-rule factors $\delta(\hat\psi_\Omega)/\delta\hat\psi(\hat x) = 1_\Omega(\hat x)/v_\Omega$ and $\delta\hat\delta(\hat z)/\delta\hat\psi(\hat x) = \hat\delta(\hat x-\hat z)-1_\Omega(\hat x)/v_\Omega$, $\hat\delta(\hat x-\hat z) := \tilde m^{-3}1[x=z]$, are not shown. They are absorbed when the line end is integrated, or bounded by $1/v_-$. In class B, $\lvert\hat\delta\rvert\leq\lvert\hat\delta\hat\psi\rvert+\lvert\delta\hat D\rvert$. A row of Table B.1 that carries an equation number of [BF80] restates that bound in Debye units, with the charges and volumes shown, and the other rows are new.

Lemma B.5. *(a) With C numerical and $\lvert m\rvert$ the charge of the factor, the factors satisfy the bounds of Table B.1. (b)* Vertex weights. *A vertex is a block B on a G_2-, vii)- or viii)-atom with its coefficient, an r_Ω with all its derivatives, an $\mathcal{E}$-factor with its coefficient and factors, or one factor $\hat F_2'$. The coefficient weight is -1 for one-body, off-$\mathfrak{D}$ and Mayer coefficients ((M1), (M3), (M4)) and $-\frac16$ for deficit coefficients on $\mathfrak{D}$ ((M2) with $c_{\mathrm{dep}}\leq b_D^{-1/6}$ for $\tau\geq\tau_0$). For a block with $k=\lvert B\rvert\geq 1$: a $G_{2,1}$-block has weight ≥ 1 in class A, ≥ 0 in class B if $k\geq 2$, and for $k=1$ in class B it is a term of weight 0 equal to $\hat\delta(\hat x)$ at the derivative point plus a term of weight $-\frac12$ with no field or numerical factor; a $G_{2,2}$-block has weight $\geq\frac12$ in class A and $\geq-\frac12$ in class B (≥ 0 if $k\geq 2$), with no numerical factor; a vii)-block has weight $\geq\frac13$; a viii)-block has weight ≥ 1 in class A and $\geq-\frac12$ in class B (≥ 0 if $k\geq 2$); r_Ω with k derivatives has weight $k/6$; $\hat F_2'$ has weight 0. So in class A every non-$\mathcal{E}$ vertex has weight ≥ 0, and $\geq\frac16$ if it carries a derivative; in class B every non-$\mathcal{E}$ vertex has weight $\geq-\frac12$, and is negative only for single-derivative blocks.*

Proof. (a) By Lemma B.1(b) each derivative of a function of $\beta_{\mathrm{GM}}^{1/2}\psi = b_D^{1/2}\hat\psi$ brings $b_D^{1/2}$, and

each derivative of $\hat\psi_\Omega$ brings $1/v_\Omega$. This gives the rows of [BF80]. For the new rows we use the following facts. In class A, $v_m(\Omega) = e^{im\beta_{\rm GM}^{1/2}\mathrm{avg}_\Omega\psi} - 1$, since $h \in P\mathbb{Z}$, and $|e^{it}-1| \le |t|$. Next, $|f_1'| = |e^{iy}-1| \le \min(|y|, 2)$. Further, $f_2^{(j)}(0) = 0$ for $j \le 3$ and $|f_2^{(4)}| \le 1$, hence $|f_2^{(k)}(y)| \le |y|^{4-k}/(4-k)!$ for $k \le 4$, $|f_2^{(k)}| \le 1$ for $k \ge 4$, $|f_2^{(2)}| = |1-\cos y| \le 2$ and $|f_2^{(3)}| = |\sin y| \le 1$. For vii), $|\partial_t^k e^{imt}| = |m|^k$. In class A the factor viii) is $f_2(mb_D^{1/2}\hat\psi(\hat z))$, and in class B its first derivative is $-mb_D^{1/2}\sin(\cdot) + m^2 b_D\hat\psi$ and its second $m^2 b_D(1-\cos(\cdot))$. Row i) is Lemma 7.31 with $\partial A_\Omega/\partial\psi(x) = \tilde m^3/v_\Omega$. For iii), $F_2 = \int d\hat x\, \hat\psi \hat F_2'$ and $\mathrm{supp}\, F_2' \subseteq \bigcup S_\beta$. (b) The coefficient weights are those of (M1) to (M4). Moreover, $c_{\rm dep} = \mathcal{O}(\log\tau)$, while $b_D^{-1/6} = e^{\frac{1}{6}\pi^2 v_{\rm Cb}(0)\tau(1+o(1))}$. For $G_{2,1}$ in class A the weight is $-1+2$ for $k \le 4$ and $-1+k/2$ for $k \ge 4$, while in class B with $k = 2, 3$ the bounded forms give $-1+1$ and $-1+\frac{3}{2}$. In class B with $k = 1$, as $\sum_{z\in\Omega}\tilde\delta(z) = 0$ and $f_2'(y) = y - \sin y$, we have $\partial_x G_{2,1}(\Omega) = \tilde m^2\tilde\delta(x) - \sum_m \rho^{\rm ref}(m) m\beta_{\rm GM}^{1/2}[\sin y_x - \mathrm{avg}_\Omega \sin y]$, whose first term is $(b_D\sum_m m^2\hat\rho^{\rm ref}(m))\hat\delta(\hat x) = \hat\delta(\hat x)$ (weight $-1+\frac{1}{2}+\frac{1}{2}$) and whose second has weight $-1+\frac{1}{2}$. For $G_{2,2}$ we split $k = k_1 + k_2$ between v_m and f_1. In class A the weight is $-1+\frac{1}{2}+\max(1, k_2/2)$ if $k_1 = 0$, else $-1+k_1/2+\max(1, k_2/2)$. In class B we distinguish three cases. If $k_2 = 0$, then $\sum_{z\in\Omega} f_1(y_z) = \sum_{z\in\Omega}(e^{iy_z}-1)$ as $\sum_z y_z = 0$, which is bounded by $2\#\Omega$ with no numerical factor, and hence the weight is $-1+k_1/2 \ge -\frac{1}{2}$. If $k_2 = 1$, the bounded form gives $-1+k_1/2+\frac{1}{2}$. If $k_2 \ge 2$, the weight is $\ge -1+k_1/2+k_2/2 \ge 0$. For a vii)-block the weight is $-\frac{1}{6}+k/2$. For a viii)-block it is $-1+2$ in class A, and $-1+\frac{1}{2}$, $-1+1$, $-1+k/2$ in class B.

A factor of an $\mathcal{E}$-factor is *gaining* if its weight is at least $\frac{1}{2}$. This is the case for all differentiated factors, in either class, for all distinguished ε's (class A), for all $\tilde\varepsilon$'s and ψ-factors in class A, and for all ψ-factors in class B. Each factor has weight ≥ 0. We select distinguished factors as in [BF80, p. 232], with the attribution to cells fixed by Rule D.

Rule D. Let e be an $\mathcal{E}$-factor of type (E.5a), $t \ge 3$. (D-a) For each $\Delta \in W(e)$ in which e has no differentiated factor, we distinguish one undifferentiated ε of e in Δ, which exists since e has a cube equal to Δ. (D-b) While e has fewer than three gaining factors, we distinguish another undifferentiated ε of e in a class-A cell, if there is one. All other undifferentiated ε's are bounded by 2. For types (E.5b) to (E.5d) no choice is made. The pinning set $\Pi(e)$ consists of the cells of the factors distinguished in (D-b) (at most three) for (E.5a), or of the cells of e (at most two) for (E.5b) to (E.5d), together with the charged cell defined below, if there is one. Hence $|\Pi(e)| \le 4$.

Let $P(e)$ be the weight of e under Rule D. An $\mathcal{E}$-factor gives $\frac{1}{6}$ to each cell of $W(e)$. Its remainder $\varrho(e) := P(e) - |W(e)|/6$ is assigned to one chosen class-B cube of e (its *charged cell*) if e has a class-B cube, and to any cell of e otherwise. We call e *charged* if $\varrho(e) < 0$. Each non-$\mathcal{E}$ vertex is assigned to its own cell.

Theorem B.6 (Net weight per lone cell)**.** *In every term, after this distribution: (i) every cell $\Delta \in X_o$ receives weight at least $\frac{1}{6}$; (ii) every class-A cell receives weight at least 0; (iii) every class-B cell receives weight at least $-N_\Delta^{\rm ch} - \frac{1}{2}n_\Delta^{(1)}$, where $N_\Delta^{\rm ch}$ is the number of $\mathcal{E}$-factors charged to Δ and $n_\Delta^{(1)} \le n_\Delta$ the number of single-derivative blocks in Δ. The value $\frac{1}{6}$ is attained, both by a lone cell whose only vertex of positive weight is r_Ω with one derivative, and by an $\mathcal{E}$-factor of type (E.5a) with $t = 3$ whose three factors are undifferentiated ε's in three cells of $W(e)$.*

Proof. We first treat the $\mathcal{E}$-factors. If all cubes of e are of class A, then $P(e) \ge |W(e)|/6$ and $P(e) \ge 0$. In general $P(e) \ge -1 + |W(e)|/2$, and hence $\varrho(e) \ge -1$. Indeed, suppose first that

all cubes are of class A. For (E.5a), (D-b) stops only when e has $g(e) \geq 3$ gaining factors, and the cells of $W(e)$ contain distinct gaining factors by (D-a), hence $g(e) \geq \max(3, |W(e)|)$ and $P(e) \geq -1 + g(e)/2 \geq g(e)/6 \geq |W(e)|/6$. For (E.5b), $P(e) \geq -1 + 1 + 1$, and for (E.5c), (E.5d), $P(e) \geq -1 + 1 + \frac{1}{2}$, both $\geq \frac{2}{6}$. In general the cells of $W(e)$ are lone, hence of class A by Lemma B.3(b), each contains a gaining factor of e, and all other factors have weight ≥ 0. (ii) A class-A cell receives its non-$\mathcal{E}$ vertices, which have weight ≥ 0 by Lemma B.5(b), credits $\frac{1}{6}$, and remainders $\varrho(e) \geq 0$ of $\mathcal{E}$-factors with no class-B cube. (i) Let $\Delta \in X_o$, with its witness from Lemma B.3(c). In case (W1) a derivative hits an atom of Δ. If this atom is r_Ω, its vertex has weight $\geq \frac{1}{6}$, and otherwise the Faà di Bruno block containing the derivative has weight $\geq \frac{1}{3}$, since the cell is of class A. In case (W2), $\Delta \in W(e)$ receives $\frac{1}{6}$ from e. All other items received are ≥ 0, as in (ii). (iii) Class-B non-$\mathcal{E}$ vertices have weight $\geq -\frac{1}{2}$, with negative weight only for single-derivative blocks. Charged remainders are ≥ -1, and uncharged ones are ≥ 0. As for attainment, the first example is (W1) with Lemma B.5(b). In the second, $P(e) = -1 + \frac{3}{2} = 3 \cdot \frac{1}{6}$, and the remainder is zero.

Each field factor of an $\mathcal{E}$-factor lies in a cell of $\Pi(e)$ or is the single (D-a) factor of a lone cell, and a non-$\mathcal{E}$ vertex carries at most four field factors per derivative. Hence the number n_j of field factors in the cell Δ_α satisfies $n_j \leq 1 + 4(n_\alpha + N_\alpha)$.

Proposition B.7 (Per-cell form of (9.824))**.** *Fix the constants of Section 7.6, with $c_{10} \leq \frac{7}{9} r_C$ in Lemma 7.37, $q_{\mathrm{pin}} := q_0 + 2$ with $q_0 := 8 + c_7''$, and c_p of Proposition 7.36. For every term κ' of the activity of a polymer X with seed Z (kinds and link-end cells fixed, and for each $\mathcal{E}$-factor e its number t_e of points and the set $\mathfrak{a}(e)$ of cells containing them; κ' contains, inside $|\cdot|_0$, the sum over all configurations of the points of e with this set of cells, that is, over the assignments of the t_e points to the cells of $\mathfrak{a}(e)$ that use every cell, over their positions in these cells and over their charges, and over the forms [GM82, (E.5b) to (E.5d)] if $t_e = 2$), every h_X and every s,*

$$\int d\mu_{C_s'} \big| e^{E(X,s)} \kappa' Q^\natural(X) \big|_0 \leq \mathcal{P}_G' \, e^{c_{13}|X| + C_{\mathfrak{D}} c_{\mathrm{dep}} N_{\mathfrak{D}}(X)} \, e^{-c' F_1^{\mathrm{loc}}(X) + \frac{c'}{16} F_1^{\mathrm{loc}}(X \setminus Z) + \frac{c'}{32} F_1^{\mathrm{loc}}(Z)} \times \prod_i \big[K_\ell \, e^{-(\bar\delta + (r_C - \bar\delta)/2)\mathrm{dist}(\Delta_i', \Delta_i'')} \big] \prod_e \mathfrak{m}_e \prod_{\Delta \subseteq X} \Psi_\Delta,$$

where $\prod_i$ runs over the links with a covariance line (types (2) to (5), Lemma A.1), $K_\ell := 2K_1''(1 + v_+/v_-)^2/\lambda_{\mathrm{PV}}$ is the line constant, $\mathfrak{m}_e := e^{-(\mu - 1 - 2c_{10}'')(t_e - 1)} e^{-r_C \hat L^E(e)}$ is the leftover Mayer weight, $\hat L^E(e)$ the Euclidean Steiner length of the cells of $\mathfrak{a}(e)$ (Lemma B.4(b)) ($\mathfrak{m}_e \leq 1$ for $\mu \geq 1 + 2c_{10}''$), and $\Psi_\Delta \leq K_0^B$ for every class-A cell, $\Psi_\Delta \leq K_0^B b_D^{1/6}$ for $\Delta \in X_o$, and $\Psi_\Delta \leq K_0^B e^{c_B(b_D)}$ for every class-B cell, with c_B as in Proposition B.10. The constant K_0^B depends only on $\lambda_{\mathrm{PV}}, C, \mu, \kappa_S, \delta_{\mathrm{GM}}, q_{\mathrm{pin}}, c_p, c'$, the constants of Theorem 6.11, Lemma 7.31 and the Wick bound, and the geometry constants $c_u, C_u, C_{\mathcal{U}}, D_{\mathcal{U}}, v_\pm$; it does not depend on b_D, L' or τ, and it contains neither K_ℓ nor $e^{c_{13}}$.

We give the proof after Lemma B.9.

The numerical factors $\delta\hat D = \hat D - \mathrm{avg}_\Omega \hat D$ and $\hat F_2'$ sit at line ends integrated freely over their cells. For a term we put $\mathcal{N} := \prod_j \int_{\Delta_j} |N_j(\hat x)| \, d\hat x$, the product over its numerical factors N_j, and $n_\Delta^N \leq n_\Delta$ denotes the number of them in the cell Δ.

Lemma B.8 (Numerical factors)**.** *All numerical factors lie in class-B cells. For every union Y of elements of $\tilde{Y}(h)$ and every $c > 0$, the numerical factors located in Y satisfy*

$$\prod_{j:\,\Delta_j\subseteq Y}\int_{\Delta_j}|N_j|\,d\hat{x} \le \prod_{\Delta\subseteq Y}\Big[\Big(\frac{C_u}{2c}\Big)^{n_\Delta^N/2}(n_\Delta^N!)^{1/2}\Big]e^{3cF_1^{\mathrm{loc}}(Y)}.$$

In particular, with $3c = c'/16$ on $X\backslash Z$ and $3c = c'/32$ on Z, $\mathcal{N} \le \prod_\Delta[\cdots]\,e^{\frac{c'}{16}F_1^{\mathrm{loc}}(X\backslash Z)+\frac{c'}{32}F_1^{\mathrm{loc}}(Z)}$.

Proof. First, $\operatorname{supp} D_\beta$, $\operatorname{supp} F'_{2,\beta} \subseteq S_\beta$ by Proposition 7.8(b), (e). Hence all numerical factors lie in class-B cells, and S_β lies in one element of $\tilde{Y}(h)$. By the Cauchy–Schwarz inequality on each cell and $\|\delta\hat{D}\|_{L^2(\Delta)} \le \|\hat{D}\|_{L^2(\Delta)}$, the left side is at most $\prod_\Delta (C_u a_\Delta)^{n_\Delta^N/2}$ with $a_\Delta := \|\hat{D}\|^2_{L^2(\Delta)} + \|\hat{F}'_2\|^2_{L^2(\Delta)}$, and $a^{n/2} \le (2c)^{-n/2}(n!)^{1/2}e^{ca}$. Finally $\sum_{\Delta\subseteq Y} a_\Delta \le (2+\varepsilon(L'))F_1^{\mathrm{loc}}(Y) \le 3F_1^{\mathrm{loc}}(Y)$ by Corollary 7.10, Theorem 7.16 and Corollary 7.18.

Lemma B.9. *All factorials produced in a term are at most $\prod_\alpha C^{n_\alpha+N_\alpha}(n_\alpha!)^{q_0}(N_\alpha!)^{q_0}$ with $q_0 := 8 + c''_7$. Consequently the pinning factor $f_{q_{\mathrm{pin}}}$ with $q_{\mathrm{pin}} = q_0 + 2$ absorbs them and leaves $\prod_\alpha (n_\alpha!)^{-2}(N_\alpha!)^{-2}$.*

Proof. We count the factorials per cell, up to factors $C^{n_\alpha+N_\alpha}$. The Wick bound $T_1 \le c_T^{\sum n_j}\prod_j n_j!$ with $n_j \le 1 + 4(n_\alpha + N_\alpha)$ gives $(n_\alpha!)^4(N_\alpha!)^4$, the expansions inside the types and Lemma B.2(c) give $(n_\alpha!)^3$, Lemma 7.31 gives $(k!)^{c''_7}$, since $k\log k \le \log k! + k$, and Lemma B.8 gives $(n_\alpha!)^{1/2}$. The total exponent is $7.5 + c''_7 \le q_0$.

Proof of Proposition B.7. We follow the argument of [BF80, (9.88) to (9.823)] on G', make it quantitative, and add the bound on the tree weights of κ'.

Assignments. For each $\mathcal{E}$-factor e we fix its form and a surjection $\pi_e : \{1,\dots,t_e\} \to \mathfrak{a}(e)$, the cell of each of its points. For the terms with these assignments we carry out all the steps below except the bound on the tree weights and the collection of the constants, with the maps l, Rule D, the pinning sets $\Pi(e)$, the witnesses and the weights $P(e)$ of these terms. In the integration over the positions we leave out the points of the $\mathcal{E}$-factors. The bounds so obtained are the same for every assignment, because the numbers m_α, n_α, N_α, n_j appear only in Lemmas B.2(c), B.4(b), B.9 and in the Wick bound, whose right sides are uniform, and $\hat{L}^E(e)$ depends only on $\mathfrak{a}(e)$. The positions and charges of the points of e appear only in $|\hat{\rho}_{t_e}|$ and $\prod_i |m_i|^{k_i}$, where k_i is the number of derivatives on the ith factor. By $|m_i|^{k_i} \le k_i!(2\kappa_S)^{-k_i}e^{2\kappa_S|m_i|}$, the factorials $k_i!$ are among those of Lemma B.9, namely those of the charge sums in the proof of Lemma B.2, and the weight $e^{2\kappa_S\sum_i|m_i|}$ does not depend on the assignment. We therefore bound the sum over the assignments, the positions and the charges of the points of e at once by (M4) with $\mathfrak{a} = \mathfrak{a}(e)$. The sum over the forms costs at most a factor 3.

Count. The sum over the maps l and the further expansions is at most the number of terms, which Lemma B.2(c) bounds, times the largest term, the "Sup" of [BF80, (9.88)].

Pinning. We insert $f_{q_{\mathrm{pin}}}$ and bound $\prod_\alpha (n_\alpha!)^{q_{\mathrm{pin}}}(N_\alpha!)^{q_{\mathrm{pin}}}$ by Lemma B.4(b), with the resulting factors absorbed as explained after that lemma. This leaves $e^{-(\bar{\delta}+(r_C-\bar{\delta})/2)\mathrm{dist}}$ per link.

Tree weights. In the term κ' of Lemma 7.37 an $\mathcal{E}$-factor e appears as $\sum_{\eta^A} e^{c_{10}L_{\eta^A}}e^{c''_{10}t_e}\mathcal{E}_{\eta^A}$, where L_{η^A} is the length [BF80, (A1.3)], measured with dist and with auxiliary points in V, of an augmented tree η^A whose marked points lie in the cells of $\mathfrak{a}(e)$, one in each, as in Lemma A.6,

and $\mathcal{E} = \sum_{\eta^A} \mathcal{E}_{\eta^A}$. Since only $\sum_{\eta^A} \mathcal{E}_{\eta^A} = \mathcal{E}$ is used, we may put $\mathcal{E}_{\eta_e^A} := \mathcal{E}$ for one such tree η_e^A of least length (a minimizer exists, since V is finite and an auxiliary vertex of degree at most 2 can be removed without increasing the length) and $\mathcal{E}_{\eta^A} := 0$ otherwise. Then e carries the number $e^{c_{10}L^d(e)}e^{c''_{10}t_e}$, $L^d(e) := L_{\eta_e^A}$, which depends only on $\mathfrak{a}(e)$, that is, on the link datum. By Lemma A.6 and the *Assignments* step, no sum over positions at a fixed η^A occurs (unlike [BF80, p. 214], where the positions are summed at a fixed η^A with the constant weights b_{η^A}). If $L_{G'}(\mathfrak{a}(e)) = \infty$, every configuration with the cells $\mathfrak{a}(e)$ has points in two components of G', so ρ_{t_e} vanishes on it by Proposition 6.3(d), and $\mathcal{E} = 0$. Such a factor is absent, as noted before Lemma A.6, so no product $0 \cdot \infty$ arises, and in what follows $L_{G'}(\mathfrak{a}(e)) < \infty$. Since $\mathrm{dist} = \tilde{m} d_{G'}$ and $L_{G'}$ is computed with $3^{-1/2} d_{G'}$, we have $L^d(e) = \sqrt{3}\,\tilde{m} L_{G'}(\mathfrak{a}(e)) = \sqrt{3}\,\hat{L}(\mathfrak{a}(e))$. Thus the Mayer decay of (M4) is $e^{-\rho_S L^d(e)}$ with $\rho_S := (1 - \delta_{\mathrm{GM}})A/(\sqrt{3}\,\tilde{m}) \geq 2r_C$, by the chain of inequalities after Lemma B.4. Also, $L^d(e) \geq \hat{L}^E(e)$, since both lengths are minima over points of the cells of e and $d_{G'}$ dominates the Euclidean distance. With the pinning factor $e^{4c_p \hat{L}^E(e)}$ of Lemma B.4(b) we obtain

$$e^{c_{10}L^d(e)} e^{4c_p \hat{L}^E(e)} e^{-\rho_S L^d(e)} \leq e^{-(\rho_S - c_{10} - 4c_p)\hat{L}^E(e)} \leq e^{-r_C \hat{L}^E(e)},$$

because $\rho_S - c_{10} \geq 0$ and $\rho_S - c_{10} - 4c_p \geq 2r_C - \frac{7}{9}r_C - \frac{2}{9}(r_C - \bar{\delta}) = r_C + \frac{2}{9}\bar{\delta}$. The bound uses L^d, not $\hat{L}^E$, because on a slit graph $d_{G'}$ is not bounded by a multiple of the Euclidean distance, and hence $e^{-r_C \hat{L}^E(e)}$ alone cannot absorb $e^{c_{10}L^d(e)}$. Finally, $t_e \geq 2$ for each $\mathcal{E}$-factor, by [GM82, (E.5a) to (E.5d)] and Appendix C. Hence $e^{c''_{10}t_e} \leq e^{2c''_{10}(t_e - 1)}$, which cancels against the factor $e^{-2c''_{10}(t_e-1)}$ of $e^{-\mu(t_e-1)}$ and leaves $e^{-(\mu - 2c''_{10})(t_e-1)}$.

Hölder and positions. At fixed positions we bound T_1 by the Wick bound, $T_2 e^{-F_1} T_3 T_4$ by Proposition 7.35(d), and the factors r_Ω by Lemma B.5. As $0 \leq \partial_{s_i} p(x, y|s) \leq 1$ in [GM82, (E.3)], $|\partial_{s_i} \hat{C}| \leq \tilde{m}^{-1}|C'|$. We bound each line by its supremum over its pair of end cells, using Lemma B.1(c), and integrate its ends against a vertex point, a cell average, an r_Ω-derivative (cost at most v_+/v_-) or a numerical factor (kept in $\mathcal{N}$). The points of each $\mathcal{E}$-factor e, summed over all assignments at once, are integrated against its coefficient by (M4) with $\mathfrak{a} = \mathfrak{a}(e)$, and the other internal points by (M1) to (M3) of Lemma B.1(d). This gives at most $\prod_i (2K''_1/\lambda_{\mathrm{PV}}) e^{-r_C \mathrm{dist}(\Delta'_i, \Delta''_i)} \prod_{\mathrm{vertices}} [\mathrm{mass} \cdot b_D^w](1 + v_+/v_-)^{2(n-1)} \mathcal{N}$.

Numerical factors and factorials. We bound these by Lemmas B.8 and B.9.

Weights. We use Theorem B.6 and $b_D \leq 1$. A cell with weight $\geq \frac{1}{6}$ gets at most $b_D^{1/6}$, and one with weight ≥ 0 at most 1. A class-B cell gets $b_D^{-N_\Delta^{\mathrm{ch}} - n_\Delta^{(1)}/2}$ with $(N_\alpha!)^{-2}(n_\alpha!)^{-2}$ from Lemma B.9 and $N_\Delta^{\mathrm{ch}} \leq N_\alpha$, since a charged $\mathcal{E}$-factor is pinned at its charged cell. Proposition B.10 bounds the product by $e^{c_B(b_D)}$.

Constants. The line constants $(2K''_1/\lambda_{\mathrm{PV}})(1 + v_+/v_-)^2 = K_\ell$ stay in $\prod_i K_\ell$, and $e^{c_{13}|X|}$, the factor of Proposition 7.35, stays as it is. Each remaining constant is per cell or per $\mathcal{E}$-factor. Recall that each $\mathcal{E}$-factor that occurs has its cells in one component of G'. The per-$\mathcal{E}$-factor constants that grow with t_e are 2 for each undistinguished ε, from Rule D, C_{M4} for each point and the factor t_e of (M4), and e^{c_p} for each factor, from Lemma B.2(c). Their product is at most $t_e(2C_{M4}e^{c_p})^{t_e}$. As $C \leq \frac{1}{4}$, we have $2C_{M4} \leq \frac{8}{3}$. Moreover, $c_p < r_C/18 < \frac{1}{540}$, since $r_C \leq \kappa_0 \leq (\varepsilon_1^{\mathrm{HK}})^{1/2} < \frac{1}{30}$ by Theorem 6.11. Therefore $q := 2C_{M4}e^{c_p}/e \leq 0.9829$, $K_t := \sup_{t \geq 1} t q^{t-1} \leq 22$ and $2C_{M4}e^{c_p}K_t = e \sup_{t \geq 1} t q^t \leq 58$. Hence $t_e(2C_{M4}e^{c_p})^{t_e} \leq 2C_{M4}e^{c_p}K_t\, e^{t_e - 1}$, and one factor $e^{-(t_e-1)}$ absorbs e^{t_e-1}. This turns the factor $e^{-(\mu - 2c''_{10})(t_e-1)}$ obtained above into $e^{-(\mu - 1 - 2c''_{10})(t_e-1)}$. The per-$\mathcal{E}$-factor constants independent of t_e, namely

K_{M4}, $4C_u$, $2C_{M4}e^{c_p}K_t \le 58$, the factor 3 for the forms when $t_e = 2$, and the pinning constant of Lemma B.4, amount to a constant per cell, as $\#\mathcal{E} \le 2|X|$. Thus the sum over the assignments contributes to K_0^B only the numerical factor $K_t \le 22$ per $\mathcal{E}$-factor, that is, at most 22^2 per cell, and the sum over the forms at most 3^2 per cell. In particular, K_0^B has the dependence stated in the proposition.

Proposition B.10. *Let $q_1 \ge 1$ be the exponent of the leftover pinning factor $\prod_\alpha (n_\alpha!)^{-q_1}(N_\alpha!)^{-q_1}$ ($q_1 = 2$ for $q_{\rm pin} = q_0 + 2$). The b_D-dependent part of the factor on a class-B cell is at most $e^{c_B(b_D)}$, where $c_B(b_D) := q_1(C_{\mathcal{E}}/b_D)^{1/q_1} + q_1(C_G b_D^{-1/2})^{1/q_1} \le q_1(C_{\mathcal{E}}^{1/q_1} + C_G^{1/q_1})b_D^{-1/q_1}$, $C_{\mathcal{E}}$ being the constant of one charged $\mathcal{E}$-factor and C_G that of one single-derivative block; both are independent of L'. For $q_1 = 2$, $c_B(b_D) \le 2(C_{\mathcal{E}}^{1/2} + C_G^{1/2})b_D^{-1/2}$. The bound gives $c_B(b_D) = o(\Phi_{\mathcal{U}})$ as $\tau \to \infty$ if and only if $q_1 > 1$.*

Proof. By Theorem B.6(iii) and the bound for a class-B cell in the proof of Proposition B.7, the cell carries at most $(C_{\mathcal{E}}/b_D)^{N^{\rm ch}}(C_G b_D^{-1/2})^{n^{(1)}}(N_\alpha!)^{-q_1}(n_\alpha!)^{-q_1}$ with $N^{\rm ch} \le N_\alpha$, $n^{(1)} \le n_\alpha$. We may assume $C_{\mathcal{E}}/b_D, C_G b_D^{-1/2} \ge 1$ and raise the exponents to N_α, n_α. For $q, a \ge 1$, $\sup_N a^N/(N!)^q \le (\sum_N a^{N/q}/N!)^q = e^{qa^{1/q}}$. For the last assertion, we use $\Phi_{\mathcal{U}} \asymp b_D^{-1}\beta_{\rm GM}^{-3}$, which follows from Proposition 7.19 as $\tilde{m}L' = K_{\rm col}\beta_{\rm GM}$, and $b_D = e^{-\pi^2 v_{\rm Cb}(0)\tau(1+o(1))}$. If $q_1 > 1$, then $c_B/\Phi_{\mathcal{U}} \le C b_D^{1-1/q_1}\beta_{\rm GM}^3 \to 0$. If $q_1 = 1$, the bound is of order b_D^{-1} and the ratio grows like $\beta_{\rm GM}^3$. As Lemma B.4 holds for all q, the choice $q_{\rm pin} = q_0 + 2$ is admissible.

The order b_D^{-1/q_1} cannot be improved within the bounds of Lemma B.5: a class-B cell can contain an $\mathcal{E}$-factor all of whose factors are undifferentiated class-B ε's, of weight exactly -1, since $\varepsilon = e^{im\beta_{\rm GM}^{1/2}(\psi+D+h)} - 1$ with $D \ne 0$ has no small bound.

Remark B.11 (One or two derivatives on G_2 in class-A cells). On [BF80, p. 231], the bounds [BF80, (9.814), (9.815)] read $(c\beta^{1/2})^n \cdot 2$ for the nth derivative of a term from v), and $c\beta(|\delta|^{2-n} + |g-h|^{2-n})$ if $n < 2$, $(c\beta^{1/2})^n$ if $n \ge 2$, for either term from vi). With these bounds a G_2-block with one or two derivatives in a class-A cell carries no net power of b_D. Indeed, a $G_{2,2}$-block whose single derivative falls on the factor vi) has weight $-1+0+1 = 0$, and a $G_{2,1}$-block with one or two derivatives has weight $-1+1 = 0$. A lone cell whose only witness is such a block then receives no power of b_D. The class-A bounds of Lemma B.5(a) supply this power. In a class-A cell $|v_m(\Omega)| \le |m|\beta_{\rm GM}^{1/2}|{\rm avg}_\Omega \psi|$ as $h \in P\mathbb{Z}$, and by charge symmetry $G_{2,1}$ is of fourth order in $\tilde{\delta}$. By Lemma B.5(b), both give weight at least $\frac{1}{2}$. Without charge symmetry the third-order form of the second term of vi) already gives $-1 + \frac{3}{2} = \frac{1}{2}$.

Proof of Lemma 7.38. We insert Proposition B.7 into Lemma 7.37. With $e^{c_2^L|X|}$ and the factor $e^{c_{13}|X|}$ of Proposition B.7, its factors are distributed cell by cell as stated, with $c_o := c_2^L + c_{13} + \log K_0^B + \log\max(1, K_\ell)$ (link i is mapped injectively to a cell of Y_{i+1}, so $\prod_i K_\ell \le \prod_\Delta \max(1, K_\ell)$) and c_B of Proposition B.10 with $q_1 = 2$. Since a class-B cell meets $\bigcup_\beta S_\beta$, it lies in a wall polymer, in X_Σ or in the seed. Factors on class-A cells carry in addition b_D^w with $w \ge 0$, by Theorem B.6(ii). Their constants are independent of L', and so are $C_{\mathcal{E}}$ and C_G, which are built from the same quantities (see Section S1.7.6). Indeed, no class-A cell carries D, F_2' or a quantity depending on $\hat{\Sigma}$, by Proposition 7.8(b), (e). This proves the assertion of the lemma on L', with $\varepsilon(L')$ coming from Lemma B.8.

Proof of Proposition 7.39. We insert Proposition B.7 into Lemma 7.37 with $c_1' = c'/4$. Of F_1^{loc} we use $c'/16$ on $X \setminus Z$ for the numerical factors of Lemma B.8 and $c'/8$ there for the wall polymers, and $c'/32$ on Z for the numerical factors. The share $c'/8$ gives each cell of a wall polymer of $X \setminus Z$ the factor $e^{-(c'/8)\Phi_{\mathcal{U}}/(1+D_{\mathcal{U}})}$, since $F_1(S_\beta) \geq \Phi_{\mathcal{U}} \# S_\beta$ by Proposition 7.19(c), a closure adds at most $D_{\mathcal{U}}$ cells per cell, and $F_1^{\mathrm{loc}}(\mathcal{S}) \geq \sum_{S_\beta \subseteq \mathcal{S}} F_1(S_\beta)$ by the definition of F_1^{loc} in Corollary 7.10. The link decay of Proposition B.7 absorbs $e^{\delta_2 d}$, as $\delta_2 < \bar{\delta}$. The Mayer cubes and the per-$\mathcal{E}$-factor constants growing with t_e are absorbed inside Proposition B.7, and the leftover Mayer weights of that proposition satisfy $\mathfrak{m}_e \leq 1$, since $\mu \geq 1 + 2c_{10}''$ by item 12 of the list in Section S1.1.1. With $K_B := K_0^B \max(1, K_\ell)$, independent of b_D, L', τ, the per-cell factors and the line constants (one per link, and the links inject into the cells) give ε_Δ (Section S1.7.6). The factor $e^{C_{\mathfrak{D}} c_{\mathrm{dep}}}$ is present only on cells meeting $\mathfrak{D}$, and the rest of the factor on $X \setminus Z$ is at most 1. The exponent $p_0 = \frac{1}{6}$ is attained by Theorem B.6.

C The cluster expansion on slit domains: proof of Proposition 7.22

In this appendix we give a condensed proof of Proposition 7.22. The expansion of [GM82, Appendix E] is an algebraic identity, built from Bernoulli interpolation of the couplings between consecutive sets, the fundamental theorem of calculus and Gaussian integration by parts, and its only analytic input is absolute convergence. On G' absolute convergence needs the Poincaré inequality on cells, as the Peierls identity of Proposition 7.4 removes the mass in the cell-fluctuation directions (Lemma C.2). On block cubes this is the bound of [BF80, (9.640) to (9.642)].

Setting. We fix h. Everything below depends on it only through its walls and the integrand of Lemma 7.21, and therefore holds equally for wall fields (Lemma 7.24). The covariance $\mathbf{C}$ is $C = (C_0^{-1} + \nu)^{-1}$ of Proposition 7.7 or C' of Lemma 7.29, positive definite on V with $\mathbf{C} \leq \alpha C_0$, $\alpha = 2$ for C and $\alpha = \alpha_D \leq \frac{7}{6}$ for C'. The integrand is $e^E Q_h$, $Q_h = e^G e^{-F_1^{\mathrm{loc}}(V)} e^{-F_2}$, with $G^\natural$ for G when $\mathbf{C} = C'$. We write $\partial_x := \partial/\partial\psi(x)$. A *sequence* is $\bar{y} = (Y_1, \dots, Y_m)$ of nonempty admissible sets (Definition 7.20), pairwise disjoint, with the first set Y_1 fixed before the expansion starts. Its union is $X(\bar{y}) := \bigcup_i Y_i$.

Interpolation. For pairwise disjoint admissible $Y_1, \dots, Y_n$ with union X, let $Y_{n+1} := V \setminus X$, $\chi_i := \mathbf{1}_{Y_i}$ and, for $s \in [0,1]^n$,

$$p(x, y \mid s) := \sum_{1 \leq i < j \leq n+1} s_i \cdots s_{j-1} \big[\chi_i(x)\chi_j(y) + \chi_i(y)\chi_j(x)\big] + \sum_{i \leq n+1} \chi_i(x)\chi_i(y), \tag{C.1}$$

and $\mathbf{C}(s)(x,y) := p(x, y \mid s)\mathbf{C}(x,y)$, as in [GM82, (E.2), (E.3)], with $\chi_{n+1} = \mathbf{1}_{V \setminus X}$. For $x \in Y_a$, $y \in Y_b$, $a \leq b$, $p(x, y \mid s) = \prod_{a \leq k < b} s_k$, and hence $\partial_{s_k} p \in [0,1]$ vanishes unless x, y lie on different sides of the cut between $\bigcup_{i \leq k} Y_i$ and $\bigcup_{i > k} Y_i$. For $\varepsilon \in \{0,1\}^n$ let $\pi(\varepsilon)$ be the partition of $\{1, \dots, n+1\}$ into the maximal runs of consecutive indices joined by the k with $\varepsilon_k = 1$, $Y_J := \bigcup_{i \in J} Y_i$ (admissible), and $\lambda_\varepsilon(s) := \prod_k s_k^{\varepsilon_k}(1 - s_k)^{1 - \varepsilon_k}$. Then

$$\mathbf{C}(s) = \sum_{\varepsilon} \lambda_\varepsilon(s)\, \mathbf{C}_\varepsilon, \qquad \mathbf{C}_\varepsilon := \sum_{J \in \pi(\varepsilon)} \mathbf{1}_{Y_J} \mathbf{C} \mathbf{1}_{Y_J}, \tag{C.2}$$

since $\sum_\varepsilon \lambda_\varepsilon(s)\mathbf{C}_\varepsilon(x,y)$ is $\mathbf{C}(x,y)$ times the probability, for independent Bernoulli(s_k) variables ε_k, that a and b lie in one run. Hence $\lambda_{\min}(\mathbf{C}) \le \mathbf{C}(s) \le \|\mathbf{C}\| \le \alpha\tilde{m}^{-2}$, $\mathbf{C}(s)(x,x) = \mathbf{C}(x,x)$, $|\mathbf{C}(s)(x,y)| \le |\mathbf{C}(x,y)|$ (so each entrywise monotone norm of $\mathbf{C}(s)$ is at most that of $\mathbf{C}$), and $\|\mathcal{M}^{1/2}\mathbf{C}(s)\mathcal{M}^{1/2}\| \le \|\mathcal{M}^{1/2}\mathbf{C}\mathcal{M}^{1/2}\|$ for each symmetric $\mathcal{M} \ge 0$ commuting with all χ_i. For a sequence $\bar{y}$ of length n, we write $\mathbf{C}^{\bar{y}}(s,s_n)$ for (C.1) built from $\bar{y}$ ($s = (s_1,\dots,s_{n-1})$), $d\mu'^{\bar{y}}_{ss_n}$ for its Gaussian measure and $d\mu^{\bar{y}}_s$ for the measure at $s_n = 0$.

The weights of E. By (S1) and [GM82, (E.5a) to (E.5d)], E is a sum $\sum_\tau e_\tau$ of terms of two kinds: for $t \ge 3$ particles, $e_\tau = \frac{1}{t!}\rho_t(\xi_1,\dots,\xi_t)\,\varepsilon_k(\xi_1)\cdots\varepsilon_k(\xi_t)$, and for $t = 2$ the three terms $\frac12\rho_2\tilde{\varepsilon}_k(\xi_1)\tilde{\varepsilon}_k(\xi_2)$, $\frac12\rho_2\tilde{\varepsilon}_k(\xi_1)\,im_2\beta_{\mathrm{GM}}^{1/2}\psi(x_2)$ and $\frac12\rho_2\,im_1\beta_{\mathrm{GM}}^{1/2}\psi(x_1)\tilde{\varepsilon}_k(\xi_2)$. Here $x_i \in V$, $m_i \ne 0$, $\varepsilon_k(\xi) = e^{im(\beta_{\mathrm{GM}}^{1/2}(\psi+g)(x)+kf(x))} - 1$ and $\tilde{\varepsilon}_k(\xi) := \varepsilon_k(\xi) - im\beta_{\mathrm{GM}}^{1/2}\psi(x)$. With $\operatorname{supp}\tau := \{x_1,\dots,x_t\}$, $I(\tau)$ is the set of $k \le n$ such that $\operatorname{supp}\tau$ meets both $\bigcup_{i\le k} Y_i$ and $\bigcup_{i>k} Y_i$. We put $w_s(\tau) := \prod_{k\in I(\tau)} s_k$, $E(Y,s) := \sum_{\operatorname{supp}\tau\subseteq Y} w_s(\tau)e_\tau$ and $E(Y) := E(Y,1)$, and $E^{\bar{y}}(Y,(s,s_n))$ denotes $E(Y,\cdot)$ built from $\bar{y}$. These functions agree with $E(X,s)$ of [GM82, p. 89] and $E(\Lambda, ss_n)$ of [GM82, p. 90, (v)], with sites for the unit cubes a_i.

Absolute convergence. Let P_V be the cell-fluctuation projection, $(P_V\varphi)|_\Omega := \varphi|_\Omega - \operatorname{avg}_\Omega\varphi$, and $\|\nu\|_0 := \sup_x \sum_y |\nu(x,y)|$. Summing the fine Poincaré inequality on cells, given by Theorem 3.20 (R6) and Proposition 6.9 (iv), with the overlap N_{ov}, we obtain $\|P_V\varphi\|^2 \le C'_P L^2\|\nabla\varphi\|^2$, $C'_P := C_W^f(k_\square/L)^2 N_{\mathrm{ov}}$, and by (L8-2) of Subsection 7.6, $C'_P(\tilde{m}L)^2 \le \eta_H/8 < \frac17$. The constants c_V, b_V, C_α below serve only to prove convergence and may depend on G'.

Lemma C.1 (Growth of $\ln Q_h$)**.** *Let $\vartheta := \psi + D$. For every admissible Y, every sequence, every s and real ψ: (a) $E(Y,s)$ is real, and $E(Y,s) = B_{Y,s}(\psi) + \frac12\langle\psi, \nu_s^Y\psi\rangle$ with $|B_{Y,s}(\psi)| \le b_V$ and $\nu_s^Y(x_1,x_2) := \mathbf{1}_Y(x_1)\mathbf{1}_Y(x_2)w_s(x_1,x_2)\nu(x_1,x_2)$, where $w_s(x_1,x_2) := \prod_{a\le k<b} s_k$ for $x_1 \in Y_a$, $x_2 \in Y_b$, $a \le b$; moreover $\|\nu_s^Y\|_{2\to2} \le \|\nu\|_0 \le \varepsilon_\nu\tilde{m}^2$, and $|\partial^\alpha E(Y,s)| \le C_\alpha(1+\|\psi\|)$ for $|\alpha| \ge 1$; (b) $G_{2,1}(Y) \le \frac12(1 + C_{\mathrm{ref}}e^{-2\kappa_S})\tilde{m}^2\|\mathbf{1}_Y P_V\vartheta\|^2$; (c) $G_1(Y) \le c_V$, $|G_{2,2}(Y)| \le c_V(1 + \|\psi\|_1)$ and $|G_{2,3}(Y)| \le c_V$; for the terms $G^{\mathfrak{D}}_{2,3}$, $G^{\natural}_{2,3}$ of $G^{\natural}$ (Lemma 7.29), $G^{\mathfrak{D}}_{2,3}(Y) \le c_V$ and $G^{\natural}_{2,3}(Y) \le c_V + \frac12\theta'\tilde{m}^2\|\psi\|^2$; (d) $F_1^{\mathrm{loc}}(Y) \ge 0$ does not depend on ψ, $F_2(Y)$ is linear in ψ, and $|\partial^\alpha \ln Q_h(Y)| \le C_\alpha(1+\|\psi\|)$ for $|\alpha| \ge 1$.*

Sketch. (a) By (S3), the terms pair into complex conjugates. With $\varepsilon_k = \tilde{\varepsilon}_k + im\beta_{\mathrm{GM}}^{1/2}\psi$ the terms with $t = 2$ add up to $\frac12\rho_2\varepsilon_k\varepsilon_k$ plus a part with charge sum $\frac12\nu$. Using (S2$^+$), we bound $B_{Y,s}$ by $\sum_{t\ge2}\frac{2^t}{t!}\sum|\rho_t|$, and its ψ-derivatives by the same sums with the weights $(\sum_i|m_i|)^n$, and the Schur test with $0 \le w_s \le 1$ gives the norm bound. (b) Here $\tilde{\delta} = P_V\vartheta$, and by (S3) $G_{2,1}$ is a sum of terms $\rho^{\mathrm{ref}}(m)[\cos y - 1 + \frac12 y^2]$ of modulus at most $\frac12|\rho^{\mathrm{ref}}(m)|y^2$, since $0 \le \cos y - 1 + \frac12 y^2 \le \frac12 y^2$, with $\beta_{\mathrm{GM}}\sum_m m^2|\rho^{\mathrm{ref}}(m)| \le (1 + C_{\mathrm{ref}}e^{-2\kappa_S})\tilde{m}^2$. Part (c) is explicit on the finite set V, with Proposition 7.5 off $\mathfrak{D}$. Part (d) follows from Corollary 7.10, Proposition 7.8, and the fact that $\ln r_\Omega$ is smooth and periodic in A_Ω.

Lemma C.2 (Margin)**.** *Assume the standing hypothesis* (7.8)*, and let κ_S be so large that $(1 + C_{\mathrm{ref}}e^{-2\kappa_S})\eta_H/8 \le \frac17$, so that (L8-2) gives $(1 + C_{\mathrm{ref}}e^{-2\kappa_S})C'_P(\tilde{m}L)^2 \le \frac17$. Put $\mathcal{M} := (1 + C_{\mathrm{ref}}e^{-2\kappa_S})\tilde{m}^2P_V + (\varepsilon_\nu + \theta')\tilde{m}^2$ for $\mathbf{C} = C'$ and $\mathcal{M} := (1 + C_{\mathrm{ref}}e^{-2\kappa_S})\tilde{m}^2P_V + \varepsilon_\nu\tilde{m}^2$ for $\mathbf{C} = C$. Then $\mathcal{M} \ge 0$ is symmetric, commutes with $\mathbf{1}_\Omega$ for every cell Ω, $\mathcal{M} \ge (1 + C_{\mathrm{ref}}e^{-2\kappa_S})\tilde{m}^2P_V + \|\nu\|_0 + \theta'\tilde{m}^2\mathbf{1}_{[\mathbf{C}=C']}$, and*

$$\langle\varphi,\mathcal{M}\varphi\rangle \le q_M\langle\varphi,\mathbf{C}^{-1}\varphi\rangle, \qquad q_M \le \tfrac16\ (\mathbf{C}=C'), \quad q_M \le \tfrac27\ (\mathbf{C}=C). \tag{C.3}$$

Moreover $\|\mathcal{M}^{1/2}\mathbf{C}(s)\mathcal{M}^{1/2}\| \le q_M$ *for every sequence of admissible sets and every* s*, and the same holds for* $\mathbf{1}_U\mathbf{C}\mathbf{1}_U$ *on* $\mathbb{R}^U$ *whenever* U *is a union of cells.*

Proof. We control the cell fluctuations by the gradient through the Poincaré inequality on cells, and the constant part by the mass. Both are dominated by $C_0^{-1} = M_Y^{-2}\Delta^2 - \Delta + \tilde m^2 \ge \tilde m^2 - \Delta$, and $C_0^{-1} \le \alpha\mathbf{C}^{-1}$. The projection P_V is block diagonal over cells, and thus $\mathcal{M}$ commutes with each $\mathbf{1}_\Omega$. Moreover, $\|\nu\|_0 \le \varepsilon_\nu \tilde m^2$ by (S5). By the Poincaré inequality $\|P_V\varphi\|^2 \le C'_P L^2 \|\nabla\varphi\|^2$,

$$\langle\varphi, \mathcal{M}\varphi\rangle \le (1 + C_{\mathrm{ref}} e^{-2\kappa_S}) C'_P (\tilde m L)^2 \|\nabla\varphi\|^2 + (\varepsilon_\nu + \theta')\tilde m^2 \|\varphi\|^2 \le \tfrac{1}{7}\langle\varphi, C_0^{-1}\varphi\rangle,$$

which is at most $\frac{\alpha}{7}\langle\varphi, \mathbf{C}^{-1}\varphi\rangle$, with $\alpha = \alpha_D \le \frac{7}{6}$ for C' and $\alpha = 2$ for C (where θ' is absent). We now turn to the last statement, for which any $q_M < 1$ suffices. Inequality (C.3) says that $\|\mathcal{M}^{1/2}\mathbf{C}\mathcal{M}^{1/2}\| \le q_M$. Each Y_i and $V \setminus X$ is a union of cells, and hence $\mathcal{M}$ commutes with each χ_i and (C.2) applies. Since $\mathbf{1}_U$ commutes with $\mathcal{M}$, we also get $\|\mathcal{M}^{1/2}\mathbf{1}_U\mathbf{C}\mathbf{1}_U\mathcal{M}^{1/2}\| \le q_M$, and (C.2) applies on $\mathbb{R}^U$.

Lemma C.3. *Call* $F : \mathbb{R}^V \to \mathbb{C}$ admissible *if* F *is* C^2 *and, for* $|\alpha| \le 2$, $|\partial^\alpha F(\psi)| \le K_F(1 + \|\psi\|)^{d_F} \exp(\frac{1}{2}\langle\psi, \mathcal{M}\psi\rangle + k_F\|\psi\|_1)$*. Under the hypotheses of Lemma C.2: (a) for every admissible* Y *and* s'*, and every finite composition* $\mathfrak{L}$ *of operators* ∂_x *and of multiplications by sub-series, with weights in* $[0,1]$*, of the term expansions of* $E(\cdot, s')$*, of* $\partial_{s_n}E(\cdot, s')$ *and of their* ψ*-derivatives (all lie in one polynomial class), the function* $e^{E(Y,s')}\mathfrak{L}Q_h(Y)$ *is admissible, uniformly in* s'*; (b) if* $t \mapsto \mathbf{C}(t)$ *is one of the covariances* (C.1) *with one* s*-variable moving and* F *is admissible, then* $\int d\mu_{\mathbf{C}(t)}|F|$ *is bounded uniformly in* t *and in the other variables,* $t \mapsto \int d\mu_{\mathbf{C}(t)}F$ *is* C^1 *on* $[0,1]$*, and*

$$\partial_t \int d\mu_{\mathbf{C}(t)}F = \frac{1}{2}\sum_{x,y\in V} \partial_t\mathbf{C}(t)(x,y) \int d\mu_{\mathbf{C}(t)}\, \partial_x\partial_y F; \tag{C.4}$$

(c) if F *also depends on* t*, is* C^1 *in* t*, and* F *and* $\partial_t F$ *are admissible uniformly in* t*, then* $\partial_t \int d\mu_{\mathbf{C}(t)}F = \int d\mu_{\mathbf{C}(t)}\partial_t F$ *plus the right side of* (C.4)*.*

Sketch. (a) We collect the exponents of Lemma C.1 and use the lower bound on $\mathcal{M}$ of Lemma C.2. (b) By Lemma C.2 and (C.2), $\mathbf{C}(t)^{-1} \ge (\mathcal{M} + 2\epsilon)/q'$ with $q' := q_M + 2\epsilon\|\mathbf{C}\| < 1$ for small $\epsilon > 0$. Therefore $|F|g_t$, where g_t is the Gaussian density, is integrable uniformly in t. Since $\partial_t g_t = \frac{1}{2}\sum_{x,y}\partial_t\mathbf{C}(t)(x,y)\partial_x\partial_y g_t$, two integrations by parts give (C.4), the step "By partial integration" of [GM82, p. 91]. (c) This follows from the product rule under the same domination.

The operators κ**.** We fix $\bar y$ of length n with $X := X(\bar y) \subsetneq V$ and write $E := E^{\bar y}(V, (s, s_n)) = \sum_\tau w(\tau)e_\tau$, $E_x := \partial_x E$. As $e^{-E}\partial_x\partial_y(e^E F) = (\partial_x + E_x)(\partial_y + E_y)F$ is symmetric in x, y, and $\partial_{s_n}\mathbf{C}^{\bar y}(x,y) = 0$ unless x, y lie on different sides of X, Lemma C.3 (c) yields $\partial_{s_n}\int d\mu'^{\bar y}_{ss_n} e^E F = \int d\mu'^{\bar y}_{ss_n} e^E \tilde\kappa(n) F$ for F not depending on s_n with $e^E F$ admissible, where

$$\tilde\kappa(n) := \partial_{s_n}E + \sum_{x\in V\setminus X}\sum_{y\in X} \partial_{s_n}\mathbf{C}^{\bar y}(x,y)(\partial_x + E_x)(\partial_y + E_y), \tag{C.5}$$

as in [GM82, (E.15), (E.16)]. As GM note on p. 92, the factor $\frac{1}{2}$ of (E.16) cancels by symmetry. Inserting $E = \sum_\tau w(\tau)e_\tau$, we write $\tilde\kappa(n) = \sum_\theta \mathfrak{t}_\theta$ with elementary terms of five types: (1) $(\partial_{s_n}w(\tau))e_\tau$; (2) $\partial_{s_n}\mathbf{C}(x,y)\partial_x\partial_y$; (3) $\partial_{s_n}\mathbf{C}(x,y)w(\tau)(\partial_x\partial_y e_\tau)$; (4) $\partial_{s_n}\mathbf{C}(x,y)w(\tau)(\partial_y e_\tau)\partial_x$,

respectively $(\partial_x e_\tau)\partial_y$; (5) $\partial_{s_n}\mathbf{C}(x,y)w(\tau)w(\tau')(\partial_x e_\tau)(\partial_y e_{\tau'})$. The support $S(\theta)$ is $\operatorname{supp}\tau$ in type (1), and $\{x,y\}$ with the supports of the τ's involved otherwise. If $\mathfrak{t}_\theta \neq 0$, then $S(\theta) \not\subseteq X$, since in type (1) $n \in I(\tau)$ and otherwise $x \in V \setminus X$. We put $Y_{n+1}(\theta) := \operatorname{sat}_h(S(\theta) \setminus X)$ and

$$\kappa(n)[Y'] := \sum_{\theta:\, Y_{n+1}(\theta)=Y'} \mathfrak{t}_\theta, \qquad \tilde{\kappa}(n) = \sum_{Y'} \kappa(n)[Y'], \tag{C.6}$$

as in [GM82, (E.11), (E.20)]. With $\bar{y}' := (\bar{y}, Y')$ and $X' := X \cup Y'$, each θ in $\kappa(n)[Y']$ has $S(\theta) \subseteq X'$, coefficients depending on $\psi|_{X'}$ only, derivatives at $x \in Y'$, $y \in X$, and E-terms of $E^{\bar{y}'}(X', (s, s_n))$ with the same weights. Thus $\kappa(n)[Y']$ depends on $s_1, \dots, s_n$ only. Moreover, $\partial_{s_n}\mathbf{C}^{\bar{y}}(x,y) = s_a \cdots s_{n-1}\mathbf{C}(x,y) = \partial_{s_n}\mathbf{C}^{\bar{y}'}(x,y)$ for $x \in Y'$, $y \in Y_a$. For $\bar{y} = (Y_1, \dots, Y_m)$ the ordered product $\kappa(\bar{y}, s) := \kappa(m-1) \circ \cdots \circ \kappa(1)$, with $\kappa(i) := \kappa(i)[Y_{i+1}]$ built from $(Y_1, \dots, Y_i)$ and each operator acting on everything to its right, is $\kappa(\bar{y}, s)$ of [GM82, (E.9), (E.10)].

Proof of Proposition 7.22. We follow the proof of [GM82, Lemma E.1, Proposition E.2, (E.2) to (E.24)], an induction in n. Each step is the fundamental theorem of calculus [GM82, (E.17)] in a new variable s_{n+1}, started from the measure and the E of the previous step by the consistency and decoupling relations [GM82, (E.21a), (E.21b)], which hold on G' by (C.1) and the definition of $w_s(\tau)$. Lemma C.3 justifies each exchange of derivatives and integrals. The following changes are needed on G'.

(1) *Atoms are sites*, whereas the atoms of [GM82, p. 89] are unit cubes. Since each Y_j is admissible, $w_s(\tau)$ depends only on which Y_j contain the points.

(2) *One finite matrix with mass at every site*, and $\chi_{n+1} = \mathbf{1}_{V \setminus X}$, so that the region outside Λ of [GM82, p. 92] does not occur. Termination occurs at $m \leq e_h + 1$, since each Y_j, $j \geq 2$, contains an element outside Y_1 ([GM82, p. 93]).

(3) *First sets may depend on h and be disconnected.* In [GM82, p. 90, (iii)] the first set is a fixed union of unit cubes. Here the factorization $Q_h = Q_h(Y_1)Q_h(V \setminus Y_1)$ of Lemma 7.21 suffices for the step $n = 1$, and the admissibility of each $\bigcup_{j \leq i} Y_j$ for the selection rule; every comparison is made at one value of h. Type (2) terms reach the element of x at any distance at which $\mathbf{C}(x,y) \neq 0$, and therefore the sets $X(\bar{y})$ are disconnected in general.

(4) *On $U = X_0^c$ the covariance is $\mathbf{1}_U\mathbf{C}\mathbf{1}_U$*: marginals of Gaussian measures are compressions, and hence $\mathcal{K}_{\bar{y}}$ involves the covariance only through the block $p(x, y \mid s)\mathbf{C}(x,y)$, $x, y \in X(\bar{y})$. Lemma C.2 covers $\mathbf{1}_U\mathbf{C}\mathbf{1}_U$.

(5) *The selection rule: $S(\theta)$ contains the bare point x* of each covariance term, so that all derivatives of a term of $\kappa(n)[Y']$ act in $X \cup Y'$. By contrast, [GM82, (E.11)] is stated for the supports of the E-terms, and for the term without E-terms [GM82, p. 91; (E.19), p. 93] take $Y_{n+1} \ni x$.

(6) *"$\chi C \chi$" is an exact block-sum decomposition*: the convex combination of [GM82, p. 88] is (C.2), with compressions to admissible blocks. The bounds on $\mathbf{C}(s)$ in Lemmas C.2, C.3 and in Section 7 (Section S1.6) follow from this decomposition. The integrals $\int d\mu_{\mathbf{C}} e^{E(Y)} Q_h(Y)$ are positive, since $E(Y)$, G, F_1^{loc}, F_2 are real by Lemma C.1, and they are finite by Lemma C.3 (b).

D Heat kernel, coarse-to-fine transfer and covariance bounds on slit domains

In this appendix we give condensed proofs of Proposition 6.9 and Theorem 6.8 and sketch the proofs of Proposition 6.10, Theorem 6.11 and Lemma D.2. Full proofs of the last three are in Section S2.4. Throughout, $G' = (V, E)$ is an output, $d = d_{G'}$, $\mathcal{E}(f) := \sum_{\{x,y\}\in E}(f(x) - f(y))^2$, and $\partial U = \partial_{G'} U$ is the set of bonds of E with exactly one end in U.

D.1 Coarse-to-fine transfer: proof of Proposition 6.9

A large fine set either fills many coarse cells, whose boundary (R2) controls, or is spread thinly inside cells, where box isoperimetry applies (canonical paths and Loomis–Whitney). Counting faces combines the two estimates. The coarse cells, the sites $c \in V_c$ of the coarse slit graph $\Gamma = (V_c, E_c)$ of §6.3, are boxes B_c with sides in $[r_0, 2r_0 - 1]$ by Definition 3.4, and by §2.5 the box graph of $B_c \cup B_{c'}$, $\{c, c'\} \in E_c$, lies in G'. Thus a coarse bond between a *heavy* cell ($|A \cap B_c| \geq |B_c|/2$) and a light one carries at least $r_0^2/4$ bonds of ∂A in $B_c \cup B_{c'}$, and $|\partial A| \geq (r_0^2/24)|\partial_\Gamma H|$ for the set H of heavy cells. Combined with (R2), in the form of Theorem 4.5(i), (ii), this proves (i), and the same argument with the expansion of K in place of (R2) proves (iii). We deduce (ii) from (i) as in Theorem 4.5(iv), and from Theorem 4.5(ii) and $r_0^3 V_* = 2V_*^f$. Part (iv) is (iii) for the blow-up of the set $\mathfrak{K}_c^\flat$ of (R6) (Lemma 4.18) with Cheeger's inequality, the overlap count inherited and $|\Omega| \geq c_1 L^3$ from (R5). Finally, (v) holds because balls in a connected graph of degree at most 6 and expansion $h \leq 1$ grow by the factor $1 + h/6$ per step up to half the vertices, so the diameter is at most $2 + (24/h)\log|\mathfrak{K}^\flat| < L'$ under (L8-2). The constants are those of §S1.5.

D.2 Nash's inequality from Faber–Krahn, and the on-diagonal bound

Lemma D.1 (Nash inequality)**.** *For $f \geq 0$ on V with $\sum f = 1$, and every $r \geq 1$ with $2r^3 \leq V_*^f$, $\|f\|_2^2 \leq A_N r^2 \mathcal{E}(f) + r^{-3}$, where $A_N := 24 \cdot 2^{2/3}/c_F^2$.*

Proof. We put $s := 1/(2r^3)$. Then $f^2 \leq 2(f - s)_+^2 + 2\min(f, s)^2$ and $\sum \min(f, s)^2 \leq s$. The function $(f - s)_+$ vanishes off $\Omega_s := \{f > s\}$, which has $|\Omega_s| < 1/s = 2r^3 \leq V_*^f$, and $\mathcal{E}((f - s)_+) \leq \mathcal{E}(f)$. By Proposition 6.9(ii), $\sum(f - s)_+^2 \leq (12/c_F^2)(2r^3)^{2/3}\mathcal{E}(f)$.

Proof of (HK0) and (HK1). We obtain (HK0) from the degree bound alone, by Davies' perturbation argument. For $\psi := -\alpha d(x, \cdot)$,

$$\langle g, e^{\psi}\Delta_{G'}e^{-\psi}g\rangle = -\mathcal{E}(g) + 2\sum_{\{x,y\}\in E}(\cosh(\psi(x) - \psi(y)) - 1)g(x)g(y) \leq 6(\cosh\alpha - 1)\|g\|^2,$$

so $\|e^{\psi}P_t e^{-\psi}\|_{2\to 2} \leq e^{6(\cosh\alpha - 1)t}$ and $p_t(x, y) = e^{\psi(y)-\psi(x)}\langle\delta_x, e^{\psi}P_t e^{-\psi}\delta_y\rangle \leq e^{-\alpha d + 6(\cosh\alpha - 1)t}$. At $\alpha = \frac{1}{4}$, $6(\cosh\frac{1}{4} - 1) < \frac{1}{4} - \frac{1}{17}$. The number N_t of jumps up to time t is dominated by a Poisson variable of mean $6t$, and $d(x, X_t) \leq N_t$.

For (HK1), the function $\psi(t) := \|P_t\delta_x\|^2 = p_{2t}(x, x)$ is nonincreasing, $\psi \leq 1$ and $\psi' = -2\mathcal{E}(P_t\delta_x)$. With $r(t)$ given by $r^{-3} = \psi/2$, the condition $2r^3 \leq V_*^f$ becomes $\psi \geq 4/V_*^f$, which holds on an interval $[0, t_*]$. On this interval Lemma D.1 implies $-\psi' \geq \psi^{5/3}/(2^{2/3}A_N) = c_N\psi^{5/3}$, hence $\psi(t) \leq (\frac{2}{3}c_N t)^{-3/2}$, while $\psi < 4/V_*^f$ after t_*. Hence

$p_s(x,x) \le \max(C_H s^{-3/2}, 4/V_*^f)$, $p_s(x,y) \le (p_s(x,x)p_s(y,y))^{1/2}$, and $4/V_*^f = \frac{64}{27} R_{\mathrm{HK}}^{-3}$ (§S1.5). The zero modes lie below the plateau, since $p_t(x,x) \to 1/|\mathfrak{K}_x| < 1/(2V_*^f)$, the last inequality by Proposition 6.9(ii).

D.3 Barlow's argument: proof of Theorem 6.8

From now on we assume $\beta_{\mathrm{GM}} \ge \beta_{\mathrm{GM}}^0$, so that $\tilde{m} \le K_0(2C_H)^{-1/3}$, $R_{\mathrm{HK}} \ge K_0/\tilde{m} \ge (2C_H)^{1/3}$, and $p_t \le C_H t^{-3/2}$ on $[1, R_{\mathrm{HK}}^2]$ by (HK1). We obtain (HK1′) as in [Bar04, Lemma 3.3(b), Prop. 3.4], (HK2) as in [Bar04, Lemmas 3.5, 3.6, Prop. 3.7], and (HK3) as the fixed-time form of [Bar04, Thm 3.8], with the following adjustments, each forced by G':

(a) [Bar04, Prop. 3.1, Cor. 3.2], the on-diagonal decay in a very good ball for $t \le cR^2/\log R$, are replaced by (HK1), which holds at all centres. Every ball is a very good ball with $N_B = 1$ [Bar04, Def. 1.7], there is no localization and no $\log R$ loss, and the window is $t \le \varepsilon_1^{\mathrm{HK}} R_{\mathrm{HK}}^2$. The exit-time bound $\frac{1}{2} + 4C_{\mathrm{mo}}^2 t/r^2$ of [Bar04, Lemma 3.5] holds at all centres for $t \le R_{\mathrm{HK}}^2/2$, and exceeds 1 for larger t when $r \le 2C_{\mathrm{mo}} R_{\mathrm{HK}}$.

(b) The walk is the variable-speed walk with counting measure on a finite, possibly disconnected graph, and [Bar04, (3.8) to (3.10)] hold without change, with $a_{xy} = 1$, $\mu \equiv 1$, as finite sums.

(c) The volume bound [Bar04, (1.1)] enters only [Bar04, Lemma 3.3(b)], through $\sum_y e^{-a\,d(x,y)} \le c_4 a^{-3}$ for $0 < a \le 2$, which follows from $d_{G'} \ge \|\cdot\|_1$: $\sum_y e^{-a\,d(x,y)} \le \sum_{z\in\mathbb{Z}^3} e^{-a\|z\|_1} = \coth(a/2)^3 \le (1+2/a)^3 \le 64a^{-3}$.

(d) [Bar04, Lemma 1.1] is replaced by (HK0), (HK1) and the Poisson domination of N_t. Let $M(t) := \mathbb{E}_x d(x, X_t)$. By (HK1), $M(t) \ge \mathbb{P}_x(X_t \ne x) \ge 1 - C_H t^{-3/2} \ge \frac{1}{2}$ for $t \ge (2C_H)^{2/3}$, and the bound by N_t gives $M(t) \le 6t$ at earlier times. For $\rho > A_1 t/4$ (A_1 of §S1.5), the event $\tau(x,\rho) < t$ forces $N_t > \rho$.

(e) (HK4) is not in [Bar04], but it follows from (HK2). With $D := d(w, X_s)$, we split the sum in $\mathbb{E}e^{\theta D} - 1 \le \theta \sum_{k\ge1} e^{\theta k}\mathbb{P}(D \ge k)$ at $k = s$ into a Gaussian part, summed with $\theta \le s^{-1/2}$, and a Poisson part, summed with $\theta \le c_{\mathrm{ex}}/2$. For $s < 1$, $D \le N_s$.

For (HK3), Chapman–Kolmogorov gives $p_t(x,y) \le 2\sup_{z,w} p_{t/2}(z,w) \sup_w \mathbb{P}_w(d(w, X_{t/2}) \ge d/2)$, and (HK1), (HK2) bound the two factors. The constants $C_{\mathrm{mo}}, c_{\mathrm{ex}}, A_1, C_{\mathrm{gs}}, c_{\mathrm{gs}}, C_M$ of §S1.5 are functions of C_H.

D.4 Resolvent decay and covariance bounds

For Proposition 6.10, (HK4) gives $\mathbb{E}_w e^{\theta_M d(w,X_s)} \le \frac{3}{2}$ for $s \le T_1 := \min(M^{-2}, \varepsilon_1^{\mathrm{HK}} R_{\mathrm{HK}}^2)$. Using blocks of length T_1, we obtain $\mathbb{E}_x e^{\theta_M d(x,X_t)} \le \frac{3}{2} e^{M^2 t/2}$, and the factor $e^{-M^2 t}$ controls this growth in $\sum_y v_{G'}^{(M)}(x,y) e^{\theta_M d(x,y)} = \int_0^\infty e^{-M^2 t} \mathbb{E}_x e^{\theta_M d(x,X_t)} dt$. For Theorem 6.11 we write $C_0 = \int_0^\infty k(t) P_t\, dt$ with $0 \le k(t) \le \min(M_Y^2 t, \varpi) e^{-r_- t}$ (§S1.5). With this representation, (H1), (H1′), (H2) follow from (HK0), (HK1), (HK3), (6.13) and the blocking bound by splitting the time integral, and (H_C) follows from the Schur test and a Neumann series in $\|\cdot\|_{r_{\mathrm{cov}}}$.

D.5 Bounds on the lattice Yukawa kernel

Lemma D.2 (Lattice Yukawa kernel)**.** *Let $0 < M \le \frac{1}{5}$ and $v^{(M)} := (-\Delta_{\mathbb{Z}^3} + M^2)^{-1}$. There is an absolute constant C_L such that for all $x \in \mathbb{Z}^3$ and all unit vectors e,*

$$v^{(M)}(x) \le C_L(1+|x|)^{-1} e^{-3M|x|/4}, \qquad |v^{(M)}(x+e) - v^{(M)}(x)| \le C_L(1+|x|)^{-2} e^{-3M|x|/4}. \tag{D.1}$$

If $\Lambda' = \prod_\mu [1, n_\mu]$ *is a box with* $\min_\mu n_\mu \geq M^{-1}$ *and* $u_x := v^{(M)}_{\Lambda'}(\cdot, x)$ *is the kernel of the box with all its bonds, then for* $x, y, y + e \in \Lambda'$*, with* C'_L *absolute,*

$$\text{(L1)} \quad u_x(y) \leq C'_L (1 + |y - x|)^{-1} e^{-M|y-x|/2},$$
$$\text{(L2)} \quad |u_x(y + e) - u_x(y)| \leq C'_L (1 + |y - x|)^{-2} e^{-M|y-x|/2}.$$

Sketch. Exponential tilts of the three one-dimensional heat kernels and the unimodality of the Poisson law give (D.1) after the time integral, the rate $3M/4$ coming from $M^2 t + \frac{9}{64}|x|^2/t \geq \frac{3}{4}M|x|$. The box kernel is the image sum $v^{(M)}_{\Lambda'}(y, x) = \sum_{g \in \mathcal{G}} v^{(M)}(y - gx)$ over the group $\mathcal{G}$ generated by the reflections in the face planes. Since $\min_\mu n_\mu \geq M^{-1}$, the images cost a factor $(1 + M|y - x|)^3$, which the loss of rate from $3M/4$ to $M/2$ absorbs.

E Comparing Wilson and Villain loops by correlation inequalities

Numbers E.n refer to the statements and subsections of this appendix, while equation numbers (E.n), and statement numbers cited with [GM82], refer to Appendix E of [GM82].

Results for the Wilson action are often obtained by comparison with the Villain action, or by redoing the Villain analysis with additional estimates [FS81a, Section 6 and Appendix B], [FS82]. For upper bounds on Wilson loops, which is the direction needed for confinement, Theorem 2.5 bounds the Wilson loop by an average of Villain loops in which a sparse random set of plaquettes is frozen. Freezing turns the problem into a Villain model on a domain with insulating walls, and this is where most of the length of this paper lies. It is natural to ask whether a better comparison avoids frozen plaquettes.

Two facts are known. First, the character coefficients $I_k(\beta)/I_0(\beta)$ of the Wilson weight, with $I_k(\beta) \sim (\beta/2)^k/k!$ as $k \to \infty$, decay more slowly than those of any Villain weight, $e^{-k^2/(2t)}$, so no Villain plaquette weight dominates the Wilson weight coefficient by coefficient (Section 1.3). Second, Ginibre's inequality [Gin70] transfers to Villain weights through complexes that realize them as limits: cable graphs for spin systems, credited to Bellissard in [FS81b, Section 3] and developed in [AHPS21, Appendix A] and [DG25, DvEG26], and carpet graphs for lattice gauge theory [CG25]. Such arguments give inequalities for all characters at once.

To our knowledge, the following box version of the first fact is new. On every finite box, an inequality between Wilson and Villain loops that holds for all charges, even with a perimeter factor, forces the Villain side to contain arbitrarily large couplings (Corollary E.4 and Proposition E.6). The proof pairs a lower bound from Ginibre's inequality with the spin-wave upper bound for Villain loops, which we extend to Villain mixtures (Theorem E.2 and Remark E.5). We complement it with three related results. Wilson loops are not monotone in the character coefficients of the plaquette weights (Theorem E.8, proved by an exact computation). If a character coefficient of a weight exceeds that of the Villain weight $\tilde{v}_t$, every representation of the weight as a mixture of Villain weights carries an explicit minimal mass of couplings above t (Proposition E.10). Freezing plaquettes has increasing returns: the effect of freezing two plaquettes can exceed the product of the effects of freezing each of them (Proposition E.12).

Section numbers of [FS81a] and [FS81b] refer to their preprints IHES/P/81/25 and IHES/P/80/10.

E.1 Setting

Complexes and loops. We use the notation of Section 2 for a finite 2-dimensional cell complex X: $E(X)$, $P(X)$, $[p:e]$, θ_p, ∂, $Z_2(X)$ and θ_C. For $n \in \mathbb{Z}^{P(X)}$ we have $\theta_{\partial n} = \sum_p n_p \theta_p$, and we set $|\partial n| = \sum_e |(\partial n)_e|$. We let $L(X) = \ker(\partial \otimes \mathbb{R}) \subseteq \mathbb{R}^{P(X)}$, with the standard inner product of $\mathbb{R}^{P(X)}$. Since ∂ is an integer matrix, $Z_2(X)$ is a lattice of full rank in $L(X)$. We write $\delta_p \in \mathbb{Z}^{P(X)}$ for the indicator of p. In a box of $\mathbb{Z}^3$ every integer 1-cycle is a boundary ∂n.

Weights. A *plaquette weight* is an even continuous function $w\colon \mathbb{T} \to [0,\infty)$ whose Fourier coefficients $c_k(w) = \int_{\mathbb{T}} w(\varphi) e^{-ik\varphi}\, d\varphi/2\pi$ are nonnegative and summable, with $c_0(w) = 1$. The Wilson weight is $w_\beta^W(\varphi) = e^{\beta\cos\varphi}/I_0(\beta)$, with $c_k(w_\beta^W) = I_k(\beta)/I_0(\beta)$. The Villain weight $\tilde{v}_t$ of Section 2 has $c_k(\tilde{v}_t) = e^{-k^2/(2t)}$ for $t > 0$, and $\tilde{v}_0 = 1$ is a removed plaquette. A weight is of *Ginibre type* if it is proportional to e^f, where $f(\varphi) = \sum_{j=1}^J \lambda_j \cos j\varphi$ with all $\lambda_j \geq 0$. The weight w_β^W is of Ginibre type. A *Villain mixture with couplings at most* τ is $w = \int_{[0,\tau]} \tilde{v}_t\, \nu(dt)$ for a probability measure ν on $[0,\tau]$. Its coefficients satisfy $c_k(w) \leq e^{-k^2/(2\tau)}$ for $k \neq 0$.

Gibbs measures. Given weights $w = (w_p)_{p\in P(X)}$, let $\mu_{X,w}$ be the probability measure on $\mathbb{T}^{E(X)}$ proportional to $\prod_p w_p(\theta_p) \prod_e d\theta_e/2\pi$, with expectation $\langle\cdot\rangle_{X,w}$. If $w_p = w_\beta^W$ for all p, this is the Wilson expectation $\langle\cdot\rangle_{X,\beta}^W$ of Section 2. For $n \in \mathbb{Z}^{P(X)}$ the Wilson loop of ∂n is $\cos\theta_{\partial n}$. We use one monotonicity property: if every w_p is of Ginibre type, with coefficients $\lambda_{p,j}$, then $\langle\cos\theta_{\partial n}\rangle_{X,w}$ is nondecreasing in every $\lambda_{p,j}$, by Ginibre's inequality (2.1).

Villain monotonicity states that $\langle\cos\theta_{\partial n}\rangle_{X,w}$ is nondecreasing in every t_p if $w_p = \tilde{v}_{t_p}$. It is proved on the lattice in [Gut80, Lemma C.1], on the cubes $\{x \in \mathbb{Z}^d : |x|_\infty \leq L\}$ in [CG25, Corollary 1.6], and on finite two-dimensional complexes in Lemma 2.3(b). We do not use it in this appendix.

Lemma E.1. *For weights w and $n \in \mathbb{Z}^{P(X)}$,*

$$\langle\cos\theta_{\partial n}\rangle_{X,w} = \frac{\hat{Z}(n)}{\hat{Z}(0)}, \qquad \hat{Z}(n) = \sum_{m\in Z_2(X)} \prod_{p\in P(X)} c_{|n_p+m_p|}(w_p).$$

Proof. Each series $w_p(\theta_p) = \sum_k c_k(w_p) e^{ik\theta_p}$ converges absolutely and uniformly. We expand the product of the weights times $e^{i\theta_{\partial n}}$ and integrate term by term. The term indexed by $l \in \mathbb{Z}^{P(X)}$ contributes $\prod_p c_{|l_p|}(w_p)$ if $\partial(l+n) = 0$, and nothing otherwise. Writing $l = m - n$ with $m \in Z_2(X)$ and replacing m by $-m$ gives $\hat{Z}(n)$. The substitution $\theta \mapsto -\theta$ shows that $\langle e^{i\theta_{\partial n}}\rangle_{X,w} = \langle\cos\theta_{\partial n}\rangle_{X,w}$. The term $m = 0$ gives $\hat{Z}(0) \geq 1$.

For Villain weights on a box, Lemma E.1 is Lemma 2.7.

E.2 All-charge comparisons

Theorem E.2. *Let $n_0 \in \mathbb{Z}^{P(X)}$ with $\partial n_0 \neq 0$, and put $\kappa = \operatorname{dist}(n_0, L(X))^2$. Then $\kappa > 0$, and the following hold.*

(a) If every w_p is a Villain mixture with couplings at most τ, then $\langle\cos k\theta_{\partial n_0}\rangle_{X,w} \leq e^{-\kappa k^2/(2\tau)}$ for every $k \in \mathbb{Z}$.

(b) If every w_p is of Ginibre type and $\partial\delta_{p_0} \neq 0$, then $\langle \cos k\theta_{p_0}\rangle_{X,w} \geq c_k(w_{p_0})$ for every $k \geq 0$. In particular $\langle \cos k\theta_{p_0}\rangle_{X,w} \geq I_k(\beta)/I_0(\beta)$ if $w_{p_0} = w_\beta^W$.

The proof of (a) uses the following elementary fact.

Lemma E.3. *Let Γ be a lattice of full rank in a finite-dimensional real inner product space L with norm $\|\cdot\|$, and let $x \in L$. Then $\sum_{m\in\Gamma} e^{-\|m+x\|^2/2} \leq \sum_{m\in\Gamma} e^{-\|m\|^2/2}$.*

Proof. By Poisson summation the left side equals $c\sum_{\xi\in\Gamma^*} e^{-2\pi^2\|\xi\|^2} e^{2\pi i\langle \xi,x\rangle}$, where Γ^* is the dual lattice and $c > 0$ depends only on Γ. All coefficients are positive, so the sum is largest at $x = 0$.

Proof of Theorem E.2. Since $\partial n_0 \neq 0$, the vector n_0 does not lie in $L(X)$, so $\kappa > 0$.

(a) Write $w_p = \int \tilde{v}_t\, \nu_p(dt)$ and $\nu = \bigotimes_p \nu_p$. We have $c_j(\tilde{v}_t) = e^{-j^2/(2t)}$, read as $\mathbf{1}_{j=0}$ when $t = 0$, and $c_j(w_p) = \int c_j(\tilde{v}_t)\, \nu_p(dt)$ by Fubini's theorem. By Lemma E.1 and Tonelli's theorem, $\hat{Z}(n) = \int \hat{Z}_t(n)\, \nu(dt)$ for $n \in \mathbb{Z}^{P(X)}$, where

$$\hat{Z}_t(n) = \sum_{m\in Z_2(X)} \prod_{p\in P(X)} e^{-(n_p+m_p)^2/(2t_p)}.$$

It suffices to show $\hat{Z}_t(kn_0) \leq e^{-\kappa k^2/(2\tau)} \hat{Z}_t(0)$ for every $t \in [0,\tau]^{P(X)}$. Let first $t \in (0,\tau]^{P(X)}$. We equip $\mathbb{R}^{P(X)}$ with the inner product $\langle x, y\rangle_t = \sum_p x_p y_p/t_p$ and its norm $\|\cdot\|_t$, so that $\hat{Z}_t(n) = \sum_{m\in Z_2(X)} e^{-\|n+m\|_t^2/2}$. Let Π_t be the orthogonal projection onto $L(X)$ for this inner product, and $\kappa_t = \|n_0 - \Pi_t n_0\|_t^2$. For $m \in Z_2(X)$ we have $\|kn_0 + m\|_t^2 = k^2\kappa_t + \|k\Pi_t n_0 + m\|_t^2$. Lemma E.3 with $\Gamma = Z_2(X)$ and $x = k\Pi_t n_0$ gives $\hat{Z}_t(kn_0) \leq e^{-k^2\kappa_t/2} \hat{Z}_t(0)$. Since $t_p \leq \tau$, we have $\|x\|_t^2 \geq |x|^2/\tau$ for all x, and hence $\kappa_t \geq \kappa/\tau$. If some coordinates of t vanish, we replace them by $s \in (0,\tau]$. As $s \downarrow 0$, every term of the resulting sums decreases to the corresponding term for t, and the inequality passes to the limit by dominated convergence.

(b) By Ginibre's inequality, $\langle \cos k\theta_{p_0}\rangle_{X,w}$ decreases when all coefficients $\lambda_{p,j}$ with $p \neq p_0$ are lowered to 0. Then $\mu_{X,w}$ is proportional to $w_{p_0}(\theta_{p_0}) \prod_e d\theta_e/2\pi$. Since $\partial\delta_{p_0}$ is a nonzero integer vector, $\theta \mapsto \theta_{p_0}$ is a continuous homomorphism of $\mathbb{T}^{E(X)}$ onto $\mathbb{T}$, so it pushes Haar measure to Haar measure. Hence the expectation equals $\int \cos(k\varphi) w_{p_0}(\varphi)\, d\varphi/2\pi = c_k(w_{p_0})$.

Corollary E.4. *Let $p_0 \in P(X)$ with $\partial\delta_{p_0} \neq 0$, and let $\beta > 0$. Let $v = (v_p)_{p\in P(X)}$ be Villain mixtures with couplings at most τ. For all constants $a, b \geq 0$ there is $k \geq 1$ with*

$$\langle \cos k\theta_{p_0}\rangle_{X,\beta}^W > a e^{bk|\partial\delta_{p_0}|} \langle \cos k\theta_{p_0}\rangle_{X,v}.$$

In particular no inequality $\langle \cos\theta_{\partial n}\rangle_{X,\beta}^W \leq a e^{b|\partial n|} \langle \cos\theta_{\partial n}\rangle_{X,v}$ holds for all $n \in \mathbb{Z}^{P(X)}$.

Proof. Let $\kappa_{p_0} = \operatorname{dist}(\delta_{p_0}, L(X))^2 > 0$. By Theorem E.2 and $I_k(\beta) \geq (\beta/2)^k/k! \geq (\beta/2k)^k$, the left side is at least $(\beta/2k)^k/I_0(\beta)$ and the right side is at most $a e^{bk|\partial\delta_{p_0}| - \kappa_{p_0}k^2/(2\tau)}$. The ratio of the two bounds tends to infinity as $k \to \infty$.

Remark E.5. (i) For $w_p = \tilde{v}_\tau$ on every plaquette, part (a) is the classical spin-wave upper bound: the Villain loop is at most the value $e^{-\kappa k^2/(2\tau)}$ of the same observable for the Gaussian spin wave. It follows from the decoupling of the Villain model into a spin wave and a Coulomb gas [GS23, Proposition 5.2], and in three dimensions it gives the bound of Glimm and Jaffe

[GJ77], see [GS23, Corollary 7.3]. In the proof above, Lemma E.3 bounds the Coulomb-gas factor by 1. Part (a) adds uniformity over Villain mixtures with couplings at most τ. (ii) For a box Λ of $\mathbb{Z}^3$, $L(\Lambda)$ is spanned by the boundaries of the cubes of Λ. These are the stars of the dual graph of Section 2.4, whose sites are the cubes and the exterior ground site, as a network with unit conductances. The matrix of the orthogonal projection onto their span is the transfer-current matrix Y [LP16, Section 2.4]. Hence $\operatorname{dist}(\delta_p, L(\Lambda))^2 = 1 - Y(p,p)$, and $Y(p,p)$ is the effective resistance across p [LP16, Section 4.2]. For a single cube, $\kappa = 5/6$.

Comparisons are often made through auxiliary complexes. Let G be a finite two-dimensional complex with plaquette weights w, and let $\gamma \in \mathbb{Z}^{P(G)}$ with $\partial\gamma \neq 0$. The *weight realized by* (G, γ, w) is the density w_G of the law of $\theta_{\partial\gamma}$ under $\mu_{G,w}$ with respect to $d\varphi/2\pi$. Its coefficients are $c_k(w_G) = \langle \cos k\theta_{\partial\gamma} \rangle_{G,w}$. If a copy of G is glued into a plaquette p of X along $\partial\gamma = \partial\delta_p$ in place of w_p, then integrating out the new edges replaces w_p by w_G. Cable graphs and carpet graphs are of this kind.

Proposition E.6. *In this setting the following hold.*

(a) If every plaquette p of G carries a weight of Ginibre type, with coefficients $\lambda_{p,j}$, then each $c_k(w_G)$ is nondecreasing in every $\lambda_{p,j}$.

(b) If every plaquette of G carries a Villain mixture with couplings at most τ, then $c_k(w_G) \leq e^{-\kappa_G k^2/(2\tau)}$ with $\kappa_G = \operatorname{dist}(\gamma, L(G))^2 > 0$.

In particular, let $\beta > 0$. If (G, γ, w) realizes w_β^W and every weight of G is of Ginibre type, then raising any of the coefficients $\lambda_{p,j}$ produces a weight w' with $c_k(w') \geq I_k(\beta)/I_0(\beta)$ for all k, and no complex whose plaquettes carry Villain mixtures with bounded couplings realizes such a weight.

Proof. Part (a) is Ginibre's inequality applied to $\cos k\theta_{\partial\gamma}$ on G, and part (b) is Theorem E.2(a) on G with $n_0 = \gamma$. The last assertion follows, since $(\beta/2)^k/(k!\, I_0(\beta))$ is not bounded by $e^{-\kappa_G k^2/(2\tau)}$ for large k.

Remark E.7. Ginibre's inequality, Villain monotonicity, complexes as in Proposition E.6, and the annealed Hartman–Watson representation with monotone rounding of the couplings (Theorem 2.5) all give inequalities for all characters at once. By Corollary E.4 none of them bounds Wilson loops by Villain loops with bounded couplings, even on one box. If the Villain side of such a comparison is a Villain mixture, its mixing measure gives positive mass to arbitrarily large couplings, and we bound the mass of large couplings from below in Section E.4. In the comparison of Theorem 2.5, which holds for every $\varepsilon > 0$, the plaquettes that carry these large couplings, those above $1/\varepsilon$, form a random set of frozen plaquettes: their couplings are raised to infinity.

E.3 Monotonicity in character coefficients

Ginibre's inequality gives monotonicity in the Fourier coefficients of the logarithm of a weight, and Villain monotonicity raises all coefficients $c_k(\tilde{v}_t)$ at once. Wilson loops are not monotone in each coefficient $c_k(w_p)$ of the weights themselves.

Let X be the boundary of the unit cube, with its 8 vertices, 12 edges and 6 faces oriented by the outward normal, and fix a face p_0. For a weight w and $x \geq 0$ let $w_x = w + 2(x - c_1(w))\cos\varphi$, so that $c_1(w_x) = x$ and $c_k(w_x) = c_k(w)$ for $k \neq \pm 1$. Put w_x on p_0 and w on the other five faces, and let $\mathcal{W}(x) = \hat{Z}(\delta_{p_0})/\hat{Z}(0)$ as in Lemma E.1. When $w_x \geq 0$, this is $\langle \cos\theta_{p_0} \rangle$.

Theorem E.8. *Let w be either the Wilson weight w_8^W or the Villain weight $\tilde v_t$ with $e^{-1/(2t)} = 15/16$. Then $\mathcal W$ is strictly decreasing on $[0,\infty)$. In particular, for $0 < s \le s_w = \frac12 \min_{\mathbb T} w$ the function $w + 2s\cos\varphi$ is a plaquette weight with c_1 larger than that of w and all other coefficients equal, and its Wilson loop is strictly smaller. Here $s_w = e^{-8}/(2I_0(8))$ for $w = w_8^W$, and $s_w > 0$ for $w = \tilde v_t$. Moreover, if every face carries $w + 2s\cos\varphi$, then $\frac{d}{ds}\langle\cos\theta_{p_0}\rangle$ is negative at $s = 0$.*

Proof. The complex X is a sphere, so $Z_2(X) = \mathbb Z\mathbf 1_{P(X)}$, where $\mathbf 1_{P(X)} = \sum_p \delta_p$. Write $c_k = c_k(w)$. Lemma E.1 with $n = \delta_{p_0}$ and $m = -h\mathbf 1_{P(X)}$ gives $\hat Z(\delta_{p_0}) = \sum_{h\in\mathbb Z} c'_{|h-1|}c^5_{|h|}$ and $\hat Z(0) = \sum_h c'_{|h|}c^5_{|h|}$, where c' denotes the coefficients of the weight on p_0. Collecting the terms that contain $c'_1 = x$ we get

$$\mathcal W(x) = \frac{A_C + xB_C}{A_0 + xB_0}, \qquad \begin{aligned} A_0 &= 1 + 2\textstyle\sum_{h\ge2} c_h^6, & B_0 &= 2c_1^5,\\ A_C &= c_1^5 + \textstyle\sum_{h\ge3} c_{h-1}c_h^5 + \sum_{h\ge1} c_{h+1}c_h^5, & B_C &= 1 + c_2^5. \end{aligned}$$

Hence $\mathcal W'(x) = S/(A_0 + xB_0)^2$ with $S = A_0B_C - A_CB_0$, a number independent of x. It remains to show $S < 0$. If every face carries $w + 2s\cos\varphi$, the derivative at $s = 0$ has the sign of $U = Z'_CZ_0 - Z_CZ'_0$, where $Z_C = A_C + c_1B_C$, $Z_0 = A_0 + c_1B_0$, $Z'_C = B_C + 5c_1^4(1 + c_2)$ and $Z'_0 = 12c_1^5$.

We evaluate S and U in exact rational arithmetic with intervals. Both are homogeneous of degree 11 in (c_k), so for w_8^W we may replace c_k by $a_k = I_k(8) = \sum_{m\ge0} 4^{2m+k}/(m!\,(m+k)!)$. The partial sums with $m < 80$ are rational lower bounds. Consecutive terms have ratio at most $16/(81(81+k))$ beyond $m = 80$, so we bound the remainder by a geometric series. Comparing the series term by term gives $I_{k+1}(8) \le 4I_k(8)/(k+1)$, and with it we bound the parts of the sums over h with $|h| > 40$ by geometric series. For $\tilde v_t$ the coefficients $c_k = (15/16)^{k^2}$ are rational, and we bound the parts with $|h| > 40$ again by geometric series. All operations are exact, with interval endpoints rounded outward. This computation, carried out by a program of the scripts archive of the paper (see the Data availability statement), gives $S < 0$ and $U < 0$ in both cases, and the Lean kernel decides the same inequalities in the formalization [Tom26].

Finally, $\min w_8^W = e^{-8}/I_0(8)$ is attained at $\varphi = \pi$, and $\tilde v_t > 0$ is continuous. For $0 < s \le s_w$ the function $w + 2s\cos\varphi$ is nonnegative, even and continuous, with nonnegative summable coefficients.

The enclosures are $S = -2.2085\ldots\cdot10^{28}$ and $U = -2.5854\ldots\cdot10^{28}$ for w_8^W, with c_k replaced by $I_k(8)$, and $S = -0.2776\ldots$ and $U = -0.3255\ldots$ for $\tilde v_t$. At $x = c_1$ the program also encloses the loop and its derivative: $\langle\cos\theta_{p_0}\rangle = 0.9464\ldots$ and $\mathcal W'(c_1) = -0.0321\ldots$ for w_8^W, and $\langle\cos\theta_{p_0}\rangle = 0.9476\ldots$ and $\mathcal W'(c_1) = -0.0342\ldots$ for $\tilde v_t$.

Remark E.9. (i) The weight w_8^W is of Ginibre type, so by Ginibre's inequality its Wilson loops increase with every coefficient of $\log w_8^W$. Raising c_1 alone changes every Fourier coefficient of the logarithm, not all in the same direction, and Theorem E.8 shows that this can lower the loop. (ii) Villain monotonicity, Lemma 2.3(b), is therefore not an instance of a general monotonicity in character coefficients, in line with the observation of [CG25, Remark 1.7] that $\log\tilde v_t$ is not positive definite. (iii) The phenomenon comes from light tails and large β or t. The Poisson kernel $P_r(\varphi) = \sum_k r^{|k|}e^{ik\varphi}$, $0 < r < 1$, has $\log P_r$ with nonnegative Fourier coefficients $r^{|k|}/|k|$ for $k \ne 0$, so Ginibre's inequality applies to it by approximation. For P_r on all faces, summing geometric series in the proof of Theorem E.8 gives $S = 1 - r^{10} > 0$. Heuristically, for w_β^W or $\tilde v_t$ the dual variable h of the proof is close to a centered Gaussian of variance about

$\beta/6$ or $t/6$. On p_0 the loop replaces h by $h-1$, whose law is close to a Gaussian with the same variance and mean $-5/6$. When the variance exceeds 1, shifting a Gaussian lowers the probability of $\{|h|=1\}$ to second order.

E.4 Villain-mixture representations

A *Villain-mixture representation* of a plaquette weight w is a probability measure ω on $(0,\infty)$ with $c_k(w)=\int e^{-k^2u/2}\,\omega(du)$ for all $k\in\mathbb{Z}$. The variable u is the inverse coupling, so that $w=\int \tilde v_{1/u}\,\omega(du)$. For w_β^W the Hartman–Watson law ω_β of Lemma 2.2 is such a representation [HW74, Theorem 1.3]. Yor [Yor80] described its density, and Patie [Pat09, Remark 2.7] relates it to the infinitely divisible laws of Hartman [Har76]. It is used in comparisons in [AHPS21, (9.10)] and in Section 2. By Lemma 2.2 it is the only probability measure that reproduces $I_\nu(\beta)/I_0(\beta)$ for every real $\nu\ge 0$.

Proposition E.10. *Let ω be a Villain-mixture representation of a weight w with coefficients c_k, let $\varepsilon>0$, and let $1\le k_1<k_2$ be integers with $c_{k_1}>e^{-k_1^2\varepsilon/2}$. Then*

$$\omega((0,\varepsilon)) \;\ge\; \bigl(c_{k_1}-e^{-k_1^2\varepsilon/2}\bigr)^{k_2^2/(k_2^2-k_1^2)}\, c_{k_2}^{-k_1^2/(k_2^2-k_1^2)}.$$

Moreover $\omega((0,\varepsilon))\le e^{k^2\varepsilon/2}c_k$ for every k.

In terms of couplings, $\omega((0,\varepsilon))$ is the mass of couplings above $1/\varepsilon$. For $\omega=\omega_\beta$ the last bound is the tail bound of Lemma 2.2.

Proof. Let $F(s)=\int_{(0,\varepsilon)} e^{-su}\,\omega(du)$ for $s\ge 0$. By Hölder's inequality $\log F$ is convex. Put $s_i=k_i^2/2$. Then $F(s_2)\le c_{k_2}$, and $F(s_1)\ge c_{k_1}-\int_{[\varepsilon,\infty)} e^{-s_1u}\omega(du)\ge c_{k_1}-e^{-s_1\varepsilon}>0$. Convexity on $[0,s_2]$ gives $\log F(s_1)\le (1-s_1/s_2)\log F(0)+(s_1/s_2)\log F(s_2)$, that is

$$\log F(0)\ge \frac{s_2\log F(s_1)-s_1\log F(s_2)}{s_2-s_1},$$

and the bounds on $F(s_1)$ and $F(s_2)$ give the claim. The upper bound is the Chernoff bound $\mathbf{1}_{u<\varepsilon}\le e^{k^2(\varepsilon-u)/2}$.

Remark E.11. (i) Only integer frequencies constrain ω, and the representation is then not unique in general. Suppose some representation has a density bounded below by a positive constant on an interval $[u_1,u_2]\subset(0,\infty)$. In the variable $x=e^{-u/2}$ the constraints read $\int x^{k^2}\,\tilde\omega(dx)=c_k$. Let $[a,b]$ be a closed interval inside $(e^{-u_2/2},e^{-u_1/2})$. Since $\sum_{k\ge1}k^{-2}<\infty$, the functions x^{k^2}, $k\ge0$, do not span a dense subspace of $C[a,b]$ [CE43], [Alm07, Theorem 29]. Hence some signed measure $\zeta\neq0$ on $[a,b]$ annihilates all of them. The image of ζ under a dilation $x\mapsto sx$ annihilates them too, since $\int (sx)^{k^2}\zeta(dx)=s^{k^2}\int x^{k^2}\zeta(dx)=0$. We average these images against a continuous probability density in s supported near 1. The average is $h(x)\,dx$ with h continuous and supported in $[e^{-u_2/2},e^{-u_1/2}]$, and it annihilates all x^{k^2}. As the support of the density shrinks to $\{1\}$, the averages converge weakly to ζ, so $h\neq0$ for a small enough support. Adding a small multiple of $h\,dx$ gives another representation. Proposition E.10 applies to all of them. (ii) Numerical. For w_β^W with $\varepsilon=1/(K\beta)$ and $\beta\to\infty$, we choose k_1 just above the crossing of $I_k(\beta)/I_0(\beta)$ and $e^{-k^2\varepsilon/2}$ and $k_2=k_1+1$. At leading order Proposition E.10 then gives mass at least $e^{-63.02\tau}$ above the coupling $\tau=K\beta$ when

$K = K_*$, while the upper bound gives $e^{-29.93\tau}$, the order of the density δ of frozen plaquettes in Lemma 2.6. Hence, at leading order, the analogue of the condition (2.5) on the density of frozen plaquettes fails for every representation when $K \le 45.3$.

E.5 Increasing returns of freezing

Let X and p_0 be as in Theorem E.8, let $\tau > 0$, and let D be a set of faces with $p_0 \notin D$. Put $\tilde{v}_\tau$ on the faces outside D and $\tilde{v}_s$ on those in D, and let $g(D) = \lim_{s\to\infty}\langle \cos\theta_{p_0}\rangle$. By monotone convergence in Lemma E.1 the limit exists and is given by the dual formula with $c_k = 1$ on D, which is finite since $p_0 \notin D$. In the notation of Section 2, $g(D)$ is the frozen expectation $g_{X,\tau}(\partial\delta_{p_0}; D)$. The effect of a frozen set is not bounded by the product of the effects of its plaquettes.

Proposition E.12. *For $\tau \ge 3$ and distinct faces $p, p' \ne p_0$,*

$$g(\{p,p'\})\, g(\emptyset) > g(\{p\})\, g(\{p'\}).$$

Hence $D \mapsto \log g(D)$ is not submodular, and $g(D)/g(\emptyset) \le \prod_{q\in D} g(\{q\})/g(\emptyset)$ fails for $D = \{p,p'\}$.

Proof. By Lemma E.1 the value $g(D)$ depends only on $d = |D|$. Write g_d for it. The dual formula gives $g_d = \sum_h e^{-((h-1)^2+(5-d)h^2)/(2\tau)} \big/ \sum_h e^{-(6-d)h^2/(2\tau)}$. Completing the square,

$$\log g_d = -\frac{5-d}{2\tau(6-d)} + \log R_d, \qquad R_d = \frac{\Theta_d(1/(6-d))}{\Theta_d(0)}, \quad \Theta_d(y) = \sum_{h\in\mathbb{Z}} e^{-(6-d)(h-y)^2/(2\tau)}.$$

By Poisson summation $\Theta_d(y)$ is a positive multiple of $1 + 2\sum_{m\ge1} e^{-2\pi^2\tau m^2/(6-d)}\cos(2\pi m y)$. This gives $(1-2\varrho)/(1+2\varrho) \le R_d \le 1$ with $\varrho = e^{-\pi^2\tau/3}/(1-e^{-\pi^2\tau/3})$. The claim is $\log g_2 - 2\log g_1 + \log g_0 > 0$. The leading terms contribute $\frac{1}{\tau}(-\frac{3}{8}+\frac{4}{5}-\frac{5}{12}) = \frac{1}{120\tau}$, and the terms $\log R_d$ contribute at least -2λ with $\lambda = \log\frac{1+2\varrho}{1-2\varrho}$. For $\tau \ge 3$ we have $\varrho < 5.2\cdot10^{-5}$ and $\lambda < 2.1\cdot10^{-4}$, so $240\tau\lambda < 1$ at $\tau = 3$. Since $\tau\lambda$ decreases in τ for $\tau \ge 3$, the claim follows.

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Declarations

Conflict of interest. The author declares no conflict of interest.
Data availability. No data were generated or analysed. The file `scripts.zip` of the Zenodo record of this paper, `https://doi.org/10.5281/zenodo.23445737`, holds the program `cube_amonotone.py`, which carries out the exact computation in the proof of Theorem E.8, the scripts behind the threshold values of Section S1.1.2, and scripts that check several statements of the paper numerically; its README describes each of them. Of the proofs in the paper, only that of Theorem E.8 depends on these scripts.

References

[AHPS21] M. Aizenman, M. Harel, R. Peled, and J. Shapiro. Depinning in integer-restricted Gaussian fields and BKT phases of two-component spin models, 2021. arXiv:2110.09498.

[Alm07] J. M. Almira. Müntz type theorems. I. *Surv. Approx. Theory*, 3:152–194, 2007. arXiv:0710.3570.

[AR95] A. Abdesselam and V. Rivasseau. Trees, forests and jungles: a botanical garden for cluster expansions. In *Constructive Physics*, volume 446 of *Lecture Notes in Physics*, pages 7–36. Springer, Berlin, 1995.

[Bar04] M. T. Barlow. Random walks on supercritical percolation clusters. *Ann. Probab.*, 32:3024–3084, 2004.

[BF77] D. Brydges and P. Federbush. The cluster expansion for potentials with exponential fall-off. *Comm. Math. Phys.*, 53:19–30, 1977.

[BF80] D. C. Brydges and P. Federbush. Debye screening. *Comm. Math. Phys.*, 73:197–246, 1980.

[BK87] D. C. Brydges and T. Kennedy. Mayer expansions and the Hamilton–Jacobi equation. *J. Stat. Phys.*, 48:19–49, 1987.

[BMK77] T. Banks, R. Myerson, and J. Kogut. Phase transitions in Abelian lattice gauge theories. *Nuclear Phys. B*, 129:493–510, 1977.

[Bry78] D. C. Brydges. A rigorous approach to Debye screening in dilute classical Coulomb systems. *Comm. Math. Phys.*, 58:313–350, 1978.

[CE43] J. A. Clarkson and P. Erdős. Approximation by polynomials. *Duke Math. J.*, 10:5–11, 1943.

[CG25] I. Chevyrev and C. Garban. Villain action in lattice gauge theory. *J. Stat. Phys.*, 192:38, 2025.

[Cha26] S. Chatterjee. A short proof of confinement in three-dimensional lattice gauge theories with a central U(1), 2026. arXiv:2602.00436, version 2 (June 2026); to appear in Adv. Nonlinear Stud.

[CNS25] S. Cao, R. Nissim, and S. Sheffield. Expanded regimes of area law for lattice Yang–Mills theories, 2025. arXiv:2505.16585.

[CPS25] S. Cao, M. Park, and S. Sheffield. Random surfaces and lattice Yang–Mills. *Commun. Amer. Math. Soc.*, 5:774–896, 2025.

[CT73] J. M. Combes and L. Thomas. Asymptotic behaviour of eigenfunctions for multiparticle Schrödinger operators. *Comm. Math. Phys.*, 34:251–270, 1973.

[DG25] P. Dario and C. Garban. Phase transitions for the XY model in non-uniformly elliptic and Poisson–Voronoi environments. *Comm. Math. Phys.*, 406:101, 2025.

[DvEG26] P. Dario, D. van Engelenburg, and C. Garban. The impact of disorder and non-convex interactions on delocalisation of height functions, 2026. arXiv:2604.24743.

[For26] M. P. Forsström. Phase transitions in the charged compact Abelian lattice Higgs model, 2026. arXiv:2602.21679.

[Frö79] J. Fröhlich. Confinement in Z_n lattice gauge theories implies confinement in SU(n) lattice Higgs theories. *Phys. Lett. B*, 83:195–198, 1979. Preprint IHES/P/79/02; equation numbers cited in this paper are those of the preprint.

[FS81a] J. Fröhlich and T. Spencer. The Kosterlitz–Thouless transition in two-dimensional Abelian spin systems and the Coulomb gas. *Comm. Math. Phys.*, 81:527–602, 1981. Preprint IHES/P/81/25; section numbers cited in this paper are those of the preprint.

[FS81b] J. Fröhlich and T. Spencer. On the statistical mechanics of classical Coulomb and dipole gases. *J. Statist. Phys.*, 24:617–701, 1981.

[FS82] J. Fröhlich and T. Spencer. Massless phases and symmetry restoration in Abelian gauge theories and spin systems. *Comm. Math. Phys.*, 83:411–454, 1982.

[Gin70] J. Ginibre. General formulation of Griffiths' inequalities. *Comm. Math. Phys.*, 16:310–328, 1970.

[GJ77] J. Glimm and A. Jaffe. Quark trapping for lattice U(1) gauge fields. *Phys. Lett. B*, 66:67–69, 1977.

[GK04] D. Galvin and J. Kahn. On phase transition in the hard-core model on $\mathbb{Z}^d$. *Combin. Probab. Comput.*, 13:137–164, 2004.

[GM81a] M. Göpfert and G. Mack. Iterated Mayer expansion for classical gases at low temperatures. *Comm. Math. Phys.*, 81:97–126, 1981. Preprint DESY 81-014 (1981); equation, section and page numbers cited in this paper refer to the preprint.

[GM81b] M. Göpfert and G. Mack. Iterated Mayer expansion for classical gases at low temperatures. Preprint DESY 81-014, DESY, 1981. Preprint version of [GM81a].

[GM81c] M. Göpfert and G. Mack. Proof of confinement of static quarks in 3-dimensional U(1) lattice gauge theory for all values of the coupling constant. Preprint DESY 81-036, DESY, 1981. Preprint version of [GM82].

[GM82] M. Göpfert and G. Mack. Proof of confinement of static quarks in 3-dimensional U(1) lattice gauge theory for all values of the coupling constant. *Comm. Math. Phys.*, 82:545–606, 1982. Preprint DESY 81-036 (1981); equation, section and page numbers cited in this paper refer to the preprint.

[GS23] C. Garban and A. Sepúlveda. Improved spin-wave estimate for Wilson loops in U(1) lattice gauge theory. *Int. Math. Res. Not. IMRN*, 2023(21):18142–18198, 2023. arXiv:2107.04021.

[Gut80] A. H. Guth. Existence proof of a nonconfining phase in four-dimensional U(1) lattice gauge theory. *Phys. Rev. D*, 21:2291–2307, 1980. Preprint CLNS 79/439 (1979); lemma numbers cited in this paper are those of the preprint.

[Har76] P. Hartman. Completely monotone families of solutions of n-th order linear differential equations and infinitely divisible distributions. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 3:267–287, 1976.

[HW74] P. Hartman and G. S. Watson. "Normal" distribution functions on spheres and the modified Bessel functions. *Ann. Probability*, 2:593–607, 1974.

[Kin86] C. King. Deconfining phase transition in the U(1) model with Wilson's action. *Comm. Math. Phys.*, 105:675–690, 1986.

[LP16] R. Lyons and Y. Peres. *Probability on Trees and Networks*. Cambridge University Press, New York, 2016.

[LSS97] T. M. Liggett, R. H. Schonmann, and A. M. Stacey. Domination by product measures. *Ann. Probab.*, 25:71–95, 1997.

[LW49] L. H. Loomis and H. Whitney. An inequality related to the isoperimetric inequality. *Bull. Amer. Math. Soc.*, 55:961–962, 1949.

[MMSP78] A. Messager, S. Miracle-Solé, and C. Pfister. Correlation inequalities and uniqueness of the equilibrium state for the plane rotator ferromagnetic model. *Comm. Math. Phys.*, 58:19–29, 1978.

[Nis25] R. Nissim. U(N) lattice Yang–Mills in the 't Hooft regime, 2025. arXiv:2510.22788.

[Ope26] OpenAI. The critical correlation exponent of the planar XY model, 2026. Manuscript dated September 24, 2026, family 216 of the repository `https://github.com/openai/math`.

[OS78] K. Osterwalder and E. Seiler. Gauge field theories on a lattice. *Ann. Physics*, 110:440–471, 1978.

[Pat09] P. Patie. Infinite divisibility of solutions to some self-similar integro-differential equations and exponential functionals of Lévy processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 45:667–684, 2009.

[Pol77] A. M. Polyakov. Quark confinement and topology of gauge theories. *Nuclear Phys. B*, 120:429–458, 1977.

[Tim13] Á. Timár. Boundary-connectivity via graph theory. *Proc. Amer. Math. Soc.*, 141:475–480, 2013.

[Tom26] Tejasvi Singh Tomar. A Lean 4 formalization of the area law for three-dimensional $U(1)$ lattice gauge theory with the Wilson action, 2026. Zenodo, `https://doi.org/10.5281/zenodo.23480809`.

[Wat39] G. N. Watson. Three triple integrals. *Quart. J. Math. Oxford Ser.*, 10:266–276, 1939.

[Yor80] M. Yor. Loi de l'indice du lacet brownien, et distribution de Hartman–Watson. *Z. Wahrsch. Verw. Gebiete*, 53:71–95, 1980.

[Zuc11] I. J. Zucker. 70+ years of the Watson integrals. *J. Stat. Phys.*, 145:591–612, 2011.

Supplementary material

S1 Constants

S1.1 Constants of Sections 2 and 8: the ordered list, the threshold K_* and the conditions defining $\beta_0(K)$

Here we give the values of the constants; the order in which they are fixed, and the reasons for it, are explained in Section 8.2. The values were evaluated by computer in high-precision arithmetic, but the proofs need only the inequalities stated.

S1.1.1 The ordered list

Each constant is fixed after all constants above it, and the constants in items 1 to 14 are independent of β, τ, K, the loop C and D, except that $\bar r_0$ of item 1 is r_0 or $2r_0-1$ according to $\mathfrak{G}(C)$, and that the depletion constant c_{dep} of item 12 depends on τ. The scales of item 11 depend on β through $\tilde m$ alone, and the derived quantities c_g' and c_u, which lie between two constants of this kind, depend on β only through the size of L_c (c_g') and through its rounding (c_u). Constants built from c_u, such as $D_{\mathcal U}$ of Appendix A and K_0^B of Appendix B, likewise lie between two constants of this kind, and we treat them as constants. Illustrative values are given at $(c_g, c_s, r_0) = (1/4, 1, 1)$. We use coarse units for the regularization and fine units in Sections 6 and 7.

0. *Absolute constants.* Among them are $c_I = 2^{2/3}$ and the Debye exponent $\pi^2 v_{\mathrm{Cb}}(0) = 2.49435508\ldots$, where $v_{\mathrm{Cb}}(0) = (-\Delta_{\mathbb{Z}^3})^{-1}(0,0) = \frac{\sqrt6}{192\pi^3}\Gamma(\frac{1}{24})\Gamma(\frac{5}{24})\Gamma(\frac{7}{24})\Gamma(\frac{11}{24}) = 0.25273101\ldots$ is half of Watson's integral [Wat39] in the normalization of [Zuc11], whose closed form is [Zuc11, (3.6)], and $\log(1/m_D) = \pi^2 v_{\mathrm{Cb}}(0)\tau - \frac12\log(8\pi^2\tau)$. The rounding rule for the Debye exponent is in Section S1.1.3. The other absolute constants are $C_{\mathrm{bl}} = 384$ (Proposition 6.9), the window count 151^3 (Definition 3.4), $\alpha = \ln 8$ and $8/31$ (Theorem 7.49 and Lemma 7.54), $c_T = 2 + \log 96$ and the threshold $14.9662 > 6\pi^2 v_{\mathrm{Cb}}(0) = 14.96613\ldots$ (Lemma 2.6 and Section 5), and the absolute constants among those of Göpfert–Mack, Brydges–Federbush and Stage 1 quoted in Sections 6, 7 and Appendices A to C. The constants of these authors and of Stage 1 that depend on the parameters C, δ_{GM}, μ, κ_S, λ_{PV} of Theorem 6.5 are fixed together with these parameters, in items 12 and 13.

1. $r_0 \geq 1$, an integer, and $\bar r_0 := 2r_0 - 1$, or $\bar r_0 := r_0$ if $\mathfrak{G}(C)$ has no exceptional layer; they are used in Definition 3.4, Lemma 5.7 and η_T.

2. $c_g := 1/4$, the constant of the property (S-face) of Definition 3.17, which the construction provides for all $c_g \leq 1 - 134\vartheta$ by Theorem 4.16. The proofs hold under $0 < c_g \leq 1/4$, so every $c_g \in (0, 1/4]$ is admissible.

3. $\eta_w = 1/10$, the hull-weight deficit of the frozen bonds (Definition 3.8).

4. $c_\rho = 1/12$, that is, $\rho_\lambda = L_c/12$ and $\lambda = 12/L_c$, the equality case of Lemma 4.6.

5. Before χ and c_L we fix $k_P/L_c = 304$, so that the windows of Definition 3.4 have sides less than $151 \cdot 2L_c < k_P$, as in (G4), and we set $r_\square := (k_P + 3)/L_c \leq 305$.

6. $c_1 := \min(c_g/2, ((1-\eta_w)c_I c_\rho/2)^3)$, which equals $2.109 \cdot 10^{-4}$ for $c_g \geq 4.22 \cdot 10^{-4}$. We set $c_{\mathrm{sh}} := c_1/216$; Table S1.1 gives their uses.

7. $c_s \in [3.3 \cdot 10^{-7}, 10]$ (illustration $c_s = 1$), with $w_* := (1+c_s)|\partial\Delta|$ (Definition 3.13) and $\lambda_U := 2c_s/\ell$ (Definition 3.11); then $2c_s\ell^2 > 1$, as $\ell \geq 308L_c \geq 1232$ (used in Theorem 4.23).

8. The forcing threshold $\vartheta = 3 \cdot 10^{-5}$ and the penalty $\mu_{\mathrm{B}} = 25$ of Definition 3.11, and the thin-set constant $\varepsilon_{\mathrm{th}} = 0.008$ of Section 4.2. They satisfy $24.6 < \mu_{\mathrm{B}}$ (Lemma 4.9), $128\vartheta < \varepsilon_{\mathrm{th}}$, $2\varepsilon_{\mathrm{th}} + 12\vartheta \leq 0.0164$ and $2/\sqrt{0.0164} > 14$ (Lemma 4.13), and $254.4\vartheta \leq \varepsilon_{\mathrm{th}}$ (Theorem 4.16); the arithmetic is in Section S1.2.

9. The weak-link constant $\chi := 2.0 \cdot 10^{-8}$ and $\chi_{\mathrm{N3}} := \chi/3 = 6.67 \cdot 10^{-9}$ of Definition 3.15. Its value depends only on items 3, 5 and 8, item 3 through the factor $12/\eta_w$ of Proposition 4.15 in the expansion bound, and this value is constrained by Theorem 4.16(c), which provides the expansion $h_\star = 2.2 \cdot 10^{-11}/L_c$ on each component of $G' \cap \hat{Q}$ at rest. Since $3k_P h_\star = 2.006 \cdot 10^{-8} \geq \chi$ (Section S1.2), (N3) holds with χ_{N3} and (R6) with χ.

10. *Consequences, fixed with the constants above and independent of L_c and ℓ.* $C_1 := 7 + 27 \cdot 24.6/\vartheta = 2.214 \cdot 10^7$, $C_{\mathrm{seal}} := 726$, $C'_\star := C_1 + 2C_{\mathrm{seal}} = 2.2141 \cdot 10^7$ and $C_{\mathrm{free}} := C'_\star(2 + 1/(2c_s) + 1/3696) = 5.536 \cdot 10^7$ (Propositions 4.15 and 4.24); $C_a := 4362 + 5808/c_s$, $C_b := 2 + 4/c_s$ and $C_c := 1 + 2/c_s$, equal to 10170, 6, 3 (Lemma 5.20), and the response load $C_{\mathrm{resp}} := 375C_a + 125C_cC'_\star = 8.307 \cdot 10^9$ of Section 5.6; finally, $C_{\mathrm{rm}} = 20$, $C_{\mathfrak{B}} := 30C'_\star + 170C_{\mathrm{resp}} + 30 = 1.413 \cdot 10^{12}$ and $C_3 := C_{\mathrm{free}} + 20(2C'_\star + 4C_{\mathrm{resp}} + 1) = 6.655 \cdot 10^{11}$ (Proposition 5.24, Theorem 5.1).

11. The constant c_L is subject to $C'_P c_L^2 \leq (1 - 4/\pi^2)/8$ and $c_L \leq \frac{1}{2}$. The first condition holds if $2916 r_\square^5 C_{\mathrm{bl}}^2 c_L^2/\chi^2 \leq (1 - 4/\pi^2)/8$, since $C'_P = C_W^f r_\square^2 N_{\mathrm{ov}} \leq 2916 r_\square^5 C_{\mathrm{bl}}^2/\chi^2$, where $C_W^f = 108 C_{\mathrm{bl}}^2/\chi^2$ is the constant of Proposition 6.9(iv) and $N_{\mathrm{ov}} \leq 27 r_\square^3$. At $r_\square = 305$ this yields the bound $8.09 \cdot 10^{-12}\chi$, and the admissible choice is $c_L \leq 8.0 \cdot 10^{-12}\chi$, that is, $c_L = 1.6 \cdot 10^{-19}$ at $\chi = 2.0 \cdot 10^{-8}$. This choice implies $\tilde{m}L \leq \frac{1}{2}$ for Lemma 4.21 and $c_L \leq 1.99 \cdot 10^{-4}\chi_{\mathrm{N3}} = 1.33 \cdot 10^{-12}$ for Lemma 7.44. The scales of Definition 3.1 are then $L_c := c_L \tilde{m}^{-1}/r_0$ rounded down to an even integer, $L := r_0 L_c$, $k_P := 304 L_c$, $\ell_D \approx \tilde{m}^{-1}$, a multiple of L, and $\ell := \ell_D/r_0 \geq k_P + 4L_c$. The derived quantities are $c'_g = c_g(1 + 1/L_c)^{-2}$ and $c_u := r_0^3 V_* \tilde{m}^3 = ((1-\eta_w)c_I \tilde{m} L/24)^3$, which lies in $[((1-\eta_w)c_I c_L/48)^3, ((1-\eta_w)c_I c_L/24)^3]$ once $c_L \tilde{m}^{-1} \geq 4r_0$ (Section S1.2). Further derived quantities are $R_{\mathrm{HK}} = (1-\eta_w)c_I c_\rho L/3$ and the constant $K_0 := (1-\eta_w)c_I c_\rho c_L/6 = 0.01984\, c_L$, whose uses are listed in Section S1.5. Since $c_L/2 \leq \tilde{m}L \leq c_L$, we have $K_0 \leq \tilde{m} R_{\mathrm{HK}} \leq 2K_0$.

12. *Stage 1 and Stage 2 constants.* These are the constants of Sections 6 and 7 and Appendices A to C. We first fix the parameters $C \in (0, \frac{1}{4}]$ and $\delta_{\mathrm{GM}} \in (0, \frac{1}{2}]$ of Theorem 6.5, where this C is not the loop; the bound $\delta_{\mathrm{GM}} \leq \frac{1}{2}$ is used for (S5) on outputs and in Lemma B.4 and Proposition B.7. The bound $C \leq \frac{1}{4}$ on this parameter implies $2C_{M4} = 2/(1-C) \leq \frac{8}{3}$, hence $q := 2C_{M4}e^{c_p}/e \leq 0.9829$ since $c_p < \frac{1}{540}$. In Proposition B.7, therefore, one factor $e^{-(t_e-1)}$ absorbs the per-$\mathcal{E}$-factor constants $t_e(2C_{M4}e^{c_p})^{t_e-1}$ up to the factor $\sup_{t \geq 1} t q^{t-1} \leq 22$, which contributes one factor per cell; here t_e is the

factor of (M4), Lemma B.1(d), which bounds the sum over all assignments of the points to their cells.

Next come $\eta_H = 0.62$, $(p_3, p', q', p_1, p_2, p_4) = (1.2, 1.3, 15.6, 20, 20, 15)$, $c_\lambda = 1/24$, $\theta_0 = 0.01$ and the link-sum rate c_{10} with $0 < c_{10} \le \frac{7}{9} r_C$, where r_C is evaluated at $\tilde{m}L = c_L/2$. We use this bound in Proposition B.7, where, for $\delta_{\mathrm{GM}} \le \frac{1}{2}$, the rate of (S2) in Debye units is at least $2r_C$ for all $\lambda_{\mathrm{PV}} \le \frac{1}{4}$, by Proposition 6.10, and pinning uses $4c_p < \frac{2}{9} r_C$ of this rate for every $\bar{\delta}$. The quantity $r_C = \min(\kappa_0/2, (2c_{\mathrm{gs}})^{1/2})$, as in Theorem 6.11, is computed with the constant K_0 of item 11, the value of $\tilde{m}R_{\mathrm{HK}}$ at $\tilde{m}L = c_L/2$, which is the worst case of $\tilde{m}L \in [c_L/2, c_L]$. It is therefore determined by items 0 to 11, and hence c_{10} is fixed independently of β, λ_{PV} and $\bar{\delta}$.

We then fix c''_{10} with $e^{c''_{10}} \ge 16e \max(1, c_{c_{10}/2})$ (Lemma A.6), and then the parameters μ and κ_S of Theorem 6.5, both large: $\mu \ge 1 + 2c''_{10}$, after c_{10} and c''_{10} (the factor $e^{-\mu(s-1)}$ of (S2) pays $e^{c''_{10} t} \le e^{2c''_{10}(t-1)}$, Proposition B.7), and κ_S so large that $(1 + C_{\mathrm{ref}} e^{-2\kappa_S})\eta_H/8 \le \frac{1}{7}$ (Lemma C.2), where $C_{\mathrm{ref}} := 8C$ with C the constant of (6.10) on $\mathbb{Z}^3$, so that $\beta_{\mathrm{GM}} \sum_m m^2 |\rho^{\mathrm{ref}}(m)| \le (1 + C_{\mathrm{ref}} e^{-2\kappa_S})\tilde{m}^2$ (as $m_D \le 2\tilde{m}$). This constant C comes from the tail bound (6.10), that is, from (S2) at $s = 1$ with $a_1 = \{x\}$ and from (6.7), whose right sides involve the parameter C and μ but not κ_S. Hence the constant C of the tail bound can be chosen independently of κ_S and λ_{PV}, and this last condition can be met. At $C = \frac{1}{4}$, for instance, $c''_{10} = 2200$, $\mu = 4401$ and $\kappa_S \ge 5.8$ are admissible; the binding condition on κ_S is (7.30).

Further, we require $\varepsilon_\nu + \theta' \le 1/7$ and $c_E \le 1/70$. The depletion constant $c_{\mathrm{dep}} := A_0^D + A_1^D \log(C'\tau/\theta_0)$ of Proposition 6.17 depends on τ, but all its occurrences are in lower bounds on τ (Theorem 7.40). Next we fix $K_{\mathrm{col}} = 2 > 27 \cdot 2.4944/(4\pi^2) = 1.706$ and $\varepsilon_0 = 0.003125$ (Corollary 7.18), and $0 < \delta_2 < \bar{\delta} < r_C$ and $c'_{\mathrm{rate}} = \frac{1}{2}(r_C - \bar{\delta})$ (Proposition 7.36). Last come the pure number $c'_0 := \frac{1}{2}(1 - 1/p_3 - c_\lambda) = 1/16$, a lower bound for the fraction c' of Proposition 7.35, then $\varepsilon_E = c'_0/8 = 1/128$, $c_w = c'/16$ and $\alpha = \ln 8$.

13. $\lambda_{\mathrm{PV}} \le \min\bigl(\frac{1}{4}, \lambda_0\bigr)$, chosen after c_L and after μ, κ_S of item 12, where $\lambda_0 = \lambda_0(\kappa_S, \mu, C, \delta_{\mathrm{GM}})$ of Theorem 6.5 comes from the Debye condition in its proof. Here M_Y is the smallest fixed point of (6.11) and $\tilde{m} := \lambda_{\mathrm{PV}} M_Y$, as in Section 6.2. The constants above are the same for every λ_{PV} allowed by the a priori bound $\lambda_{\mathrm{PV}} \le \frac{1}{4}$, whereas those of Göpfert–Mack, Brydges–Federbush and Stage 1 that depend on $(\lambda_{\mathrm{PV}}, \kappa_S, \mu, C, \delta_{\mathrm{GM}})$, among them the threshold β^0_{GM} of Theorem 6.5, are fixed here, after λ_{PV}.

14. *The cut-rule constants.* Those of Definition 3.18 are $\theta_{\mathrm{NF}} := c'_0 c_g/(90(c_g{+}4)) = 4.08{\cdot}10^{-5}$, $c_3^{\mathrm{NF}} := c'_0 c_g^2/(64(c_g + 4)) = 1.44 \cdot 10^{-5}$ and $\kappa_{\mathrm{NF}} := 4(c_g + 4)/c_g^2 = 272$ at $c_g = 1/4$. They belong to the regularization but come after item 12, because they are built from the pure number c'_0, and the constants above are independent of θ_{NF} and c_3^{NF}.

15. $K > K_* = 377.38$, which implies $c_\delta^{\mathrm{sp}}(K)/(2K) > 14.9662$ (see Section S1.1.2). Proposition 5.11 and Theorem 5.1 use this threshold.

16. $\beta_0(K)$, determined by the conditions of Section S1.1.4.

Among the scales determined by these constants are $\tilde m^{-1}$, with $\tilde m \in [m_D/2, 2m_D]$, and L_c, L, k_P, ℓ as in item 11. These constants further determine $V_* = (1-\eta_w)^3\rho_\lambda^3/2$, $V_{\mathcal U} = \tilde m^{-3}$, $R_* = (J+1)w' + 6L_c = O((\ell/c_s)\log(\ell/c_s))$ of Theorem 4.23 together with $r' = 2R_* + 2$, and $R_{\rm resp} = 10\ell$ together with $R' = R_* + 2R_{\rm resp}$. The remaining scales are $L' = 2(\beta_{\rm GM}/a)\tilde m^{-1}$, $R_1(\beta) = (2r_\square + 5)L$, $F_{\rm unit} = \sigma_F L^2 = (\pi^2 c_g'/18)(\tilde m L)^3/(\tilde m \beta_{\rm GM})$ of Definition 3.18, $R_{\rm NF} = Le^{c_3^{\rm NF} F_{\rm unit}}$, and the box Λ° of side $M'_\Lambda \in \{M_\Lambda, M_\Lambda + 1\}$, where $M_\Lambda = \max(3\max(R,T), r_0(\ell+3))$.

S1.1.2 The threshold K_* of Lemma 2.6

Lemma 2.6 gives, with a_* the solution of $a = K\log(2a)$,

$$c_\delta(K) = \frac{|\log\delta|}{\beta} \ge c_\delta^{\rm sp}(K) - \frac{1}{4a_*} - O\Big(\frac{\log\beta}{\beta}\Big), \qquad c_\delta^{\rm sp}(K) := a_*\Big(\tfrac12\log(2a_*) - 1\Big) + 1. \quad \text{(S1.1)}$$

The function $K \mapsto c_\delta^{\rm sp}(K)/(2K)$ is increasing. Indeed $da_*/dK = a_*\log(2a_*)/(a_* - K)$ and $a_* - K = K(\log(2a_*) - 1)$, which yield

$$\frac{dc_\delta^{\rm sp}}{dK} = \Big(\tfrac12\log(2a_*) - \tfrac12\Big)\frac{da_*}{dK} = \tfrac12\log^2(2a_*),$$

and $K\cdot\frac12\log^2(2a_*) - c_\delta^{\rm sp}(K) = K\log(2a_*) - 1 > 0$. Hence $\frac{d}{dK}\frac{c_\delta^{\rm sp}(K)}{K} > 0$. The threshold of Lemma 2.6 and (2.5) is 14.9662, that is, $6\pi^2 v_{\rm Cb}(0) = 14.96613\ldots$ rounded up. The equation $c_\delta^{\rm sp}(K)/(2K) = 14.9662$ has the unique root $K = 377.3635$, and $K_* := 377.38$ lies above it. Hence $c_\delta^{\rm sp}(K)/(2K) > 14.9662$ for all $K \ge K_*$. Numerically, at $K = K_*$,

$$\frac{c_\delta^{\rm sp}(K_*)}{2K_*} = 14.96639, \qquad a_* = 3321.43, \qquad \frac{1}{4a_*} = 7.5\cdot10^{-5}.$$

The computation behind (2.4) (proof of Lemma 2.6). We put $\nu := a_*\beta$ in Lemma 2.4(iv). Then $\nu^2\varepsilon/2 = a_*^2\beta/(2K)$. Let $h(\mu) := (\beta/2)^\mu/\Gamma(\mu+1)$ and $\mu_n := (n^2+\nu^2)^{1/2} \ge \nu$. From the series $I_\mu(\beta) = \sum_{k\ge0}(\beta/2)^{\mu+2k}/(k!\,\Gamma(\mu+k+1))$ and $\Gamma(\mu+k+1) = \Gamma(\mu+1)\prod_{i=1}^k(\mu+i) \ge \Gamma(\mu+1)(\mu+1)^k$, we get $I_\mu(\beta) \le h(\mu)\sum_{k\ge0}(\beta^2/(4(\mu+1)))^k/k! = h(\mu)e^{\beta^2/(4(\mu+1))} \le h(\mu)e^{\beta/(4a_*)}$ for $\mu \ge \nu$. Since $\log\Gamma$ is convex, its difference quotients increase. Thus for $\mu \ge \nu$

$$\log\Gamma(\mu+1) - \log\Gamma(\nu+1) \ge (\mu-\nu)\big(\log\Gamma(\nu+1) - \log\Gamma(\nu)\big) = (\mu-\nu)\log\nu,$$

hence $h(\mu) \le h(\nu)(\beta/(2\nu))^{\mu-\nu} \le h(\nu)2^{-(\mu-\nu)}$, because $\nu = a_*\beta \ge \beta$. As $\mu_n - \nu \ge 0$ for $|n| \le \nu$ and $\mu_n - \nu \ge |n| - \nu$ otherwise,

$$\sum_{n\in\mathbb Z} I_{\mu_n}(\beta) \le e^{\beta/(4a_*)}h(\nu)\Big(2\nu + 1 + 2\sum_{m>\nu}2^{-(m-\nu)}\Big) \le (2a_*\beta+5)\,e^{\beta/(4a_*)}h(\nu).$$

Stirling's inequality $\Gamma(x+1) \ge (x/e)^x$ gives $\log h(\nu) \le -a_*\beta(\log 2a_* - 1)$. Finally, for $\beta \ge \pi^{-2}$, using $\cos\theta \ge 1 - \theta^2/2$ on $[0, \beta^{-1/2}] \subseteq [0,\pi]$, we obtain $I_0(\beta) = \frac1\pi\int_0^\pi e^{\beta\cos\theta}d\theta \ge \frac{1}{\pi\sqrt\beta}e^{\beta-1/2}$. Collecting these bounds, we obtain

$$\log\delta \le -\beta\Big[a_*(\log 2a_* - 1) - \frac{a_*^2}{2K} + 1 - \frac{1}{4a_*}\Big] + \log(2a_*\beta+5) + \log(\pi\sqrt\beta) + \frac12,$$

and $a_*^2/(2K) = \frac{1}{2}a_* \log 2a_*$ by the definition of a_*, which gives (2.4), with exactly the function $c_\delta^{\mathrm{sp}}(K)$ and the correction $1/(4a_*)$ of (S1.1).

The term $O(\log\beta/\beta)$ is absorbed in the term $O(\log\tau + \log r_0)$ of Lemma 5.12. The margins $c_\delta^{\mathrm{sp}}(K)/(2K) - 14.9662$ that appear in η_T are:

K	377.38	10^3	$3\cdot 10^3$	10^4
$c_\delta^{\mathrm{sp}}(K)/(2K)$	14.9664	19.521	25.289	32.372
margin	$1.9\cdot 10^{-4}$	4.55	10.32	17.41

Numerical values. To more digits, $a_*(K_*) = 3321.434$ and $c_\delta^{\mathrm{sp}}(K_*)/(2K_*) = 14.966392$; moreover $1/(8K_*a_*) = 9.97\cdot 10^{-8}$, and the root is 377.363524, or 377.363533 with the correction $-1/(4a_*)$. The correction $1/(4a_*)$ in (S1.1) moves $c_\delta/(2K)$ by 10^{-7} and the root by less than 10^{-4}. The entropy exponent of Lemma 5.12 is $6\pi^2 v_{\mathrm{Cb}}(0) < 6\cdot 2.4943551 = 14.96613\ldots < 14.9662$, so the threshold is computed without the rounding of the Debye exponent up to 2.4944 used elsewhere; with the rounded exponent the threshold is 14.9664, whose root 377.3807 exceeds K_*.

S1.1.3 The string tension and the rounding rule

The slit-domain tension of Theorem 7.57 is $\sigma_V = \varepsilon_E\sigma_F = (c_g'\tilde{m}L/9216)\tilde{m}/\tau$, with $c_g' = c_g(1 + 1/L_c)^{-2}$ and with $\tilde{m}L > c_L - 2r_0\tilde{m}$ by Section 3.1. For $L_c \geq 35$, one of the conditions of Section S1.1.4(e), we have $c_L\tilde{m}^{-1} \geq 35r_0$, and hence $c_g'\tilde{m}L \geq (36/35)^{-2}(1 - 2/35)\, c_g c_L \geq 0.89\, c_g c_L \geq (9/16)c_g c_L$. In (8.1), $18432\cdot 16/9 = 32768 = 2^{15}$ exactly and $(8\pi^2)^{1/2}/2^{16} = 1.3559\cdot 10^{-4}$. We take $c_g = 1/4$ and $c_L = 8.0\cdot 10^{-12}\cdot 2.0\cdot 10^{-8} = 1.6\cdot 10^{-19}$, as in items 9 and 11, although any smaller admissible c_L also works; at these values (8.1) gives $\sigma(\beta) \geq 5.4\cdot 10^{-24}(K\beta)^{-1/2}e^{-2.49436K\beta}$.

Throughout, and in particular in Section 8.1, we round as follows. The Debye exponent is $\pi^2 v_{\mathrm{Cb}}(0) = 2.49435508\ldots < 2.4943551$. In a lower bound on $\tilde{m}$, and hence on a tension, the exponent is rounded up, to 2.49436, and prefactors are rounded down, to $1.35\cdot 10^{-4}$. In an upper bound on a scale the exponent is rounded up to 2.4944: since $\tilde{m} \geq m_D/2$, $\log(1/\tilde{m}) \leq 2.4943551\tau - \frac{1}{2}\log(8\pi^2\tau) + \log 2 \leq 2.4944\tau$ for all $\tau \geq 1$. We use this form for $K_{\mathrm{col}} > 27\cdot 2.4944/(4\pi^2)$, in the logarithmic step of the proof of Lemma 6.18, and in the bound $\log(1/\tilde{m})/\beta_{\mathrm{GM}} \leq 2.4944/(4\pi^2) = 0.063184 < 0.0632$ of Remark 7.58. The bounds that determine K_* use the exponent without rounding. There the entropy $6\log R_*$ of a root and a part in Lemma 5.12, with $\ell \approx \tilde{m}^{-1}/r_0$ and with R_*, r' and R' of Section 5, is bounded with $\pi^2 v_{\mathrm{Cb}}(0) < 2.4943551$, as in Sections S1.1.2 and S1.4 (for the threshold with the rounded exponent, see the end of Section S1.1.2). Where the exponent serves as a lower bound on $\log(1/\tilde{m})$, it is rounded down: the hypothesis of Lemma 6.18 has 2.494τ, because it is used to bound $\tilde{m} \leq 2m_D \leq 17.8\,\tau^{1/2}e^{-2.494\tau}$ from above.

S1.1.4 The conditions defining $\beta_0(K)$

We fix the constants of items 0 to 15. As explained in Section 8.2, each condition below is a lower bound on $\tau = K\beta$. Then $\beta_0(K)$ is τ_0/K for the largest of the thresholds τ_0.

(a) *Section 5.* The conditions used in Proposition 5.9, Lemma 5.10, Proposition 5.11, Lemma 5.17, Proposition 5.19, Lemma 5.23, Proposition 5.24, Lemmas 5.25, 5.26 and

Theorem 5.1 are

$$\varepsilon_c \le \tfrac{1}{2}, \qquad \eta_T^{(1)} \le 1, \qquad \varepsilon'' \le \tfrac{1}{2}, \qquad \eta_T \le 1, \qquad 2^{29}\cdot 26e\cdot \varepsilon''^{1/(3C_{\mathfrak{B}})} \le \tfrac{1}{2},$$
$$2\big(2\lceil R'/L_c\rceil + 5\big)^3\, \varepsilon''^{\,1/(48\varepsilon_D(1+8L_c)(C'_\star+3C_{\mathrm{resp}}+6))} \le \tfrac{1}{2},$$
$$8\varepsilon_D(1+8L_c)(C'_\star+3C_{\mathrm{resp}}+6)(48C_{\mathrm{resp}}+2) \le 1 \quad \text{(that is, (5.9))},$$
$$R_{\mathrm{NF}} \ge \max\big(3456\bar r_0 L_c,\ 2048\bar r_0 L_c/(c_{\mathrm{sh}}\theta_{\mathrm{NF}}F_{\mathrm{unit}})\big).$$

Here $\varepsilon_D = 3.2768\cdot 10^7 \bar r_0 L_c/(c_{\mathrm{sh}}\theta_{\mathrm{NF}}F_{\mathrm{unit}}R_{\mathrm{NF}})$ is doubly exponentially small. The factor 2^{29} contains the factor 2^b for the sizes of the components in Lemma 5.26. The binding condition is the first of (5.10), $|\log\varepsilon''| \ge 75.16\, C_{\mathfrak{B}} \approx 1.1\cdot 10^{14}$, while $\eta_T \le 1$ needs about $1.7\cdot 10^{13}$ and $\eta_T^{(1)} \le 1$ needs about $3.6\cdot 10^8$; all three values are computed in Section S1.4.

(b) *Section 6.* The conditions are $\tilde m \le K_0(2C_H)^{-1/3}$; $M_Y \le m_W$ and $\tilde m \le \min(1/68, m_W/2)$ for Theorem 6.11, with $m_W = \min(1, c_{\mathrm{ex}}/2)$ of Proposition 6.10; $M_Y \le \frac{1}{5}$, for Lemma D.2 on Λ° in the proof of Theorem 6.15; $m_D/2 \le \tilde m \le 2m_D$; the coupling condition of Lemma 6.18, $\tau \ge 2$ and $2.494\tau \ge \ln(C_*^{\mathrm{dep}}/\theta_0) + 2.5\ln\tau$, whose exponent is rounded as in Section S1.1.3; and the threshold $\beta_{\mathrm{GM}} \ge \beta^0_{\mathrm{GM}}(\lambda_{\mathrm{PV}}, \kappa_S, \mu, C, \delta_{\mathrm{GM}})$ of Theorem 6.5.

(c) *Section 7 and Appendices A to C.* The conditions are the slack in $\varepsilon_\nu + \theta' \le 1/7$ and $c_E \le 1/70$, which needs the terms $\beta_{\mathrm{GM}}M_Y$ and $e^{-\mu}$, $e^{-2\kappa_S}$ to be small; $\varepsilon(L') \le \varepsilon_0$ of Corollary 7.18; (β1) and (β3) of (S1.5); the cell-diameter conditions of Lemma 7.44 and Proposition 6.9, of the form $2 + C(k_\square/\chi)\log k_\square < L'$; the conditions of Proposition 7.39 and Theorem 7.40, the latter a condition $0.4157\,\tau \ge C_{\mathfrak{D}}A_1^D\log(C'\tau/\theta_0) + \alpha + O(\log\tau)$; $c_{16}e^{c_2} \le 1/8$ of Lemma S1.2; the conditions of Lemma 7.48; $\Theta_4 \le \frac{1}{4}$ of Proposition 7.53; and $F_{\mathrm{unit}} \ge (8C'/c')(1+\ln\beta_{\mathrm{GM}})$ of Proposition 7.52.

(d) *Section 8.* The conditions are $\eta_T + \eta_S \le \frac{1}{2}$ and $\sigma_V \le 1$. Both deficits are explicit upper bounds that are eventually decreasing in τ: $\eta_T = O_K(1/\tau)$, because, by Lemma 5.12, $|\log\varepsilon''| \ge \gamma_T(K)\tau - O(\log\tau + \log r_0)$ with $\gamma_T(K) > 0$ for $K > K_*$, and $\eta_S = O(\tilde m\beta_{\mathrm{GM}}\log\tau)$.

(e) *Rounding of L_c in Section 3.* The conditions are $L_c \ge 4$, which implies $c'_g \ge (16/25)c_g \ge (9/16)c_g$ for Lemma 4.19; $L_c \ge 35$, used for $\lambda = 12/L_c \le 1$ in Proposition 4.24, in Lemma 7.44 and for $c'_g\tilde m L \ge (9/16)c_g c_L$ in Section S1.1.3 and Section 8.1; and $c_L\tilde m^{-1} \ge 4r_0$, which gives $\tilde m L \ge c_L/2$ (the bounds on c_u and on $C_{\mathcal U}$, Lemma 4.21).

(f) *Section 2.* The conditions are $\beta \ge \pi^{-2}$, for Lemma 2.6; (2.5), that is, $\beta \ge \beta_1(K)$; and $\delta < 1$, which follows from (2.4) for all large β and is used in Step (1.4), with $\delta' = \delta$ in Proposition 2.9.

In addition, the range of Remark 8.2, where Phase 2 is empty, is nonempty once $\sqrt{3}(r_0(\ell + 3) + 1) \le R_{\mathrm{NF}}$. This condition belongs to that remark rather than to Theorem 1.2, and it holds for all large τ because $r_0\ell \approx \tilde m^{-1}$ is exponential and R_{NF} is doubly exponential in τ.

All these thresholds are finite but very large (Section 8.2).

S1.2 Constants of Section 3 and Subsections 4.1 and 4.2

We group the constants by the main-text statement in which they appear.

Definition 3.1: the scales. The screening mass satisfies $m_D/2 \le \tilde{m} \le 2m_D$, so that $\log(1/\tilde{m}) = \pi^2 v_{\mathrm{Cb}}(0)\,\tau + O(\log\tau)$ with $\pi^2 v_{\mathrm{Cb}}(0) = 2.49435508\ldots$, and $\tilde{m}^{-1}$ is the Debye length. The scales are fixed at item 11 of the ordered list in Section S1.1.1. Since $c_L \le 1/308$, (3.1) follows from $\ell \ge \tilde{m}^{-1}/r_0 \ge L_c/c_L$. Of the conditions $L_c \ge 4$ and $c_L\tilde{m}^{-1} \ge 4r_0$ of Section S1.1.4(e), the first implies $c_g' := c_g(1+1/L_c)^{-2} \ge \frac{9}{16}c_g$, and by the second, $L > c_L\tilde{m}^{-1} - 2r_0 \ge c_L\tilde{m}^{-1}/2$, that is, $\tilde{m}L \ge c_L/2$.

Definition 3.4: windows. A window is the union of the blocks at block distance at most 75 from a block. It therefore has at most 151 blocks per direction, each of coarse side at most $2L_c - 1$, and its sides are at most $151(2L_c-1) < 302L_c < k_P$. A block lies in the windows of the at most $151^3 = 3.44\cdot 10^6$ blocks at block distance at most 75 from it, and $151^3 < 27\cdot 304^3 = 7.6\cdot 10^8 = 27(k_P/L_c)^3$, the overlap bound of (R6).

Definition 3.11, Lemma 4.9, Theorem 4.17: the cost of a fill. A block Q has $|\partial Q| \le 6(2L_c-1)^2 < 24L_c^2$ and $|Q| < 8L_c^3$. Hence, with $\lambda_U = 2c_s/\ell$, $c_s \le 10$ and $\ell \ge 308L_c$, we have $\lambda_U|Q| < 16c_sL_c^3/\ell \le 0.52L_c^2$ and $|\partial Q| + \lambda_U|Q| < 24.6L_c^2 < \mu_{\mathrm{B}}L_c^2 = 25L_c^2$. Also $\lambda_U = 2c_s/\ell \le 20/(308L_c) < \lambda = 12/L_c$. A removal cell Δ with sides $a,b,c \in [\ell, 3\ell)$ has $|\Delta|/|\partial\Delta| = abc/(2(ab+bc+ca)) \le (abc)^{1/3}/6 < \ell/2$ by the arithmetic-geometric mean inequality, so $\lambda_U|\Delta| < c_s|\partial\Delta|$ and $|\partial\Delta| + \lambda_U|\Delta| < w_*$. The number of forced blocks is at most $27|\mathbb{M}|/(\vartheta L_c^2)$, since a mark lies in at most 27 sets $3Q$. Hence filling them costs at most $27\cdot 24.6/\vartheta = 664.2/\vartheta$ per mark.

Lemma 4.4: constants that depend on the hull weight. Frozen bonds have the hull weight $2-\eta_w$ of Definition 3.8, and the hull weight affects the constants in two ways. Through Lemma 4.4 it determines $(a_\eta, b_\eta) = (1+1/(1-\eta_w), 1/(1-\eta_w))$ and the factor $1/(1-\eta_w)$ in the bound on $N_2(W)$. Table S1.1 lists the constants that depend on η_w through Lemma 4.4, with their formula in η_w and their value at $\eta_w = 1/10$; the other constants, among them the constant c_0 of the density bound, the bound on c_L, K_*, $\sigma(\beta)$ and $R_1(\beta)$, do not depend on η_w through Lemma 4.4. In the cell analysis the hull weight appears, by Lemma 4.8(b), as the deficit η_w of a short segment through a frozen bond, which yields the factor $120 = 12/\eta_w$ of Proposition 4.15. Through this factor, $C_{\mathrm{seal}} = 726$, the number 134 of non-good lines per unit of $m_Q + m_{Q'}$ and the 158 in the expansion bound behind $h_\star$ depend on η_w (see the arithmetic in Section S1.2), and so do the constants built from them, such as $C_\star'$, C_{free} and C_{resp}, and the constraint on the weak-link constant χ (items 9 and 10 of Section S1.1.1).
At $\eta_w = 1/10$, c_1 equals its second entry, with the factor $(1-\eta_w)^3 = 0.729$, when $c_g \ge 4.22\cdot 10^{-4}$, and $c_1 = c_g/2$ otherwise. Hence the factor $(1-\eta_w)^{-3} = 1.372$ in the table is an upper bound. The Nash constant $A_N = 24\cdot 2^{2/3}/c_F^2$ of Appendix D depends on η_w only through c_F.

Theorem 4.5. We have $V_* = (c_I\rho_\lambda/(2b_\eta))^3$, since $(c_I/2)^3 = \frac{1}{2}$. Hence $b_\eta\lambda V_*^{1/3} = c_I/2$. In (iii), $\alpha := c_I/(12a_\eta) = 0.0626\ldots \le \frac{1}{4}$, and the two expressions in the minimum are equal at $r = 24(a_\eta/b_\eta)\rho_\lambda = 45.6\rho_\lambda$.

Lemma 4.21. We set

$$C_{\mathcal{U}} := 8\sqrt{3}\,(2 + 6/c_g)/(c_1c_L^2),$$

which bounds $2\sqrt{3}(2+6/c_g)/(c_1(\tilde{m}L)^2)$ when $\tilde{m}L \ge c_L/2$.

Table S1.1: Constants depending on the hull weight; values at $\eta_w = 1/10$, $c_I = 2^{2/3}$, $\rho_\lambda = L_c/12$.

quantity	formula	value at $\eta_w = 1/10$	used in
(a_η, b_η)	$(1 + 1/(1-\eta_w),\ 1/(1-\eta_w))$	$(19/9, 10/9)$	Lemma 4.4, (R2)
isoperimetry for $\lvert W\rvert \le V_*$	$(c_I/(2a_\eta))\lvert W\rvert^{2/3}$	$(9c_I/38)\lvert W\rvert^{2/3} = 0.376\lvert W\rvert^{2/3}$	Theorem 4.5(i)
V_*	$(1-\eta_w)^3\rho_\lambda^3/2$	$0.3645\rho_\lambda^3$	Definition 3.1, (R2)
volume growth constant	$(c_I/(48a_\eta))^3$	$3.844 \cdot 10^{-6}$	Theorem 4.5(iii)
saturation radius of (iii)	$24a_\eta(1-\eta_w)\rho_\lambda$	$45.6\rho_\lambda$	Theorem 4.5(iii)
Faber–Krahn constant	$c_I^2/(48a_\eta^2)$	$c_I^2/213.93$	Theorem 4.5(iv)
$N_2(W)$ factor	$1/(1-\eta_w)$	$10/9$	Lemmas 4.4, 6.13
c_1 (second entry)	$((1-\eta_w)c_I/24)^3$	$2.109 \cdot 10^{-4}$	(R5)
c_u	$((1-\eta_w)c_I\tilde{m}L/24)^3$	$(0.0595\,\tilde{m}L)^3$	(R7)
R_{HK}, K_0	$(1-\eta_w)c_I\rho_\lambda/3$, $K_0 = (1-\eta_w)c_Ic_\rho c_L/6$	$\tilde{m}R_{\mathrm{HK}} = 0.0397\tilde{m}L \in [K_0, 2K_0]$, $K_0 = 0.01984c_L$	Theorem 6.8
c_F (fine isoperimetry)	$(c_I/(2a_\eta))/152.4$	$2.467 \cdot 10^{-3}$	Proposition 6.9
$C_{\mathcal{U}}$, N_{near}, A_2	through $1/c_1$ and $1/c_u$, at most the factor $(1-\eta_w)^{-3}$	factor at most 1.372	(R7), Section 7
c_{sh}	$c_1/216$	$9.77 \cdot 10^{-7}$	Lemma 5.15

Proposition 4.26: the constant c_0. The fine plaquettes dual to the insulating bonds with an end in a removal cell Δ, those of frozen bonds counted twice, number at most $\bar{r}_0^2(3+c_s)|\partial\Delta|$, since a coarse bond is dual to at most $\bar{r}_0^2$ fine plaquettes. Hence they number at most $c_0/(z\tau)$ with $c_0 := z\tau\,\bar{r}_0^2(3+c_s)\max_\Delta|\partial\Delta|$, where z is the monopole activity of Section 6, $m_D^2 = 2\beta_{\mathrm{GM}}z$ and $z\tau = m_D^2/(8\pi^2)$. Using $\bar{r}_0 < 2r_0$, $r_0\ell = \ell_D < \tilde{m}^{-1} + L \le (1+c_L)\tilde{m}^{-1}$, $m_D \le 2\tilde{m}$, the bound $|\partial\Delta| < 54\ell^2$ of (G3) and $c_L \le \frac{1}{2}$, we obtain

$$c_0 < \frac{4\tilde{m}^2}{8\pi^2} \cdot 4 \cdot 54\,(3+c_s)(1+c_L)^2\tilde{m}^{-2} \le \frac{243}{\pi^2}(3+c_s) < 25(3+c_s),$$

independently of β. On a grid without exceptional layers and with cubic removal cells, $c_0 = 6(3+c_s)(m_D\ell_D)^2/(8\pi^2)$.

Lemma 4.10, Proposition 4.12, Lemmas 4.13, 4.14, Proposition 4.15, Theorem 4.16: the cell analysis. With $\vartheta = 3 \cdot 10^{-5}$ and $\varepsilon_{\mathrm{th}} = 0.008$:

- *Giant.* $12 + 6.4 \cdot 9 \cdot 2 = 127.2 < 128$, and $(1-3\vartheta)L_c^3 > 384\vartheta L_c^3 = 3 \cdot 128\vartheta L_c^3$ since $384\vartheta = 0.01152$.
- *Thin sets.* $2\varepsilon_{\mathrm{th}} + 12\vartheta = 0.01636 \le 0.0164 \le \frac{1}{4}$, and $2/\sqrt{0.0164} = 15.62 \ge 15.6 > 14$. Non-giant components are thin since $128\vartheta = 0.00384 < \varepsilon_{\mathrm{th}}$.

- *Seal.* Lemma 4.8(b) with supp $\le 6(m_Q + m_{Q'})$ and $\eta_w = 1/10$ gives $120(m_Q + m_{Q'})$ lines, plus $m_Q + m_{Q'}$ lines with a mark, in total 121 per unit of $m_Q + m_{Q'}$, and $C_{\mathrm{seal}} = 6 \cdot 121 = 726$.

- *Interfaces.* $1 + 121 + 12 = 134$ non-good lines per unit of $m_Q + m_{Q'} < \vartheta L_c^2$, so $\mathrm{good}(I) \ge (1 - 134\vartheta)|F'| = 0.99598|F'| \ge c_g|F'|$.

- *Expansion.* For two large sides, $\varepsilon_{\mathrm{th}} - 127.2\vartheta = 0.004184 \ge \varepsilon_{\mathrm{th}}/2$, so $\mathrm{cut} \ge \varepsilon_{\mathrm{th}} L_c^2/25.6 = 3.125{\cdot}10^{-4} L_c^2$. For adjacent in/out blocks, $\mathrm{cut} \ge (1 - 158\vartheta - 2\varepsilon_{\mathrm{th}})L_c^2/3 = 0.32642 L_c^2$. Both are at least $3.1 \cdot 10^{-4} L_c^2$, and $|A| < k_P^3/2 = 1.4047 \cdot 10^7 L_c^3$. Hence $h_\star \ge 3.1 \cdot 10^{-4}/(1.4047 \cdot 10^7 L_c) = 2.207 \cdot 10^{-11}/L_c \ge 2.2 \cdot 10^{-11}/L_c$. Then $3k_P h_\star = 912 \cdot 2.2 \cdot 10^{-11} = 2.006 \cdot 10^{-8} \ge \chi = 2.0 \cdot 10^{-8}$, and $\chi_{\mathrm{N3}} = \chi/3 = 6.67 \cdot 10^{-9} \le k_P h_\star = 6.69 \cdot 10^{-9}$.

Theorem 4.16 shows that the construction provides c_1-type volume $(1 - 3\vartheta)L_c^3$ per cell, one interface per block face and six neighbouring cells per cell. These facts imply (R5) and (R8) with the constants c_1, $1/c_g$ and $6/c_g$, which we keep.

S1.3 Constants of Sections 4.3 and 4.4

We first list the sizes on the loop-adapted grid and then the constants, grouped by main-text statement.

Sizes on the loop-adapted grid. Table S1.2 lists the sizes on $\mathfrak{G}(C)$ and, for orientation only, their values on the uniform grid, in which all coarse cells have side r_0 and all block spacings are L_c; the constant c_1 in the table is that of item 6 of Section S1.1.1. The opacity property (S-face) of Definition 3.17 is normalized per face, with the threshold $c_g|F(I)|$, and hence the good fraction and the cell degree are independent of the face area.

Lemma 4.22(c) and Theorem 4.23: the reach. In Lemma 4.22(c), $\lambda^{-1} + \lambda_U^{-1} = L_c/12 + \ell/(2c_s) \le \ell/c_s$ because $c_s \le 10 < 6\ell/L_c$, and $w \ge 120\ell/c_s$ gives the factor $30\ell/(c_s w) \le \frac{1}{4}$. In Theorem 4.23, $w' \le 121\ell/c_s + 2L_c \le 122\ell/c_s$, since $2L_c \le \ell/c_s$, and for $\rho \ge 4L_c$ we have $g(\rho) \le 54\rho^2 + 54c_s\rho^3/\ell$. Therefore $g((J+1)w') \le 10^8(J+1)^3(\ell/c_s)^2$, and J is at most the least integer with $5^J > 10^8(J+1)^3(\ell/c_s)^2$, that is, $J = 2\log_5(\ell/c_s) + O(\log\log(\ell/c_s))$. Hence

$$R_* = (J+1)w' + 6L_c \le 122(J+1)\ell/c_s + 6L_c = O\big((\ell/c_s)\log(\ell/c_s)\big);$$

with $\log \ell = \pi^2 v_{\mathrm{Cb}}(0)\,\tau + O(\log\tau)$ this is $R_* \approx 380\,\tau\,\ell/c_s$, and $\log R_* = \log(1/\tilde{m}) + O(\log\tau)$, the only property of R_* that the threshold K_* uses, in Lemma 5.12. In the clearing-out step, each site of $\mathbb{X}^1$ weighs at least λ_U in f, because $\lambda \ge \lambda_U$. Further, $|B_{w'}(x)| \ge \ell^3$ since $w' \ge \ell$ and each side of Λ° is at least ℓ, and $\lambda_U \ell^3 = 2c_s\ell^2 > 1$.

Proposition 4.24: the area per part. The competitor costs at most $(6 + \lambda)N(P)$ for the marks, $24.6L_c^2$ for each of at most $27N(P)/(\vartheta L_c^2)$ forced blocks, and 2 for each of at most $C_{\mathrm{seal}}N(P) = 726N(P)$ seal bonds. We therefore set

$$C_1 := 7 + 27{\cdot}24.6/\vartheta = 7 + 664.2/\vartheta = 2.214{\cdot}10^7, \qquad C_\star' := C_1 + 2C_{\mathrm{seal}} = C_1 + 1452 = 2.2141{\cdot}10^7$$

at $\vartheta = 3 \cdot 10^{-5}$, and $C_{\mathrm{free}} := C_\star'(2 + 1/(2c_s) + 1/3696) = 5.536 \cdot 10^7$ at $c_s = 1$, where $1/(\lambda\ell) = L_c/(12\ell) \le 1/3696$ by (3.1). The marks of a set $3Q$ are within distance $3(2L_c - 1) < 6L_c$ of

Table S1.2: Sizes on the loop-adapted grid $\mathfrak{G}(C)$, in coarse units unless stated, with the values on the uniform grid for orientation and the statements in which each size is used.

quantity	on $\mathfrak{G}(C)$	uniform grid	used in
block side	$[L_c, 2L_c - 1]$	L_c	(G1)
block and cell volume	$< 8L_c^3$	L_c^3	(R5); Lemma 4.21 ($\tilde{m}L \le \frac{1}{2}$)
face area $\lvert F' \rvert$	$[L_c^2, 4L_c^2)$	L_c^2	(S-face), Lemma 4.9
block diameter	$< 2\sqrt{3}L_c$	$\sqrt{3}L_c$	Cor. 4.7
window side; windows per site	$< k_P$; $\le 151^3$	$151L_c$; 151^3	(G4), (R6)
removal cell side	$[\ell, 3\ell)$	ℓ	Lemma 5.21
$\lvert \partial\Delta \rvert$	$[6\ell^2, 54\ell^2)$	$6\ell^2$	(R9), c_0 of Section S1.2
reach R_*	Theorem 4.23	same	Theorem 3.21
k_P	$304L_c$	same	(G4), (R6)
c_1; good fraction; cell degree	$2.109 \cdot 10^{-4}$; c_g; $6/c_g$	same	(R5), (R8)
fine good fraction	$c_g(1 + 1/L_c)^{-2}$	c_g	Lemma 4.19
q_m	$3(\bar{r}_0 + 1)^3\delta^{1/2} \le 24r_0^3\delta^{1/2}$	$3(r_0 + 1)^3\delta^{1/2}$	Lemma 5.7
fine area of a coarse face	$\le \bar{r}_0^2 < 4r_0^2$	r_0^2	Section 5 ($\bar{r}_0^2$ in η_T)
c_I for $\lvert W \rvert \le V_*$	$2^{2/3}$	$2^{2/3}$	Theorem 4.5

each other, and a bond of $\bar{S}^1$ has both ends within $4L_c - 3$ of a mark of its two blocks, so $R_* + 4L_c < r' = 2R_* + 2$.

Lemma 4.25. The comparison base is $\mathbb{X}^1 \cup R_t$ rather than the base $B_t = \mathbb{M} \cup \mathbb{U}^1 \cup R_t$. A bond from a block of $\mathbb{U}^1$ to hull material of $\mathbb{X}^1$ lies in $\partial\mathbb{U}^1$ but not in $\mathrm{ins}^1 = \partial\mathbb{X}^1$, and the marks do not bound such bonds, so only the comparison with $\mathbb{X}^1 \cup R_t$ gives a bound by ins^1.

S1.4 Constants of Section 5

Theorem 5.1. The constants of (5.2) are

$$c_T'' := 1 + \log(64\kappa_{\mathrm{pl}}), \qquad \kappa_{\mathrm{pl}} := 12e, \qquad C_3 := C_{\mathrm{free}} + 20(2C_\star' + 4C_{\mathrm{resp}} + 1) \quad \text{(Proposition 5.24)},$$

with $C_\star'$, C_{free} and C_{resp} as in item 10 of Section S1.1.1. The quantity ε'' is defined in (5.8), with $R' = R_* + 2R_{\mathrm{resp}} = R_* + 20\ell$ and $r' = 2R_* + 2$. The conditions used in the proof are those of item (a) of Section S1.1.4, which hold for large τ because $|\log \varepsilon''| \ge \gamma_T(K)\tau - O(\log\tau + \log r_0)$ with $\gamma_T(K) > 0$ for $K > K_*$ (Lemma 5.12), the constants C_3 and $C_{\mathfrak{B}}$ are free of β and K, the quantities ε_D, M_m and R_{NF}^{-1} are doubly exponentially small in τ, and $\log d_0 = O(\log\tau)$. The constants C_{resp}, $C_{\mathfrak{B}}$ and C_3 are fixed before K, and hence they affect $\beta_0(K)$ but not K_*. As for the order of η_T, we have $C_{\mathrm{free}} = O(C_\star'(1 + 1/c_s))$, hence $C_3 = O((C_\star' + C_{\mathrm{resp}})(1 + 1/c_s))$, and Lemma 2.6 shows that $\gamma_T(K)$ is of the order of $\log^2 K$ as $K \to \infty$.

Lemma 5.12. Since $R_* \geq w' \geq 12\ell$ by Theorem 4.23, we have $2R_* + 1 \leq 3R_*$ and $2r' + 1 = 4R_* + 5 \leq 5R_*$. With the bound $q_m \leq 24 r_0^3 \delta^{1/2}$ of Lemma 5.7 we obtain

$$\begin{aligned}\log \varepsilon_c &\leq \log(8e) + 3\log 3 + 3\log 5 + \log 24 + 6\log R_* + 3\log r_0 - \tfrac{1}{2}|\log\delta| \\ &\leq 14.4 + 6\log R_* + 3\log r_0 - \tfrac{1}{2}|\log\delta|.\end{aligned}$$

By Theorem 4.23, $R_* = O((\ell/c_s)\log(\ell/c_s))$ (Section S1.3), and $\ell \leq \ell_D = O(\tilde{m}^{-1})$ by Definition 3.1. With $\log(1/\tilde{m}) = \pi^2 v_{\mathrm{Cb}}(0)\,\tau + O(\log\tau)$ and $\pi^2 v_{\mathrm{Cb}}(0) < 2.4943551$, these bounds and the bound on the entropy exponent in Section S1.1.2 give $6\log R_* \leq 14.9662\,\tau + O(\log\tau)$. Also $\frac{1}{2}|\log\delta| = c_\delta(K)\beta/2 = \tau c_\delta(K)/(2K)$. This is the first inequality of (5.6). By Lemma 2.6, $c_\delta(K) \geq c_\delta^{\mathrm{sp}}(K) - 1/(4a_*) - O(\log\beta/\beta)$. The function $\gamma_T(K) := (c_\delta^{\mathrm{sp}}(K) - 1/(4a_*))/(2K) - 14.9662$ is increasing in K, and its zero is given, together with the margins, in Section S1.1.2. Hence $\gamma_T(K) > 0$ for $K > K_*$. The $O(\log\beta/\beta)$ term contributes $O(\log\tau)$ after multiplication by $\tau/(2K)$. For ε'' we use $R_{\mathrm{resp}} = 10\ell$ from Lemma 5.21, $L_c \leq \ell$ and $R_* = O(\ell\tau)$. Hence $\log(2R' + 4L_c + 1) \leq \log(1/\tilde{m}) + O(\log\tau)$, and the same computation applies.

Lemma 5.17. In the proof of (c), with $\rho = \rho(s)$, we have $|x_c - x_Q|_\infty \leq \lceil\rho\rceil/2 \leq \rho/2 + 1$. In (ii), the ends of the other cut bond are at distance more than $73\rho - 1$ from x_c by (U2), hence more than $73\rho - \rho/2 - 2 > 9\rho$ from x_Q. In (iii), two bonds of $B_\infty(x_c, 10\rho)$ have ends within coarse distance 20ρ, and $\sqrt{3}\bar{r}_0(20\rho + 1) < 128\bar{r}_0\rho = s$, so (U1) applies. For the last claim of (c), the two chosen bonds have ends within coarse distance $\rho + 2$, and $\sqrt{3}\bar{r}_0(\rho + 3) \leq s$. The range $27L_c \leq \rho \leq \mathrm{side}_{\min}/24$ holds. The lower bound follows from $R_{\mathrm{NF}} \geq 3456\bar{r}_0 L_c$. For the upper bound, $\varphi_s \geq 2$ forces s below the lattice diameter of Λ°, which is at most $\sqrt{3}\bar{r}_0\,\mathrm{side}_{\max} \leq 2\sqrt{3}\bar{r}_0\,\mathrm{side}_{\min}$, the last inequality by Lemma 3.3(d), and hence $\rho < 2\sqrt{3}\,\mathrm{side}_{\min}/128$.

Lemmas 5.20 and 5.21. In the proof of Lemma 5.20, summing $121(m_q + m_{q'})$ over the six faces of each block q of Δ_0, we count at most $121(6m(\Delta_0) + 6m(U)) \leq 1452\,m$ seal bonds with an end in Δ_0 and, summing over the blocks of U, at most $121(6m(U) + 6m(U_2(\Delta_0))) \leq 1452\,m$ seal bonds with both ends in U. With $u_3 \leq c_s^{-1}(i + 2904m + 2n)$ we obtain

$$\mathrm{units}(\Delta_0) \leq 1452m + 2u_3 + i + 2\cdot 1452m + 2n + 6m \leq \Big(4362 + \frac{5808}{c_s}\Big)m + \Big(2 + \frac{4}{c_s}\Big)n + \Big(1 + \frac{2}{c_s}\Big)i,$$

since $1452 + 2904 + 6 = 4362$ and $2 \cdot 2904 = 5808$. At $c_s = 1$, $(C_a, C_b, C_c) = (10170, 6, 3)$, and the loads of Lemma 5.21 are $125C_a = 1.27 \cdot 10^6$, $125C_b = 750$ and $125C_c = 375$. Thus

$$C_{\mathrm{resp}} := 375C_a + 125C_cC'_\star = 3.81 \cdot 10^6 + 375 \cdot 2.2141 \cdot 10^7 = 8.307 \cdot 10^9;$$

the term $125C_cC'_\star$, the load of the Phase-1 insulation owned by a part, dominates. The range is $R_{\mathrm{resp}} = 10\ell \geq 9\ell + 1$.

Proposition 5.22. A touched Y_Δ costs at most $2|\partial\Delta| + 2|\mathrm{ins}^1 \cap E(\Delta)| + 4.5|\partial\Delta| = 6.5|\partial\Delta| + 2|\mathrm{ins}^1 \cap E(\Delta)|$. A touched $Y = Y_{\Delta'}$ costs $2|\partial Y| + |Y|/\ell$, which by the bound on $|\partial Y|$ is at most $4\,\#\{\text{bonds of types (a) to (c) with an end in } \Delta'\} + 13|Y|/\ell$, and $13|Y|/\ell \leq (26L_c/\ell)m(\Delta')$. Since a bond has two ends, it is charged by at most two cells of the third kind, at most 4 each. A unit (Δ, b) is charged at most 6.5 by Y_Δ and at most 4 by the cell of the other end of b. A bond of ins^1 inside a removed cell is charged at most 2 by Y_Δ. All coefficients are at most 20.

Lemma 5.23 and Proposition 5.24. The constants are

$$C_{\mathfrak{B}} := 30C'_\star + 170C_{\rm resp} + 30, \qquad M_m := 16\varepsilon_D(1+8L_c)(C'_\star + 3C_{\rm resp} + 6),$$
$$C_3 := C_{\rm free} + 20(2C'_\star + 4C_{\rm resp} + 1).$$

To obtain $C_{\mathfrak{B}}$, we consider the proof of Lemma 5.23(c). There the bonds of ins[1] number at most $C'_\star N_{\mathcal{Q}}$, as in (b), the cut bonds number at most $A_{\mathcal{F}}$, and the units number at most $2C_{\rm resp}(N_{\mathcal{Q}}+A_{\mathcal{F}})$ by (a), each of these being charged by at most two blocks. The marks involved, at most $N_{\mathcal{Q}}$, belong to parts of $\mathcal{Q}_{\rm att}$, each charged by at most seven blocks. Summing, we obtain

$$|\mathfrak{B}_{\mathcal{F}}| \le 2\big(C'_\star N_{\mathcal{Q}} + A_{\mathcal{F}} + 2C_{\rm resp}(N_{\mathcal{Q}} + A_{\mathcal{F}})\big) + 7N_{\mathcal{Q}} \le (2C'_\star + 4C_{\rm resp} + 10)N_{\mathcal{Q}}$$

by Lemma 5.23(b), and $2C'_\star + 4C_{\rm resp} + 10 \le C_{\mathfrak{B}}$.

For M_m, we have $m \le 2A_{\mathcal{F}} \le 16\varepsilon_D(1+8L_c)(C'_\star + 3C_{\rm resp} + 6)N_{\mathcal{Q}} = M_m N_{\mathcal{Q}}$ by Lemma 5.23(d) and (b).

The size of $C_{\rm resp}$. We have $C_{\rm resp} \ge 125C_cC'_\star \ge 125C'_\star$, in particular $C_{\rm resp} \ge C'_\star + 20$.

For C_3, we note that $20(u_{\rm touch}+c_{\rm touch}+i_{\rm touch}) \le 20(C_{\rm resp}+C'_\star)N_{\mathcal{Q}} + 20(C_{\rm resp}+1)A_{\mathcal{F}}$ in the proof of Proposition 5.24, and that $26L_c/\ell \le 26/308 < 1/11$. With $A_{\mathcal{F}} \le N_{\mathcal{Q}}/(48C_{\rm resp}+2)$, all terms with $A_{\mathcal{F}}$ together are at most $N_{\mathcal{Q}}$, and

$$\#(\Xi' \cap \mathfrak{F})/\bar r_0^2 \le \big(C_{\rm free} + 20C'_\star + 20C_{\rm resp} + 2\big)N_{\mathcal{Q}} \le C_3 N_{\mathcal{Q}}.$$

The binding condition. The first condition of (5.10) reads $|\log\varepsilon''| \ge 3C_{\mathfrak{B}}\ln(2^{30}\cdot 26e) = 75.16\,C_{\mathfrak{B}}$. At the illustrative values $(c_g, c_s, r_0) = (1/4, 1, 1)$, the constants of item 10 are $C'_\star = 2.2141\cdot 10^7$, $C_{\rm free} = 5.536\cdot 10^7$ and $C_{\rm resp} = 8.307\cdot 10^9$, hence $C_3 = 6.655\cdot 10^{11}$ and $C_{\mathfrak{B}} = 1.413\cdot 10^{12}$, and this condition becomes $|\log\varepsilon''| \ge 1.06\cdot 10^{14}$. The condition $\eta_T \le 1$ needs $|\log\varepsilon''| \ge 3c''_T\bar r_0^2 C_3 = 1.73\cdot 10^{13}\bar r_0^2$, with $c''_T = 8.644$, and $\eta_T^{(1)} \le 1$ needs $|\log\varepsilon_c| \ge c_T\bar r_0^2 C_{\rm free} = 3.63\cdot 10^8\bar r_0^2$. Each of these is a lower bound on τ, by Lemma 5.12.

S1.5 Constants of Section 6 and Appendix D

Masses (Section 6). The Coulomb kernel at the origin is

$$v_{\rm Cb}(0) = \tfrac12 W_S = (\sqrt6/(192\pi^3))\Gamma(\tfrac{1}{24})\Gamma(\tfrac{5}{24})\Gamma(\tfrac{7}{24})\Gamma(\tfrac{11}{24}) = 0.2527310098\ldots,$$

where $W_S = \pi^{-3}\int_{[0,\pi]^3}(3-\sum_i\cos x_i)^{-1}dx$ is Watson's integral [Wat39], in the Gamma-function form of [Zuc11, (3.6)]. Numerically, $\pi^2 v_{\rm Cb}(0) = 2.4943550871\ldots$. Since $m_D = (8\pi^2\tau)^{1/2}e^{-\pi^2 v_{\rm Cb}(0)\tau}$ and $2(8\pi^2)^{1/2} = 17.77153\ldots$, it follows that

$$\tilde m \le 2m_D \le 17.7716\,\tau^{1/2}e^{-2.494\tau} \qquad (m_D/2 \le \tilde m \le 2m_D). \tag{S1.2}$$

Here 2.494 is $\pi^2 v_{\rm Cb}(0)$ rounded down to a lower bound by the rule of Section S1.1.3, and the hypothesis of Lemma 6.18 has the same 2.494 on its left.

Theorem 6.5: the Debye condition. In [GM82, (4.24)], at $A = (1-\delta)M$, [GM82] use $\|v^2\|_A \le 2(\delta M)^{-2}$ [GM82, Lemma 5, (3.26)], and the Debye condition [GM82, p. 34] reads $1 \le \frac14\delta^2 C^3(1-C)^8 e^{-2\kappa'}M^2/m_D^2$, where $\kappa' = \kappa_S + \frac12\mu$ by the choice made on [GM82, p. 35]. On G' the bound $\|v^2_{G'}\|_A \le 3M_Y^{-2}$ holds by Proposition 6.10 on outputs for $A \le \sqrt3\,\kappa_Y M_Y$,

and by $\|v^2_{G'}\|_0 \le M_Y^{-2}$ on all slit graphs. It replaces $2(\delta M)^{-2}$, that is, $\frac12\delta^2$ by $\frac13$, and hence $\frac14\delta^2$ becomes $\frac16$:

$$1 \le \tfrac16 C^3(1-C)^8 e^{-2\kappa_S-\mu} M_Y^2/m_D^2.$$

Since $M_Y \ge m_D/(2\lambda_{\rm PV})$ on I_Y, $M_Y^2/m_D^2 \ge 1/(4\lambda_{\rm PV}^2)$, and the condition holds for all $M_Y \in I_Y$ once

$$\lambda_{\rm PV}^2 \le C^3(1-C)^8 e^{-2\kappa_S-\mu}/24. \tag{S1.3}$$

In [GM82, (4.23)] the bound $\|v^1_{G'}\|_A \le 3M_1^{-2}$, with $M_1 = -4\pi\beta_{\rm GM}^{-1}\ln(1-C)$, is a power of $\beta_{\rm GM}$, which $e^{\beta_{\rm GM}\varepsilon_1}$ absorbs for $\beta_{\rm GM} \ge \beta^0_{\rm GM}$. Neither condition depends on G'.

The scales of §6.3: the rounding of L. By Definition 3.1, $L = r_0 L_c$ with L_c the largest even integer $\le c_L\tilde m^{-1}/r_0$, and $c_L\tilde m^{-1} \ge 4r_0$. Hence

$$c_L/(2\tilde m) \le L \le c_L/\tilde m. \tag{S1.4}$$

In fine units $r_0\rho_\lambda = L/12 = c_\rho L$, where $c_\rho = 1/12$, while $c_I = 2^{2/3}$, $V_* = (1-\eta_w)^3\rho_\lambda^3/2$ in coarse units and $V_*^f = r_0^3 V_*/2$, so $R_{\rm HK} = (1-\eta_w)c_I r_0\rho_\lambda/3$ and $4R^3_{\rm HK}/V_*^f = 64/27$ exactly. The constant

$$K_0 := (1-\eta_w)c_I c_\rho c_L/6 \qquad (= 0.2381\, c_\rho c_L = 0.01984\, c_L \text{ at } \eta_w = 0.1)$$

satisfies $K_0 \le \tilde m R_{\rm HK} \le 2K_0$ by (S1.4). We use K_0 only through the lower bound $\tilde m R_{\rm HK} \ge K_0$, in the following places.

(a) In Proposition 6.10, $(\varepsilon_1^{\rm HK})^{1/2} M_Y R_{\rm HK} \ge (\varepsilon_1^{\rm HK})^{1/2} K_0/\lambda_{\rm PV}$, so the rate of (6.13) at $M = M_Y$ is at least $\kappa_Y M_Y$.

(b) In (H2) of Theorem 6.11, at the mass $r_-^{1/2} \ge \tilde m$ the rate of (6.13) is at least $\kappa_W \tilde m \min(1, (\varepsilon_1^{\rm HK})^{1/2}\tilde m R_{\rm HK}) \ge \kappa_0\tilde m$, using $(\varepsilon_1^{\rm HK})^{1/2}K_0 < 1$.

(c) In (H1) and (H1′) of Theorem 6.11, $R_{\rm HK}^{-3} r_-^{-1} \le \tilde m^{-2} R_{\rm HK}^{-3} \le \tilde m K_0^{-3} = \lambda_{\rm PV} K_0^{-3} M_Y$.

(d) In Lemma 6.14, $R_{\rm HK}^{-1} + M_Y^{-2} R_{\rm HK}^{-3} \le \tilde m K_0^{-1} + \lambda_{\rm PV}^2 \tilde m K_0^{-3} = \eta_G$.

(e) In Appendix D, $\tilde m \le K_0(2C_H)^{-1/3}$ gives $R_{\rm HK} \ge (2C_H)^{1/3}$.

By (S1.4), $M_Y r_0\rho_\lambda = c_\rho M_Y L \in [c_\rho c_L/(2\lambda_{\rm PV}),\, c_\rho c_L/\lambda_{\rm PV}]$. We will use the lower bound in (f) to (h) below.

Theorem 6.8. We set $c_N := c_F^2/(24\cdot 2^{4/3}) = 1.006\cdot 10^{-7}$, $C_H := (c_N/3)^{-3/2} = 1.628\cdot 10^{11}$ and $\varepsilon_1^{\rm HK} := (\log 2)^2/512 = 9.38\cdot 10^{-4}$. In the proof, given in Appendix D, the constant of Lemma D.1 is $A_N := 24\cdot 2^{2/3}/c_F^2$, so that $c_N = 1/(2^{2/3}A_N)$, and (HK0) uses $6(\cosh\frac14 - 1) = 0.1885 < \frac14 - \frac{1}{17} = 0.1912$. The constants $C_{\rm mo} \ge 1$ of the moment bound, $c_{\rm ex} := \min((\log 2)/(2A_1), \frac12)$ with $A_1 := 512C_{\rm mo}^2/(\log 2)^2$ (so $\varepsilon_1^{\rm HK} = C_{\rm mo}^2/A_1$), $C_{\rm gs} := 8C_H e^{c_{\rm ex}/2}$, $c_{\rm gs} := c_{\rm ex}/8$ and C_M, which contains the factor $e^{1/(2c_{\rm ex})}$, depend only on C_H. Hence $\kappa_W = \min(1, 1/(2C_M))$ of Proposition 6.10 is extremely small and fixed before $\beta_{\rm GM}$; what matters is that it is a constant independent of τ, K, C and D.

Proposition 6.9. The box inequalities of the sketch in §D.1 are as follows. For a box graph B with sides in $[n_{\min}, n_{\max}]$, $\varrho := n_{\max}/n_{\min}$, and $U \subseteq B$, canonical paths give $|\partial_B U| \geq 4|U||B \setminus U|/(n_{\max}|B|) \geq 2\min(|U|, |B \setminus U|)/n_{\max}$, and the Loomis–Whitney inequality gives $|\partial_B U| \geq 3|U|^{2/3} - 3|U|/n_{\min}$. Hence $|\partial_B U| \geq c_{\mathrm{box}}(\varrho)|U|^{2/3}$ for $|U| \leq |B|/2$, with $c_{\mathrm{box}}(\varrho) := 6/(3\varrho + 2)$, as one sees by minimizing over $|U|^{1/3} = t\, n_{\min}$. In the face count, for a heavy cell c and a light neighbour c', the set $D := B_c \cup B_{c'}$ has $n_{\max}(D) < 4r_0$ and $|U||D \setminus U|/|D| \geq r_0^3/4$ for $U := A \cap D$, so D carries at least $\gamma_D r_0^2$ bonds of ∂A, $\gamma_D = \frac{1}{4}$. Summing over $\partial_\Gamma H$, where $\Gamma = (V_c, E_c)$ is the coarse slit graph of §6.3, counts the bonds inside a coarse cell at most six times. Hence $|\partial A| \geq (\gamma_D r_0^2/6)|\partial_\Gamma H|$. With the constant $a := 9c_I/38$ of (R2), and using $c_{\mathrm{box}}(\varrho) > \frac{3}{4}$ for $\varrho < 2$ and $\gamma_D = \frac{1}{4}$, the two cases of (i) give $|\partial A| \geq 2^{-2/3} c_{\mathrm{box}} |A|^{2/3} \geq 0.47|A|^{2/3}$ and $|\partial A| \geq (\gamma_D a/6) 16^{-2/3} |A|^{2/3} = (a/152.39\ldots)|A|^{2/3}$, so $c_F := a/152.4 = 2.467 \cdot 10^{-3}$. The constant $C_{\mathrm{bl}} := 384$ comes from $\gamma_D r_0^3/(12 v_{\max}) \geq 1/384$ with $v_{\max} < 8r_0^3$. Further, $C_W^f := 108 C_{\mathrm{bl}}^2/\chi^2$, $k_\square := r_0(k_P + 3)$, $h^f := \chi/(3C_{\mathrm{bl}} k_\square)$, $12/(h^f)^2 = C_W^f k_\square^2$ and $N_{\mathrm{ov}} \leq 27(k_P/L_c)^3$. For (v), (L8-2) reads $C_W^f r_\square^2 N_{\mathrm{ov}}(\tilde{m}L)^2 \leq \eta_H/8$ and $C_W^f r_\square^2 (\tilde{m}L)^2 = 12\tilde{m}^2/(h^f)^2$, so $1/h^f \leq 96^{-1/2}\tilde{m}^{-1} \leq 0.103\,\tilde{m}^{-1}$, since $\eta_H < 1 \leq N_{\mathrm{ov}}$. Moreover $\log k_\square \leq \log(1/\tilde{m}) \leq 2.4944\,\tau = 0.0632\,\beta_{\mathrm{GM}}$. Also $|\mathfrak{K}^\flat| \leq |V \cap \hat{Q}(Q)| \leq k_\square^3$: a window has fewer than k_P coarse sites per direction (G4), all of fine side r_0 except at most one exceptional layer per special interval (Section S2.2), of fine side at most $2r_0 - 1$, and it meets at most three special intervals per direction, so its fine sides are at most $r_0 k_P + 3(r_0 - 1) < k_\square$. Hence the diameter bound $2 + (24/h^f) \cdot 3 \log k_\square \leq 2 + 24 \cdot 0.103 \cdot 3 \cdot 0.0632\,\beta_{\mathrm{GM}}/\tilde{m} < 2\beta_{\mathrm{GM}}/\tilde{m} = L'$ holds.

Theorem 6.11. Let $\varpi := (1 - 4\lambda_{\mathrm{PV}}^2)^{-1/2} \in [1, 1.16]$ and $r_\pm := \frac{1}{2} M_Y^2 (1 \pm (1 - 4\lambda_{\mathrm{PV}}^2)^{1/2})$, so that $r_- = 2\tilde{m}^2/(1 + (1 - 4\lambda_{\mathrm{PV}}^2)^{1/2}) \in [\tilde{m}^2, 1.08\,\tilde{m}^2]$ for $\lambda_{\mathrm{PV}} \leq \frac{1}{4}$. The constants, which depend only on $(c_F, c_\rho, c_L, \lambda_{\mathrm{PV}})$, are

$$
\begin{aligned}
&K_1 := \tfrac{1}{2} + 2C_H(1 + \varpi) + \varpi C_H \lambda_{\mathrm{PV}} K_0^{-3}, \qquad K_3 := 3\varpi\tilde{m}^2/r_- \leq 3.5,\\
&K_1' := \max\big\{eK_1,\ \varpi\big(313 + C_{\mathrm{gs}}(2\pi/c_{\mathrm{gs}})^{1/2} + 4C_H(\varepsilon_1^{\mathrm{HK}})^{-3/2}\lambda_{\mathrm{PV}}K_0^{-3}\big)\big\},\\
&K_1'' := \max(eK_1, K_1').
\end{aligned}
$$

The bound $de^{-3d/68} \leq M_Y d^2 e^{-3d/68} \leq (136/3)^2 e^{-2} M_Y < 279\, M_Y$, valid for $d \geq M_Y^{-1}$, together with $279 \leq 313$, accounts for the number 313.

Section 6.4: the depletion constants. Both A_0^{AD} and $C_F := C_{\mathrm{Gr}} C_L'$ are upper-bound constants, taken ≥ 1; we also take $C_{\mathrm{Gr}} \geq 1$ and $C_L' \geq C_{\mathrm{im}}$, the constant of the far images in step (iv) of the proof of Theorem 6.16, so that $C_F \geq C_{\mathrm{im}}$. Let $\varepsilon_0 := \theta_0/(2\pi^2\tau)$, $\varepsilon_F := \varepsilon_0/(4C_F)$,

$$
L_g := 3 + \log_2^+\big(8\lambda_{\mathrm{PV}}/(c_\rho c_L)\big), \qquad C_R := 256\pi^2 A_0^{\mathrm{AD}} C_F L_g, \qquad C_S := C A_0^{\mathrm{AD}}\big(1 + \lambda_{\mathrm{PV}}/(c_\rho c_L)\big),
$$

so that $32 A_0^{\mathrm{AD}} L_g/\varepsilon_F = C_R\tau/\theta_0$ and $R_{\mathrm{dep}} = \lceil C_R\tau/\theta_0 \rceil$. Moreover, $2/\varepsilon_F = 16\pi^2 C_F \tau/\theta_0$ and $R_{\mathrm{dep}} \leq 2C_R\tau/\theta_0$. Since $A_0^{\mathrm{AD}} \geq 1$ and $L_g \geq 3$, $C_R \geq 768\pi^2 C_F \geq 24\pi^2 C_{\mathrm{im}}$.

(f) In Theorem 6.15, the shells $2^k \leq r(y) < 2^{k+1}$ with $2^{k+1} \leq r_0\rho_\lambda$ contribute at most $8A_0^{\mathrm{AD}}$ each to $S_2(x)$ by Lemma 6.13, and hence at most $CA_0^{\mathrm{AD}} \log(2/M_Y)$ up to $r \leq M_Y^{-1}$. The shells beyond M_Y^{-1} are summed with $e^{-M_Y r/2}$, and the growth term gives $\sum_k (16A_0^{\mathrm{AD}}/(r_0\rho_\lambda)) 2^k e^{-M_Y 2^k/2} \leq CA_0^{\mathrm{AD}}/(M_Y r_0 \rho_\lambda) \leq 2CA_0^{\mathrm{AD}}\lambda_{\mathrm{PV}}/(c_\rho c_L)$. Since $\log(2/M_Y) \geq 1$, $S_2(x) \leq C_S \log(2/M_Y)$.

(g) In step (ii) of the proof of Theorem 6.16, with $R \le r_0\rho_\lambda$, the far pairs contribute at most $16A_0^{\mathrm{AD}}/R + (16A_0^{\mathrm{AD}}/(r_0\rho_\lambda))(2 + \log_2^+(4/(M_Y r_0\rho_\lambda))) \le (16A_0^{\mathrm{AD}}/R)(3 + \log_2^+(8\lambda_{\mathrm{PV}}/(c_\rho c_L))) = 16A_0^{\mathrm{AD}}L_g/R$, which is at most $\varepsilon_F/2$ for $R = R_{\mathrm{dep}}$. In the second sum, the shells with $2^{k+1}R > r_0\rho_\lambda$ give $\sum_{j\ge 0} e^{-a2^j}$ with $a > M_Y r_0\rho_\lambda/4$, which is at most $2 + \log_2^+(1/a)$.

(h) For Lemma 6.18, see the proof below.

As a function of R_{dep}, the covering number N_{cov} of Proposition 6.17 takes the same value for all $R_{\mathrm{dep}} \le \ell_D$, so it does not depend on θ_0. It is a very large constant, and it raises β_{GM}^0 through a single condition, that of Theorem 7.40.

Proof of Lemma 6.18. The hypothesis says $C_*^{\mathrm{dep}}\tau^{5/2}e^{-2.494\tau} \le \theta_0$, and we use the bound (S1.2) throughout.

We first prove $R_{\mathrm{dep}} \le r_0\rho_\lambda$. Since $A_0^{\mathrm{AD}}, C_F \ge 1$ and $L_g \ge 3$, $C_R \ge 768\pi^2 > 7.5\cdot 10^3$. Hence $y := C_R\tau/\theta_0 \ge 1.5\cdot 10^6$ and $R_{\mathrm{dep}} = \lceil y \rceil \le (1+10^{-6})y$. By (S1.4), $r_0\rho_\lambda = c_\rho L \ge c_\rho c_L/(2\tilde m)$, so it suffices that $2(1+10^{-6})C_R\tau\tilde m \le c_\rho c_L\theta_0$. Using $\tau \ge 2$,

$$\begin{aligned} 2(1+10^{-6})C_R\tau\tilde m &\le 2(1+10^{-6})\,17.7716\,C_R\tau^{3/2}e^{-2.494\tau} \le 17.8\,C_R\tau^{5/2}e^{-2.494\tau} \\ &\le c_\rho c_L C_*^{\mathrm{dep}}\tau^{5/2}e^{-2.494\tau} \le c_\rho c_L\theta_0. \end{aligned}$$

Next, $R_{\mathrm{dep}} \le \ell_D$ because $r_0\rho_\lambda = c_\rho L < L \le \ell_D$.

Finally, for (F), $\log(2/M_Y) = \ln(2\lambda_{\mathrm{PV}}/\tilde m) \le \ln(1/m_D) = \pi^2 v_{\mathrm{Cb}}(0)\tau - \frac12\ln(8\pi^2\tau) \le 2.4944\,\tau$, using $\lambda_{\mathrm{PV}} \le \frac14$ and $\tilde m \ge m_D/2$. With $\eta_G = \tilde m(K_0^{-1} + \lambda_{\mathrm{PV}}^2 K_0^{-3})$,

$$C_F\eta_G C_S\log(2/M_Y) \le C_F C_S(K_0^{-1} + \lambda_{\mathrm{PV}}^2K_0^{-3})\cdot 17.7716\,\tau^{1/2}e^{-2.494\tau}\cdot 2.4944\,\tau,$$

and this is at most $\varepsilon_0/4 = \theta_0/(8\pi^2\tau)$ because $8\pi^2\cdot 17.7716\cdot 2.4944 < 3500.2 \le 3.6\cdot 10^3$ and $C_*^{\mathrm{dep}} \ge 3.6\cdot 10^3 C_F C_S(K_0^{-1} + \lambda_{\mathrm{PV}}^2K_0^{-3})$.

S1.6 Constants of Subsections 7.0 to 7.5 and Appendix C

Proposition 7.5. Off $\mathfrak{D}$, (S4′) gives $|\rho_1(\pm 1, x) - \rho^{\mathrm{ref}}(1)| \le \rho^{\mathrm{ref}}(1)(1 - e^{-\theta_0} + Ce^{-\mu})$, and $\sum_{|m|\ge 2}(|\rho_1(m,x)| + |\rho^{\mathrm{ref}}(m)|)m^2 \le 2Ce^{-2\kappa_S}m_D^2/\beta_{\mathrm{GM}}$. Since $2\beta_{\mathrm{GM}}\rho^{\mathrm{ref}}(1) = \tilde m^2 - \beta_{\mathrm{GM}}\sum_{|m|\ge 2} m^2\rho^{\mathrm{ref}}(m) \le (1 + Ce^{-2\kappa_S})\tilde m^2$ and $m_D \le 2\tilde m$, we obtain, after enlarging C,

$$\sum_m |\rho_1(m,x) - \rho^{\mathrm{ref}}(m)|\, m^2 \le \theta'\,\tilde m^2/\beta_{\mathrm{GM}} \qquad (x \notin \mathfrak{D}),$$

with θ' of (7.7). On $\mathfrak{D}$ we have $|\rho_1(\pm 1, x)| \le \rho^{\mathrm{ref}}(1)(1 + Ce^{-\mu})$ and, since $e^{-\Phi} \le 1$, $\rho_1(\pm 1, x) - \rho^{\mathrm{ref}}(1) \le Ce^{-\mu}\rho^{\mathrm{ref}}(1)$. These bounds, the same tails and $2\beta_{\mathrm{GM}}\rho^{\mathrm{ref}}(1) \le 2\tilde m^2$ give the constants of part (b). At $\theta_0 = 0.01$ we have $1 - e^{-\theta_0} = 0.00995\ldots$, and hence the standing hypothesis (7.8) holds once $\varepsilon_\nu + C(e^{-\mu} + e^{-2\kappa_S}) \le 0.1329$.

The partial fractions (7.16) **and Lemma 7.15.** If $\lambda_{\mathrm{PV}} \le \frac14$, then $\varpi = (1-4\lambda_{\mathrm{PV}}^2)^{-1/2} \in [1, 1.155]$, $c_+ \le \varpi$, $c_- \le 1.26\,\lambda_{\mathrm{PV}}^2$, $\tilde m^2 \le r_- \le 1.08\,\tilde m^2$ and $r_+ \ge 0.932\,M_Y^2$. In part (b), with $\alpha = \tilde m/3$, the bound $\cosh x - 1 \le \frac{x^2}{2}\cosh x \le 0.5281\,x^2$ for $x \le \frac13$ gives $6(\cosh\alpha - 1) \le 0.3521\,\tilde m^2$. With this α, part (a) gives $\|e^{\rho} R_{r_-} e^{-\rho}\| \le (r_- - 0.3521\tilde m^2)^{-1} \le 1.544\,\tilde m^{-2}$ and $\|e^{\rho} R_{r_+} e^{-\rho}\| \le (r_+ - 0.3521\tilde m^2)^{-1} \le 1.1\,M_Y^{-2} = 1.1\,\lambda_{\mathrm{PV}}^2 \tilde m^{-2}$, and hence $\|e^{\rho}\mathcal{G}e^{-\rho}\| \le (1.155\cdot 1.544 + 1.26\cdot 1.1\,\lambda_{\mathrm{PV}}^4)\tilde m^{-2} \le 1.80\,\tilde m^{-2}$. The matrix $e^{\rho}\nabla^* e^{-\rho}$ has entries $\pm e^{\rho(x)-\rho(b)}$ with $|\rho(x)-\rho(b)| \le \frac13$, at most 6 per row and 2 per column. Hence its norm is at most $\sqrt{12}\,e^{1/3} \le 4.835$, and $1.80\cdot 4.835 \le 8.71$. In part (c), $\|[\Delta,\chi]\| \le 6\Lambda_\chi$ and $\|\Delta\| \le 12$, so $\|[C_0^{-1},\chi]\| \le M_Y^{-2}\cdot 2\cdot 12\cdot 6\,\Lambda_\chi + 6\Lambda_\chi = \Lambda_\chi(144 M_Y^{-2} + 6) \le 150\,\Lambda_\chi M_Y^{-2}$, since $M_Y \le 1$.

Theorem 7.16. The constant is $C_\varepsilon := 3.8\cdot 10^9$. With $d(A', V_B) \ge L'/2 - 2$, $\Lambda_\chi = 4/L'$, $M_Y^{-4} = \lambda_{\mathrm{PV}}^4 \tilde m^{-4}$ and $\tilde m \le 1$, Lemma 7.15 (b), (c) give $\|F'_{2,\beta}\|^2 \le (150\cdot \frac{4}{L'} M_Y^{-2})^2 (8.71\,\tilde m^{-2})^2 e^{4/3} e^{-\tilde m L'/3}\|w_\beta\|^2 \le 1.71\cdot 10^6\cdot 16\,e^{4/3}\,\tilde m^2 (\tilde m L')^{-2}\lambda_{\mathrm{PV}}^4 \tilde m^{-8} e^{-\tilde m L'/3}\|w_\beta\|^2$. Theorem 7.13 and $c'_g \ge \frac{9}{16} c_g$ give

$$\sum_{\beta : S_\beta \subseteq X} \|w_\beta\|^2 \le 18\,F_1^{\mathrm{loc}}(X)/(c'_g \tilde m^2 L/2) \le 32\,F_1^{\mathrm{loc}}(X)/(c_g \tilde m^2 L/2).$$

The product of the numerical factors is $1.71\cdot 10^6\cdot 16\cdot 32\,e^{4/3} \le 3.4\cdot 10^9 \le C_\varepsilon$.

Corollary 7.18. The Hölder exponents and the constant of Subsection 7.6 are $p_2 = 20$, $p_3 = 1.2$, $p' = 1.3$, $q' = 15.6$ and $c_\lambda = \frac{1}{24}$, so $\varepsilon_0 = \min\{(1 - \frac{1}{1.2} - \frac{1}{24})/40,\ 2(1.3/15.6)^2\} = \min\{0.003125,\ 0.0139\} = 0.003125$. Dropping $(\tilde m L')^{-2} \le 1$ in (7.17), we obtain $\varepsilon(L') \le \varepsilon_0$ as soon as

$$\tilde m L' \ge 27\log(1/\tilde m) + 3\log\bigl(C_\varepsilon \lambda_{\mathrm{PV}}^4/(c_g \varepsilon_0\,\tilde m L/2)\bigr).$$

The second term is bounded, since $c_L/2 \le \tilde m L \le c_L$ by Definition 3.1. For the first, Definition 3.1 and Section S1.7.3 give $\log(1/\tilde m) \le 2.4944\,\tau = (2.4944/4\pi^2)\beta_{\mathrm{GM}}$ for $\tau \ge \tau_0$, so the right side is at most $1.706\,\beta_{\mathrm{GM}} + O(1)$, while $\tilde m L' = K_{\mathrm{col}}\beta_{\mathrm{GM}} = 2\beta_{\mathrm{GM}}$.

Proposition 7.19. The constant of part (c) is

$$\Phi_{\mathcal{U}} := \frac{\pi c_g c_u}{24}\,\frac{(\tilde m L)^3}{(\tilde m \beta_{\mathrm{GM}})(\tilde m L')^3 (11/10)^3}.$$

Dividing (a) by (b) gives the share $\frac{\pi c'_g c_u}{12}(\tilde m L)^3/((\tilde m\beta_{\mathrm{GM}})(\tilde m L')^3(1+\eta_{\mathrm{col}})^3)$, which is at least $\Phi_{\mathcal{U}}$ because $c'_g \ge \frac{9}{16}c_g \ge \frac12 c_g$ and $\eta_{\mathrm{col}} \le \frac{1}{10}$ for $\beta_{\mathrm{GM}} \ge \beta_{\mathrm{GM}}^0$. The latter holds since L and D_E are $O(\tilde m^{-1})$, while $L' = 2\beta_{\mathrm{GM}}\tilde m^{-1}$. With $\tilde m L \ge c_L/2$ and $\tilde m L' = 2\beta_{\mathrm{GM}}$, $\Phi_{\mathcal{U}} \ge \pi c_g c_u c_L^3/(24\cdot 64\cdot 1.331\,\tilde m \beta_{\mathrm{GM}}^4)$, and $\tilde m\beta_{\mathrm{GM}}^4 \to 0$ because $\log(1/\tilde m) = \pi^2 v_{\mathrm{Cb}}(0)\tau + O(\log\tau)$ and $\beta_{\mathrm{GM}} = 4\pi^2\tau$.

Structural facts of the expansion (Appendix C). The bounds on the activities in Subsections 7.6 and 7.7 and in Appendices A and B use the expansion of Proposition 7.22 only through the following facts S1 to S14, and none of them uses translation invariance, an infinite lattice, or the shape of the atoms. Each fact is followed by the place where it is proved; "change n" is item (n) of the changes in the proof of Proposition 7.22 in Appendix C. These facts are not the properties (S1) to (S5) of Theorem 6.5.

S1 $\mathbf{C}_s(x,x) = \mathbf{C}(x,x)$ and $|\mathbf{C}_s(x,y)| \le |\mathbf{C}(x,y)|$; the weighted norms and $\|\mathbf{C}_s\|_{2\to 2}$ are bounded by those of $\mathbf{C}$ (change 6).

S2 $\mathbf{C}_s = \sum_\varepsilon \lambda_\varepsilon \mathbf{C}_\varepsilon$, where each $\mathbf{C}_\varepsilon$ is block diagonal with admissible blocks (change 6).

S3 $\log \int d\mu_{\mathbf{C}}\, e^{\frac{1}{2}\langle \psi+D, M(\psi+D)\rangle}$ is convex in $\mathbf{C}$ for $M \ge 0$ ([BF80, (9.67) to (9.611)]; Lemma 7.33, Step 3).

S4 The integrand of $\mathcal{K}_{\bar y}$ depends on $\psi|_{X(\bar y)}$ only, and covariance derivatives have both ends in $X(\bar y)$ (Proposition 7.22).

S5 Each Y_{i+1} is a union of elements, each meeting $S(\theta_i) \setminus X_i$, the part of the support of its link outside X_i (selection rule (C.6)).

S6 Link i carries $\prod_{k=T(i+1)}^{i-1} s_k$ (covariance link from $Y_{T(i+1)}$) or $\prod_{k=j_{\min}}^{i-1} s_k$ (E-term link); all other s-dependence is through weights in $[0,1]$ and through $\mathbf{C}_s$ ((C.1), (C.5)).

S7 There are five types of terms per link ((C.5)).

S8 The link ends $x \in Y_{i+1}$ and $y \in Y_{T(i+1)}$ are counted in $\mathcal{U}$-cells (selection rule (C.6)).

S9 The sequences consist of disjoint admissible sets, and there are finitely many of them (change 2).

S10 $Z(Y) > 0$, and the one-set activities are positive (Proposition 7.22).

S11 The first set may depend on h (change 3).

S12 The first set may be disconnected (change 3).

S13 The expansion is on $U = X_0^c$ with $\mathbf{1}_U \mathbf{C} \mathbf{1}_U$, and the activities do not depend on U (change 4).

S14 Every term converges absolutely (Lemmas C.2 and C.3).

The facts are used as follows: S1 in Lemma 7.29 and Appendices A and B; S2 and S3 in Lemma 7.33; S4 in Lemma 7.37 and Theorem 7.27(b), (c); S5 and S6 in Lemma 7.37 (Appendix A); S7 and S8 in Appendix A; S9 in Lemma 7.37 and Proposition 7.47; S10 in Lemmas 7.41 and 7.42 and Proposition 7.53; S11 in Theorem 7.27 and Lemma 7.48; S12 in Theorem 7.27(c) and Lemma 7.48; S13 in Theorems 7.27(b) and 7.49 and in Lemma 7.37 (Appendix A); and S14 throughout.

S1.7 Section 7.6 and Appendix B

S1.7.1 The Hölder exponents and (L8-1)

Section S1.1.1 fixes $\eta_H = 0.62$, $p_1 = p_2 = 20$, $p_3 = 1.2$, $p_4 = 15$, $p' = 1.3$ and $q' = 15.6$. These values satisfy $\sum_i 1/p_i = 1$ and $1/p' + 1/q' = 1/p_3$. With the bound $\alpha_D \le 7/6$ of Lemma 7.29(c) we get $\hat\eta = \alpha_D p' \eta_H \le \frac{7}{6} \cdot 1.3 \cdot 0.62 = 0.94033\ldots < 0.941$, so (L8-1) holds. Moreover, $\eta_* \le \bar\eta \le 1 - 4/\pi^2 + 2C_{\hat\rho} e^{-2\kappa_S} < 0.62$ for κ_S so large that $C_{\hat\rho} e^{-2\kappa_S} < 0.0126$, since $1 - 4/\pi^2 = 0.5947\ldots$.

S1.7.2 Lemma 7.32(c): the shell constant $C_{\rm sh} = 92$

With $r_* = \frac12 + \frac{\sqrt3}{2}$, $\int_{1/2}^{D}(r+r_*)^2 r^{-3}dr \le \log(2D) + 4r_* + 2r_*^2$ and $(r+r_*)/r \le 1 + 2r_*$ for $r \ge \frac12$, the shell count of the proof gives, for $D \ge 1$,

$$\sum_{d(b,C)\le D} d(b,C)^{-2} \le 6\pi|C|\Big[\log(2D) + 4r_* + 2r_*^2 + \tfrac12(1+r_*)^2\Big] + 16\pi\big[(1+r_*)^3 + 2(1+2r_*)^3\big]D.$$

The first bracket is at most $\log(2D) + 12$, since $4r_* + 2r_*^2 + \frac12(1+r_*)^2 = 11.9951\ldots$, and $16\pi[(1+r_*)^3 + 2(1+2r_*)^3] = 16\pi\cdot 117.207\ldots = 5891.4\ldots \le 5892$. At $D = D_{\mathcal N} = \sqrt3\, k_\square + 1 \le 2k_\square$ and $|C| \ge 8k_\square \ge 4D_{\mathcal N}$ the sum is at most $6\pi|C|\big(\log k_\square + \log 4 + 12 + 1473/(6\pi)\big) \le 6\pi|C|(\log k_\square + 92)$, since $\log 4 + 12 + 1473/(6\pi) = 91.53\ldots$. Multiplying by $(2\pi)^2$ gives $S(D_{\mathcal N}) \le 24\pi^3|C|(\log k_\square + 92)$.

S1.7.3 Lemma 7.33: the constants c_8 and $c_{\rm near}$

We set

$$c_8 := \frac{\hat\eta(1+4/\eta_H)K_1C_u}{2\lambda_{\rm PV}(1-\nu_0)}, \qquad c_{\rm near} := \frac{2\theta_P C_{\rm BS}(\log k_\square + C_{\rm sh})}{\beta_{\rm GM}} + \frac{1024\pi^2\hat\eta}{\eta_H c_1}\,\frac{(2r_\square+5)^2c_L^2}{\beta_{\rm GM}}.$$

Uniformity of $c_{\rm near}$. Since $k_\square = r_\square L \le r_\square c_L/\tilde m$, $\log k_\square \le \log(r_\square c_L) + \log(1/\tilde m)$ with $\log(r_\square c_L) < 0$. With $\tilde m \ge m_D/2$, $m_D = (8\pi^2\tau)^{1/2}e^{-\pi^2 v_{\rm Cb}(0)\tau}$, $\pi^2 v_{\rm Cb}(0) < 2.4943551$ and $\beta_{\rm GM} = 4\pi^2\tau$,

$$\frac{\log(1/\tilde m)}{\beta_{\rm GM}} \le \frac{2.4943551}{4\pi^2} - \frac{\frac12\log(8\pi^2\tau) - \log 2}{4\pi^2\tau} < 0.0632 \qquad (\tau \ge 1).$$

Hence $2\theta_P C_{\rm BS}\log k_\square/\beta_{\rm GM} < 0.1264\,\theta_P C_{\rm BS}$, which gives the bound on $c_{\rm near}$ in Lemma 7.33. The coefficient θ_P in that bound satisfies $\theta_P \le (4\hat\eta/\eta_H)C_P'c_L^2 \le \frac12$ by (L8-2), uniformly in $\beta_{\rm GM}$.

S1.7.4 Lemma 7.34: the condition (7.30)

At $p_4 = 15$ and $p_3 = 1.2$, so that $c_\lambda = \frac14(1-1/p_3) = \frac{1}{24}$, and with $\alpha_D \le \frac76$ and $c_E \le \frac{1}{70}$, we have $2\alpha_D p_4 c_E \le 2\cdot\frac76\cdot\frac{15}{70} = \frac12$ and $p_4\alpha_D c_E^2/(1-2\alpha_D p_4 c_E) + c_E \le \frac{15\cdot(7/6)}{4900\cdot(1/2)} + \frac{1}{70} = 0.00714\ldots + 0.01428\ldots = 0.02142\ldots \le \frac{1}{24}$. In Proposition 7.35(d), $c_0' = \frac12(1 - 1/p_3 - c_\lambda) = \frac12(\frac16 - \frac{1}{24}) = \frac{1}{16}$.

S1.7.5 Theorem 7.40: the threshold τ_0

With $\pi^2 v_{\rm Cb}(0) = 2.49435508\ldots$ and $p_0 = \frac16$, the lone-cell condition in the proof of Theorem 7.40 reads

$$0.4157\,\tau \ge C_{\mathfrak D}A_1^D\log(C'\tau/\theta_0) + \alpha + O(\log\tau),$$

where $c_{\rm dep} = A_0^D + A_1^D\log(C'\tau/\theta_0)$ is the depletion constant of Proposition 6.17. The threshold τ_0 is explicit, and it is large because A_1^D is large, as θ_0 is small and the covering number of Proposition 6.17 is large. We need only that this threshold exists, and the same holds for the threshold of Lemma S1.2. For the wall-polymer cells, Proposition 7.19 gives $\Phi_{\mathcal U} \ge (\pi c_g c_u c_L^3/24)/(64\beta_{\rm GM}^3(11/10)^3 b_D)$.

S1.7.6 Lemma 7.38 and Proposition 7.39: constants and fractions

Constants on class-A cells. They, and the constants $C_{\mathcal{E}}$, C_G of Proposition B.10, are independent of L' and depend only on the following quantities:

- c_7, c_7', c_7'' of Lemma 7.31 (through $v_\pm$, η_H, κ_S, that is, on the grid scale L);
- C_k, θ', c_{dep} of Lemma B.1(d);
- $\mu, \kappa_S, \delta_{\mathrm{GM}}, C$ and the Mayer decay;
- $K_1'', K_3, \alpha_D K_1, \lambda_{\mathrm{PV}}, r_C$ of the lines;
- $C_r(c_p)$ and $c(q_{\mathrm{pin}}, c_p, K_g, C_{\mathcal{U}})$ of the count and the pinning;
- c_T of the Wick bound;
- c_{13} and $C_{\mathfrak{D}} c_{\mathrm{dep}}$ of the Hölder split;
- the geometry constants $c_u, C_u, C_{\mathcal{U}}, D_{\mathcal{U}}, v_\pm, K_g$.

Fractions of F_1^{loc}. They are those of Remark S1.1 in Section S1.8. Proposition 7.39 uses the fractions for the numerical factors, which are $c'/16$ on $X \setminus Z$ (Lemma B.8) and $c'/32$ on Z, and the fraction $c'/8$ on $X \setminus Z$ for the wall polymers.

Cells. Put $K_B := K_0^B \max(1, K_\ell)$, with the line constant K_ℓ of Proposition B.7. Each cell carries the per-cell constants $e^{c_2^L + c_{13}}$ and at most the factor $\max(1, K_\ell)$ of the line constants, as the links inject into the cells. Since $C_u \geq 1$, $c_2^L + c_{13} \geq 0$ and $\alpha > 0$, it follows that $e^{c_2^L + c_{13}} \leq e^{\alpha} e^{(c_2^L + c_{13}) C_u}$. With $K_{\mathrm{cell}} = K_B e^{(c_2^L + c_{13}) C_u} \max(1, \lambda_{\mathrm{PV}}^{1/6})$ and $b_D^{1/6} = \lambda_{\mathrm{PV}}^{1/6} (b_D/\lambda_{\mathrm{PV}})^{1/6}$, Proposition B.7 shows that a non-seed lone cell carries at most $K_{\mathrm{cell}} e^{\alpha + C_{\mathfrak{D}} c_{\mathrm{dep}}} (b_D/\lambda_{\mathrm{PV}})^{1/6}$, that a non-seed cell of a wall polymer carries at most $K_{\mathrm{cell}} e^{\alpha + C_{\mathfrak{D}} c_{\mathrm{dep}} + c_B(b_D)} e^{-(c'/8)\Phi_{\mathcal{U}}/(1+D_{\mathcal{U}})}$, and that a seed cell carries at most $K_{\mathrm{cell}} e^{\alpha + C_{\mathfrak{D}} c_{\mathrm{dep}}}$, times $e^{c_B(b_D)}$ if it is of class B.

S1.8 Constants of Sections 7.7 to 7.9 and Appendix A

Lemma 7.44. Since $t_1 \leq 2 + (36 k_P/\chi_{\mathrm{N3}}) \ln k_P$, we have $D_{\mathrm{cell}} \leq 35 r_0 + 504\, r_0 k_P \ln k_P/\chi_{\mathrm{N3}}$. In fine units $r_0 k_P = 304 L \leq 304 c_L/\tilde{m}$ and $k_P < 1/\tilde{m}$, and therefore $\ln k_P \leq \ln(1/\tilde{m}) < 0.0632\, \beta_{\mathrm{GM}}$, as in the proof of Lemma 7.33. Hence $D_{\mathrm{cell}} \leq 35 r_0 + 504 \cdot 304 \cdot 0.0632\, (c_L/\chi_{\mathrm{N3}})\, \beta_{\mathrm{GM}}/\tilde{m} \leq 35 r_0 + 1.927\, \beta_{\mathrm{GM}}/\tilde{m}$ for $c_L \leq 1.99 \cdot 10^{-4} \chi_{\mathrm{N3}}$, which is less than $L' = 2\beta_{\mathrm{GM}}/\tilde{m}$ for $\beta_{\mathrm{GM}} \geq \beta_{\mathrm{GM}}^0$, since $L_c \geq 35$ gives $35 r_0 \leq L$. The choice of c_L through $C_P' c_L^2 \leq (1 - 4/\pi^2)/8$ places it far below the threshold $1.99 \cdot 10^{-4} \chi_{\mathrm{N3}}$.

Constants of the seeded sums (before Lemma 7.48). The constant C_9 exists because blocks have side at least L, a block has fewer than $8/c_1$ cells and a cell has at most $6/c_g$ interfaces. We set[3]

$$c_S := \log K_{\mathrm{cell}} + \alpha + C_{\mathfrak{D}} c_{\mathrm{dep}} + 1 + c_4' + \log 2, \qquad \eta_{\mathrm{seed}} := 6 D_I e^{-(c'/4) F_{\mathrm{unit}}}.$$

Since the constants $\log K_{\mathrm{cell}}$, $D_{\mathcal{U}}$ and α are fixed numbers and $c_{\mathrm{dep}} = O(\log \tau)$ by Proposition 6.17, we have $c_S = O(\log \tau)$ and $c_2 = c_S + 2 = O(\log \tau)$. Moreover $\log D_I = O(\log \tau)$, as $g/L = O(\tau)$, while F_{unit} is exponentially large in τ, so that η_{seed} is doubly exponentially small.

[3] The summand $1 + c_4' + \log 2$ of c_S is already contained in $\log K_{\mathrm{cell}}$: in the proof of Proposition A.7 the factor $e^{(1 + c_4' + \log 2)|Z|}$ is bounded by $e^{a|Z|}$, which enters $c_2^L = a + \alpha$ and hence K_{cell}. Counting it twice only enlarges c_S, which the later arguments allow.

Proposition 7.52 (the count), Step 4. By Definition 3.18, $c_3^{\mathrm{NF}}\kappa_{\mathrm{NF}} = c_0'/16$ and $\theta_{\mathrm{NF}} = c_0'c_g/(90(c_g+4))$, and hence

$$\frac{\ln 16}{\ln 2}\, c_3^{\mathrm{NF}}\kappa_{\mathrm{NF}} = 4c_3^{\mathrm{NF}}\kappa_{\mathrm{NF}} = \frac{c_0'}{4}, \qquad 8\theta_{\mathrm{NF}}\ln 16 = \frac{8\ln 16}{90}\,\frac{c_g}{c_g+4}\,c_0' \le \frac{c_0'}{8}.$$

With $n \le \nu$, $n_0 \le (4/c_g')\nu$ and $j_g = O(\ln \beta_{\mathrm{GM}})$, the terms $C(n+n_0+\nu)$ and $(1+j_g)\kappa_{\mathrm{NF}}\nu\ln 16$ of Steps 2 and 3 are at most $C'(1+\ln\beta_{\mathrm{GM}})\nu$ with C' depending only on C_9, c_g, c_g', c_L, $C_{\mathcal{U}}$ and κ_{NF} (constants of Section S1.1.1; c_g' through $M \le (1+4/c_g')\nu$, and c_L, $C_{\mathcal{U}}$ through j_g). Hence $\ln \# \le \frac{3}{8}c_0'F_{\mathrm{unit}}\nu + C'(1+\ln\beta_{\mathrm{GM}})\nu \le \frac{c'}{2}F_{\mathrm{unit}}\nu$ as soon as $F_{\mathrm{unit}} \ge (8C'/c_0')(1+\ln\beta_{\mathrm{GM}})$, which holds for $\beta_{\mathrm{GM}} \ge \beta_{\mathrm{GM}}^0$ because $F_{\mathrm{unit}} = (\pi^2 c_g'/18)(\tilde{m}L)^3/(\tilde{m}\beta_{\mathrm{GM}}) \ge (\pi^2 c_g'/144)c_L^3/(\tilde{m}\beta_{\mathrm{GM}})$ ($\tilde{m}L \ge c_L/2$) is exponentially large. In addition, $c' \ge c_0'$.

Lemma 7.56 and Theorem 7.57. Under the weight $2-\eta_w$ of the hull functional the constant c_u of (R7) is the value fixed in Section S1.1.1, and only the value of $C_X = 3C_{\mathcal{U}}/c_u$ depends on it. Since small components of G' are $\mathcal{U}$-cells of Debye volume at least c_u, they are counted in Lemma 7.56. In Theorem 7.57, with $\sigma_F = (\pi^2 c_g'/18)(\tilde{m}L)\tilde{m}/\beta_{\mathrm{GM}}$ from Definition 3.18 and with $\beta_{\mathrm{GM}} = 4\pi^2\tau$,

$$\sigma_V = \varepsilon_E\,\frac{\pi^2 c_g'\tilde{m}L}{18}\,\frac{\tilde{m}}{\beta_{\mathrm{GM}}} = \varepsilon_E\,\frac{c_g'\tilde{m}L}{72}\,\frac{\tilde{m}}{\tau} = \frac{c_g'\tilde{m}L}{9216}\,\frac{\tilde{m}}{\tau} \quad \text{at } \varepsilon_E = \tfrac{1}{128} = \tfrac{c_0'}{8}, \qquad \eta_S = \frac{c_2 C_X \tilde{m}^2}{\sigma_V},$$

which is the displayed expression, and it is $O(\tilde{m}\beta_{\mathrm{GM}}\log\tau)$ because $c_2 = O(\log\tau)$.

Remark 7.58 (the perimeter constant). By Lemma 7.33, $c_{\mathrm{near}} \le 0.1264\,\theta_P C_{\mathrm{BS}} + O(1/\beta_{\mathrm{GM}})$ with $C_{\mathrm{BS}} = 24\pi^3$, uniformly for $\beta_{\mathrm{GM}} \ge \beta_{\mathrm{GM}}^0$. The coefficient $0.1264\,\theta_P C_{\mathrm{BS}}$ decreases as c_L decreases, since $\theta_P \propto c_L^2$. Since $c_2 = O(\log\tau)$, $\tilde{m} = e^{-\pi^2 v_{\mathrm{Cb}}(0)\tau(1+o(1))}$ and $R_1(\beta) = (2r_\square+5)L \ge (2r_\square+5)c_L/(2\tilde{m})$, the terms $3c_2C_XC_{\mathcal{U}}\tilde{m}$ and $9c_2C_XC_{\mathcal{U}}^2/R_1(\beta)$ of c_N tend to 0.

Lemma A.6 and the order of c_{10}, c_{10}'', μ. We fix c_{10} first, independently of $\bar{\delta}$, with $c_{10} \le \frac{7}{9}r_C$. Then we fix c_{10}'', which depends only on c_{10} and on the growth constants $C_{\mathcal{U}}$, c_u, with $e^{c_{10}''} \ge 16e\max(1, c_{c_{10}/2})$, and finally we fix the Mayer parameter μ with $\mu \ge 1 + 2c_{10}''$.

Proposition A.7 (the constant c_2^L). The constant $c_2^L = a+\alpha$, where $a = \log(2eD_{\mathcal{U}}) + 4 + c_4' + a''$ with $e^{a''} \ge 4C_\varphi$, is built from $D_{\mathcal{U}}$, c_4', C_φ and α. Of these, c_4' depends on τ, but it is doubly exponentially small and decreasing in τ, hence bounded uniformly for $\tau \ge \tau_0$, and in this sense c_2^L is uniform in L', β_{GM} and τ. Lemma S1.2 provides a finite threshold τ_0, which grows with c_2^L.

Proposition A.8: the conditions on β_{GM}^0. The conditions on β_{GM}^0 used in Proposition A.8 are

$$\begin{aligned} &(\beta 1)\ \ \tfrac{c'}{32}\Phi_{\mathcal{U}} \ge (1+D_{\mathcal{U}})\big(c_S + c_B(b_D)\big), \\ &(\beta 3)\ \ 3eD_I e^{-(c'/4)F_{\mathrm{unit}}} \le \tfrac{1}{2}, \qquad (\beta 4)\ \ c_{16}^D \le 1. \end{aligned} \tag{S1.5}$$

Proposition A.8: the booking of F_1^{loc}. In the proof of the core estimate (A.10), we book the factor $e^{-(3c'/4-\varepsilon)F_1^{\mathrm{loc}}(X)}$ left by Proposition 7.35 and the extraction according to Remark S1.1, as follows.

- *Type-iii factors, per region.* The factors $|D|$ and $|F_2'|$ at derivatives in a region involve D and F_2' on that region only, because $\operatorname{supp} D_\beta$ lies in the element of S_β. For a wall component with N factors $|D|$, n_Δ of them in the cell Δ, let $a_\Delta := \|D\|_{L^2(\Delta)}$, so that $\sum_\Delta a_\Delta^2 \le 2F_1^{\mathrm{loc}}/\tilde m^2$. The weighted arithmetic-geometric mean inequality and the identity $\sup_{F>0}(2F/(\tilde m^2 N))^{N/2}e^{-\delta F} = (e\delta\tilde m^2)^{-N/2}$ give, for every $\delta>0$,
$$\prod_\Delta a_\Delta^{n_\Delta}\, e^{-\delta F_1^{\mathrm{loc}}} \le (\delta\tilde m^2)^{-N/2}\prod_\Delta (n_\Delta!)^{1/2},$$
and F_2' is handled in the same way. These constants enter K_{cell} and the pinning factor of Proposition B.7 and Lemma B.8. We take $\delta = c'/16$ on $X\setminus Z$ and $\delta = c'/32$ on Z.

- *(M-Σ) on $X\setminus Z$.* We use $c'/8$ of $F_1^{\mathrm{loc}}(X\setminus Z)$. A wall polymer $\mathcal{S}$ is the union of the closures of its components, $|\operatorname{cl}(S_\beta)| \le (1+D_{\mathcal{U}})|S_\beta|$, and $F_1(S_\beta) \ge \Phi_{\mathcal{U}}|S_\beta|$ by Proposition 7.19(c). Hence $F_1^{\mathrm{loc}}(\mathcal{S}) \ge \Phi_{\mathcal{U}}|\mathcal{S}|/(1+D_{\mathcal{U}})$, and each $\mathcal{U}$-cell of a wall polymer of $X\setminus Z$ receives the factor $e^{-(c'/8)\Phi_{\mathcal{U}}/(1+D_{\mathcal{U}})}$ of ε_Δ in Proposition 7.39. Lone cells of $X\setminus Z$ carry $\varepsilon_\Delta = K_{\mathrm{cell}}e^{\alpha+C_{\mathfrak{D}}c_{\mathrm{dep}}}(b_D/\lambda_{\mathrm{PV}})^{1/6}$. The remaining fraction $9c'/16-\varepsilon \ge c'/16$ of $F_1^{\mathrm{loc}}(X\setminus Z)$ contributes a factor bounded by 1.

- *Seed.* We use $c'/32$ of $F_1^{\mathrm{loc}}(Z,N_Z)$ for the $\mathcal{U}$-cells of Z^Σ. Each carries at most $K_{\mathrm{cell}}e^{\alpha+C_{\mathfrak{D}}c_{\mathrm{dep}}+c_B(b_D)} \le e^{c_S+c_B(b_D)}$ by Proposition 7.39, and $F_1^{\mathrm{loc}}(Z^\Sigma) \ge \Phi_{\mathcal{U}}|Z^\Sigma|/(1+D_{\mathcal{U}})$ pays it under (β1). Lone seed cells carry at most $K_{\mathrm{cell}}e^{\alpha+C_{\mathfrak{D}}c_{\mathrm{dep}}} \le e^{c_S}$. What remains is
$$e^{-(c'-c'/4-\varepsilon-c'/32-c'/32)F_1^{\mathrm{loc}}(Z,N_Z)} = e^{-(11c'/16-\varepsilon)F_1^{\mathrm{loc}}(Z,N_Z)}.$$

Each non-seed cell carries at most $\varepsilon_\Delta \le \frac12 c_{16}^D \le \frac12$ by (β4). In (a) and (b) of Lemma 7.48, $c'/8$ of the fraction $11c'/16-\varepsilon$ left on the seed is used for the entropy of the first set, and at least $c'/16$ is left unused. In (c′) of that lemma this fraction pays $\varepsilon \le c'/8$ and $c_w = c'/16$ and leaves at least $c'/2$.

Remark S1.1 (The division of F_1). The proofs of Lemma 7.48 and Proposition 7.39 divide F_1^{loc} as follows. On the non-seed elements ($X\setminus Z$) they use the fraction $1-c'$ for the Hölder split and [GM82, Lemma 13] (T_2,T_3,T_4) in Proposition 7.35, $c'/4$ for the h-entropy c_1' of Proposition A.7, ε for the extraction of Lemma 7.48 ($\varepsilon \le c'/2$; the chain uses $\varepsilon \le \varepsilon_E$), $c'/16$ for the type-iii factors $|D|$, $|F_2'|$ of Proposition A.8, and $c'/8$ for the cells of wall polymers, (M-Σ) in Proposition 7.39; the remaining $9c'/16-\varepsilon \ge c'/16$ is discarded. On the seed elements (Z) they use $1-c'$, $c'/4$ and ε in the same way, $c'/32$ for the type-iii factors and $c'/32$ for the seed cells of Z^Σ under (β1); the remaining $11c'/16-\varepsilon$ is retained. In each region the fractions add up to 1, and no fraction is used twice. Since the walls on Z are fixed on $\mathcal{N}_\sigma(X)$ and no wall lies on an interface between Z and $X\setminus Z$, the weight $e^{c_1'F_1^{\mathrm{loc}}(X\setminus Z)}$ suffices in the sum over N_X in the proof of Proposition A.7, and the fraction for the h-entropy on Z is optional. We keep it, at the cost of $c'/4$ on the seed that would otherwise be left unused, so that Lemma 7.37 can be stated with $e^{c_1'F_1^{\mathrm{loc}}(X,N_X)}$.

Lemma S1.2 (Constant check)**.** *There is τ_0 such that for $\tau \ge \tau_0$ the conditions* (S1.5) *hold and $c_{16}e^{c_2} \le \frac18$.*

Proof. Each condition compares an energy that is exponentially large in τ with a cost that is logarithmic in τ or of smaller exponential order. Since $\tilde{m} \le 2m_D$ and $m_D = (8\pi^2\tau)^{1/2}e^{-\pi^2 v_{\mathrm{Cb}}(0)\tau}$, $\log(1/b_D) \ge \pi^2 v_{\mathrm{Cb}}(0)\,\tau - \frac{3}{2}\log\tau - C$, with $\pi^2 v_{\mathrm{Cb}}(0) = 2.49435508\ldots$. For the constants of the seeded sums we showed (Section S1.8) that $\log D_I = O(\log\tau)$ and, by Proposition 6.17, that $c_S = O(\log\tau)$ and $c_2 = c_S + 2 = O(\log\tau)$. By Proposition 7.19, $\Phi_{\mathcal{U}} \ge c\,b_D^{-1}\beta_{\mathrm{GM}}^{-3}$ with $c > 0$ fixed, while $c_B(b_D) \le C b_D^{-1/2}$ by Lemma 7.38. Moreover, since $\tilde{m}L \ge c_L/2$, we have $F_{\mathrm{unit}} = (\pi^2 c_g'/18)(\tilde{m}L)^3(\tilde{m}\beta_{\mathrm{GM}})^{-1} \ge (\pi^2 c_g'/144)c_L^3(\tilde{m}\beta_{\mathrm{GM}})^{-1}$. Hence $(\beta 1)$ and $(\beta 3)$ hold for $\tau \ge \tau_0$, and $(\beta 4)$ is the content of Theorem 7.40. For the product, the first branch of c_{16}^D gives

$$\log\bigl(2K_{\mathrm{cell}}e^{\alpha + C_{\mathfrak{D}}c_{\mathrm{dep}}}(b_D/\lambda_{\mathrm{PV}})^{1/6}e^{c_2}\bigr) = O(\log\tau) - \tfrac{1}{6}\pi^2 v_{\mathrm{Cb}}(0)\,\tau,$$

and the second gives $O(\log\tau) + c_B(b_D) - (c'/8)\Phi_{\mathcal{U}}/(1 + D_{\mathcal{U}})$, which tends to $-\infty$ faster. Consequently, $c_{16}^D e^{c_2} = \exp(O(\log\tau) - \pi^2 v_{\mathrm{Cb}}(0)p_0\tau(1 + o(1)))$ with $p_0 = 1/6$. Also $\eta_{\mathrm{seed}}e^{c_2} = 6D_I e^{c_S+2}e^{-(c'/4)F_{\mathrm{unit}}}$ is doubly exponentially small. Both are at most $\frac{1}{16}$ for $\tau \ge \tau_0$.

S2 Full proofs of steps sketched in the main text

Part S4 gives more detailed versions of some of these proofs: those of Section S2.3 in Section S4.2, those of Section S2.4 in Section S4.3, those of Section S2.5 in Sections S4.4 and S4.5, those of Section S2.6 in Section S4.8, in a version that includes the bounds for every primitive-admissible wall field, and those of Section S2.7 also in Section S4.8.

S2.1 Full proofs for Section 2

Proof of Lemma 2.3. We verify Ginibre's hypothesis [Gin70] for the Villain weights directly. A continuous function K on $\mathbb{T}^m \times \mathbb{T}^m$ is a *positive kernel* if it is a uniform limit of finite sums $\sum_i u_i(\alpha)u_i(\beta)$ with continuous real u_i. Positive kernels form a cone closed under products and under pullback by homomorphisms $\mathbb{T}^{m'} \to \mathbb{T}^m$, and $\iint s(\alpha)s(\beta)K(\alpha,\beta)\,d\alpha\,d\beta \ge 0$ for every continuous real s. A continuous $w : \mathbb{T} \to \mathbb{R}$ is *admissible* if $(a,b) \mapsto w(a+b)w(a-b)$ is a positive kernel, and we write $w_1 \preceq w_2$ if $(a,b) \mapsto w_1(a+b)w_2(a-b) - w_1(a-b)w_2(a+b)$ is one.

Step 1 (Ginibre's argument). Let $w_p > 0$ be admissible, $\mu \propto \prod_p w_p(\theta_p)\,d\theta$, and B, B' integer vectors. The map $(\alpha,\beta) \mapsto (\alpha+\beta, \alpha-\beta)$ of $\mathbb{T}^E \times \mathbb{T}^E$ onto itself preserves Haar measure, and $\prod_p w_p(\theta_p(\alpha)+\theta_p(\beta))\,w_p(\theta_p(\alpha)-\theta_p(\beta))$ is a positive kernel in (α,β). Doubling the variables and using $\cos(x+y) \pm \cos(x-y) \in \{2\cos x\cos y,\ -2\sin x\sin y\}$ gives $\langle\cos B\cdot\theta\rangle_\mu \ge 0$ and $\langle\cos B\cdot\theta\,\cos B'\cdot\theta\rangle_\mu \ge \langle\cos B\cdot\theta\rangle_\mu\langle\cos B'\cdot\theta\rangle_\mu$, as in [Gin70, Propositions 2, 3]. If moreover $w_q \preceq w_q'$ for one plaquette q, with $w_q' > 0$, then the same change of variables in $\iint(\cos B\cdot\theta' - \cos B\cdot\theta)\,w_q(\theta_q)w_q'(\theta_q')\prod_{p\neq q}w_p(\theta_p)w_p(\theta_p')\,d\theta\,d\theta'$, symmetrized under $\beta \mapsto -\beta$, shows that replacing w_q by w_q' does not lower $\langle\cos B\cdot\theta\rangle$.

Step 2 (the Villain weights). Let $t, t_1, t_2 > 0$. Expanding $\tilde{v}_t(a \pm b)$ in Fourier series and putting $p = n+m$, $r = n-m$ (so $p \equiv r \bmod 2$ and $n^2+m^2 = \frac{1}{2}(p^2+r^2)$) gives

$$\tilde{v}_t(a+b)\tilde{v}_t(a-b) = \tfrac{1}{2}\bigl(c(a)c(b) + \tilde{c}(a)\tilde{c}(b)\bigr),$$
$$c(a) := \sum_p e^{-p^2/4t}\cos pa, \qquad \tilde{c}(a) := \sum_p (-1)^p e^{-p^2/4t}\cos pa,$$

so $\tilde v_t$ is admissible. For $t_1 \le t_2$, with $\sigma := \frac{1}{8t_1} + \frac{1}{8t_2}$ and $\kappa := \frac{1}{4t_1} - \frac{1}{4t_2} \ge 0$,

$$\tilde v_{t_1}(a+b)\tilde v_{t_2}(a-b) - \tilde v_{t_1}(a-b)\tilde v_{t_2}(a+b) = 2 \sum_{p \equiv r \ (2)} e^{-\sigma(p^2+r^2)} \sinh(\kappa p r)\, \sin pa\, \sin rb,$$

a positive kernel: expand $\sinh(\kappa pr) = \sum_{j \text{ odd}} \kappa^j p^j r^j / j!$ and $\mathbf{1}_{p\equiv r} = \frac12(1+(-1)^{p+r})$; the series converges absolutely, as $\sigma - \kappa/2 = 1/(4t_2) > 0$. Hence $\tilde v_{t_1} \preceq \tilde v_{t_2}$. Finally $\tilde v_t(a-b) - \tilde v_t(a+b) = 2\sum_n e^{-n^2/2t} \sin na \sin nb$, so $1 \preceq \tilde v_t$, and the constant 1 (a removed plaquette) is admissible.

Step 3 (conclusion). Step 1 with $w_p = \tilde v_{t_p}$ gives (a), with $\langle \sin\theta_C \rangle = 0$ by $\theta \mapsto -\theta$. Step 1 with $\tilde v_{t_1} \preceq \tilde v_{t_2}$, one plaquette at a time, gives (b), and with $1 \preceq \tilde v_t$ and the admissibility of 1 it gives (c).

Step 4 (frozen limits). For fixed couplings off D, the expectation is a nondecreasing function of the couplings on D with values in $[0,1]$, and therefore has a limit. This limit inherits monotonicity in the other couplings. If $D \subseteq D'$ and $s \ge \tau$, then $t^{\tau,D,s} \le t^{\tau,D',s}$ coordinatewise, so $\langle \cos\theta_C \rangle^V_{t^{\tau,D,s}} \le \langle \cos\theta_C \rangle^V_{t^{\tau,D',s}} \le g_{X,\tau}(C;D')$, and $g_{X,\tau}(C;D) \le g_{X,\tau}(C;D')$. Monotonicity in τ passes to the limit in the same way.

Proof of Lemma 2.8. Let $Q_s(k) := \frac{1}{2\tau}|\mathbf{1}_{F^c}(n_0+\partial k)|^2 + \frac{1}{2s}|\mathbf{1}_F(n_0+\partial k)|^2$, so that by Lemma 2.7 the left side of (2.7) at finite s is N_s/D_s with $N_s := \sum_k e^{-Q_s(k)}$ and D_s the same sum for $n_0 = 0$. We write $k = w' + w$ with $w' \in W'$, $w = \sum_{\mathfrak{K}} a_{\mathfrak{K}} \mathbf{1}_{\mathfrak{K}} \in W$. Since $\mathbf{1}_{F^c}\partial w = 0$,

$$N_s = \sum_{w' \in W'} e^{-\frac{1}{2\tau}|\mathbf{1}_{F^c}(n_0+\partial w')|^2} S_s(w'), \qquad S_s(w') := \sum_{a \in \mathbb{Z}^{\mathcal{K}_F}} e^{-\frac{1}{2s}|b+Ba|^2},$$

with $b := \mathbf{1}_F(n_0+\partial w') \in \mathbb{R}^F$ and $Ba := \partial(\sum_{\mathfrak{K}} a_{\mathfrak{K}} \mathbf{1}_{\mathfrak{K}}) \in \mathbb{R}^F$. The matrix $M := B^{\mathsf T}B$ is positive definite because ∂ is injective. Completing the square, we get $|b+Ba|^2 = (a-a_0)\cdot M(a-a_0) + |Pb|^2$ with $a_0 := -M^{-1}B^{\mathsf T}b$ and $P := 1 - BM^{-1}B^{\mathsf T}$ the orthogonal projection onto $(\operatorname{im} B)^\perp$. Poisson summation in a gives

$$S_s(w') = (2\pi s)^{|\mathcal{K}_F|/2} (\det M)^{-1/2}\, e^{-|Pb|^2/(2s)} \big(1 + r_s(w')\big),$$
$$|r_s(w')| \le \rho_s := \sum_{0 \ne m \in \mathbb{Z}^{\mathcal{K}_F}} e^{-2\pi^2 s|m|^2/\|M\|},$$

uniformly in w', and $\rho_s \to 0$ exponentially. The prefactor is the same in N_s and D_s and cancels. In the remaining sums the factor $e^{-|Pb|^2/(2s)}(1+r_s)$ tends to 1, is bounded by 2 for large s, and differs from 1 by at most $|\mathbf{1}_F(n_0+\partial w')|^2/(2s) + \rho_s$. If $\mathbf{1}_{F^c}\partial w' = 0$ for some $w' \in W' \otimes \mathbb{R}$, then w' is constant on each component of the unfrozen graph and vanishes on the component of the ground, so $w' \in (W \otimes \mathbb{R}) \cap (W' \otimes \mathbb{R}) = \{0\}$. Hence $|\mathbf{1}_{F^c}(n_0+\partial w')|^2 \ge c|w'|^2 - C$ for some $c > 0$, and $e^{-\frac{1}{2\tau}|\mathbf{1}_{F^c}(n_0+\partial w')|^2}$ times any polynomial in w' is summable over W'. Dominated convergence gives (2.7), and the bound on the factor gives the rate $O(1/s)$. Since the summands of (2.7) are W-invariant functions of $k \in \mathbb{Z}^{\Lambda'_3}$, the sums do not depend on the complement.

S2.2 Section 3: the loop-adapted grid

Construction of $\mathfrak{G}(C)$ (Definition 3.4). With $\Lambda^\circ = \prod_\mu [a_\mu, b_\mu]$ as in Definition 3.2, we construct the objects in the following order.

1. *Special planes.* In direction μ the special planes are the lattice planes $\{x_\mu = s\}$, $s \in \mathcal{S}_\mu$, with $\mathcal{S}_1 = \{a_1, r_1, r_1+R, b_1\}$, $\mathcal{S}_2 = \{a_2, t_1, t_1+T, b_2\}$, $\mathcal{S}_3 = \{a_3, z_0, b_3\}$. A *special interval* is the fine interval between two consecutive special planes of one direction.

2. *Sites.* A special interval of fine length D' is divided into $\lfloor D'/r_0 \rfloor$ consecutive intervals: $\lfloor D'/r_0 \rfloor - 1$ of length r_0, followed, at the end with the larger coordinate, by one of length $r_0 + (D' \bmod r_0) \in [r_0, \bar{r}_0]$. The sites (coarse cells) are the boxes $\prod_\mu I_\mu$ with I_μ such intervals. Their fine sides lie in $[r_0, \bar{r}_0]$, and equal r_0 outside at most one exceptional layer per special interval. The sites form, combinatorially, a box of $\mathbb{Z}^3$. We denote by E its nearest-neighbour bonds, and the coarse face dual to a bond is the common face of its two sites.

3. *Blocks.* In coarse index units, a special interval of coarse length n is divided into $p := \lfloor n/L_c \rfloor$ consecutive intervals, and with $r := n - pL_c$ the first $(r \bmod p)$ of them have length $L_c + \lceil r/p \rceil$ and the others $L_c + \lfloor r/p \rfloor$. Their endpoints are the *block planes.* Blocks are the boxes between consecutive block planes of all three directions, and a *block face* is the common face of two adjacent blocks.

4. *Removal cells.* In each direction the block intervals, in increasing order, are grouped greedily into consecutive groups of total coarse length $\geq \ell$, and a final remainder of length $< \ell$ is merged into the last group. Removal cells Δ are the boxes $\prod_\mu J_\mu$ with J_μ such groups, and $|\partial\Delta|$ is the number of bonds of $\mathbb{Z}^3$ (in coarse index units) with exactly one end in Δ, bonds leaving Λ° included.

5. *Windows.* We index the blocks by their position in each direction. The block distance is the ℓ^∞ distance of these indices. For each block Q, $\hat{Q}(Q)$ is the union of the blocks at block distance at most 75 from Q, a block-aligned box, and $\mathcal{Q}$ is the set of windows.

6. *Tube zones and tubes.* Let F' be a block face between blocks Q, Q', in the plane P'. Its tube zone $Z(F')$ is the set of sites of $Q \cup Q'$ among the $L_c/2$ layers nearest to P' on each side. For a bond b across F', its tube τ_b is the segment of the line of b formed by the L_c sites of that line in $Z(F')$. We write $|F'|$ for the number of coarse bonds across F'.

Where the rules leave a choice, we put the exceptional coarse layer at the end with the larger coordinate and the longer block intervals first. These conventions depend only on C and the constants, and any others with this property would do.

Proof of Lemma 3.3(a) to (d). (a) The quantity M_Λ is determined by R, T, r_0 and ℓ, and the centre is determined by C. The grid is built from Λ° and the planes of C by rules that involve only the constants.

(b) By (e) and $m \geq \min(R, T) \geq R_1 \geq L$, the special planes a_μ, b_μ are at fine distance $\geq L$ from the planes of C, and $R, T \geq R_1 \geq L$.

(c) Step 2 cuts a special interval of fine length D' into $\lfloor D'/r_0 \rfloor$ coarse intervals. Hence a side of Λ° made of at most three special intervals has at least $M'_\Lambda/r_0 - 3 \geq \ell$ coarse cells. Step 4 tiles each side of coarse length $\geq \ell$ by groups of block intervals, so Λ° is a union of removal cells. Removal cells are unions of blocks by construction. The bound $\ell \geq k_P + 4L_c$ is (3.1).

(d) Each n_μ lies in $[M'_\Lambda/r_0 - 3,\ M'_\Lambda/r_0]$, and $M'_\Lambda/r_0 \geq \ell + 3 \geq 6$. Hence $\max_\mu n_\mu \leq M'_\Lambda/r_0 \leq 2(M'_\Lambda/r_0 - 3) \leq 2 \min_\mu n_\mu$.

Proof of (G1) to (G5). (G1) By Lemma 3.3(b), a special interval has fine length $D' \geq L = r_0 L_c$, hence coarse length $n = \lfloor D'/r_0 \rfloor \geq L_c$ and $p \geq 1$. Since $r < L_c$, each block interval has length in $[L_c, 2L_c - 1]$. A block lies in one special interval in each direction, so it meets at most one exceptional layer per direction, and its fine side lies in $[r_0 L_c, (2L_c - 1)r_0 + (r_0 - 1)] = [L, 2L - 1]$.

(G2) This holds because the construction is a tensor product and the endpoints of special intervals are block planes.

(G3) A greedy group closes as soon as its length reaches ℓ, and a block interval has length at most $2L_c - 1$. Hence a group has length in $[\ell, \ell + 2L_c - 2]$. Adding a final remainder of length at most $\ell - 1$ keeps it at most $2\ell + 2L_c - 3 < 3\ell$, by (3.1). Every side of Λ° is $\geq \ell$ by Lemma 3.3(c), so there is at least one group per direction. Then $|\partial\Delta| = 2(s_1 s_2 + s_2 s_3 + s_3 s_1)$ with $s_\mu \in [\ell, 3\ell)$.

(G4) A window has at most 151 blocks per direction, each of coarse side at most $2L_c - 1$ by (G1), so its sides are at most $151(2L_c - 1) < k_P$. If $u \in Q$ and $|y - u|_\infty \leq 75 L_c$, the segment from u to y in each direction crosses at most $1 + \lfloor (75L_c - 1)/L_c \rfloor = 75$ block planes, since consecutive parallel block planes are at least L_c apart. Thus the block of y is at block distance at most 75 from Q, and $Q_\infty(u, 75L_c) \cap \Lambda^\circ \subseteq \hat{Q}(Q)$. An adjacent block is at block distance 1 and therefore lies in the window. Finally, a block lies in $\hat{Q}(Q)$ only for the at most 151^3 blocks Q at block distance at most 75 from it.

(G5) This holds because blocks have at least L_c layers and parallel block planes are at least L_c apart.

S2.3 Section 5

The dual paths of Lemma 5.2. In fine coordinates with $\Xi = [0, R] \times [0, T] \times \{0\}$, the dual paths are the closed paths of fine cubes in the planes $y = j + \frac{1}{2}$ around the rectangles of cube centres $[i + \frac{1}{2}, 2R - i - \frac{1}{2}] \times [i - R + \frac{1}{2}, R - i - \frac{1}{2}]$ in (x, z), $0 \leq j < T$, $0 \leq i < R$. These paths are disjoint, and in these coordinates the path of (i, j) crosses Ξ once, at $x = i + \frac{1}{2}$.

Proof of Lemma 5.8. By [GK04, Lemma 2.1 and the sentence after it, p. 141], the first number is at most the number $\binom{Dn}{n}/((D-1)n+1) = \binom{Dn}{n-1}/n$ of n-vertex subtrees containing the root of the infinite rooted D-ary tree, $D := D_{\max}$, and $\binom{Dn}{n-1}/n \leq D^{n-1} n^{n-1}/n! \leq (eD)^{n-1}$ as $n! \geq e(n/e)^n$. For the second, we choose the first plaquette at an edge of C of each component among at most $4|C|$ plaquettes ($2^{4|C|}$ choices), the sizes of the components in the order of these roots (2^a), and each component by the first count with $D_{\max} = 12$.

Proof of Lemma 5.10. We put $\lambda := \log(2^{k+1}\kappa_{\mathrm{pl}})$ and $\omega := |\log \varepsilon|/F_* \geq c_T(k)/\eta \geq c_T(k) = 1 + \lambda$. By Lemma 5.8, $\mathbb{P}_D(E) \leq 2^{4|C|+j} \sum_{a \geq RT} e^{-a(\omega-\lambda)+\omega(1-\eta)RT} \leq 2^{4|C|+j} \frac{e}{e-1} e^{RT(\lambda - \eta\omega)} \leq 2^{4|C|+j+2} e^{-RT}$, as $\omega - \lambda \geq 1$. Since $j \leq 3$, $\mathbb{E}_D e^{\sigma X} \leq e^{\sigma\eta RT} + e^{\sigma RT} \mathbb{P}_D(E) \leq e^{\sigma\eta RT}(1 + 2^{4|C|+5}) \leq e^{\sigma\eta RT + 4|C| + 4}$.

Proof of Lemma 5.14(a), (c) to (e). We fix the cut j and write $U := U_j$, $U' := U'_j$, with U^α and $\mathcal{W}(\alpha)$ as in Definition 5.13.

(a) Let (u, v) be a bond of $\mathfrak{S}^*_j$ with $u \in U^\alpha$ and $v \in W$. If $v \in U$, then v is box-adjacent to u and lies in U^α, which is impossible. Hence $v \notin U$. If (u, v) is a G'_{t_j}-bond, then $v \in \mathfrak{K}_j$

because $\mathfrak{K}_j$ is a component, so $v \in U'$ and the bond is a cut bond. Otherwise $v \in \mathbb{X}_{t_j}$ (a hole face) or the bond is frozen.

(c) Let $W \in \mathcal{W}(\alpha)$ and $z \in W \cap U'$. We take a G'_{t_j}-path in $\mathfrak{K}_j$ from z to U^α and let y be the site just before its first entry into U^α. The segment from z to y is a box path avoiding U^α, so $y \in W$. Also $y \in \mathfrak{K}_j$, and $y \notin U$ because a site of U box-adjacent to U^α lies in U^α. Thus $y \in U'$, and the bond from y into U^α is a cut bond in $\partial(U^\alpha, W)$. A bond (u, v) with $u \in U$, $v \notin U$ determines α (the box-component of u in U) and W (the box-component of v in $\Lambda^\circ \setminus U^\alpha$), so distinct pieces are disjoint. Each piece contains one of the A_j cut bonds.

(d) For a cut bond (u, z) with $u \in U^\alpha$, $z \in U'$, the box-component of z in $\Lambda^\circ \setminus U^\alpha$ belongs to $\mathcal{W}(\alpha)$. The same argument with U and U' exchanged gives the piece of $\mathfrak{S}_j^{**}$.

(e) Let u^* be the last site of U on the path, $u^* \in U^{\alpha^*}$. The rest of the path lies in $\Lambda^\circ \setminus U \subseteq \Lambda^\circ \setminus U^{\alpha^*}$ and is box-connected. Hence it lies in one box-component W^* of $\Lambda^\circ \setminus U^{\alpha^*}$, and $W^* \in \mathcal{W}(\alpha^*)$ because the path ends in U'. The bond leaving u^* lies in $\partial(U^{\alpha^*}, W^*)$.

Proof of Lemma 5.15. *Shells.* We partition Λ° into *shell boxes* as a tensor product, cutting each coordinate axis of Λ° into intervals of lengths in $[6L_c, 12L_c)$ (possible since every coarse side of Λ° is at least $\ell \geq 308L_c$, Lemma 3.3(c)). We index the boxes by $\iota \in \prod_\mu [0, n_\mu^{\mathrm{sh}})$ and put $\delta(b, b') := |\iota - \iota'|_\infty$ and $n_{\min} := \min_\mu n_\mu^{\mathrm{sh}}$. For a box b_0 and $i \geq 1$ the *shell* $\mathrm{Sh}_i(b_0)$ is the union of the boxes b with $\delta(b, b_0) \in \{3i, 3i+1, 3i+2\}$.

Shell claim: if $\mathrm{Sh} = \mathrm{Sh}_i(b_0)$ *with* $3i + 2 \leq (n_{\min} - 1)/2$ *contains a whole cell of* U_j *and a whole cell of* U'_j *(cells at time* t_j*), then each of* $\mathfrak{S}_j^*$ *and* $\mathfrak{S}_j^{**}$ *has at least* $(c_1/24)L_c^2$ *bonds with both ends in* Sh. We write $U := U_j$. Let A' be the set of sites joined to U by a box path with no bond of $\mathfrak{S}_j^*$, and $B' := \Lambda^\circ \setminus A'$. Then $U'_j \subseteq B'$ by Lemma 5.14(e), and every box bond between A' and B' lies in $\mathfrak{S}_j^*$. We put $W := A' \cap \mathrm{Sh}$. The sets W and $\mathrm{Sh} \setminus W$ contain a cell each, hence at least $c_1 L_c^3$ sites each, and it suffices to find $(c_1/24)L_c^2$ box bonds inside Sh between them. The boxes of Sh form a face-connected family: since $3i+2 \leq (n_{\min}-1)/2$, each index sphere of radius $3i$, $3i+1$, $3i+2$ around b_0 keeps, in each direction, at least one of its two faces inside the index box, so its clipped faces form a connected surface, and consecutive spheres are face-adjacent. Put $f_b := |W \cap b|/|b|$.

Case 1: $f_b \leq \frac{1}{2}$ *for all boxes* b *of* Sh. The box Cheeger inequality of the proof of Lemma 5.15 in each box (sides less than $12L_c$) gives at least $|W|/(24L_c) \geq c_1 L_c^2/24$ bonds.

Case 2: $f_b \geq \frac{1}{2}$ *for all* b. The same argument applies to $\mathrm{Sh} \setminus W$.

Case 3: otherwise. There are face-adjacent boxes b_1, b_2 with $f_{b_1} \geq \frac{1}{2} > f_{b_2}$. In the box $b_1 \cup b_2$ (sides less than $24L_c$) both W and its complement have at least $\frac{1}{2}(6L_c)^3 = 108L_c^3$ sites, and the same inequality gives at least $108L_c^3/(48L_c) > 2L_c^2 > (c_1/24)L_c^2$ bonds.

The same argument with U and U'_j exchanged gives the bound for $\mathfrak{S}_j^{**}$.

Counting shells. Let b_0 be the shell box of x. The U_j-end of the cut bond of Lemma 5.15(i) lies within $\rho/2+2$ of x, hence at index distance less than $\rho/(12L_c)+2$ from b_0, and the far site of (ii) lies at index distance more than $3\rho/(4L_c) - 1$. A c-path in U_j from the former to the latter changes the index distance by at most 1 per step, and cells lie in one block, of side less than the box sides. Hence each shell $\mathrm{Sh}_i(b_0)$, $i \in I(\rho) := \{i \in \mathbb{Z} : \rho/(36L_c) + 1 \leq i \leq \rho/(4L_c) - 1\}$, contains a whole cell of U_j, and likewise of U'_j. These shells are disjoint, lie within 9ρ of x, satisfy $3i + 2 \leq (n_{\min} - 1)/2$, and number $|I(\rho)| \geq \rho/(9L_c)$ for $\rho \geq 27L_c$. In this range, the shell claim applied in each shell gives $(\rho/(9L_c))(c_1/24)L_c^2 = c_{\mathrm{sh}}\rho L_c$ bonds of $\mathfrak{S}_j^*$, and as many of $\mathfrak{S}_j^{**}$.

Proof of Lemma 5.26(b). We take the breadth-first spanning tree from o of the graph defining anchoring, with fixed orders. Along this tree, the block set is determined by the following data:

- the sizes of the components (2^b choices);
- the anchors of the children of o in $Z(\Xi')$, $|Z(\Xi')| \le 2a$, and of the children of each component C among its blocks (2^{2a+k_o} and $2^{|C|+k_C}$ choices, $k_o + \sum_C k_C = m$);
- the positions of the root blocks relative to their anchors ($(2d_0+1)^3$ choices each);
- the components, $*$-connected sets of given sizes containing their root blocks ($(26e)^{|C|}$ choices by Lemma 5.8 with $D_{\max} = 26$).

With $\sum_C |C| = b$ the product is $2^b \cdot 2^{2a+b+m}(2d_0+1)^{3m}(26e)^b$.

S2.4 Full proofs for Section 6 and Appendix D

We use the conventions of Appendix D.

Proof of Proposition 6.10. Diffusive displacement is needed only up to the time $T_1 = \min(M^{-2}, \varepsilon_1^{\mathrm{HK}} R_{\mathrm{HK}}^2)$. Beyond this time the walk is controlled by blocks of length T_1, in each of which it moves a distance of order $T_1^{1/2}$; the resulting bound grows like a ballistic motion at speed of order $T_1^{-1/2}$, and the mass term controls this growth.

We write $\sum_y v_{G'}^{(M)}(x,y) e^{\theta d(x,y)} = \int_0^\infty e^{-M^2 t}\, \mathbb{E}_x e^{\theta d(x,X_t)} dt$. If $T_1 = M^{-2}$ then $\theta_M = \kappa_W M = \kappa_W T_1^{-1/2}$. If $T_1 = \varepsilon_1^{\mathrm{HK}} R_{\mathrm{HK}}^2 < M^{-2}$ then $\theta_M = \kappa_W M^2 T_1^{1/2}$. In both cases $\theta_M \le \min(T_1^{-1/2}, c_{\mathrm{ex}}/2)$, as $M \le m_W$, and $C_M \theta_M T_1^{1/2} \le C_M \kappa_W M^2 T_1 \le \frac12 M^2 T_1 \le \frac12$. Hence (HK4) gives $\sup_w \mathbb{E}_w e^{\theta_M d(w,X_s)} \le q := 1 + C_M \theta_M T_1^{1/2} \le \frac32$ for $s \le T_1$. For $t = kT_1 + s$ with $0 \le s < T_1$, $d(x, X_t) \le \sum_{i \le k} d(X_{(i-1)T_1}, X_{iT_1}) + d(X_{kT_1}, X_t)$, and the Markov property gives

$$\mathbb{E}_x e^{\theta_M d(x,X_t)} \le q^{k+1} \le q \exp\bigl(C_M \theta_M t / T_1^{1/2}\bigr) \le q\, e^{M^2 t/2}.$$

Hence the integral is at most $2q/M^2 \le 3M^{-2}$, which is (6.13).

At $M = M_Y$, $(\varepsilon_1^{\mathrm{HK}})^{1/2} M_Y R_{\mathrm{HK}} \ge (\varepsilon_1^{\mathrm{HK}})^{1/2} K_0/\lambda_{\mathrm{PV}}$ by (6.12), which gives (6.14). The bound for $v_{G'}^2$ follows from the inequalities $0 \le v_{G'}^2 \le v_{G'}$ of Lemma 6.2(iii), and that for $v_{G'}^1$ follows from (6.13) at $M = M_1$, since $M \mapsto \theta_M$ is nondecreasing. Finally, $\kappa_0 \le \kappa_W$ because $(\varepsilon_1^{\mathrm{HK}})^{1/2} K_0 < 1$. If $(\varepsilon_1^{\mathrm{HK}})^{1/2} K_0 \ge \lambda_{\mathrm{PV}}$, then $\kappa_Y M_Y = \kappa_W \tilde m/\lambda_{\mathrm{PV}} \ge \kappa_0 \tilde m/\lambda_{\mathrm{PV}}$. Otherwise $\kappa_Y M_Y = \kappa_0 \tilde m/\lambda_{\mathrm{PV}}^2$. In both cases $\kappa_Y M_Y \ge 4\kappa_0 \tilde m$, because $\lambda_{\mathrm{PV}} \le \frac14$.

Proof of Theorem 6.11. Since C_0 is a difference of two resolvents, and hence a positive mixture of heat kernels, every bound on it reduces to the heat kernel. With $r_\pm := \frac12 M_Y^2 (1 \pm (1 - 4\lambda_{\mathrm{PV}}^2)^{1/2})$, the roots of $s^2 + M_Y^2 s + M_Y^2 \tilde m^2$ are $-r_\pm$, and partial fractions give

$$C_0 = F(-\Delta_{G'}), \qquad F(s) = \frac{M_Y^2}{(s+r_-)(s+r_+)} = \varpi\Bigl(\frac{1}{s+r_-} - \frac{1}{s+r_+}\Bigr),$$
$$C_0 = \varpi\bigl(v_{G'}^{(r_-^{1/2})} - v_{G'}^{(r_+^{1/2})}\bigr) = \int_0^\infty k(t) P_t\, dt,$$

with $k(t) = \varpi(e^{-r_- t} - e^{-r_+ t})$ and $0 \le k(t) \le \min(M_Y^2 t, \varpi) e^{-r_- t}$. Since $F(0) = \tilde m^{-2}$, $C_0 1_{\mathfrak{K}} = \tilde m^{-2} 1_{\mathfrak{K}}$, that is, the zero modes are part of C_0. We put $R := R_{\mathrm{HK}}$, so that $R^{-3} r_-^{-1} \le \tilde m^{-2} R^{-3} \le \tilde m K_0^{-3} = \lambda_{\mathrm{PV}} K_0^{-3} M_Y$ by (6.12).

(H1). Since $p_t \le 1$ on $[0,1]$ and $p_t \le C_H t^{-3/2} + C_H R^{-3}$ on $[1,\infty)$ by (HK1), we have

$$\begin{aligned} C_0(x,x) &\le \int_0^1 M_Y^2 t \, dt + C_H \int_1^\infty \min(M_Y^2 t, \varpi) t^{-3/2} dt + C_H R^{-3} \int_0^\infty \varpi e^{-r_- t} dt \\ &\le \tfrac12 M_Y + 2C_H(1+\varpi) M_Y + \varpi C_H \lambda_{\mathrm{PV}} K_0^{-3} M_Y, \end{aligned}$$

where the middle integral is split at $t = M_Y^{-2}$, whatever the order of M_Y^{-2} and R^2.

(H2). Entrywise, $C_0 \le \varpi v_{G'}^{(r_-^{1/2})}$, because $v_{G'}^{(r_+^{1/2})} \ge 0$ by Lemma 6.2. Since $\tilde m \le m_W/2$, the mass $r_-^{1/2} \le 1.04\,\tilde m$ is at most m_W, and (6.13) at this mass has $\theta \ge \kappa_W \tilde m \min(1, (\varepsilon_1^{\mathrm{HK}})^{1/2} \tilde m R) \ge \kappa_0 \tilde m$, since $\tilde m R \ge K_0$ and $(\varepsilon_1^{\mathrm{HK}})^{1/2} K_0 < 1$. Hence the row sums of $C_0 e^{\kappa_0 \tilde m d}$ are at most $3\varpi/r_- = K_3 \tilde m^{-2}$, and C_0 is symmetric.

(H1′). First, $C_0(x,y) \ge 0$ because $k \ge 0$ and $p_t \ge 0$. If $d := d(x,y) < M_Y^{-1}$, then $C_0(x,y) \le (C_0(x,x) C_0(y,y))^{1/2} \le K_1 M_Y \le e K_1 M_Y e^{-\kappa_0' \tilde m d}$, since $\kappa_0' \tilde m d \le \lambda_{\mathrm{PV}} \le 1$. If $d \ge M_Y^{-1}$, we use $C_0(x,y) \le \varpi \int_0^\infty e^{-r_- t} p_t(x,y)\, dt$ and split the time axis. (i) For $t \le d$, by (HK0) the contribution is at most $\varpi d e^{-d/17} \le 313 \varpi M_Y e^{-d/68} \le 313 \varpi M_Y e^{-\kappa_0' \tilde m d}$, because $d e^{-3d/68} \le M_Y d^2 e^{-3d/68} < 279\, M_Y$ for $d \ge M_Y^{-1}$, and $\kappa_0' \tilde m \le 1/68$. (ii) For $d \le t \le T_2 := 2\varepsilon_1^{\mathrm{HK}} R^2$ we have $t \ge d \ge 1$, and (HK3) together with $r_- t + c_{\mathrm{gs}} d^2/(2t) \ge (2 c_{\mathrm{gs}} r_-)^{1/2} d$ bounds the contribution by

$$\begin{aligned} \varpi C_{\mathrm{gs}} e^{-(2c_{\mathrm{gs}} r_-)^{1/2} d} \int_0^\infty t^{-3/2} e^{-c_{\mathrm{gs}} d^2/(2t)} dt &= \varpi C_{\mathrm{gs}} (2\pi/c_{\mathrm{gs}})^{1/2} d^{-1} e^{-(2c_{\mathrm{gs}} r_-)^{1/2} d} \\ &\le \varpi C_{\mathrm{gs}} (2\pi/c_{\mathrm{gs}})^{1/2} M_Y e^{-\kappa_0' \tilde m d}. \end{aligned}$$

(iii) For $t \ge T_2$, as in the proof of (HK3), $p_t(x,y) \le 2 \sup p_{t/2} \cdot \sup_w \mathbb{P}_w(d(w, X_{t/2}) \ge d/2)$. Here $\sup p_{t/2} \le C_H (\varepsilon_1^{\mathrm{HK}})^{-3/2} R^{-3}$ by (HK1), and Markov's inequality, combined with the blocking bound from the proof of Proposition 6.10 in §S2.4 at the mass $r_-^{1/2}$ and with its rate $\theta \ge \kappa_0 \tilde m$, gives $\mathbb{P}_w(d(w, X_{t/2}) \ge d/2) \le q\, e^{-\theta d/2} e^{r_- t/4}$. Thus the contribution is at most $4\varpi C_H (\varepsilon_1^{\mathrm{HK}})^{-3/2} R^{-3} r_-^{-1} e^{-\kappa_0 \tilde m d/2} \le 4\varpi C_H (\varepsilon_1^{\mathrm{HK}})^{-3/2} \lambda_{\mathrm{PV}} K_0^{-3} M_Y e^{-\kappa_0' \tilde m d}$. The three parts give K_1'.

(H_C). By the Schur test, $\|\nu\|_{2\to 2} \le \|\nu\|_0 \le \varepsilon_\nu \tilde m^2$, and $C_0^{-1} \ge \tilde m^2$. Hence $C_0^{-1} + \nu \ge (1 - \varepsilon_\nu) C_0^{-1} > 0$, C exists, is positive definite, and $C \le (1-\varepsilon_\nu)^{-1} C_0$, which gives $C(x,x) \le 2K_1 M_Y$ because $\varepsilon_\nu \le 1/(2K_3) < \frac12$. Since $r_{\mathrm{cov}} \le \kappa_0 \tilde m$, $\|C_0\|_{r_{\mathrm{cov}}} \le K_3 \tilde m^{-2}$, and the Neumann series $C = \sum_{n\ge 0} (-1)^n C_0 (\nu C_0)^n$ converges in $\|\cdot\|_{r_{\mathrm{cov}}}$ with $\|C\|_{r_{\mathrm{cov}}} \le K_3 \tilde m^{-2} \sum_n (K_3 \varepsilon_\nu)^n \le 2K_3 \tilde m^{-2}$. For the pointwise bound we write $C = C_0 - C_0 \nu C$ and use $e^{r_{\mathrm{cov}} d(x,y)} \le e^{r_{\mathrm{cov}} d(x,z)} e^{r_{\mathrm{cov}} d(z,w)} e^{r_{\mathrm{cov}} d(w,y)}$ and the bound $\sup_z C_0(x,z) e^{r_{\mathrm{cov}} d(x,z)} \le K_1'' M_Y$, which follows from (H1), (H1′) and $r_{\mathrm{cov}} \le \kappa_0' \tilde m$. This gives $|C(x,y)| e^{r_{\mathrm{cov}} d(x,y)} \le K_1'' M_Y (1 + \|\nu\|_{r_{\mathrm{cov}}} \|C\|_{r_{\mathrm{cov}}}) \le K_1'' M_Y (1 + 2K_3 \varepsilon_\nu) \le 2K_1'' M_Y$. A matrix C_s dominated entrywise by $|C|$, with $C_s(x,x) \le C(x,x)$, inherits the three bounds.

Proof of Lemma D.2. The heat kernel of $\mathbb{Z}^3$ factorizes into three one-dimensional kernels, and an exponential tilt of each turns the Gaussian bound into a bound with exponential decay. The time integral then produces the factor $(1+|x|)^{-1}$ from times of order $|x|^2$ and the decay from the mass.

Let $q_t(k) := e^{-2t} I_k(2t)$ be the transition probability of the continuous-time walk on $\mathbb{Z}$ that jumps to each neighbour at rate 1, so that the kernel of $e^{t\Delta_{\mathbb{Z}^3}}$ is $p_t(x) = \prod_\mu q_t(x_\mu)$ and $v^{(M)}(x) = \int_0^\infty e^{-M^2 t} p_t(x)\, dt$. For $\lambda \geq 0$, $q_t^\lambda(k) := q_t(k) e^{\lambda k - 2t(\cosh\lambda - 1)}$ is the law at time t of $N_+ - N_-$ with independent Poisson variables of means te^λ and $te^{-\lambda}$. Conditioning on N_-, we get $\sup_k q_t^\lambda(k) \leq \sup_j \mathbb{P}(N_+ = j) \leq \sqrt{2}(1+t)^{-1/2}$, by Stirling's formula. Conditioning on N_-, q_t^λ is a mixture of translates of the Poisson law π of mean te^λ, which is unimodal ($\pi(j)/\pi(j-1) = te^\lambda/j$); hence $\sum_k |q_t^\lambda(k+1) - q_t^\lambda(k)| \leq \sum_k |\pi(k+1) - \pi(k)| = 2\max_j \pi(j) \leq 2\sqrt{2}(1+t)^{-1/2}$. Writing $q_t^\lambda = q_{t/2}^\lambda * q_{t/2}^\lambda$, we obtain $\sup_k |q_t^\lambda(k+1) - q_t^\lambda(k)| \leq 8(1+t)^{-1}$. Hence, for $k \geq 0$,

$$q_t(k) \leq \sqrt{2}(1+t)^{-1/2} e^{-\lambda k + 2t(\cosh\lambda - 1)},$$
$$|q_t(k+1) - q_t(k)| \leq \big(8(1+t)^{-1} + \sqrt{2}\lambda(1+t)^{-1/2}\big) e^{-\lambda k + 2t(\cosh\lambda - 1)},$$

the second because $q_t(k+1) - q_t(k) = e^{-\lambda k + 2t(\cosh\lambda - 1)}[e^{-\lambda} q_t^\lambda(k+1) - q_t^\lambda(k)]$ and $1 - e^{-\lambda} \leq \lambda$. For negative k the bounds follow by symmetry, at the cost of a factor e^λ in the difference bound.

We let $r := |x| \geq 1$, choose $\lambda_\mu := s|x_\mu|/r$ with $s \in [0, \frac{1}{2}]$, and use $\sum_\mu 2(\cosh\lambda_\mu - 1) \leq \sum_\mu \lambda_\mu^2 \cosh\lambda_\mu \leq s^2 \cosh s$ and $\sum_\mu \lambda_\mu |x_\mu| = sr$. Then $p_t(x) \leq 2^{3/2}(1+t)^{-3/2} \exp(-sr + ts^2 \cosh s)$. For $t \leq r$ the choice $s = \frac{1}{2}$ makes the exponent at most $-r/2 + r\cosh(\frac{1}{2})/4 \leq -0.218\, r$. For $t \geq r$ the choice $s = r/(2t) \leq \frac{1}{2}$ makes the exponent at most $-r^2/(4t)\,(2 - \cosh\frac{1}{2}) \leq -0.218\, r^2/t$, and

$$M^2 t + 0.218\, r^2/t \geq \big(M^2 t + \tfrac{9}{64} r^2/t\big) + 0.077\, r^2/t \geq \tfrac{3}{4} M r + 0.077\, r^2/t.$$

Hence

$$\begin{aligned} v^{(M)}(x) &\leq 2^{3/2} e^{-0.218 r} \int_0^r (1+t)^{-3/2} dt + 2^{3/2} e^{-3Mr/4} \int_0^\infty t^{-3/2} e^{-0.077 r^2/t} dt \\ &\leq 2^{5/2} e^{-0.218 r} + \frac{2^{3/2}\sqrt{\pi}}{\sqrt{0.077\, r}} e^{-3Mr/4}, \end{aligned}$$

and $e^{-0.218 r} \leq e^{-3Mr/4} e^{-0.068 r}$ because $\frac{3}{4} M \leq 0.15$. This gives the first bound in (D.1). For the difference in direction $e = \pm e_\mu$ only the factor $q_t(x_\mu)$ changes, and the same choices of λ give $|p_t(x+e) - p_t(x)| \leq C(1+t)^{-3/2}\big((1+t)^{-1/2} + s\big) \exp(-sr + ts^2 \cosh s)$. For $t \leq r$ this is bounded as before. For $t \geq r$ we take $s = r/(2t)$ and use $\int_0^\infty (t^{-2} + r t^{-5/2}) e^{-0.077 r^2/t} dt \leq C r^{-2}$. For $r < 1$ both bounds are trivial.

For the box, the reflections in the six face planes (at half-integer positions) generate a group $\mathcal{G}$ that acts coordinatewise, and $v_{\Lambda'}^{(M)}(y, x) = \sum_{g \in \mathcal{G}} v^{(M)}(y - gx)$. Indeed, the right side is $\mathcal{G}$-invariant, so across each face its two values agree and the Neumann condition holds, and the orbit of x meets Λ' only at x. Coordinatewise $|y_\mu - (gx)_\mu| \geq |y_\mu - x_\mu|$ for $x_\mu, y_\mu \in [1, n_\mu]$, so $|y - gx| \geq r_{yx} := |y - x|$, and $\#\{g : |y - gx| \leq \rho\} \leq 8(\rho M + 1)^3$ because $n_\mu \geq M^{-1}$. For a decreasing f, summation by parts gives $\sum_g f(|y - gx|) \leq 8\big[(1 + M r_{yx})^3 f(r_{yx}) + \int_{r_{yx}}^\infty 3M(1 + M\rho)^2 f(\rho)\, d\rho\big]$. With $f(\rho) = C_L(1+\rho)^{-j} e^{-3M\rho/4}$, $j \in \{1, 2\}$, this is at most $C(1 + M r_{yx})^3 (1 + r_{yx})^{-j} e^{-3M r_{yx}/4} \leq C'(1 + r_{yx})^{-j} e^{-M r_{yx}/2}$. Since the differences $u_x(y+e) - u_x(y)$ are the sums of the differences of the images, (L1) and (L2) follow.

S2.5 Full proofs for Subsection 7.5

Proof of Lemma 7.24. The integrand of Proposition 7.7 depends on the configuration only through the field $\psi + g + k\beta_{\mathrm{GM}}^{-1/2} f$, read modulo $P\mathbb{Z}$ at each site and in each cell average, and through the walls. We write $\vartheta := \psi + D$.

(a) Let $h \in \mathcal{H}$ and $k \in \{0, 1\}$. By (7.10), $g + k\beta_{\mathrm{GM}}^{-1/2} f = h + D$. Since h takes values in $P\mathbb{Z} = 2\pi\beta_{\mathrm{GM}}^{-1/2}\mathbb{Z}$, $e^{im\beta_{\mathrm{GM}}^{1/2} h(x)} = 1$, so each phase $e^{im(\beta_{\mathrm{GM}}^{1/2}(\psi+g)(x)+kf(x))}$ of E', ε_k, $G_{2,2}$ and $G_{2,3}$ equals $e^{im\beta_{\mathrm{GM}}^{1/2}\vartheta(x)}$. In $G_1 = \sum_\Omega \ln r_\Omega(A_\Omega)$ the function r_Ω of (7.4) depends on A_Ω only through $A_\Omega + k\beta_{\mathrm{GM}}^{-1/2}\bar{f}_\Omega$ and is P-periodic in it. This quantity is $\mathrm{avg}_\Omega\,\vartheta + h|_\Omega$, as h is constant on Ω. Hence $r_\Omega(A_\Omega)$ is r_Ω at $k = 0$ evaluated at $\mathrm{avg}_\Omega\,\vartheta$. Likewise $\tilde{\delta} = (\psi + g + k\beta_{\mathrm{GM}}^{-1/2} f) - \mathrm{avg}_\Omega(\psi + g + k\beta_{\mathrm{GM}}^{-1/2} f) = \vartheta - \mathrm{avg}_\Omega\,\vartheta$ on Ω. The terms added in $G^\natural$ (Lemma 7.29) are functions of ϑ and ψ through the same phases. The term $F_1(S_\beta)$ is given by (7.13), and $F_2(Y) = \langle \mathbf{1}_Y\psi, F_2'\rangle$ with $F_2' = \sum_\beta F_{2,\beta}'$ and $F_{2,\beta}' = -[C_0^{-1}, \chi_\beta]C_0 s_\beta$ as in Proposition 7.8 (e), both functions of w alone. Hence the two definitions of $Q_N(Y)$ and of $E(Y, s)$ agree.

(b) After the replacement, the action is a sum of three kinds of terms. The cell terms are $\ln r_\Omega(\mathrm{avg}_\Omega\,\vartheta)$. The site terms are the terms of $G_{2,1}$, $G_{2,2}$, $G_{2,3}$, of $G^\natural$ and of E, functions of ψ and ϑ at the sites of their support (the E-terms with weights $w_s(\tau)$). The wall-component terms $F_1(S_\beta)$ and $\langle\psi, F_{2,\beta}'\rangle$ are supported in S_β by Proposition 7.8 (e). On a union Y of wall components and of $\mathcal{U}$-cells outside $\hat{\Sigma}$, Proposition 7.8 (f) shows that $D|_Y = \sum_{S_\beta \subseteq Y} D_\beta|_Y$ depends on N only through $N|_{E(Y)}$. The proof of Lemma 7.21 needs no more than these facts. An admissible set contains each cell and each wall component entirely or not at all, which gives the factorization and the locality for every wall field. Definition 7.20 and the proof of Lemma 7.23 use only the wall sites, the wall components and Proposition 7.8 (a), all of which are defined from N. On $U \neq V$ we take closures in U, which is consistent because the wall polymers of N have their closures in U.

Proposition 7.22 is proved in Appendix C from the factorization, the admissibility of the sets of a sequence and absolute convergence. Absolute convergence rests on Lemma C.1, whose bounds are pointwise in (ψ, ϑ), and on the Poincaré inequality on cells of Lemma C.2, which does not involve the configuration. Here Lemma C.1 (b) is a bound on $G_{2,1}$ by $\|\mathbf{1}_Y P_V \vartheta\|^2$.

In Theorem 7.13, the bound (7.14) of Corollary 7.10 (c) with $g - H = D$ and $\nabla g + k\beta_{\mathrm{GM}}^{-1/2} R_f = w + \nabla D$ reads $F_1^{\mathrm{loc}}(Y) \geq \frac{1}{2}\tilde{m}^2\|D\|_Y^2 + \frac{1}{2}\|w + \nabla D\|_{E(Y)}^2$, which follows from (7.13) by dropping the term with M_Y^{-2}, the supports of different β being disjoint. In the same theorem, the tube argument uses only that w vanishes on the bonds inside cells and is constant on interfaces (Lemma 7.12). The proof of Proposition 7.5 bounds $\cos(mu) - 1$ at each site, for an arbitrary real number u. Here $u = \beta_{\mathrm{GM}}^{1/2}(\varphi + k\beta_{\mathrm{GM}}^{-1/2} f - h)(x)$, which after the translation $\varphi = \psi + g$ equals $\beta_{\mathrm{GM}}^{1/2}\vartheta(x)$.

S2.6 Full proofs for Subsections 7.7 to 7.9

Proof of Proposition 7.46. The bounds of Subsection 7.6 and Appendices A, B are stated for the configurations $\mathcal{N}(X)$, $\mathcal{N}_\sigma(X)$ produced by labels. We go through the uses of the label in their proofs and show that each of them survives for long wall fields. To this end we fix $k = 0$ and a configuration $N_X \in \mathcal{N}^{\mathrm{long}}(X)$ (or $\mathcal{N}_\sigma^{\mathrm{long}}(X)$), with the integrand of Lemma 7.24.

(i) *Sums over configurations.* They enter only through the following. Lemma 7.30 (b) maps N_X to its interface values $(N_I)_{I\subseteq X} \in \mathbb{Z}^{n_I(X)}$ and drops the realizability constraints.

A wall field is constant on interfaces and vanishes off them. Hence the map is injective on $\mathcal{N}^{\mathrm{long}}(X)$, and the bound holds there as well. In the proof of Proposition A.8, step (2) splits by the first set $\mathrm{sat}_{N_X}(S_0)$ through Lemma 7.23, which holds in wall form by Lemma 7.24 (b), and the seed configuration it produces is long by Lemma 7.26 (a). Step (3) of that proof uses Lemma 5.8 to count wall polymers with their walls through their wall interfaces and values at the Euclidean link scale g. This is again a count of integer functions on interfaces, combined with $F_1^{\mathrm{loc}}(\mathcal{S}) \geq 2F_{\mathrm{unit}} \sum_I N_I^2$ from Lemma 7.30 (a). The sum over link data and sets in Propositions A.4 and A.7 does not involve the configuration beyond these.

(ii) *Bounds at a fixed configuration.* Each is a bound on a Gaussian integral whose integrand is a function of ψ, D, w and F_2', that is, of ψ and N_X, by Lemma 7.24. The label enters these proofs only in the following places.

- Appendix B uses the label in the phases of class-A cells, through $e^{im\beta_{\mathrm{GM}}^{1/2}h(x)} = 1$. Under Lemma 7.24 the phase is $e^{im\beta_{\mathrm{GM}}^{1/2}\vartheta(x)}$ by definition. The same holds for the cell function $\hat{R}_v$ of Lemma 7.31, evaluated at $\mathrm{avg}_\Omega(\psi + D)$, and for Lemma 7.41, where $D = 0$ on the wall-free cell Δ.

- Theorem 7.13 enters through Lemma 7.30 (a), and it holds for every wall field by Lemma 7.24 (b).

- In Step 1 of Lemma 7.33, the identity (7.28) is stated in terms of D_Y, w_Y and s_Y for unions of elements and involves no extension of h. At $k = 0$ the near-cell term vanishes.

- The functions $G^\natural$ of Lemma 7.29 are functions of $\vartheta = \psi + D$ and ψ, bounded pointwise.

- Lemma 7.32 (a) is used in Step 2 of Lemma 7.33. Its proof first shows that $w = w_Y$ on $E(\mathfrak{K}_\Omega^\flat)$, using only (L8-4) and Proposition 7.8, and then puts $a_\Omega := h$ on $\mathfrak{K}_\Omega^\flat$, using $P_\Omega h = 0$ and $\nabla h = w$ on $E(\mathfrak{K}_\Omega^\flat)$. For a long configuration, Lemma 7.26 (a) shows that the field $N_X \mathbf{1}_{E(Y)}$ on the union Y of elements is long, and Lemma 7.45 gives $h' : \mathfrak{K}_\Omega^\flat \to P\mathbb{Z}$, constant on Ω, with $\nabla h' = w_Y$ on $E(\mathfrak{K}_\Omega^\flat)$. We apply Proposition 6.9 (iv) to $\vartheta + h'$. Since $\nabla(\vartheta + h') = \nabla\vartheta + w_Y$ on $E(\mathfrak{K}_\Omega^\flat)$ and $P_\Omega(\vartheta + h') = P_\Omega\vartheta$, we obtain $\|P_\Omega\vartheta\|^2 \leq C_W^f k_\square^2 \|\nabla\vartheta + w_Y\|^2_{E(\mathfrak{K}_\Omega^\flat)}$, which is (a) at $k = 0$. Step 2 of Lemma 7.33 then goes through without change.

Absolute convergence at a fixed configuration (Proposition 7.22) follows as in Section S2.5, the estimates used there being pointwise in (ψ, D). Lemma 7.32 (a) is the only place where the bounds use that the configuration is long: for a wall field with a nonzero period on a short cycle of $\mathfrak{K}_\Omega^\flat$, or on a same-cell closure through Ω, no primitive of w_Y constant on Ω exists. With (i) and (ii), the proofs of Propositions 7.35, 7.39, Lemmas 7.37, 7.38 and Proposition A.8 hold with the same constants, and so does Lemma 7.48.

S2.7 Full proofs for Appendix A

Proof of Proposition A.8, step (3). We use the notation of Lemma 7.48 and Proposition A.8, and we put $w(\mathcal{S}) := e^{-(c'/8)F_1^{\mathrm{loc}}(\mathcal{S},N_\mathcal{S})}$. Since $Z^o \subseteq S_0$ and $11c'/16 - \varepsilon \geq c'/8$, steps (1) and (2) show that the left side of Lemma 7.48(a) is at most $\mathcal{P}' e^{c_S|S_0|} \sum_\sigma \prod_\mathcal{S} w(\mathcal{S})$, the sum

over $\sigma \in \Sigma_{\rm sat}(S_0)$ and the product over the wall polymers $\mathcal{S}$ of σ. Assigning the disjoint wall polymers of σ to their first cells in S_0 bounds the sum by $\prod_{\Delta\subseteq S_0}\bigl(1+\sum_{\mathcal{S}\ni\Delta} w(\mathcal{S})\bigr)$, and

$$\sum_{\mathcal{S}\ni\Delta} w(\mathcal{S}) \le 6D_I e^{-(c'/4)F_{\rm unit}} = \eta_{\rm seed}$$

under (β3). Indeed, Lemmas 7.6(a) and 7.23 show that a wall polymer with its walls is determined by its wall interfaces and their values. These interfaces form a set connected at Euclidean scale g containing one of the D_I interfaces within distance g of Δ, so the lattice-animal count of Lemma 5.8, together with $F_1^{\rm loc}(\mathcal{S}) \ge 2F_{\rm unit}\sum_I N_I^2$ from Lemma 7.30(a), gives $\sum_{n\ge1} D_I(eD_I)^{n-1}(3e^{-(c'/4)F_{\rm unit}})^n$. This gives (a).

In (b), the terms with $\sigma = (S_0, 0)$ and $X \ne S_0$ gain c_{16}^D, and those whose σ contains a wall polymer contribute $(1+\eta_{\rm seed})^{|S_0|} - 1 \le \eta_{\rm seed} e^{2|S_0|}$, so the sum without the term $X = S_0$, $N_X = 0$ is at most $(c_{16}^D + \eta_{\rm seed})\mathcal{P}' e^{(c_S+2)|S_0|} = c_{16}\mathcal{P}' e^{c_2|S_0|}$.

In (c′) we have $\mathcal{P}' = 1$ and $Z^o = \emptyset$, and (A.10) with $\varepsilon \le c'/8$ and $c_w = c'/16$ leaves $e^{-(c'/2)F_1^{\rm loc}(Z,N_Z)}$.