

GOLDBACH NUMBERS IN INTERVALS OF LENGTH $4.45 \log^2 x$ UNDER THE RIEMANN HYPOTHESIS

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ABSTRACT. Assuming the Riemann Hypothesis, we prove that for every real $x \geq 3$ the interval $(x, x + 4.45 \log^2 x)$ contains an even integer that is a sum of two odd primes. The proof combines a weighted mean square bound for the Chebyshev prime counting error in short intervals with a piecewise linear weight in a bilinear identity for short interval prime counts. The deduction from explicit analytic inputs, stated as hypotheses, is checked in Lean 4.

1. INTRODUCTION

Theorem 1.1. *Assume the Riemann Hypothesis. For every real $x \geq 3$ the interval $(x, x + 4.45 \log^2 x)$ contains a Goldbach number, that is, an even integer that is a sum of two odd primes.*

Assuming the Riemann Hypothesis, Kátai [6] and Montgomery and Vaughan [7] proved that for some constant $C > 0$ every interval $(x, x + C \log^2 x)$ with x sufficiently large contains a Goldbach number, and Goldston [4] gave a proof comparing the measure of prime-free short intervals with Selberg’s mean square bound. Cully-Hugill and Dudek [3, Theorem 1.3] made this result explicit with $C = 9696$ for all $x \geq 2$. Chirre and Hagen [2] reduced the constant to $C = 123$. The binary Goldbach conjecture itself remains open; the result here concerns Goldbach numbers in short intervals under the Riemann Hypothesis.

Our proof follows the mean square framework of Chirre and Hagen [2] and makes three changes. First, in the mean square over $[1, x]$, we replace the sharp cutoff by the smooth weight $W(u) = \min(1, u^{-4})$, which produces the positive kernel $4/(4 + (\gamma_i - \gamma_j)^2)$ and bounds the inner sum over zero ordinates directly by $\log |\gamma_i|$. Second, we bound the diagonal sum using the exact modulus squared of the short interval coefficient, a degree-6 Taylor majorant for $1 - \cos$, and a geometric partition of the scaled mid-range $[1, 139.984375]$ into 2161 pieces, reducing the mean square constant from 2.2258 in [2] to 1.0069. Third, instead of deducing the theorem directly from the mean square on an interval $[ax, x]$ with $\delta = h/(2x)$ as in [2, Section 5], we integrate $A(t)A(x - t) = 0$ against the weight $\varphi(t) = 11t/(5x)$ on $[x/64, 5x/11]$ and $\varphi(t) = 1$ on $[5x/11, x/2]$, which produces a nonincreasing weight on the quadratic error term and allows direct Abel summation over 256 equal steps. The resulting constants are new as far as we know.

2. NOTATION AND INPUTS

Throughout the paper, $\Lambda(n)$ denotes the von Mangoldt function, $\psi(x) = \sum_{n \leq x} \Lambda(n)$, and $\theta(x) = \sum_{p \leq x} \log p$, where p runs over primes. Under the Riemann Hypothesis, the nontrivial zeros of the Riemann zeta function $\zeta(s)$ take the form $\rho = 1/2 + i\gamma$ with $\gamma \in \mathbb{R}$. We write

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$N(T)$ for the number of nontrivial zeros $\rho = \beta + i\gamma$ with $0 < \gamma \leq T$, counted with multiplicity. For $\delta > 0$ and $\rho = 1/2 + i\gamma$, we define the short interval coefficient

$$c_\delta(\rho) = c_\delta(\gamma) = \frac{(1 + \delta)^{1/2+i\gamma} - 1}{1/2 + i\gamma}. \quad (2.1)$$

For $x \geq 2$ and $\delta > 0$, we define the short interval mean squares

$$J_\psi(x, \delta) = \int_1^x (\psi((1 + \delta)y) - \psi(y) - \delta y)^2 dy, \quad J_\theta(x, \delta) = \int_1^x (\theta((1 + \delta)y) - \theta(y) - \delta y)^2 dy. \quad (2.2)$$

We state the six inputs used in the formal proof together with their published sources, and we prove the second input, Lemma 2.2, directly.

Lemma 2.1 ([8, Equation (12.1)], Hypothesis **ExplicitA**). *Assume the Riemann Hypothesis. For every $\delta > 0$ and almost every $y \geq 2$,*

$$\psi((1 + \delta)y) - \psi(y) - \delta y = - \lim_{T \rightarrow \infty} \sum_{|\gamma| \leq T} c_\delta(\rho) y^{1/2+i\gamma} + \frac{1}{2} \log \frac{1 - y^{-2}}{1 - ((1 + \delta)y)^{-2}}. \quad (2.3)$$

Equation (2.3) is the difference of von Mangoldt's explicit formula [8, Equation (12.1)] at $(1 + \delta)y$ and y for $y \geq 2$ when neither endpoint is a prime power. In the Lean formalization, **SmoothT2.ExplicitA** states (2.3) as the limit of $-\sum_{\rho \in Z_n} c_\delta(\rho) y^\rho$ along truncations Z_n , for almost every $y \geq 2$, with $B(y) = \frac{1}{2} \log \frac{1 - y^{-2}}{1 - ((1 + \delta)y)^{-2}}$ included as the function **Bf**; the main theorem assumes it, under RH, for each $0 < \delta \leq 10^{-13}$.

Lemma 2.2 (Hypothesis **HadamardK**). *Assume the Riemann Hypothesis. For every nontrivial zero ordinate γ of $\zeta(s)$ and every finite set of nontrivial zero ordinates γ' ,*

$$\sum_{\gamma'} \frac{4}{4 + (\gamma - \gamma')^2} \leq \log |\gamma|. \quad (2.4)$$

Proof. Since each summand is positive, it suffices to bound the sum over all nontrivial zeros $\rho' = 1/2 + i\gamma'$ counted with multiplicity. Setting $s = 5/2 + i\gamma$, we have

$$\sum_{\rho'} \frac{4}{4 + (\gamma - \gamma')^2} = 2 \sum_{\rho'} \operatorname{Re} \frac{1}{s - \rho'}. \quad (2.5)$$

By the partial fraction decomposition of $\xi'(s)/\xi(s)$ [8, Corollary 10.14],

$$2 \sum_{\rho'} \operatorname{Re} \frac{1}{s - \rho'} = \operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{s}{2} + 1 \right) - \log \pi + 2 \operatorname{Re} \frac{\zeta'}{\zeta}(s) + 2 \operatorname{Re} \frac{1}{s - 1}. \quad (2.6)$$

For $\operatorname{Re} z > 0$, Binet's integral formula gives $\frac{\Gamma'}{\Gamma}(z) = \log z - \frac{1}{2z} - \int_0^\infty f(t) e^{-tz} dt$, where $f(t) = \frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1}$ satisfies $f(0) = 0$ and $f'(t) = \frac{1}{t^2} - \frac{1}{4 \sinh^2(t/2)}$ decreases monotonically from $f'(0) = \frac{1}{12}$ to 0 on $(0, \infty)$. Integrating by parts twice gives $\int_0^\infty f(t) e^{-tz} dt = \frac{1}{12z^2} + \frac{1}{z^2} \int_0^\infty f''(t) e^{-tz} dt$, which has modulus at most $\frac{1}{12|z|^2} + \frac{1}{|z|^2} \int_0^\infty (-f''(t)) dt = \frac{1}{6|z|^2} \leq \frac{\operatorname{Re} z}{2|z|^2} = \operatorname{Re} \frac{1}{2z}$ whenever $\operatorname{Re} z \geq \frac{1}{3}$, so $\operatorname{Re} \frac{\Gamma'}{\Gamma}(z) \leq \log |z|$. Applying this at $z = \frac{9}{4} + \frac{i\gamma}{2}$ with $|\gamma| \geq 14$ from Lemma 2.3 yields

$$\operatorname{Re} \frac{\Gamma'}{\Gamma} \left(\frac{9}{4} + \frac{i\gamma}{2} \right) \leq \log \left| \frac{9}{4} + \frac{i\gamma}{2} \right| \leq \log |\gamma| - \log 2 + \frac{81}{8\gamma^2}. \quad (2.7)$$

On the line $\operatorname{Re}(s) = 5/2$, the Dirichlet series for ζ'/ζ yields $\operatorname{Re} \frac{\zeta'}{\zeta}(s) \leq -\frac{\zeta'}{\zeta}(5/2) = 0.28874 \dots < 0.2888$, while $2 \operatorname{Re} \frac{1}{s-1} = \frac{3}{9/4+\gamma^2} < \frac{3}{\gamma^2}$. For $|\gamma| \geq 14$, we obtain

$$-\log(2\pi) + 2(0.2888) + \frac{105}{8 \cdot 14^2} = -1.83787 \dots + 0.5776 + 0.06696 \dots < -1.1933 < 0, \quad (2.8)$$

so the right-hand side of (2.6) is at most $\log |\gamma| - 1.1933 < \log |\gamma|$. $\square$

Lemma 2.3 ([11, Chapter XV], Hypothesis `Ordinates14`). *Every nontrivial zero $\rho = \beta + i\gamma$ of $\zeta(s)$ satisfies $|\gamma| \geq 14$.*

The lowest nontrivial zero has ordinate $\gamma_1 = 14.134725 \dots > 14$.

Lemma 2.4 ([1, Lemma 1], Hypothesis `BPTN`). *For all $T \geq 2\pi e$, the zero counting function $N(T)$ satisfies*

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi e} + \frac{7}{8} + Q(T), \quad (2.9)$$

where

$$|Q(T)| \leq 0.137 \log T + 0.443 \log \log T + 1.588 + \frac{0.2}{T}. \quad (2.10)$$

We use the constants of Lemma 1 of the published version [1]. The Lean formalization only uses Lemma 2.4 for $T \geq 10^{13}$, as counts over both signs of the ordinates.

Lemma 2.5 ([10, Corollary 1]). *Assume the Riemann Hypothesis. For every real $u \geq 599$,*

$$|\theta(u) - u| < \frac{\sqrt{u} \log^2 u}{8\pi}. \quad (2.11)$$

In particular, for every real $u \geq 10^{16}$,

$$|\theta(u) - u| \leq 10^{-6}u. \quad (2.12)$$

Inequality (2.12) follows immediately from (2.11) because $u^{-1/2} \log^2 u$ is decreasing for $u > e^4$ and equals $5.4005 \dots \times 10^{-7} \cdot 8\pi < 8\pi \cdot 10^{-6}$ at $u = 10^{16}$. In the Lean formalization, inequality (2.11) is not assumed; it is proved under RH for $u \geq 10^{11}$ from the explicit formula and the zero counting bound of Hasanalizade, Shen, and Wong [5, Corollary 1.2], together with the numerical fact, taken as a hypothesis in Lean, that every nontrivial zero has ordinate of absolute value at least 14.

Lemma 2.6 ([9]). *Every even integer $2n$ in the range $4 \leq 2n \leq 4 \cdot 10^{18}$ is a sum of two primes.*

3. THE MEAN SQUARE BOUND

We bound the mean square on $[1, x]$ of a finite Dirichlet polynomial supported on nontrivial zeros of $\zeta(s)$.

Lemma 3.1. *Assume the Riemann Hypothesis. Let $\gamma_1, \dots, \gamma_M$ be a finite family of nontrivial zero ordinates of $\zeta(s)$, let $c_1, \dots, c_M \in \mathbb{C}$, and set*

$$P(y) = \sum_{i=1}^M c_i y^{1/2+i\gamma_i}. \quad (3.1)$$

For every $x \geq 1$,

$$\int_1^x |P(y)|^2 dy \leq x^2 \sum_{i=1}^M |c_i|^2 \sum_{j=1}^M \frac{4}{4 + (\gamma_i - \gamma_j)^2} \leq x^2 \sum_{i=1}^M |c_i|^2 \log |\gamma_i|. \quad (3.2)$$

Proof. Define $W(u) = \min(1, u^{-4})$ for $u > 0$. Since $W(u) \geq 0$ for all $u > 0$ and $W(y/x) = 1$ for $1 \leq y \leq x$, the substitution $y = xu$ gives

$$\begin{aligned} \int_1^x |P(y)|^2 dy &\leq \int_0^\infty |P(y)|^2 W(y/x) dy \\ &= x^2 \sum_{i=1}^M \sum_{j=1}^M c_i \bar{c}_j x^{i(\gamma_i - \gamma_j)} \int_0^\infty u^{1+i(\gamma_i - \gamma_j)} W(u) du. \end{aligned} \quad (3.3)$$

For any $\Delta \in \mathbb{R}$, splitting the integral at $u = 1$ yields

$$\int_0^\infty u^{1+i\Delta} W(u) du = \int_0^1 u^{1+i\Delta} du + \int_1^\infty u^{-3+i\Delta} du = \frac{1}{2+i\Delta} + \frac{1}{2-i\Delta} = \frac{4}{4+\Delta^2}. \quad (3.4)$$

Since $\frac{4}{4+(\gamma_i - \gamma_j)^2} > 0$ is symmetric in i and j , applying $|c_i \bar{c}_j x^{i(\gamma_i - \gamma_j)}| \leq \frac{1}{2}(|c_i|^2 + |c_j|^2)$ in (3.3) gives the first inequality in (3.2). Applying Lemma 2.2 to the inner sum over j gives the second inequality. $\square$

We next bound the diagonal sum $\sum_\gamma |c_\delta(\gamma)|^2 \log |\gamma|$.

Lemma 3.2 (Theorem diag_u6806). *Assume the Riemann Hypothesis. For every δ in the range $0 < \delta \leq 10^{-13}$,*

$$\sum_\gamma |c_\delta(\gamma)|^2 \log |\gamma| \leq C_{\text{diag}} \delta \log^2 \left(\frac{2.0001}{\delta} \right), \quad C_{\text{diag}} = 1.0068. \quad (3.5)$$

Proof. By symmetry of the nontrivial zeros with respect to the real axis, $\sum_\gamma |c_\delta(\gamma)|^2 \log |\gamma| = 2 \sum_{\gamma > 0} |c_\delta(\gamma)|^2 \log \gamma$, and every term has $\gamma \geq 14$ by Lemma 2.3. Let $\eta = \log(1 + \delta)$ and $\Lambda = \log(1/\delta)$. From (2.1), the exact modulus squared of $c_\delta(\gamma)$ is

$$|c_\delta(\gamma)|^2 = \frac{(1 + \delta) + 1 - 2\sqrt{1 + \delta} \cos(\gamma\eta)}{1/4 + \gamma^2} = \frac{(\sqrt{1 + \delta} - 1)^2 + 2\sqrt{1 + \delta}(1 - \cos(\gamma\eta))}{1/4 + \gamma^2}. \quad (3.6)$$

For all real t , the alternating Taylor series for $\cos t$ satisfies the degree-6 majorant

$$1 - \cos t \leq \frac{t^2}{2} - \frac{t^4}{24} + \frac{t^6}{720}. \quad (3.7)$$

Write $v = \delta|\gamma|$ for the scaled variable, and set

$$X = \delta\Lambda^2, \quad Y = \delta\Lambda, \quad E = \delta^2(\Lambda + 5)^2.$$

The certified bound is organized in two cases, split at $\delta = 2^{-188}$.

Pieces. Let $0 < A < B$ and suppose that $Q\delta^2$ majorizes $|c_\delta(\gamma)|^2$ for $A/\delta < |\gamma| \leq B/\delta$; such a Q follows from (3.6) and a Taylor majorant of $1 - \cos$ such as (3.7) on the piece. With λ the piece parameter, Lemma 2.4 gives

$$\sum_{A/\delta < |\gamma| \leq B/\delta} |c_\delta(\gamma)|^2 \log |\gamma| \leq 0.31831 Q(B - A)(X + (2\lambda - 1.8)Y + (\lambda^2 - 1.8\lambda)\delta) + 2QE. \quad (3.8)$$

The piece endpoints are dyadic rationals. For example, the first piece has $A = 1$, $B = 1.0029296875$, $Q = 0.9194789$, $\lambda = 0.0029296875$, and the last piece has $A = 139.6982421875$, $B = 139.984375$, $Q = 0.0001399$, $\lambda = 4.9456581952875$.

Case $2^{-188} < \delta \leq 10^{-13}$. On the head range $|\gamma| \leq 1/\delta$ the sum is at most $\delta\Lambda^2/\pi \leq 0.31831 X$. On the range $v > 1$ it is at most

$$0.696652249211 X + 0.294654626964 Y + 0.112752623673 \delta + 785.028327376 E.$$

This range consists of 2161 pieces covering $1 < v \leq 139.984375$, each bounded by (3.8), a further range $139.984375 < v \leq 599.734375$, and a tail $v > 599.734375$ bounded by $0.002182674424 X + 0.027980492816 Y + 0.089673014185 \delta$. Table 1 lists the coefficients of $X = \delta \Lambda^2$. They sum to 1.014962249211, and the bound $1.0068 \delta \log^2(2.0001/\delta)$ follows because $\log(2.0001/\delta) = \Lambda + \log 2.0001$ with $\log 2.0001 \geq 0.6931$, and $\Lambda \leq 130.3116699904$ when $\delta > 2^{-188}$; together these absorb the lower order terms.

Case $\delta \leq 2^{-188}$. The head range is $|\gamma| \leq 10^{-20}/\delta$, followed by a dyadic chain of 64 steps, which gives

$$1.006510467199 X - 0.263090165331 Y + 0.112752623673 \delta + 924.581825936 E \leq 1.0068 \delta \log^2(2.0001/\delta).$$

All of these bounds are certified by exact rational arithmetic and checked in Lean as `diag.u6806`. $\square$

TABLE 1. Coefficients of $\delta \Lambda^2$ in the case $2^{-188} < \delta \leq 10^{-13}$ of Lemma 3.2.

Range	Method	Coefficient of $\delta \Lambda^2$
Head $ \gamma \leq 1/\delta$	Bound $\delta \Lambda^2/\pi$	0.31831
$1 < v \leq 139.984375$	2161 pieces, (3.8)	0.690304599302
$139.984375 < v \leq 599.734375$	Certified chain	0.004164975485
$v > 599.734375$	Tail bound	0.002182674424
Total		1.014962249211

The lower order terms and $\Lambda \leq 130.3116699904$ bring the total to 1.0068 relative to $\delta \log^2(2.0001/\delta)$.

Combining Lemma 2.1, Lemma 3.1, and Lemma 3.2 with the L^2 convergence of the truncated explicit formula as proved in [2, Section 3] yields the short interval mean square bound.

Proposition 3.3 (Theorem `theorem2_u6806`). *Assume the Riemann Hypothesis. For all $x \geq 10^{13}$ and $0 < \delta \leq 10^{-13}$,*

$$J_\psi(x, \delta) \leq J \delta \log^2\left(\frac{2.0001}{\delta}\right) x^2, \quad J = 1.0069. \quad (3.9)$$

Proof. Write $V(y) = \psi((1+\delta)y) - \psi(y) - \delta y$ and let

$$B(y) = \frac{1}{2} \log \frac{1 - y^{-2}}{1 - ((1+\delta)y)^{-2}} \quad (3.10)$$

be the secondary term of Lemma 2.1, so $V(y) - B(y)$ is the zero sum. By Lemma 2.1, the L^2 limit argument of [2, Section 3], and Lemmas 3.1 and 3.2, we have

$$\int_2^x (V(y) - B(y))^2 dy \leq 1.0068 \delta \log^2\left(\frac{2.0001}{\delta}\right) x^2. \quad (3.11)$$

For $y \geq 2$ and $0 < \delta \leq 1$, we have $|B(y)| \leq \delta$, so $\int_2^x B(y)^2 dy \leq \delta^2 x$. Since $(u+b)^2 \leq (1+\eta)u^2 + (1+1/\eta)b^2$ for every $\eta > 0$, taking $\eta = 10^{-5}$ with $L = \log(2.0001/\delta)$ gives

$$\int_2^x V(y)^2 dy \leq (1 + 10^{-5}) \cdot 1.0068 \delta L^2 x^2 + (1 + 10^5) \delta^2 x. \quad (3.12)$$

On $[1, 2]$, since $\psi(y) = 0$ for $y < 2$, the difference $\psi((1+\delta)y) - \psi(y)$ is nonzero only for $y \in [2/(1+\delta), 2]$, an interval of length at most 2δ , where it is at most $\log 2$; with $u =$

$\psi((1+\delta)y) - \psi(y) \geq 0$ and $v = \delta y \geq 0$ we have $(u-v)^2 \leq u^2 + v^2$, so $\int_1^2 V(y)^2 dy \leq 2\delta(\log 2)^2 + \frac{7}{3}\delta^2 \leq 3\delta$. For $x \geq 10^{13}$ and $0 < \delta \leq 10^{-13}$, we have $L \geq \log(2 \cdot 10^{13}) > 30$. Dividing by $\delta L^2 x^2$, the first term is $1.0068(1+10^{-5}) = 1.00681007$, the second term is at most $10^5 \cdot 10^{-13}/(900 \cdot 10^{13}) < 10^{-30}$, and the third is at most $3/(900 \cdot 10^{26}) < 10^{-28}$. The sum is below 1.0069, so $J_\psi(x, \delta) \leq 1.0069 \delta L^2 x^2$. $\square$

We now pass from $J_\psi(X, \delta)$ to $J_\theta(X, \delta)$. Write $D(y) = (\psi - \theta)((1+\delta)y) - (\psi - \theta)(y) \geq 0$ and $u = \psi((1+\delta)y) - \psi(y) - \delta y$, so $\theta((1+\delta)y) - \theta(y) - \delta y = u - D(y)$. Applying $(u-D)^2 \leq (1+\eta)u^2 + (1+1/\eta)D^2$ with $\eta = 0.001$ gives $J_\theta(X, \delta) \leq 1.001J_\psi(X, \delta) + 1001 \int_1^X D(y)^2 dy$. By Fubini's theorem, $\int_1^X D(y) dy \leq \delta X(\psi - \theta)((1+\delta)X)$. When $\delta\sqrt{y} < 1$, every interval $(y^{1/k}, ((1+\delta)y)^{1/k}]$ with $k \geq 2$ has length at most $\frac{1}{2}\delta\sqrt{y} < 1$ and contains at most one prime, so $D(y) \leq \log^2((1+\delta)y)/\log 2$. Multiplying this pointwise bound by the L^1 bound and using $(\psi - \theta)((1+\delta)X) \leq 2\sqrt{(1+\delta)X} \log((1+\delta)X)$ shows that $1001 \int_1^X D(y)^2 dy \leq 0.0019 \delta \log^2(2.0001/\delta) X^2$ for $X \geq 5 \cdot 10^{17}$ and $0 < \delta \leq 10^{-13}$. Thus, if $J_\psi(X, \delta) \leq J\delta \log^2(2.0001/\delta) X^2$, then for $X \geq 5 \cdot 10^{17}$, $0 < \delta \leq 10^{-13}$, and $\delta\sqrt{X} < 1$, we have

$$J_\theta(X, \delta) \leq c \delta \log^2\left(\frac{2.0001}{\delta}\right) X^2, \quad c = 1.001J + 0.0019. \quad (3.13)$$

With $J = 1.0069$ from Proposition 3.3, this gives $c = 1.0098069$.

4. THE BILINEAR ARGUMENT

To prove Theorem 1.1, we split the range of x at $2 \cdot 10^{18}$. For $3 \leq x \leq 2 \cdot 10^{18}$, the interval length satisfies $4.45 \log^2 x \geq 4.45 \log^2 3 > 5.36$, so $(x, x + 4.45 \log^2 x)$ contains an even integer $2n$ in $[6, 4 \cdot 10^{18}]$, which is a sum of two odd primes by Lemma 2.6. The analytic argument runs for $x \geq 1.1 \cdot 10^{18}$, where $\log x \geq 41$.

Let $x \geq 1.1 \cdot 10^{18}$ and fix H with $4 \leq H \leq 61.5$. Put $h = H \log^2 x$ and $\delta = h/x$, so $0 < \delta \leq 10^{-13}$ and $\delta\sqrt{x} < 1$. Let

$$A(t) = \theta((1+\delta)t) - \theta(t), \quad E(t) = A(t) - \delta t. \quad (4.1)$$

As observed by Goldston [4, p. 285], if no Goldbach number lies in $(x, x+h]$, then for every $t \in [x/64, x/2]$ the intervals $(t, (1+\delta)t]$ and $(x-t, (1+\delta)(x-t)]$ cannot both contain an odd prime, and since $t \geq x/64 > 10^{16} > 2$, we have

$$A(t)A(x-t) = 0 \quad \text{for all } t \in [x/64, x/2]. \quad (4.2)$$

We weight (4.2) by the continuous piecewise linear function

$$\varphi(t) = \begin{cases} \frac{11t}{5x}, & \frac{x}{64} \leq t \leq \frac{5x}{11}, \\ 1, & \frac{5x}{11} \leq t \leq \frac{x}{2}. \end{cases} \quad (4.3)$$

Substituting $A(t) = \delta t + E(t)$ and $A(x-t) = \delta(x-t) + E(x-t)$ into $0 = \int_{x/64}^{x/2} \varphi(t) A(t) A(x-t) dt$ gives the weighted identity

$$\delta^2 \int_{x/64}^{x/2} \varphi(t) t(x-t) dt + \delta \int_{x/64}^{x/2} \varphi(t) (tE(x-t) + (x-t)E(t)) dt = - \int_{x/64}^{x/2} \varphi(t) E(t) E(x-t) dt. \quad (4.4)$$

By the arithmetic-geometric mean inequality, for every $t \in [x/64, x/2]$ we have

$$|\varphi(t) E(t) E(x-t)| \leq \frac{1}{2} E(t)^2 + \frac{1}{2} \varphi(t)^2 E(x-t)^2. \quad (4.5)$$

Integrating (4.5) over $t \in [x/64, x/2]$ and substituting $u = x - t$ in the second term yields

$$\int_{x/64}^{x/2} |\varphi(t)E(t)E(x-t)| dt \leq \int_{x/64}^{63x/64} w(u)E(u)^2 du, \quad (4.6)$$

where the combined weight $w(u)$ on $[x/64, 63x/64]$ is

$$w(u) = \begin{cases} \frac{1}{2}, & \frac{x}{64} \leq u \leq \frac{6x}{11}, \\ \frac{121}{50} \left(1 - \frac{u}{x}\right)^2, & \frac{6x}{11} \leq u \leq \frac{63x}{64}. \end{cases} \quad (4.7)$$

At the junction $u = 6x/11$, both expressions in (4.7) equal $1/2$. Thus $w(u)$ is positive, continuous, and nonincreasing on $[x/64, 63x/64]$.

Proof of Theorem 1.1. Assume $x \geq 1.1 \cdot 10^{18}$, fix H with $4 \leq H \leq 61.5$, put $h = H \log^2 x$ and $\delta = h/x$, and suppose that no Goldbach number lies in $(x, x+h]$. Evaluating the main term on the left-hand side of (4.4) with $\varphi(t)$ from (4.3) gives $\delta^2 x^3 M$, where

$$\begin{aligned} M &= \frac{11}{5} \left[\frac{(5/11)^3 - (1/64)^3}{3} - \frac{(5/11)^4 - (1/64)^4}{4} \right] + \left[\frac{(1/2)^2 - (5/11)^2}{2} - \frac{(1/2)^3 - (5/11)^3}{3} \right] \\ &= 0.0567215028 \dots \end{aligned} \quad (4.8)$$

To bound the cross terms in (4.4), write $R(u) = \theta(u) - u$, so $E(u) = R((1+\delta)u) - R(u)$. Substituting $u = x - t$ in $\int_{x/64}^{x/2} \varphi(t)tE(x-t) dt$ splits the cross-term integral into four smooth pieces; the first two are written in the variable t and the last two in $u = x - t$: $[x/64, 5x/11]$ with weight $\frac{11}{10} \cdot \frac{2}{x} t(x-t)$, $[5x/11, x/2]$ with weight $x-t$, $[6x/11, 63x/64]$ with weight $\frac{11}{10} \cdot \frac{2}{x} (x-u)^2$, and $[x/2, 6x/11]$ with weight $x-u$. On each piece $[ax, bx]$ with normalized weight $g(u)$, changing variables $v = (1+\delta)u$ in $\int_{ax}^{bx} g(u)R((1+\delta)u) du$ gives

$$\int_{ax}^{bx} g(u)E(u) du = \int_{ax}^{bx} \left(\frac{1}{1+\delta} g\left(\frac{v}{1+\delta}\right) - g(v) \right) R(v) dv + \left(\int_{bx}^{(1+\delta)bx} - \int_{ax}^{(1+\delta)ax} \right) g(u)R((1+\delta)u) du. \quad (4.9)$$

Since $1 - (1+\delta)^{-k} \leq k\delta$ for $k = 1, 2, 3$ and $0 < u \leq x$, the interior factor $|\frac{1}{1+\delta} g(\frac{v}{1+\delta}) - g(v)|$ is bounded on $[0, x]$ by $2(2+3)\delta x = 10\delta x$ for $\frac{2}{x}u(x-u)$, by $(1+2)\delta x = 3\delta x$ for $x-u$, and by $2(1+4+3)\delta x = 16\delta x$ for $\frac{2}{x}(x-u)^2$. Since $u \geq x/64 > 10^{16}$ throughout all four intervals, Lemma 2.5 gives $|R(v)| \leq 10^{-6}v \leq 10^{-6}x$ on $[ax, bx]$ of length $(b-a)x$, and $|R((1+\delta)u)| \leq 10^{-6}(1+\delta)u \leq 2 \cdot 10^{-6}x$ on the boundary intervals $[bx, (1+\delta)bx]$ and $[ax, (1+\delta)ax]$ of lengths δbx and δax . Summing the four bounds shows that the left-hand side of (4.4) is at least $\delta^2 x^3 (M - K)$, where

$$\begin{aligned} K &= \frac{1}{10^6} \left[\frac{11}{10} \left(10 \left(\frac{5}{11} - \frac{1}{64} \right) + 2 \cdot \frac{5}{11} + \frac{2}{64} \right) + \left(3 \left(\frac{1}{2} - \frac{5}{11} \right) + 1 + 2 \cdot \frac{5}{11} \right) \right. \\ &\quad \left. + \frac{11}{10} \left(16 \left(\frac{63}{64} - \frac{6}{11} \right) + 2 \cdot \frac{63}{64} + 2 \cdot \frac{6}{11} \right) + \left(3 \left(\frac{6}{11} - \frac{1}{2} \right) + 2 \cdot \frac{6}{11} + 1 \right) \right] \\ &= 2.1226 \cdot 10^{-5}. \end{aligned} \quad (4.10)$$

To bound the right-hand side of (4.6), take 256 equal steps

$$s_j = \frac{6}{11} + \frac{309j}{180224}, \quad w_j = \frac{121}{50} (1 - s_j)^2 \quad \text{for } j = 0, \dots, 256, \quad (4.11)$$

so $s_0 = 6/11$, $s_{256} = 63/64$, and $w_0 = 1/2$. The weight $w(u)$ on $E(u)^2$ in (4.7) is $1/2$ on $[x/64, 6x/11]$ and at most w_j on $[s_j x, s_{j+1} x]$. Since $s_j x \geq (6/11)x \geq 6 \cdot 10^{17} > 5 \cdot 10^{17}$, w is nonincreasing, and $\int_1^{s_j x} E(u)^2 du \leq J_\theta(s_j x, \delta) \leq c \delta \log^2(2.0001/\delta) s_j^2 x^2$ by (3.13), Abel summation bounds the right-hand side of (4.6) by $c \delta \log^2(2.0001/\delta) x^2 U$, where

$$U = \sum_{j=1}^{255} (w_{j-1} - w_j) s_j^2 + w_{255} s_{256}^2 = 0.2492140 \dots < 0.24922. \quad (4.12)$$

Since $h \geq 2.0001$, we have $\log(2.0001/\delta) \leq \log x$. Using $\delta = H \log^2 x/x$ and dividing by $\delta x^2 \log^2 x$, no Goldbach number in $(x, x+h]$ forces $H(M-K) \leq c \cdot 0.24922$. Thus it is enough that

$$(1.001J + 0.0019) \cdot 0.24922 < H(M-K). \quad (4.13)$$

With $J = 1.0069$ from Proposition 3.3, the left-hand side of (4.13) is 0.2516641, and $H = 4.44$ gives right-hand side 0.2517492; the smallest H that works is $H_{\min} = 4.43850$. This gives a Goldbach number n with $x < n \leq x + 4.44 \log^2 x$, and the extra $0.01 \log^2 x$ in the theorem gives the strict upper bound in $(x, x + 4.45 \log^2 x)$. $\square$

At $J = 1.0069$, this family of piecewise linear weights cannot go much below about 4.43. Lowering the order of the interval length below $\log^2 x$ from the Riemann Hypothesis alone would require an upper bound on the pair correlation of zeros of $\zeta(s)$ that is not known; under the Riemann Hypothesis together with Montgomery's pair correlation conjecture, Goldston [4] already reaches $(2 + \epsilon) \log x$.

5. NUMERICAL VERIFICATION

All numerical constants in Lemma 3.2, Proposition 3.3, and the proof of Theorem 1.1 were certified by rational arithmetic over finite grids and verified inside the Lean 4 kernel. In Lemma 3.2, the range $1 < v \leq 139.984375$ of the scaled variable $v = \delta|\gamma|$ is split into 2161 pieces with dyadic rational endpoints, each bounded by (3.8) with the explicit zero count from Lemma 2.4. In Section 4, the interval $[6/11, 63/64]$ for u/x is partitioned into 256 equal steps; the main term M in (4.8), the cross-term bound K in (4.10), and the Abel summation factor U in (4.12) are evaluated by exact rational arithmetic. Table 2 records the certified parameters and margin checks around $H_{\min} = 4.43850$.

TABLE 2. Certified constants and margin checks for the main theorem.

Quantity	Certified value	Role in proof
End of the 2161 pieces	139.984375	In $v = \delta \gamma $, Lemma 3.2
Pieces	2161	Dyadic rational endpoints on $1 < v \leq 139.984375$
Diagonal constant C_{diag}	1.0068	Bound in (3.5) for $0 < \delta \leq 10^{-13}$
Mean square constant J	1.0069	Bound in (3.9) for $x \geq 10^{13}$
Transfer constant c	1.0098069	$c = 1.001J + 0.0019$ in (3.13) for $X \geq 5 \cdot 10^{17}$
Abel summation steps	256	Equal steps in u/x from $6/11$ to $63/64$ in (4.11)
Main term constant M	0.0567215028...	Exact integral in (4.8) with weight φ
Cross-term constant K	$2.1226 \cdot 10^{-5}$	Cross-term bound in (4.10)
Abel constant U	$0.2492140 \dots < 0.24922$	Upper bound factor in (4.12)
Minimum admissible $H_{\min}$	4.43850	Threshold in (4.13); 4.44 gives $0.2517492 > 0.2516641$

APPENDIX A. FORMAL VERIFICATION

The deduction of Theorem 1.1 from the inputs in Section 2 is formalized in Lean 4 with Mathlib in the files `ChirreHagen2025/U6806Theorem2Sp.lean` and `Routes/U1508Compose445.lean`. The diagonal estimate of Lemma 3.2 is formalized as `diag_u6806`, the short interval mean square bound of Proposition 3.3 as `theorem2_u6806`, and the main result Theorem 1.1 as `goldbach_445_hsw_gZ` in `U1508Compose445.lean`. The Lean formalization takes `SmoothT2.HadamardK` as a hypothesis, whereas the paper proves it in Lemma 2.2.

The headline theorem in Lean 4 has the following statement:

```
theorem goldbach_445_hsw_gZ {Z : ℕ → Finset ZIdx}
  (hE : RiemannHypothesis → ∀ δ, 0 < δ → δ ≤ 1 / 10 ^ 13 → SmoothT2.ExplicitA gZ Z δ)
  (hH : RiemannHypothesis → SmoothT2.HadamardK gZ) (h14 : SmoothT2.Ordinates14 gZ)
  (hB : RiemannHypothesis → Lemma26B.BPTN gZ)
  (hEv : EvenGoldbachUpTo (4 * 10 ^ 18))
  (hN : HSWN70 gZ) (hHd : HSWHead14 gZ) :
  RiemannHypothesis → ∀ x : ℝ, 3 ≤ x →
    ∃ n : ℕ, IsGoldbachNumber n ∧ x < n ∧ (n : ℝ) < x + 4.45 * Real.log x ^ 2
```

Here `gZ` is the family of ordinates of the nontrivial zeros of $\zeta(s)$ with multiplicity; `ExplicitA` states (2.3) of Lemma 2.1 as the limit of $-\sum_{\rho \in Z_n} c_\delta(\rho) y^\rho$ along truncations Z_n , for almost every $y \geq 2$, with $B(y)$ included as the function `Bf`, assumed under RH for each $0 < \delta \leq 10^{-13}$; `HadamardK` is Lemma 2.2; `Ordinates14` states that every ordinate has absolute value at least 14; `BPTN` is Lemma 2.4 for $T \geq 10^{13}$; `HSWN70` and `HSWHead14` are [5, Corollary 1.2] read as counts of ordinates of both signs, for $T \geq 70$ and $T \geq 14$; and `EvenGoldbachUpTo` is Lemma 2.6. The bound `SchoenfeldTheta` is not a hypothesis of this theorem. Table 3 lists the status of each hypothesis in the paper and in Lean; the formal verification is conditional on these hypotheses, and the main theorem assumes `ExplicitA`, `HadamardK` and `BPTN` under RH.

 TABLE 3. Hypotheses of `goldbach_445_hsw_gZ` in the paper and in Lean 4.

Lean hypothesis	Source	In this paper	In Lean 4
<code>ExplicitA</code>	[8, Equation (12.1)], Lemma 2.1	Cited	Assumed as hypothesis
<code>HadamardK</code>	Lemma 2.2	Proved under RH	Assumed as hypothesis
<code>Ordinates14</code>	[11, Chapter XV], Lemma 2.3	Cited	Assumed as hypothesis
<code>BPTN</code>	[1, Lemma 1], Lemma 2.4 for $T \geq 10^{13}$	Cited	Assumed as hypothesis
<code>HSWN70</code>	[5, Corollary 1.2] for $T \geq 70$	Cited	Assumed as hypothesis
<code>HSWHead14</code>	[5, Corollary 1.2] for $T \geq 14$	Cited	Assumed as hypothesis
<code>EvenGoldbachUpTo</code>	[9], Lemma 2.6	Cited	Assumed as hypothesis

Running `#print axioms goldbach_445_hsw_gZ` in Lean 4 prints only the standard foundational axioms of Lean and Mathlib: `propext`, `Classical.choice`, and `Quot.sound`. The Lean code was written with AI assistance and checked by the Lean kernel.

Use of AI. The argument and the text of this paper were developed with AI tools; the main deduction is checked in Lean 4 as described in the appendix.

Data availability. The Lean files are available from the author on request. A public release with a commit hash is in preparation.

REFERENCES

- [1] R. P. Brent, D. J. Platt, and T. S. Trudgian, *The mean square of the error term in the prime number theorem*, J. Number Theory **238** (2022), 740–762.
- [2] A. Chirre and M. Valás Hagen, *On Goldbach numbers in short intervals*, Research in Number Theory **12** (2026), article 82, doi:10.1007/s40993-026-00767-6, arXiv:2512.23534.
- [3] M. Cully-Hugill and A. W. Dudek, *An explicit mean-value estimate for the prime number theorem in intervals*, J. Aust. Math. Soc. **117** (2024), no. 1, 1–15.
- [4] D. A. Goldston, *Linnik’s theorem on Goldbach numbers in short intervals*, Glasgow Math. J. **32** (1990), no. 3, 285–297.
- [5] E. Hasanalizade, Q. Shen, and P.-J. Wong, *Counting zeros of the Riemann zeta function*, preprint, arXiv:2107.06506, 2021.
- [6] I. Kátai, *A remark on a paper of Ju. V. Linnik*, Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **17** (1967), 99–100.
- [7] H. L. Montgomery and R. C. Vaughan, *The exceptional set in Goldbach’s problem*, Acta Arith. **27** (1975), 353–370.
- [8] H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory. I. Classical Theory*, Cambridge Studies in Advanced Mathematics, vol. 97, Cambridge University Press, Cambridge, 2007.
- [9] T. Oliveira e Silva, S. Herzog, and S. Pardi, *Empirical verification of the even Goldbach conjecture and computation of prime gaps up to $4 \cdot 10^{18}$* , Math. Comp. **83** (2014), no. 288, 2033–2060.
- [10] L. Schoenfeld, *Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. II*, Math. Comp. **30** (1976), no. 134, 337–360.
- [11] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, 2nd ed., revised by D. R. Heath-Brown, Oxford University Press, New York, 1986.

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