

RECURRENT TUBES AND CONNECTEDNESS-PRESERVING BIJECTIONS OF EUCLIDEAN SPACES

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ABSTRACT. For every $n \geq 2$ we construct a connectedness-preserving bijection F of $\mathbb{R}^n$ whose inverse does not preserve connectedness. The restriction of F off the closed coordinate ray E is a homeomorphism onto the complement of $L = K \setminus \{\infty\}$, where $K \subset \mathbb{S}^n$ is a nondegenerate continuum with empty interior. The discontinuity sets of F and F^{-1} are exactly E and L , and both maps can be chosen to have exact Baire class 2. The preimage of a fixed straight segment can be any prescribed uncountable closed subset of E with empty relative interior, including the middle-thirds Cantor set. The construction is uniform in the dimension. Homeomorphisms with compact support move finitely many collar points to a prescribed net, and narrowing the resulting collars produces a recurrent tube. Compact arcs on one retained boundary line supply dense connected tracks, while a gate lemma proves preservation of arbitrary connected sets. This answers W. Wong's question in the negative. We also prove that every component of the closure of the discontinuity set of a planar connectedness-preserving bijection is unbounded, and construct Borel counterexamples on manifolds with a standard spherical end.

1. INTRODUCTION

A map $f : X \rightarrow Y$ between topological spaces is *connectedness-preserving* if $f(C)$ is connected for every connected subset $C \subset X$. Continuous maps have this property, as do many discontinuous maps; for real-valued functions on the line it is the classical Darboux property [5, 9], and *Darboux map* and Wong's *connected map* are synonyms for the term adopted here. In 2016 Willie Wong asked the following question on MathOverflow [20].

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a bijection which is connectedness-preserving.
Must f^{-1} be connectedness-preserving?

The question concerns arbitrary connected subsets of $\mathbb{R}^n$, including those that are neither path connected nor compact. For $n = 1$ the answer is affirmative, since an injective map of the line with the intermediate value property is monotone and continuous, and a bijection with these properties is a homeomorphism. For $n \geq 2$ the question was open. Its equivalent formulation in terms of continuity follows from two classical results. By a theorem of Tanaka [18] (see also Pervin and Levine [16]), a bijection of $\mathbb{R}^n$ that preserves connectedness in both directions is a homeomorphism; by invariance of domain [11, Theorem 2B.3], a continuous bijection of $\mathbb{R}^n$ is a homeomorphism. Wong's question is therefore equivalent to:

Is every connectedness-preserving bijection of $\mathbb{R}^n$ continuous?

Among the positive results, Banach and Banach [2] proved that connectedness-preserving injections are continuous in several compact situations: from a compact

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connected metric space into a one-manifold, from a closed surface into a surface, and from a closed 3-manifold M with finite $H_1(M; \mathbb{Z})$ into a 3-manifold. An earlier note of the author [15] removes the homology hypothesis and extends this to every dimension: every connectedness-preserving injection from a connected closed n -manifold into an n -manifold is a homeomorphism onto a component, which answers the questions about $\mathbb{T}^3$ and $\mathbb{S}^4$ raised in [2]. These results establish rigidity in the compact setting and direct attention to the noncompactness of $\mathbb{R}^n$. Gro-Tsen [10] constructed a connectedness-preserving bijection $\mathbb{R} \rightarrow \mathbb{R}^2$ whose inverse is not connectedness-preserving, by arranging that every nondegenerate interval is mapped onto a set meeting every nonempty perfect subset of the plane; such sets are automatically connected. The same answer gives a self-bijection of $\mathbb{R}^2$ preserving connectedness of *path-connected* sets, and no self-map of $\mathbb{R}^n$ preserving every connected subset was known. Its image-density argument will be used below: an image is connected whenever it contains a connected subset dense in that image.

We answer Wong's question in the negative in every dimension $n \geq 2$, with prescribed inverse sets and exact Baire-class control. The scope results show that every component of the closure of the planar discontinuity set is unbounded and give a Borel counterexample on every manifold with a standard spherical end.

Theorem 1.1. *Let $n \geq 2$, put $E = [0, \infty) \times \{0\}^{n-1}$, and let $Z_0 \subset E$ be an uncountable closed set with empty interior in E . There are a nondegenerate continuum $K \subset \mathbb{S}^n$ containing infinity and having empty interior, and a bijection $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, with the following properties. Write $L = K \setminus \{\infty\}$.*

- (i) $F(C)$ is connected for every connected $C \subset \mathbb{R}^n$.
- (ii) $F|_{\mathbb{R}^n \setminus E}$ is a homeomorphism onto $\mathbb{R}^n \setminus L$.
- (iii) $\text{Disc}(F) = E$ and $\text{Disc}(F^{-1}) = L$.
- (iv) For $a_0 = (1, 0, \dots, 0) \in \mathbb{R}^{n-1}$ and the straight segment

$$C_* = \{(a_0, s) : 0 \leq s \leq 1\},$$

one has $F^{-1}(C_*) = Z_0$.

The choices can be made so that F and F^{-1} , as Euclidean-valued maps, have exact Baire class 2.

Every such Z_0 is disconnected, since a connected subset of a real interval containing two points contains the interval between them. Thus F^{-1} fails to preserve connectedness on the straight segment C_* . In particular one may take $Z_0 = \mathcal{C} \times \{0\}^{n-1}$. The regular image $\mathbb{R}^n \setminus L$ is dense and open; the proof also identifies the complete spherical cluster sets at both exceptional loci.

The idea. The construction combines a homeomorphism $h : \mathbb{R}^n \setminus E \rightarrow U$ onto a dense open set with a bijection $b : E \rightarrow L$, where $L = \mathbb{R}^n \setminus U$. Every nondegenerate interval in E has an image under b that contains a connected subset dense in L , so b preserves connectedness. To prove the same property for $F = h \cup b$, it remains to treat connected sets meeting both E and its complement.

Lemma 3.1 reduces this step to a *gate condition*: for each $p \in L$, each neighborhood O of p and each $z \notin E$, there is an open neighborhood N of the whole ray that omits z and whose frontier is mapped into O . A separation of $F(C)$ would then induce a nonempty proper clopen subset of C . The following three features supply the gates and the dense connected subsets of L .

Recurrence. A homeomorphism supported in the connected open complement of the current ball moves finitely many center points of its full end-disk collar to prescribed net points. Narrowing the collar after this homeomorphism has been chosen gives small sections by uniform continuity. A scalar push enlarges the ball while retaining the earlier lateral boundary and disk coordinates. The complementary compactum $K = \mathbb{S}^n \setminus U$ contains this boundary, and arbitrarily late whole sections converge to each point of K . Each open section is the image, under the source coordinates of Lemma 4.1, of the frontier of a neighborhood of the whole ray.

The source collapse. The coordinate homeomorphism Ψ extends to a continuous map q from $\mathbb{R}^n$ into a closed cone by sending all of E to the cone vertex. Approach to any finite point of E sends the height coordinate to infinity, so late tube sections correspond to frontiers of neighborhoods of E . The map q is nonproper: the vertex has the unbounded inverse image E , and approaches to the remaining cone boundary come from source infinity.

Tracks on one boundary line. Recurrence makes every tail of one retained boundary line dense in K . Choose pairwise disjoint compact arcs on that line so that the i -th arc is $\frac{1}{i}$ -dense in K , and partition the arcs into countably many infinite classes. Lemma 7.1 shows that the union in each class is connected and dense in K . These pairwise disjoint unions provide the tracks in every dimension $n \geq 2$.

Organization. Section 2 fixes conventions and gives the inverse-component obstruction and the two-sided continuity theorem. Sections 3 and 4 establish the gate lemma for arbitrary exceptional sets and the source-tail lemma, then give the coordinates for the ray. Sections 5–6 construct the recurrent tube by finite point moving and full topological collars. Sections 7–8 construct the connected tracks and the bijection with a prescribed inverse set, identify both exceptional cluster sets, and establish exact Baire class 2. Section 9 gives the finite-component obstruction, the planar restriction on the closure of the discontinuity set, and the Borel extension to manifolds with a standard spherical end, followed by further questions.

2. CONVENTIONS AND ELEMENTARY OBSERVATIONS

We work in ZFC and consider arbitrary connected subsets. The construction requires $n \geq 2$, while the elementary lemmas apply in the settings specified in their statements. We write

$$\mathbb{S}^n = \mathbb{R}^n \cup \{\infty\}$$

for the one-point compactification, equipped with the chordal metric d induced by stereographic projection. Its restriction to $\mathbb{R}^n$ induces the usual Euclidean topology.

For $A \subset X$, the symbols $\text{cl}_X(A)$, $\text{Int}_X(A)$ and $\text{Fr}_X(A)$ denote closure, interior and frontier in the explicitly indicated ambient space. Let

$$\mathbb{D}^{n-1} = \{a \in \mathbb{R}^{n-1} : |a| \leq 1\}, \quad \mathbb{B}^{n-1} = \{a \in \mathbb{R}^{n-1} : |a| < 1\}.$$

Thus $\text{Fr}_{\mathbb{R}^{n-1}}(\mathbb{D}^{n-1}) = \mathbb{S}^{n-2}$, which is the two-point set $\{-1, 1\}$ when $n = 2$. An arc is a homeomorphic image of a compact nondegenerate interval. For a map F , $\text{Disc}(F)$ denotes its set of discontinuity points. A *perfect* set is a nonempty closed set without isolated points. Manifolds are Hausdorff, second countable and without boundary unless a boundary is specified; a closed manifold is compact and has no boundary. A continuum is a nonempty compact connected metric space. We write $\mathfrak{c} = 2^{\aleph_0}$ and

$\mathcal{C} = \{\sum_{k \geq 1} 2\epsilon_k 3^{-k} : \epsilon_k \in \{0, 1\}\}$. Source coordinates are $(x, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$, while tube coordinates are $(a, t) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

A Polish space is a separable completely metrizable space, and a standard Borel space is a measurable space isomorphic to the Borel structure of a Polish space. We write Σ_2^0 for the F_σ sets, Π_2^0 for the G_δ sets and Σ_3^0 for countable unions of G_δ sets. A map is Σ_i^0 -measurable if every open preimage belongs to Σ_i^0 . Baire class 0 means continuous, class 1 means a pointwise limit of continuous maps, and class 2 means a pointwise limit of class-1 maps. Exact class 2 excludes class 1. A metrizable space is called zero-dimensional if it has a base of clopen sets. All Baire-class assertions use Euclidean codomains, and the continuous approximants may take values throughout those codomains.

A closed n -ball $M \subset \mathbb{S}^n$ is *flat* if its complementary closure $\mathbb{S}^n \setminus \text{Int}_{\mathbb{S}^n}(M)$ is also a closed n -ball. Brown's generalized Schoenflies theorem [3, Theorem 5] gives this property for a bicollared boundary sphere, whereas the Alexander horned ball [1] provides a nonflat example. In the construction below, explicit ambient isotopies establish flatness. For a cylinder $\mathbb{D}^{n-1} \times [0, \ell]$, the *lateral boundary* is $\mathbb{S}^{n-2} \times [0, \ell]$ and the *end disk* is $\mathbb{D}^{n-1} \times \{\ell\}$.

Lemma 2.1 (Connected adherence and closed inverse components). *If A is connected and $A \subset B \subset \text{cl}_X(A)$, then B is connected. If $g : X \rightarrow Y$ preserves connectedness and its image subspace $g(X)$ is T_1 , then*

$$g(\text{cl}_X(A)) \subset \text{cl}_Y(g(A))$$

for every nonempty connected $A \subset X$. Moreover, every connected component of $g^{-1}(H)$ is closed in X whenever H is closed in Y .

Proof. A separation of B would put its connected dense subset A on one side and leave a nonempty relatively open subset disjoint from A , which is impossible. In particular, $A \cup \{x\}$ is connected whenever $x \in \text{cl}_X(A)$. If $g(x) \notin \text{cl}_Y(g(A))$, the singleton $\{g(x)\}$ would be open in $g(A) \cup \{g(x)\}$ and closed there by the T_1 property of $g(X)$, contradicting connectedness preservation.

Finally, if A is a component of $g^{-1}(H)$, adherence gives $g(\text{cl}_X(A)) \subset \text{cl}_Y(g(A)) \subset H$. The connected set $\text{cl}_X(A)$ therefore lies in the same component, so $\text{cl}_X(A) = A$. $\square$

Closed inverse components give the following local criterion for continuity.

Corollary 2.2 (Inverse-component obstruction). *Let $g : X \rightarrow Y$ preserve connectedness, with $g(X)$ a T_1 subspace. Then g is continuous at x if $g(x)$ has an open neighborhood base whose complementary inverse images have component families locally finite at x ; here this means that some neighborhood of x meets only finitely many members of each such family. In particular, finitely many components suffice. If $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$ and g is discontinuous at x , there are $r > 0$ and a sequence $x_j \rightarrow x$ lying in pairwise different components of*

$$g^{-1}(\mathbb{R}^n \setminus B(g(x), r)).$$

Each of those components is closed and has closure missing x .

Proof. For a basic neighborhood O of $g(x)$, Lemma 2.1 makes every component of $g^{-1}(Y \setminus O)$ closed in X , and each component misses x . A neighborhood meeting only finitely many components can therefore be shrunk to avoid their finite closed union, proving continuity at x .

In the Euclidean case, discontinuity supplies $r > 0$ such that $x \in \text{cl}_{\mathbb{R}^n}(g^{-1}(\mathbb{R}^n \setminus B(g(x), r)))$. After finitely many components have been chosen, their closed union misses x , so the next point may be chosen in the remaining inverse image within distance $1/j$ of x . Induction gives the asserted sequence in pairwise different components. $\square$

The finite-complement mechanism in the next proposition is the one underlying the classical results of Tanaka and Pervin–Levine [18, 16].

Proposition 2.3. *Let Y be a T_1 space with an open neighborhood base whose members have complements with finitely many connected components. A bijection $f : X \rightarrow Y$ preserving connectedness in both directions is continuous. If X has the same separation and base properties, f is a homeomorphism. In particular, a bijection of $\mathbb{R}^n$ preserving connectedness in both directions is a homeomorphism. Every injective real-valued connectedness-preserving map on a nondegenerate real interval is strictly monotone and is a homeomorphism onto its image; consequently, for bijections of $\mathbb{R}$, one direction already suffices.*

Proof. Let O be a member of the stated base in Y . Every component H of $Y \setminus O$ is closed in Y , and $f^{-1}(H)$ is connected because f^{-1} preserves connectedness. Lemma 2.1 then makes $f^{-1}(H)$ closed in X . Since there are finitely many such components, $f^{-1}(O)$ is open, proving continuity. Applying the argument in reverse proves the homeomorphism assertion. The Euclidean case follows because complements of open balls are connected when $n \geq 2$ and are unions of two closed rays when $n = 1$.

For the interval assertion, if $x < y < z$ and $f(y)$ is not between $f(x)$ and $f(z)$, a value between $f(y)$ and both endpoint values occurs once on each of $[x, y]$ and $[y, z]$, contradicting injectivity. Thus f preserves betweenness and is strictly monotone. The intermediate value property excludes jumps, including one-sided jumps at any included endpoint, and hence gives continuity. The image is an interval, and the same order argument gives continuity of the inverse. $\square$

3. A GATE CONDITION FOR ARBITRARY CONNECTED SETS

Frontiers of neighborhoods of an exceptional set provide the following criterion for connectedness preservation between arbitrary topological spaces.

Lemma 3.1 (Gate lemma). *Let X, Y be topological spaces, $g : X \rightarrow Y$ a map, and $E \subset X$ any subset. Suppose $g|_{X \setminus E}$ is continuous and $g|_E$ preserves connectedness. For each $p \in g(E)$ choose an open neighborhood base $\mathcal{O}_p$ at p . Suppose that for every $O \in \mathcal{O}_p$ and $z \in X \setminus E$ there is an open $N \subset X$ such that*

$$E \subset N, \quad z \notin N, \quad g(\text{Fr}_X(N)) \subset O. \quad (3.1)$$

Then g preserves connectedness.

Proof. Let $C \subset X$ be connected. The restriction hypotheses settle the cases $C \subset E$ and $C \cap E = \emptyset$, so suppose that C meets both sets and that $g(C) = P \sqcup Q$ is a separation. There are points $x \in C \cap E$ and $z \in C \setminus E$ whose images lie on opposite sides of the separation. Indeed, if all points of $C \setminus E$ map to one side, some point of $C \cap E$ maps to the other; if their images meet both sides, choose a point opposite the image of any point of $C \cap E$.

Relabel so that $g(x) \in P$ and $g(z) \in Q$, and choose $O \in \mathcal{O}_{g(x)}$ with $O \cap g(C) \subset P$. For the corresponding gate N , set

$$A = \{a \in C \setminus N : g(a) \in Q\}.$$

The set A contains z , omits x , and misses $\text{Fr}_X(N)$. It is therefore the inverse image of Q under the restriction to $C \setminus \text{cl}_X(N)$, and continuity of $g|_{X \setminus E}$ makes it relatively open in C . It is also relatively closed, because $C \setminus N$ is closed in C and the restriction of g to that set is continuous. Thus A is a nonempty proper clopen subset of C , a contradiction. $\square$

Remark 3.2 (Testwise gates and the converse limit). For a fixed connected set C , the proof uses only $g(C \cap \text{Fr}_X(N)) \subset O$. It also applies within C , using a relatively open neighborhood N_C of $C \cap E$ with $z \notin N_C$ and $g(\text{Fr}_C(N_C)) \subset O$. Preservation of connectedness on E is used only when $C \subset E$, so a mixed set C may have disconnected intersection with E . Frontier containment is essential: for the step map $g(x) = 0$ when $x \leq 0$ and $g(x) = 1$ when $x > 0$, with $E = \{0\}$, the images of frontiers of small intervals about 0 meet every $O \ni 0$, yet $g(\mathbb{R})$ is disconnected.

When $g(X)$ is T_1 , connectedness preservation gives the necessary approach condition $g(x) \in \text{cl}_Y(g(A))$ for connected $A \subset X \setminus E$ accumulating at $x \in E$, by Lemma 2.1. The full gate condition gives the stronger inclusion $g(E) \subset \text{cl}_Y(g(A))$ for every such nonempty A : a gate aimed at any $p \in g(E)$ and excluding a point of A has frontier meeting A , because A is connected and meets the open neighborhood of x contained in the gate. Connectedness preservation alone does not imply the gate condition, even for the identity of $\mathbb{R}^n$ with E marked as the standard ray. The segment from $(1, 1, 0, \dots, 0)$ to $(1, 0, \dots, 0)$ avoids $B(0, 1/2)$ but crosses the frontier of every neighborhood of E excluding its first endpoint. Here the marked set is larger than the discontinuity locus.

Remark 3.3 (Testing a dense set of regular points). In Lemma 3.1, it is enough to test a dense set of regular points if E is closed, $X \setminus E$ is locally connected, Y is regular Hausdorff, and g is injective. Let D be dense in $X \setminus E$ and suppose the gates are known for $z \in D$. Given $p \in g(E)$, $y \notin E$ and an open neighborhood O of p , choose an open $O' \ni p$ with $\text{cl}_Y(O') \subset O$ and $g(y) \notin \text{cl}_Y(O')$. Continuity on the open locally connected complement gives a connected open neighborhood W of y there whose image misses $\text{cl}_Y(O')$.

Take $z \in D \cap W$, choose a member of the prescribed neighborhood base at p contained in O' , and use its gate N excluding z . The frontier of N misses W , so connectedness of W and $z \notin N$ imply $W \subset X \setminus \text{cl}_X(N)$. Thus this gate excludes y from its closure.

4. THE SOURCE COLLAPSE AND ITS GATES

The tube is parametrized by $\mathcal{T} = \mathbb{D}^{n-1} \times [0, \infty)$, with interior $\mathbb{B}^{n-1} \times (0, \infty)$. The following lemma gives the source neighborhoods whose frontiers will map to tube sections.

Lemma 4.1 (Source tails). *Let A be closed in a topological space X , and let*

$$\Phi : X \setminus A \longrightarrow \mathbb{B}^{n-1} \times (0, \infty)$$

be a homeomorphism with height coordinate t . Suppose that for each $x \in A$ and $T > 0$ there is a neighborhood W of x such that $t(z) > T$ for every $z \in W \setminus A$. Then,

for every $T > 0$, the set

$$N_T = A \cup \{z \in X \setminus A : t(z) > T\}$$

is open, and

$$\text{Fr}_X(N_T) = \Phi^{-1}(\mathbb{B}^{n-1} \times \{T\}). \quad (4.1)$$

If $z \notin A$ and $T > t(z)$, then $z \notin \text{cl}_X(N_T)$.

Proof. The height hypothesis makes N_T open at every point of A , while the coordinate homeomorphism makes its restriction to the open complement of A an open tail. In that complement, the frontier is exactly the level T , since every level point is approached from both sides in the height coordinate. As $A \subset \text{Int}_X(N_T)$, this is also the frontier in X . Finally, a point of $X \setminus A$ with height less than T has an open neighborhood in the complement on which all heights are less than T , so it lies outside $\text{cl}_X(N_T)$. $\square$

We apply the lemma to the ray $E = [0, \infty) \times \{0\}^{n-1}$. For $(x, y) \notin E$, put

$$R = (x^2 + |y|^2)^{1/2}, \quad v = \left(\frac{R - x}{2}\right)^{1/2}, \quad u = \frac{y}{2v}, \quad a = \frac{u}{\sqrt{1 + |u|^2}}, \quad t = v^{-1}. \quad (4.2)$$

Here $v, t > 0$ and $|a| < 1$. The map $\Psi(x, y) = (a, t)$ is a homeomorphism $\mathbb{R}^n \setminus E \rightarrow \mathbb{B}^{n-1} \times (0, \infty)$: the parabolic change inverts by $(x, y) = (|u|^2 - v^2, 2vu)$, and the remaining inverse is $u = a/\sqrt{1 - |a|^2}$, $v = t^{-1}$. Thus

$$x = \frac{|a|^2}{1 - |a|^2} - \frac{1}{t^2}, \quad y = \frac{2a}{t\sqrt{1 - |a|^2}}. \quad (4.3)$$

These formulas are continuous on their indicated domains, and approach to a finite point of E forces $v \rightarrow 0$ and hence $t \rightarrow \infty$. Lemma 4.1 therefore applies with $A = E$ and $\Phi = \Psi$; henceforth N_T denotes these Euclidean tail neighborhoods.

The map

$$(a, t) \mapsto \left(\frac{a}{1+t}, \frac{t}{1+t}\right)$$

identifies $\mathcal{T}$ with the closed cone

$$Q = \{(b, s) : 0 \leq s \leq 1, |b| \leq 1 - s\}$$

minus its vertex $\theta = (0, 1)$. Composing with Ψ and assigning all of E to θ gives a continuous map $q : \mathbb{R}^n \rightarrow Q$ with exact range $\text{Int}_{\mathbb{R}^n}(Q) \cup \{\theta\}$. The cone is a closed n -ball by radial rescaling from an interior point. Since $q^{-1}(\theta) = E$ is unbounded, q is nonproper.

Although q collapses E to one point and is injective elsewhere, it is not a quotient map onto its image. To see this, let $W = \{z : \text{dist}(z, E) < 1\}$, which is open and saturated for q , and take $v \downarrow 0$ and $u = v^{-2}a_0$, where a_0 is any unit vector in $\mathbb{R}^{n-1}$. Then $t = v^{-1} \rightarrow \infty$, so $q(z) \rightarrow \theta$, whereas $x = v^{-4} - v^2 > 0$ eventually and $\text{dist}(z, E) = |y| = 2/v \rightarrow \infty$. Thus $q(W)$ is not a neighborhood of θ , although its inverse image is open.

Every other cone boundary point is approached from source infinity: approach to a boundary point different from θ has bounded t , and either $t \rightarrow 0$ or $|a| \rightarrow 1$. The

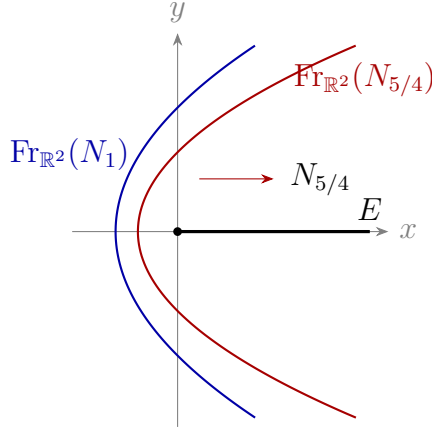


FIGURE 1. Gate frontiers in the source plane. At height $t = T$ the equation is $x = T^2 y^2 / 4 - T^{-2}$. Each is a proper line, with the whole positive ray on its right-hand side. Only finite portions are drawn.

identity

$$\|(x, y)\| = |u|^2 + v^2 = \frac{|a|^2}{1 - |a|^2} + t^{-2}$$

therefore makes the source norm tend to infinity. Distinct such boundary points give distinct limits in Q , preventing a continuous extension with a single value at source infinity.

5. FINITE EXTENSIONS THAT RETAIN THE LATERAL BOUNDARY

Each finite extension uses an embedded product chart obtained by moving finitely many points of an existing collar through the open complement and then narrowing its sections. An ambient isotopy establishes flatness of the enlarged ball, while retained disk coordinates ensure that successive parameterizations agree. The support of a homeomorphism is the closure of the set of points it moves.

A *marked end disk* of a closed set $M \subset \mathbb{R}^n$ consists of an embedding

$$\xi : \mathbb{D}^{n-1} \times [-\alpha_0, \beta_0] \longrightarrow \mathbb{R}^n, \quad \alpha_0, \beta_0 > 0,$$

with $F_0 = \xi(\mathbb{D}^{n-1} \times \{0\})$ and

$$M \cap \xi(\mathbb{D}^{n-1} \times [-\alpha_0, \beta_0]) = \xi(\mathbb{D}^{n-1} \times [-\alpha_0, 0]).$$

The marking includes the whole disk, its rim, and its coordinates.

5.1. A visiting chart from a full collar.

Lemma 5.1 (Finite point moving). *Let $\Omega \subset \mathbb{R}^n$ be connected and open, with $n \geq 2$. Given two ordered lists of distinct points $p_1, \dots, p_m$ and $q_1, \dots, q_m$ in Ω , there is a homeomorphism of $\mathbb{R}^n$, with compact support contained in Ω , carrying each p_i to q_i .*

Proof. An open connected subset of $\mathbb{R}^n$ is polygonally connected, since the points reachable from one point by polygonal paths form a nonempty set that is both open and relatively closed. For $n \geq 2$, removing finitely many points preserves polygonal connectedness: a polygonal path can be detoured within small disjoint balls about the forbidden points, using two radial segments and a polygonal route around the center of each punctured ball.

A closed ball admits a homeomorphism fixing its boundary and moving its center to any specified interior point. To construct it, map each segment from the old center to a boundary point linearly onto the segment from the new center to the same boundary point. This is a homeomorphism because every ray from the new center meets the sphere once, and extension by the identity gives a homeomorphism of $\mathbb{R}^n$ supported on the ball.

Proceeding inductively, join the current image of p_i to q_i by a compact polygonal path in $\Omega \setminus \{q_1, \dots, q_{i-1}\}$. Subdivide the path into sufficiently short steps and use the preceding ball homeomorphisms to move the point along it, choosing their closed supporting balls in this open set so that previously placed points remain fixed. The resulting finite composition completes the induction and has compact support in Ω . $\square$

Lemma 5.2 (Relative topological visiting chart). *Let $M \subset \mathbb{R}^n$ be closed with connected complement, $n \geq 2$, and let F_0 be a marked end disk. Given distinct points $q_1, \dots, q_m \in \mathbb{R}^n \setminus M$, numbers $\varepsilon > \text{diam}_d(F_0)$ and $\eta > 0$, there are $\alpha > 0$ and $\beta > L > 0$ and an embedding*

$$\chi : \mathbb{D}^{n-1} \times [-\alpha, \beta] \longrightarrow \mathbb{R}^n \quad (5.1)$$

whose zero slice agrees pointwise with the marked disk and which meets M exactly in its nonpositive fibers. Its center arc on $[0, L]$ visits each q_i in the prescribed order, every section with $0 \leq s \leq \beta$ has spherical diameter less than ε , and every section with s near L has diameter less than η .

Proof. Reparametrize the given collar as $\xi : \mathbb{D}^{n-1} \times [-1, 2] \rightarrow \mathbb{R}^n$, without changing its zero slice, and set $L = 1$. Choose increasing parameters $s_i \in (1/3, 2/3)$. Lemma 5.1, applied to $\Omega = \mathbb{R}^n \setminus M$, gives a homeomorphism h with compact support in Ω such that

$$h(\xi(0, s_i)) = q_i.$$

For $m = 0$, take h to be the identity. Since the compact support of h lies in $\mathbb{R}^n \setminus M$, the map fixes an open neighborhood of M , and hence the whole collar strip $\xi(\mathbb{D}^{n-1} \times [0, \delta])$ for some $\delta \in (0, 1/4)$. After decreasing δ , uniform continuity of ξ and the strict inequality $\text{diam}_d(F_0) < \varepsilon$ also give

$$\text{diam}_d(\xi(\mathbb{D}^{n-1} \times [0, \delta])) < \varepsilon.$$

Choose a continuous positive function w that equals 1 on $[-1, 0]$, decreases to a constant $\lambda \in (0, 1)$ within $[0, \delta]$, and equals λ on $[\delta, 2]$. Define

$$\chi(a, s) = h(\xi(w(s)a, s)).$$

The map χ is an embedding because each of $(a, s) \mapsto (w(s)a, s)$, ξ , and h is an embedding. It agrees with the entire old collar on the nonpositive fibers, and its positive fibers are disjoint from M because h and its inverse fix M . The center arc passes through every q_i , independently of w .

The sections in $[0, \delta]$ lie in the displayed small collar strip, where h is the identity. On $[\delta, 2]$, uniform continuity of $h \circ \xi$ on its compact domain makes

$$\sup_{\substack{\delta \leq s \leq 2 \\ a \in \mathbb{D}^{n-1}}} d(h(\xi(\lambda a, s)), h(\xi(0, s))) < \frac{1}{2} \min\{\varepsilon, \eta\}$$

when λ is sufficiently small, so the triangle inequality gives the remaining diameter bounds. $\square$

5.2. The finger extension.

Lemma 5.3 (Topological finger extension). *Let M be a closed ball in a Hausdorff n -manifold X without boundary and let $\chi : \mathbb{D}^{n-1} \times [-\alpha, \beta] \rightarrow X$ be an embedded full product chart, with $\alpha, \beta > 0$, that meets M exactly in its nonpositive fibers. For $0 < L < \beta$, the ball can be enlarged along the center arc $\chi(0, [0, L])$ to a ball M' by an ambient isotopy supported on the chart. Every old boundary point outside the relative interior of the marked end disk stays fixed, and the whole relative interior of that disk becomes interior to M' . Retained disk coordinates extend pointwise over the added portion, and the new end disk has a full two-sided product collar. If the original ball is flat in a sphere, the enlarged ball is flat as well. The new sections inherit any diameter bound for the corresponding sections of χ , and the end disk can be made arbitrarily small when χ is supplied by Lemma 5.2.*

Proof. Fix $0 < c < 1$ and define on $\mathbb{D}^{n-1}$

$$\ell(a) = \begin{cases} L, & |a| \leq c, \\ L(1 - |a|)/(1 - c), & c \leq |a| \leq 1. \end{cases}$$

On each axial fiber set

$$H_a(s) = \begin{cases} -\alpha + \frac{\alpha + \ell(a)}{\alpha}(s + \alpha), & -\alpha \leq s \leq 0, \\ \ell(a) + \frac{\beta - \ell(a)}{\beta}s, & 0 \leq s \leq \beta. \end{cases} \quad (5.2)$$

These increasing interval homeomorphisms depend continuously on a , fix both endpoints, and are the identity for $|a| = 1$, so the product map is a homeomorphism of the compact chart domain fixing its boundary. Invariance of domain identifies the chart interior as an ambient open set; conjugating by χ and extending by the identity therefore gives an ambient homeomorphism H . Replacing ℓ by $\tau\ell$, $0 \leq \tau \leq 1$, gives the required ambient isotopy.

The enlarged ball is

$$M' = H(M) = M \cup \chi\{(a, s) : 0 \leq s \leq \ell(a)\}.$$

Thus $M \subset M'$, and flatness is preserved whenever it applies because the ambient homeomorphism carries the complementary closure of M onto that of M' . Every point of the relative interior of the end disk becomes interior to M' , since ℓ is positive there. The rim and all earlier boundary points outside the relative interior of the face remain fixed: inside the chart, the negative fibers with $|a| < 1$ and $-\alpha < s < 0$ are interior to M , and every other such old boundary point lies on the chart boundary.

Retain the old parameter values and append

$$e_{\text{new}}(a, s) = \chi(r(s)a, s), \quad r(s) = 1 - (1 - c)s/L, \quad 0 \leq s \leq L. \quad (5.3)$$

For each $s \in [0, L]$, the inequality $s \leq \ell(b)$ is equivalent to $|b| \leq r(s)$. Thus these slices fill the added part, agree with the entire old end disk at $s = 0$, and put their lateral points on the new boundary. Their diameters do not exceed those of the corresponding full chart sections.

Choose $0 < \rho < \min\{L, \beta - L\}$. The new end disk and its coordinates are given by $a \mapsto \chi(ca, L)$, and its full collar is

$$\xi_{\text{next}}(a, u) = \chi(ca, L + u), \quad |u| \leq \rho.$$

For $u \leq 0$ one has $r(L + u) = c - (1 - c)u/L \geq c$, so every negative collar fiber lies in M' , including its rim. For $u > 0$, the axial parameter exceeds $L \geq \ell(a)$, so every positive fiber lies outside M' , proving the collar condition. $\square$

Remark 5.4. The extension requires positive height on the whole relative interior of the marked end disk, since a finger on a proper subdisk would leave an annulus on the old boundary. It also requires a full collar at the new end disk, including its rim. The ambient isotopy (5.2) establishes flatness, while the retained coordinates (5.3) make successive parameterizations agree.

6. A TUBE RECURRENT TO ITS WHOLE COMPLEMENTARY COMPACTUM

For every $n \geq 2$, begin with the closed cylinder

$$M_0 = \mathbb{D}^{n-1} \times [0, 1] \subset \mathbb{R}^{n-1} \times \mathbb{R}.$$

Its end disk $\mathbb{D}^{n-1} \times \{1\}$ has a full two-sided product collar, its initial parameterization is the identity, and radial normalization of the convex ball establishes flatness. Each finite finger extension preserves flatness and supplies the parameterized collar for the next stage. The open complement remains connected because it is the interior of a closed complementary ball with the point ∞ removed and $n \geq 2$.

Fix a decreasing sequence $\varepsilon_j \downarrow 0$, and make a preliminary extension whose end disk has spherical diameter less than ε_1 . At stage j , apply Lemma 5.2 with section bound ε_j and terminal bound ε_{j+1} , followed by Lemma 5.3. Every section of the stage then has diameter less than ε_j , including the entrance taper, and the end disk meets the bound required for the next stage. Reparameterize each stage onto a unit axial interval so that stage parameters tend to infinity.

At stage j , let N be the current complementary ball and choose a finite ε_j -net P_j of N . Since the interior of N is dense in N , choose distinct finite points $v_p \in \text{Int}_{\mathbb{S}^n}(N)$, one for each $p \in P_j$, with $d(p, v_p) < \varepsilon_j$, and prescribe them in Lemma 5.2. At the corresponding visit, the section center is v_p and every point of that section is within ε_j of it. The triangle inequality, applied successively to a point of N , its net point, v_p , and any point of the corresponding section, gives the bound $3\varepsilon_j$.

Enumerate the stages, including the preliminary one. For the resulting nested finite-stage balls M_k , define

$$U = \bigcup_k \text{Int}_{\mathbb{S}^n}(M_k), \quad K = \mathbb{S}^n \setminus U. \quad (6.1)$$

Each finite construction has bounded Euclidean support and fixes infinity, so the complementary closures $N_k = \mathbb{S}^n \setminus \text{Int}_{\mathbb{S}^n}(M_k)$ are nested nonempty compact connected sets containing infinity.

Proposition 6.1. *The construction supplies a continuous injection*

$$e : \mathcal{T} = \mathbb{D}^{n-1} \times [0, \infty) \longrightarrow \mathbb{R}^n$$

with the following properties.

- (a) *Its restriction to $\mathbb{D}^{n-1} \times (0, \infty)$ is a homeomorphism onto U .*
- (b) *The base and all finite-stage lateral boundaries belong to K . The set K is a nondegenerate continuum, contains infinity, and has empty interior in $\mathbb{S}^n$.*
- (c) *For every $p \in K$ there are $t_l \rightarrow \infty$ with*

$$\sup_{a \in \mathbb{D}^{n-1}} d(e(a, t_l), p) \longrightarrow 0. \quad (6.2)$$

The open section $e(\mathbb{B}^{n-1} \times \{t_l\})$ lies in U , and its cross-sectional boundary lies in K .

Proof. At each stage, retain the old parameterization and append (5.3), rescaled to a unit axial interval. The maps agree on the entire common end disk, and each finite closed parameter cylinder maps continuously and injectively onto its finite-stage ball; compactness makes this map an embedding. Any two parameter points lie in a common finite cylinder, proving global injectivity, while continuity follows locally from the finite parameterizations.

The boundary of each finite-stage ball consists of the base, the lateral boundary, and the end disk; at the next stage, the relative interior of the end disk becomes interior and its rim remains on the lateral boundary. The local finite charts therefore identify $\mathbb{B}^{n-1} \times (0, \infty)$ homeomorphically with the union in (6.1), since every interior parameter point lies in the interior of a sufficiently long finite-stage ball.

Nested compactness makes $K = \bigcap_k N_k$ compact and connected, and it contains the unchanged initial base and lateral boundary as well as infinity. For each $p \in K$, every stage j has an entire cross-section within $3\varepsilon_j$ of p , since K lies in every complementary ball. This gives (6.2), with parameters tending to infinity by the unit-interval convention. As the section centers belong to U , the same estimate makes U dense in $\mathbb{S}^n$, and hence K has empty interior. $\square$

Remark 6.2. Later parts of the lateral boundary accumulate on earlier ones, so only the interior restriction of e is a homeomorphism; the balls N_k share retained boundary and need not lie in one another's interiors.

7. DENSE CONNECTED TRACKS ON ONE LATERAL LINE

The construction uses the following general connectedness criterion.

Lemma 7.1 (Convergent connected pieces). *Let K be a topological space, and let $A_j \subset K$ be nonempty connected sets such that every nonempty open subset of K meets all sufficiently late A_j . Then every infinite subunion of the A_j is connected and dense in K .*

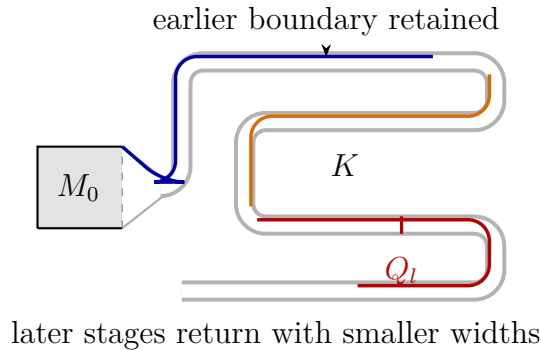


FIGURE 2. A schematic two-dimensional section of finite stages of the tube. The initial taper uses the whole old end disk, shown dashed. Colored portions indicate subarcs on one retained line of the lateral boundary; an actual stage visits a full finite net of the complement. The drawing is not to scale. A late cross-section Q_l is small, while the tail beyond it still returns near all of K .

Proof. Every infinite subsequence has the stated eventual-intersection property, so its union is dense in K . If this union had a separation, each connected piece would lie on one side, and one side would contain infinitely many pieces. Those pieces have dense union in K , so that side is dense in the full subunion; being relatively closed, it would equal the full subunion, a contradiction. $\square$

Fix the angular parameter $a_0 = (1, 0, \dots, 0) \in \mathbb{S}^{n-2}$ and put

$$f(t) = e(a_0, t), \quad t \geq 0.$$

The curve $f([0, \infty))$ is a line of the lateral boundary, meaning the image of a fixed angular parameter ray. By Proposition 6.1, f is continuous and injective with values in $K \setminus \{\infty\}$, and every tail $f([s, \infty))$ is dense in K : for $p \in K$, the parameters $t_l \rightarrow \infty$ of (6.2) satisfy $d(f(t_l), p) \leq \sup_{a \in \mathbb{D}^{n-1}} d(e(a, t_l), p) \rightarrow 0$.

Choose compact intervals $J_1, J_2, \dots \subset (1, \infty)$ inductively as follows. Given J_{i-1} (with $\sup J_0 = 1$), let P_i be a finite $\frac{1}{2^i}$ -net of K , and for each $p \in P_i$ use tail density to pick $t_p > \sup J_{i-1} + 1$ with $d(f(t_p), p) < \frac{1}{2^i}$; put $J_i = [\min_p t_p, \max_p t_p + 1]$. Then the J_i are separated by gaps of length at least one, $\inf J_i \rightarrow \infty$, and each

$$A_i = f(J_i)$$

is a compact arc in $K \setminus \{\infty\}$ which is $\frac{1}{i}$ -dense in K . Fix any map $c : \{1, 2, \dots\} \rightarrow \{1, 2, \dots\}$ with every fiber infinite, and set

$$D_m = \bigcup_{c(i)=m} A_i.$$

The arcs A_i are pairwise disjoint because the J_i are disjoint and e is injective. Since A_i is $\frac{1}{i}$ -dense in K , every infinite subfamily satisfies the hypothesis of Lemma 7.1.

Proposition 7.2. *The tracks $D_1, D_2, \dots$ are pairwise disjoint connected F_σ subsets of $\mathbb{R}^n$, each of cardinality $\mathfrak{c}$, with*

$$\text{cl}_{\mathbb{S}^n}(D_m) = K.$$

Each is unbounded, and each is disjoint from the explicit straight segment

$$C_* = \{(a_0, s) : 0 \leq s \leq 1\} \subset K. \quad (7.1)$$

The constituent arcs are exactly the path components of each track. No track is a G_δ subset of $\mathbb{R}^n$.

Proof. Every nonempty relatively open subset of K contains $\{q \in K : d(p, q) < r\}$ for some $p \in K$ and $r > 0$, and meets every A_i with $i > 1/r$. For each m the family $\{A_i : c(i) = m\}$ is infinite, so Lemma 7.1 gives connectedness and density of D_m in K , while disjointness of the arcs gives disjointness across colors.

Each D_m has cardinality $\mathfrak{c}$ because it contains a compact arc, and it is F_σ in $\mathbb{R}^n$ because it is a countable union of compact sets. It is unbounded because its spherical closure contains infinity, and it is disjoint from the initial lateral boundary segment $C_* = f([0, 1])$ because the intervals J_i lie in $(1, \infty)$.

Every constituent arc of a fixed color is closed in that track and has empty relative interior, since every nonempty relatively open subset meets all sufficiently late other arcs of that color. The track is therefore meagre in itself, being a countable union of relatively nowhere-dense closed sets. If it were G_δ in $\mathbb{R}^n$, it would be Polish, contrary to the Baire category theorem [13, Theorems 3.11 and 8.4].

The image of a path in D_m meeting two constituent arcs would be a compact Euclidean continuum partitioned into countably many disjoint closed sets, contrary to Sierpiński's theorem [17]. Since each constituent arc is path connected, the arcs are exactly the path components of the track. $\square$

Remark 7.3. Compactness is essential in the path-component argument: connected nonclosed planar unions of disjoint compact arcs occur in [8, Example 3.9]. The reserved segment C_* supplies the path-connected set whose inverse image will be disconnected.

8. THE BIJECTION ON THE RAY AND THE COMPLETION

Let $L = K \setminus \{\infty\}$. Since L is closed in $\mathbb{R}^n$ and each connected set D_m is dense in L , the following assignment ensures that the image of every nondegenerate source interval contains a connected dense subset of L .

Lemma 8.1 (Cantor-set assignment). *Let $I \subset \mathbb{R}$ be a nondegenerate interval, let L be a Borel subspace of a Euclidean space, and let $D_1, D_2, \dots \subset L$ be pairwise disjoint Borel sets of cardinality $\mathfrak{c}$, with D_m connected and dense in L for every $m \geq 2$. Let $Z_0 \subset I$ be uncountable and relatively closed, with empty interior in I , and let $T \subset L$ be a Borel set of cardinality $\mathfrak{c}$ disjoint from every D_m . There is a Borel isomorphism $b : I \rightarrow L$ such that $b^{-1}(T) = Z_0$ and the image of every nondegenerate interval in I contains one of the D_m with $m \geq 2$. In particular b preserves every connected subset of I ; if every D_m with $m \geq 2$ is unbounded, every nondegenerate interval has unbounded image.*

Proof. Reserve an affine compact Cantor set $Z \subset \text{Int}_{\mathbb{R}}(I) \setminus Z_0$, and enumerate the bounded rational open intervals B_r contained in $\text{Int}_{\mathbb{R}}(I)$. Every nondegenerate interval in I contains one of these intervals.

Inductively choose an affine compact Cantor set $X_r \subset B_r$ disjoint from Z_0, Z and the preceding X_s . A finite union of relatively closed nowhere-dense sets leaves a nonempty open interval in B_r , so this is possible. Set $X_0 = Z_0$ and choose bijections

$$X_0 \longrightarrow T, \quad X_r \longrightarrow D_{r+1} \quad (r \geq 1).$$

The unassigned source contains Z , and the unassigned target contains D_1 , so both remainders have cardinality $\mathfrak{c}$. A bijection between them completes the assignments to a bijection $b : I \rightarrow L$ with $b^{-1}(T) = Z_0$.

Every nondegenerate interval $J \subset I$ contains some X_r , and hence

$$D_{r+1} \subset b(J) \subset L = \text{cl}_L(D_{r+1}).$$

Lemma 2.1 makes $b(J)$ connected; it is unbounded if D_{r+1} is. Connected subsets of I are intervals or singletons, so b preserves them all.

The spaces I and L are standard Borel, and all the assigned pieces and both remainders are uncountable Borel subspaces. In particular Z_0 has cardinality $\mathfrak{c}$, as does every uncountable standard Borel space. The Borel isomorphism theorem [13, Corollary 13.4 and Theorem 15.6] allows every assignment to be a Borel isomorphism. Countable assembly makes both b and b^{-1} Borel. $\square$

Remark 8.2. The reserves Z and D_1 ensure that both remainders have cardinality $\mathfrak{c}$ for the general target T . The dense tracks give preservation of connected subsets by b , as needed for the gate argument, and the Borel assignments give measurability. This assignment uses the dense-connected-image principle of [10]; disjoint Cantor

kernels in rational intervals also occur in [12, Examples 3.3–3.4]. The connected tracks and the remainder assignments give a bijection with a prescribed inverse set.

This construction requires infinitely many whole tracks, because pairwise disjoint nondegenerate source intervals have disjoint images under an injection, and each nonempty track can belong to at most one of those images. Subdividing one of two reservoirs into infinitely many dense connected subtracks produces such an infinite family.

Remark 8.3. Lemma 8.1 applies to any Borel target of cardinality $\mathfrak{c}$ disjoint from the chosen tracks. For example, given $T_0 \geq 1$, choose the intervals J_i in Section 7 after T_0 . Tail density gives the same tracks, which are disjoint by injectivity of e from

$$P_{T_0} = e\left((\mathbb{D}^{n-1} \times \{0\}) \cup (\mathbb{S}^{n-2} \times [0, T_0])\right).$$

Thus any Borel $T \subset P_{T_0}$ of cardinality $\mathfrak{c}$ may replace C_* in Lemma 8.1, and any nondegenerate continuum there gives a connected inverse witness, with the reserved track supplying the target remainder. If T is closed and Z_0 satisfies Theorem 1.1, the Polish-piece argument of Proposition 8.6 allows both completed Euclidean maps to have exact Baire class 2. The existence of suitable tracks for other prescribed targets is the subject of Question 9.13.

Remark 8.4 (Closed prescribed sets with intervals). With I, L and the tracks as in Lemma 8.1, let $Z_0 \subset I$ be any proper relatively closed set, let $T \subset L$ be Borel and disjoint from every track, and suppose a Borel isomorphism $b_0 : Z_0 \rightarrow T$ is given that preserves every interval contained in Z_0 . Reserve a compact Cantor set in $\text{Int}_{\mathbb{R}}(I \setminus Z_0)$, and place the kernels X_r only in the rational intervals contained in that open set. The same assignments to D_{r+1} and the same two reserves give a Borel completion $b : I \rightarrow L$ extending b_0 . An interval contained in Z_0 has connected image by hypothesis; any other nondegenerate interval meets $I \setminus Z_0$ in a nondegenerate interval, and its image contains a dense connected track. Thus the completion preserves connectedness.

For $I = E$ and $Z_0 = [0, 1] \times \{0\}^{n-1}$, one may take b_0 to be a homeomorphism onto C_* . Proposition 8.5 gives the two exceptional loci, and Proposition 2.3 shows that the inverse of the completed Euclidean map fails to preserve connectedness. Here C_* itself is no longer a witness; for an arbitrary given b_0 , the conclusion is Borel measurability.

The preliminary assignment is essential: $([0, 1] \cup \{2\}) \times \{0\}^{n-1}$ cannot map bijectively onto a segment while preserving connected subsets, because the interval restriction would be a strictly monotone continuous injection with compact interval image, and one remaining point cannot fill the complement of a proper compact subinterval.

Proposition 8.5 (Exceptional cluster sets). *Put $h = e \circ \Psi : \mathbb{R}^n \setminus E \rightarrow U$. For every bijection $b : E \rightarrow L$, let $F = h \cup b$. Its discontinuity sets are*

$$\text{Disc}(F) = E, \quad \text{Disc}(F^{-1}) = L.$$

In the one-point compactifications, the cluster sets along the regular regions are

$$\text{Clust}_{\mathbb{R}^n \setminus E}(h, x) = K \quad (x \in E), \quad \text{Clust}_U(h^{-1}, p) = E \cup \{\infty\} \quad (p \in L). \quad (8.1)$$

Here a cluster point is a limit of map values along a sequence of domain points converging to, and distinct from, the tested point. The same cluster equalities hold for the full maps F and F^{-1} .

Proof. Fix $x = (\rho, 0) \in E$, choose $u \in \mathbb{R}^{n-1}$ with $|u|^2 = \rho$, and set $a = u/\sqrt{1+\rho} \in \mathbb{B}^{n-1}$. The inverse coordinates (4.3) give

$$\Psi^{-1}(a, T) = (\rho - T^{-2}, 2u/T) \longrightarrow (\rho, 0).$$

For every $q \in K$, choose recurrent whole sections with parameters $T_j \rightarrow \infty$ converging to q as in (6.2), so that $e(a, T_j) \rightarrow q$. The choice $u = 0$ includes the endpoint of E . A cluster point in U is excluded by continuity of h^{-1} , which would make the source sequence converge to a point outside E .

Conversely, recurrence of the same whole sections at a fixed $p \in L$ gives every finite point of E as an inverse cluster point by the preceding choices. To obtain infinity, choose a unit vector $v \in \mathbb{R}^{n-1}$ and put $u_T = Tv$ and $a_T = u_T/\sqrt{1+T^2}$. Equation (4.3) gives

$$\Psi^{-1}(a_T, T) = (T^2 - T^{-2}, 2v) \longrightarrow \infty,$$

so infinity is also an inverse cluster point, since recurrence controls the entire section, including the varying a_T . A finite inverse cluster point outside E is excluded by continuity of h there. Values at exceptional arguments lie in L for F and in E for F^{-1} , so they add no cluster points outside the two displayed compact sets.

Both maps are continuous on their regular open regions. Each displayed exceptional cluster set contains infinity and a finite point, so neither map is continuous at an exceptional point. This proves both discontinuity equalities, independently of the choice of b . $\square$

Proposition 8.6 (Baire class of the completion). *For the ray E , the closed set $L = K \setminus \{\infty\}$ and the tracks constructed above, the assignment in Lemma 8.1 with target C_* and any uncountable closed $Z_0 \subset E$ of empty interior in E can be made so that b and b^{-1} are Σ_3^0 -measurable. Regarded as Euclidean-valued maps, they and the completed maps F, F^{-1} have exact Baire class 2. No choices satisfying the whole-track assignments in that proof give either completed map Baire class 1.*

Proof. Kuratowski's isomorphism theorem gives a bijection between any two uncountable Polish spaces for which both directions have F_σ preimages of open sets in the given subspace topologies [14, p. 212, corollary].

Keep the pieces $X_0 = Z_0$, X_r , the source reserve Z , and the target reserve D_1 from Lemma 8.1. Write each used track as its disjoint union of compact arcs,

$$D_{r+1} = \bigcup_{j \geq 1} A_{r,j},$$

and give that set the Polish disjoint-sum topology $\widehat{D}_{r+1} = \bigsqcup_{j \geq 1} A_{r,j}$. Choose the bijection $X_r \rightarrow \widehat{D}_{r+1}$ by Kuratowski's theorem. The inclusion of this disjoint sum into Euclidean space is continuous, so the forward map $X_r \rightarrow D_{r+1}$ has F_σ open preimages. For the inverse, an open preimage intersected with each compact arc is F_σ in that arc, hence F_σ in $\mathbb{R}^n$; its countable union over the arcs remains F_σ . Thus each kernel-to-track assignment has the required bound in both actual subspace topologies. Apply the same theorem directly to the closed Polish pieces Z_0 and C_* .

The remainders

$$R_E = E \setminus \bigcup_{r \geq 0} X_r, \quad R_L = L \setminus \left(C_* \cup \bigcup_{r \geq 1} D_{r+1} \right)$$

are G_δ subsets of the closed Euclidean spaces E, L , and hence Polish [13, Theorem 3.11]. The two reserves make them uncountable, so their Polish topologies also admit a bijection that is Σ_2^0 -measurable in both directions. An F_σ set relative to an ambient G_δ set is an ambient Σ_3^0 set: write it as a countable union of intersections of that G_δ set with ambient closed sets. Countable assembly therefore makes b, b^{-1} Σ_3^0 -measurable. Adding the continuous restrictions on the open regular regions preserves this bound for F, F^{-1} .

The scalar Lebesgue–Hausdorff correspondence [7, Theorem 1.1 and its proof] gives Baire class at most 2 for bounded scalar maps with Σ_3^0 open preimages. To pass to unbounded Euclidean-valued maps, first consider an unbounded scalar map f , set $q(t) = t/(1 + |t|)$, and take Baire-class-1 maps $g_j \rightarrow q \circ f$. If c_j clamps the line to $[-1 + 1/(j + 1), 1 - 1/(j + 1)]$, then $q^{-1} \circ c_j \circ g_j$ is Baire class 1 and converges pointwise to f . Apply this to each coordinate and take finite tuples of approximants. Thus all four maps have Baire class at most 2 as Euclidean-valued maps.

For the lower bound, b is unbounded on every nonempty relatively open subset of the complete metric space E : each contains a nondegenerate interval, whose assigned image contains an unbounded whole track. The inverse is likewise unbounded on every nonempty relatively open subset $W \subset L$. Given $M > 0$, choose an enumerated rational interval $B_r \subset (M, \infty)$. Density gives $y \in W \cap D_{r+1}$, and $b^{-1}(y) \in X_r \subset B_r$, so $|b^{-1}(y)| > M$.

If continuous Euclidean-valued maps u_j converge pointwise on a nonempty complete metric space X , then

$$X = \bigcup_{m \geq 1} \bigcap_{j \geq 1} \{x \in X : |u_j(x)| \leq m\}.$$

The sets in the union are closed, so Baire’s theorem [13, Theorem 8.4] gives one with nonempty interior, on which the limit is bounded. Consequently neither b nor b^{-1} is Baire class 1, and the same holds for F and F^{-1} because restriction preserves Baire class 1. This proves exactness and the impossibility assertion for all choices satisfying the same assignments. $\square$

Proof of Theorem 1.1. Take e, U, K from Proposition 6.1, and put $L = K \setminus \{\infty\}$. Apply Lemma 8.1 with $I = E$, identified with the ray, with the prescribed Z_0 , with $T = C_*$, and with the tracks of Proposition 7.2. Define

$$F(z) = \begin{cases} e(\Psi(z)), & z \in \mathbb{R}^n \setminus E, \\ b(z), & z \in E. \end{cases} \quad (8.2)$$

The first restriction is a homeomorphism onto U , and the second is a bijection onto L ; since these disjoint ranges exhaust $\mathbb{R}^n$, the map F is bijective with the asserted regular restriction.

Fix $p \in L$, an open neighborhood O of p in $\mathbb{R}^n$, and $z \notin E$. Equation (6.2) supplies arbitrarily late T with $e(\mathbb{B}^{n-1} \times \{T\}) \subset O$, since spherical convergence to a finite point implies this Euclidean containment. For a sufficiently late such T , Lemma 4.1 gives

$$E \subset N_T, \quad z \notin \text{cl}_{\mathbb{R}^n}(N_T), \quad F(\text{Fr}_{\mathbb{R}^n}(N_T)) = e(\mathbb{B}^{n-1} \times \{T\}) \subset O.$$

The gate lemma proves that F preserves every connected subset of $\mathbb{R}^n$. Proposition 8.5 supplies both exact discontinuity sets. The set K has all the stated properties by Proposition 6.1.

The palette assignment gives the prescribed preimage $F^{-1}(C_*) = Z_0$, which is disconnected because it is uncountable and contains no nondegenerate interval. Thus F^{-1} does not preserve connectedness. Choosing the palette as in Proposition 8.6 gives exact Baire class 2 for both completed maps. $\square$

9. CONSEQUENCES AND SCOPE

The local chart construction uses connected open complements and full collars. Its application to a self-map of another manifold requires compatible global source and target data. We first derive an obstruction from adherence, then study products, the planar discontinuity locus, and localization to a standard spherical end.

Why the tube must return. The finite-complement argument of Tanaka and Pervin–Levine yields the following local obstruction.

Lemma 9.1 (Finite-component obstruction). *Let $f : X \rightarrow Y$ preserve connectedness, where Y is regular and T_1 . Let $G \subset X$ and suppose $h = f|_G : G \rightarrow V = f(G)$ is a bijection whose inverse preserves connectedness. Fix $x \in X$ and suppose $f(x)$ has arbitrarily small closed neighborhoods B such that $V \setminus B$ is a finite union of connected sets. Then f is continuous at x along G : every neighborhood O of $f(x)$ contains $f(W \cap G)$ for some neighborhood W of x . If G is dense and this conclusion holds at every point of X , then f is continuous.*

Proof. Choose $B \subset O$ with $f(x) \in \text{Int}_Y(B)$ and write $V \setminus B = \bigcup_{i=1}^k A_i$ with each A_i connected. Since every $h^{-1}(A_i)$ is connected, Lemma 2.1 gives

$$f(\text{cl}_X(h^{-1}(A_i))) \subset \text{cl}_Y(A_i) \subset Y \setminus \text{Int}_Y(B).$$

Thus x is outside each of these finitely many closures, whose complement is a neighborhood W with $f(W \cap G) \subset B \subset O$.

For the last assertion, let O be an open neighborhood of $f(x)$ and choose open O_0 with $f(x) \in O_0 \subset \text{cl}_Y(O_0) \subset O$. Continuity along G gives an open $W \ni x$ with $f(W \cap G) \subset O_0$. For every $y \in W$, density of G and continuity along G at y imply $f(y) \in \text{cl}_Y(f(W \cap G))$: otherwise an open neighborhood of $f(y)$ disjoint from $f(W \cap G)$, together with continuity along G at y , would give a nonempty open subset of W missing G . Consequently $f(W) \subset \text{cl}_Y(O_0) \subset O$. $\square$

The same proof allows a connected cover whose preimages form a family locally finite at x , in the sense of Corollary 2.2. If continuity along G fails, then for every sufficiently small such B the preimages of the components of $V \setminus B$ accumulate collectively at x , although the closure of each individual connected preimage misses x ; that family cannot be locally finite there. For the recurrent tube, fix $p \in L$ and a sufficiently small closed ball B centered at p , and choose $q \in K \setminus B$. Recurrence supplies successively later sections alternating between those wholly inside $\text{Int}_{\mathbb{R}^n}(B)$ and those near q and outside B . The centers of the latter sections belong to different components of $U \setminus B$: a connected subset joining two such centers would have an interval as its t -projection and would cross an intervening removed whole section. Thus $U \setminus B$ has infinitely many components, while whole-section recurrence also provides the gates used in the construction.

Corollary 9.2 (Closed countable exceptional sets). *Let $n \geq 2$ and let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a connectedness-preserving bijection that is continuous off a closed countable set*

E_0 . Then f is a homeomorphism. In particular, the closure of the discontinuity locus of a discontinuous connectedness-preserving bijection of $\mathbb{R}^n$ is uncountable.

Proof. The set $G = \mathbb{R}^n \setminus E_0$ is dense and open, so invariance of domain makes $f|_G$ a homeomorphism onto an open set V ; bijectivity then gives $\mathbb{R}^n \setminus V = f(E_0)$, a closed countable set.

Every connected open subset $W \subset \mathbb{R}^n$, $n \geq 2$, remains path connected after removal of a countable set S . First, for two points a, b of a convex open ball outside S , choose a point v of that ball outside S and outside all lines through a or b and a point of S . Such a v exists because a countable union of lines and points cannot fill a nonempty open ball when $n \geq 2$, by the Baire category theorem [13, Theorem 8.4]. The segments $[a, v]$ and $[v, b]$ then avoid S and stay in the ball. For general W , the set of points reachable from a fixed point by a finite chain of overlapping open balls contained in W is open and has open complement in W , so it is all of W . Choose transition points outside S in consecutive overlaps and concatenate the paths just constructed inside the balls.

Apply this fact to the connected exterior domain $\mathbb{R}^n \setminus \overline{B}$, for every closed ball $\overline{B}$ centered at $f(x)$, and to the countable set $f(E_0)$. It follows that $V \setminus \overline{B}$ is path connected, so Lemma 9.1 gives continuity along G at every point and hence continuity everywhere. Invariance of domain then makes the continuous bijection a homeomorphism. The last assertion follows by taking $E_0 = \text{cl}_{\mathbb{R}^n}(\text{Disc}(f))$ if that closure were countable. $\square$

Products and one-point extensions. Juhász and van Mill proved the full product obstruction [12, Theorem 2.2]: if every X_i is nonempty and connected, every Y_i is T_1 , and at least two maps $f_i : X_i \rightarrow Y_i$ are nonconstant, then $\prod_i f_i$ preserves connectedness if and only if every factor f_i is continuous. In particular, a product with two discontinuous factors, such as $F \times F$, fails to preserve connectedness, as does a product of a discontinuous factor with a nonconstant identity map. For the latter case, retain the assumptions that X is connected and Y is T_1 , and consider the following comb. If $f : X \rightarrow Y$ is discontinuous, choose $a \in \text{cl}_X(S)$ and an open neighborhood O of $f(a)$ disjoint from $f(S)$. For distinct z_0, z_1 in a connected T_1 space Z , the set

$$(S \times Z) \cup (X \times \{z_1\}) \cup \{(a, z_0)\}$$

is connected: the horizontal copy of connected X meets every connected vertical fiber, and (a, z_0) belongs to their closure. Its image under $f \times \text{id}_Z$ has the point $(f(a), z_0)$ isolated by $O \times (Z \setminus \{z_1\})$ and closed by the T_1 hypotheses, so it is disconnected. The source connectedness hypothesis is essential: the discontinuous map $f : \mathbb{Q} \rightarrow \{0, 1\}$ given by $f(0) = 1$ and $f = 0$ elsewhere preserves connectedness, as does $f \times \text{id}_Z$, since every connected subset of $\mathbb{Q} \times Z$ has constant first coordinate. The product obstruction rules out obtaining the maps constructed here as product lifts of lower-dimensional maps.

The one-point extension of F to $\mathbb{S}^n$ also fails to preserve connectedness. For fixed T , the frontier (4.1) is a properly embedded copy of $\mathbb{R}^{n-1}$ by (4.3), so adjoining ∞ gives a sphere whose image consists of the bounded open section $e(\mathbb{B}^{n-1} \times \{T\})$ and an isolated point at infinity. This agrees with the compact-manifold restrictions of [2] and the exclusion of such bijections on closed manifolds in every dimension [15].

A planar restriction on the discontinuity locus. The next consequence uses the closed-surface injection theorem of Banach and Banach [2, Theorem 1.4(2)].

Lemma 9.3 (Jordan-interior rigidity). *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a connectedness-preserving bijection. If J is a Jordan curve and $f|_J$ is continuous, then f maps the bounded complementary domain of J homeomorphically onto the bounded complementary domain of $f(J)$.*

Proof. The restriction to the compact curve J is an embedding, so $J' = f(J)$ is a Jordan curve. Write D, E for the bounded and unbounded domains complementary to J , and D', E' for those complementary to J' . The two connected sets $f(D)$ and $f(E)$ are disjoint and cover $\mathbb{R}^2 \setminus J'$. Each lies in one target complementary domain, and both domains must be covered, so they match D', E' exactly, initially in either order.

Collapse J to a point in the closed disk $\overline{D}$, obtaining a sphere S and a quotient map $q : \overline{D} \rightarrow S$. If $f(D) = D'$, collapse J' in $\overline{D'}$ to obtain a target sphere S' . If $f(D) = E'$, first add infinity to $\overline{E'}$, making a closed disk in $\mathbb{S}^2$, and then collapse J' to obtain S' . Denote the target quotient by q' . The Jordan–Schoenflies theorem [19, Theorem 3.1] gives the stated closed disks; collapsing the boundary of a closed disk gives a sphere. The assignment

$$\hat{f}(q(x)) = q'(f(x)), \quad x \in \overline{D},$$

is well defined and injective: precisely the points of each boundary curve are identified.

The source quotient q is closed and has connected fibers, so it has connected inverse images of arbitrary connected subsets of its target. To see this, its restriction over any subset is still closed, since the image of $A \cap q^{-1}(C)$, for closed $A \subset \overline{D}$, is $q(A) \cap C$. A separation of $q^{-1}(C)$ cannot split a connected fiber, so the two parts would have disjoint nonempty closed images covering C , contradicting connectedness of C . It follows that $\hat{f}$ preserves every connected subset: apply q^{-1} , then f , then the continuous map q' . Banach–Banach’s theorem makes the injection $\hat{f} : S \rightarrow S'$ continuous, and compactness and invariance of domain make it a homeomorphism onto all of S' . The exterior matching is impossible because its image omits $q'(\infty)$. In the bounded matching, restriction away from the collapsed boundary points gives the asserted homeomorphism $D \rightarrow D'$. $\square$

Theorem 9.4 (Planar discontinuity locus). *For every connectedness-preserving bijection $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, every connected component of $\text{cl}_{\mathbb{R}^2}(\text{Disc}(f))$ is unbounded.*

Proof. We first show that if a closed set $A \subset \mathbb{R}^2$ has a bounded component C , there is a Jordan curve in $\mathbb{R}^2 \setminus A$ whose bounded complementary domain contains C . Choose R with $C \subset B(0, R)$ and put $A_R = A \cap \overline{B}(0, R)$, so that C is also a component of the compact set A_R . In a compact Hausdorff space, a component is the intersection of all clopen sets containing it. Indeed, the intersection Q of the clopen sets containing a fixed point is connected: if Q were disconnected, normality of the compact Hausdorff space would separate its two compact parts by disjoint ambient open sets U, V . Compactness would then give a finite intersection P of those clopen sets with $Q \subset P \subset U \cup V$; the part of P on the side containing the fixed point would be a further clopen set omitting the other nonempty part of Q , a contradiction. Thus Q is exactly the component.

For each point of $A_R \cap \partial B(0, R)$, choose a clopen subset of A_R containing C and omitting that point. Compactness of this boundary intersection gives a finite

intersection P of such clopen sets with

$$C \subset P \subset A_R \cap B(0, R).$$

The compact set P is positively separated from both $A_R \setminus P$ and $\partial B(0, R)$. Choose finitely many small open round disks covering P , with their closures contained in a neighborhood of P missing $A \setminus P$. Slight changes of their radii preserve the covering and separation and make their boundary circles transverse, with no triple intersection. The boundary of their union therefore consists of finitely many disjoint Jordan curves: at each boundary point the exposed circle arcs form a single local arc, including at the transverse corners. The connected set C lies in one component of this union, and that component has an outer boundary Jordan curve enclosing it. Every boundary point misses P , since P is covered by the open union, and misses $A \setminus P$ by the separation choice, proving the isolation fact.

Apply the fact to $A = \text{cl}_{\mathbb{R}^2}(\text{Disc}(f))$ and a hypothetical bounded component C . The resulting Jordan curve J misses A , so $f|_J$ is continuous and Lemma 9.3 makes f continuous throughout its bounded interior D . Since D is open, it also misses the closure of the discontinuity locus, contrary to $C \subset D \cap A$. $\square$

The corresponding questions for higher dimensions and for a nonclosed discontinuity locus appear below as Questions 9.8 and 9.9. In $\mathbb{R}^n$, $n \geq 3$, the isolation boundary of a bounded component need not be a sphere.

Manifolds with a standard spherical end.

Lemma 9.5 (Extension of a local differentiable map). *Let φ be a C^1 local diffeomorphism near $y_0 \in \mathbb{R}^n$, with $\varphi(y_0) = x_0$. There is a global C^1 diffeomorphism $A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ agreeing with φ near y_0 and affine outside an arbitrarily small ball about y_0 contained in its domain.*

Proof. Put $P = D\varphi(y_0)$, $T(y) = x_0 + P(y - y_0)$, and $\psi = T^{-1} \circ \varphi$, so that $\psi(y_0) = y_0$ and $D\psi(y_0) = I$. Take a smooth cutoff χ , supported in the ball of radius r about y_0 , equal to one on the concentric ball of radius $r/2$, and satisfying $|D\chi| \leq C/r$ for a fixed constant C . Choose r so small that $\|D\psi - I\| \leq \delta$ throughout the closed ball, where $(1 + C)\delta < 1/2$. The mean-value estimate gives $|\psi(y) - y| \leq \delta|y - y_0|$. Define

$$\Phi(y) = y + \chi(y)(\psi(y) - y)$$

on the ball and $\Phi(y) = y$ outside it. This is globally C^1 and satisfies $\|D\Phi - I\| \leq (1 + C)\delta < 1/2$. Writing $\Phi = \text{id} + v$, the equation $\Phi(y) = z$ is the fixed-point equation $y = z - v(y)$ for a contraction of $\mathbb{R}^n$, so Φ is bijective with Lipschitz inverse. The inverse function theorem makes its inverse C^1 , so $A = T \circ \Phi$ has the required properties. The affine factor allows either orientation, and the resulting map is proper. $\square$

Theorem 9.6 (A standard spherical end). *Let M be a connected noncompact boundaryless topological n -manifold, $n \geq 2$. Suppose a closed subset $W \subset M$ admits a homeomorphism*

$$\theta : \mathbb{R}^n \setminus \text{Int } B \longrightarrow W,$$

where B is a round closed ball, $\theta(\partial B) = \text{Fr}_M(W)$, and $\theta(\mathbb{R}^n \setminus B) = \text{Int}_M(W)$. Then M admits a bijection $H : M \rightarrow M$, Borel in both directions, that preserves every connected subset and whose inverse H^{-1} fails to preserve connectedness on a topological arc. Its discontinuity set is a closed properly embedded ray $E_M \subset \text{Int}_M(W)$,

and H is a homeomorphism from $M \setminus E_M$ onto the complement of a closed nowhere-dense subset L_M . Moreover, $\text{Disc}(H^{-1}) = L_M$, and the witnessing arc has a Cantor set on E_M as its inverse image.

Proof. Use the Euclidean witness F with its prescribed Cantor preimage. Its regular restriction $h = e \circ \Psi$ is a smooth local diffeomorphism at $x_0 = (-4, 0, \dots, 0)$: the initial tube parameterization is the identity, and at $(a, t) = (0, 1/2)$ the source inverse (4.3) has derivatives

$$\partial_t x = 16, \quad D_a x = 0, \quad D_a y = 4I, \quad \partial_t y = 0.$$

Thus its derivative is invertible, with $h(x_0) = (0, \dots, 0, 1/2)$. Apply Lemma 9.5 to the local inverse of h , using cutoff support inside the initial tube, and put $F_0 = A \circ F$, which equals the identity near x_0 . Choose concentric round closed balls $B \subset \text{Int } B_2$ about x_0 , disjoint from E , on a neighborhood of which F_0 is the identity. A Euclidean similarity identifies the exterior of this chosen B with the exterior in the hypothesis.

Bijectivity and the identity on B_2 show that F_0 preserves $\mathbb{R}^n \setminus \text{Int } B$ and that $L_0 = A(L)$ misses B_2 . Define $H = \theta \circ F_0 \circ \theta^{-1}$ on W and let H be the identity on $M \setminus \text{Int } W$. These definitions agree near the seam in M : the set $\theta(\mathbb{R}^n \setminus \text{Int } B_2)$ is closed in the closed subset W , hence in M , and its complement is an open neighborhood of $\text{Fr}_M(W)$ on which both definitions are the identity. Thus H is a bijection, and its regular restrictions give a homeomorphism

$$M \setminus E_M \longrightarrow M \setminus L_M, \quad E_M = \theta(E), \quad L_M = \theta(L_0).$$

The sets E_M, L_M are closed, L_M is nowhere dense, and E_M is a proper ray because W is closed and θ is a homeomorphism onto it.

To prove preservation of connected sets meeting both the end and the core, we transport the gates. The source height t is bounded on B_2 , so for $T > \max_{B_2} t$ the closure of the source gate N_T misses B_2 . Fix $p_M \in L_M$, an open neighborhood $O \ni p_M$ in M , and $z_M \notin E_M$. Choose a Euclidean open neighborhood O_0 of $\theta^{-1}(p_M)$, contained in $\mathbb{R}^n \setminus B$, with $\theta(O_0) \subset O$. Recurrence supplies arbitrarily late T with

$$F_0(\text{Fr}_{\mathbb{R}^n}(N_T)) = A(e(\mathbb{B}^{n-1} \times \{T\})) \subset O_0.$$

If $z_M \in \text{Int}_M(W)$, require also $T > t(\theta^{-1}(z_M))$. The transported set $N = \theta(N_T)$ is open in M , contains E_M , and excludes z_M even from its closure. Its closure stays away from the seam, and closedness of W prevents additional frontier points outside W ; hence

$$\text{Fr}_M(N) = \theta(\text{Fr}_{\mathbb{R}^n}(N_T)), \quad H(\text{Fr}_M(N)) \subset O.$$

The restriction to E_M preserves connectedness by conjugation, so Lemma 3.1 proves that H preserves every connected subset of M .

The topological arc $\theta(A(C_*))$ has inverse image $\theta(\mathcal{C} \times \{0\}^{n-1})$, a disconnected Cantor set. Both exact discontinuity loci are transported from Proposition 8.5: every point of E_M or L_M lies in the interior of the end chart, while the map and its inverse are the identity near the seam and regular elsewhere. Finally, H and its inverse are Borel by conjugation and finite Borel assembly with the identity. $\square$

A punctured coordinate ball supplies the required closed end in a punctured closed manifold. For the interior of a compact connected manifold with a spherical boundary component, Brown's collar theorem [4, Theorem 2] supplies a collar $\mathbb{S}^{n-1} \times [0, \varepsilon)$; after deleting the boundary, $\mathbb{S}^{n-1} \times (0, \varepsilon/2]$ is the required closed end. In dimension two, this includes the interiors of compact connected surfaces with nonempty boundary.

Further questions. The Euclidean construction has exact Baire class 2 in both directions, the planar theorem constrains the closure of the discontinuity set, and the spherical-end theorem gives Borel examples on its stated class of manifolds. These results lead to the following questions.

Question 9.7 (Baire class one). For any $n \geq 2$, does there exist a discontinuous connectedness-preserving bijection of $\mathbb{R}^n$ of Baire class 1? Can both directions have class 1?

Question 9.8 (Higher-dimensional closed locus). For $n \geq 3$, must every connected component of $\text{cl}_{\mathbb{R}^n}(\text{Disc}(f))$ be unbounded for a connectedness-preserving bijection $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$?

Question 9.9 (The nonclosed locus). Can a discontinuous connectedness-preserving bijection of $\mathbb{R}^n$, $n \geq 2$, have a zero-dimensional actual discontinuity set when that set is not assumed closed?

Question 9.10 (General noncompact manifolds). Does every connected noncompact boundaryless n -manifold, $n \geq 2$, admit a Borel connectedness-preserving self-bijection whose inverse does not preserve connectedness?

Question 9.11 (Necessity of gates). If a connectedness-preserving bijection $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ has a nonempty closed nowhere-dense actual discontinuity set E and a dense regular image $f(\mathbb{R}^n \setminus E)$, must it satisfy the whole- E gate condition of Lemma 3.1?

Question 9.12 (Other exceptional sets). Can a connectedness-preserving bijection of $\mathbb{R}^3$ whose inverse does not preserve connectedness have a knotted proper locally flat ray as its exact discontinuity set? Can a line or a closed half-space in $\mathbb{R}^n$, $n \geq 2$, occur for such a bijection through a different construction?

Question 9.13 (Avoiding prescribed targets). Which closed sets $T \subset L$ of cardinality $\mathfrak{c}$ admit countably many pairwise disjoint connected Borel tracks dense in L and disjoint from T , so that the prescribed-target assignment applies?

Question 9.14 (Prescribed source sets). Which uncountable closed subsets Z_0 of the ray admit a connectedness-preserving Borel bijection onto a straight segment, as required by Remark 8.4?

Question 9.15 (Path-connected planar tracks). Can the planar recurrent tube be chosen with countably many pairwise disjoint path-connected Borel tracks dense in L , while retaining whole-section recurrence?

A change of coordinates covers ambiently equivalent proper rays. Locally flat rays may be knotted in dimension three [6, Introduction], while a line has complement of homotopy type $\mathbb{S}^{n-2}$ and hence cannot replace the ray in the same source coordinates. A half-space has an open-cell complement but lacks the interval structure used by the palette. These distinctions motivate Question 9.12; the standard spherical-end case of Question 9.10 is supplied by Theorem 9.6.

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APPENDIX A. PROVENANCE OF THE IDEAS

This appendix records the sources of the arguments used in the paper and the contributions developed within the program.

From the literature. The two-sided continuity mechanism is due to Tanaka and Pervin–Levine [18, 16]; Proposition 2.3 states its finite-complement form. Invariance of domain is used as in [11]. The planar argument combines the Jordan–Schoenflies theorem [19] with the closed-surface continuity theorem of Banach and Banach [2]. Brown’s boundary-collar theorem [4] supplies the compact-core examples for the spherical-end extension. The closed-manifold rigidity result cited in the introduction is from the author’s earlier note [15] (June 2026).

The dense-connected-image principle occurs in Gro-Tsen’s answer [10], and rational-interval Cantor kernels occur in [12, Examples 3.3–3.4]. The assignment here combines these with bijective allocation to the retained connected tracks and a prescribed inverse set. Sierpiński’s partition theorem [17] identifies the path components of the tracks; García-Máñez’s example [8] provides the comparison in Remark 7.3. The product obstruction is due to Juhász and van Mill [12].

The Borel isomorphism theorem and the Polish-space and category results [13], Kuratowski’s refinement controlling open preimages [14], and the scalar Baire correspondence [7] supply the descriptive-set-theoretic inputs.

Within the program. The gate condition and nonproper ray collapse led to the recurrent-tube construction. The planning layer proposed allocating arcs of one retained boundary line to countably many classes, making the construction uniform in the dimension. The connected-pieces argument and the selection of arcs from tail density give the dense connected tracks, while the initial cylinder retains the straight segment used for the prescribed inverse set.

The finite construction uses point moving and full topological collars. Its scalar push preserves flatness, and its retained disk coordinates give coherence through the iteration. The completion identifies both exceptional cluster sets and allows prescribed uncountable closed inverse sets of empty relative interior with exact Baire class 2 in both Euclidean directions. The scope arguments give the planar restriction on the closure of the discontinuity set and the spherical-end extension, obtained by normalizing the map to fix a regular ball and transferring the exterior construction to the end.

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