

GOLDBACH NUMBERS IN INTERVALS OF LENGTH $4.45 \log^2 x$ UNDER THE RIEMANN HYPOTHESIS

STEVE HUBBARD

ABSTRACT. Assuming the Riemann Hypothesis, we prove that for every real $x \geq 3$ the interval $(x, x + 4.45 \log^2 x)$ contains an even integer that is a sum of two odd primes. The proof combines a weighted mean square bound for the Chebyshev prime counting error in short intervals with a piecewise linear weight in a bilinear identity for short interval prime counts. The deduction is checked in Lean 4 conditionally on the explicit formula `ExplicitA`, the zero sum bound `HadamardK`, the zero count `BPTN`, the HSW zero counts, the bound $|\gamma| \geq 14$, and the Goldbach verification to $4 \cdot 10^{18}$, all stated as hypotheses.

1. INTRODUCTION

Theorem 1.1. *Assume the Riemann Hypothesis. For every real $x \geq 3$ the interval $(x, x + 4.45 \log^2 x)$ contains a Goldbach number, that is, an even integer that is a sum of two odd primes.*

Assuming the Riemann Hypothesis, Kátaı [6] and Montgomery and Vaughan [8] proved that for some constant $C > 0$ every interval $(x, x + C \log^2 x)$ with x sufficiently large contains a Goldbach number, and Goldston [4] gave a proof comparing the measure of prime-free short intervals with Selberg’s mean square bound. Cully-Hugill and Dudek [3, Theorem 1.3] made this result explicit with $C = 9696$ for all $x \geq 2$. Chirre and Hagen [2] reduced the constant to $C = 123$. The binary Goldbach conjecture itself remains open; the result here concerns Goldbach numbers in short intervals under the Riemann Hypothesis.

Our proof follows the mean square framework of Chirre and Hagen [2] and makes three changes. First, in the mean square over $[1, x]$, we replace the sharp cutoff by the smooth weight $W(u) = \min(1, u^{-4})$, which produces the positive kernel $4/(4 + (\gamma_i - \gamma_j)^2)$ and bounds the inner sum over zero ordinates directly by $\log |\gamma_i|$. Second, we bound the diagonal sum using the exact modulus squared of the short interval coefficient, Taylor majorants for $1 - \cos$, and a partition of the scaled mid-range $[1, 139.984375]$ into 2161 pieces with dyadic rational endpoints, reducing the mean square constant from 2.2258 in [2] to 1.0069. Third, instead of deducing the theorem directly from the mean square on an interval $[ax, x]$ with $\delta = h/(2x)$ as in [2, Section 5], we integrate $A(t)A(x - t) = 0$ against the weight $\varphi(t) = 11t/(5x)$ on $[x/64, 5x/11]$ and $\varphi(t) = 1$ on $[5x/11, x/2]$, which produces a nonincreasing weight on the quadratic error term and allows direct Abel summation over 256 equal steps. The resulting constants are new as far as we know.

In comparison with [2], each of these three changes replaces a step in their argument to tighten the resulting constants. First, the smooth weight W replaces their unweighted mean square expansion on $[1, x]$ and makes every off-diagonal term carry the positive kernel $4/(4 + (\gamma_i - \gamma_j)^2)$, so the inner sum is bounded directly by Lemma 2.2. Second, the exact modulus with the 2161 certified pieces replaces their bound for $|c_\delta(\gamma)|^2$; together with the smooth weight

2020 *Mathematics Subject Classification.* Primary 11P32; Secondary 11M26, 11N05, 11Y35.

Key words and phrases. Goldbach numbers, short intervals, Riemann Hypothesis, explicit bounds.

W , this lowers the mean square constant from 2.2258 in [2] to 1.0069 here. Third, the weight φ with 256 step Abel summation replaces their direct deduction from the unweighted short interval mean square, combining with the smaller mean square constant to reduce the interval length constant from $C = 123$ in [2] to 4.45.

The deduction of Theorem 1.1 from seven stated hypotheses is checked in Lean 4 with Mathlib as the theorem `goldbach_445_hsw_gZ`. Lean checks the smooth mean square bound, the 2161-piece diagonal certificate, the transfer from ψ to θ , the 256-step Abel summation in the bilinear identity, and the combination with the finite Goldbach verification conditionally on the seven hypotheses `ExplicitA`, `HadamardK`, `Ordinates14`, `BPTN`, `EvenGoldbachUpTo`, `HSWN70`, and `HSWHead14`. Among these, `HadamardK` is proved in this paper (Lemma 2.2) but assumed as a hypothesis in Lean, and `ExplicitA`, `HadamardK`, and `BPTN` are assumed under the Riemann Hypothesis. See Table 3 in Appendix A for the full list of hypotheses and their status, and Table 4 for representative pieces of the diagonal certificate.

2. NOTATION AND INPUTS

Throughout the paper, $\Lambda(n)$ denotes the von Mangoldt function, $\psi(x) = \sum_{n \leq x} \Lambda(n)$, and $\theta(x) = \sum_{p \leq x} \log p$, where p runs over primes. Under the Riemann Hypothesis, the nontrivial zeros of the Riemann zeta function $\zeta(s)$ take the form $\rho = 1/2 + i\gamma$ with $\gamma \in \mathbb{R}$. We write $N(T)$ for the number of nontrivial zeros $\rho = \beta + i\gamma$ with $0 < \gamma \leq T$, counted with multiplicity. For $\delta > 0$ and $\rho = 1/2 + i\gamma$, we define the short interval coefficient

$$c_\delta(\rho) = c_\delta(\gamma) = \frac{(1 + \delta)^{1/2+i\gamma} - 1}{1/2 + i\gamma}. \quad (2.1)$$

For $x \geq 2$ and $\delta > 0$, we define the short interval mean squares

$$J_\psi(x, \delta) = \int_1^x (\psi((1 + \delta)y) - \psi(y) - \delta y)^2 dy, \quad J_\theta(x, \delta) = \int_1^x (\theta((1 + \delta)y) - \theta(y) - \delta y)^2 dy. \quad (2.2)$$

We state the inputs used in the formal proof together with their published sources, and we prove the second input, Lemma 2.2, directly.

Lemma 2.1 ([9, Equation (12.1)], Hypothesis `ExplicitA`). *Assume the Riemann Hypothesis. For every $\delta > 0$ and almost every $y \geq 2$,*

$$\psi((1 + \delta)y) - \psi(y) - \delta y = - \lim_{T \rightarrow \infty} \sum_{|\gamma| \leq T} c_\delta(\rho) y^{1/2+i\gamma} + \frac{1}{2} \log \frac{1 - y^{-2}}{1 - ((1 + \delta)y)^{-2}}. \quad (2.3)$$

Equation (2.3) is the difference of von Mangoldt's explicit formula [9, Equation (12.1)] at $(1 + \delta)y$ and y for $y \geq 2$ when neither endpoint is a prime power. In the Lean formalization, `SmoothT2.ExplicitA` states (2.3) as the limit of $-\sum_{\rho \in Z_n} c_\delta(\rho) y^\rho$ along truncations Z_n , for almost every $y \geq 2$, with $B(y) = \frac{1}{2} \log \frac{1 - y^{-2}}{1 - ((1 + \delta)y)^{-2}}$ included as the function `Bf`; the main theorem assumes it, under RH, for each $0 < \delta \leq 10^{-13}$.

Lemma 2.2 (Hypothesis `HadamardK`). *Assume the Riemann Hypothesis. For every nontrivial zero ordinate γ of $\zeta(s)$ and every finite family of nontrivial zero ordinates γ' counted with multiplicity,*

$$\sum_{\gamma'} \frac{4}{4 + (\gamma - \gamma')^2} \leq \log |\gamma|. \quad (2.4)$$

Proof. Since each summand is positive, it suffices to bound the sum over all nontrivial zero ordinates γ' counted with multiplicity, where $\rho' = 1/2 + i\gamma'$. Setting $s = 5/2 + i\gamma$, we have

$$\sum_{\gamma'} \frac{4}{4 + (\gamma - \gamma')^2} = 2 \sum_{\gamma'} \operatorname{Re} \frac{1}{s - \rho'}. \quad (2.5)$$

By the partial fraction decomposition of $\xi'(s)/\xi(s)$ [9, Corollary 10.14],

$$2 \sum_{\gamma'} \operatorname{Re} \frac{1}{s - \rho'} = \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{s}{2} + 1 \right) - \log \pi + 2 \operatorname{Re} \frac{\zeta'}{\zeta}(s) + 2 \operatorname{Re} \frac{1}{s - 1}. \quad (2.6)$$

For $\operatorname{Re} z > 0$, Binet's integral formula gives $\frac{\Gamma'}{\Gamma}(z) = \log z - \frac{1}{2z} - \int_0^\infty f(t) e^{-tz} dt$, where $f(t) = \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}$ satisfies $f(0) = 0$ and $f'(t) = \frac{1}{t^2} - \frac{1}{4 \sinh^2(t/2)}$ decreases monotonically from $f'(0) = \frac{1}{12}$ to 0 on $(0, \infty)$. Integrating by parts twice gives $\int_0^\infty f(t) e^{-tz} dt = \frac{1}{12z^2} + \frac{1}{z^2} \int_0^\infty f''(t) e^{-tz} dt$, which has modulus at most $\frac{1}{12|z|^2} + \frac{1}{|z|^2} \int_0^\infty (-f''(t)) dt = \frac{1}{6|z|^2} \leq \frac{\operatorname{Re} z}{2|z|^2} = \operatorname{Re} \frac{1}{2z}$ whenever $\operatorname{Re} z \geq \frac{1}{3}$, so $\operatorname{Re} \frac{\Gamma'}{\Gamma}(z) \leq \log |z|$. Applying this at $z = \frac{9}{4} + \frac{i\gamma}{2}$ with $|\gamma| \geq 14$ from Lemma 2.3 yields

$$\operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{9}{4} + \frac{i\gamma}{2} \right) \leq \log \left| \frac{9}{4} + \frac{i\gamma}{2} \right| \leq \log |\gamma| - \log 2 + \frac{81}{8\gamma^2}. \quad (2.7)$$

On the line $\operatorname{Re}(s) = 5/2$, the Dirichlet series for ζ'/ζ yields $\operatorname{Re} \frac{\zeta'}{\zeta}(s) \leq -\frac{\zeta'}{\zeta}(5/2) = 0.28874 \dots < 0.2888$, while $2 \operatorname{Re} \frac{1}{s-1} = \frac{3}{9/4 + \gamma^2} < \frac{3}{\gamma^2}$. For $|\gamma| \geq 14$, we obtain

$$-\log(2\pi) + 2(0.2888) + \frac{105}{8 \cdot 14^2} = -1.83787 \dots + 0.5776 + 0.06696 \dots < -1.1933 < 0, \quad (2.8)$$

so the right-hand side of (2.6) is at most $\log |\gamma| - 1.1933 < \log |\gamma|$; the lemma states the weaker bound $\log |\gamma|$ because that is the form used later. $\square$

Lemma 2.3 ([12, Chapter XV], Hypothesis `Ordinates14`). *Every nontrivial zero $\rho = \beta + i\gamma$ of $\zeta(s)$ satisfies $|\gamma| \geq 14$.*

The lowest nontrivial zero has ordinate $\gamma_1 = 14.134725 \dots > 14$.

Lemma 2.4 ([1, Lemma 1], Hypothesis `BPTN`). *For all $T \geq 2\pi e$, the zero counting function $N(T)$ satisfies*

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} + Q(T), \quad (2.9)$$

where

$$|Q(T)| \leq 0.137 \log T + 0.443 \log \log T + 1.588 + \frac{0.2}{T}. \quad (2.10)$$

We use the constants of Lemma 1 of the published version [1]. The Lean formalization only uses Lemma 2.4 for $T \geq 10^{13}$, as counts over both signs of the ordinates.

Lemma 2.5 ([11, Corollary 1]). *Assume the Riemann Hypothesis. For every real $u \geq 599$,*

$$|\theta(u) - u| < \frac{\sqrt{u} \log^2 u}{8\pi}. \quad (2.11)$$

In particular, for every real $u \geq 10^{16}$,

$$|\theta(u) - u| \leq 10^{-6} u. \quad (2.12)$$

Inequality (2.12) follows immediately from (2.11) because $u^{-1/2} \log^2 u$ is decreasing for $u > e^4$ and equals $5.4005 \cdots \times 10^{-7} \cdot 8\pi < 8\pi \cdot 10^{-6}$ at $u = 10^{16}$. In the Lean formalization, inequality (2.11) is not assumed; it is proved under RH for $u \geq 10^{11}$ from the explicit formula and the zero counting bound of Hasanlizada, Shen, and Wong [5, Corollary 1.2] (assumed as the hypotheses `HSWN70` and `HSWHead14`), together with the numerical fact, taken as the hypothesis `Ordinates14` in Lean, that every nontrivial zero has ordinate of absolute value at least 14.

Lemma 2.6 ([10], Hypothesis `EvenGoldbachUpTo`). *Every even integer $2n$ in the range $4 \leq 2n \leq 4 \cdot 10^{18}$ is a sum of two primes.*

3. THE MEAN SQUARE BOUND

We bound the mean square on $[1, x]$ of a finite Dirichlet polynomial supported on nontrivial zeros of $\zeta(s)$.

Lemma 3.1. *Assume the Riemann Hypothesis. Let $\gamma_1, \dots, \gamma_M$ be a finite family of nontrivial zero ordinates of $\zeta(s)$, let $c_1, \dots, c_M \in \mathbb{C}$, and set*

$$P(y) = \sum_{i=1}^M c_i y^{1/2+i\gamma_i}. \quad (3.1)$$

For every $x \geq 1$,

$$\int_1^x |P(y)|^2 dy \leq x^2 \sum_{i=1}^M |c_i|^2 \sum_{j=1}^M \frac{4}{4 + (\gamma_i - \gamma_j)^2} \leq x^2 \sum_{i=1}^M |c_i|^2 \log |\gamma_i|. \quad (3.2)$$

Proof. Define $W(u) = \min(1, u^{-4})$ for $u > 0$. Since $W(u) \geq 0$ for all $u > 0$ and $W(y/x) = 1$ for $1 \leq y \leq x$, the substitution $y = xu$ gives

$$\begin{aligned} \int_1^x |P(y)|^2 dy &\leq \int_0^\infty |P(y)|^2 W(y/x) dy \\ &= x^2 \sum_{i=1}^M \sum_{j=1}^M c_i \overline{c_j} x^{i(\gamma_i - \gamma_j)} \int_0^\infty u^{1+i(\gamma_i - \gamma_j)} W(u) du. \end{aligned} \quad (3.3)$$

For any $\Delta \in \mathbb{R}$, splitting the integral at $u = 1$ yields

$$\int_0^\infty u^{1+i\Delta} W(u) du = \int_0^1 u^{1+i\Delta} du + \int_1^\infty u^{-3+i\Delta} du = \frac{1}{2+i\Delta} + \frac{1}{2-i\Delta} = \frac{4}{4+\Delta^2}. \quad (3.4)$$

Since $\frac{4}{4+(\gamma_i - \gamma_j)^2} > 0$ is symmetric in i and j , applying $|c_i \overline{c_j} x^{i(\gamma_i - \gamma_j)}| \leq \frac{1}{2}(|c_i|^2 + |c_j|^2)$ in (3.3) gives the first inequality in (3.2). Applying Lemma 2.2 to the inner sum over j gives the second inequality. $\square$

We next bound the diagonal sum $\sum_\gamma |c_\delta(\gamma)|^2 \log |\gamma|$.

Lemma 3.2 (Theorem `diag_u6806`). *Assume the Riemann Hypothesis; in Lean, Theorem `diag_u6806` is conditional on the hypotheses `BPTN` and `Ordinates14`. For every δ in the range $0 < \delta \leq 10^{-13}$,*

$$\sum_\gamma |c_\delta(\gamma)|^2 \log |\gamma| \leq C_{\text{diag}} \delta \log^2 \left(\frac{2.0001}{\delta} \right), \quad C_{\text{diag}} = 1.0068. \quad (3.5)$$

Proof. By symmetry of the nontrivial zeros with respect to the real axis, $\sum_{\gamma} |c_{\delta}(\gamma)|^2 \log |\gamma| = 2 \sum_{\gamma > 0} |c_{\delta}(\gamma)|^2 \log \gamma$, and every term has $\gamma \geq 14$ by Lemma 2.3. Let $\eta = \log(1 + \delta)$ and $\Lambda = \log(1/\delta)$. From (2.1), the exact modulus squared of $c_{\delta}(\gamma)$ is

$$|c_{\delta}(\gamma)|^2 = \frac{(1 + \delta) + 1 - 2\sqrt{1 + \delta} \cos(\gamma\eta)}{1/4 + \gamma^2} = \frac{(\sqrt{1 + \delta} - 1)^2 + 2\sqrt{1 + \delta}(1 - \cos(\gamma\eta))}{1/4 + \gamma^2}. \quad (3.6)$$

For all real t , the alternating Taylor series for $\cos t$ gives the degree-6 majorant (`one_sub_cos_1e`)

$$1 - \cos t \leq \frac{t^2}{2} - \frac{t^4}{24} + \frac{t^6}{720}. \quad (3.7)$$

Write $v = \delta|\gamma|$ for the scaled variable, and set

$$X = \delta\Lambda^2, \quad Y = \delta\Lambda, \quad E = \delta^2(\Lambda + 5)^2.$$

The certified bound is organized in two cases, split at $\delta = 2^{-188}$.

Pieces. The bound on an individual piece is formalized as the lemma `piece_sum` in `LeanLab/ChirreHagen2025/U1263Core.lean`, conditional on `BPTN` (Lemma 2.4) and `Ordinates14` (Lemma 2.3). Let $0 < \delta \leq 10^{-13}$, let $1 \leq A \leq B$, let $\log B \leq \lambda \leq 5$, let $Q \geq 0$, and assume the majorant hypothesis on the piece that $|c_{\delta}(\gamma)|^2 \leq Q\delta^2$ whenever $A/\delta < |\gamma| \leq B/\delta$. Then `piece_sum` states that for any finite set of zero ordinates,

$$\sum_{A/\delta < |\gamma| \leq B/\delta} |c_{\delta}(\gamma)|^2 \log |\gamma| \leq 0.31831 Q(B - A)(X + (2\lambda - 1.8)Y + (\lambda^2 - 1.8\lambda)\delta) + 2QE. \quad (3.8)$$

In the structure of this Lean lemma, the zeros on the piece are counted with Lemma 2.4, $\log |\gamma| \leq \Lambda + \lambda$ there because $|\gamma| \leq B/\delta$ and $\log B \leq \lambda$, and the term $2QE$ collects the lower order part of the count.

To obtain Q on a piece $A/\delta < |\gamma| \leq B/\delta$, write $t = |\gamma| \log(1 + \delta)$ for the phase; the lemma `phase_range` gives $A(1 - 10^{-13}) < t \leq B$ on the piece. The majorant hypothesis $|c_{\delta}(\gamma)|^2 \leq Q\delta^2$ is established by one of two Lean routes in `U1263Core.lean`: `pt_norm` for pieces nearest $v = 1$, and `pt_raw` elsewhere. In `pt_norm`, for $0 \leq \text{lo} \leq A(1 - 10^{-13})$, Q satisfies $2(1 + 10^{-13})(1/2 - \text{lo}^2/24 + B^4/720) + 10^{-26} \leq Q$. In `pt_raw`, given an upper bound $1 - \cos t \leq u$ for $A(1 - 10^{-13}) < t \leq B$, Q satisfies

$$\frac{10^{-26} + 2(1 + 10^{-13})u}{A^2} \leq Q. \quad (3.9)$$

The bound u in `pt_raw` comes from `hu_poly`, the degree-6 majorant (3.7) after shifting t by $2k\pi$, or from `U6743.hu_pi`, the majorant $1 - \cos t \leq 2 - s^2/2 + s^4/24$ with s the shift of t by an odd multiple of π ; both use $3.141592 < \pi < 3.141593$. The piece endpoints are dyadic rationals; the piece endpoints and λ were chosen by a numerical search and certified piece by piece. Here λ is any number with $\log B \leq \lambda \leq 5$, certified in Lean through a dyadic bound on $\log B$; Q is a rational number at least the `pt_raw` or `pt_norm` bound. For example, the first piece has $A = 1$, $B = 1.0029296875$, $Q = 0.9194789$, $\lambda = 0.0029296875$, and the last piece has $A = 139.6982421875$, $B = 139.984375$, $Q = 0.0001399$, $\lambda = 4.9456581952875$.

Case $2^{-188} < \delta \leq 10^{-13}$. On the head range $|\gamma| \leq 1/\delta$ the sum is at most $\delta\Lambda^2/\pi \leq 0.31831 X$. On the range $v > 1$ it is at most

$$0.696652249211 X + 0.294654626964 Y + 0.112752623673 \delta + 785.028327376 E.$$

This range consists of 2161 pieces covering $1 < v \leq 139.984375$, each bounded by (3.8), a further range $139.984375 < v \leq 599.734375$, and a tail $v > 599.734375$ bounded by $0.002182674424 X + 0.027980492816 Y + 0.089673014185 \delta$. Table 1 lists the coefficients of

$X = \delta\Lambda^2$. They sum to 1.014962249211, and the bound $1.0068\delta\log^2(2.0001/\delta)$ follows because $\log(2.0001/\delta) = \Lambda + \log 2.0001$ with $\log 2.0001 \geq 0.6931$, and $\Lambda \leq 130.3116699904$ when $\delta > 2^{-188}$; together these absorb the lower order terms.

Case $\delta \leq 2^{-188}$. The head range is $|\gamma| \leq 10^{-20}/\delta$, followed by a dyadic chain of 64 steps, which gives

$$1.006510467199 X - 0.263090165331 Y + 0.112752623673 \delta + 924.581825936 E \leq 1.0068\delta\log^2(2.0001/\delta).$$

All of these bounds are certified by exact rational arithmetic and checked in Lean as `diag_u6806`, conditional on `BPTN` and `Ordinates14`. $\square$

TABLE 1. Coefficients of $\delta\Lambda^2$ in the case $2^{-188} < \delta \leq 10^{-13}$ of Lemma 3.2.

Range	Method	Coefficient of $\delta\Lambda^2$
Head $ \gamma \leq 1/\delta$	Bound $\delta\Lambda^2/\pi$	0.31831
$1 < v \leq 139.984375$	2161 pieces, (3.8)	0.690304599302
$139.984375 < v \leq 599.734375$	Certified chain	0.004164975485
$v > 599.734375$	Tail bound	0.002182674424
Total		1.014962249211

The lower order terms and $\Lambda \leq 130.3116699904$ bring the total to 1.0068 relative to $\delta\log^2(2.0001/\delta)$.

Combining Lemma 2.1, Lemma 3.1, and Lemma 3.2 with the L^2 convergence of the truncated explicit formula as proved in [2, Section 3] yields the short interval mean square bound.

Proposition 3.3 (Theorem `theorem2_u6806`). *Assume the Riemann Hypothesis; in Lean, Theorem `theorem2_u6806` is conditional on the hypotheses `ExplicitA`, `HadamardK`, `Ordinates14`, and `BPTN`. For all $x \geq 10^{13}$ and $0 < \delta \leq 10^{-13}$,*

$$J_\psi(x, \delta) \leq J \delta \log^2\left(\frac{2.0001}{\delta}\right) x^2, \quad J = 1.0069. \quad (3.10)$$

Proof. Write $V(y) = \psi((1+\delta)y) - \psi(y) - \delta y$ and let

$$B(y) = \frac{1}{2} \log \frac{1 - y^{-2}}{1 - ((1+\delta)y)^{-2}} \quad (3.11)$$

be the secondary term of Lemma 2.1, so $V(y) - B(y)$ is the zero sum. By Lemma 2.1, the L^2 limit argument of [2, Section 3], and Lemmas 3.1 and 3.2, we have

$$\int_2^x (V(y) - B(y))^2 dy \leq 1.0068\delta\log^2\left(\frac{2.0001}{\delta}\right) x^2. \quad (3.12)$$

For $y \geq 2$ and $0 < \delta \leq 1$, we have $|B(y)| \leq \delta$, so $\int_2^x B(y)^2 dy \leq \delta^2 x$. Since $(u+b)^2 \leq (1+\eta)u^2 + (1+1/\eta)b^2$ for every $\eta > 0$, taking $\eta = 10^{-5}$ with $L = \log(2.0001/\delta)$ gives

$$\int_2^x V(y)^2 dy \leq (1 + 10^{-5}) \cdot 1.0068\delta L^2 x^2 + (1 + 10^5)\delta^2 x. \quad (3.13)$$

On $[1, 2]$, since $\psi(y) = 0$ for $y < 2$, the difference $\psi((1+\delta)y) - \psi(y)$ is nonzero only for $y \in [2/(1+\delta), 2)$, an interval of length at most 2δ , where it is at most $\log 2$; with $u = \psi((1+\delta)y) - \psi(y) \geq 0$ and $v = \delta y \geq 0$ we have $(u-v)^2 \leq u^2 + v^2$, so $\int_1^2 V(y)^2 dy \leq 2\delta(\log 2)^2 + \frac{7}{3}\delta^2 \leq 3\delta$. For $x \geq 10^{13}$ and $0 < \delta \leq 10^{-13}$, we have $L \geq \log(2 \cdot 10^{13}) > 30$.

Dividing by $\delta L^2 x^2$, the first term is $1.0068(1 + 10^{-5}) = 1.00681007$, the second term is at most $10^5 \cdot 10^{-13} / (900 \cdot 10^{13}) < 10^{-30}$, and the third is at most $3 / (900 \cdot 10^{26}) < 10^{-28}$. The sum is below 1.0069, so $J_\psi(x, \delta) \leq 1.0069 \delta L^2 x^2$. $\square$

We now pass from $J_\psi(X, \delta)$ to $J_\theta(X, \delta)$. Write $D(y) = (\psi - \theta)((1 + \delta)y) - (\psi - \theta)(y) \geq 0$ and $u = \psi((1 + \delta)y) - \psi(y) - \delta y$, so $\theta((1 + \delta)y) - \theta(y) - \delta y = u - D(y)$. Applying $(u - D)^2 \leq (1 + \eta)u^2 + (1 + 1/\eta)D^2$ with $\eta = 0.001$ gives $J_\theta(X, \delta) \leq 1.001 J_\psi(X, \delta) + 1001 \int_1^X D(y)^2 dy$. By Fubini's theorem, $\int_1^X D(y) dy \leq \delta X(\psi - \theta)((1 + \delta)X)$. When $\delta\sqrt{y} < 1$, every interval $(y^{1/k}, ((1 + \delta)y)^{1/k}]$ with $k \geq 2$ has length at most $\frac{1}{2}\delta\sqrt{y} < 1$ and contains at most one prime, so $D(y) \leq \log^2((1 + \delta)y) / \log 2$. Multiplying this pointwise bound by the L^1 bound and using $(\psi - \theta)((1 + \delta)X) \leq 2\sqrt{(1 + \delta)X} \log((1 + \delta)X)$ shows that $1001 \int_1^X D(y)^2 dy \leq 0.0019 \delta \log^2(2.0001/\delta) X^2$ for $X \geq 5 \cdot 10^{17}$ and $0 < \delta \leq 10^{-13}$. Thus, if $J_\psi(X, \delta) \leq J \delta \log^2(2.0001/\delta) X^2$, then for $X \geq 5 \cdot 10^{17}$, $0 < \delta \leq 10^{-13}$, and $\delta\sqrt{X} < 1$, we have

$$J_\theta(X, \delta) \leq c \delta \log^2\left(\frac{2.0001}{\delta}\right) X^2, \quad c = 1.001J + 0.0019. \quad (3.14)$$

With $J = 1.0069$ from Proposition 3.3, this gives $c = 1.0098069$.

4. THE BILINEAR ARGUMENT

To prove Theorem 1.1, we split the range of x at $2 \cdot 10^{18}$. For $3 \leq x \leq 2 \cdot 10^{18}$, the interval length satisfies $4.45 \log^2 x \geq 4.45 \log^2 3 > 5.36$, so $(x, x + 4.45 \log^2 x)$ contains an even integer $2n$ in $[6, 4 \cdot 10^{18}]$, which is a sum of two odd primes by Lemma 2.6. The analytic argument runs for $x \geq 1.1 \cdot 10^{18}$, where $\log x \geq 41$.

Let $x \geq 1.1 \cdot 10^{18}$ and fix H with $4 \leq H \leq 61.5$. Put $h = H \log^2 x$ and $\delta = h/x$, so $0 < \delta \leq 10^{-13}$ and $\delta\sqrt{x} < 1$. Let

$$A(t) = \theta((1 + \delta)t) - \theta(t), \quad E(t) = A(t) - \delta t. \quad (4.1)$$

As observed by Goldston [4, p. 285], if no Goldbach number lies in $(x, x + h]$, then for every $t \in [x/64, x/2]$ the intervals $(t, (1 + \delta)t]$ and $(x - t, (1 + \delta)(x - t)]$ cannot both contain an odd prime, and since $t \geq x/64 > 10^{16} > 2$, we have

$$A(t)A(x - t) = 0 \quad \text{for all } t \in [x/64, x/2]. \quad (4.2)$$

We weight (4.2) by the continuous piecewise linear function

$$\varphi(t) = \begin{cases} \frac{11t}{5x}, & \frac{x}{64} \leq t \leq \frac{5x}{11}, \\ 1, & \frac{5x}{11} \leq t \leq \frac{x}{2}. \end{cases} \quad (4.3)$$

Substituting $A(t) = \delta t + E(t)$ and $A(x - t) = \delta(x - t) + E(x - t)$ into $0 = \int_{x/64}^{x/2} \varphi(t) A(t) A(x - t) dt$ gives the weighted identity

$$\delta^2 \int_{x/64}^{x/2} \varphi(t) t(x - t) dt + \delta \int_{x/64}^{x/2} \varphi(t) (tE(x - t) + (x - t)E(t)) dt = - \int_{x/64}^{x/2} \varphi(t) E(t) E(x - t) dt. \quad (4.4)$$

By the arithmetic-geometric mean inequality, for every $t \in [x/64, x/2]$ we have

$$|\varphi(t) E(t) E(x - t)| \leq \frac{1}{2} E(t)^2 + \frac{1}{2} \varphi(t)^2 E(x - t)^2. \quad (4.5)$$

Integrating (4.5) over $t \in [x/64, x/2]$ and substituting $u = x - t$ in the second term yields

$$\int_{x/64}^{x/2} |\varphi(t)E(t)E(x-t)| dt \leq \int_{x/64}^{63x/64} w(u)E(u)^2 du, \quad (4.6)$$

where the combined weight $w(u)$ on $[x/64, 63x/64]$ is

$$w(u) = \begin{cases} \frac{1}{2}, & \frac{x}{64} \leq u \leq \frac{6x}{11}, \\ \frac{121}{50} \left(1 - \frac{u}{x}\right)^2, & \frac{6x}{11} \leq u \leq \frac{63x}{64}. \end{cases} \quad (4.7)$$

At the junction $u = 6x/11$, both expressions in (4.7) equal $1/2$. Thus $w(u)$ is positive, continuous, and nonincreasing on $[x/64, 63x/64]$.

Proof of Theorem 1.1. Assume $x \geq 1.1 \cdot 10^{18}$, fix H with $4 \leq H \leq 61.5$, put $h = H \log^2 x$ and $\delta = h/x$, and suppose that no Goldbach number lies in $(x, x+h]$. Evaluating the main term on the left-hand side of (4.4) with $\varphi(t)$ from (4.3) gives $\delta^2 x^3 M$, where

$$\begin{aligned} M &= \frac{11}{5} \left[\frac{(5/11)^3 - (1/64)^3}{3} - \frac{(5/11)^4 - (1/64)^4}{4} \right] + \left[\frac{(1/2)^2 - (5/11)^2}{2} - \frac{(1/2)^3 - (5/11)^3}{3} \right] \\ &= 0.0567215028 \dots \end{aligned} \quad (4.8)$$

To bound the cross terms in (4.4), write $R(u) = \theta(u) - u$, so $E(u) = R((1+\delta)u) - R(u)$. Substituting $u = x - t$ in $\int_{x/64}^{x/2} \varphi(t)tE(x-t) dt$ splits the cross-term integral into four smooth pieces; the first two are written in the variable t and the last two in $u = x - t$: $[x/64, 5x/11]$ with weight $\frac{11}{10} \cdot \frac{2}{x} t(x-t)$, $[5x/11, x/2]$ with weight $x-t$, $[6x/11, 63x/64]$ with weight $\frac{11}{10} \cdot \frac{2}{x} (x-u)^2$, and $[x/2, 6x/11]$ with weight $x-u$. On each piece $[ax, bx]$ with normalized weight $g(u)$, the substitution $v = (1+\delta)u$ transforms $\int_{ax}^{bx} g(u)R((1+\delta)u) du$ into $\int_{(1+\delta)ax}^{(1+\delta)bx} \frac{1}{1+\delta} g(\frac{v}{1+\delta}) R(v) dv$ with new limits of integration $(1+\delta)ax$ and $(1+\delta)bx$, which gives

$$\begin{aligned} \int_{ax}^{bx} g(u)E(u) du &= \int_{ax}^{bx} \left(\frac{1}{1+\delta} g\left(\frac{v}{1+\delta}\right) - g(v) \right) R(v) dv \\ &\quad + \left(\int_{bx}^{(1+\delta)bx} - \int_{ax}^{(1+\delta)ax} \right) \frac{1}{1+\delta} g\left(\frac{v}{1+\delta}\right) R(v) dv. \end{aligned} \quad (4.9)$$

Since $1 - (1+\delta)^{-k} \leq k\delta$ for $k = 1, 2, 3$ and $0 < u \leq x$, the interior factor $|\frac{1}{1+\delta} g(\frac{v}{1+\delta}) - g(v)|$ is bounded on $[0, x]$ by $2(2+3)\delta x = 10\delta x$ for $\frac{2}{x}u(x-u)$, by $(1+2)\delta x = 3\delta x$ for $x-u$, and by $2(1+4+3)\delta x = 16\delta x$ for $\frac{2}{x}(x-u)^2$. Since $u \geq x/64 > 10^{16}$ throughout all four intervals, Lemma 2.5 gives $|R(v)| \leq 10^{-6}v \leq 10^{-6}x$ on $[ax, bx]$ of length $(b-a)x$, and $|R(v)| \leq 10^{-6}v \leq 10^{-6}(1+\delta)x \leq 2 \cdot 10^{-6}x$ on the boundary intervals $[bx, (1+\delta)bx]$ and $[ax, (1+\delta)ax]$ of lengths δbx and δax . Summing the four bounds shows that the left-hand side of (4.4) is at least $\delta^2 x^3(M - K)$, where

$$\begin{aligned} K &= \frac{1}{10^6} \left[\frac{11}{10} \left(10 \left(\frac{5}{11} - \frac{1}{64} \right) + 2 \cdot \frac{5}{11} + \frac{2}{64} \right) + \left(3 \left(\frac{1}{2} - \frac{5}{11} \right) + 1 + 2 \cdot \frac{5}{11} \right) \right. \\ &\quad \left. + \frac{11}{10} \left(16 \left(\frac{63}{64} - \frac{6}{11} \right) + 2 \cdot \frac{63}{64} + 2 \cdot \frac{6}{11} \right) + \left(3 \left(\frac{6}{11} - \frac{1}{2} \right) + 2 \cdot \frac{6}{11} + 1 \right) \right] \\ &= \frac{1}{10^6} \left(\frac{469}{80} + \frac{45}{22} + \frac{3549}{320} + \frac{49}{22} \right) = 2.122585 \dots \cdot 10^{-5} < 2.1226 \cdot 10^{-5}. \end{aligned} \quad (4.10)$$

To bound the right-hand side of (4.6), take 256 equal steps

$$s_j = \frac{6}{11} + \frac{309j}{180224}, \quad w_j = \frac{121}{50}(1 - s_j)^2 \quad \text{for } j = 0, \dots, 256, \quad (4.11)$$

so $s_0 = 6/11$, $s_{256} = 63/64$, and $w_0 = 1/2$. The weight $w(u)$ on $E(u)^2$ in (4.7) is $1/2$ on $[x/64, 6x/11]$ and at most w_j on $[s_j x, s_{j+1} x]$. Since $s_j x \geq (6/11)x \geq 6 \cdot 10^{17} > 5 \cdot 10^{17}$, w is nonincreasing, and $\int_1^{s_j x} E(u)^2 du \leq J_\theta(s_j x, \delta) \leq c \delta \log^2(2.0001/\delta) s_j^2 x^2$ by (3.14), Abel summation bounds the right-hand side of (4.6) by $c \delta \log^2(2.0001/\delta) x^2 U$, where

$$U = \sum_{j=1}^{255} (w_{j-1} - w_j) s_j^2 + w_{255} s_{256}^2 = 0.2492140 \dots < 0.24922. \quad (4.12)$$

Since $h \geq 2.0001$, we have $\log(2.0001/\delta) \leq \log x$. Using $\delta = H \log^2 x / x$ and dividing by $\delta x^2 \log^2 x$, no Goldbach number in $(x, x + h]$ forces $H(M - K) \leq c \cdot 0.24922$. Thus it is enough that

$$(1.001J + 0.0019) \cdot 0.24922 < H(M - K). \quad (4.13)$$

With $J = 1.0069$ from Proposition 3.3, the left-hand side of (4.13) is 0.2516641, and $H = 4.44$ gives right-hand side 0.2517492; the smallest H that works is $H_{\min} = 4.43850$. This gives a Goldbach number n with $x < n \leq x + 4.44 \log^2 x$, and the extra $0.01 \log^2 x$ in the theorem gives the strict upper bound in $(x, x + 4.45 \log^2 x)$. $\square$

By (4.13), $H_{\min} = (1.001J + 0.0019) \cdot 0.24922 / (M - K)$, so $H_{\min}$ is proportional to c and the mean square constant J is the main limiting input; at $J = 1.0069$ the weight scan gives about 4.43, and the 4.43 figure is a numerical scan of this weight family at $J = 1.0069$, not a proven lower limit. Lowering the order of the interval length below $\log^2 x$ from the Riemann Hypothesis alone would require an upper bound on the pair correlation of zeros of $\zeta(s)$ that is not known; under the Riemann Hypothesis together with Montgomery's pair correlation conjecture [7], Goldston [4] already reaches $(2 + \epsilon) \log x$.

5. NUMERICAL VERIFICATION

All numerical constants in Lemma 3.2, Proposition 3.3, and the proof of Theorem 1.1 are certified in the Lean 4 kernel conditionally on the seven hypotheses of `goldbach_445_hsw_gZ`. The certification uses finite grids with explicit rational parameters: in Lemma 3.2, the case $2^{-188} < \delta \leq 10^{-13}$ uses 2161 pieces on $1 < v \leq 139.984375$, a further range $139.984375 < v \leq 599.734375$, and a tail $v > 599.734375$, while the case $\delta \leq 2^{-188}$ uses a head range $|\gamma| \leq 10^{-20}/\delta$ and a dyadic chain of 64 steps; in Section 4, the interval $[6/11, 63/64]$ is split into 256 equal Abel steps in u/x . Every constant is a decimal or rational number checked by `norm_num` and `linarith` in the Lean kernel, so no floating point arithmetic enters the certificate and no rounding error analysis is needed; transcendental quantities enter only through certified bounds such as $3.141592 < \pi < 3.141593$, $\log 2.0001 \geq 0.6931$, and dyadic bounds on $\log B$. The Lean build completes successfully in 9183 jobs with exit code 0, and `#print axioms` for `goldbach_445_hsw_gZ`, `theorem2_u6806`, and `goldbach_445_of_J_hsw` lists only the standard foundational axioms `propext`, `Classical.choice`, and `Quot.sound`. Table 2 records the certified parameters and margin checks around $H_{\min} = 4.43850$.

APPENDIX A. FORMAL VERIFICATION

The deduction of Theorem 1.1 from the seven stated hypotheses is formalized in Lean 4 with Mathlib at <https://github.com/sthubbard/lean-lab-preprints>, commit 74c2d1a. The diagonal estimate of Lemma 3.2 is formalized conditionally on `BPTN` and `Ordinates14`

TABLE 2. Certified constants and margin checks for the main theorem.

Quantity	Certified value	Role in proof
End of the 2161 pieces	139.984375	In $v = \delta \gamma $, Lemma 3.2
Pieces	2161	Dyadic rational endpoints on $1 < v \leq 139.984375$
Diagonal constant C_{diag}	1.0068	Bound in (3.5) for $0 < \delta \leq 10^{-13}$
Mean square constant J	1.0069	Bound in (3.10) for $x \geq 10^{13}$
Transfer constant c	1.0098069	$c = 1.001J + 0.0019$ in (3.14) for $X \geq 5 \cdot 10^{17}$
Abel summation steps	256	Equal steps in u/x from $6/11$ to $63/64$ in (4.11)
Main term constant M	0.0567215028...	Exact integral in (4.8) with weight φ
Cross-term constant K	$2.1226 \cdot 10^{-5}$	Cross-term bound in (4.10)
Abel constant U	$0.2492140 \dots < 0.24922$	Upper bound factor in (4.12)
Minimum admissible $H_{\min}$	4.43850	Threshold in (4.13); 4.44 gives $0.2517492 > 0.2516641$

as `diag_u6806`, the short interval mean square bound of Proposition 3.3 conditionally on `ExplicitA`, `HadamardK`, `Ordinates14`, and `BPTN` as `theorem2_u6806`, and the main result Theorem 1.1 conditionally on all seven hypotheses as `goldbach_445_hsw_gZ` in `LeanLab/Routes/U1508Compose445.lean`. The proof of Lemma 2.2 (`HadamardK`) uses the partial fraction expansion of $\xi'(s)/\xi(s)$ over all zeros and Binet's formula for the digamma function. These ingredients were not formalized in this project, so `HadamardK` enters Lean as a hypothesis.

The headline theorem is in file `LeanLab/Routes/U1508Compose445.lean`, namespace `LeanLab.Routes.U1508`, and has the exact statement:

```

theorem goldbach_445_hsw_gZ {Z : ℕ → Finset ZIdx}
  (hE : RiemannHypothesis → ∀ δ, 0 < δ → δ ≤ 1 / 10 ^ 13 →
    SmoothT2.ExplicitA gZ Z δ)
  (hH : RiemannHypothesis → SmoothT2.HadamardK gZ)
  (h14 : SmoothT2.Ordinates14 gZ)
  (hB : RiemannHypothesis → Lemma26B.BPTN gZ)
  (hEv : EvenGoldbachUpTo (4 * 10 ^ 18))
  (hN : HSWN70 gZ) (hHd : HSWHead14 gZ) :
  RiemannHypothesis → ∀ x : ℝ, 3 ≤ x →
    ∃ n : ℕ, IsGoldbachNumber n ∧ x < n ∧
      (n : ℝ) < x + 4.45 * Real.log x ^ 2

```

Here `gZ` is the family of ordinates of the nontrivial zeros of $\zeta(s)$ with multiplicity; `ExplicitA` states (2.3) of Lemma 2.1 as the limit of $-\sum_{\rho \in Z_n} c_\delta(\rho)y^\rho$ along truncations Z_n , for almost every $y \geq 2$, with $B(y)$ included as the function `Bf`, assumed under RH for each $0 < \delta \leq 10^{-13}$; `HadamardK` is Lemma 2.2; `Ordinates14` states that every ordinate has absolute value at least 14; `BPTN` is Lemma 2.4 for $T \geq 10^{13}$; `HSWN70` and `HSWHead14` are [5, Corollary 1.2] read as counts of ordinates of both signs, for $T \geq 70$ and $T \geq 14$; and `EvenGoldbachUpTo` is Lemma 2.6. The bound `SchoenfeldTheta` is not a hypothesis of this theorem. Table 3 lists the status of each hypothesis in the paper and in Lean; the formal verification is conditional on these seven hypotheses, and the main theorem assumes `ExplicitA`, `HadamardK`, and `BPTN` under RH.

Structure of the Lean formalization. The formal proof is organized along the following chain of Lean 4 files and theorems; items 1 to 4 are in `LeanLab/ChirreHagen2025/` and items 5 and 6 in `LeanLab/Routes/`:

- (1) `U1263Core.lean` (namespace `LeanLab.ChirreHagen2025.U1263`) proves `piece_sum` (the piece bound (3.8)), `pt_raw`, `pt_norm`, and `one_sub_cos_le`, while `U6743Core.lean` provides `U6743.hu_pi`:

```

lemma piece_sum (hN : BPTN γ) (h14 : Ordinates14 γ) {δ : ℝ} (hδ : 0 < δ)

```

TABLE 3. Hypotheses of `goldbach_445_hsw_gZ` in the paper and in Lean 4.

Lean hypothesis	Source	In this paper	In Lean 4
<code>ExplicitA</code>	[9, Equation (12.1)], Lemma 2.1	Cited	Assumed as hypothesis
<code>HadamardK</code>	Lemma 2.2	Proved under RH	Assumed as hypothesis
<code>Ordinates14</code>	[12, Chapter XV], Lemma 2.3	Cited	Assumed as hypothesis
<code>BPTN</code>	[1, Lemma 1], Lemma 2.4 for $T \geq 10^{13}$	Cited	Assumed as hypothesis
<code>HSWN70</code>	[5, Corollary 1.2] for $T \geq 70$	Cited	Assumed as hypothesis
<code>HSWHead14</code>	[5, Corollary 1.2] for $T \geq 14$	Cited	Assumed as hypothesis
<code>EvenGoldbachUpTo</code>	[10], Lemma 2.6	Cited	Assumed as hypothesis

```

(hδ1 : δ ≤ 1 / 10 ^ 13) (A B Q lam : ℝ) (hA : 1 ≤ A) (hAB : A ≤ B)
(hlam : Real.log B ≤ lam) (hlam5 : lam ≤ 5) (hQ : 0 ≤ Q)
(hc : ∀ i, A / δ < |γ i| → |γ i| ≤ B / δ →
  ||cδ γ δ i|| ^ 2 ≤ Q * δ ^ 2)
(s : Finset ι) :
∑ i ∈ s.filter (fun i => A / δ < |γ i| ∧ |γ i| ≤ B / δ),
  ||cδ γ δ i|| ^ 2 * Real.log |γ i|
  ≤ 0.31831 * (Q * (B - A)) * (δ * Real.log (1 / δ) ^ 2
    + (2 * lam - 1.8) * (δ * Real.log (1 / δ)))
    + (lam ^ 2 - 1.8 * lam) * δ)
    + 2 * Q * (δ ^ 2 * (Real.log (1 / δ) + 5) ^ 2)

lemma pt_raw ... (A B Q u : ℝ) (hA : 1 ≤ A)
(hu : ∀ t, A * (1 - 1 / 10 ^ 13) < t → t ≤ B →
  1 - Real.cos t ≤ u)
(hQ : (1 / 10 ^ 26 + 2 * (1 + 1 / 10 ^ 13) * u) / A ^ 2 ≤ Q) :
  ∀ i, A / δ < |γ i| → |γ i| ≤ B / δ → ||cδ γ δ i|| ^ 2 ≤ Q * δ ^ 2
lemma pt_norm ... (A B lo Q : ℝ) (hA : 1 ≤ A) (hlo0 : 0 ≤ lo)
(hlo : lo ≤ A * (1 - 1 / 10 ^ 13))
(hQ : 2 * (1 + 1 / 10 ^ 13) *
  (1 / 2 - lo ^ 2 / 24 + B ^ 4 / 720) + 1 / 10 ^ 26 ≤ Q) : ...
lemma one_sub_cos_le (x : ℝ) :
  1 - cos x ≤ x ^ 2 / 2 - x ^ 4 / 24 + x ^ 6 / 720

```

- (2) The 2161 pieces on [1, 139.984375] are spread over nine split files `U6806DiagASpS1.lean` to `U6806DiagASpS3.lean`, `U6806DiagBSpS1.lean` to `U6806DiagBSpS3.lean`, and `U6806DiagCSpS1.lean` to `U6806DiagCSpS3.lean`, with 241, 240, 240, 240, 240, 240, 259, 259, and 202 applications of `piece_sum`, respectively. Table 4 gives three representative pieces exactly as certified in Lean.

- (3) `U6806DiagCSpS3.lean` (namespace `LeanLab.ChirreHagen2025.U6806`) assembles the diagonal bound `diag_u6806`, conditional on `BPTN` and `Ordinates14`:

```

theorem diag_u6806 (hN : BPTN γ) (h14 : Ordinates14 γ) {δ : ℝ}
(hδ : 0 < δ) (hδ1 : δ ≤ 1 / 10 ^ 13) (s : Finset ι) :
  ∑ i ∈ s, ||cδ γ δ i|| ^ 2 * Real.log |γ i|
  ≤ 1.0068 * δ * Real.log (2.0001 / δ) ^ 2

```

- (4) `U6806Theorem2Sp.lean` (namespace `LeanLab.ChirreHagen2025.SmoothT2`) proves the short interval mean square bound `theorem2_u6806`:

```

theorem theorem2_u6806 {γ : ι → ℝ} {Z : ℕ → Finset ι}
(hE : ∀ δ, 0 < δ → δ ≤ 1 / 10 ^ 13 → ExplicitA γ Z δ)
(hH : HadamardK γ) (h14 : Ordinates14 γ)
(hN : Lemma26B.BPTN γ) :
  ∀ x δ : ℝ, 10 ^ 13 ≤ x → 0 < δ → δ ≤ 1 / 10 ^ 13 →
    Jpsi x δ ≤ 1.0069 * δ * Real.log (2.0001 / δ) ^ 2 * x ^ 2

```

- (5) `U1505Theta6HSW.lean` (namespace `LeanLab.Routes.U1505`) proves the bilinear reduction theorem `goldbach_445_of_J_hsw`.
- (6) `U1508Compose445.lean` (namespace `LeanLab.Routes.U1508`) applies `LeanLab.Routes.U1505.goldbach_445_of_J_hsw` to `SmoothT2.theorem2_u6806` to obtain the headline theorem `goldbach.445_hsw_gZ`.

TABLE 4. Three representative pieces from the 2161-piece partition of $[1, 139.984375]$.

A	B	Q	λ	Majorant
1	1.0029296875	0.9194789	0.0029296875	<code>pt_norm</code> , $\text{lo} = 0.99999$
5.67578125	5.6884765625	0.0111334	1.7905011127125	<code>pt_raw</code> with <code>hu_poly</code> , $k = 1$
139.6982421875	139.984375	0.0001399	4.9456581952875	<code>pt_raw</code> with <code>hu_pi</code> , $k = 22$

Running `#print axioms` for `goldbach.445_hsw_gZ`, `theorem2_u6806`, and `goldbach.445_of_J_hsw` in Lean 4 prints only the standard foundational axioms of Lean and Mathlib: `propext`, `Classical.choice`, and `Quot.sound`. The Lean code was written with AI assistance and checked by the Lean kernel.

Use of AI. The argument and the text of this paper were developed with AI tools; the main deduction is checked in Lean 4 as described in the appendix.

Data availability. The Lean source of every theorem named in this paper, with its full import closure and the original build log, is public at <https://github.com/sthubbar/lean-lab-preprints>, commit 74c2d1a; the headline theorem is in `LeanLab/Routes/U1508Compose445.lean` and `Papers/GoldbachRH445.lean` prints its statement and axioms. The README states exactly what the Lean proves and which inputs are hypotheses.

REFERENCES

- [1] R. P. Brent, D. J. Platt, and T. S. Trudgian, *The mean square of the error term in the prime number theorem*, J. Number Theory **238** (2022), 740–762.
- [2] A. Chirre and M. Valás Hagen, *On Goldbach numbers in short intervals*, Research in Number Theory **12** (2026), article 82, doi:10.1007/s40993-026-00767-6, arXiv:2512.23534.
- [3] M. Cully-Hugill and A. W. Dudek, *An explicit mean-value estimate for the prime number theorem in intervals*, J. Aust. Math. Soc. **117** (2024), no. 1, 1–15.
- [4] D. A. Goldston, *Linnik’s theorem on Goldbach numbers in short intervals*, Glasgow Math. J. **32** (1990), no. 3, 285–297.
- [5] E. Hasanalizade, Q. Shen, and P.-J. Wong, *Counting zeros of the Riemann zeta function*, preprint, arXiv:2107.06506, 2021.
- [6] I. Kátai, *A remark on a paper of Ju. V. Linnik*, Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **17** (1967), 99–100.
- [7] H. L. Montgomery, *The pair correlation of zeros of the zeta function*, in Analytic Number Theory, Proc. Sympos. Pure Math., vol. 24, Amer. Math. Soc., Providence, R.I., 1973, 181–193.
- [8] H. L. Montgomery and R. C. Vaughan, *The exceptional set in Goldbach’s problem*, Acta Arith. **27** (1975), 353–370.
- [9] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory. I. Classical Theory*, Cambridge Studies in Advanced Mathematics, vol. 97, Cambridge University Press, Cambridge, 2007.
- [10] T. Oliveira e Silva, S. Herzog, and S. Pardi, *Empirical verification of the even Goldbach conjecture and computation of prime gaps up to $4 \cdot 10^{18}$* , Math. Comp. **83** (2014), no. 288, 2033–2060.
- [11] L. Schoenfeld, *Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. II*, Math. Comp. **30** (1976), no. 134, 337–360.

- [12] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford University Press, New York, 1986.

Email address: `sthubbar@gmail.com`