

ALL FINITE CARDINALITIES OF THE HYPERSPACE OF REGULAR SUBCONTINUA

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ABSTRACT. For a metric continuum X , let $\mathcal{D}(X)$ be the family of subcontinua A of X with $A = \text{cl}_X \text{int}_X A$. Ordoñez asked which finite numbers occur as $|\mathcal{D}(X)|$. Camargo, Ordoñez and Ramírez excluded 2, 4, 5, 8, 9 and asked whether every $k \geq 256$ occurs. We prove that a positive integer k equals $|\mathcal{D}(X)|$ for some metric continuum X if and only if $k \notin \{2, 4, 5, 8, 9, 16\}$. The finite cardinalities of $\mathcal{D}(X)$ are exactly the numbers of vertex sets inducing connected subgraphs of finite connected graphs; every finite connected graph is realized by copies of the Knaster continuum glued at points. We determine the values of this graph invariant without computer assistance. A short argument excludes 16, and an explicit recursive family of graphs realizes every $k \geq 17$ with the least possible number of vertices. Consequently, for $k \geq 32$ the least number of minimal regular subcontinua of a continuum X with $|\mathcal{D}(X)| = k$ is $\lfloor \log_2 k \rfloor + 1$. Programs check the explicit finite graph counts, and Theorem A is formally verified in Lean.

1. INTRODUCTION

A *continuum* is a nonempty compact connected metric space. For a continuum X , let $C(X)$ be its hyperspace of nonempty subcontinua with the Hausdorff metric. A subcontinuum A of X is *regular* if $A = \text{cl}_X \text{int}_X A$, the closure in X of the interior of A in X . The hyperspace

$$\mathcal{D}(X) = \{A \in C(X) : A = \text{cl}_X \text{int}_X A\}$$

was introduced by Ordoñez [9]. Since $X \in \mathcal{D}(X)$, always $|\mathcal{D}(X)| \geq 1$. For example, every nondegenerate closed subinterval of $[0, 1]$ is regular in $[0, 1]$, so $\mathcal{D}([0, 1])$ is infinite. At the other extreme, a continuum is *indecomposable* if it is not the union of two proper subcontinua. Every proper subcontinuum of an indecomposable continuum has empty interior [7, Chapter XI], so then $\mathcal{D}(X) = \{X\}$ [9, Theorem 3.7]. The hyperspace $\mathcal{D}(X)$ is compact if and only if it is finite [9, Corollary 4.13]. Further properties of $\mathcal{D}(X)$ and of the hyperspace of subcontinua with empty interior are studied in [10]. Ordoñez posed the following problem [9, Problem 4.14], repeated as [2, Question 3.5] and [11, Pregunta 3.1.6].

Question 1.1. *For which $n \in \mathbb{N}$ does there exist a continuum X such that $\mathcal{D}(X)$ has exactly n elements?*

Chains, cycles and bouquets of n indecomposable continua give $n(n+1)/2$, $n(n-1)+1$ and $2^n - 1$ regular subcontinua [9, Proposition 4.15], [11, Proposición 3.1.7]. An element of $\mathcal{D}(X)$ is *minimal* if it contains no other element of $\mathcal{D}(X)$, that is, if it is minimal with respect to inclusion; we call it a *minimal regular subcontinuum*. Camargo, Ordoñez and Ramírez [2] showed that a finite $\mathcal{D}(X)$ is determined by its minimal elements, which have pairwise disjoint interiors and cover X [2, Theorem 3.16]; this reduces the count to a problem about graphs. They proved $|\mathcal{D}(X)| \notin \{2, 4, 5, 8, 9\}$ [2, Proposition 3.18],¹ and a computer search over graphs with at most eight

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¹The abstract of [2] and the outline at the end of its Section 1 also list 3 among the excluded values. This is a misprint: 3 occurs, for instance for two indecomposable continua meeting in one point [9, Proposition 4.15], [2, Proposition 3.17].

vertices showed that 16 does not occur while every integer from 17 to 255 does [2, Remark 3.20]. They asked:

Question 1.2 ([2, Question 3.21]). *If $k \geq 256$, does there exist a continuum X such that $|\mathcal{D}(X)| = k$?*

The thesis [11, Pregunta 3.1.13] asks whether $|\mathcal{D}(X)| = 2^n$ for some $n > 0$. Both questions are special cases of the problem of characterizing the spaces homeomorphic to some $\mathcal{D}(X)$ [2, Problem 3.22].

For a finite simple graph G , a *connected set* of G is a nonempty set of vertices inducing a connected subgraph, and $c(G)$ denotes the number of connected sets. Singletons are connected sets and the empty set is not; distinct vertex sets are counted separately even when they induce isomorphic subgraphs. For a tree, $c(G)$ is the number of subtrees.

Theorem A. *For a positive integer k , the following are equivalent.*

- (1) *There is a continuum X with $|\mathcal{D}(X)| = k$.*
- (2) *There is a finite connected graph G with $c(G) = k$.*
- (3) *$k \notin \{2, 4, 5, 8, 9, 16\}$.*

Moreover, for every $k \geq 32$ the continuum in (1) can be chosen as the union of $\lfloor \log_2 k \rfloor + 1$ copies of the Knaster continuum, any two of which meet in at most one point. No continuum with $|\mathcal{D}(X)| = k$ has fewer minimal regular subcontinua.

Corollary B. (a) *The answer to Question 1.2 is yes for every $k \geq 256$; in fact $|\mathcal{D}(X)| = k$ occurs for every $k \geq 17$.*

- (b) *A finite metric space S is homeomorphic to $\mathcal{D}(X)$ for some continuum X if and only if S is nonempty and $|S| \notin \{2, 4, 5, 8, 9, 16\}$. This settles the finite case of [2, Problem 3.22].*
- (c) *There is a continuum X with $|\mathcal{D}(X)| = 2^m$ if and only if $m = 0$ or $m \geq 5$. This answers [11, Pregunta 3.1.13].*

The equivalence of (1) and (2) is Theorem 3.10. The implication (1) $\Rightarrow$ (2) is the reduction of [2]. For (2) $\Rightarrow$ (1) we glue copies of the Knaster continuum along the edges of G (Construction 3.4). The key step, Lemma 3.7, shows that a regular subcontinuum meeting the interior of a piece contains that piece. Its proof is direct; it does not use the discreteness hypothesis of [2, Theorem 3.16], and it does not depend on where the attachment points lie. The equivalence of (2) and (3) is purely combinatorial (Sections 4 and 5). The value 16 is excluded by a short argument about spanning trees. Every $k \geq 17$ is realized by a recursive construction. The construction uses cones, disjoint unions, a family of *near-cones*, and seven small graphs. This part needs no computer, and it realizes k with the least possible number of vertices. Theorem A is formally verified in Lean (Section 7).

For trees, the analogous problem was solved by Czaparka, Székely and Wagner [3]: every positive integer except 34 explicit values, the largest being 89, is the number of subtrees of some tree. Their theorem gives a second, computer-assisted route to the values $k \geq 90$; we recomputed its computer-assisted parts (Section 7). The invariant $c(G)$ is a specialization of the subgraph component polynomial of Tittmann, Averbouch and Makowsky [14], which counts the vertex sets of G by their size and by the number of components of the induced subgraph: $c(G)$ is the number of vertex sets inducing exactly one component. To our knowledge, the inverse problem for $c(G)$ over general graphs has not been treated before. A companion note [6] illustrates the construction on the examples $|\mathcal{D}(X)| = 1, 3, 6, 7, 11, 17$.

2. CONNECTED SETS IN GRAPHS

Graphs are finite and simple; they need not be connected, and the graph with no vertices is allowed. For a graph G , we write $|G| = |V(G)|$ for its order, and for $F \subseteq V(G)$, $G[F]$ is the induced subgraph. A *connected set* is a nonempty F with $G[F]$ connected. We write $c(G)$ for

the number of connected sets, $c_v(G)$ for the number of connected sets containing a vertex v , and $T(n) = n(n+1)/2$. If G is a tree, a connected subgraph of G is the induced subgraph on its vertex set, so $c(G)$ is the number of subtrees of G in the sense of [13, 3].

For integers $a \leq b$ we write $[a, b] = \{a, a+1, \dots, b\}$. For sets A, B of integers and an integer t we write $t + A = \{t + a : a \in A\}$ and $A + B = \{a + b : a \in A, b \in B\}$. A vertex is *universal* if it is adjacent to every other vertex. The *cone* over a graph H is obtained by adding one new vertex adjacent to every vertex of H , which is then universal. A *pendant vertex* is a vertex of degree one. A *spider* consists of a vertex, its *centre*, and paths, its *legs*, each joined to the centre by an edge at one of its ends and otherwise disjoint from each other; the *length* of a leg is its number of vertices, not counting the centre.

Lemma 2.1. *Let G, H and K be graphs.*

- (a) *If G is the disjoint union of H and K , then $c(G) = c(H) + c(K)$.*
- (b) *If H is a spanning subgraph of G , then $c(H) \leq c(G)$. If some edge of G is not an edge of H , then $c(H) < c(G)$.*
- (c) *Suppose that $G = G_1 \cup G_2$ for subgraphs with $V(G_1) \cap V(G_2) = \{v\}$ and $E(G) = E(G_1) \cup E(G_2)$. Then*

$$c(G) = c_v(G_1) c_v(G_2) + c(G_1 - v) + c(G_2 - v).$$

- (d) (Cone.) *If G is the cone over K , then $c(G) = 2^{|K|} + c(K)$.*
- (e) (Almost-cone.) *Let u be a vertex of K of degree d in K . If G is obtained from K by adding a vertex w adjacent to every vertex of K except u , then*

$$c(G) = c(K) + 2^{|K|} - 2^{|K|-1-d}.$$

- (f) *If G is connected with $n \geq 1$ vertices, then $T(n) \leq c(G) \leq 2^n - 1$. The lower bound is attained by the path P_n and the upper bound by the complete graph K_n .*

Proof. (a) A connected set lies in one component.

(b) Every connected set of H is connected in G . If uv is an edge of G but not of H , then $\{u, v\}$ is connected in G but not in H .

(c) No edge joins $G_1 - v$ and $G_2 - v$, so a connected set avoiding v lies in $G_1 - v$ or in $G_2 - v$. Let F be a connected set containing v , and put $F_i = F \cap V(G_i)$. A path in $G[F]$ between two vertices of F_i stays in $V(G_i)$: it could leave $V(G_i)$ only through v , and it would have to return through v , which a path cannot do. So each F_i is a connected set of G_i containing v . Conversely, the union of a connected set of G_1 and one of G_2 , both containing v , is connected. This gives a bijection between the connected sets of G containing v and such pairs (F_1, F_2) .

(d) and (e) We have $c(G) = c(K) + c_w(G)$, where w is the added vertex and $c_w(G)$ is the number of sets $U \subseteq V(K)$ such that $\{w\} \cup U$ is connected. This happens if and only if every component of $K[U]$ contains a neighbour of w . In (d) this holds for all $2^{|K|}$ sets U . In (e) it fails exactly when $\{u\}$ is a component of $K[U]$, that is, when $u \in U$ and U contains no neighbour of u . There are $2^{|K|-1-d}$ such sets.

(f) Let S be a spanning tree of G . For vertices x, y , possibly equal, the vertex set of the x - y path in S is connected in S , hence in G . Its two ends are the vertices of degree at most one in S restricted to this set, so distinct pairs $\{x, y\}$ give distinct sets. This gives $n + \binom{n}{2} = T(n)$ connected sets. The connected sets of P_n are its $T(n)$ intervals. There are only $2^n - 1$ nonempty sets, and all of them are connected in K_n . $\square$

We record three families of formulas that follow from Lemma 2.1.

Example 2.2. (i) The star $K_{1,d}$ satisfies $c(K_{1,d}) = 2^d + d$. A spider whose legs have lengths $\ell_1, \dots, \ell_s$ satisfies $c = \prod_i (\ell_i + 1) + \sum_i T(\ell_i)$: a connected set containing the centre takes an initial segment of each leg, possibly empty, and a connected set avoiding the centre is an interval of one leg.

- (ii) Attach $t_v \geq 0$ pendant vertices at each vertex v of a complete graph K_a . A connected set is either a single pendant vertex, or meets the clique in a nonempty set Q , together with any pendant vertices attached at Q . Hence $c = \prod_v (1 + 2^{t_v}) - 1 + \sum_v t_v$.
- (iii) Let Φ be a disjoint union of stars $K_{1,s_1}, \dots, K_{1,s_p}$ with $s_i \geq 1$ and at most n vertices in all, and let $K_n - \Phi$ be the complete graph on n vertices with the edges of a copy of Φ removed. A set U is disconnected in $K_n - \Phi$ if and only if $U = A \cup B$ with A, B nonempty and disjoint and all pairs in $A \times B$ edges of Φ . This complete bipartite subgraph of Φ is connected, so it lies in one star, and then A or B is the centre alone. Hence

$$c(K_n - \Phi) = 2^n - 1 - \sum_i (2^{s_i} - 1).$$

3. FROM FINITE HYPERSPACES TO GRAPHS

Throughout this section, closures, interiors and boundaries without a subscript are taken in the ambient continuum, which is X in Sections 3.1 and 3.2 and X_G in Section 3.3. Relative operations carry a subscript, as in cl_K , int_{N_i} or bd_K .

3.1. Unions of continua. A subset A of a space is *regular closed* if $A = \text{cl int } A$; no connectedness is required. Thus $\mathcal{D}(X)$ consists of the regular closed subcontinua of X . Let $A_1, \dots, A_k$ be subcontinua of a continuum. Their *intersection graph* Γ has vertex set $\{1, \dots, k\}$, with $i \sim j$ when $i \neq j$ and $A_i \cap A_j \neq \emptyset$. For $\emptyset \neq F \subseteq \{1, \dots, k\}$ put $A_F = \bigcup_{i \in F} A_i$.

Lemma 3.1. (a) A_F is connected if and only if F is a connected set of Γ .

(b) A finite union of regular closed sets is regular closed.

(c) If the sets A_i are regular closed and nonempty, with pairwise disjoint interiors, then $F \mapsto A_F$ is injective on nonempty sets F .

Proof. (a) If $\Gamma[F]$ is disconnected, split $F = F_1 \sqcup F_2$ with no edge between F_1 and F_2 . Then A_F is the union of the disjoint nonempty closed sets A_{F_1} and A_{F_2} . If $\Gamma[F]$ is connected, add the sets A_i one at a time, each meeting the union already built. A union of two intersecting connected sets is connected.

(b) If $R_1, \dots, R_p$ are regular closed, then $R_j = \text{cl int } R_j \subseteq \text{cl int } \bigcup_l R_l$ for each j . The reverse inclusion holds because the union is closed.

(c) Suppose $A_F = A_{F'}$ and $i \in F \setminus F'$. The nonempty open set $\text{int } A_i$ is contained in $A_{F'} = \text{cl } \bigcup_{j \in F'} \text{int } A_j$, so it meets $\text{int } A_j$ for some $j \in F'$. Since $j \neq i$, this contradicts the disjointness of interiors. $\square$

3.2. Necessity.

Proposition 3.2. Let X be a continuum with $\mathcal{D}(X)$ finite. Then $|\mathcal{D}(X)| = c(\Gamma)$ for a finite connected graph Γ , namely the intersection graph of the minimal elements of $\mathcal{D}(X)$.

Proof. A finite subset of the metric space $C(X)$ is discrete, so [2, Theorem 3.16] applies. It gives minimal elements $N_1, \dots, N_k$ of $\mathcal{D}(X)$ with pairwise disjoint interiors and $X = \bigcup_i N_i$. By [2, Proposition 3.15], every $K \in \mathcal{D}(X)$ satisfies

$$(1) \quad K = \bigcup \{N_i : K \cap \text{int } N_i \neq \emptyset\}.$$

This is equation (3.1) of [2]. Proposition 3.15 of [2] gives the inclusion $\supseteq$; we include the reverse inclusion. Each N_i is regular, so $X = \text{cl } \bigcup_i \text{int } N_i$, and the open set $\bigcup_i \text{int } N_i$ is dense. Hence $\text{int } K \cap \bigcup_i \text{int } N_i$ is dense in $\text{int } K$. Every point of it lies in some N_i with $K \cap \text{int } N_i \neq \emptyset$. Taking closures and using $K = \text{cl int } K$ gives $\subseteq$.

Let Γ be the intersection graph of $N_1, \dots, N_k$. By Lemma 3.1, the map $F \mapsto N_F$ sends connected sets of Γ injectively into $\mathcal{D}(X)$. By (1) it is onto: for $K \in \mathcal{D}(X)$ the set $F = \{i : K \cap \text{int } N_i \neq \emptyset\}$

is nonempty, $K = N_F$, and F is connected by Lemma 3.1(a). Finally, Γ is connected because $N_{\{1, \dots, k\}} = X$ is connected. The N_i are all the minimal elements of $\mathcal{D}(X)$: if $M \in \mathcal{D}(X)$ is minimal, then (1) gives $N_i \subseteq M$ for some i , so $M = N_i$. $\square$

Remark 3.3 (Where discreteness is used). The proofs of [2, Propositions 3.14 and 3.15 and Theorem 3.16] use the following statement about a regular subcontinuum Y of X :

if $\mathcal{D}(Y)$ is finite, then every proper element of $\mathcal{D}(Y)$ is contained in an inclusion-maximal proper element of $\mathcal{D}(Y)$.

For a finite family this is immediate, and if $\mathcal{D}(Y) = \{Y\}$ it is vacuous. When $\mathcal{D}(X)$ is finite, so is $\mathcal{D}(Y)$ for every $Y \in \mathcal{D}(X)$, because $\mathcal{D}(Y) = \{A \in \mathcal{D}(X) : A \subseteq Y\}$ [11, Lema 3.2.8]. In [2], Theorem 3.16 assumes that $\mathcal{D}(X)$ is discrete, and this hypothesis enters its proof and the proofs of Propositions 3.14 and 3.15 only through [2, Corollary 3.13], which provides the statement above. That corollary is applied to $\mathcal{D}(Y)$, where Y is X or a regular subcontinuum of X , such as $N_1 \cup N_2$, $A \cup N$ or K_1 in the notation of [2], and always to a proper element of $\mathcal{D}(Y)$. Here “maximal” must mean maximal in $\mathcal{D}(Y) \setminus \{Y\}$, as in [11, Definición 3.1.10]; read literally, [2, Definition 3.8] would make Y itself maximal. With this reading we checked the proofs of [2, Lemma 3.9, Propositions 3.14 and 3.15, and Theorem 3.16] line by line for finite $\mathcal{D}(X)$. The Lean formalization of Section 7 also proves Proposition 3.15 and Theorem 3.16 of [2] for finite $\mathcal{D}(X)$ directly.

3.3. Realization.

Construction 3.4. Let Q be a nondegenerate indecomposable continuum, for instance the Knaster bucket-handle continuum (Figure 1). Recall that every proper subcontinuum of Q has empty interior in Q [7, Chapter XI], [11, Teorema 1.1.17], and that Q has no isolated points, so every nonempty open subset of Q is infinite. Let G be a finite connected graph with vertex set $\{1, \dots, k\}$ and edge set E . If $k = 1$, put $X_G = Q$. Otherwise proceed in four steps.

- (1) Take a copy Q_i of Q for each vertex i , and let Y be the disjoint union of $Q_1, \dots, Q_k$; it is a compact metric space.
- (2) For each edge $ij \in E$ choose a point $p_{ij} \in Q_i$ and a point $p_{ji} \in Q_j$, so that the points chosen in each copy Q_i , one for each edge at i , are distinct.
- (3) Let $\sim$ identify p_{ij} with p_{ji} for every edge, and let $X_G = Y/\sim$, with quotient map q .
- (4) Put

$$N_i = q(Q_i), \quad B_i = \{q(p_{ij}) : ij \in E\}.$$

The points of B_i are the *attachment points* of N_i : at $q(p_{ij})$ the pieces N_i and N_j are joined. An identification merges two points into one; it adds no arc between them. The distinctness in step (2) ensures that each class of $\sim$ is a singleton or a pair $\{p_{ij}, p_{ji}\}$ with points in two different copies: no three pieces share a point, and no two points of the same copy are identified.

Lemma 3.5. *Let $k \geq 2$. In Construction 3.4:*

- (a) X_G is a continuum;
- (b) q restricts to an embedding of each Q_i , so N_i is homeomorphic to Q ;
- (c) for $i \neq j$, $N_i \cap N_j = \{q(p_{ij})\}$ if $ij \in E$, and $N_i \cap N_j = \emptyset$ otherwise;
- (d) $\text{int } N_i = N_i \setminus B_i$, and this set is dense in N_i ;
- (e) each N_i belongs to $\mathcal{D}(X_G)$, and the sets $\text{int } N_i$ are pairwise disjoint.

Proof. Let P be the finite set of chosen points and d a metric on Y . The classes of $\sim$ are the pairs $\{p_{ij}, p_{ji}\}$ and the singletons. A subset of Y is *saturated* if it contains every class of $\sim$ that it meets. By the definition of the quotient topology, q maps saturated open sets onto open sets.

(a) *Compactness.* $X_G = q(Y)$ is a continuous image of the compact space Y .

The map q is closed. For closed $C \subseteq Y$, the saturation $q^{-1}(q(C))$ is C together with finitely many points, hence closed.

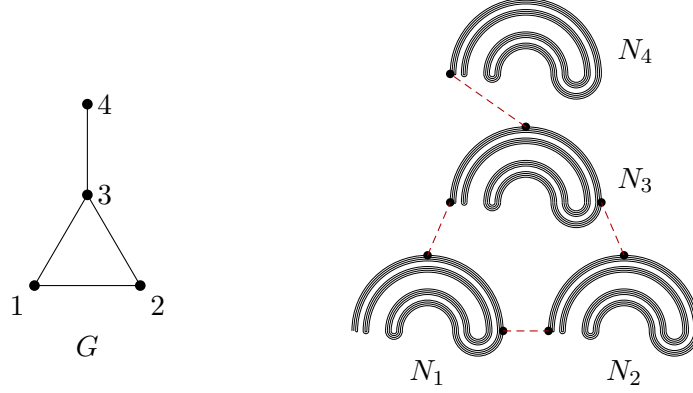


FIGURE 1. Construction 3.4 for the paw G , a triangle with one pendant vertex. Each piece N_i is a copy of the Knaster bucket-handle continuum, drawn at a finite stage of its construction. Dashed segments indicate point identifications: the two dots at the ends of a segment become one point of X_G , and the segments are not part of X_G . The attachment points on a piece are distinct and may be chosen anywhere.

Hausdorff. Let $\xi \neq \eta$ be two classes, and let $\varepsilon > 0$ be smaller than half the least distance between distinct points of the finite set $\xi \cup \eta \cup P$. The ε -neighbourhoods of ξ and of η are disjoint. Each of them contains no point of P outside ξ , respectively η , so they are saturated, and their images are disjoint open sets.

Metrizability. Let $\mathcal{B}$ be a countable base of Y closed under finite unions. Every fibre of q is finite, so if $z \in W$ with W open in X_G , some $V \in \mathcal{B}$ satisfies $q^{-1}(z) \subseteq V \subseteq q^{-1}(W)$, and then $z \in X_G \setminus q(Y \setminus V) \subseteq W$. As q is closed, these sets form a countable base of X_G . A compact Hausdorff space with a countable base is metrizable by Urysohn's theorem.

Connectedness. The sets N_i are connected, $N_i \cap N_j \neq \emptyset$ for $ij \in E$, and G is connected, so $X_G = \bigcup_i N_i$ is connected (Lemma 3.1(a)).

(b) No two points of Q_i are identified, and a continuous injection of a compact space into a Hausdorff space is an embedding.

(c) A point of $N_i \cap N_j$ is a class meeting Q_i and Q_j ; the only such class is $\{p_{ij}, p_{ji}\}$, which exists exactly when $ij \in E$.

(d) The set $Q_i \setminus P$ is open in Y and saturated, so its image $N_i \setminus B_i$ is open in X_G . A point $q(p_{ij}) \in B_i$ is not interior to N_i : each of its neighbourhoods contains the images of infinitely many points of Q_j near p_{ji} , and only one point of Q_j maps into N_i . Density holds because B_i is finite and N_i has no isolated points.

(e) By (d), $\text{cl int } N_i = N_i$. If $i \neq j$, then $\text{int } N_i \cap \text{int } N_j \subseteq (N_i \cap N_j) \setminus B_i = \emptyset$ by (c). $\square$

We use the boundary bumping theorem in the following form.

Lemma 3.6 (Boundary bumping). *Let K be a continuum and let A be a nonempty proper closed subset of K . Then every component of A meets $\text{bd}_K A$.*

Proof. This is a form of the boundary bumping theorem [7, Chapter V]; we include the short proof. Since A is closed, $\text{bd}_K A = A \setminus \text{int}_K A$. Suppose that a component C of A satisfies $C \subseteq \text{int}_K A$. In the compact space A , the component C is the intersection of the subsets of A that are open and closed in A and contain C [4, Theorem 6.1.23]. The compact set $A \setminus \text{int}_K A$ is covered by the complements of these sets, so finitely many of them have an intersection D with $C \subseteq D \subseteq \text{int}_K A$, and D is open and closed in A . Then D is closed in K . Also $D = O \cap A$ for some open $O \subseteq K$, so $D = O \cap \text{int}_K A$ is open in K . As $\emptyset \neq D \subseteq A \neq K$, this contradicts the connectedness of K . $\square$

The next lemma is the key step. The intersection $K \cap N_i$ of a regular subcontinuum K with a piece need not be connected, so the indecomposability of N_i cannot be applied to it directly. Boundary bumping shows that $K \cap N_i$ has only finitely many components, each containing an attachment point. If every component were a proper subcontinuum of N_i , their union would be nowhere dense in N_i , while the regularity of K forces $K \cap N_i$ to contain a nonempty open subset of N_i .

Lemma 3.7 (Key lemma). *Let $K \in \mathcal{D}(X_G)$ and $1 \leq i \leq k$. If $K \cap \text{int } N_i \neq \emptyset$, then $N_i \subseteq K$.*

Proof. For $k = 1$ this is the indecomposability of Q , so let $k \geq 2$. Since K is regular, $\text{int } K$ is dense in K ; as $\text{int } N_i$ is an open set meeting K , it meets $\text{int } K$. So $W = \text{int } K \cap \text{int } N_i$ is nonempty. It is open in X_G , hence open in N_i . Put $A = K \cap N_i$, a closed subset of K containing W .

If $K \subseteq N_i$, then K is a subcontinuum of N_i with nonempty interior in N_i . Since N_i is indecomposable, $K = N_i$.

Suppose $K \not\subseteq N_i$. Then A is a nonempty proper closed subset of the continuum K . Every point of A outside B_i belongs to $K \cap \text{int } N_i$, which is open in K and contained in A . Consequently $\text{bd}_K A \subseteq B_i$. By Lemma 3.6, every component of A meets B_i . Distinct components are disjoint, so A has at most $|B_i|$ components, and each of them is a subcontinuum of N_i . If all of them were proper subcontinua of N_i , each would be nowhere dense in N_i by indecomposability, and so would their finite union A . This contradicts $W \subseteq A$. Thus some component of A equals N_i , and $N_i \subseteq K$. $\square$

Proposition 3.8. *For every finite connected graph G ,*

$$\mathcal{D}(X_G) = \left\{ \bigcup_{i \in F} N_i : F \text{ a connected set of } G \right\},$$

and distinct connected sets give distinct elements. In particular $|\mathcal{D}(X_G)| = c(G)$.

Proof. The case $k = 1$ is $\mathcal{D}(Q) = \{Q\}$. Let $k \geq 2$. By Lemma 3.5(c), the intersection graph of $N_1, \dots, N_k$ is G . By Lemma 3.1 and Lemma 3.5(e), for every connected set F the set N_F is a regular subcontinuum, and $F \mapsto N_F$ is injective. Conversely, let $K \in \mathcal{D}(X_G)$ and $F = \{i : K \cap \text{int } N_i \neq \emptyset\}$. By Lemma 3.7, $N_F \subseteq K$. The set $\bigcup_i \text{int } N_i = X_G \setminus (B_1 \cup \dots \cup B_k)$ is the complement of a finite set in a space without isolated points, so $\text{int } K \cap \bigcup_i \text{int } N_i$ is dense in $\text{int } K$. This set is contained in N_F . Therefore $K = \text{clint } K \subseteq N_F$. So $K = N_F$; moreover $F \neq \emptyset$, and F is connected by Lemma 3.1(a). $\square$

Remark 3.9 (Position of the attachment points). Recall that the *composant* of a point x in a continuum is the union of all proper subcontinua containing x . Lemma 3.7 does not depend on where the attachment points lie, but the choice of attachment points does affect which proper subcontinua X_G has. If two attachment points $q(p_{ij})$ and $q(p_{il})$ of N_i lie in one composant of N_i , then a proper subcontinuum C of N_i contains both, and $L = N_j \cup C \cup N_l$ is a subcontinuum of X_G that is not a union of pieces. It is not regular. The set $\text{int } L \setminus (N_j \cup N_l)$ is open in X_G and contained in C , hence open in N_i , and it is empty because C is nowhere dense in N_i . So $\text{clint } L \subseteq N_j \cup N_l$ misses the nonempty set $C \setminus (N_j \cup N_l)$. Thus attachment points in a common composant create additional subcontinua, but the regular subcontinua are still exactly the connected unions of pieces. What matters is that each B_i is finite: this bounds the number of components of $K \cap N_i$ in Lemma 3.7. In the construction of [1], with countably many attachment points on a piece, the composants do matter.

Theorem 3.10. *The set of finite cardinalities of hyperspaces $\mathcal{D}(X)$, over all continua X , equals the set of values $c(G)$ over all finite connected graphs G .*

Proof. Combine Propositions 3.2 and 3.8. $\square$

4. THE VALUES UP TO SIXTEEN

For $m \geq 0$ let

$$\begin{aligned}\Lambda_m &= \{c(K) : K \text{ a graph with exactly } m \text{ vertices}\}, \\ V_m &= \{c(G) : G \text{ a connected graph with exactly } m \text{ vertices}\},\end{aligned}$$

so that $\Lambda_0 = \{0\}$ and $V_m \subseteq \Lambda_m$. By Lemma 2.1(f), $V_m \subseteq [T(m), 2^m - 1]$.

Lemma 4.1. $V_1 = \{1\}$, $V_2 = \{3\}$, $V_3 = \{6, 7\}$ and $V_4 = \{10, 11, 12, 13, 14, 15\}$.

Proof. Up to isomorphism, the connected graphs on at most three vertices are K_1 , K_2 , P_3 and K_3 , and the six connected graphs on four vertices are the path, the star, the paw (a triangle with one pendant vertex), the cycle, the diamond (K_4 minus an edge) and K_4 . Lemma 2.1 and Example 2.2 give 10, 11, 12, 13, 14 and 15. For the cycle, the connected sets are the 12 proper cyclic intervals of vertices together with the whole vertex set. These agree with [2, Figure 3]. $\square$

Proposition 4.2. No connected graph G satisfies $c(G) = 16$.

Proof. Let G be connected with n vertices and $c(G) = 16$. By Lemma 2.1(f), $T(n) \leq 16$, so $n \leq 5$, and $2^n - 1 \geq 16$, so $n \geq 5$. Thus $n = 5$. The trees on five vertices are the path P_5 , the spider with legs of lengths 2, 1 and 1, and the star $K_{1,4}$. By Example 2.2(i) they have $c = 15$, $3 \cdot 2 \cdot 2 + 3 + 1 + 1 = 17$ and $2^4 + 4 = 20$. Let S be a spanning tree of G . If S is the spider or the star, then $c(G) \geq c(S) \geq 17$ by Lemma 2.1(b).

Suppose $S = P_5$, with vertices $v_1, \dots, v_5$ in order, and $G \neq P_5$. Then G contains P_5 plus a chord $v_a v_b$ with $b \geq a + 2$. A set that is connected in $P_5 + v_a v_b$ but not in P_5 consists of two intervals of the path, one containing v_a and the other containing v_b , separated by at least one vertex. So these sets are $\{v_\alpha, \dots, v_x\} \cup \{v_y, \dots, v_\beta\}$ with $\alpha \leq a \leq x$ and $x + 2 \leq y \leq b \leq \beta$. The outer ends can be chosen in a ways ($1 \leq \alpha \leq a$) and in $6 - b$ ways ($b \leq \beta \leq 5$). The inner ends must satisfy $a \leq x$, $x + 2 \leq y \leq b$, which leaves $T(b - a - 1)$ choices. Thus the chord adds $a(6 - b)T(b - a - 1)$ connected sets. For the six possible chords $(a, b) = (1, 3), (2, 4), (3, 5), (1, 4), (2, 5), (1, 5)$ this number is 3, 4, 3, 6, 6, 6, so it is always at least 3. Hence $c(G) \geq c(P_5 + v_a v_b) \geq 18$ by Lemma 2.1(b). In every case $c(G) \neq 16$. $\square$

Table 1 lists representatives of the 21 isomorphism classes of connected graphs on five vertices; with Lemma 4.1 it covers all 31 classes of connected graphs with at most five vertices. It confirms that $V_5 = \{15\} \cup [17, 24] \cup [26, 31]$.

Corollary 4.3. The values of c on connected graphs that are at most 16 are 1, 3, 6, 7 and the integers from 10 to 15.

Proof. Graphs with at least six vertices have $c \geq 21$ by Lemma 2.1(f). Apply Lemma 4.1, Proposition 4.2 and the bound $c \geq 15$ for five vertices. $\square$

5. EVERY INTEGER AT LEAST SEVENTEEN

The construction works with all graphs, not only connected ones. The reason is the cone of Lemma 2.1(d). If H has $m - 1$ vertices and $c(H) = x$, then the cone over H is a connected graph on m vertices with $c = 2^{m-1} + x$. So the natural auxiliary object is the set Λ_m of values of c on all graphs with m vertices. Cones alone do not suffice. Every graph on $m - 1$ vertices has at least $m - 1$ connected sets, its singletons, so cones give only values $2^{m-1} + x$ with $x \geq m - 1$. They miss the $m - 1$ values from 2^{m-1} to $2^{m-1} + m - 2$, at the bottom of the interval $[2^{m-1}, 2^m - 1]$. The near-cones defined below fill this gap.

	edges	c		edges	c
1	04 13 23 24	15	12	03 04 12 14 23 34	24
2	04 14 23 34	17	13	04 12 13 14 23 24 34	24
3	04 12 13 23 34	18	14	03 04 13 14 23 24	26
4	04 13 23 24 34	19	15	03 04 12 14 23 24 34	26
5	04 14 24 34	20	16	03 04 13 14 23 24 34	27
6	04 14 23 24 34	21	17	03 04 12 13 14 23 24	27
7	04 13 14 23 24	21	18	03 04 12 13 14 23 24 34	28
8	03 04 12 14 23	21	19	02 03 04 12 13 14 24 34	29
9	04 12 13 23 24 34	22	20	02 03 04 12 13 14 23 24 34	30
10	03 04 12 14 24 34	22	21	all ten edges	31
11	04 13 14 23 24 34	23			

TABLE 1. Representatives of the 21 isomorphism classes of connected graphs on five vertices, with their values of c . Each representative is the canonical labeled graph of Appendix A, on the vertex set $\{0, 1, 2, 3, 4\}$; the edge ij is written ij . The value 16 does not occur; the value 25 first occurs on six vertices.

5.1. Three rules and seven small graphs.

Definition 5.1. Let $d \geq 1$, $r \geq 0$, and let R be a graph with r vertices. The *near-cone* $N(d, R)$ is obtained as follows: take a star $K_{1,d}$ with centre u and leaves $\ell_1, \dots, \ell_d$ and a disjoint copy of R , and add a vertex w adjacent to every vertex except u (Figure 2). Formally, its vertices are $u, \ell_1, \dots, \ell_d, w$ and the vertices of R , and its edges are those of R , the edges $u\ell_t$ and $w\ell_t$ for $1 \leq t \leq d$, and the edges joining w to every vertex of R . Thus $N(d, R)$ is connected and has $m = d + r + 2$ vertices.

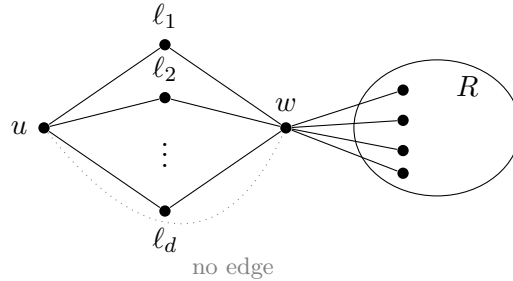


FIGURE 2. The near-cone $N(d, R)$. The vertex w is joined to $\ell_1, \dots, \ell_d$ and to every vertex of R ; the centre u of the star is joined only to the ℓ_t .

Lemma 5.2. Let $m \geq 1$.

- (U) If $x \in V_a$ and $y \in \Lambda_{m-a}$ with $1 \leq a \leq m$, then $x + y \in \Lambda_m$. In particular $\Lambda_{m-1} + 1 \subseteq \Lambda_m$.
- (C) $2^{m-1} + \Lambda_{m-1} \subseteq V_m$.
- (N) If $d \geq 1$, $r \geq 0$ and $d + r + 2 = m$, then $2^{m-1} - 2^r + 2^d + d + \Lambda_r \subseteq V_m$. More precisely,

$$c(N(d, R)) = 2^{m-1} - 2^r + 2^d + d + c(R).$$

Proof. (U) is Lemma 2.1(a) (add a disjoint connected graph, for instance an isolated vertex), and (C) is Lemma 2.1(d).

For (N), let $G = N(d, R)$ and count its connected sets according to whether they contain w . The connected sets avoiding w are those of $G - w = K_{1,d} \sqcup R$; there are $2^d + d + c(R)$ of them,

by Lemma 2.1(a) and Example 2.2(i). The sets containing w are the 2^{m-1} sets $\{w\} \cup U$ with $U \subseteq V(G) \setminus \{w\}$. Since w is adjacent to every vertex other than u , and u is adjacent only to the leaves, such a set is disconnected exactly when U contains u but no leaf ℓ_t . There are 2^r such sets U : they contain u , no leaf, and an arbitrary set of vertices of R . Hence $c(G) = 2^d + d + c(R) + 2^{m-1} - 2^r$. This is also the case $K = K_{1,d} \sqcup R$ of Lemma 2.1(e). $\square$

Lemma 5.3. *The seven graphs of Table 2 and Figure 3 have the stated values. Hence $19 \in V_5$; $25, 32, 35, 57 \in V_6$; and $67, 90 \in V_7$.*

c	m	graph	count
19	5	triangle xyz with one pendant vertex at x and one at y	Ex. 2.2(ii): $3 \cdot 3 \cdot 2 - 1 + 2$
32	6	triangle xyz with two pendant vertices at x and one at y	Ex. 2.2(ii): $5 \cdot 3 \cdot 2 - 1 + 3$
35	6	K_4 minus an edge ab , with one pendant vertex at a and one at a vertex z of degree 3	see the proof
57	6	K_6 minus two disjoint paths with three vertices each	Ex. 2.2(iii): $63 - 3 - 3$
67	7	$K_{2,3}$ with two pendant vertices at a vertex x of the larger side	Lem. 2.1(c): $13 \cdot 4 + 13 + 2$
90	7	K_6 minus two edges va, vb , with a pendant vertex at v	Lem. 2.1(c): $29 \cdot 2 + 31 + 1$
25	6	spider with legs of lengths 1, 2, 2	Ex. 2.2(i): $2 \cdot 3 \cdot 3 + 1 + 3 + 3$

TABLE 2. The sporadic graphs, drawn in Figure 3. The six graphs above the line fill the gaps left by the rules (U), (C) and (N) in Lemma 5.4 (Remark 5.5). The spider below the line realizes 25 on six vertices (Corollary 5.8).

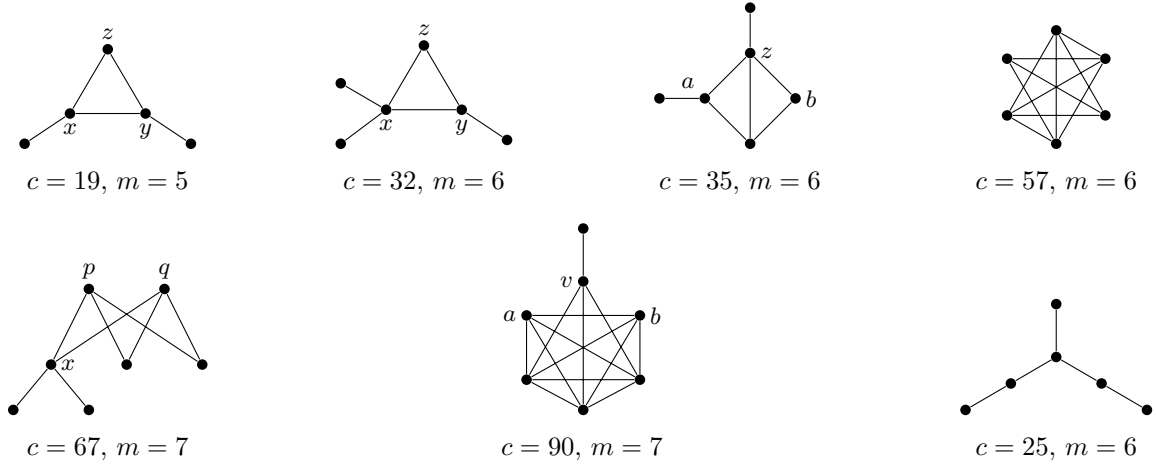


FIGURE 3. The seven sporadic graphs of Table 2, with their values of c and their numbers m of vertices. In the graph with $c = 57$, the hexagon is drawn without four of its sides; these form the two removed paths with three vertices each.

Proof. The values 19, 25, 32 and 57 follow from Example 2.2; for 57, Φ consists of two copies of $K_{1,2}$.

For $c = 35$: a connected set is either one of the two pendant vertices, or a connected set Q of $K_4 - ab$ together with any of the pendant vertices attached at Q . The connected sets of $K_4 - ab$ are all nonempty subsets of its vertex set other than $\{a, b\}$, hence 14 sets. Summing $2^{|Q \cap \{a, z\}|}$ over all nonempty Q gives $3 \cdot 3 \cdot 2 \cdot 2 - 1 = 35$; removing $Q = \{a, b\}$ subtracts 2, and the pendant vertices add 2, giving 35.

For $c = 67$: glue $K_{2,3}$ and the star $K_{1,2}$ at x (Lemma 2.1(c)). Let $\{p, q\}$ be the smaller side and $\{x, y, z\}$ the larger one. The connected sets of $K_{2,3}$ containing x are $\{x\}$ and the sets $\{x\} \cup S \cup S'$ with $\emptyset \neq S \subseteq \{p, q\}$ and $S' \subseteq \{y, z\}$; there are $1 + 3 \cdot 4 = 13$. Also $K_{2,3} - x = C_4$ has $c = 13$, and the star contributes $c_x = 4$ and $c(K_{1,2} - x) = 2$.

For $c = 90$: let $H = K_6 - va - vb$ and glue H with an edge at v . In H , the sets containing v that are disconnected are $\{v, a\}$, $\{v, b\}$ and $\{v, a, b\}$, so $c_v(H) = 32 - 3 = 29$; and $H - v = K_5$ has $c = 31$. The edge contributes $c_v = 2$ and 1. $\square$

5.2. The base.

Lemma 5.4. (a) $\Lambda_3 \supseteq \{3, 4, 6, 7\}$, $\Lambda_4 \supseteq [4, 8] \cup [10, 15]$, $\Lambda_5 \supseteq [5, 24] \cup [26, 31]$, $\Lambda_6 \supseteq [6, 25] \cup [27, 63]$ and $\Lambda_7 \supseteq [7, 127]$.

(b) $V_5 \supseteq [17, 24] \cup [26, 31]$, $V_6 \supseteq [32, 63]$ and $V_7 \supseteq [64, 127]$.

Proof. We apply the rules of Lemma 5.2, starting from $V_1 = \Lambda_1 = \{1\}$ and $\Lambda_0 = \{0\}$, one order at a time. At each order we use only the inclusions already established at smaller orders, and we record only inclusions. We give the order $m = 5$ in full; Table 3 lists every order.

For $m = 5$, the rule (C) applied to $\Lambda_4 \supseteq [4, 8] \cup [10, 15]$ gives $16 + ([4, 8] \cup [10, 15]) = [20, 24] \cup [26, 31]$. The rule (N) with $(d, r) = (1, 2)$ gives $2^4 - 2^2 + 2^1 + 1 + \Lambda_2 \supseteq 15 + \{2, 3\} = \{17, 18\}$. The sporadic graph with $c = 19$ (Lemma 5.3) supplies the remaining value, so $V_5 \supseteq [17, 24] \cup [26, 31]$. By (U), $\Lambda_4 + 1 \supseteq [5, 9] \cup [11, 16]$ and $7 + 3 = 10 \in V_3 + \Lambda_2$. Together with V_5 , this gives $\Lambda_5 \supseteq [5, 24] \cup [26, 31]$. $\square$

m	cones (C)	near-cones (N), with (d, r)	sporadic	$V_m \supseteq$	$\Lambda_m \supseteq$ by (U) and V_m
2	$2 + \{1\} = \{3\}$	—	—	$\{3\}$	$\{1\} + 1$; so $\{2, 3\}$
3	$4 + \{2, 3\} = \{6, 7\}$	—	—	$\{6, 7\}$	$\{2, 3\} + 1$; so $\{3, 4, 6, 7\}$
4	$8 + \{3, 4, 6, 7\} = \{11, 12, 14, 15\}$	$(1, 1)$: 10; $(2, 0)$: 13	—	$[10, 15]$	$\{3, 4, 6, 7\} + 1$, $3 + 3$; so $[4, 8] \cup [10, 15]$
5	$16 + \Lambda_4$: $[20, 24] \cup [26, 31]$	$(1, 2)$: $15 + \{2, 3\} = \{17, 18\}$	19	$[17, 24] \cup [26, 31]$	$[5, 9] \cup [11, 16]$, $7 + 3$; so $[5, 24] \cup [26, 31]$
6	$32 + \Lambda_5$: $[37, 56] \cup [58, 63]$	$(1, 3)$: $27 + \{3, 4, 6, 7\} = \{30, 31, 33, 34\}$; $(2, 2)$: $34 + \{2, 3\} = \{36, 37\}$	32, 35, 57	$[32, 63]$	$[6, 25] \cup [27, 32]$; so $[6, 25] \cup [27, 63]$
7	$64 + \Lambda_6$: $[70, 89] \cup [91, 127]$	$(1, 4)$: $51 + [13, 15] = [64, 66]$; $(2, 3)$: $62 + \{6, 7\} = \{68, 69\}$	67, 90	$[64, 127]$	$[7, 26] \cup [28, 64]$, $24 + 3$; so $[7, 127]$

TABLE 3. The steps of the proof of Lemma 5.4. In the cone and near-cone columns, Λ_r stands for the part of Λ_r established in the previous rows. The column (U) lists $\Lambda_{m-1} + 1$ and the sums $x + y$ with $x \in V_a$, $y \in \Lambda_{m-a}$ that are needed; for instance $3 + 3 \in V_2 + \Lambda_2$ and $24 + 3 \in V_5 + \Lambda_2$.

Remark 5.5. Two statements must be distinguished here: a value may fail to be produced by the rules (C) and (N), and a value may fail to be realized by any graph. Lemma 5.4 is sharp in the second sense: $5 \notin \Lambda_3$, $9 \notin \Lambda_4$, $25 \notin \Lambda_5$, $26 \notin \Lambda_6$ and $16, 25 \notin V_5$, by the computation of Section 7.

In the first sense, even with every sporadic graph of smaller order available, the rules (C) and (N) produce on five vertices every value in $[17, 31]$ except 19 and 25, on six vertices every value in $[32, 63]$ except 32, 35 and 57, and on seven vertices every value in $[64, 127]$ except 67 and 90 (Section 7). All these values except 25 do lie in V_m , and the six sporadic graphs above the line in Table 2 realize them. The remaining sporadic graph realizes 25 on six vertices.

5.3. The induction.

Theorem 5.6. *For every $m \geq 8$:*

$$(G_m) \quad V_m \supseteq [2^{m-1}, 2^m - 1];$$

$$(H_m) \quad \Lambda_m \supseteq [m, 2^m - 1].$$

Together with Lemma 5.4, (G_m) holds for all $m \geq 6$ and (H_m) for all $m \geq 7$.

The two assertions play different roles. The assertion (H_m) provides auxiliary graphs, possibly disconnected, with every count from m to $2^m - 1$; the assertion (G_m) provides connected graphs for the upper half of the range. We prove (G_m) from earlier (H) -statements, and then obtain (H_m) by combining (G_m) with the isolated-vertex rule (U). In the proof of (G_m) the interval splits as

$$[2^{m-1}, 2^m - 1] = \underbrace{[2^{m-1}, 2^{m-1} + m - 2]}_{\text{near-cones}} \cup \underbrace{[2^{m-1} + m - 1, 2^m - 1]}_{\text{cones}}.$$

Proof. We use induction on $m \geq 8$. Assume $(H_{m'})$ for $7 \leq m' < m$; for $m' = 7$ this is Lemma 5.4.

Proof of (G_m) . By (C) and (H_{m-1}) , $V_m \supseteq 2^{m-1} + [m-1, 2^{m-1} - 1] = [2^{m-1} + m - 1, 2^m - 1]$. It remains to realize $2^{m-1} + j$ for $0 \leq j \leq m - 2$. By (N), it suffices to find $d \geq 1$ and $r \geq 0$ with $d + r + 2 = m$ and

$$(2) \quad x_j := 2^r - 2^d - d + j \in \Lambda_r.$$

The parameters must satisfy two constraints. For the largest $j = m - 2$ we need $x_j \leq 2^r - 1$, which holds exactly when $2^d + d \geq m - 1$. For the smallest $j = 0$ we need $x_j \geq r$, which requires 2^d to be small compared with 2^r . For $m \geq 12$ we take the least d with $2^d + d \geq m - 1$: this meets the first constraint and keeps d close to $\log_2 m$, so that $r = m - 2 - d$ is large. For $8 \leq m \leq 11$ this choice leaves r too small, and we use Lemma 5.4(a) instead:

- $m = 8$: $(d, r) = (2, 4)$, $x_j = 10 + j \in [10, 15]$ for $j \leq 5$; $(3, 3)$, $x_6 = 3$.
- $m = 9$: $(2, 5)$, $x_j = 26 + j \in [26, 31]$ for $j \leq 5$; $(3, 4)$, $x_6 = 11$ and $x_7 = 12$.
- $m = 10$: $(3, 5)$, $x_j = 21 + j \in [21, 29] \setminus \{25\}$ for $j \neq 4$; $(2, 6)$, $x_4 = 62$.
- $m = 11$: $(3, 6)$, $x_j = 53 + j \in [53, 62]$.

Now let $m \geq 12$, let d be the least integer $d \geq 1$ with $2^d + d \geq m - 1$, and put $r = m - 2 - d$. Then $d \geq 3$. If $d = 3$, then $m = 12$ and $r = 7$. If $d \geq 4$, minimality of d gives $2^{d-1} + d - 1 < m - 1$, so $r = m - 2 - d \geq 2^{d-1} - 1 \geq 7$ and $r \geq d + 1$. In both cases $r \geq 7$ and $r \geq d + 1$. Then

$$x_j \leq 2^r - 2^d - d + m - 2 \leq 2^r - 1,$$

because $m - 1 \leq 2^d + d$. Also

$$x_j \geq 2^r - 2^d - d \geq 2^r - 2^{r-1} - (r - 1) = 2^{r-1} - r + 1 \geq r,$$

because $r \geq d + 1 \geq 4$. So $x_j \in [r, 2^r - 1] \subseteq \Lambda_r$ by (H_r) , which holds since $7 \leq r < m$.

Proof of (H_m) . By (U) and (H_{m-1}) , $\Lambda_m \supseteq [m - 1, 2^{m-1} - 1] + 1 = [m, 2^{m-1}]$. By (G_m) , $\Lambda_m \supseteq V_m \supseteq [2^{m-1}, 2^m - 1]$. $\square$

Example 5.7. Take $k = 128 = 2^7$. Then $m = 8$ and $j = 0$, and the proof uses $(d, r) = (2, 4)$, with $x_0 = 2^4 - 2^2 - 2 = 10 = c(P_4)$. So the near-cone $N(2, P_4)$, formed by the path P_4 and the vertices u, ℓ_1, ℓ_2, w , is a connected graph with 8 vertices and

$$c(N(2, P_4)) = 2^7 - 2^4 + 2^2 + 2 + 10 = 128.$$

Seven vertices are not enough, since then $c \leq 2^7 - 1$.

Corollary 5.8. *Every integer $k \geq 17$ equals $c(G)$ for some connected graph G . If $k \geq 32$, or $17 \leq k \leq 31$ and $k \neq 25$, then G can be chosen with $\lfloor \log_2 k \rfloor + 1$ vertices, the least possible number. The value 25 needs six vertices.*

Proof. Let $m = \lfloor \log_2 k \rfloor + 1$, so $2^{m-1} \leq k \leq 2^m - 1$. If $m \geq 6$, then $k \in V_m$ by Theorem 5.6. If $m = 5$ and $k \neq 25$, then $k \in V_5$ by Lemma 5.4. The spider of Table 2 gives $25 \in V_6$, and $25 \notin V_5$ by Table 1. Fewer than m vertices are impossible, because $c \leq 2^{m-1} - 1 < k$ on $m - 1$ vertices. $\square$

Remark 5.9 (An algorithm). The proofs are constructive. For $m \geq 7$ and $v \in [m, 2^m - 1]$, the following recursive procedure returns a graph $\text{Re}(m, v)$ with m vertices and $c = v$, which is connected when $v \geq 2^{m-1}$.

- (1) If $m = 7$, return the graph given by the proof of Lemma 5.4.
- (2) If $v < 2^{m-1}$, return $\text{Re}(m - 1, v - 1)$ together with an isolated vertex.
- (3) If $v \geq 2^{m-1} + m - 1$, return the cone over $\text{Re}(m - 1, v - 2^{m-1})$.
- (4) Otherwise put $j = v - 2^{m-1}$, choose (d, r) as in the proof of Theorem 5.6, and return $N(d, R)$. Here $R = \text{Re}(r, x_j)$ if $r \geq 7$, and R is the graph given by the proof of Lemma 5.4 if $r \leq 6$.

Each step makes at most one recursive call, on fewer vertices. Apart from that call, it uses $O(m)$ operations on integers with at most m bits and writes $O(m^2)$ edges. Hence $\text{Re}(m, v)$ is computed in time polynomial in m . For $k \geq 64$, the graph $\text{Re}(m, k)$ with $m = \lfloor \log_2 k \rfloor + 1 \geq 7$ is connected, has the least possible number of vertices, and is found in time polynomial in $\log k$. For $32 \leq k \leq 63$ the procedure does not apply, since $m = 6$; a connected graph with six vertices and $c = k$ is given by the proof of Lemma 5.4(b). The procedure cannot simply be extended to $m = 6$, because $26 \notin \Lambda_6$. For example, $c = 10^{100}$ is realized on 333 vertices. Section 7 reports a check of this procedure.

6. PROOFS OF THE MAIN RESULTS

Proof of Theorem A. (1) $\Leftrightarrow$ (2) is Theorem 3.10. (2) $\Rightarrow$ (3) is Corollary 4.3, which excludes exactly 2, 4, 5, 8, 9, 16 below 17. (3) $\Rightarrow$ (2): for $k \leq 15$ use Lemma 4.1, and for $k \geq 17$ use Corollary 5.8.

For the last assertions, let $k \geq 32$ and let G be the graph of Corollary 5.8, with $m = \lfloor \log_2 k \rfloor + 1$ vertices. Then X_G is a union of m Knaster continua, any two meeting in at most one point (Lemma 3.5(c)), and $|\mathcal{D}(X_G)| = k$ by Proposition 3.8. Conversely, if $|\mathcal{D}(X)| = k$ and $\mathcal{D}(X)$ has n minimal elements, then $k = c(\Gamma) \leq 2^n - 1$ by Proposition 3.2 and Lemma 2.1(f). Hence $n \geq m$. $\square$

Proof of Corollary B. (a) is the case $k \geq 256$ of Theorem A. For (b), a finite metric space is discrete, so two finite metric spaces are homeomorphic exactly when they have the same cardinality; also $\mathcal{D}(X) \neq \emptyset$. Apply Theorem A. For (c), $2^m \in \{2, 4, 8, 16\}$ for $1 \leq m \leq 4$, while $2^0 = 1$ and $2^m \geq 32$ for $m \geq 5$ are allowed values. $\square$

7. COMPUTATIONAL VERIFICATION

The proofs above do not use a computer. We checked every explicit finite graph count and certificate in this paper, and the computer-assisted results of [2], with programs written for this purpose. No program uses SageMath or nauty.

Counting. We compute $c(G)$ in three ways.

- (i) By testing all $2^n - 1$ nonempty vertex sets.
- (ii) By the ESU enumeration scheme [15], which lists each connected set once, in Python and in C.
- (iii) By a recurrence. For a nonempty vertex set S let $g(S) = 1$ if $G[S]$ is connected and $g(S) = 0$ otherwise. The component of $G[S]$ containing $\min S$ is the only set $T \subseteq S$ with $\min S \in T$, $G[T]$ connected, and no edge between T and $S \setminus T$. So the sum of $g(T)$ over all sets $T \subseteq S$ with $\min S \in T$ and no edge between T and $S \setminus T$ equals 1. Hence $g(S) = 1 - \sum g(T)$, the sum running over such sets $T \neq S$.

Generation of graphs on at most eight vertices. Program A takes the graphs on at most seven vertices from the atlas of Read and Wilson [12], as distributed with NetworkX [5]. Deleting a vertex from a graph on eight vertices leaves a graph on seven vertices, which need not be connected. So A extends each of the 1044 graphs on seven vertices, connected or not, by every nonempty neighbourhood of a new vertex. It removes isomorphic copies in two independent ways: by Weisfeiler–Lehman hashing followed by an exact isomorphism test, and by comparing canonical graph6 strings (Appendix A). Both give the 11117 connected graphs on eight vertices of OEIS A001349 [8]. Program B, written in C, uses no atlas and no isomorphism test. It runs over all labeled graphs on at most seven vertices, and over the graphs on eight vertices obtained by extending the labeled graphs on seven vertices with nonincreasing degree sequence; every graph has such a labeling. Its numbers of labeled connected graphs agree with OEIS A001187.

Results for small orders. Program A counts with methods (i) and (ii), program B with (iii), and they give the same set V_n for every $n \leq 8$. Every value in $[1, 255]$ except 2, 4, 5, 8, 9, 16 occurs, and every value in $[17, 127]$ occurs on at most seven vertices. This confirms [2, Remark 3.20]; we found no discrepancy.² For each $n \leq 8$, V_n contains every integer in $[T(n), 2^n - 1]$ except 16, 25 ($n = 5$); 22, 23, 26 ($n = 6$); 29, 30, 31, 35 ($n = 7$); and 37, 38, 39, 40, 43, 50, 55 ($n = 8$). Program A selects the witnesses of Appendix A; program B rederives each witness on at most seven vertices by brute force over all labeled graphs, and confirms the least edge counts on eight vertices.

The construction of Section 5. We implemented Lemma 5.4 and the procedure of Remark 5.9 literally, and checked the gaps stated in Remark 5.5. An independently written checking program re-checked each of the 353 derivations for $m \leq 7$. For every $k \in [17, 70000]$, and for 300 random $k < 2^{22}$, we built the resulting graph and counted its connected sets with the C version of method (ii). For every $8 \leq m \leq 16$ and every $v \in [m, 2^m - 1]$, the graph $\text{Re}(m, v)$ has exactly m vertices, is connected when $v \geq 2^{m-1}$, and has $c = v$; in total 130708 graphs were checked.

Subtrees. We also re-verified the computer-assisted parts of [3]: the list of 34 exceptions, from all trees with at most 18 vertices, generated and counted in two independent ways; the caterpillar search; and the base case of [3, Proposition 3]. The details are in the supplementary material.

Formalization in Lean. The supplementary material also contains a formalization in Lean 4 with Mathlib, in the directory `lean`. It proves that a positive integer is a value of c on a finite connected graph if and only if it lies outside $\{2, 4, 5, 8, 9, 16\}$, which is the equivalence of (2) and (3) in Theorem A, together with Corollary 5.8, including the exception 25. The proof follows Sections 2, 4 and 5: the rules (U), (C) and (N), the sporadic graphs, the 353 derivations of Lemma 5.4, and Theorem 5.6. The values 16 and 25 on five vertices are excluded by a kernel computation over all 2^{10} labeled graphs on five vertices, using a counting procedure proved equal to c .

The topological part is formalized as well. For pieces with the properties of Lemma 3.5, the formalization proves Lemmas 3.1, 3.6 and 3.7, Proposition 3.8 and the counting argument of Proposition 3.2, including the inclusion of (3.1) that we supplied. It proves Proposition 3.15 and Theorem 3.16 of [2] for spaces with finite $\mathcal{D}(X)$, along the lines of their proofs. Finiteness gives the maximal elements directly, and the formal proof uses only that $\mathcal{D}(X)$ is finite and that X is connected. The Knaster continuum is constructed as the inverse limit of the tent map $t \mapsto 1 - |2t - 1|$ on $[0, 1]$, a standard model of the bucket handle. The formalization proves that it is an infinite indecomposable metrizable continuum without isolated points. Construction 3.4 is realized without a quotient, as a subspace of $Q \times \mathbb{R}^k$. The copy indexed by i is the graph of $x \mapsto g_i(x)e_i$, where e_i is the i th unit vector and g_i is a continuous function on Q vanishing exactly at the attachment points of that copy. Two copies then meet exactly at their common attachment points. This space is homeomorphic to X_G for a particular choice of attachment points, since the induced map from

²The program in [2, Remark 3.20] contains a de-duplication loop that compares the methods `vertex_iterator` without calling them, so it removes nothing. The result is unaffected, because `connected_subgraph_iterator` lists each connected induced subgraph once by default.

X_G is a continuous bijection from a compact space onto a Hausdorff space. The formal proof does not use this: it proves Lemma 3.5 for this space directly.

Theorem A, including the least number of pieces, and Corollary B(a) and (c) are thus proved with no hypotheses. Not formalized are Corollary B(b), which concerns the topology of $\mathcal{D}(X)$, and the computations of this section. The formalization uses neither `sorry` nor `native_decide`, and its results depend only on the axioms `propext`, `Classical.choice` and `Quot.sound`. It was replayed through the kernel with `leanchecker` and checked by the independent type checker `nanoda`.

Supplementary material. The programs, their outputs and the Lean formalization form the supplementary material of this paper, version 2.3. It is available, together with the companion note [6] and the first prompt given to Claude, at <https://github.com/LucasAlves8Sp/ALL-FINITE-CARDINALITIES-OF-THE-HYPERSPACE-OF-REGULAR-SUBCONTINUA>. One script reruns all computations in about five minutes and reproduces every output file exactly. The file `CHECKS.md` maps each computational statement of this paper to the program that checks it and to its output, and explains how each generator is complete and how it removes duplicates. The file `MANIFEST.txt` lists SHA-256 checksums of all files.

AI DISCLOSURE

Claude Opus 5.5 (Anthropic) was used extensively as a generative research and writing tool, and most of the work was done by it. In an iterative process directed by the author, the model carried out most of the exploratory proof development, found the near-cones of Section 5, wrote and ran the programs of Section 7, and generated most of the first draft of the manuscript and the accompanying Lean 4 formalization. The author formulated the problem and the objectives, namely Questions 3.5 and 3.21 of [2], provided the mathematical context and constraints, and steered the iterations through prompts and feedback. The author also supplied ideas and suggested strategies: the reduction to graphs through [2, Theorem 3.16], the realization by Knaster continua glued at points, the exclusion of 16, and the theorem of [3] on subtrees as a fallback. The author selected and revised the outputs and reviewed the final arguments.

The manuscript then went through many rounds of writing, revision and checking. In each round the author, with ChatGPT-Astra (OpenAI), reviewed the paper, the proofs and the supplementary material and sent feedback, and Claude revised the text, the programs and the formalization; AI was used again for proofreading and review. The explicit finite graph counts are checked by the programs of the supplementary material, and Theorem A is formally verified in Lean. No independent expert has yet verified the proofs. The author takes full responsibility for the content.

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APPENDIX A. WITNESSES FOR THE VALUES FROM 17 TO 127

A *witness* for an integer k is a connected graph G with $c(G) = k$. We record witnesses as graph6 strings. A *graph6 string* is a standard printable encoding of a finite labeled graph with vertex set $\{0, \dots, n-1\}$: its first character encodes n , and the remaining characters encode the entries above the diagonal of the adjacency matrix, read column by column, six bits per character.

Canonical strings. A graph6 string depends on the labeling. For a fixed n all graph6 strings have the same length, and we compare them lexicographically by their ASCII characters. The *canonical graph6 string* of a graph is the least graph6 string among all its labelings with vertex set $\{0, \dots, n-1\}$; two graphs are isomorphic if and only if their canonical strings coincide. A labeling attaining the canonical string need not be unique, since automorphisms can give several, but the labeled graph on $\{0, \dots, n-1\}$ that it produces is unique; we call it the *canonical labeled graph*. Table 1 and this appendix use canonical labeled graphs.

Witness rule. For each $k \in [17, 127]$ the table gives a witness: a connected graph G with $c(G) = k$ of least possible order n ; among those, of least edge count; and among those, the one with the least canonical graph6 string. Equivalently, if n is the least order of a connected graph with $c = k$, the witness is the labeled graph on $\{0, \dots, n-1\}$ with $c = k$ and least edge count whose graph6 string is least. The table lists k , n and the canonical string. Program B of Section 7 rederives every entry by brute force over all labeled graphs on at most seven vertices. Edge lists are in the supplementary file `witnesses_small.csv`.

k	n	graph6	k	n	graph6	k	n	graph6
17	5	D@s	54	6	EB~o	91	7	F?Nv_
18	5	DJc	55	6	EFz_	92	7	F?Fno
19	5	DBk	56	6	EFzg	93	7	F@Q~o
20	5	D?{	57	6	EL~o	94	7	F?Fnw
21	5	D@{	58	6	EFzw	95	7	F?]~_
22	5	DJk	59	6	EF~w	96	7	F@Vn_
23	5	DB{	60	6	E]~o	97	7	F@N^W
24	5	DLs	61	6	E]~w	98	7	F?Nvo
25	6	E@QW	62	6	E~~w	99	7	F@vf_
26	5	DFw	63	6	E~~w	100	7	F?B~o
27	5	DF{	64	7	F?Dn0	101	7	F?B~w
28	5	DN{	65	7	F??~o	102	7	F?F~w
29	5	D]{	66	7	F?C^W	103	7	F?N~o
30	5	D~{	67	7	F?KuW	104	7	F?~vo
31	5	D~{	68	7	F?Cng	105	7	F@~vo
32	6	E?NW	69	7	F?@ w	106	7	F@vv_
33	6	E@Ug	70	7	F??Fw	107	7	F?~^o
34	6	EJ]G	71	7	F??Nw	108	7	FBnn0

k	n	graph6	k	n	graph6	k	n	graph6
35	6	E@Uw	72	7	F?Cfw	109	7	F@vvo
36	6	E?No	73	7	F??^w	110	7	FBzv_
37	6	E?Bw	74	7	F?C^w	111	7	F@vvw
38	6	E?Fw	75	7	F@Q^G	112	7	F?^v_
39	6	E@Rw	76	7	F?Cnw	113	7	F?^vg
40	6	E?Nw	77	7	F??^w	114	7	F@^vg
41	6	E@Nw	78	7	F?Dno	115	7	F?^vw
42	6	E?^o	79	7	F@Um_	116	7	F?^~w
43	6	E@Vw	80	7	F?LVw	117	7	F@^~w
44	6	E?^w	81	7	F@Q^W	118	7	FFzng
45	6	E@vg	82	7	F?Dnw	119	7	FFzfw
46	6	EBn_	83	7	F?@^o	120	7	FFznw
47	6	E@vo	84	7	F?D^o	121	7	FFz^o
48	6	EBz_	85	7	F?@^w	122	7	FFz^w
49	6	E@vw	86	7	F?Fng	123	7	FF^~w
50	6	EJfw	87	7	F@Umg	124	7	F]^vw
51	6	E?^o	88	7	F?NNg	125	7	F]^~w
52	6	E?^w	89	7	F?N^_	126	7	F^~^w
53	6	E@^w	90	7	F@]v_	127	7	F^~^w

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