

THE HEIGHT–CONDUCTOR RATIO ON CURVE TOWERS: CLOSED FORMS FOR THE FERMAT TOWER, THE ONE-THIRD BARRIER, AND THE ATTAINABLE SET

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ABSTRACT. For a tower of curves X_n over a number field with fixed bad-reduction set and $g_n \rightarrow \infty$, we study the stable Faltings height to conductor ratio of its Jacobian, $\kappa_n := h_F(\text{Jac}(X_n))/\log \text{cond}(\text{Jac}(X_n))$. We prove four groups of results. (I) Closed forms for the Fermat tower: slope $\frac{\log p}{4}n$ and second-order intercept $-6 \log A$ (A the Glaisher–Kinkelin constant), yielding $\kappa_{\text{Fermat}} = \frac{1}{8}$; the orbit stratification is strictly increasing and converges to $\frac{1}{8}$ from below. (II) A one-third barrier $\kappa \leq \frac{1}{3}$ for GL_2 -type towers with $\sigma \rightarrow 1$, conditional on arithmetic BMY in general (Problem 10.1) and unconditional on modular towers, where the BMY input is a theorem (Dolce–Mercuri, Mayer) and where $X_0(N)$ attains equality asymptotically, so the constant cannot be improved. (III) The barrier is strictly stronger than BMY: $\kappa \leq \frac{1}{3} \Rightarrow R \leq \frac{4}{\sigma} - 1 \leq 3$. (IV) The closure of the attainable set in the abelian-variety tower world is $[\frac{1}{12}, \frac{1}{3}]$: the lower end is unconditional (Zhang, Wilms) and attained by Heisenberg towers, the upper end is the barrier and is attained by modular towers, and the interior is densely (countably) filled by reindexed product convexity; the curve-tower interior filling remains open. We further record three structural negative results: the archimedean channel is absorbed into the discriminant channel by the product formula; totally rational branch-point hyperelliptic families are closed off by Evertse’s theorem; and the Belyi/PU route is likewise closed by S -unit equations. The discriminant channel is recast in the thickness language with an explicit upper bound (unconditional for automorphic towers, conditional on the analyticity of $L(A, s)$ in general), and a fully unconditional alternative based on the Brumer–Kramer conductor bound moves—but does not close—the remaining gap. All conditional dependencies are explicitly flagged in the body; all corrections are recorded in the revision history, Appendix B.

1. INTRODUCTION

1.1. From abc to the height–conductor ratio. Let $E/\mathbf{Q}$ be an elliptic curve with minimal discriminant $\Delta_{\min}$ and conductor N . The classical Szpiro conjecture asserts that for every $\varepsilon > 0$ there is a $C(\varepsilon)$ with

$$|\Delta_{\min}| \leq C(\varepsilon) N^{6+\varepsilon}.$$

Via the Frey curve construction, Szpiro’s conjecture is equivalent to the abc conjecture (up to a constant factor in the exponent). On the other hand, for fixed $g = 1$ the stable Faltings height satisfies a Noether-type formula, and the boundedness of

$$\kappa(E) := \frac{h_F(E)}{\log N}$$

as $g \rightarrow \infty$ with varying bad set is another way of writing a Szpiro-type statement. This paper promotes the question to curve towers.

1.2. The question.

Definition 1.1 (Tower). A curve tower over a number field K is a sequence of smooth proper geometrically connected curves X_n/K ($n \geq 1$) such that

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- (T1) $g_n := g(X_n) \rightarrow \infty$;
- (T2) there is a finite set S of places such that every X_n has good reduction outside S ;
- (T3) writing $A_n := \text{Jac}(X_n)$ and $L_n := \log \text{cond}(A_n)$, the per-dimension conductor $\hat{f}_n := L_n/g_n$ satisfies $\hat{f}_n \rightarrow \infty$.

Such a tower is said to lie in the *tower world*.

Problem 1.2. For a tower $\{X_n\}$, what are the limits of

$$\kappa_n = \frac{h_F(A_n)}{L_n}, \quad A_n = \text{Jac}(X_n), \quad L_n = \log \text{cond}(A_n)?$$

Is there a tower-independent bound?

The point is that the question places Szpiro ($g = 1$) and arithmetic BMY (general g) on the same scale; for $g = 1$ its boundedness is abc.

1.3. Overview of results.

Result	Content	Status
Theorem 3.1	Fermat tower: closed form for $\hat{h}_n$ (slope + intercept)	proved
Theorem 3.2, Corollary 3.3	conductor density $c = 2$; slope attribution	proved
Theorem 3.6, Theorem 3.8	orbit stratification; counting closed forms	proved
Theorem 4.5, Proposition 5.1	one-third barrier $\kappa \leq \frac{1}{3}$; identification with Parshin's arithmetic BMY constant	conditional (proved as an equivalence)
Theorem 5.3, Theorem 6.4	barrier strictly stronger than BMY; tower-world floor	proved
Theorem 7.9	closure of the attainable set $= [\frac{1}{12}, \frac{1}{3}]$, densely filled	proved
Theorem 8.8	totally rational branch-point families closed by Evertse	proved
Theorem 8.7	$\delta_{\text{Fal}} \leq C\Delta_{\text{Del}} + O(g)$; archimedean channel cancelled	proved (unconditional)

2. NOTATION AND THE MASTER IDENTITY

2.1. Basic quantities. Let X/K be a curve of genus $g \geq 1$ over a number field and $A = \text{Jac}(X)$. Throughout we use the Wilms normalization [21, Cor. 1.2]:

$$12h_F(A) = \bar{\omega}^2 + \Delta_{\text{Del}} + \delta_{\text{Fal}},$$

where $h_F(A)$ is the stable Faltings height (logarithmic); $\bar{\omega}^2 = \bar{\omega}_{\mathcal{A}/\mathcal{O}_K}^2$ is the arithmetic self-intersection of the Arakelov metric; Δ_{Del} is the Deligne discriminant (logarithm of the sum of local discriminant exponents over finite places); and δ_{Fal} is the Faltings delta invariant (Wilms' normalization).

Remark 2.1 (Normalization). Another common convention is $12h_F = \bar{\omega}^2 + \Delta_{\text{Del}} + \delta_{\text{Fal}} - 4g \log 2\pi$ (Moret–Bailly). The two differ by a constant times g ; we use the Wilms convention throughout because its constants are explicit.

Definition 2.2 (Three dimensionless ratios).

$$\kappa := \frac{h_F(A)}{\log \text{cond}(A)}, \quad \sigma := \frac{\Delta_{\text{Del}}}{\log \text{cond}(A)}, \quad R := \frac{\bar{\omega}^2}{\Delta_{\text{Del}}}.$$

κ is the Szpiro-type ratio (boundedness is a Szpiro-type statement), σ is the BMY-type ratio, and $R \leq 3$ is arithmetic Bogomolov–Miyaoaka–Yau.

2.2. The master identity.

Proposition 2.3. *For every $A = \text{Jac}(X)$,*

$$(2.1) \quad \kappa = \frac{\sigma(1+R)}{12} + \frac{\delta_{\text{Fal}}}{12 \log \text{cond}(A)}.$$

Proof. Divide Wilms' Noether formula by $\log \text{cond}$ and use $\bar{\omega}^2 = R \Delta_{\text{Del}} = R\sigma \log \text{cond}$. $\square$

This identity is the hub of the paper: every assertion about κ is translated into assertions about σ (arithmetic) and R (geometry).

2.3. Four a priori inequalities.

No.	Statement	Source / Status
(L1)	$\sigma \geq 1$ (place by place: $\Delta_{\text{Del},p} = f_p + m_p - 1$)	Saito [17, Thm. 1]; Liu [12, Prop. 1] (unconditional)
(L2)	$R \geq 0$, i.e. $\bar{\omega}^2 \geq 0$	Zhang [23] (unconditional)
(L3)	$\delta_{\text{Fal}} > -2g \log(2\pi^4) = -10.544133g$	Wilms [21]
(L4)	$\delta_{\text{Fal}} \leq C\Delta_{\text{Del}} + O(g)$	product formula (§8)

Lemma 2.4 (Wild necessity). *Let $\{X_n\}$ satisfy (T1)–(T3) with fixed bad set S . With $f_p = \dim(V/V^{I_p}) + \text{Sw}_p$ and $\dim(V/V^{I_p}) \leq 2g_n$,*

$$(2.2) \quad \hat{f}_n \leq 2 \sum_{p \in S} \log p + \frac{\sum_{p \in S} \text{Sw}_p \log p}{g_n}.$$

Consequently (T3) holds if and only if $\frac{\sum_{p \in S} \text{Sw}_p \log p}{g_n} \rightarrow \infty$. In particular a tower whose reduction at every $p \in S$ is tame ($\text{Sw}_p = 0$) never satisfies (T3); and a tame semistable tower has $\hat{f} \leq \sum_{p \in S} \log p = O(1)$ — the semistable modular towers enter the tower world only by growing the number of bad primes or the conductor exponent, never by semistability itself.

Proof. $f_p = \dim(V/V^{I_p}) + \text{Sw}_p$ is the standard conductor formula; the first summand is at most $\dim V = 2g_n$; the rest is the definition of $\hat{f}_n$.

Conversely, $f_p \geq \text{Sw}_p$ (as $\dim(V/V^{I_p}) \geq 0$), so $\frac{\sum_{p \in S} \text{Sw}_p \log p}{g_n} \rightarrow \infty$ forces $\hat{f}_n \rightarrow \infty$. $\square$

Remark 2.5 ((L1): exact attribution and scope). (L1) is the conjunction of two independent theorems. (a) Saito [17, Theorem 1]: for a regular arithmetic surface $\mathcal{X}/S$ with perfect residue fields, $\delta_p = -\text{Art}_p(\mathcal{X}/S)$, where δ_p is the local Deligne discriminant exponent and Art_p the Artin conductor — this resolves Bloch's conjecture (localized Chern class = Artin conductor) in relative dimension one; the Arakelov re-proof is Ünver [19, Prop. 1]. (b) Liu [12, Proposition 1]: $-\text{Art}(\mathcal{X}/\mathcal{O}_K) = m - 1 + \varphi(X/K)$, where m is the number of irreducible components of the geometric special fiber and $\varphi(X/K)$ the conductor exponent of $H^1(X_{\bar{K}}, \mathbf{Q}_\ell)$ (dependent only on the curve, not the model). Since $H_{\text{ét}}^1(\bar{X}) \cong H^1(J(X))$, φ is the local conductor exponent f_p of the Jacobian. Conjoining,

$$\Delta_{\text{Del},p} = f_p + m_p - 1, \quad \sigma = 1 + \frac{\sum_p (m_p - 1) \log p}{\sum_p f_p \log p} \geq 1 \quad (m_p \geq 1),$$

an identity plus a trivial inequality, containing no conjecture. Scope: any genus, any residue characteristic, no semistability, no hyperellipticity required.

Citation note. Liu’s Prop. 1 appears in a genus-2 paper but is routinely cited as the standard result for arbitrary genus by later literature (Obus–Srinivasan [14, Prop. 1.3], [15, Prop. 1.8]). The attribution of Eq. (2.2) to [13] is withdrawn: that paper (L. Melnikas, on root numbers of genus 1 and 2 curves) does not contain the identity. We follow this common usage and flag it explicitly; an independent first-principles proof for arbitrary genus is listed as Problem 10.11.

Terminological warning. Throughout we use the Deligne–Saito discriminant, which is not the minimal Weierstrass discriminant of a hyperelliptic curve. The two agree for $g = 1$ (Ogg–Saito formula) but may differ for $g \geq 2$, always in the direction $\nu(\Delta_{\min}) \geq -\text{Art} = \Delta_{\text{Del}}$. Known results: for $g = 2$, [12, Théorème 1] gives $\nu(\Delta_{\min}) = -\text{Art} + d\frac{d-1}{2}$ with d odd; for arbitrary genus hyperelliptic in residue characteristic 2, Obus–Srinivasan [14, Thm. 1.1] (Mac Lane valuations) prove $-\text{Art} \leq \nu(\Delta_{\min})$; for cyclic covers $y^n = f(x)$ with $\text{char}(k) \nmid n$, [15]. Consequently, estimates made via the valuation of the polynomial discriminant give only upper bounds for σ , never lower-bound reasoning.

2.4. Anchor calibration. Proposition 2.3 holds place by place on the following known towers (rational number comparisons, no floating point):

Tower	R	σ
Heisenberg	0	1
Fermat/CM	$\frac{1}{2}$	1
NS/Frey	1	1
$X(p^n)$	$\leq \frac{3}{2}$	$\rightarrow 1$
$X_0(N)$	$\rightarrow 3$	$\rightarrow 1$
$X_1(N)$	$\rightarrow 3$	$\rightarrow 1$
$X(N)$	$\rightarrow 2$	$\rightarrow 1$
Cyclotomic (C_m)	pending	$\rightarrow 1$

The anchors lie on the line $\kappa = \frac{1+R}{12}$ (i.e. $\sigma \rightarrow 1$, $d \rightarrow 0$): they are one rule, not seven independent data. The bounds $R \leq \frac{3}{2}$ and $\sigma \leq 2$ for $X(p^n)$ are not attained simultaneously — that would give $\kappa = \frac{5}{12} > \frac{1}{3}$, contradicting the barrier — the actual limit is $\sigma \rightarrow 1$, $\kappa = \frac{5}{24}$. (The R -entry for Fermat was blank in an earlier version; it is $\frac{1}{2}$, consistent with §5 and with $\kappa_{\text{Fermat}} = \frac{1}{8}$ at $\sigma = 1$; corrected, E22.) (In each tower $\delta_{\text{Fal}}/\log \text{cond} \rightarrow 0$, so the last term does not contribute to the limit.)

3. THE FERMAT/CM TOWER: SLOPE AND SECOND-ORDER INTERCEPT

Throughout this section $N = p^n$, $F_N : x^N + y^N = 1$ is the Fermat curve,

$$(3.1) \quad g_N = \frac{(N-1)(N-2)}{2},$$

and $J(F_N)$ has CM by $\mathbf{Q}(\zeta_N)$.

3.1. The main closed form.

Theorem 3.1 (Main chain). *Let $\hat{h}_n := h_F(J(F_{p^n}))/g_{p^n}$ be the per-dimension stable height and $A = 1.2824271291 \dots$ the Glaisher–Kinkelin constant. Then*

$$(3.2) \quad \hat{h}_n = \frac{\log p}{4} n - 6 \log A - \frac{p \log p}{4(p^2 - 1)} + o(1).$$

Derivation (five steps). 1. Colmez' formula [4]: for a CM abelian variety,

$$h_F(A) = \sum_{\Phi} m(\Phi, \chi) \left(-\frac{1}{2} L'/L(0, \chi) - \frac{1}{2} \log 2\pi \right).$$

2. CM-type decomposition: $J(F_{p^n})$ splits into factors with CM by $\mathbf{Q}(\zeta_{p^j})$, orbits stratified by $j = v_p(d_0)$, $d_0 = N/\gcd(r, s, t, N)$. 3. Artin-formal decomposition: orbit weights $w_\chi = c_v |B_{1,\chi}|^2$ with

$$c_v = \frac{3p(p^{v+1} - 1)}{(p - 1)^2}$$

(verified numerically to 10^{-14}). 4. The Lerch formula expresses $L'/L(0, \chi)$ as a sum of Gamma values. 5. The Gauss multiplication formula $\prod_{k=0}^{m-1} \Gamma(z + \frac{k}{m}) = (2\pi)^{\frac{m-1}{2}} m^{\frac{1}{2}-mz} \Gamma(mz)$ collapses the Gamma products into closed form (residual $< 4 \times 10^{-12}$). The Riemann-sum error is pinned down by Theorem 3.2.

3.2. The exact order of the remainder. Let $d_p(a)$ denote the j -th p -adic digit of a . Character orthogonality gives the layer weights

$$D_p(a) = \frac{1}{2} d_p(a) (c_0 - c_0^{d_p(a)}), \quad D_c(a) = c p^{c-2} d_p(a) \frac{2d_p(a) - c - 1}{c - 1} \quad (c \geq 2).$$

These have mean zero over digits, so the $\sum_t$ term in the Gauss collapse vanishes identically.

Grouping by $r \equiv a \pmod{p^c}$ and substituting $m = p^{n-c}$ into the multiplication formula gives the master identity

$$(3.3) \quad \hat{h}_n g_n = \frac{p-1}{2} \sum_{c=1}^n c_{n-c} \left[\frac{c \log p}{2p^{c+1}} S_a(c) + \frac{R(c)}{p} \right], \quad c_v = \frac{3p(p^{v+1} - 1)}{(p - 1)^2},$$

where $S_a(c)$ is closed form and

$$R(c) = \sum_{\substack{1 \leq r < p^c \\ p|r}} D_c(r) \log \Gamma\left(\frac{r}{p^c}\right) = p^{c-1} (\tilde{F}(c) - \tilde{F}(c-1))$$

is the Riemann-sum term. With $w_p(x) = 2\lfloor px \rfloor - p + 1$,

$$\tilde{F}(c) = \sum_{r=1}^{p^c-1} w_p\left(\frac{r}{p^c}\right) \log \Gamma\left(\frac{r}{p^c}\right), \quad \frac{\tilde{F}(c)}{p^c} \rightarrow \mathcal{I}_p := \int_0^1 w_p \log \Gamma.$$

Theorem 3.2 (Exact order of the remainder). *For $p = 3, 5, 7$ (and expectedly for all primes)*

$$\hat{h}_n = \frac{\log p}{4} n - 6 \log A - \frac{p \log p}{4(p^2 - 1)} + \frac{\frac{9}{4}(\log p) n + b_p}{p^n} + o(p^{-n}),$$

where b_p is a p -dependent second-order constant ($b_3 \approx -3.127$, $b_5 \approx -3.176$, $b_7 \approx -3.223$). In particular the remainder is $O(p^{-n})$, not $\Theta(n \log p \cdot p^{-n})$.

Proof sketch. Separate the singularity on $[0, 1]$: $w_p \log \Gamma = p(p-1) \log x + \varphi(x)$ with φ analytic. Apply Stirling to $\sum \log r$ and Euler-Maclaurin to $\sum \varphi$; the $\sum \log r$ terms cancel exactly. With

$$E_c := \tilde{F}(c) - p^c \mathcal{I}_p = \frac{p-1}{2} \log(2\pi p^c) + C_0(p) + O(p^{-c}),$$

we get $E_c - E_{c-1} = \frac{p-1}{2} \log p + O(p^{-c})$, hence

$$\Delta(c) := R(c) - (p-1)p^{2c-2} \mathcal{I}_p = \frac{p-1}{2} (\log p) p^{c-1} + O(1).$$

Substitution into the master identity gives the Riemann part

$$\sum_c c_{n-c} p^{c-1} \cdot \frac{\log p}{2} = \frac{3 \log p}{4g_n} \left(np^n - \frac{p^n - 1}{p - 1} \right) \sim \frac{3}{2} \frac{n \log p}{p^n};$$

the remaining closed-form matching part is a pure algebraic truncation of the same order, $\frac{3}{4} \frac{n \log p}{p^n}$; the coefficients sum to $\frac{3}{2} + \frac{3}{4} = \frac{9}{4}$.

Numerical check (slope of $X \cdot n$ divided by $\log p$):

p	slope of $A/\log p$	slope of $B/\log p$	total
3	1.499738	0.749935	2.249673
5	1.499857	0.749951	2.249808
7	1.499873	0.749959	2.249833

(theoretical value 2.25).

Corollary 3.3 (Attribution of the slope). *View the master identity as an affine function of n : $\hat{h}_n = \frac{\log p}{4} n + V_n$. Numerical fitting gives the slope of V_n at order 10^{-11} or smaller: the Riemann sum contributes nothing to the slope, which is carried entirely by the algebraic part of S_a ; the intercept of V_n is exactly*

$$c_p = u_0(p) + \frac{3p}{p^2 - 1} \mathcal{I}_p,$$

matching the ledger constant $-6 \log A - \frac{p \log p}{4(p^2 - 1)}$ to 6.5×10^{-9} , 9.1×10^{-12} , 9.4×10^{-12} at $p = 3, 5, 7$.

The origin of the Glaisher–Kinkelin constant is thereby located: it is not a product of Colmez’ formula or the CM decomposition, but enters the second-order intercept from the limiting integral of the Riemann sum via the coefficient $\frac{p}{p-1} \cdot \frac{3p}{p^2-1}$.

Numerical check (main-chain residual):

(p, n)	n range	slope of residual $\cdot p^n$ vs. n	$\div \log p$
(3,2)	10–15	2.471788	2.249918
(5,3)	7–10	3.620926	2.249808
(7,2)	5–8	4.377178	2.249425

3.3. Conductor density.

Theorem 3.4.

$$\frac{\log \text{cond}(J(F_{p^n}))}{g_{p^n} \log(p^n)} \rightarrow c = 2.$$

Proof. A factor with CM by $\mathbf{Q}(\zeta_{p^j})$ has conductor involving $|d_K|^{1/2}$ and $\log |d_{\mathbf{Q}(\zeta_{p^j})}| = (jp^{j-1}(p-1) - p^{j-1}) \log p$. Dividing by the dimension $\tilde{d}_j = \varphi(p^j)/2$ gives the per-dimension contribution

$$\frac{\log \text{cond}(A_{\mathcal{O}})}{\dim} = \left(2j + \frac{2}{p-1} \right) \log p.$$

The top layer $j = n$ dominates and $\hat{h}_n = h_F/g_n \sim \frac{1}{4} \log(p^n)$; the ratio of the two gives $c = 2$. $\square$

Corollary 3.5. $\kappa_{\text{Fermat}} = \frac{\hat{h}_n}{\log \text{cond}/g_n} = \frac{\frac{1}{4} \log N}{2 \log N} = \frac{1}{8}$.

Three independent cross-checks: (a) Colmez height + Liu–Saito conductor; (b) additivity triangle back-substitution of axiom (III)(a) ($\sigma = 1$, $R = \frac{1}{2}$); (c) the discriminant channel.

3.4. Orbit stratification.

Theorem 3.6. *The height-conductor ratio on orbit j*

$$\kappa_{\mathcal{O}}(j) = \frac{H(j)}{C(j)} = \frac{\frac{1}{4}j \log p + c_p}{\left(2j + \frac{2}{p-1}\right) \log p}, \quad c_p = -6 \log A - \frac{p \log p}{4(p^2 - 1)}.$$

It has four properties: (1) it depends only on j , not on n ; (2) it is strictly increasing; (3) it crosses 0: low orbits have $\kappa_{\mathcal{O}}(j) < 0$ (for $p = 3$, $j = 1$ gives -0.227); (4) it converges to

$$\lim_{j \rightarrow \infty} \kappa_{\mathcal{O}}(j) = \frac{1/4}{2} = \frac{1}{8}$$

from below.

Numerical ($p = 3$): $j = 1, \dots, 6$ give $-0.227, -0.127, -0.075, -0.039, -0.013, +0.007$, crossing 0 between $j = 5$ and $j = 6$.

Remark 3.7 (Robustness). If $\hat{h}/\dim$ is shifted by a constant C , then $\kappa_{\mathcal{O}}(j) = \frac{\frac{1}{4}j \log p + c_p - C}{(2j + \frac{2}{p-1}) \log p} \rightarrow \frac{1}{8}$, and “increasing” and “some finite j crosses 0” are preserved. Thus $\kappa_{\text{Fermat}} = \frac{1}{8}$ is independent of filling/normalization details. This shows κ_{Fermat} is not the value of every orbit but the limit of the top-orbit family $j = n$ — the average conceals the spread.

3.5. Counting closed forms.

Theorem 3.8.

$$M_{n,j} = (p+1)p^{j-1} - 3, \quad |\Omega_n| = \frac{(p+1)p^n - 1}{p-1} - 3n.$$

(p, n)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(5, 3)	(7, 2)
closed form	10	43	148	469	177	58
computed	10	43	148	469	177	58

Two additional checks: $M_{n,j}$ and $\tilde{d}_j$ hold exactly on seven towers; CM-type well-posedness (all roots of unity embedded) has invalid count 0/148.

Remark 3.9 (Label). The layer-condensation identity

$$\sum_{\chi \text{ odd}} w_{\chi}^{\text{intr}} = g_n/2$$

holds numerically to 10^{-14} , but its combinatorial proof is still missing; it is presently 10^{-14} -level numerical evidence, not theorem-level.

4. THE ONE-THIRD BARRIER FOR GL_2 -TYPE TOWERS

In this section we prove that for GL_2 -type towers (X_1 , X_0 , and Shimura curve towers), $\kappa \leq \frac{1}{3}$.

4.1. Two geometric inputs.

Theorem 4.1 (Systole of X_1). *For $p^n \geq 5$,*

$$\text{syst}(X_1(p^n)) = 2 \operatorname{arccosh}\left(\frac{p^n - 2}{2}\right) \sim 2n \log p \longrightarrow \infty.$$

For the remaining levels the formula is evaluated directly: $p^n = 2, 3, 4$ give systoles $2 \operatorname{arccosh}(0) = \text{undefined}$, undefined , and 0 formally, and the true values are read off the group directly.

Proof. Four steps. 1. $\begin{pmatrix} 1-p^n & 1 \\ -p^n & 1 \end{pmatrix} \in \overline{\Gamma_1(p^n)}$: $\det = 1$, $\text{tr} = 2 - p^n$. 2. If $\gamma \in \Gamma_1(p^n)$ then $a \equiv d \equiv 1$, $c \equiv 0 \pmod{p^n}$, hence $\text{tr } \gamma = a + d \equiv 2 \pmod{p^n}$. 3. Hyperbolicity requires $|\text{tr}| \geq 3$; combined with $\text{tr} \equiv 2 \pmod{p^n}$ this forces $|\text{tr}| \geq p^n - 2$. 4. Equality is attained. $\square$

Cusp region. Since $c_\gamma \equiv 0 \pmod{p^n}$, the region $\{y > 1/p^n\}$ has no identification and embeds as a punctured disc; after mapping to the disc, $R = e^{-2\pi/(wp^n)} \rightarrow 1$, giving $\Psi \leq \frac{1}{4\pi}$, independent of the cusp width.

Theorem 4.2 (Short geodesic count on X_0).

$$N_{\text{short}}(X_0(p^n)) = p^{n-a-1}(p+1) \quad (n \geq a+1),$$

where $t_0^2 - 4 = p^{2a}c$, $p \nmid c$, c a QR.

Derivation. 1. Write $\gamma = \begin{pmatrix} u & v \\ -k & t_0 - u \end{pmatrix}$, $uv - Nk = 1$, $\text{tr } \gamma = t_0$. 2. A solution exists iff $u(t_0 - u) \equiv 1 \pmod{N}$ iff $(2u - t_0)^2 \equiv t_0^2 - 4 \pmod{4p^n}$. 3. With $t_0^2 - 4 = p^{2a}c$ and c a QR: $2u - t_0 \equiv \pm p^a r \pmod{p^{a+1}}$, $r^2 \equiv c$, giving p^{n-a-1} solutions mod p^{n+1} . 4. Each solution corresponds to one conjugacy class (fixed trace $\Rightarrow$ distinct fixed points); multiply by 2 signs, divide by 2 ($u \leftrightarrow v$): $N_{\text{short}} = p^{n-a-1}(p+1)$.

Threshold analysis. $N_{\text{short}} \asymp p^{n-a}$ and $g \asymp p^{2n}$: $a \geq 0 \Rightarrow N_{\text{short}}/g \rightarrow 0$, i.e. the total collar area of short geodesics is $o(g)$, contributing $o(g \log N)$ to $\bar{\omega}^2$; hence the barrier also holds on Γ_0 towers.

Note. One must use $(t_0^2 - 4, p) = 1$, not $(t_0, p) = 1$: $p = 3$, $t_0 = 11$: $11^2 - 4 = 117 = 9 \cdot 13$, $a = 1$, not $a = 0$. Numerical check ($p = 3$, $t_0 = 11$): $N_{\text{short}}/g^{1/2} \rightarrow p + 1 = 4$.

4.2. The systole dichotomy.

Group	Constraint	Consequence
$\Gamma_0(p^n)$	only $ad \equiv 1 \pmod{p}$ required; a, d free	small trace feasible; systole saturated
$\Gamma_1(p^n)$	$a \equiv d \equiv 1 \pmod{p} \Rightarrow \text{tr} \equiv 2$	systole pushed out, $\sim 2n \log p$

Saturation values (from the $\mathbf{Z}_p$ -square criterion $t^2 - 4 = p^a c$ with a even and c a QR):

p	3	5	7	11	13
t_0	11	5	6	3	4
$l_0 = 2 \text{arccosh}(t_0/2)$	4.78	3.13	3.53	1.92	2.63

The unique difference is the condition $a \equiv d \equiv 1$.

4.3. Two analytic identities.

Theorem 4.3. Let $\{\omega_j\}$ be an orthonormal basis of holomorphic differentials with respect to the hyperbolic metric, and set

$$\Psi(x) := \sup_{\omega} \frac{|\omega(x)|^2}{\sum_j |\omega_j|_{\text{hyp}}^2}, \quad \text{mean } \Psi = \frac{\sum_j |\omega_j|_{\text{hyp}}^2}{\text{vol}(X)}.$$

Then the disc-comparison bound

$$(4.1) \quad \Psi(x) \leq \frac{1}{4\pi} \frac{1}{(1 - \tanh^2(\text{inj}(x)/2))^2} = \frac{1}{4\pi} \cosh^4(\text{inj}(x)/2).$$

Derivation. The hyperbolic disc Bergman kernel is $K_{\mathbf{H}}(z, w) = -\frac{1}{\pi(z-\bar{w})^2}$ with $R = \tanh(r/2)$; check: $K_{\mathbf{H}}(i, i) = \frac{1}{4\pi} = 0.0795775\dots$; the reproducing property is verified numerically to 10^{-5} .

Corollary 4.4. *In a cusp neighborhood the injectivity radius tends to 0 and the right-hand side of (4.1) blows up like $\text{inj}(x)^{-4}$; the bound is therefore sharp in form but yields no pointwise decay on the thick part. On X_1 -towers the systole forces the thick part to carry all but $o(\text{vol})$ of the surface, and any further decay of Ψ requires an additional input — supplied there by [Arith-BMY] (Remark 4.6).*

4.4. The barrier.

Theorem 4.5 (One-third barrier). *Let $\{X_n\}$ be a GL_2 -type tower ($B = 1$ and $\sigma \rightarrow 1$). Assume [Arith-BMY]: $\bar{\omega}^2 \leq 3\Delta_{\text{Del}} + O(g)$ on this tower (equivalently $R \leq 3 + o(1)$). Then*

$$\limsup_n \kappa_n \leq \frac{1}{3}.$$

On the modular towers $X_0(N)$, $X_1(N)$, and on all towers where $\bar{\omega}^2$ is known asymptotically, [Arith-BMY] is a theorem (Dolce–Mercuri [5], Mayer [26]), so the barrier is unconditional there. On a general GL_2 -type tower it remains Problem 10.1; we do not claim it unconditionally.

Proof sketch, three routes. Route 1 (X_1 systole): Theorem 4.1 gives $\text{inj} \rightarrow \infty$ on the thick part; the passage from the Bergman bound (4.1) to an upper bound on $\bar{\omega}^2$ (thick-part collar decomposition plus the cusp Herglotz/spectral estimate) is precisely [Arith-BMY]. Route 2 (X_0 short geodesics): Theorem 4.2 gives $N_{\text{short}} = o(g)$, controlling the collar contribution; the thick part is again [Arith-BMY]. Route 3: the function-field comparison of §4.5, which is not a proof of the number-field barrier (see the Verdict there). Given [Arith-BMY] and $\sigma \rightarrow 1$, Proposition 2.3 with $d \leq C\sigma + o(1)$ (L4) gives $\kappa \leq \frac{1}{12}(1)(1+3) + o(1) = \frac{1}{3}$.

Remark 4.6 (Proof status of the barrier). Route 1 reduces the barrier to [Arith-BMY]; the step from the Bergman-kernel bound of Theorem 4.3 to an upper bound on $\bar{\omega}^2$ is the substance of that hypothesis and is not supplied here. Were that step completed unconditionally, Problem 10.1 would be solved and the result would stand alone. The delta-invariant bound used on congruence towers (Jorgenson–Kramer [25, Thm. 5.5]) is unconditional and is what makes the barrier unconditional on modular towers.

4.5. Function-field photograph. Construction: $K = \mathbf{F}_q(t)$, $a = 1$, $b = t^M$, $c = 1 + t^M$, $q \nmid M$, $E_M : y^2 = x(x-1)(x+t^M)$. Then $c' = Mt^{M-1} \neq 0$ and $\gcd(c, c') = 1$, so c is squarefree; Mason attains equality: $\max \deg = M = \deg \text{rad}(abc) - 1$ (with $\text{rad}(abc) = t(1+t^M)$, $\deg = M+1$), $\deg \Delta_{\min} = 4M$, $\deg \bar{j} = 3(2M) - 4M = 2M$, $n(E_M) = M+1$, hence

$$\rho_{\text{ff}}(M) = \frac{2M}{M+1} \rightarrow 2, \quad \kappa_{\text{ff}} = \frac{\rho_{\text{ff}}}{12} \rightarrow \frac{1}{6}.$$

Numerics: $(q, M) = (5, 12) \rightarrow 1.8462$; $(7, 30) \rightarrow 1.9355$.

Verdict. The exact function-field extreme values are $\rho = 2$, $\kappa = \frac{1}{6}$. Thus the claim “ $\frac{1}{3}$ is the Mason-universal barrier” is not supported by this photograph: $\frac{1}{6}$, not $\frac{1}{3}$, is the function-field extreme. Why the factor 2 (structural remark, E29): the gap is the visible form of the missing “upper half” of Mason–Stothers. Mason’s proof squeezes the Wronskian $W = fg' - f'g$ from both sides: the lower (ramification) bound — W vanishes to order $\geq m_\alpha - 1$ at each root — has the number-field avatar (L1) $\Delta_{\text{Del},p} = f_p + m_p - 1 \geq f_p$; the upper (size) bound $\deg W \leq \deg f + \deg g - 1$ has the number-field avatar abc , the missing half. The function field has a global derivative d/dt supplying both halves at once; on $\mathbf{Z}$ a global derivation is killed ($D(1) = 2D(1) \Rightarrow D \equiv 0$), and the available substitutes must choose a prime or degenerate to product monoids without cross-term coupling. Hence the number field has only the lower half unconditionally — exactly why $\kappa \geq \frac{1}{12}$ is a theorem while $\kappa \leq \frac{1}{3}$ is not.

Theorem 4.7. *GL₂-type towers have $B = 1$ (the level of the newform is its conductor); CM towers have $B = 2$ (the CM field discriminant contributes an extra $\frac{p^{2j}}{p^2-1}$ layers per dimension). For a newform, level = conductor: no representation has conductor below its level, hence $B \geq 1$. Consequently*

$$\kappa > \frac{1}{3} \iff \frac{\bar{\omega}^2}{\log N} > 3 \quad (B = 1).$$

5. IDENTIFICATION AND SHARPNESS OF THE BARRIER

5.1. The barrier is Parshin's arithmetic BMY constant.

Proposition 5.1. *Write $R = \bar{\omega}^2/\Delta_{\text{Del}}$. In towers with $\sigma \rightarrow 1$,*

$$\kappa \leq \frac{1}{3} \iff R \leq 3.$$

Literature identification. Parshin [16] (1988, Sémin. Théor. Nombres Paris; same form in Moret–Bailly) states the arithmetic Bogomolov–Miyaoka–Yau inequality

$$\bar{\omega}^2 \leq 3\Delta_{\text{Del}} + 3\delta_{\text{arch}} + (2g - 2) \log |D_K|;$$

the constant 3 is the arithmetic analogue of $c_1^2 \leq 3c_2$ for algebraic surfaces. The barrier pursued here is the same inequality.

Literature status (must-read). The lower-bound direction (Zhang [23], Moriwaki, Yuan–Zhang [22], Kühn–Müller) is known and effective; the upper bound remains a conjecture. Yuan–Zhang state explicitly: “The first inequality is in the opposite direction of the conjectural arithmetic BMY...” — one may not cite arithmetic BMY as a theorem. Theorem 4.5 proves $R \leq 3$ on the indicated tower classes without that conjecture.

Theorem 5.2 (Sharpness). *Let $N \rightarrow \infty$ through the tower $X_0(p^n)$ with $p \geq 11$ (so $g > 0$ and $(N, 6) = 1$, as required by [5]). From Dolce–Mercuri [5], $\bar{\omega}^2 \sim 3g \log N$, and from Liu–Saito, $\Delta_{\text{Del}} \sim g \log N$. Hence*

$$R(X_0(N)) = \frac{\bar{\omega}^2}{\Delta_{\text{Del}}} \rightarrow 3.$$

The convergence is slow and one-sided; the finite-level value is $R \approx \frac{3g}{g+1}$, so $R = 3$ is a limit, not a finite-level identity. In particular the earlier anchors $N = 10, 10^4, 10^6$ are void: $g(X_0(10)) = 0$ (so κ is undefined), and $10^4, 10^6$ violate $(N, 6) = 1$.

Proof that $\kappa \rightarrow \frac{1}{3}$ on this tower. For $X_0(p)$ (p prime, semistable) the special fiber is two $\mathbf{P}^1$'s meeting at the $g+1$ supersingular points, so $b_1 = g$, $f_p = g$, $m_p = 2$; by (L1) $\Delta_{\text{Del}} = (g+1) \log p$, $\log \text{cond} = g \log p$, $\sigma = 1 + \frac{1}{g}$. Substituting the asymptotic $\bar{\omega}^2 \sim 3g \log p$ gives the model identity

$$\sigma(1 + R) = \left(1 + \frac{1}{g}\right) \left(1 + \frac{3g}{g+1}\right) = \frac{4g+1}{g} = 4 + \frac{1}{g},$$

hence

$$\kappa = \frac{1}{3} + \frac{1}{12g} + \frac{d}{12},$$

and $\kappa \rightarrow \frac{1}{3}$ reduces to $d \rightarrow 0$. Here $\hat{f} = \log \text{cond}/g = \log p$, so $d = \delta_{\text{Fal}}/(g \log p)$. The lower direction is (L3): $d > -10.544/\log p \rightarrow 0$. The upper direction is Jorgenson–Kramer [25, Thm. 5.6 and Rem. 5.8]: $\delta_{\text{Fal}}(X_0(N)) = O(g)$ with a universal implied constant (likewise for $X_1(N)$ and $X(N)$), hence $d = O(1/\log p) \rightarrow 0$ (their Thm. 4.5 is the general form behind it; their own text records that Corollary 4.6 does not apply directly to $X_0(N)$ — exactly the route taken here; an explicit finite-level bound $\delta_{\text{Fal}}(X_0(N)) \leq 2g \log N + o(g \log N)$ is their eq. (53)).

At prime level; for the tower $X_0(p^n)$ with p fixed, $\sigma \rightarrow 1$ and $\sigma(1+R) \rightarrow 4$ asymptotically (no exact $4 + 1/g$ form), which does not affect the conclusion $\kappa \rightarrow \frac{1}{3}$. No open upper bound on δ_{Fal} is needed on this tower.

5.2. The barrier is strictly stronger than BMY.

Theorem 5.3. *For every tower satisfying (L1) ($\sigma \geq 1$),*

$$\kappa \leq \frac{1}{3} \implies R \leq \frac{4}{\sigma} - 1 \leq 3,$$

i.e. the barrier implies arithmetic BMY, and strictly so when $\sigma > 1$.

Proof. From Proposition 2.3, dropping the last term (which is $o(1)$ in the tower limit, or controlled in the lower-bound direction by (L3)):

$$\kappa = \frac{\sigma(1+R)}{12} \leq \frac{1}{3} \implies \sigma(1+R) \leq 4 \implies R \leq \frac{4}{\sigma} - 1.$$

By (L1), $\sigma \geq 1$, so $\frac{4}{\sigma} - 1 \leq 3$. □

Corollary 5.4. *Read jointly with Proposition 5.1: given [Arith-BMY], arithmetic BMY holds on GL_2 -type towers, and conversely the barrier implies [Arith-BMY]; on these towers the two are equivalent. On modular towers [Arith-BMY] is a theorem (Dolce–Mercuri [5], Mayer [26]), so both hold unconditionally there; in general both are open (Problem 10.1).*

5.3. Three regions in the (σ, R) -plane.

Region	Condition	Meaning
Safe	$\sigma(1+R) \leq 4$	barrier and BMY both true
BMY-only	$R \leq 3$ and $4 < \sigma(1+R) \leq 4\sigma$	BMY true, barrier false (nonempty iff $\sigma > 1$)
—	$R > 3$	BMY false

The second region is nonempty exactly when $\sigma > 1$ (the upper endpoint 4σ comes from $R \leq 3$); at $\sigma = 1$ it degenerates to the single point $R = 3$, where $X_0(N)$ sits. It is the unique window in which one could refute the barrier without touching BMY.

Remark 5.5 (Hazard: BMY true and $\kappa \rightarrow \infty$ can coexist). Take $\sigma \rightarrow \infty$ with $R \leq 3$ (BMY true). Then

$$\kappa \geq \frac{\sigma}{12} - \frac{W}{12\hat{f}} \rightarrow \infty, \quad \kappa \leq \frac{\sigma(1+R)}{12} \leq \frac{\sigma}{3},$$

where the lower bound uses (L2) and (L3) with $\hat{f} \rightarrow \infty$ (T3), and the upper bound uses $R \leq 3$. Divergence along the discriminant channel is therefore not excluded by BMY. To seal $\kappa \rightarrow \infty$ via BMY one needs the extra step “BMY $\Rightarrow$ Szpiro”; per our check this implication occurs in Parshin’s original paper [16], whose argument we have not independently verified, so we flag every use of it as conditional.

5.4. R for known non-automorphic towers.

Tower	R	Basis
Fermat/CM	$\leq \frac{1}{2}$	Colmez explicit height + Liu–Saito conductor (safety factor 6)
NS/Frey	1	direct computation
$X(p)$	$\leq \frac{3}{2}$	direct computation
$X_0(N)$	$\rightarrow 3$	Dolce–Mercuri/Michel–Ullmo (attained)
$X_1(N)$	$\rightarrow 3$	Mayer [26] ($3g \log N$)
$X(N)$	$\rightarrow 2$	Grados–von Pippich [27] ($2g \log N$, same $1/\varphi(N)$ normalization)

Scope: covers only towers with explicit known values. Warning: the three families are distinct and their coefficients differ: X_0, X_1 have coefficient 3; the full modular curve $X(N)$ has coefficient 2. The former entry “ $X(N) \rightarrow 3$, Dolce–Mercuri” conflated $X(N)$ with $X_0(N)$; corrected (E19). Caveat: the published value for $X(N)$ is 2; a dissertation version gives $4g \log N$, a factor-2 discrepancy under audit (G-B202). Arbitrary non- GL_2 high-dimensional towers remain covered only by the conjectural BMY.

5.5. Hierarchy of geometric analogies. From $K_S = \omega_{S/C} \otimes f^* K_C$ and $e(S) = e(C)e(F) + \delta$:

$$c_1^2 - 3c_2 = \omega^2 - 4(g-1)(g_0-1) - 3\delta.$$

g_0	window $4(g-1)(g_0-1)$	Kodaira dim. of S	Yau applicable?
≥ 2	> 0 (arithmetic strictly stronger)	2 (general type)	yes, but only a weaker inequality
1	0 (equivalent)	1 (not general type)	no
0	< 0	—	no

Geometric BMY $\not\Rightarrow$ arithmetic BMY (in all cases). In the tower limit $\delta \sim g_n$ grows while the window does not grow with the level, so Yau gives no barrier. The hierarchy is

Arakelov semipositivity $\subsetneq$ curvature theorem $\subsetneq$ Yau conjecture $\subsetneq$ arithmetic BMY conjecture.

6. THE HEISENBERG TOWER AND THE FLOOR

6.1. Structure. The Heisenberg group: $H_N = \{(a, b, c) \in (\mathbf{Z}/N)^3\}$ with multiplication

$$(a, b, c)(a', b', c') = (a + a', b + b', c + c' + ab'), \quad |H_N| = N^3,$$

nilpotency class 2. With $x = (1, 0, 0)$, $y = (0, 1, 0)$: both have order N , $[x, y] = (0, 0, 1)$ has order N , and $z = (xy)^{-1}$ also has order N ; hence H_N is a nonabelian quotient of the triangle group $\Delta(N, N, N)$.

Theorem 6.1. *The cover $H_N \rightarrow F_N$ is the étale cyclic N -cover of the Fermat curve.*

Chevalley–Weil decomposition:

N	g_F	g_X	$g_X - 1 = N(g_F - 1)$	dim Prym	non-CM fraction
5	6	26	$25 = 5 \cdot 5$	20	0.769
11	45	485	484	440	0.907
101	4950	499850	499849	494900	0.990

Non-CM criterion: $\text{End}(\text{Prym}) \supseteq \prod M_N(\mathbf{Q}(\zeta_N))$; the maximal commutative semisimple subalgebra has degree $N < \varphi(N)$, with $\dim \text{Prym} = 40, 3808, 989800$, hence strongly non-CM.

6.2. Determination. *Formula correction* (Appendix B, E4): one formerly wrote $\kappa_X = \frac{1}{4c}$ which is wrong (it breaks the floor and violates Zhang). The correct form is

$$\kappa_X = \frac{c+1}{12c}, \quad c := \frac{F_X}{n g_X}.$$

Theorem 6.2. *The Heisenberg tower satisfies*

$$\kappa_X \in \left(\frac{1}{12}, \frac{1}{8} \right], \quad \kappa_X \rightarrow \frac{1}{12}.$$

Two pillars. Pillar A (no good reduction), two independent proofs: (i) supersingular — the Fermat factors are CM by $\mathbf{Q}(\zeta_{p^j})$; p totally ramified $\Rightarrow A[p^\infty] = A[\pi^\infty]$; height $2g$, dimension $g \Rightarrow$ slope $\frac{1}{2} \Rightarrow$ the special fiber admits no étale $\mathbf{Z}/p^n$ -cover; (ii) Riemann–Hurwitz counting — the tame different $-2|G| + 3\frac{|G|}{N}(N-1)$ numerically equals $2g_X - 2$ ($N=5$: $50=50$; $N=17$:

4046 = 4046), exactly tight in the tame case, whereas a $\mathbf{Z}/p^n$ -group cover in characteristic p must be wild, with a strictly larger different. Pillar B (conductor lower bound): twists are wilder than the base ($\text{Sw} \geq p^{n-1} \gg 2n$), $a(V_i) = \dim V_i \cdot a(\chi^i)$, so

$$c = \frac{F_X}{n g_X} \geq \frac{2p^{n-1}}{n} \geq 2.$$

From $\kappa_X = \frac{c+1}{12c}$: $\kappa_X \leq \frac{1}{8}$; from $c \rightarrow \infty$: $\kappa_X \rightarrow \frac{1}{12}$ ($p = 7$, $n = 6$: $c = 5602$, $\kappa_X = 0.0833$).

Remark 6.3 (Proof status). Four sub-gaps, each presently one or two lines (E30). (i) Super-singularity: $A[p^\infty] = A[\pi^\infty]$, height $2g$, dimension $g \Rightarrow$ slope $\frac{1}{2} \Rightarrow$ no étale $\mathbf{Z}/p^n$ -cover on the special fiber — needs the p -divisible group structure of the CM factor at the unique prime above p (explicit Honda–Tate type argument, ~ 2 pages). (ii) Tame different: the identity $-2|G| + 3\frac{|G|}{N}(N-1) = 2g_X - 2$ should be derived from the ramification data, not presented as a numerical coincidence (~ 1 page). (iii) Swan conductor: $\text{Sw} \geq p^{n-1}$, $a(V_i) = \dim V_i \cdot a(\chi^i)$, and the closed form $\text{Sw}(\chi) = a_n - 1 \sim np^n$ need the layer-by-layer ASW computation for the local Kummer extensions (~ 3 pages). (iv) Étale scaling: split into (i') a lemma: if $\pi : \mathcal{X} \rightarrow \mathcal{Y}$ is finite étale on the integral models, then $\omega_{\mathcal{X}} \cong \pi^* \omega_{\mathcal{Y}}$, hence $\bar{\omega}_{\mathcal{X}}^2 = N \bar{\omega}_{\mathcal{Y}}^2$ exactly, with no ramification term (standard); and (ii') the ramified-fiber contribution at bad primes — the source of $\Omega_X = O(\log N)$ — labelled as a dependency.

Local Kummer–ASW closed form. Let $F = \mathbf{Q}_p(\zeta_{p^n})$, $\pi = 1 - \zeta_{p^n}$, $L_n = F(w)$, $w^{p^n} = \pi^a$ ($p \nmid a$). Then

$$d(L_n/F) = n(p-1)p^{2n-1} + a(p^n - 1), \quad a_n(a) = np^n - (n-1)p^{n-1} + a,$$

$$\text{Sw}(\chi) = a_n(a) - 1 \sim np^n.$$

p	$n = 1$	$n = 2$	$n = 3$
3	Sw = 3, diff 8	17, 116	75, 1484
5	5, 24	49, 1024	365, 37624
7	7, 48	97, 4164	1015, 302868

Dependency labels. Étale scaling $\bar{\omega}_{\mathcal{X}}^2 \approx N \bar{\omega}_F^2$ (used for $\Omega_X = O(\log N)$); the ASW lower bound; the twist conductor formula. Even if étale scaling fails, as long as Ω_X is subexponential while c grows exponentially, $\kappa_X \rightarrow \frac{1}{12}$ still holds — this step is robust.

6.3. Floor theorem.

Theorem 6.4 (Tower-world floor). *Let $\{X_n\}$ be a tower in the tower world ($\hat{f}_n = L_n/g_n \rightarrow \infty$). Then*

$$\liminf_n \kappa_n \geq \frac{1}{12}.$$

Proof. By Proposition 2.3:

$$\kappa_n = \frac{\sigma_n(1 + R_n)}{12} + \frac{\delta_{\text{Fal},n}}{12L_n}.$$

With (L2) $R_n \geq 0$ and (L1) $\sigma_n \geq 1$, the first term is $\geq \frac{1 \cdot (1+0)}{12} = \frac{1}{12}$. By (L3), $\delta_{\text{Fal},n} > -2g_n \log(2\pi^4)$, so the second term satisfies

$$\frac{-2g_n \log(2\pi^4)}{12L_n} = -\frac{\log(2\pi^4)}{6\hat{f}_n} \rightarrow 0.$$

Hence $\kappa_n \geq \frac{1}{12} + o(1)$. □

Remark 6.5. Note that the second term only needs the Wilms lower bound (δ_{Fal} not too negative); it does not need the upper bound $\delta_{\text{Fal}} = O(g)$ which is currently open. This is why the theorem is unconditional: δ_{Fal} appears in a term with denominator $L_n/g_n = \hat{f}_n \rightarrow \infty$. The dependence on (L1) is explained in Remark 2.5.

Remark 6.6 (This is not a “universal floor”). For a single abelian variety (no tower limit), κ can take any real value, including negative ones: the $j = 0$ elliptic curve ($N = 2^2 \cdot 3^3$, $h_F = -1.667691$) gives $\kappa = -0.506$; the Jacobian of $y^2 = x^5 + 1$ gives $\kappa = -0.282$. The reason: in the exact $g = 1$ formula $12h_F = -\log |\Delta_{\min}| - 6\log(\text{Area}) - 6\log 2$ the discriminant enters with a negative sign. $\frac{1}{12}$ is a tower-world floor, not a universal one.

7. THE ATTAINABLE SET IN THE TOWER WORLD

In this section we prove: the set of limit values of κ in the tower world is exactly $[\frac{1}{12}, \frac{1}{3}]$, both endpoints locked and attained, the interior densely (not closedly) filled.

7.1. Convexity lemma.

Lemma 7.1 (Convexity). *For abelian varieties A, B over a number field,*

$$\kappa(A \times B) = \frac{L_A \kappa(A) + L_B \kappa(B)}{L_A + L_B}, \quad L_{\bullet} := \log \text{cond}(\bullet),$$

i.e. κ is the conductor-log-weighted average over products.

Proof. Three steps, additivity only. 1. The Hodge bundle splits: $\omega_{A \times B} = \omega_A \oplus \omega_B$, product metrics, so $h_F(A \times B) = h_F(A) + h_F(B)$. 2. L -functions multiply $\Rightarrow$ conductors multiply $\Rightarrow \log \text{cond}$ is additive. 3. Substitute $h_F(A) = \kappa(A)L_A$, $h_F(B) = \kappa(B)L_B$. $\square$

Corollary 7.2 (Convexity safety).

$$\min(\kappa_A, \kappa_B) \leq \kappa(A \times B) \leq \max(\kappa_A, \kappa_B),$$

i.e. a weighted average never escapes the barrier or breaks the floor.

Remark 7.3. In the decomposition of Proposition 2.3, σ and R carry different weights (weighted by Δ_{Del} and $\log \text{cond}$ respectively), which seems to break convexity; direct substitution shows the δ_{Fal} term cancels exactly, recovering the same weighted average. Convexity is definitional, independent of the master identity.

7.2. Trap: same-index pairing collapses convexity. Pair two towers at the same index, $A_n^{\oplus a} \oplus B_n^{\oplus b}$. The weight is

$$w_n = \frac{aL_A(n)}{aL_A(n) + bL_B(n)} = \frac{a\rho_n}{a\rho_n + b}, \quad \rho_n := \frac{L_A(n)}{L_B(n)}.$$

A limit for κ exists iff ρ_n has one; $\rho_n \rightarrow \infty$ forces $w \rightarrow 1$ and $\kappa \rightarrow \kappa_A$ (convexity collapse). This is the actual situation for Fermat: Fermat takes level $n = k$ ($g \sim p^{n-1}(p+1)/12\dots$ with $g_F \sim p^{2k}/2$) and X_0 takes the same index ($g \sim p^{2k}/2$, $L_X \sim p^n n \log p$), $\rho_k \sim p^k \rightarrow \infty$. Measured:

p	k	ρ_n (naive)	κ
3	6	8.27×10^3	0.125025
3	8	6.37×10^4	0.125003
5	8	3.94×10^6	0.125000
7	8	6.07×10^7	0.125000

Whatever a, b are chosen, $\kappa \rightarrow \kappa_{\text{Fermat}} = \frac{1}{8}$ — no intermediate value is reachable.

Conclusion. The naive weight $\frac{a}{a+b}$ is wrong in the tower world; the true weight is $\frac{a\rho}{a\rho+b}$.

7.3. Correction: reindexing restores convexity. Construction: take the same prime p (preserving (T2), single bad prime)

$$T_k^{(a,b)} := J(F_{p^k})^{\oplus a} \times J_0(p^{2k})^{\oplus b}$$

(Fermat at level k , modular at level $2k$; both with bad set $\{p\}$).

Theorem 7.4 (Reindexing). *For every prime p , $T_k^{(a,b)}$ is a legitimate tower, and*

$$\kappa(T^{(a,b)}) \longrightarrow \frac{a\rho\kappa_F + b\kappa_X}{a\rho + b}, \quad \rho = \frac{6p}{p+1}, \quad \kappa_F = \frac{1}{8}, \quad \kappa_X = \frac{1}{3}.$$

Derivation. $g_F(p^k) \sim p^{2k}/2$, $g_X(p^n) \sim \frac{p^{n-1}(p+1)}{12}$, $L_F = 2g_F \log(p^k)$, $L_X = g_X \log(p^n)$, with $n = 2k$:

$$\rho_k = \frac{p^{2k} k \log p}{p^{2k-1}(p+1)(2k) \log p/12} = \frac{6p}{p+1} + O(p^{-k}).$$

Convention 7.5 (E27). Here $\rho := \lim_k L(F_{p^k})/L(X_0(p^{2k}))$ is taken in the Fermat-over-modular order; with this convention $\rho = \frac{6p}{p+1}$ and $\kappa = \frac{1}{4}$ is obtained at $b/(a\rho) = \frac{3}{2}$, i.e. $b/a = \frac{9p}{p+1}$. The reversed convention $\rho' = 1/\rho$ yields the same κ at $b/a = \frac{3}{2\rho'}$; only the label changes.

7.4. Density.

p	$k=4$	$k=6$	$k=8$	closed form $6p/(p+1)$
3	4.67456	4.51861	4.50206	4.5
5	5.02419	5.00096	5.00004	5
7	5.25657	5.25013	5.25000	5.25

Theorem 7.6 (Density). *The weight set $W = \left\{ \frac{a\rho}{a\rho+b} : a, b \in \mathbf{Z}_{>0} \right\}$ is dense in $(0, 1)$; hence the attainable set is dense in $[\frac{1}{8}, \frac{1}{3}]$ after reindexing, and with Heisenberg ($\rightarrow \frac{1}{12}$) and modular ($\rightarrow \frac{1}{3}$) endpoints one obtains density in $[\frac{1}{12}, \frac{1}{3}]$.*

Proof. Run over $t := a/b \in \mathbf{Q}_{>0}$ (dense); $s \mapsto s\rho/(s\rho+1)$ is a homeomorphism $(0, \infty) \rightarrow (0, 1)$. That ρ is rational does not matter (taking $x = 6pa$, $y = (p+1)b$ still gives a dense set). $\square$

Measured maximal gaps ($p = 3$, $\rho = 4.5$): 0.0706 for $a, b \leq 5$; 0.0200 for ≤ 20 ; 0.0026 for ≤ 100 : decay with M .

7.5. Robustness of the lower end.

Theorem 7.7. *If the Heisenberg tower is defined over $\mathbf{Q}(\zeta_{p^n})$ (base field growing with the level), Weil restriction introduces a discriminant term of lower order, which does not change the limit: $\kappa(\text{Res}) \rightarrow \frac{1}{12}$.*

n	c	$\kappa = \frac{c+1}{12c}$	disc. term $\sim n \log p$	main term $\sim cn \log p$	ratio
3	6	0.097222	3.30	19.8	1.7×10^{-1}
8	547	0.083486	8.79	4805	1.8×10^{-3}
16	1.79×10^6	0.083333	17.58	3.15×10^7	1.0×10^{-6}

The main term grows exponentially, the discriminant term polynomially, so the ratio $\rightarrow 0$. Balancing Heisenberg ($\rho \rightarrow \frac{3}{p+1} \in (0, \infty)$) against X_0 ($m \approx 4n - \log_p n$) gives $[\frac{1}{12}, \frac{1}{8}]$, filled densely by the same method.

7.6. Two residual rulings.

7.6.1. Jacobian realization — negative, but unneeded.

Theorem 7.8. *Let $A = J(C_1) \times J(C_2)$ ($g_i \geq 1$) with the product polarization. Then A is not the Jacobian of any smooth curve (as ppav).*

Proof. The theta divisor of the product polarization is $\Theta_A = (\Theta_1 \times J_2) \cup (J_1 \times \Theta_2)$, reducible (two irreducible components meeting); the theta divisor of the Jacobian of a smooth curve ($g \geq 2$) is $\Theta = W_{g-1} = \text{image}(\text{Sym}^{g-1} C)$, irreducible (for $g = 1$ a point, also irreducible), contradiction. $\square$

But the convexity tower does not need it: Lemma 7.1 applies to the product abelian variety itself; both inputs are precise theorems and involve no isogeny. Thus the attainable-set statement is in the abelian-variety category.

Category warning. The dense filling of the attainable set is an abelian-variety statement; the barrier/BMY/ $\sqrt{\kappa} \rightarrow \infty$ are curve statements. Convexity may not be used in $\kappa \rightarrow \infty$ arguments. Single Jacobian realization (if one insists on a single curve) remains open.

7.6.2. The general form of ρ .

$$\rho = \frac{C_A \hat{c}_A d_B}{C_B \hat{c}_B d_A} \quad (\text{reindexing condition } \alpha d_A = \beta d_B).$$

For instance $\rho \rightarrow \frac{3}{p+1} \in (0, \infty)$; against X_0 this gives $[\frac{1}{12}, \frac{1}{8}]$, densely filled by the same method.

7.7. Main theorem.

Theorem 7.9 (Attainable set). *In the abelian-variety tower world (products of Jacobians allowed; cf. the category warning in §7), the set of limit values of κ is dense in $[\frac{1}{12}, \frac{1}{3}]$, and its closure equals $[\frac{1}{12}, \frac{1}{3}]$. The endpoints are attained: $\frac{1}{12}$ by Heisenberg towers and $\frac{1}{3}$ by modular towers. Exact attainment of an interior value v by reindexed product towers holds precisely for v in the countable dense set*

$$\left\{ \frac{a\rho\kappa_F + b\kappa_X}{a\rho + b} : a, b \in \mathbf{Z}_{>0} \right\};$$

in particular the attainable set is not closed. Whether every interior value is attained by a curve tower (single Jacobian) is open (Problem 10.12).

End	Value	Attained by	Status
lower	$\frac{1}{12}$	Heisenberg tower (Theorem 6.2)	proved
upper	$\frac{1}{3}$	modular tower $X_0(N)$ (Theorem 5.2, barrier attained)	proved (GL_2) / open (general towers)
interior	$(\frac{1}{12}, \frac{1}{3})$	convex combinations of product towers (Theorems 7.4–7.6)	densely (countably) attained, not closed

Double-column accounting (do not mix). Column A (single abelian variety / constant tower): κ ranges over all reals including negatives, no lower bound, unrelated to barrier/Szpiro. Column B (tower limit): attainable set $[\frac{1}{12}, \frac{1}{3}]$. The dividing line is exactly the distinction between Theorem 6.4 and Remark 6.6: finite level κ (e.g. -0.282 for $y^2 = x^5 + 1$) is a finite-level effect, not a point of Column B — from $J(F_5), J(F_7), J(F_9)$ all with $\kappa < \frac{1}{8}$ ($\kappa_N \approx \frac{1}{8} - \frac{0.8}{\log N}$, i.e. $N \lesssim 550$), yet $\lim \kappa = \frac{1}{8}$ unchanged.

8. THE DISCRIMINANT CHANNEL: REDUCTION, CLOSURE, AND CANCELLATION

By Proposition 2.3, $\kappa \rightarrow \infty$ has two routes: (D) (discriminant, $\sigma \rightarrow \infty$) and (C) (curvature, σ bounded and $R \rightarrow \infty$). This section treats (D) and cancels a third route once proposed.

8.1. The local identity and the $g = 1$ combinatorial equivalence.

Proposition 8.1 (Ogg–Saito/Liu; the place-by-place form of (L1)). *For a regular arithmetic surface of any genus,*

$$\Delta_{\text{Del},p} = f_p + m_p - 1,$$

where f_p is the conductor exponent and m_p the number of components of the special fiber of the minimal regular model. For $g = 1$ this equals the minimal Weierstrass discriminant exponent (Ogg’s formula $\nu_p(\Delta_{\min}) = f_p + m_p - 1$).

Check (full Kodaira table):

Type	I_0	I_n	II	III	IV	I_0^*	I_n^*	IV*	III*	II*
$\nu(\Delta)$	0	n	2	3	4	6	$n+6$	8	9	10
f	0	1	2	2	2	2	2	2	2	2
m	1	n	1	2	3	5	$n+5$	7	8	9
$f+m-1$	$0\sqrt{}$	$n\sqrt{}$	$2\sqrt{}$	$3\sqrt{}$	$4\sqrt{}$	$6\sqrt{}$	$n+6\sqrt{}$	$8\sqrt{}$	$9\sqrt{}$	$10\sqrt{}$

Corollary 8.2. *For $g = 1$, $\sigma = 1 + \frac{m-1}{f}$, so additive (potentially good) reduction types all satisfy $\sigma \leq 5$ ($I_0^* \dots II^*$ attains) and the unique unbounded source is multiplicative (I_n , $\sigma = n$). Therefore*

$$\text{Szpiro } (g = 1) \iff n \leq 6 + \varepsilon \iff \text{boundedness of long } (-2)\text{-chains.}$$

Remark 8.3 (Outside semistability the conductor strictly exceeds the “effective conductor”). Liu and Saito define the effective conductor

$$\text{Effcond}(X_K/K) := \text{length}_{\mathcal{O}_K} \text{Coker}(H^0(X, \Omega) \rightarrow H^0(X, \omega))$$

(the conducteur efficace of Pesenti–Szpiro) and prove $\text{Effcond} \leq \text{Art}$. Combined with $-\text{Art} = \hat{m} - 1 + \hat{f}$ from Proposition 8.1, this gives a theorem-level reading of the mechanism of (D): outside semistability, the gap between “component-count contribution” and “effective conductor” is absorbed by torsion; in particular, lengthening a (-2) -chain increases m and hence Δ_{Del} while f is unchanged — this is exactly the escape window for the discriminant, requiring only Proposition 8.1, no Ogg formula and no conjecture.

Combinatorial dictionary in the semistable world. Semistable: (dual graph) $\#E = m$, $\#V = b_1 + 1$, $\nu(\Delta) = \#\{\text{nodes}\}$, $= b_1(\Gamma_{\text{dual}})$, and

$$\text{Szpiro} \iff \#\text{nodes} \leq (6 + \varepsilon) \cdot b_1.$$

8.2. Mestre forcing law and bounds for stable fibers.

Theorem 8.4. *Let the bad set of a tower be fixed S , $q_{\max} := \max S$. By Mestre’s inequality [24] (requiring analyticity of $L(A, s)$) and the combinatorial bound $m \leq 2g - 2$ for the stable fiber:*

$$\sigma \leq 1 + \frac{2g-3}{g} \cdot \frac{\log q_{\max}}{\log B_M} = 1 + c_S \cdot \frac{\log q_{\max}}{\log B_M}, \quad c_S = \frac{2g-3}{g} \leq 2.$$

Proof sketch. By (L1), $\sigma = 1 + \frac{\sum_p (m_p - 1) \log p}{\sum_p f_p \log p}$. Bounding each $m_p - 1 \leq 2g - 3$ (stable fiber, $m \leq 2g - 2$) and $\log p \leq \log q_{\max}$, the numerator is at most $(2g - 3) \log q_{\max} \cdot \#S$; Mestre’s bound $f \geq \log B_M$ place by place (conditional on the analyticity of $L(A, s)$; automatic on automorphic towers) gives the stated $\sigma \leq 1 + c_S \frac{\log q_{\max}}{\log B_M}$ with $c_S = \frac{2g-3}{g}$. The hypothesis that the fiber is stable, not merely semistable, is essential: semistability permits (-2) -chains, in which case m is unbounded (E5).

$q_{\max}$	7	11	101	313	317	10^6
$\log B_M / \log q_{\max}$	1.199	0.973	0.505	0.406	0.405	0.169
bound $\sigma \leq 1 + 2 \frac{\log q_{\max}}{\log B_M}$	2.668	3.055	4.954	5.924	5.935	12.837

Remark 8.5 (Convention unification, E43). All constants in this table use $\log B_M = 2.334374$ ($B_M \approx 10.323$); the previous version printed values computed with $\log 10$ in place of $\log B_M$ (they differ by 1.3%, since $B_M \approx 10$) and stated $c_S = \frac{2g-3}{g}$ without reconciling the two — the table above is the single consistent convention, matching Theorem 10.13.

Two corollaries: $\hat{f} \rightarrow \infty \Rightarrow \sigma \rightarrow \infty$ would contradict $f \leq g$; hence $\hat{f} \geq c_S \frac{\log q_{\max}}{\log B_M} > 1$... this yields $c_S \leq 1$, place-by-place consistent with Fontaine (“no nonzero abelian variety with good reduction outside $\hat{f} < 1$ ”), with existence allowed for $q_{\max} \in \{2, 3, 5, 7\}$. Mestre reproduces Fontaine independently.

For $q_{\max} = 11$, Szpiro holds automatically (no abc needed). The threshold lies between 313 and 317: $\text{rad } N_S \leq 313 \Rightarrow \sigma < 6$.

Remark 8.6 (Limited range). Theorem 8.4 covers only “semistable stable fibers” (minimal regular stable, no (-2) -chains). Semistability does not imply stability: semistability allows rational components, i.e. (-2) -chains, in which case m is unbounded. The (-2) -chains of the minimal regular model come from resolving the stable model; the chain length n is exactly the part by which σ exceeds its stable bound. One must distinguish: stable model $\# \text{singularities} = e_2$; minimal regular model $m \leq 2g - 2$. We formerly conflated the two; corrected (Appendix B, E5).

8.3. Cancellation of the archimedean channel. The third route once proposed was $d := \delta_{\text{Fal}} / \log \text{cond} \rightarrow \infty$.

Theorem 8.7. δ_{Fal} has logarithmic singularities along the discriminant divisor, and by the product formula the finite discriminant is the non-archimedean compensation of the same quantity:

$$\delta_{\text{Fal}} = -c_g \log |\Delta_{\text{disc}}|_{\infty} + O(g), \quad \Delta_{\text{Del}} = - \sum_p \log |\Delta_{\text{disc}}|_p = - \log |\Delta_{\text{disc}}|_{\infty} + O(1).$$

Hence

$$\delta_{\text{Fal}} \leq C \Delta_{\text{Del}} + O(g) \implies d \leq C\sigma + o(1),$$

i.e. the third channel is fully absorbed into the discriminant channel: $d \rightarrow \infty$ forces $\sigma \rightarrow \infty$. The archimedean channel is cancelled.

Singularity order. The logarithmic singularity of δ_{Fal} along the discriminant divisor is explicit in Wilms [21]: near a degeneration with vanishing cycle of length δ , $\delta_{\text{Fal}} \sim -(\delta + \varepsilon) \log |t| - 6 \log \det \text{Im } \Omega(t)$; the logarithm’s coefficient is tied to the vanishing-cycle length, which is why the finite discriminant is the exact non-archimedean compensation of the same quantity.

Sharp calibration for $g = 1$. From $\eta(iy) \sim e^{-\pi y/12}$ and $-24 \log |\eta| \sim 2\pi y$: $\delta_{\text{Fal}} \sim 2\pi \text{Im } \tau = \log |j|$. The Legendre family gives $N \Rightarrow \lambda = 1/N$, $\delta_{\text{Fal}} \sim 2 \log N$, $\Delta_{\text{Del}} \sim 4 \log N$, $\log \text{cond} \sim \log N$:

$$\frac{\delta_{\text{Fal}}}{\Delta_{\text{Del}}} \rightarrow \frac{1}{2}, \quad d \rightarrow 2, \quad \sigma \rightarrow 4.$$

The third channel not only fails to diverge but converges to a constant, and it is half of the discriminant channel.

Correction (Appendix B, E3): one formerly wrote $\delta_{\text{Fal}} \sim \frac{1}{3} \log |j|$; the coefficient is $2\pi \text{Im } \tau = \log |j|$. The earlier argument “ S -unit equations bound σ for fixed S , hence no tower exists” is correct at fixed genus but does not constitute an escape along the third channel.

8.4. Two closure theorems (negative results).

Theorem 8.8 (Evertse closure). *Let $y^2 = f(x)$, $f \in \mathbf{Q}[x]$, $\deg f = 2g + 1$. If the bad set is fixed, S , then all prime factors of $\text{disc}(f) = \prod_{i < j} (r_i - r_j)^2$ lie in S . Normalize $r_1 = 0$, $r_2 = 1$, $r_3 = c$; for $k \geq 4$, the root r_k : with $x = \frac{r_k}{c}$, $y = -\frac{r_k - c}{c}$ one gets $x + y = 1$ with x, y non- S -units, and by Evertse's theorem [6] ($x + y = 1$ in S -units) the number of solutions is finite with an explicit bound N_S depending only on $|S|$:*

$$n \leq 3 + N_S, \quad g \leq \frac{N_S + 2}{2}.$$

Totally rational branch-point hyperelliptic curves cannot form a tower with $g \rightarrow \infty$ at fixed bad set.

Theorem 8.9 (Belyi/PU route closed). *More generally, if the branch points α_i are S -units (the natural condition for PU/Belyi uniformization), the discriminant is an S -unit $\Rightarrow$ each difference has S -unit norm $\Rightarrow$ after normalization one again lands in an S -unit equation $\Rightarrow$ finitely many solutions $\Rightarrow g \nrightarrow \infty$. The whole route is closed.*

The numerical side is likewise negative. After correcting the root-collision count ($\lfloor n/3 \rfloor$), resifting all tuples for $p = 3$, $n = 5, 7, 9, 11$, the optimum is 3.5×10^7 (at $n = 11$) with a declining trend; while for fixed S , $\Theta(w)/w \approx 0.28$ is linear, so enlargement cannot repair it.

Corollary 8.10 (Sharpening of the search space). *A candidate tower for the discriminant channel must satisfy at least one of:*

Condition	Explanation
branch points not totally rational	$\text{disc}(f)$ being an S -unit does not force each difference to be an S -unit (they are algebraic numbers with S -unit norm), Evertse's argument fails
non-hyperelliptic	no such factorization of branch points/discriminant

All known towers (Fermat, $X_0(p^n)$, Heisenberg) satisfy one, and all have $\sigma \rightarrow 1$, consistent with the theorems.

Cost (honest label). *Once one leaves the hyperelliptic world, there is no integral-point/branch-point formula to compute, and the decomposition $\bar{\omega}^2 \Delta_{\text{Del}} \sim \prod (\alpha_i - \alpha_j)^2$ is lost, so both the discriminant and curvature channels may lose computability simultaneously. This is mutually exclusive with “finding towers”, and is evidence of negative expected return on this route.*

8.5. Unconditional exclusion of $\kappa \rightarrow \infty$ on Fermat/wild towers.

Theorem 8.11. *On Fermat/wild CM towers, $\kappa \rightarrow \infty$ is excluded unconditionally.*

Proof. By Proposition 2.3, $\kappa = \frac{\sigma(1+R)}{12} + \frac{d}{12}$. Use $R \geq 0$ (L2) for the first term and, for d , the lower bound (L3) $\delta_{\text{Fal}} > -Wg$, $W = 2 \log(2\pi^4)$, which gives $d = \delta_{\text{Fal}}/(g\hat{f}) > -W/\hat{f}$:

$$\kappa \geq \frac{\sigma}{12} - \frac{W}{12\hat{f}}.$$

By (T3) $\hat{f} \rightarrow \infty$, so $\kappa \geq \frac{\sigma}{12} - o(1)$; hence $\sigma \rightarrow \infty$ forces $\kappa \rightarrow \infty$. (The former proof quoted the upper bound (L4) $d \leq C\sigma + o(1)$ which is the wrong direction for a lower estimate; corrected, E20.) Now $\kappa_{\text{Fermat}} = \frac{1}{8}$ is a proved theorem (Corollary 3.5), so $\sigma \leq 12\kappa + o(1) = \frac{3}{2} + o(1)$ is a proved theorem: σ is bounded on this tower. $\square$

Remark 8.12 (Scope). This covers Fermat/CM towers; an upper bound on σ for general wild towers remains open (see §10).

9. THE CURVATURE CHANNEL

By Proposition 2.3 and Theorem 8.7,

$$\text{no } \kappa \rightarrow \infty \iff \sigma = O(1) \text{ \& } R = O(1),$$

i.e. Szpiro and arithmetic Bogomolov control the two factors separately. This section treats the second factor, $R = O(1)$.

9.1. An unconditional reduction theorem.

Theorem 9.1.

$$\bar{\omega}^2 \leq 12h_F + 2 \log(2\pi^4) g = 12h_F + 10.544133 g.$$

Three inputs (all theorems). 1. Noether (Wilms): $12h_F = \bar{\omega}^2 + \Delta_{\text{Del}} + \delta_{\text{Fal}}$; 2. $\Delta_{\text{Del}} \geq 0$: a sum of non-archimedean local terms, each ≥ 0 ; 3. Wilms [21, Cor. 1.2]: for every compact connected Riemann surface X , $g \geq 1$,

$$\delta_{\text{Fal}}(X) > -2g \log(2\pi^4), \quad \delta_{\text{Fal}} = -24H + 2\varphi - 8g \log 2\pi.$$

Derivation: solve Noether for $\bar{\omega}^2 = 12h_F - \Delta_{\text{Del}} - \delta_{\text{Fal}}$; use $\Delta_{\text{Del}} \geq 0$ for $-\Delta_{\text{Del}} \leq 0$ and the Wilms lower bound for $-\delta_{\text{Fal}} < 2g \log(2\pi^4)$. Unconditional: no abc, no BMY, no conjecture.

Corollary 9.2 (Reduction).

$$\text{Faltings-Shafarevich } (h_F = O_S(g)) \implies \text{arithmetic Bogomolov } (\bar{\omega}^2 = O(g)).$$

Meaning: the task “bound $\bar{\omega}^2$ above” is replaced by “bound h_F above” — the direction changes and so does the toolbox (Faltings finiteness, moduli spaces, Belyi bounds). But see Remark 9.2: the tools are still missing.

9.2. A necessary correction. A statement made in three rounds — “fixed bad set S implies $\bar{\omega}^2 \leq C(S)g$ ” — is false. Counterexample: $S = \{p\}$ fixed, while Dolce–Mercuri [5] give $\bar{\omega}^2 \sim 3g \log N$ for $X_0(p^n)$: $\bar{\omega}^2 \rightarrow \infty$, not $O(1)$. The correct statement is

$$S \text{ fixed \& } \hat{f} = \frac{\log \text{cond}}{g} = O(1) \implies \bar{\omega}^2 = O_S(g).$$

Since $\hat{f} = O(1)$ is exactly the region where κ diverges, the correction affects no existing conclusion; the hypothesis must be added (Appendix B, E11).

9.3. A non-circular cross-check. Solving Proposition 2.3 against (L3) (write $\hat{f} = \log \text{cond} / g$):

$$R < \frac{12\kappa}{\sigma} - 1 + \frac{2 \log(2\pi^4)}{\sigma \hat{f}}.$$

Substituting the Fermat values $\kappa = \frac{1}{8}$, $\sigma = 1$, $\hat{f} \rightarrow \infty$:

$$R \longrightarrow \frac{12 \cdot \frac{1}{8}}{1} - 1 = \frac{1}{2},$$

matching the independently obtained $R \leq \frac{1}{2}$ from Colmez’ explicit height plus the Liu–Saito conductor, with disjoint inputs: the latter uses CM theory and L -values, the former only Noether and the pure-analytic Wilms lower bound. The limiting R -values of all four towers (Fermat, X_0 , NS, Heisenberg) align with independently known values.

Difference from a withdrawn “check”: an earlier version solved the identity for $\kappa = \frac{1}{8}$ and read off R : that contains no information (circular). The direction here is opposite — the independent Wilms lower bound constrains R — hence non-circular.

9.4. The reverse direction is unreachable. From $12h_F = \bar{\omega}^2 + \Delta_{\text{Del}} + \delta_{\text{Fal}}$, obtaining $\bar{\omega}^2 = O_S(g)$ requires bounding h_F above; and the whole chain already exists in the literature: this corollary appears in Parshin’s 1988 paper [16]. We merely named the intermediate step. In the $\hat{f} = O(1)$ region, arithmetic Bogomolov is equivalent to “no $\kappa \rightarrow \infty$ ” (since $\bar{\omega}^2 = O(g) \iff R\sigma = O(1)$), so it is not an independent seal alongside Szpiro — we formerly estimated so, and retract it (Appendix B, E12).

9.5. What survives. Arithmetic Bogomolov is strictly weaker than BMY: if $\sigma = O(1)$ then BMY gives $\bar{\omega}^2 \leq 3\Delta_{\text{Del}}$ ($= 3\sigma \log \text{cond}$, i.e. Bogomolov); the converse fails. Refuting $\kappa \rightarrow \infty$ does not require full BMY, only the scale $R = O(1)$, not the constant. Theorem 9.1 is unconditional and reduces the problem to an upper bound on the height.

Kept judgment: arithmetic Bogomolov $R = O(1)$ is more likely true than BMY — it is true in the function field (Deligne’s semipositivity of the Hodge bundle) and is a strict weakening of BMY. The two can be separated; they need not share a fate.

10. OPEN PROBLEMS AND THE FUNCTION-FIELD COMPARISON

10.1. Function-field comparison: a missing pair of inputs.

Channel	Function-field mechanism	Number-field status
$R = O(1)$ ($\bar{\omega}^2/g = O(1)$)	Deligne semipositivity	missing = Bogomolov
$\sigma = O(1)$ ($\Delta_{\text{Del}}/g = O(1)$)	Mason	missing = Szpiro/abc
spectral interpretation of the finite Δ	Werner [20], Wei	partial

The function field seals both channels with two independent theorems; the number field has neither. The precise form of “the function field has Mason, the number field only the abc conjecture” is: Mason is only one of the two.

On the spectral row: the interpretation of the Néron local height pairing via the Laplacian of the dual graph on semistable curves is established in the published literature (Werner [20], Wei) for Mumford curves; a recent preprint extends it to arbitrary semistable reduction and attempts an identification with boundary two-point functions in p -adic string theory. In that language, $\sigma = \# \text{nodes}/b_1$, so Szpiro is equivalent to the boundedness of the scale of the two-point function on the unit-torus rank. This is an independent paraphrase and confirmation of channel (D). But the equivalence is published only for the Mumford-graph case; the extension preprint gives no new bound and is not peer-reviewed. We do not use it as input to any derivation.

10.2. Open problems.

Problem 10.1 (General arithmetic BMY). Does $\bar{\omega}^2 \leq 3\Delta_{\text{Del}} + O(g)$ hold for all towers? = the curvature channel; in towers with $\sigma \rightarrow 1$ it is the same question as the barrier. We record it as Hypothesis [Arith-BMY]; the known classes:

class	status	source
$X_0(N)$, $(N, 6) = 1$, squarefree, $g > 0$	theorem	Michel–Ullmo [29]/Abbes–Ullmo [30]
$X_0(p^n)$, $p > 3$	theorem	Dolce–Mercuri [5]
$X_1(N)$, N squarefree, ≥ 2 factors ≥ 4	theorem	Mayer [26]
$X(N)$	theorem (co-eff. 2, not 3)	Grados–von Pippich [27]
general GL_2 -type tower	open	—

In every known class $\sigma \rightarrow 1$, so $R \rightarrow c$ with c the coefficient (2 or 3), and BMY is asymptotically attained only in the coefficient-3 families.

Problem 10.2. Can $\kappa_n \rightarrow \infty$ occur? (Discriminant channel $\sigma \rightarrow \infty$ or curvature channel.) All known towers have $\sigma \rightarrow 1$, $R \leq 3$.

Problem 10.3. Search space: the tower must either have non-totally-rational branch points or be non-hyperelliptic (Corollary 8.10); this condition is mutually exclusive with computability.

Problem 10.4 ($g = 1$ barrier-breaker). Threshold $|c_4| \geq 10^{47}$, the known record is short by a factor ~ 10 (lattice reduction). Update (E25): an explicit extremal family is now available at $m = 4$ (Davenport-extremal, $\deg f = 8$, $\deg g = 12$, $\deg h = 5 = m + 1$), giving $C(4) = \frac{5m-1}{m+1} = \frac{19}{5} = 3.8$ for $\delta_{\text{Fal}}/\Delta_{\text{Del}}$; but convergence is extremely slow — with the $-6 \log \log$ correction the ratio first exceeds 3 only at parameter $t_0 \sim 10^{16.4}$ ($\log |\Delta_{\min}| \approx 218$), and reaching 3.5 requires $t_0 \sim 10^{50}$. The extremal ratio is a limit, not a finite-level value. Caveat: the underlying asymptotics are unreliable at small parameters.

Problem 10.5 (Siegel/L-value channel). In all $g > 1$ cases it does not bind, and is “provable but not computable”. Rated low priority.

Problem 10.6 (Mestre’s hypothesis). Mestre’s conductor bound $f \geq \log B_M$ depends on the analyticity of $L(A, s)$ (Langlands-type hypothesis). Bypassable on automorphic towers; conditional on general towers.

Problem 10.7 (Normalization tension). Tension between the two normalizations of h_F (period formula vs. Noether): affects the threshold reading in the $g = 1$ branch.

Problem 10.8 (Curve realization of convexity towers). Single Jacobian realization of the interior filling: the abelian-variety version is complete (Theorem 7.8).

Problem 10.9. Towers neither non-hyperelliptic nor Belyi: no attainable construction currently.

Problem 10.10 (E19: resolved, negatively). Partially ramified orbits with $\bar{B} > 0$ but $B \ll g$. Status: resolved, negatively. Instances with $\bar{B} > 0$, $\bar{B}/g \rightarrow 0$ and $\nu_p/\bar{B} \rightarrow \infty$ do exist (explicit diagonal families: $\bar{B} = 1$ fixed, $\nu_p = 1 + 4t$, $g = 4, 5, 7, 10, 13$, all squarefree with $h \bmod p$ nonreduced). The mechanism is forced: a single tame Eisenstein factor contributes $\nu_p = d - 1 = \bar{B}$ so the ratio is 1; divergence requires collisions (resultants of positive p -adic valuation) between unramified factors. But the bad set cannot be kept fixed: forcing the collisions makes $|S|$ grow ($7 \rightarrow 26$ in the measured family), violating (T2). Hence this corner produces no tower and does not refute the discriminant channel. Caveat: $\nu_p/\bar{B}$ is a ratio of the model discriminant to the lower bound; it does not by itself force the conductor exponent or σ to diverge.

Problem 10.11 (Independent proof of (L1)). An independent first-principles proof of (L1) for arbitrary genus: Liu 1994 Prop. 1 is commonly cited for arbitrary genus, but no explicit independent proof has been found (Remark 2.5).

Problem 10.12 (Single-curve realization of the interior filling). Does there exist a curve tower (single Jacobians, as in Definition 1.1) whose κ -limit is an interior value of $(\frac{1}{12}, \frac{1}{3})$? The product constructions work in the abelian-variety category (Theorem 7.8); the curve category is open.

10.3. The final form of the discriminant channel: thickness language and an explicit bound. *Thickness formula.* Let X^{st} be the stable model; the local equation of the special fiber at node i is $xy = p^{N_i}$, and N_i is the thickness of the node. The minimal regular model resolves these singularities, inserting $N_i - 1$ copies of $\mathbf{A}^1$ at each.

$$(10.1) \quad (\text{D}) \iff \frac{\nu}{f} \rightarrow \infty \quad \text{and} \quad \frac{\sum_p \text{Sw}_p}{g} \rightarrow \infty$$

(the second condition is exactly (T3), by the wild-necessity lemma). Thickness alone is insufficient: the $\mathbf{P}^1$ -cactus below has arbitrarily large $\sum N_i$ yet $\hat{f} = \log p = O(1)$, hence lies outside the tower world. One checks $b_1 = \# \text{nodes}$, $m = m^{\text{st}} + \sum_i (N_i - 1)$, $\nu_p = \sum_i N_i$, $f_p = b_1(\Gamma_p) + \text{Sw}_p$ (check: n nodes of thickness 1 gives $\nu = n$; or a single self-node of thickness n gives $\nu = n$; the two readings agree). The stable fiber has $\#V \leq 2g - 2$, so the unique source of $\#V \gg g$ is the thickness itself, i.e. channel (D) is the higher-genus Szpiro conjecture in thickness language.

An extremal candidate passing Noether. Take the stable fiber to be a $\mathbf{P}^1$ -cactus: one rational curve with g self-nodes, thicknesses $N_1, \dots, N_g$. Then $f = b_1 = g$ (torus rank, $\text{Sw}_p = 0$), $\nu = \sum_i N_i$, $\sigma = \frac{\sum_i N_i}{g}$. Its minimal regular model (one vertex with hanging self-chains of lengths N_i) passes Noether place by place: $\#V = 1$, $B = g$, $L = 2B - 2 = 2g - 2$, $F \cdot b_1 = 1$, $\nu = B(L + 1)$, $\sigma = L + 1$ arbitrary in N_i . On known towers $N_i = O(1)$, $\sigma \rightarrow 1$ — consistent with §2.4.

Semistability hypothesis. The thickness identity $\nu_p = \sum_i N_i$ and the reduction $f_p = b_1 + \text{Sw}_p$ are used only in the semistable setting, where $\dim(V/V^{I_p}) = t$; outside semistability the correct local identity is (L1) $\Delta_{\text{Del},p} = f_p + m_p - 1$, and the two are not interchangeable — e.g. for $X_1(p)$ ($p > 7$) the minimal regular model has geometrically integral special fiber (Conrad–Edixhoven–Stein [28, Thm. 1.1.2]), so $m_p = 1$ and $\Delta_{\text{Del},p} = f_p > 0$, while the semistable thickness reading would give $\nu_p = 0$. All statements in this section are to be read under semistability (E26).

A new mechanism specific to $g \geq 2$. For $g = 1$, $b_1 = 1$, so a semistable singular fiber can only be an n -gon, $b_1 = n$; every bad prime has $f_p \geq 1$ and the discriminant has no “free lunch”. For $g \geq 2$ there are tree-like fibers with $b_1 = 0$ (e.g. two elliptic curves meeting at one point): then $f_p = 0$ (the Jacobian in fact has good reduction at p) while $\nu_p = m_p - 1 > 0$. The discriminant can be supported at primes invisible to the conductor. Szpiro intuition does not transplant to higher genus.

Theorem 10.13. *Write $\hat{f} = \log \text{cond} / g$. From $12\kappa = \Omega + \sigma + d$ and three inputs — (L2) Zhang $\Omega \geq 0$; (L3) Wilms $\delta_{\text{Fal}} > -Wg$, $W = 2 \log(2\pi^4) = 10.544133$; the conductor bound $\hat{f} \geq \log B_M$*

$$d = \frac{\delta_{\text{Fal}}}{g\hat{f}} \geq -\frac{W}{\log B_M} \implies \sigma \leq 12\kappa + \frac{W}{\log B_M}.$$

With Mestre’s bound $\hat{f} \geq \log B_M$ where $\log B_M = 2.334374$ (i.e. $B_M \approx 10.323$): $\sigma \leq 12\kappa + \frac{W}{\log B_M} = 12\kappa + 4.5169$; for GL_2 towers $\sigma \leq 8.5169$ (and $\sigma \leq 4$ when $\hat{f} \rightarrow \infty$). Notation warning: 2.334374 is $\log B_M$, not B_M (E43).

Channel (D) is closed for all GL_2 (automorphic/CM) towers, consistent with all five anchors of §2.4.

The third input is conditional. Mestre’s lower bound requires, per four-source verification, that $L(J, s)$ be an analytic L -function: Tchamitchian [18] (assumption added in v2/v3), Mestre [24] (zbMATH abstract), Kowalski (MO #10860), and the Persiflage blog. Version-level evidence: the paper’s v1 abstract read “Any abelian variety over $\mathbf{Q}$ of dimension g has conductor greater than $10^{0.323g}$ ” (no hypothesis); v2/v3 add the hypothesis. Even the specialist author once omitted it — Mestre must always be cited with the hypothesis. Hence Theorem 10.13 is unconditional on automorphic towers, conditional on general towers.

10.4. The structural mismatch between tools and loopholes. This is the last and methodologically most important observation of the paper.

What Mestre covers (automorphic towers) is exactly the class of the five anchors of §2.4, where Theorem 10.13 already gives $\sigma \leq 8.5169$; the (D) loophole requires average thickness (§10.3), which is necessarily non-semistable and expected non-automorphic — exactly outside

Mestre's range; and the unconditional conductor bound is only $\hat{f} \geq O(1)$ (Fontaine: no abelian varieties above $\hat{f} < 1$; Abrashkin-type results). Thus $\hat{f} \rightarrow 0$ cannot be excluded, and then $d = \delta_{\text{Fal}}/(g\hat{f}) \rightarrow -\infty$ is not ruled out, so all constraint from (L2)+(L3) is lost. (§10.6 gives an unconditional bound for hyperelliptic Jacobians, repairing this over a wide range; but it can vanish identically at genuine bad primes, so the gap is moved, not closed.)

The inability to rule out $\kappa \rightarrow \infty$ is not because the crack is wide, but because the only tool is effective on the half where the answer is already known. This sentence explains why §8–§10.6 keep refining the reduction everywhere without producing a new theorem. This is not a temporary technical gap but a structural one. We propose to leave (D) and Problem 10.1 as open problems for follow-up, and judge accordingly: the expected return of further search along the discriminant channel is negative.

10.5. A theorem-level closure: cyclotomic towers. Let p be an odd prime with fixed bad set $\{2, p\}$.

Cluster picture and two exact quantities. $C_m : y^2 = x^{p^m} + 1$, $g_m = \frac{p^m-1}{2}$. The root set is $\{-\zeta_{p^m}^k\}$, so pairwise valuations depend only on $v(1 - \zeta_{p^m}) = \frac{1}{p^{M-1}(p-1)}$; the cluster picture is a full v -adic tree: clusters of size p^s , each in number p^{m-s} ($s = 1, \dots, m$), depth $\frac{1}{p^{s-1}(p-1)}$. p odd $\Rightarrow$ all cluster sizes odd $\Rightarrow$ no even clusters $\Rightarrow$ torus rank $t = 0$. Hence

$$f_p = 2g + \text{Sw}_p, \quad \nu_p(\Delta) = mp^m,$$

i.e. the tower is potentially good, and Sw_p is the only unknown.

Conductor lower bound, verified. By the explicit form of the conductor formula for induced representations, $a(\text{Ind } \psi) = \text{val}(\mathfrak{d}_{L/k}) \dim \psi + f_{L/k} a(\psi)$; for Fermat–Jacobi quotients the conductor is $|\text{disc } \mathbf{Q}(\mu_n)| \cdot N_{K/\mathbf{Q}}(\mathbf{m})$; Cain [3] gives the formula, and Rohrlich [31] proves $f \mid p^2$, so each Hecke character contributes $c_j \leq 2$. The value $v_p(d_{\mathbf{Q}(\zeta_{p^j})}) = p^{j-1}(j(p-1) - 1)$ differs from the lower bound by exactly 1 or 2, cross-confirming Rohrlich's bound. The earlier concern “reflex fields might overestimate $\bar{B}$ ” is excluded: Weil's (1952) Jacobi-sum Grössencharaktere are defined on $\mathbf{Q}(\zeta_{p^j})$ itself; Gallese–Goodson–Lombardo [9] prove each simple factor nondegenerate; Cain's formula uses exactly the discriminant of $\mathbf{Q}(\zeta_{p^j})$.

$$f_p \geq S_m := \sum_{j=1}^m v_p(d_{\mathbf{Q}(\zeta_{p^j})}) = \frac{p^m(m(p-1) - 2) + 2}{p-1},$$

$$S_m \leq f_p \leq S_m + 2m \implies \frac{mp^m}{S_m + 2m} \leq \sigma_p \leq \frac{mp^m}{S_m},$$

both ends tending to 1. Quantitatively (E42):

$$\sigma_p \leq \frac{mp^m(p-1)}{p^m(m(p-1) - 2) + 2} \rightarrow 1 + \frac{2}{m(p-1) - 2} \quad \left(\rightarrow \frac{m}{m-1} \text{ at } p=3 \right),$$

and $S_m/(2g_m) \rightarrow m - \frac{2}{p-1}$ (with $g_m = \frac{p^m-1}{2}$ the superelliptic genus, not the Fermat genus).

Numerics: for $p=3$, $\sigma \leq 1.4727, 1.3279, 1.1997$ at $m=3, 4, 6$, against the asymptotic $\frac{m}{m-1} = 1.5, 1.333, 1.200$. At $q=2$ no patch is needed: the denominator takes a partial sum (safe in the discard direction), and the contribution $\nu_2 = 4g \sim 4 \log 2$ is $o(mp^m)$. With the unconditional lower bound (L1), $\sigma \geq 1$:

$$\boxed{\sigma \rightarrow 1 \quad (\text{theorem level, no estimate needed})}$$

Meaning: Theorem 10.13 gives only a constant upper bound on σ ; this section gives the asymptotic. The two are complementary, and the route here bypasses Mestre entirely — on this tower class (D) is sealed not by “a tool covering the half where the answer is known” but by

an independent conductor formula. This sharpens the meta-observation of §10.4: there is more than one rescue mechanism.

Honest labels. Cain’s formula is for a single Fermat factor; the decomposition of f comes from eigenvalue analysis; the literature gives only “each simple factor nondegenerate” without listing CM fields one by one; only $c_j \leq 2$ is known. Cain states the composite case only in a sketch at $n = 2$, so taking $c_j \leq 2$ factor by factor is a corollary, not a direct citation.

10.6. An unconditional conductor bound, and why it still does not suffice. *The Brumer–Kramer bound.* Let K be a p -adic field, p an odd prime; $h(x) \in K[x]$ a separable polynomial of odd degree $\deg h = 2g + 1$, $B = \text{Jac}(y^2 = h)$. Factor $h = h_1 \cdots h_r$ with root fields $M_i = K(\theta_i)$. Then [1, Lemma 6.3]

$$f_p(B/K) \geq \bar{B} := \sum_{i=1}^r v_K(\mathfrak{d}_{M_i/K}) f_{M_i/K},$$

with equality when h is irreducible and totally ramified. The key point: it uses only the Artin conductor of $B[2]$ — no L -functions, no modularity, no analytic continuation — and it is fully computable from the factorization of h over K and the relative discriminants of the root fields. Three hard restrictions: p odd; the shape $y^2 = h$; $\deg h$ odd and separable.

Comparison with §10.5: $\bar{B} = S_m$. On the cyclotomic tower $y^2 = x^{p^m} + 1$,

$$x^{p^m} + 1 = (x + 1) \prod_{j=1}^m \Phi_{2p^j}(x), \quad M_j = \mathbf{Q}_p(\zeta_{p^j}),$$

totally ramified, $v_p(\mathfrak{d}_{M_j/\mathbf{Q}_p}) = p^{j-1}(j(p-1) - 1)$;

$$\bar{B} = \sum_j p^{j-1}(j(p-1) - 1) = S_m.$$

m	g	ν_p	S_m	$\sigma \leq \nu_p/S_m$	$S_m/2g$
3	13	81	55	1.473	2.12
4	40	324	244	1.328	3.05
6	364	4374	3646	1.200	5.01

Identical to §10.5 place by place. But this route needs neither Cain, Rohrlich, nor Weil, nor reflex-field arguments — the risk drops from “a four-section corollary chain” to “one standard 1994 lemma”. Thus $\sigma \rightarrow 1$ is a theorem-level conclusion confirmed by two independent paths.

Counterexample: $\bar{B}$ can vanish identically at a genuine bad prime (Appendix B, E10). Take $u = 2$ (a non-square in $\mathbf{F}_3$), $p = 3$,

$$h = (x^2 - u)(x^2 - 2p^m x + (p^{2m} - u))(x - 1).$$

The two quadratic orbits have discriminants $4u = 8$ and $4u = 8$, both unramified, so $\bar{B} = 0$. But they collide mod 3 ($h_1 \equiv h_2 \equiv x^2 + 1$), and $p = 3$ is a genuine bad prime:

m	$\nu_3(\text{Res})$	$\nu_3(\text{disc } h)$	$\bar{B}$	$\nu_3/\bar{B}$
1	2	4	0	∞
2	4	8	0	∞
3	6	12	0	∞

The criterion is clean: every M_i/K unramified $\Rightarrow \bar{B} = 0$. So this is not “the bound is too weak” but “the bound can vanish outright”. And the mechanism scales with genus: take

$$h = \prod_{i=1}^s (x^2 - u_i) \prod_{i=1}^s (x^2 - 2p^m x + p^{2m} - u_i)(x - c), \quad g = 2s, \nu_p = 2gm, \bar{B} = 0,$$

$n \asymp g \Rightarrow \nu_p \asymp g^2$, of which Brumer–Kramer knows nothing. Measured at $p = 5$, $s = 2$, $g = 4$; $m = 1, 2, 3$ give $\nu_5 = 8, 16, 24$, confirmed place by place.

Neighboring-bad-set tension. Why does this not save (D)? Because $\text{Res} = p^{2m}(p^{2m} - 4u_i)$, and keeping unramifiedness requires u_i to be a unit mod p , so $p^{2m} - 4u_i \equiv -4u_i \not\equiv 0 \pmod{p}$: an integer of this size must acquire new prime factors beyond p . Measured: $p = 7$, $m = 3$ gives $p^{2m} - 4u = 117641 = 47 \times 2503$. The larger p, m , the more new primes, so the bad set must grow. Quantifier note (E36): for fixed u and $n \rightarrow \infty$ a Bang–Zsigmondy-type argument for $p^{2n} - 4u$ should yield a provable growth theorem (a publishable lemma); when u varies with m the statement is open and may be a counterexample nursery. The tension is accordingly labelled: theorem-to-be for fixed u , open in general. Thus the hyperelliptic family trichotomizes:

Case	Criterion	Consequence
(i) split completely	—	Evertse closure (Theorem 8.8); no tower
(ii) every orbit unramified	$\bar{B} = 0$	neighboring-bad-set tension: bad set grows with m
(iii) some orbit ramified	$\bar{B} > 0$	the bound bites

Cases (i) and (ii) are completely different mechanisms (S -unit equations vs. conductor-bound degeneration), yet are captured by the same criterion $\bar{B} = 0$ — the most unexpected finding of this round.

Three comparison families: BK is region-optimal, not universal:

Family	What BK gives	Comparison with other tools
strongest wild X_1 twist $y^2 = x^5 + p^{2n}$	$\bar{B} = 404$ vs $2g = 80$ $\sigma \leq \text{mod } 2, n \leq 4g + 1$	strongest weaker than Noether (Zhang+Noether give $O(1)$)

Verdict: the gap is moved, not closed. BK does not close the gap of §10.4; it only moves it from (D) into a weak version: for hyperelliptic Jacobians, case (iii) now has an unconditional bound independent of Mestre; but case (ii) shows the bound can vanish at genuine bad primes, and then the bad set must grow, so the conclusion retreats to “a weak version with growing bad set” rather than the original problem. The remaining exposure is Problem 10.10.

Honest labels. The three hard restrictions of [1, Lemma 6.3] (p odd; $y^2 = h$; $\deg h$ odd separable) are not removable; $\nu_p = 2gm$ is measured only up to $s = 2$ (a larger p is needed, since $\mathbf{F}_3$ has only one non-square class, not accommodating several distinct u_i); $s \geq 3$; the tension of §10.3 is presently a mechanism argument plus numerical confirmation, not a theorem; Brumer–Kramer’s upper bound (their Theorem 6.2) we could not reproduce — the definition of λ is missing from the available text, and the most natural guess (digit sum) at $p = 3$, $g = 40$ gives 20824, conflicting with the lower bound. Hence that upper bound is unusable; only the lower-bound direction is used here, which is unaffected.

10.7. A meta-observation. Three “geometrically dangerous” instances are all rescued by arithmetic quantities, and each rescue covers only automorphic/CM towers:

Dangerous instance	Geometric side	Rescue (scope)
dense systole, $\#\Theta \rightarrow \infty$	fail ($R \leq \frac{3}{2}$ allowed)	CM towers only
CM + thin-dense ($c > 1$)	fail	CM towers only
long (-2) -chains ($\sigma \rightarrow \infty$)	extreme allowed	needs analyticity of $L(A, s)$

The geometric side (Bergman/systole) has zero effect on all three instances; the arithmetic side rescues three times, but all unconditionally only on automorphic towers. $\kappa \rightarrow \infty$ is not excluded not because the crack is wide, but because all our tools are effective only on arithmetic towers.

10.8. A tension concerning BMY $\Rightarrow$ effective Mordell. BMY implies effective Mordell. If $\text{GRH} \Rightarrow \text{BMY}$ extended to general towers, then $\text{GRH} \Rightarrow \text{effective Mordell}$ — suspiciously strong. Our judgment: $\text{GRH} \Rightarrow \text{BMY}$ most likely holds only for CM/automorphic families (where L -value formulas exist), while general BMY is exactly Problem 10.1, out of range. This is a judgment, not a conclusion; we register it.

APPENDIX A. FALSIFICATION OF AN EXTERNAL FRAMEWORK: THE RIGIDITY-SPECTRUM TEST

This appendix is independent of the main line, but records a complete falsifiability test, as a methodological specimen. It may be read separately or skipped.

A.1. Background. A recent phenomenological framework (self-described as “not a theorem, analogous to critical phenomena and renormalization group in physics”) postulates a universal set of dimensionless constants

$$\text{LOW} = 0.035, \quad \text{BAND} = [0.62, 0.71], \quad \text{SAT} \approx 0.97 \sim 1,$$

called the rigidity spectrum, and asserts that “most arithmetic problems fall in this band”, with the screening criterion “nonvanishing leading term of a functional equation at the critical point”. All five anchor towers of §2.4 satisfy this screening criterion (all are standard automorphic/CM objects). Hence, if the framework is self-consistent, they should land in the band. Taking our κ as the candidate for its “modulus” provides the test.

A.2. Pre-registration (written before computing).

No.	Prediction	Outcome
P1	No raw κ lands in the band	✓ (0/5)
P2	Scaling preserves the spread ratio	✓ (band ratio 1.145)
P3	The most natural normalization 3κ spreads the five towers over $[0.25, 1]$ rather than clustering	✓ (band catches 1/5)
P4	With σ as modulus the five towers degenerate to the saturation end; band still empty	✓
P5	The band contains several mutually unrelated constants, suggesting a density artifact of constant-scanning	✓ (4/10)

A.3. Results. T_0 (raw): $\kappa \in \{0.083, 0.125, 0.167, 0.208, 0.333\}$; hits 0/5. The maximum misses the lower edge of the band by 0.29.

$S1$ (structural exclusion — the strongest): scaling preserves the spread ratio, and

$$\frac{\kappa_{\max}}{\kappa_{\min}} = \frac{1/3}{1/12} = 4, \quad \frac{0.71}{0.62} = 1.1452, \quad 4 > 1.145.$$

Hence no c can send all five values into the band simultaneously. This is independent of the specific values of κ , far stronger than case-by-case comparison — consequently “the normalization was not chosen correctly” is not an available explanation.

S2 (affine transformation works but is ad hoc): with the affine form $a + b\kappa$, the pair $(a, b) = (0.59, 0.36)$ sends the five towers to 0.62, 0.635, 0.65, 0.665, 0.71, all in the band: indeed $0.59 + 0.36 \cdot \frac{1}{12} = 0.62$ and $0.59 + 0.36 \cdot \frac{1}{3} = 0.71$. But the two coefficients have no mathematical meaning in either framework; adopting them is ad hoc fitting.

T_1 – T_4 (four motivated transformations): the most natural normalization by the proved barrier gives $3\kappa \in \{0.25, 0.375, 0.5, 0.625, 1.0\}$: the five towers spread uniformly over $[0.25, 1]$, and the band touches only one value (0.625, just 0.005 above the lower edge). Alternative candidate moduli: $\sigma \rightarrow 1$ (the five towers collapse to one point); $R \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 3\}$. Band fill rates: 0/5 throughout.

A.4. Two side findings. *Constant density.* $[0.62, 0.71]$ contains $\ln 2 = 0.6931$, $1 - 1/e = 0.6321$, $1/\sqrt{2} = 0.7071$, $2/3 = 0.6667$ — four constants of completely unrelated origin. A window of width 0.09 containing four unrelated constants is more plausibly a density artifact of constant-scanning. $1/\varphi = 0.6180$ misses the band by 0.002, showing the band edge is highly sensitive to fitting choices.

The spectrum may not be independent. The framework states $\Phi_{\text{Tate}} \approx 0.7 \in \text{BAND}$ and $\Phi_{\text{cusp}} \approx 0.07$, while $0.07/2 = 0.035 = \text{LOW}$. If both hold, the “rigidity spectrum” is really $\{\Phi_{\text{cusp}}/2, \Phi_{\text{Tate}}, 1\}$: two amplitudes plus a trivial critical point, not three independent eigenvalues.

Multiple comparisons. $3 \times 4 + 10 = 22$ variables, 4 transformations, 10 constants, 4 chances: best result 1/5. After correction this is not a signal.

A.5. Verdict and the disposition of this paper.

Item	Verdict
κ vs. the spectrum	falsified (0/5 + structural exclusion)
σ, R vs. the spectrum	falsified (band empty)
Whether the spectrum is independent of Φ	doubtful; pending confirmation by the framework’s side

We do not merge the framework: the main line is theorem-driven, the framework is data-driven, their correctness standards differ, and merging would dilute the proved part. But two independent convergence points are retained as cross-checks: the framework’s “dangerous direction = bands with multiplicative parameter (−2)-chains” $\leftrightarrow$ our independent identification of the (D) loophole (additive reduction + long (−2)-chains, §10.3); and the framework’s “Ogg rewrite $\Rightarrow$ Szpiro form” $\leftrightarrow$ our independent discovery of the Ogg even-graph Euler characteristic $\nu = \#E$, $f = b_1$, $m = \#V - 1$ (§8). Two programs, completely different methods, pointing to the same conclusion — such independent convergences are worth more than the framework’s own fits.

Final word (to the framework’s side): what it lacks is not more fitting but one a priori quantitative prediction about a new object — written down before measurement. Do it once, and the nature of the framework changes.

Honest labels. The band edges are taken from a paraphrase, not verified against the original verbatim; this test assumes κ is a legitimate candidate for the “modulus”; if the framework defines it otherwise, the test must be rerun under its definition with the five towers’ values. If the reply is “these five towers belong to another sector”, that reply makes the spectrum unfalsifiable — the most common failure mode of phenomenological frameworks.

APPENDIX B. REVISION HISTORY

This is the fourth version (2026-10-29). The third version (final, 2026-10-09) added §10.6 and E9, E10; the present revision incorporates a certified external revision list of 33 items (F1–F33),

recorded as E16–E47 below and flagged inline, and upgrades the Jorgenson–Kramer input to [25, Thm. 5.6 and Rem. 5.8]. The second version (2026-10-08) added: §3.2 (exact order of the remainder of the main chain; the first version’s $\Theta(n \log p \cdot p^{-n})$ was wrong); the revision history itself; the thickness language for the discriminant channel. The third version adds §10.6 (the Brumer–Kramer unconditional bound) and E9, E10 below. All corrections:

No.	Correction	Disposition
E1	Remainder order $O(n \log p \cdot p^{-n})$	should be $O(p^{-n})$; slope attribution in Corollary 3.3
E2	Conductor density $c \rightarrow$	should be $c = 2$ (direct Liu–Saito recomputation)
E3	$\delta_{\text{Fal}} \sim \log j $	coefficient is $2\pi \operatorname{Im} \tau = \log j $
E4	$\kappa_X = \frac{1}{4c}$	breaks the floor; violates Zhang
E5	stable vs. minimal regular model conflated	(-2) -chains make m unbounded; must distinguish
E6	convexity weight $\frac{a}{a+b}$	true weight $\frac{a\rho}{a\rho+b}$ (§7.2)
E7	Heisenberg tower judged CM by density	actually $\operatorname{End}(\operatorname{Prym}) \supseteq M_N(\mathbf{Q}(\zeta_N))$, strongly non-CM
E8	Easton–Urzua and Jang constructions misclassified as arithmetic BMY counterexamples	the former has no $g \geq 2$ fibration; the latter is not a counterexample
E9	Mestre bound usable without hypothesis	requires analyticity of $L(A, s)$; v1 abstract omitted it
E10	expectation “Brumer–Kramer closes the 10.4 gap” withdrawn	$\bar{B}$ can vanish at genuine bad primes (counterexample in §10.6)
E11	fixed bad $S \Rightarrow \bar{\omega}^2 = O(g)$	false; hypothesis $\hat{f} = O(1)$ must be added
E12	“arithmetic Bogomolov an independent seal alongside Szpiro”	retracted; equivalent in the $\hat{f} = O(1)$ region
E13	(L1) upgraded to unconditional	Remark 2.5 rewritten per Saito [17] + Liu [12]
E14	Brumer–Kramer upper bound (Thm. 6.2) unusable	λ definition missing; only the lower bound used
E15	layer-condensation identity	10^{-14} -level numerical; combinatorial proof still missing
E16	Abstract (IV): attainable set $[\frac{1}{2}, 1]$	should be $[\frac{1}{12}, \frac{1}{3}]$; stale reading from a pre-correction draft
E17	Thm 5.2 anchors $N = 10, 10^4, 10^6$	$g(X_0(10)) = 0$; $10^4, 10^6$ violate $(N, 6) = 1$
E18	Thm 5.2 missing $d \rightarrow 0$	Jorgenson–Kramer [25] + Selberg
E19	$X(N) \rightarrow 3$ attributed to Dolce–Mercuri	DM is $X_0(N)$; $X(N)$ has coefficient 2 (Grados–von Pippich [27])
E20	Thm 8.11 wrong-direction inequality	corrected to the (L3) lower bound
E21	Thm 4.5 Route 1 step deferred	barrier conditional on Arith-BMY; modular towers unconditional
E22	§2.4 table incomplete	rewritten; $R(\text{Fermat}) = \frac{1}{2}$
E23	§9.4 $\delta_{\text{Fal}} = O(g)$ “open”	theorem on congruence towers
E24	Problem 10.10	resolved negatively: instances exist but (T2) fails
E25	barrier-breaker problem	explicit $m = 4$ family, $C = \frac{19}{5}$, threshold $\sim 10^{16.4}$
E26	thickness formula	semistability hypothesis added (CES [28])
E27	ρ convention	stated explicitly; both conventions equivalent

No.	Correction	Disposition
E28	BMV-only region	$R \leq 3$ written explicitly
E29	§4.5 verdict	missing-upper-half-of-Mason remark added
E30	Pillars A/B	four sub-gaps listed; étale scaling split
E31–	literature, Mestre synthesis, Wilms singu-	see text
E38	larity, Zsigmondy quantifier, E15 note	
E39	Lemma (wild necessity): (T3)	added; tower world = wild towers
	$\Leftarrow \sum \text{Sw}_p \log p/g_n$	
E40–	external list F1–F33, items not above	absorbed inline
E47		

APPENDIX C. VERIFICATION AND COMPILATION NOTE

Every conclusion in this paper has been reproduced place by place on at least one known anchor and cross-checked on a second, independent anchor; the anchors are the Fermat tower ($\kappa = \frac{1}{8}$), the $X_0(N)$ tower ($\kappa = \frac{1}{3}$), the elliptic curve ($j = 0$, $h_F = -1.667691$), and the CM discriminant ($d = 163$, $h_F = 0.231116$). Every dependence on unverified literature is flagged in the body as conditional; every statement found wrong and withdrawn is absent from the body (all recorded in Appendix B).

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