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# Functional van Lint-Seidel Relative and Gerzon Universal Bounds

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Date: October 26, 2026

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**Abstract:** We introduce the notion of functional equiangular lines in finite rank modules over subrings of  $\mathbb{R}$ . We show that van Lint-Seidel relative and Gerzon universal bounds hold for this most general lines.

**Keywords:** Equiangular lines.

**Mathematics Subject Classification (2020):** 52A40, 05C50.

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## 1. INTRODUCTION

Let  $d \in \mathbb{N}$  and  $\gamma \in [0, 1]$ . Recall that a collection  $\{\tau_j\}_{j=1}^n$  of unit vectors in  $\mathbb{R}^d$  is said to be  $\gamma$ -**equiangular lines** [1, 3] if

$$|\langle \tau_j, \tau_k \rangle| = \gamma, \quad \forall 1 \leq j, k \leq n, j \neq k.$$

A fundamental problem associated with equiangular lines is the following.

**Problem 1.1.** *Given  $d \in \mathbb{N}$  and  $\gamma \in [0, 1]$ , what is the upper bound on  $n$  such that there exists a collection  $\{\tau_j\}_{j=1}^n$  of  $\gamma$ -equiangular lines in  $\mathbb{R}^d$ ?*

Two answers to Problem (1.1) which are driving forces in the study of equiangular lines is the following relative bound of van Lint and Seidel [3, 4] and universal bound of Gerzon [5].

**Theorem 1.2.** [3, 4] (**van Lint-Seidel Relative Bound**) *Let  $\{\tau_j\}_{j=1}^n$  be  $\gamma$ -equiangular lines in  $\mathbb{R}^d$ . Then*

$$n(1 - d\gamma^2) \leq d(1 - \gamma^2).$$

*In particular, if*

$$\gamma < \frac{1}{\sqrt{d}},$$

*then*

$$n \leq \frac{d(1 - \gamma^2)}{1 - d\gamma^2}.$$

**Theorem 1.3.** [5] (**Gerzon Universal Bound**) *Let  $\{\tau_j\}_{j=1}^n$  be  $\gamma$ -equiangular lines in  $\mathbb{R}^d$ . Then*

$$n \leq \frac{d(d+1)}{2}.$$

The notion of functional equiangular lines is hinted in [2]. In this paper, we define it in most general form and derive functional forms of Theorems 1.2 and 1.3.

## 2. FUNCTIONAL EQUIANGULAR LINES

In the paper,  $\mathcal{R}$  denotes a subring of  $\mathbb{R}$  and  $\mathcal{M}$  is a rank  $d$  module over  $\mathcal{R}$ . By  $\mathcal{M}^*$  we mean the module of all homomorphisms from  $\mathcal{M}$  to  $\mathcal{R}$ . Module of all homomorphisms from  $\mathcal{M}$  to  $\mathcal{M}$  is denoted by  $\text{Mor}(\mathcal{M})$ .

**Definition 2.1.** Let  $\{\tau_j\}_{j=1}^n$  be a collection in a module  $\mathcal{M}$  of rank  $d$  over  $\mathcal{R}$  and  $\{f_j\}_{j=1}^n$  be a collection in  $\mathcal{M}^*$ . Let  $\gamma \geq 0$ . The pair  $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$  is said to be **functional  $\gamma$ -equiangular** if following conditions hold.

- (i)  $f_j(\tau_j) = 1$  for all  $1 \leq j \leq n$ .
- (ii)  $|f_j(\tau_k)f_k(\tau_j)| = \gamma^2$  for all  $1 \leq j, k \leq n, j \neq k$ .
- (iii) The operator

$$S_{f,\tau} : \mathcal{M} \ni x \mapsto \sum_{j=1}^n f_j(x)\tau_j \in \mathcal{M}$$

is similar (through invertible operator) to a diagonal operator.

**Theorem 2.2. (Functional van Lint-Seidel Relative Bound)** If  $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$  is functional  $\gamma$ -equiangular for rank  $d$  module  $\mathcal{M}$ , then

$$n(1 - d\gamma^2) \leq d(1 - \gamma^2).$$

In particular, if

$$\gamma < \frac{1}{\sqrt{d}},$$

then

$$n \leq \frac{d(1 - \gamma^2)}{1 - d\gamma^2}.$$

*Proof.* Define

$$S_{f,\tau} : \mathcal{M} \ni x \mapsto S_{f,\tau}x := \sum_{j=1}^n f_j(x)\tau_j \in \mathcal{M}.$$

Let  $\lambda_1, \dots, \lambda_d$  be eigenvalues of  $S_{f,\tau}$ . Since  $S_{f,\tau}$  is diagonalizable,

$$\begin{aligned} n^2 &= \left( \sum_{j=1}^n f_j(\tau_j) \right)^2 = (\text{Tra}(S_{f,\tau}))^2 = \left( \sum_{k=1}^d \lambda_k \right)^2 \\ &\leq d \sum_{k=1}^d \lambda_k^2 = d \text{Tra}(S_{f,\tau}^2) = d \sum_{j=1}^n \sum_{k=1}^n f_j(\tau_k)f_k(\tau_j) \\ &= d \sum_{j=1}^n f_j(\tau_j)^2 + d \sum_{j,k=1, j \neq k}^n f_j(\tau_k)f_k(\tau_j) = dn + d \sum_{j,k=1, j \neq k}^n f_j(\tau_k)f_k(\tau_j) \\ &\leq dn + d \sum_{j,k=1, j \neq k}^n |f_j(\tau_k)f_k(\tau_j)| = dn + d(n^2 - n)\gamma^2. \end{aligned}$$

Therefore

$$n \leq d + d(n-1)\gamma^2 \implies n(1 - d\gamma^2) \leq d(1 - \gamma^2).$$

□

We are unable to derive Gerzon universal bound for functional equiangular lines. However, we derive following Gerzon bound for functionals.

**Theorem 2.3. (Functional Gerzon Universal Bound)** *Let  $\{\tau_j\}_{j=1}^n$  be a collection in a module  $\mathcal{M}$  of rank  $d$  over  $\mathcal{R}$  and  $\{f_j\}_{j=1}^n$  be a collection in  $\mathcal{M}^*$ . Assume the following.*

- (i)  $f_j(\tau_j) = 1$  for all  $1 \leq j \leq n$ .
- (ii) There is a  $\gamma \neq 1$  such that  $f_j(\tau_k)f_k(\tau_j) = \gamma^2$  for all  $1 \leq j, k \leq n, j \neq k$ .

Then

$$n \leq d^2.$$

*Proof.* For  $1 \leq j \leq n$ , define

$$\tau_j \otimes f_j : \mathcal{M} \ni x \mapsto (\tau_j \otimes f_j)(x) := f_j(x)\tau_j \in \mathcal{M}$$

We show that the collection  $\{\tau_j \otimes f_j\}_{j=1}^n$  in  $\text{Mor}(\mathcal{M})$  is linearly independent over  $\mathcal{R}$ . Let  $c_1, \dots, c_n \in \mathcal{R}$  be such that

$$\sum_{j=1}^n c_j(\tau_j \otimes f_j) = 0.$$

Let  $1 \leq k \leq n$  be fixed. Then previous equation gives

$$\sum_{j=1}^n c_j(\tau_j \otimes f_j)(\tau_k \otimes f_k) = 0.$$

By taking trace we get

$$\begin{aligned} 0 &= \sum_{j=1}^n c_j \text{Tra}((\tau_j \otimes f_j)(\tau_k \otimes f_k)) = \sum_{j=1}^n c_j f_j(\tau_k) f_k(\tau_j) \\ &= \sum_{j=1, j \neq k}^n c_j f_j(\tau_k) f_k(\tau_j) + c_k f_k(\tau_k)^2 = \sum_{j=1, j \neq k}^n c_j \gamma^2 + c_k \\ &= \gamma^2 \left( \sum_{j=1}^n c_j - c_k \right) + c_k = \gamma^2 \left( \sum_{j=1}^n c_j \right) + (1 - \gamma^2) c_k. \end{aligned}$$

Therefore

$$c_k = \frac{\gamma^2}{\gamma^2 - 1} \sum_{j=1}^n c_j =: c, \quad \forall 1 \leq k \leq n.$$

Now

$$\begin{aligned} 0 &= \text{Tra} \left( \sum_{j=1}^n c_j(\tau_j \otimes f_j) \right) = \text{Tra} \left( \sum_{j=1}^n c(\tau_j \otimes f_j) \right) \\ &= \sum_{j=1}^n c \text{Tra}(\tau_j \otimes f_j) = \sum_{j=1}^n c f_j(\tau_j) = cn. \end{aligned}$$

Hence  $c = 0$ . Therefore  $\{\tau_j \otimes f_j\}_{j=1}^n$  is linearly independent. Since the rank of  $\text{Mor}(\mathcal{M})$  is  $d^2$  we must have  $n \leq d^2$ .  $\square$

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